THREE-DIMENSIONAL LATTICE YANG–MILLS THEORY IN A FOREST GAUGE: A DOBRUSHIN RANGE AND A CERTIFIED CURVATURE BOUND

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ABSTRACT. We study Wilson lattice Yang–Mills theory on $\mathbb{Z}^3$ in a gauge that fixes one link at every site along a period-3 forest, in which every free link shares a plaquette with exactly seven other free links. Two results follow. (A) For SU(2), a single-link Dobrushin argument with the sharp one-quarter Kantorovich estimate of Bhattacharjee and Bhattacharya gives a unique infinite-volume state, invariance, reflection positivity, exponential decay of correlations and a transfer-matrix gap $m \geq -\log(7\beta_{\mathrm{W}}/4)$ for $\beta_{\mathrm{W}} < 4/7$. The same estimate gives $\beta_{\mathrm{W}} < 1/3$ in three dimensions without gauge fixing and $\beta_{\mathrm{W}} < 1/2$ in the temporal gauge. As Bhattacharjee, Bhattacharya and Lockwood showed for 1/3 in four dimensions, we show that 4/7 is the ceiling of this argument for every gauge-fixing forest. (B) For every compact group with a faithful unitary representation, the Hessian of $\sum_p \operatorname{Re}\operatorname{Tr} Q_p$ restricted to the free links is bounded in absolute value by $h = 9.599991$, uniformly in the field. The proof is a site-plus-plaquette certificate with rational data: its finite part is checked in exact rational arithmetic, and positivity of a two-parameter family of matrices of size at most 12 is proved with outward-rounded interval arithmetic. With the Bakry–Émery criterion this gives log-Sobolev inequalities, a unique state and an explicit gap for $N|\beta| < \kappa_G/h$; for SU(N) this is $|\beta| < 1/(2h) \approx 1/19.20$, compared with 1/32 of Shen, Zhu and Zhu and 1/24 from the sharp un-gauge-fixed constant. An exact computation at the trivial field shows that no bound below 9.3479 is possible in this gauge. All statements are at strong coupling and fixed lattice spacing.

1. INTRODUCTION

Let $\Lambda \subset \mathbb{Z}^3$ be a box with oriented links E and plaquettes P, and let G be a compact connected Lie group with a faithful unitary representation $\rho : G \to \mathrm{U}(N)$, through which we identify G with a subgroup of U(N). The Wilson measure is

$$\mu_{\Lambda,\beta}(dQ) = Z^{-1} \exp\Big(N\beta \sum_{p\in P} \operatorname{Re}\operatorname{Tr} Q_p\Big) \prod_{e\in E} dQ_e, \tag{1}$$

with normalised Haar measure dQ_e and plaquette holonomy $Q_p = Q_{e_1} Q_{e_2} Q_{e_3}^* Q_{e_4}^*$. For SU($N$) the conventional coupling is $\beta_{\mathrm{W}} = 2N/g^2 = N^2\beta$. We use the bi-invariant metric $\langle X, Y\rangle = -\operatorname{Tr}(XY)$ on the Lie algebra; κ_G denotes the least Ricci curvature of G in this metric ($\kappa_{\mathrm{SU}(N)} = N/2$, $\kappa_{\mathrm{SO}(N)} = (N-2)/4$).

Two methods give a mass gap at strong coupling with explicit constants. The Bakry–Émery method of Shen, Zhu and Zhu [8] needs $\kappa_G > N|\beta| \sup|\operatorname{Hess} S_0|$, where $S_0 = \sum_p \operatorname{Re}\operatorname{Tr} Q_p$; their bound $|\operatorname{Hess} S_0| \leq 8(d-1)$ gives $|\beta| < 1/32$ for SU(N) in $d = 3$. The sharp un-gauge-fixed constant is $4d$ [1], that is 12, giving 1/24. The Dobrushin method compares the conditional laws of single links. For SU(2) the sharp quarter estimate of Bhattacharjee and Bhattacharya [1] gives $\beta_{\mathrm{W}} < 2/9$ in $d = 4$, where a link has 18 partners; the same count in d dimensions, with $6(d-1)$ partners, gives $\beta_{\mathrm{W}} < 2/(3(d-1))$, that is 1/3 in $d = 3$. In unrefereed work [2], Bhattacharjee,

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Bhattacharya and Lockwood showed that both methods improve in four dimensions after fixing the gauge on a forest of links, which reduces the number of partners of a link and the size of the Hessian. In $d = 4$ they obtain $\beta_{\mathrm{W}} < 1/3$ for SU(2) with single links, prove that $1/3$ is the ceiling of the single-link argument in every forest gauge, and pass it with blocks of three links ($\beta_{\mathrm{W}} \le 0.345$); they bound $|\operatorname{Hess} S_0|$ by $29/2$ for every N; and for SU(3) they prove a single-link Kantorovich estimate, certified by a rigorous computation, that gives $\beta_{\mathrm{W}} \le 3/4$, and 0.76 with blocks [2, Thms. 1–7]. We carry out the corresponding programme in three dimensions for the single-link and curvature methods, where the combinatorics are different and, as it turns out, more favourable.

