

A DESCENT-FORCED VARIANT OF THE COLLATZ MAP: Definition, Termination, and the Minimal-Counterexample Method

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Abstract

We study a deterministic procedure, denoted $\mathcal{P}$, built from three operations: $3n + 1$ on odd values, division by two (d2) on even values, and a correction step -1 that is applied whenever the current odd value exceeds the previous odd value. The correction keeps the sequence of odd values below its starting point, so the “lower part” of every start is maintained by construction. We prove that for each odd start $n \geq 3$ the procedure reaches an odd value $G(n) < n$ in a number of operations determined by a 2-adic valuation, that G has a closed form, and that iterating G reaches 1 in at most $\lfloor \log_{4/3}(n - 1) \rfloor + 1$ descents. We then show that $\mathcal{P}$ coincides with the Collatz map only until its first correction, and we apply the minimal-counterexample (contradiction) method. The argument closes if and only if a single merging statement holds: for every $y \equiv 2 \pmod{3}$ the Collatz orbits of y and $2y + 1$ meet. This statement is equivalent to the Collatz conjecture, so $\mathcal{P}$ locates the difficulty precisely but does not remove it. Worked examples are tabulated.

1 Introduction

The Collatz (or $3x + 1$) map sends an even integer x to $x/2$ and an odd integer x to $3x + 1$. The Collatz conjecture asserts that every positive integer eventually reaches 1 and hence enters the cycle $4 \rightarrow 2 \rightarrow 1$. Surveys of the problem are given in [4, 5]. Computer searches have confirmed the conjecture for all starting values up to at least 2^{60} [1], a limit that has since been extended [2].

A standard reformulation uses the *stopping time*: the number of steps until the orbit of n first falls below n . If every $n > 1$ has a finite stopping time, then strong induction shows that every n reaches 1, because all smaller numbers are already known to do so [7, 4]. This is the idea behind the “lower part” of an orbit: when a start n is examined, every number below n is taken to be proven, and the orbit of n needs to be followed only until it reaches a number below n .

This paper examines a procedure that enforces this lower-part idea directly. Whenever an odd value is larger than the previous odd value, one is subtracted so that the value becomes even and can be halved. We call this the *correction step*. Our aims are:

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1. to define the procedure $\mathcal{P}$ precisely and prove that it terminates, with an explicit bound (Section 3);
2. to state exactly how $\mathcal{P}$ relates to the Collatz map (Section 4);
3. to illustrate the procedure in tables (Section 5);
4. to apply the minimal-counterexample method and identify the one statement that would be needed to transfer the termination of $\mathcal{P}$ to the Collatz map (Section 6).

The termination of $\mathcal{P}$ is unconditional. The relevance of $\mathcal{P}$ to the Collatz conjecture is governed entirely by the statement isolated in Section 6.

2 Notation and preliminaries

Let $\mathbb{N} = \{1, 2, 3, \dots\}$. For $x \in \mathbb{N}$ let $\nu_2(x)$ denote the exponent of the largest power of 2 dividing x , and let $\text{odd}(x) = x/2^{\nu_2(x)}$ be the odd part of x . The Collatz map is

$$C(x) = \begin{cases} x/2, & x \text{ even,} \\ 3x + 1, & x \text{ odd.} \end{cases}$$

The *orbit* of x is $\{C^j(x) : j \geq 0\}$. We write $x \rightsquigarrow 1$ if the orbit of x contains 1. The set of counterexamples is $\mathcal{C} = \{x \in \mathbb{N} : x \not\rightsquigarrow 1\}$, and the Collatz conjecture states that $\mathcal{C} = \emptyset$.

Lemma 1 (Reduction to a lower part). *Let $n \geq 2$ and suppose $m \rightsquigarrow 1$ for every $m < n$. If the orbit of n contains some value below n , then $n \rightsquigarrow 1$.*

Proof. If $C^j(n) = m < n$, then $m \rightsquigarrow 1$ by hypothesis, and the orbit of n contains the orbit of m . $\square$

Since $C(x) = 3x + 1$ is even for odd x , a $3x + 1$ move is always followed by at least one halving. We use three named operations, listed in Table 1.

Name	Action	Applied to
3n+1	$x \mapsto 3x + 1$	odd x (not a rise)
d2	$x \mapsto x/2$	even x
-1	$x \mapsto x - 1$	odd x larger than the previous odd value

Table 1: The three operations of the procedure.

