

Arithmetic Unitarity V

Boosts, Rotations and the Missing Surface

Hodge Index, Ghost Detection, Krein and Toeplitz Signatures,
Newton over Hodge, and Passivity

Thermodynamics of Arithmetic Action, Lemmas 86–118

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Grok Bot is an AI assistant; see the Statement of Intellectual Honesty.

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Abstract

This volume collects TAA Lemmas 86–118 of the programme *Thermodynamics of Arithmetic Action*. Volumes I–IV (ai.viXra:2610.0045v2, 2610.0054, 2610.0052, 2610.0056, 2610.0055) recast the Riemann Hypothesis (RH) as the absence of negative-norm (“ghost”) directions for the Weil form, and traced the missing positivity to the Hodge index theorem on $C \times C$, which ζ lacks. Over finite fields Frobenius is a rotation in the Rosati metric; a ghost would be a boost. Volume V asks what a surface $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$ would have to supply. Central-value positivity (Gross–Zagier) turns out to be a parity statement about real zeros. Tested next: the standard conjectures and correspondence rings, the Connes–Consani self-intersection with synthetic ghost zeros, ample classes, two scalings and weight gradings. Certified (Arb) results: positivity of the Weil form of ζ on explicit spline spaces up to support $a = 8$, and certified detection of hypothetical off-line zeros. A compatible complex structure commuting with Frobenius exists exactly when the Frobenius torus is compact; the volume then compares Krein, Toeplitz and Newton-over-Hodge signatures, the modular cusp, BRST and thermodynamic geometry, and ends with Lax–Phillips passivity: the doubled modular system is passive unconditionally. A statement at the start lists exactly what is claimed, and where. Three proposed errata await the author’s decision; they are listed, not applied. Formalization and computations were AI-assisted, and this is disclosed. **RH is NOT proved:** structures built from the zeros restate Weil positivity, those built without them are blind to off-line zeros, and an RH-independent ample class for $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$ is still missing.

RH is NOT proved.

Nothing in this volume proves the Riemann Hypothesis, implies it, or reduces it to an easier statement.

Statement of Intellectual Honesty

The Riemann Hypothesis is NOT proved in this volume.
Nothing here implies it, and no chapter reduces it to an easier statement.
Neither GRH nor the Birch–Swinnerton-Dyer conjecture is proved here.

1. No claim of proof. No statement in this volume is, or should be quoted as, a proof of RH, or as progress towards one beyond the modest items listed in “What this volume does and does not advance” below. Every RH-equivalent condition in the notes is labelled [Restatement]. Results about function fields, finite “Weil-polynomial” models, the Davenport–Heilbronn function f_{DH} , or hypothetical (“ghost”) zeros inserted into ζ ’s zero set are statements about *those* objects. They do not transfer to ζ unless a chapter says how, and the transfer is then labelled.

2. Authorship and AI assistance. The questions, hunches and direction of every chapter are J. W. McGreevy’s, and each note records the request it answers (tags such as [t488]). The written notes are a different matter. Their wording, proofs, literature checks, figures and all computations (scripts in Python with mpmath, NumPy and SciPy; python-flint/Arb ball arithmetic; PARI/GP; SymPy) were produced by an AI assistant (Grok Bot) at his direction. That is why every note carries the byline “J.W. McGreevy; formalization by Grok Bot”. The assembly of this volume (selection, front matter, introduction, appendices) was also done by Grok Bot and checked against the notes. The submitting author is J. W. McGreevy, who is responsible for the decision to publish. The material has not been refereed or checked by an independent human expert. Read it as a research record, not as vetted mathematics.

3. Pasted texts and chat claims were critiqued, not trusted. Many chapters respond to claims made in the working conversation: J.W.’s hunches stated as claims, pasted texts, or the assistant’s own earlier chat replies. These were treated as *claims to check*, never as sources. Each note ends with a “Corrections” section that records which claims were right, which were wrong, and how they were fixed (consolidated in Appendix A). Where a claim could not be verified, the note says so. Where a later note proposes a correction to an earlier note that J.W. has not yet accepted, the earlier note is reproduced unchanged and the proposed correction is listed, with its exact effect, in Appendix B, §B.1.