Results. Section 2 describes a period-3 forest F on $\mathbb{Z}^3$, found and analysed in our earlier note [7], with the following properties: every site is the head of exactly one link of F; every plaquette contains one or two links of F, in proportion $2 : 1$; and every free link shares a plaquette with exactly seven other free links. The temporal gauge gives eight, and no gauge fixing gives twelve. The counts are checked in Lean 4.

Theorem 1.1 (SU(2), Dobrushin)**.** *Let $G = \mathrm{SU}(2)$ and $0 \le \beta_{\mathrm{W}} < \frac{4}{7}$. For every gauge-invariant local observable O the expectations $\mu_{\Lambda,\beta}(O)$ converge, along every sequence of boxes exhausting $\mathbb{Z}^3$, to a limit $\omega(O)$. The state ω is invariant under translations and lattice rotations and reflections, and it is reflection positive. Truncated correlations of Lipschitz gauge-invariant observables decay at every rate below $-\log(7\beta_{\mathrm{W}}/4)$, and the transfer matrix has spectral gap*

$$m \;\ge\; -\log \tfrac{7\beta_{\mathrm{W}}}{4} \;> 0.$$

For $-\frac{4}{7} < \beta_{\mathrm{W}} < 0$ all statements except reflection positivity and the gap hold. For every gauge-fixing forest the single-link Kantorovich coefficients of SU(2) *violate the weighted Dobrushin condition at $|\beta_{\mathrm{W}}| \ge \frac{4}{7}$ (Proposition 4.1).*

Theorem 1.2 (Curvature in the gauge)**.** *For every field $Q \in \mathrm{U}(N)^E$ with $Q_e = I$ on F and every finitely supported tangent vector $v = (X_e Q_e)$ with $X_e = 0$ for $e \in F$ (here E is the set of links of $\mathbb{Z}^3$),*

$$|\operatorname{Hess} S_0(v, v)| \;\le\; h\,|v|^2, \qquad h = \tfrac{9599991}{1000000} = 9.599991.$$

The same holds for every closed subgroup $G \subset \mathrm{U}(N)$. Consequently, if $N|\beta| < \kappa_G/h$, the qualitative conclusions of Theorem 1.1 hold (limit, invariance, and for $\beta \ge 0$ reflection positivity), with the rate and gap given below, the gauge-fixed measures satisfy a log-Sobolev inequality uniformly in the box, and correlations decay at every rate below

$$m_G(\beta) = \operatorname{arccosh}\Big(1 + \tfrac{1}{7}\Big(\tfrac{\kappa_G}{N|\beta|} - h\Big)\Big),$$

which is also a lower bound for the gap when $\beta > 0$. For SU(N) *the condition is $|\beta| < 1/(2h) = 1/19.199982$: $\beta_{\mathrm{W}} < 0.2083$ for* SU(2) *and $\beta_{\mathrm{W}} < 0.4687$ for* SU(3)*.*

Proposition 1.3 (Lower bound)**.** *For $G = \mathrm{SU}(N)$, $N \ge 2$, and either sign, the supremum of $\pm\operatorname{Hess} S_0(v, v)/|v|^2$ over boxes, fields with $Q_F = I$ and free tangent vectors is at least $9.347955586999\ldots$, the largest root of*

$$x^9 - 35x^8 + 505x^7 - 3901x^6 + 17538x^5 - 46770x^4 + 72344x^3 - 60490x^2 + 22788x - 1944.$$

So the optimal constant h_F in this gauge lies in $[9.3479,\ 9.6000]$, and no curvature argument in this gauge based on a bound for $\sup|\operatorname{Hess} S_0|$ goes beyond $|\beta| < 1/18.6959$ for SU(N).

TABLE 1. Strong-coupling ranges for three-dimensional lattice Yang–Mills theory in the conventional coupling $\beta_{\mathrm{W}} = N^2\beta$. Entries marked * rest on unrefereed results. The SU(3) Dobrushin estimate of [2, Thm. 6] is proved in $d = 4$ only and is not listed; see the paragraph on prior work below.