3 The procedure and its main properties

Definition 2 (Procedure $\mathcal{P}$). Let $n \geq 3$ be odd. The procedure keeps a pair (x, p) , where x is the current value and p is the previous odd value. Initially $(x, p) = (n, n)$. Repeat the following until the stopping condition holds.

1. If x is even, apply d2.
2. If x is odd, record $p \leftarrow x$ after performing one of:

- if $x > p$ (a *rise*), apply -1 (the result is even);
- otherwise apply $3n+1$.

Stopping condition: the current value x is odd and $x < n$. In that case n is *proven by* x .

The value of p used in the comparison is the previous odd value as it stood *before* the correction, as in the worked tables below.

Definition 3 (Descent map). For odd $n \geq 3$ let $\varepsilon(n) = +1$ if $n \equiv 1 \pmod{4}$ and $\varepsilon(n) = -1$ if $n \equiv 3 \pmod{4}$, so that $4 \mid 3n + \varepsilon(n)$. Define

$$G(n) = \text{odd}\left(\frac{3n + \varepsilon(n)}{4}\right).$$

Proposition 4 (Closed form of one descent). *Let $n \geq 3$ be odd. Then $\mathcal{P}$ halts with value $G(n) < n$, and:*

- (a) *If $n \equiv 1 \pmod{4}$, the operations are $3n+1$ followed by $\nu_2(3n+1) \geq 2$ copies of $d2$. All of these are Collatz moves, and the number of operations is $1 + \nu_2(3n+1)$.*
- (b) *If $n \equiv 3 \pmod{4}$, write $n = 4k + 3$. The operations are $3n+1$, $d2$, -1 , followed by $1 + \nu_2(3k+2)$ copies of $d2$, and the number of operations is $4 + \nu_2(3k+2)$.*

Proof. (a) Let $n = 4k + 1$. Then $3n + 1 = 4(3k + 1)$. The first operation is $3n+1$. All intermediate values $(3n + 1)/2^i$ with $i < \nu_2(3n + 1)$ are even, so no stopping occurs and no rise occurs. The first odd value is $(3n + 1)/2^{\nu_2(3n+1)} = \text{odd}(3k + 1) = G(n)$, which is at most $(3n + 1)/4 < n$.

(b) Let $n = 4k + 3$. Then $3n + 1 = 12k + 10$, and $3n+1$ gives the even value $12k + 10$. Applying $d2$ gives $x = 6k + 5$, which is odd and larger than n , so it is a rise (the previous odd value is n) and the stopping condition fails. The operation -1 gives $6k + 4 = 2(3k + 2)$. Subsequent halvings produce even values until the odd value $\text{odd}(3k + 2) = G(n)$ is reached after $\nu_2(6k + 4) = 1 + \nu_2(3k + 2)$ copies of $d2$. Finally $G(n) \leq 3k + 2 < 4k + 3 = n$. $\square$

Theorem 5 (Termination with an explicit bound). *Define the full procedure $\mathcal{P}^*$ by running $\mathcal{P}$ again from each landing value until the value 1 is reached. Then for every odd $n \geq 3$ the landing values $n = m_0 > m_1 > m_2 > \dots$ with $m_{t+1} = G(m_t)$ are strictly decreasing odd integers, and $m_t = 1$ for some*

$$t \leq \lfloor \log_{4/3}(n - 1) \rfloor + 1.$$

In particular $\mathcal{P}^$ terminates for every start.*

Proof. First we check that restarting is consistent. When $\mathcal{P}$ lands on an odd $m < n$, the previous odd value p satisfies $p \geq n > m$ (in case (a), $p = n$; in case (b), $p = 6k + 5 > n$). Processing m therefore applies $3n+1$ and sets $p \leftarrow m$, exactly as a fresh start at m would. Hence the run of $\mathcal{P}^*$ is the concatenation of descents, and the landing values satisfy $m_{t+1} = G(m_t)$.

By Proposition 4, $m_{t+1} \leq (3m_t + 1)/4$ whenever $m_t \geq 3$. Setting $a_t = m_t - 1 \geq 0$ gives

$$a_{t+1} \leq \frac{3m_t + 1}{4} - 1 = \frac{3}{4} a_t,$$

so $a_t \leq (3/4)^t a_0 = (3/4)^t (n - 1)$. The a_t are non-negative integers, so $a_t < 1$ forces $a_t = 0$, i.e. $m_t = 1$. This occurs once $(3/4)^t (n - 1) < 1$, which holds for $t = \lfloor \log_{4/3}(n - 1) \rfloor + 1$. $\square$

Remark 6. The contraction factor $3/4$ comes from $3 < 2^2$. For the analogous map $x \mapsto qx + 1$ with $q \geq 5$ one obtains the factor $q/4 > 1$, and the argument above does not apply. Termination of $\mathcal{P}$ is therefore a property of the arithmetic of the pair $(3, 2)$ together with the correction step, and the correction is what guarantees the descent.