4. Numerics. Computations are [Numerical] unless they are explicitly labelled computer-assisted (Arb ball arithmetic with rigorous inference). The certified items are listed in §A below. Floating-point and multiprecision results are reported with their stated tolerances. They are not proofs, and checks over finite ranges say nothing beyond those ranges. Many chapters insert synthetic off-line zeros (“ghosts”) into a list of computed zeros of ζ to test whether a structure would notice them; such tests concern the modified zero list, and the conventions for inserting them are stated in the chapters (see §B.1, item E8, for the one convention still pending a decision).

5. References. The notes distinguish [CITED] (checked at the source), [CITED-sec] (checked through a named secondary source) and [BIB] (bibliographic record verified, content not re-read). Each note’s references are recorded in the project’s per-lemma reference logs (`lit/lemmaNN/verified_refs.md`). A source that could not be verified was dropped or explicitly marked. Several notes cite unrefereed preprints; they are marked as such where they are used.

6. Facsimile chapters. Chapters 86–118 are the compiled notes themselves, reproduced page by page (see “Status of this volume”). A statement “Thm. $s.n$ of Ch. k ” is Theorem $s.n$ of TAA Lemma k . Each page carries the note’s own page number and the volume page number in its footer.

Claim labels

Every claim in the notes carries a label. The local tag variants map to seven volume classes:

Volume class	Meaning	Tags used in the chapters
Theorem	Proved in the note, or a published theorem used as stated.	[THEOREM]; [CITED], [CITED-sec], [BIB] for cited results; “computer-assisted, rigorous” for certified finite computations (Ch. 92); [TAA-L] for a result of an earlier TAA lemma
Restatement	Equivalent to RH, GRH or a known statement; no new content.	[RESTATEMENT]
Analogy	A structural parallel; proves nothing.	[ANALOGY]
Conjecture	A stated expectation, with its evidence.	[CONJECTURE]
Open	A question or gap explicitly left open.	[OPEN]; “Questions for J.W.”
Numerical	Floating-point or multiprecision computation; not a proof.	[NUMERICAL]
Heuristic	A plausibility argument or estimate, weaker than a conjecture.	[HEURISTIC]

What this volume does and does not advance

Summary. This volume does not advance RH for ζ . Its thread is a single question asked in many forms: over $\mathbb{F}_q$, Weil’s proof gets positivity from a surface $C \times C$, whose Hodge index theorem (equivalently the Castelnuovo–Severi inequality) says that the intersection form has exactly one “timelike” direction, and the Rosati involution then makes Frobenius a *rotation* (a compact torus, $|\alpha| = \sqrt{q}$). An off-line zero is the opposite: a *boost* (hyperbolic, $|\alpha| \neq \sqrt{q}$). For ζ the surface $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ is missing. The chapters test candidate replacements: central signs, saddle cones, the standard conjectures, correspondence rings, Yoneda-type specifications, the Connes–Consani self-intersection, two scalings, Cartan and Hodge circles, compatible complex structures and spinors, Krein and Toeplitz signatures, Newton and Hodge polygons place by place, the cusp of the modular surface, BRST/no-ghost structures, thermodynamic geometry, and Lax–Phillips passivity. The recurring outcome is a *dichotomy*: a structure built from the zeros (or from Weil’s form) is RH-equivalent, and a structure built without them is blind to off-line zeros. The contributions are of four modest kinds. (A) A few certified computations. (B) Elementary theorems that we did not find stated in the literature we checked; the searches were not exhaustive, so each is “new to our knowledge” at most, and none bears directly on RH. (C) Re-packagings of known results, credited to their authors. (D) Negative results: proofs or computations showing that a proposed source of positivity cannot work. Items are labelled with the classes above. Locations give the chapter, the statement number inside the note, the note page and the volume page.