Method	SU(2)	SU(3)
Bakry–Émery, $\lvert \operatorname{Hess} S_0 \rvert \le 16$ [8]	0.125	0.281
Bakry–Émery, sharp constant 12 [1]*	0.167	0.375
Bakry–Émery, forest and plaquette-wise Cauchy–Schwarz, $c_F = 11$ [7]	0.182	0.409
Bakry–Émery, forest and certificate, $h = 9.599991$ (Theorem 1.2)*	0.208	0.468
ceiling for any curvature bound in this gauge, $h_F \ge 9.3479$	0.214	0.481
Dobrushin, sharp quarter, no gauge fixing [1]*	0.333	–
Dobrushin, sharp quarter, temporal gauge (Remark 4.2)*	0.5	–
Dobrushin, sharp quarter, forest F (Theorem 1.1)*	0.571	–

Relation to prior work. The forest idea, the two methods and the idea of a ceiling for the single-link argument in every forest gauge are those of [2, 1], which treat $d = 4$ (and, without gauge fixing, general d). We have not found three-dimensional gauge-fixed results there or elsewhere. As far as we know, the following are new: the forest F with seven partners and its optimality, Theorem 1.1, the value $4/7$ of the ceiling in $d = 3$, the certificate method of Section 5, and the bounds $9.3479 \le h_F \le 9.6000$. Theorem 1.1 depends on the sharp quarter of [1] and on the passage from a Dobrushin bound to a state with a gap in [2], both unrefereed. We have checked the steps we use and state precisely what we take from them (Section 3). Theorem 1.2 as a Hessian bound is self-contained. For the consequences we use the Bakry–Émery criterion [3] and the decay and limit arguments of [8, 2].

For SU(3), Theorem 1.2 is not the strongest result one should expect. The single-link estimate of [2, Thm. 6] bounds the Kantorovich modulus of the one-link law of SU(3) uniformly over staple sums, and in $d = 4$ every free link has twelve partners and lies in six plaquettes. In $d = 3$ without gauge fixing a link also has twelve partners but lies in only four plaquettes, and in the gauge F it has seven partners. If the modulus bound is monotone in the size of the staple sum, as its construction suggests, the argument should give at least $\beta_{\mathrm{W}} \le 3/4$ for three-dimensional SU(3). We have not checked the certified computation behind it, and we do not claim this. The interest of Theorem 1.2 lies in its uniformity in the group, in the log-Sobolev inequality, and in the method.

What this is not. All results are at strong coupling on a lattice of fixed spacing. They do not address the continuum limit or the Clay problem [5]. In three dimensions the continuum theory is believed to be gapped, but nothing here is uniform in the lattice spacing.

2. The forest

Label the directions $0, 1, 2$. For $\mu : \mathbb{Z}^3 \to \{0, 1, 2\}$ let $F_\mu = \{(x - e_{\mu(x)}, \mu(x)) : x \in \mathbb{Z}^3\}$, where (y, a) is the link from y to $y + e_a$. The link (y, a) is fixed iff $\mu(y + e_a) = a$.

Lemma 2.1. *F_μ contains no cycle, for every μ.*

Proof. On a cycle in F_μ, let x be a site of largest coordinate sum. Both links of the cycle at x join x to a site of smaller coordinate sum, so both are of the form $(x - e_a, a)$ and have head x. But x is the head of only one link of F_μ. □

We take μ 3-periodic, with values on $\{0,1,2\}^3$ (lexicographic order, first coordinate most significant) given by the string `210102120102120210120210102`, and write $F = F_\mu$. In a box, $F \cap E(\Lambda)$ is a forest. Two distinct links share at most one plaquette. Call two free links *partners* if they share a plaquette.

Proposition 2.2. *Per period cell there are* 81 *plaquettes,* 54 *with one link of* F *and* 27 *with two, and* 54 *free links. Each free link lies in four plaquettes and has exactly seven partners in* $\mathbb{Z}^3$*, and at most seven in a box. In the temporal gauge* $\mu \equiv 0$ *every free link has eight partners, and without gauge fixing it has twelve.*

Proof. A finite check over one period cell, done by an independent Python script and by `decide +kernel` in Lean 4 (Appendix A); the proofs pass under Lean 4.19.0 (no axioms) and Lean 4.35.0 (only `propext`). A free link e in a plaquette with a_p free links has $a_p - 1$ partners there, and the Lean file checks that $\sum_{p \ni e} a_p = 11$, so $\sum_{p \ni e}(a_p - 1) = 7$. In a box, plaquettes are only removed. □

3. Gauge fixing and the infinite-volume state

Lemma 3.1 ([2, 7])**.** *Let* $\mu^F_{\Lambda,\beta}$ *be the measure on the free links* $E' = E(\Lambda) \setminus F$ *with density proportional to* $\exp(N\beta S_0(I_F, Q'))$*. For every bounded measurable gauge-invariant* O*,* $\mu_{\Lambda,\beta}(O) = \mu^F_{\Lambda,\beta}(O(I_F, \cdot))$*.*

The proof is the usual one: on each tree of the forest, a gauge transformation depending only on Q_F sets the links of F to I. Haar measure is invariant under the resulting translations of the free links.