4 Relation to the Collatz map

Proposition 7 (Agreement until the first correction). *Let $n \geq 3$ be odd. The values produced by $\mathcal{P}$ coincide with the Collatz orbit of n up to, but not including, the first application of -1 . If $n \equiv 1 \pmod{4}$, no correction occurs before the descent ends, so the descent is a segment of the Collatz orbit of n . If $n \equiv 3 \pmod{4}$, the first correction occurs at the third operation, at the value $x = (3n + 1)/2$.*

Proof. Until a correction, the operations of $\mathcal{P}$ are exactly C . The two cases follow from Proposition 4. $\square$

For $n = 4k + 3$, the Collatz successor of $x = 6k + 5$ is $3x + 1 = 18k + 16$, whereas $\mathcal{P}$ moves to $x - 1 = 6k + 4$. After this point the values of $\mathcal{P}$ are not guaranteed to lie on the Collatz orbit of n . Writing $y = 3k + 2$, we have $x = 2y + 1$ and $x - 1 = 2y$, so the effect of the correction followed by $d2$ is to replace $x = 2y + 1$ by y .

5 Worked examples

Table 2 gives the complete trace of $\mathcal{P}$ for $n = 27$. The previous odd value is shown in the last column.

Step	Operation	Value	Odd?	Previous odd p (before update)
0	start	27	yes	27
1	$3n+1$	82	no	27
2	d2	41	yes ($41 > 27$, rise)	27
3	-1	40	no	41
4	d2	20	no	41
5	d2	10	no	41
6	d2	5	yes ($5 < 27$)	41
Stop: 27 is proven by 5.				

Table 2: Trace of $\mathcal{P}$ for $n = 27$. In the closed form, $27 = 4 \cdot 6 + 3$, so $G(27) = \text{odd}(3 \cdot 6 + 2) = \text{odd}(20) = 5$ after $4 + \nu_2(20) = 6$ operations.

Table 3 compares $\mathcal{P}$ with the Collatz map for several starts. The “Collatz” columns give the number of Collatz steps until the orbit first falls below the start, and the highest value reached before that. For $n \equiv 1 \pmod{4}$ the two procedures agree. For $n \equiv 3 \pmod{4}$ they differ, and the difference grows with the length of the Collatz climb.

n	$n \bmod 4$	$G(n)$	Ops in $\mathcal{P}$	Peak in $\mathcal{P}$	Collatz steps	Collatz lands on	Collatz peak
9	1	7	3	28	3	7	28
25	1	19	3	76	3	19	76
7	3	5	4	22	11	5	52
15	3	11	4	46	11	10	160
27	3	5	6	82	96	23	9232
31	3	23	4	94	91	23	9232

Table 3: Comparison of $\mathcal{P}$ with the Collatz map. For $n \equiv 1 \pmod{4}$ the two coincide. For $n \equiv 3 \pmod{4}$ the peak of $\mathcal{P}$ is $3n + 1$, while the Collatz orbit can climb far higher before falling below n .

6 Application: the minimal-counterexample method

We now show how $\mathcal{P}$ is to be used in a proof by contradiction, and exactly where the argument requires an additional statement.

Proposition 8 (Properties of a minimal counterexample). *Suppose $\mathcal{C} \neq \emptyset$ and let $n_0 = \min \mathcal{C}$. Then:*

- (i) n_0 is odd and $n_0 \geq 3$;
- (ii) the Collatz orbit of n_0 contains no value below n_0 ;
- (iii) $n_0 \equiv 3 \pmod{4}$.

Proof. (i) If n_0 were even, then $n_0/2 < n_0$ would be a counterexample too, since its orbit is contained in that of n_0 . Also $1 \rightsquigarrow 1$.

(ii) If the orbit of n_0 contained some $m < n_0$, then $m \notin \mathcal{C}$ by minimality, so $m \rightsquigarrow 1$ and $n_0 \rightsquigarrow 1$ (Lemma 1), a contradiction.