A. Certified computer-assisted results [Theorem, computer-assisted]

- A1. *Semi-local positivity of ζ ’s Weil form on explicit spline spaces.* For 21 support parameters $a \in \{\frac{1}{4}, \dots, 8\}$ the Weil form, computed from the prime side (all prime powers up to e^{2a} , pole and archimedean terms), is positive definite on the cubic-spline space $S(a, n) \cap V_0$, with certified lower bounds for the smallest eigenvalue; the largest case is $a = 8$, $n = 157$ (Arb balls of radius $\leq 2 \cdot 10^{-70}$; congruence plus Gershgorin). Ch. 92, Thm. 2.1 (note p. 4, vol. p. 77). These are finite-dimensional spaces: the certificate says nothing about all test functions of support a , and nothing about RH. (For all test functions of a given support, the proved range is still Yoshida’s $a = \frac{1}{2} \log 2$; an unrefereed 2026 computer-assisted claim, cited there, reaches $a = \frac{1}{2} \log 19$.)
- A2. *Certified detection of hypothetical zeros.* For each row of Table 2 of Ch. 92 there is an explicit test function ψ with $Q(\psi) + G_{\gamma, \delta}(\psi) < 0$, where $G_{\gamma, \delta}$ is the contribution an extra off-line quartet at (γ, δ) would make; so such a quartet would be detected at the stated support (e.g. $(\gamma, \delta) = (85.699, 0.3085)$ at $a = 7/5$). Ch. 92, Thm. 3.2 (note p. 7, vol. p. 80). This is a statement about modified zero sets, not about ζ .
- A3. *Davenport–Heilbronn central value.* $f_{\text{DH}}(\sigma) > 0$ on $[\frac{1}{2}, 2]$ (Arb, 3000 subintervals), hence $\Lambda_{\text{DH}}(\frac{1}{2}) > 0$ although f_{DH} has off-line zeros. Ch. 86, §5 (D) (note p. 8, vol. p. 8). This illustrates B1 (central signs are blind to complex off-line zeros).

B. Elementary theorems, new to our knowledge [Theorem]

Each is a short argument from standard facts. We claim the formulation, not depth, and we may have missed prior statements.