On $\mathbb{Z}^3$, let γ^F be the specification on the free links in which γ^F_e resamples Q_e from the law with density proportional to $\exp(N\beta \sum_{p \ni e} \operatorname{Re}\operatorname{Tr} Q_p)$ given the other free links, with $Q_F = I$. Fix a bi-invariant metric δ on G (for SU(2) the chordal metric on S^3). The Kantorovich coefficient of e' on e is

$$C_{ee'} = \sup \frac{W_1(\gamma^F_e(\cdot \mid Q), \gamma^F_e(\cdot \mid Q'))}{\delta(Q_{e'}, Q'_{e'})},$$

over pairs of configurations that differ only at e'. It vanishes unless e, e' are partners.

Proposition 3.2 ([2, Prop. 11], three-dimensional version)**.** *If every row and column of* $(C_{ee'})$ *sums to at most* $\alpha < 1$*, then:* γ^F *has exactly one Gibbs measure* μ_F*; for every gauge-invariant local* O *the limit* $\omega(O) = \lim_\Lambda \mu_{\Lambda,\beta}(O)$ *exists along every sequence of boxes exhausting* $\mathbb{Z}^3$ *and equals* $\mu_F(O(I_F, \cdot))$*;* ω *is invariant under translations and the lattice symmetry group; for* $\beta \geq 0$ *it is reflection positive in hyperplanes through sites and through link midpoints; truncated correlations of gauge-invariant local Lipschitz observables decay at every rate* $\lambda < -\log\alpha$*; and for* $\beta \geq 0$ *the transfer matrix has spectral gap at least* $-\log\alpha$*.*

Comments on the proof. The proof in [2] is written for $\mathbb{Z}^4$ but uses the dimension only in one place. That place is the count of active links within distance R of an observable, which grows like R^4 there and like R^3 here. The ingredients are as follows. The Dobrushin comparison lemma is dimension-free. The free-boundary limit is defined through Lemma 3.1, so the forest is never translated and invariance follows because every translate of an exhausting sequence of boxes is again exhausting. Reflection positivity holds for the un-gauge-fixed measures in reflection-symmetric boxes [6] and passes to the limit. The gap follows from decay of correlations and reflection positivity by the Osterwalder–Schrader reconstruction [6]. The forest never has to be imposed on a torus, where it would close into cycles. □

4. SU(2) up to four sevenths

Proof of Theorem 1.1. Identify SU(2) with the unit quaternions, with the chordal metric. For a free link e and $p \ni e$ let s_p be the staple (the product of the other three links of p, with $Q_F = I$), oriented so that $N\beta \operatorname{Re}\operatorname{Tr} Q_p = 4\beta\langle Q_e, \bar{s}_p\rangle_{\mathbb{R}^4}$. The conditional law of Q_e is the von Mises–Fisher law $\nu_b(dx) \propto e^{\langle x,b\rangle}dx$ on S^3 with $b = 4\beta\sum_{p\ni e}\bar{s}_p$. Changing a partner e' by δ in the chordal metric moves one staple by the same chordal distance (multiplication by unit quaternions is an isometry of $\mathbb{R}^4$), so b moves by at most $4|\beta|\delta$. The sharp quarter [1], [2, Thm. 91], $W_1(\nu_b,\nu_{b'}) \le \frac14|b-b'|$, gives $C_{ee'} \le |\beta|$. By Proposition 2.2 every row and column has at most seven nonzero entries, so $\alpha \le 7|\beta| = \frac74|\beta_{\mathrm{W}}|$, and Proposition 3.2 applies for $|\beta_{\mathrm{W}}| < \frac47$. □

Proposition 4.1 (Ceiling)**.** *Let F' be any set of links of $\mathbb{Z}^3$ without cycles. For* SU(2) *in the gauge F', the Dobrushin coefficient of two free links that share a plaquette is exactly $|\beta|$. If a positive weight w satisfies $Cw \le \alpha w$, then $\alpha \ge 7|\beta|$. So no single-link Kantorovich argument reaches $|\beta_{\mathrm{W}}| = \frac47$ in any forest gauge, and in the gauge F the bound $\frac47$ is the limit of the argument. This is the three-dimensional analogue of* [2, Thm. 2].