(iii) If $n_0 = 4k + 1$ with $k \geq 1$, the orbit contains $3k + 1 < n_0$ by Proposition 4(a), contradicting (ii). $\square$

Applying $\mathcal{P}$. Let n_0 be a minimal counterexample, so $n_0 = 4k + 3$. By Proposition 4(b), $\mathcal{P}$ applied to n_0 lands on the odd value $m = G(n_0) < n_0$ after finitely many operations. By minimality, $m \rightsquigarrow 1$. A contradiction would follow if n_0 could be connected to m by Collatz moves. The Collatz orbit of n_0 contains $x = 6k + 5 = 2y + 1$, where $y = 3k + 2$ and $m = \text{odd}(y)$, so m and y lie in the same orbit. The only step at which $\mathcal{P}$ leaves the Collatz orbit is $x \rightarrow x - 1$. This leads to the following statement.

Definition 9 (Merging statement (M)). For every integer $y \geq 2$ with $y \equiv 2 \pmod{3}$, the Collatz orbits of y and of $2y + 1$ have a common element.

Theorem 10 (Equivalence). *Statement (M) is equivalent to the Collatz conjecture.*

Proof. ($\Leftarrow$) If every positive integer reaches 1, then the orbits of y and $2y + 1$ both contain 1.

($\Rightarrow$) Assume (M). We prove $n \rightsquigarrow 1$ for all n by strong induction. The case $n = 1$ is trivial. Let $n \geq 2$ and assume $m \rightsquigarrow 1$ for all $m < n$.

- If n is even, $n/2 < n$ and $n/2 \rightsquigarrow 1$, so $n \rightsquigarrow 1$.

- If $n = 4k + 1$ with $k \geq 1$, the orbit of n contains $3k + 1 < n$, so $n \rightsquigarrow 1$.
- If $n = 4k + 3$, put $y = 3k + 2$. Then $y \equiv 2 \pmod{3}$, $y \geq 2$ and $y < n$, so $y \rightsquigarrow 1$ by hypothesis. The orbit of n contains $(3n + 1)/2 = 6k + 5 = 2y + 1$. By (M) the orbits of y and $2y + 1$ share an element z . Since $y \rightsquigarrow 1$, the orbit of z contains 1, and so does the orbit of $2y + 1$ and hence of n .

This completes the induction. □

Conclusion of the contradiction argument. Under (M), the minimal counterexample n_0 of Proposition 8 cannot exist: the argument in the third case above shows $n_0 \rightsquigarrow 1$, contradicting $n_0 \in \mathcal{C}$. Without (M), the termination of $\mathcal{P}$ (Theorem 5) does not yield a contradiction, because the descent produced by $\mathcal{P}$ is not a descent of the Collatz orbit of n_0 .

Remark 11 (How the procedure should be applied). In a proof, $\mathcal{P}$ should be used as follows. (1) Assume a minimal counterexample n_0 and deduce $n_0 \equiv 3 \pmod{4}$ (Proposition 8). (2) Run $\mathcal{P}$ to obtain the explicit smaller value $G(n_0)$, which is proven by minimality. (3) Prove that the single corrected step $2y + 1 \rightarrow y$ is a Collatz connection, that is, prove (M) for the relevant y . Steps (1) and (2) are rigorous. Step (3) is equivalent to the whole conjecture by Theorem 10. Partial information about step (3) is available from sieve results: the density of starts whose Collatz orbit stays above its start tends to zero [7, 3], and almost all orbits reach arbitrarily slowly growing bounds [6]. Computationally, (M) holds for every y with $2y + 1$ below the verified bound [1, 2], since both orbits then reach 1.

7 Discussion

The procedure $\mathcal{P}$ maintains the lower bound exactly as intended: the odd values it produces strictly decrease, and each start is reduced to a smaller one in a number of operations given by a 2-adic valuation, with $O(\log n)$ descents in total (Theorem 5). The cost is that, for $n \equiv 3 \pmod{4}$, the correction step replaces the Collatz successor of $x = 2y + 1$ by y , and the Collatz climb that would normally follow is not traversed. Theorem 10 shows that this replacement is harmless to the Collatz conjecture if and only if the conjecture itself is true, so $\mathcal{P}$ is a precise reformulation and not a reduction. A proof along these lines would need a direct argument for (M), for example an explicit sequence of Collatz moves from $2y + 1$ to the orbit of y for every $y \equiv 2 \pmod{3}$.

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