- B1. *Central sign = parity of real zeros.* If Λ is real on $[\frac{1}{2}, \infty)$, positive for large σ , with a functional equation of sign $+1$, the sign of its leading Taylor coefficient at $\frac{1}{2}$ is $(-1)^{N_{\mathbb{R}}}$, $N_{\mathbb{R}}$ the number of real zeros in $(\frac{1}{2}, \infty)$. So Kohnen–Zagier and Gross–Zagier positivity are parity statements about real zeros and cannot see complex off-line quartets. Ch. 86, Thm. 3.1, Cor. 3.3 (note p. 4, vol. p. 4).
- B2. *Saddle light cones.* At an on-line saddle of $\log |\Lambda|$ the level-set cones are exactly $\pm 45^\circ$; off-line they are rotated by $-\frac{1}{2} \arg c$; and Kim’s saddle identity takes a light-cone form (a [Restatement] in the limit). Ch. 87, Thms. 3.1–3.2 (note p. 4, vol. p. 17).
- B3. *Ghost quartets in the Connes–Consani self-intersection.* An off-line quartet added to the zeros contributes a form of real signature $(2, 2)$, so on any finite span $n_+ \leq 2k$ for k added quartets, Ch. 91, Prop. 1.3 (note p. 3, vol. p. 66).
- B4. *Detection thresholds.* The supports at which a ghost is detected form an up-ray, and detection on a space where Q is positive is decided by the inertia of a 4×4 matrix (one quartet gives at most two negative directions) Ch. 92, Thm. 3.1 (note p. 6, vol. p. 79). Function-field calibration: for true curves $Q \geq 0$, and a weak ghost (replacement model) is detected iff $N \geq g$, i.e. $a^* = g \log q$ Ch. 92, Thm. 4.1 (note p. 10, vol. p. 83). The conjectured law $a^* = \frac{1}{2} \log(\gamma/2\pi) + O(1)$ is [Conjecture].
- B5. *A ceiling for two-scaling pairings.* Every pairing pulled back from a single scaling has inertia bounded by that of the single-scaling span Ch. 95, Thm. 2.1 (note p. 4, vol. p. 116). Over $\mathbb{F}_p$ the Hecke lattice $\langle H, V, \Delta, T_p \rangle$ has signature $(1, 2)$ for every trace a (blind to Ramanujan), while the Frobenius lattice $\langle H, V, \Delta, \Gamma_F \rangle$ has signature $(1, 3)$ iff $a^2 < 4p$ Ch. 95, Thms. 3.1–3.2 (note p. 6, vol. p. 118). Riemann–Roch plus Nakai gives Hodge index Ch. 95, Thm. 4.2 (note p. 8, vol. p. 120).
- B6. *One boost.* An element of $O(1, n)$ has at most one eigenvalue of modulus > 1 , and it is real; its inverse is the only one of modulus < 1 . Over $\mathbb{F}_q$, Hodge index plus “one boost” gives Hasse’s bound. Ch. 98, Thms. 4.1–4.2 (note p. 6, vol. p. 149).
- B7. *Compatible complex structures for Frobenius.* A polarizing J commuting with F exists iff the Frobenius torus is compact ($q/\alpha = \bar{\alpha}$) Ch. 100, Thm. 4.1 (note p. 5, vol. p. 169); the spinor form of this criterion Ch. 104, Thm. 1.1 (note p. 2, vol. p. 201); the block-by-block count of commuting complex structures, compatible complex structures and anti-symplectic involutions by eigenvalue type (k_u, k_r, k_c) Ch. 110, Thm. 1.1 (note p. 3, vol. p. 243); and the minimal-boost theorem (over all metrics, the least possible log-norm of F is the largest off-circle rapidity) Ch. 113, Thm. 2.2 (note p. 3, vol. p. 263).
- B8. *Discrete Weil criterion.* Given the explicit formula, RH implies that the hat-basis Toeplitz matrices of the Weil form are positive definite for every step h , and failure of RH makes them indefinite for all h outside a countable aliasing set. Ch. 112, Thm. 3.1 (note p. 4, vol. p. 256). Ch. 112 asks for prior literature on this exact form.
- B9. *One subsidiary condition.* With $W(g)$ Weil’s functional, $\text{RH} \iff W(g) \leq 0$ for all g with $\hat{g}(1) = 0$ (equivalently, with $\hat{g}(0) = 0$). Ch. 115, Thm. 2.4 (note p. 4, vol. p. 288). A [Restatement] (given Weil’s explicit formula and criterion); no priority is claimed.
- B10. *Lax–Phillips passivity.* The doubled Lax–Phillips system of the modular surface is passive unconditionally, so passivity cannot detect off-line zeros (the decay rates $(1 - \beta)/2$ do depend on them) Ch. 118, Thm. 3.1 (note p. 4, vol. p. 319). A time-reversal-symmetric Blaschke-type scattering function, unimodular on the line, is passive iff all its resonances lie on the line (lossless); for the undoubled ζ this is a [Restatement] of RH Ch. 118, Thm. 4.1 (note p. 5, vol. p. 319).

p. 320). A truncated Euler product P_N is positive-real on $\operatorname{Re} s = \sigma$ iff $\sum_{p \leq N} \arcsin p^{-\sigma} \leq \frac{\pi}{2}$; the thresholds σ_N increase to van de Lune's $\sigma_0 = 1.19234\dots$ Ch. 118, Prop. 4.4 (note p. 6, vol. p. 321). The exact Lorentzian decomposition of the modular scattering density (widths $\frac{1}{4} \pm \frac{\delta}{2}$) is in Ch. 105, Thm. 3.2 (note p. 3, vol. p. 209).

- B11. *Shift dominance*. $\text{RH} \iff |\xi(s+c)| > |\xi(s-c)|$ on $\operatorname{Re} s > \frac{1}{2}$ for every $c > 0$, and the doubled (Lax–Phillips/Lagarias–Suzuki) statement is the unconditional case $c = \frac{1}{2}$. Ch. 117, Prop. 3.2 (note p. 4, vol. p. 309). A [Restatement], possibly known.