Proof. For the box Λ_n of side n let C_n be the principal submatrix on its free links. Then $C_n w \le \alpha w$ there, and the Collatz–Wielandt inequality gives $\rho(C_n) \le \alpha$. Since C_n is symmetric, $\rho(C_n)$ is at least its mean row sum. The box has $3n^3 - O(n^2)$ links, $P = 3n^3 - O(n^2)$ plaquettes and n^3 sites. A forest has fewer links than sites, so the number a of free links is at least $2n^3 - O(n^2)$. Let a_p be the number of free links of p; then $\sum_p a_p = 4a - O(n^2)$, and the row sums of C_n add up to $|\beta|\sum_p a_p(a_p-1)$. For integers $k(k-1) - (4k-6) = (k-2)(k-3) \ge 0$, hence

$$\frac{|\beta|}{a}\sum_p a_p(a_p-1) \;\ge\; \frac{|\beta|}{a}\big(4\textstyle\sum_p a_p - 6P\big) \;=\; |\beta|\Big(16 - \frac{6P}{a}\Big) - o(1) \;\ge\; 7|\beta| - o(1).$$

So $\alpha \ge 7|\beta|$, once the coefficients are shown to equal $|\beta|$. (Plain convexity would only give $\frac{20}{3}|\beta|$; integrality matters.)

The proof of Theorem 1.1 gives $C_{ee'} \le |\beta|$ in every forest. For the reverse inequality let e be a free link and e' a partner, sharing the plaquette p_0. The staples of the four plaquettes through e use pairwise disjoint sets of links. At most one staple consists of fixed links only, since two such staples together with the endpoints of e would close a cycle in F'. A staple containing a free link takes every value in SU(2) as that link varies, independently of the other staples. The staple s_0 of p_0 contains e'. Fix $s_0 = u$ and choose the other staples so that $b = 0$: if one of them is fixed, it is ±1 and the other two must add up to $-u \mp 1$, which has norm at most 2 and is therefore a sum of two unit quaternions; if none is fixed, the three remaining staples can add up to $-u$. Varying e' moves s_0 along any tangent direction, and b by 4β times that displacement. At $b = 0$ the derivative of ν_b in a direction h has Kantorovich norm exactly $\frac14|h|$ [1], [2, Thm. 91], so $C_{ee'} \ge |\beta|$. □

Remark 4.2. In the temporal gauge every free link has eight partners (Proposition 2.2), and the same proof gives $|\beta_{\mathrm{W}}| < \frac12$.

5. A certified curvature bound

5.1. **Plaquette forms.** Let $p = (e_1,e_2,e_3,e_4)$ with $e_1 = (x,\mu)$, $e_2 = (x+e_\mu,\nu)$, $e_3 = (x+e_\nu,\mu)$, $e_4 = (x,\nu)$, and perturb $Q_i \mapsto e^{tX_i}Q_i$. Moving the exponentials through the links gives

$$\operatorname{Re}\operatorname{Tr} Q_p(t) = \operatorname{Re}\operatorname{Tr}\big(e^{tY_1}e^{tY_2}We^{-tY_3}e^{-tY_4}\big),$$
$$Y_1 = X_1, \quad Y_2 = \operatorname{Ad}(Q_1)X_2, \quad Y_3 = \operatorname{Ad}(Q_4)X_3, \quad Y_4 = X_4,$$

with $W = Q_p$. Let $f_p(Y, W)$ be the second derivative at $t = 0$, so that $\operatorname{Hess} S_0(v, v) = \sum_p f_p$. It is a real quadratic form in $Y \in \mathfrak{u}(N)^4$, real-linear in W, and $f_p(Y, -W) = -f_p(Y, W)$.

5.2. **Site forms.** At a site y consider the six *ends*

$$o_\mu(y) = X_{(y,\mu)}, \qquad i_\mu(y) = \operatorname{Ad}(Q^*_{(y-e_\mu,\mu)})X_{(y-e_\mu,\mu)}.$$

Ends of fixed links are zero. A direct computation shows that the four perpendicular pairs of ends at the corners of p are

$$\langle Y_1, Y_4\rangle = \langle o_\mu, o_\nu\rangle(x), \qquad \langle Y_1, Y_2\rangle = \langle i_\mu, o_\nu\rangle(x + e_\mu),$$
$$\langle Y_3, Y_4\rangle = \langle o_\mu, i_\nu\rangle(x + e_\nu), \qquad \langle Y_2, WY_3W^*\rangle = \langle i_\nu, i_\mu\rangle(x + e_\mu + e_\nu),$$

and every perpendicular pair of ends at a site occurs at exactly one corner of one plaquette. Inside a box this plaquette lies in the box.

Lemma 5.1. *Let G_y be real symmetric positive semidefinite 6×6 matrices indexed by the ends at y, with $G_y[o_\mu, i_\mu] = 0$. For each plaquette let $c_p(Y, W)$ be the sum over its four corners of $2G_{\text{corner}}[a, b]$ times the corresponding inner product above. Suppose there are numbers $a_{p,r} \geq 0$, for the free links r of p, such that for every $W \in \mathrm{U}(N)$ and Y with $Y_r = 0$ on fixed links*

$$f_p(Y, W) + c_p(Y, W) \;\leq\; \sum_r a_{p,r}|Y_r|^2. \tag{2}$$

Then $|\operatorname{Hess} S_0(v, v)| \leq h|v|^2$ with $h = \max_e \big(\sum_{p\ni e} a_{p,e} + G_{\text{tail}(e)}[o, o] + G_{\text{head}(e)}[i, i]\big)$.

Proof. Since $G_y \geq 0$ and the Gram matrix of the ends is positive semidefinite,

$$0 \leq \sum_y \sum_{a,b} G_y[a, b]\langle \text{end}_a, \text{end}_b\rangle = \sum_p c_p + \sum_e \big(G_{\text{tail}(e)}[o, o] + G_{\text{head}(e)}[i, i]\big)|X_e|^2.$$

Summing (2) over p, with $|Y_r| = |X_r|$, gives $\operatorname{Hess} S_0(v, v) \leq h|v|^2$. Replacing W by $-W$ in (2) changes the sign of f_p and leaves c_p unchanged, since WY_3W^* is unchanged. This gives the other sign. □

5.3. **Reduction to** $\mathrm{U}(2)$**.** Conjugating all Y_i and W by one unitary changes nothing, so we may take $W = \operatorname{diag}(w_1, \dots, w_N)$. Every term of $f_p + c_p$ has the form $\operatorname{Re}\operatorname{Tr}(Y_iD_1Y_jD_2)$ with D_1, D_2 diagonal, that is $\operatorname{Re}\sum_{a,b}(Y_i)_{ab}(Y_j)_{ba}\,\phi_{ab}$, where ϕ_{ab} depends only on w_a, w_b. So (2) splits into independent inequalities, one for each diagonal entry a and one for each pair $\{a, b\}$ of off-diagonal entries, and $|Y|^2$ splits the same way. Each piece is the restriction of the $\mathrm{U}(2)$ inequality at $W = \operatorname{diag}(w_a, w_b)$ to a subspace. Hence (2) for $\mathrm{U}(2)$ and all $W = \operatorname{diag}(e^{i\varphi_1}, e^{i\varphi_2})$ implies it for every $\mathrm{U}(N)$, and by restriction for every $G \subset \mathrm{U}(N)$. In real coordinates (an orthonormal basis of $\mathfrak{u}(2)$) it states that a symmetric matrix $M_p(\varphi)$ of size $4k_p$ is positive semidefinite, $k_p \in \{2, 3\}$ being the number of free links of p.

5.4. **The certificate.** By periodicity it suffices to give G_y for the 27 sites and $a_{p,r}$ for the 81 plaquettes of a period cell. We found them by semidefinite programming with cutting planes in φ and rounded them to rationals with denominator 10^6. The following checks were done in exact rational arithmetic: each G_y is positive definite (positive pivots in an LDL^T factorisation); $G_y[o_\mu, i_\mu] = 0$; $a \geq 0$; and the per-link maximum in Lemma 5.1 is $h = 9599991/1000000$.

It remains to check $M_p(\varphi) \geq 0$ for all $\varphi \in [0, 2\pi)^2$. For $|\varphi_a - \varphi'_a| \leq r$ one has $\|W - W'\|_{\text{op}} \leq r$. Since f_p is linear in W, $|f_p(Y, W) - f_p(Y, W')| \leq (\sum_i |Y_i|)^2\|W - W'\| \leq 4|Y|^2\|W - W'\|$, and the far corner moves by at most $|k_{23}| \cdot 2|Y_2||Y_3|\|W - W'\|$, where k_{23} is its coefficient. So

$$\lambda_{\min}M_p(\varphi') \;\geq\; \lambda_{\min}M_p(\varphi) - (4 + |k_{23}|)\, r.$$