C. Re-packagings of known results [Theorem], credited

- C1. *The standard conjectures as common root* (Ch. 88): over $\mathbb{F}_q$ the BSD regulator lattice and the RH positivity lattice coincide for the constant surface $E \times E \rightarrow E$ (Theorem A, Ch. 88, Thm. 3.1 (note p. 3, vol. p. 32)). Ch. 89 shows that Theorem A is classical (Silverman V.1.1–V.1.2, Shioda 2008, Milne 1968); the corresponding errata to Ch. 88 are *pending* (§B.1). The four-way identity Rosati index = saddle count = Néron–Severi index is a composition of earlier lemmas, Ch. 88, Thm. 4.1 (note p. 4, vol. p. 33). Over $\mathbb{Q}$ the comparison is [Analogy].
- C2. *Correspondence rings* (Ch. 89): $\text{NS}(C \times C) \cong \mathbb{Z}^2 \oplus \text{End}(J)$, Frobenius idempotents as $\mathbb{Q}$ -divisors, and the localization of the ghost count per $\dagger$ -orbit of irreducible factors, Ch. 89, Thms. 2.1, 3.1, 4.1 (note p. 2, vol. p. 42); positivity sits in the product part, Ch. 89, Thm. 6.1 (note p. 7, vol. p. 47).
- C3. *Category theory and $\operatorname{Spec} \mathbb{Z}$* (Ch. 90): Yoneda needs naturality; $\operatorname{Spec} \mathbb{Z}$ is terminal, so the naive square collapses; $C_K \cong C_K^1 \times \mathbb{R}_{>0}$ for every number field, so this splitting carries no RH content, Ch. 90, Thm. 6.4 (note p. 7, vol. p. 58).
- C4. *Two partners of a zero* (Ch. 93): duality ($\rho \leftrightarrow 1 - \rho$) and conjugation ($\rho \leftrightarrow \bar{\rho}$); RH says they coincide, Ch. 93, Thm. 1.1 (note p. 2, vol. p. 89).
- C5. *Rosati as a Cartan involution* (Ch. 99): Rosati is the Cartan involution restricted to the commutant of $h(i)$; Deligne's polarizability criterion via Milne, Ch. 99, Thm. 2.1 (note p. 3, vol. p. 159).
- C6. *Logic* (Ch. 107): RH is Π_1^0 (Lagarias), so a proof of its independence from ZFC would prove it, Ch. 107, Thm. 1.2 (note p. 2, vol. p. 222).
- C7. *Krein stability, Toeplitz–Carathéodory, Newton over Hodge* (Chs. 111–113): Krein–Gelfand–Lidskii strong stability (via Kollár–Miller), Verblunsky's theorem (via Simon), Weyl's inequality and Mazur's inequality (Newton above Hodge), with Kottwitz–Rapoport and Fargues–Fontaine as cited. Over $\mathbb{F}_q$: Rosati $\Rightarrow$ flat Hodge polygon $\Rightarrow$ RH, Ch. 113, Thm. 2.1 (note p. 3, vol. p. 263).
- C8. *Lorentzian polynomials and the cusp* (Ch. 114): the bivariate polynomials $f_{d,n}$ built from the Taylor coefficients of ξ are Lorentzian for all d, n iff the Turán inequalities hold, which they do unconditionally (Csordas–Norfolk–Varga), so this Lorentzian family is strictly weaker than RH, Ch. 114, Thm. 1.1 (note p. 3, vol. p. 272).
- C9. *Thermodynamics and scattering* (Chs. 116, 118): the Bost–Connes system, Fisher–Ruppeiner geometry, Mrugała's contact metric, Bisognano–Wichmann; Lax–Phillips and Faddeev–Pavlov, Sz.-Nagy–Foiás, Pusz–Woronowicz passivity, Bagchi's strong recurrence. All cited as stated in the chapters.

D. Negative results (what cannot supply the positivity)

These are the volume's main content. They are labelled [Theorem] where proved, otherwise [Numerical].

- D1. *Effectivity is not the missing positivity*. Over $\mathbb{Q}$ there is an unconditional off-diagonal positivity for nonnegative test functions with separated supports Ch. 94, Thm. 3.1 (note p. 6, vol. p. 321).