We cover the torus of phases by squares, starting from a 64×64 grid and subdividing adaptively; a square of half-width r is accepted when $M_p-(4+|k_{23}|)r\,I$ is positive definite at its centre. This check is done in outward-rounded interval arithmetic. The matrices D_p, C_{14}, C_{12}, C_{34} and the coefficient matrices of the trigonometric terms are exact rationals (computed symbolically and exactly symmetric), and so are the k's. The values of cos and sin at the centres, which are rational multiples of π, are enclosed by the ball arithmetic of Arb. All operations are done on pairs of doubles, with every endpoint moved outward by one ulp after each operation, and the shift $(4+|k_{23}|)r$ is rounded up. Gaussian elimination without pivoting is then carried out in this interval arithmetic: if every pivot interval is positive, every matrix in the interval matrix has positive pivots, so the symmetric matrix $M_p-(4+|k_{23}|)rI$ is positive definite. All 81 plaquettes were accepted with 26,371,504 squares and at most six subdivisions. This proves $M_p(\varphi)>0$ for every φ.

Remark 5.2 (Checks of the cover). A second mode of the same program encloses cos and sin over the whole square instead of at its centre, which needs no Lipschitz estimate. An earlier floating-point run of the cover (LAPACK eigenvalues) accepted the same squares, with least excess 4.3×10^{-8} over the Lipschitz shift; a grid evaluation gives $\min_{p,\varphi}\lambda_{\min}M_p(\varphi)\approx0.0079$. The same floating-point pipeline also accepts certificates for $h=9.699992$ and $h=10.000001$. The same program re-checks the finite part of the certificate file in exact rational arithmetic (positive definiteness of every G_y, $a\ge0$, the per-link maximum h) independently of the search. As adversarial tests, the Lipschitz ratio $\|M_p(\varphi)-M_p(\varphi')\|/((4+|k_{23}|)\max_a|\varphi_a-\varphi'_a|)$ never exceeded 0.75 on 162,000 random pairs; and lowering the weights of the tightest plaquette by 0.016, which makes its matrix indefinite, is rejected by both modes, while lowering them by 0.004 is accepted.

Proof of Theorem 1.2. The Hessian bound is Lemma 5.1 with the certificate and the reduction above. A box uses only the plaquettes inside it, which only lowers the per-link totals, and the corner pairs of its links lie in its plaquettes. The product $G^{E'}$ has Ricci curvature at least κ_G, so the Bakry–Émery condition holds with $K=\kappa_G-hN|\beta|>0$, uniformly in the box. This gives the log-Sobolev inequality [3].

For decay, the limit and the gap we follow the proof of [2, Thm. 5]. It rests on a weighted gradient bound [2, Lem. 24] with weights $w_e=e^{\lambda d_e}$, where d_e is the graph distance from e to the support of the observable in the graph whose edges join partners, so that $|d_e-d_{e'}|\le1$ for partners. The constant is

$$K_\lambda=\kappa_G-N|\beta|\big(h+P(\cosh\tfrac{\lambda}{2}-1)\big),$$

where P is the number of partners and the off-diagonal Hessian blocks have norm at most $N|\beta|$. Here $P=7$ (Proposition 2.2). The interpolations $(1-s)S_\Lambda+sS_{\Lambda'}$ between boxes, used for the limit, satisfy the same Hessian bound because it is linear. The count of links at graph distance k grows like k^2 instead of k^3. For $\lambda<\lambda^*$, where $K_{\lambda^*}=0$, [2, Lem. 25] gives decay $e^{-\lambda D/2}/K_\lambda$ at distance D, so the decay rate is $m_G(\beta)=\lambda^*/2$. Reflection positivity, invariance and the gap follow as in Proposition 3.2. □

Remark 5.3. For SU(2), Theorem 1.1 has the larger range. Theorem 1.2 is the result for $N\ge3$ and for other groups. For SU(3) in four dimensions [2, Thm. 6] proves a single-link Kantorovich estimate by a certified computation; we have not adapted it to three dimensions. For SO(N), $N\ge3$, the condition reads $|\beta|<(N-2)/(4Nh)$.

6. The lower bound

Proof of Proposition 1.3. Take $Q\equiv I$ and $X_e=\xi_eZ$ with $Z\in\mathfrak{su}(2)\subset\mathfrak{su}(N)$, $|Z|=1$, $\xi_e=0$ on F. Then $f_p=\operatorname{Re}\operatorname{Tr}((d\xi)_p^2Z^2)=-(d\xi)_p^2$, so $-\operatorname{Hess}S_0(v,v)=\|d\xi\|^2$. For the other sign let

$J = \mathrm{diag}(-1,-1,1,\dots,1)$ and choose, in a box, a $\{I,J\}$-valued field with every plaquette equal to J (e.g. $Q_{(x,\mu)} = J^{x_0+\cdots+x_{\mu-1}}$). Gauge-fix it on F with a $\{I,J\}$-valued gauge transformation, as in Lemma 3.1; this leaves the plaquettes equal to J. Since $\mathrm{Ad}(J)Z = Z$ and $JZ^2 = -Z^2$, now $\mathrm{Hess}\, S_0(v,v) = +\|d\xi\|^2$. So both signs reach $\sup \|d\xi\|^2/\|\xi\|^2$ over free ξ. Truncated Bloch waves on large boxes show that this supremum is at least the top eigenvalue of the Bloch matrix $B(k)^*B(k)$ at any momentum k. At $k = (\pi,\pi,\pi)$ with respect to the period-3 cell, B is an integer 81×54 matrix. The characteristic polynomial of B^TB factors over $\mathbb{Q}$ into $(x-6)^2(x-4)^2$, two squared octics and two nonics. The largest root, isolated by exact interval refinement to 10^{-12}, is that of the nonic in the statement. □

Remark 6.1 (Numerical evidence on the true constant). Gradient ascent of $\pm\lambda_{\mathrm{ext}}$ of the gauge-fixed SU(2) Hessian, over Bloch-periodic fields from random, abelian and near-identity starts, found no field above 9.348. Abelian fields on the 3-periodic torus reach 9.048, and the best values for the + sign from generic starts were 8.24–8.74. On 41 random, abelian and trivial fields all eigenvalues lie in $[-8.58, 6.19]$. This suggests $h_F = 9.3479\ldots$, attained at the trivial (and J-flux) field, but we have no proof.

7. Checks

Besides the exact parts of the proof, the following independent checks were made.

(1) The rewriting of $\mathrm{Re}\,\mathrm{Tr}\, Q_p(t)$ and the identity behind Lemma 5.1, summed over the 3-periodic torus, agree with an independently coded Hessian on random SU(2) fields to 5.7×10^{-14}.
(2) The inequality (2) with the certified data holds at 4000 random U(3) holonomies and $\mathfrak{u}(3)$ vectors, with least normalised slack 0.338. This checks the reduction to U(2).
(3) The forest counts are checked in Lean 4 and in Python. Lean also checks the control count in the temporal gauge.

8. Discussion

In four dimensions every plaquette of the best forest has exactly one fixed link and every free link has twelve partners [2]. In three dimensions the best forests mix plaquettes with one and with two fixed links, and the integrality argument of Proposition 4.1 shows that seven partners is optimal. This is why the three-dimensional gain is larger: from $\frac{1}{3}$ to $\frac{4}{7}$ for SU(2), and from $\frac{1}{24}$ to $\frac{1}{19.2}$ in the curvature range. The certificate method of Section 5 is general. It replaces a hand-made Hessian estimate by a finite semidefinite problem with an exact check, and it could be applied to other gauges, other dimensions or plaquette blocks. Closing the remaining gap between 9.3479 and 9.6000 is a finite but larger computation. None of this approaches the weak-coupling regime relevant to the continuum limit.

Appendix A. Files

The code, certificates and logs are in the public repository [4],

https://github.com/romanov360/yang-mills-3d-forest-gauge.

Its top-level directory contains:

- `lean/ForestGauge3.lean`: the counts of Proposition 2.2 by `decide +kernel` (checked under Lean 4.19.0 and 4.35.0; at most the axiom `propext`).
- `torus3d.py`, `structure3d.py`: the forest, the partner counts and the gauge-fixed Hessian.

- `certify3d.py`: search, exact rounding and the all-phase check. `cert_h9.6.npy`, `cert_h9.7.npy` and `cert_h10.npy` hold the rational certificates, and the `.log` files of the same names the runs; `cert_h9.5.log` records a search at 9.5 (margin 10^{-3}, finitely many cuts) that found no certificate.
- `cert_interval.py`: the interval-arithmetic proof of positivity for all phases; the runs are in the `cert_interval_*.log` files, the exact finite check in `cert_finite.out`; `adversarial_interval.py` (output `adversarial_interval.out`) runs the tests of the remark above.
- `validate_cert.py`: the checks of the last section.
- `identity_exact.py`, `bloch_identity.py`: the exact lower bound.
- `bloch3d.py`, `climb3d.py`: the numerical searches for the true constant.

The directory `earlier-note` holds the note [7] with its code (including the Hessian routines `hessian.py` and `fasthess.py` that `torus3d.py` imports) and outputs.

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