- p. 105), but it is ghost-blind: numerically, an on-line pair can break effectivity while keeping Hodge index, and a ghost quartet keeps every separated pair positive, so effectivity and Hodge index are independent over $\mathbb{Q}$ [Numerical], Ch. 94, Thm. 3.2 (note p. 6, vol. p. 105).
- D2. *Two scalings add no positivity* (B5); and in random models, Ramanujan bounds at every prime do not imply global Weil positivity ([Numerical], Ch. 95).
- D3. *Renormalizations and gradings*. The prime/archimedean split of the explicit formula is RH-blind, Ch. 96, Thm. 2.3 (note p. 4, vol. p. 127); no renormalized Δ^2 in the tested schemes is finite, of Hodge sign and ghost-sensitive, Ch. 96, Thm. 5.6 (note p. 8, vol. p. 131); a weight grading presupposes $\text{Re } \rho$, Ch. 97, Thm. 2.2 (note p. 4, vol. p. 137).
- D4. *Complex structures and spinors for ζ* . No RH-independent compact Hodge circle acts on the zeros (Ch. 100); a spinor J is RH-equivalent if built from Weil’s form and ghost-blind otherwise, Ch. 104, Thm. 2.4 (note p. 4, vol. p. 203); for any algebra containing the scaling flow, “Rosati positivity” of the Weil trace is Weil positivity itself, Ch. 106, Cor. 3.4 (note p. 3, vol. p. 216).
- D5. *Scattering-side structures*. Lax–Phillips closure cannot turn resonances into eigenvalues, Ch. 105, Thm. 5.2 (note p. 5, vol. p. 211); the geometric reflection of the cusp is ghost-blind, Ch. 108, Prop. 6.2 (note p. 5, vol. p. 231); the Popa–Zagier hyperbolic-class weights exclude ghosts rather than detect them (the reading proposed in Ch. 109, Q2), Ch. 109, Prop. 2.3 (note p. 3, vol. p. 236).
- D6. *Stability and places*. Along the de Bruijn–Newman heat flow the graphical Krein signatures of the real zeros alternate, so Krein strong-stability theory gives no protection, Ch. 111, Prop. 3.1 (note p. 4, vol. p. 250); for q -palindromic models over $\mathbb{F}_q$, ghosts have the same ℓ -adic, p -adic and product-formula data as true Weil numbers, so only the archimedean place sees RH, Ch. 113, Thm. 5.1 (note p. 5, vol. p. 265).
- D7. *Thermodynamic and passive structures*. The complexified Fisher metric is flat for every holomorphic function, hence ghost-blind, Ch. 116, Prop. 2.5 (note p. 4, vol. p. 299); the stability of the doubled Lagarias–Suzuki and Lax–Phillips systems is unconditional and hence ghost-blind, Ch. 117, Prop. 3.2 (note p. 4, vol. p. 309), Ch. 118, Thm. 3.1 (note p. 4, vol. p. 319); KMS passivity of the primon gas concerns ζ only at real $\beta > 1$ Ch. 118, Prop. 5.1 (note p. 8, vol. p. 323); a Darlington realization of the arithmetic target from primes is exactly circular, Ch. 118, Prop. 4.3 (note p. 6, vol. p. 321).
- D8. *Central signs* cannot see complex off-line zeros (B1, A3).

What remains open

- (1) *The missing surface*. No object is known over $\mathbb{Q}$ that plays the role of $C \times C$ with an ample class whose Hodge index theorem (an arithmetic Castelnuovo–Severi inequality) is proved independently of the zeros. Every candidate tested here either restates Weil positivity or is blind to off-line zeros [Open].
- (2) *Rotation versus boost*. Over $\mathbb{F}_q$, RH is the statement that Frobenius is a rotation in the Rosati metric, and a ghost is a boost (Chs. 98–100, 104, 110, 113). For ζ no RH-independent compact circle, compatible complex structure or unitary dilation acting on the zeros is known [Open].
- (3) *The undoubled system*. The modular surface gives a passive Lax–Phillips system with ζ ’s zeros as resonances only in the doubled variable (at $\rho/2$), where passivity is unconditional (the half-width $c = \frac{1}{2}$). An undoubled passive realization with the zeros themselves as resonances would be lossless iff RH holds; none is known (Chs. 105, 108, 117, 118) [Open].

The “Questions for J.W.” sections of the chapters are collected in Appendix B. The same appendix lists the proposed errata that await the author’s decision.

Status of this volume

Assembly. This volume was assembled on 25 October 2026 from the latest compiled versions of TAA Lemmas 86–118, as they stand in the project directory. These include the sections added after a first draft, such as Lemma 98 Section (E) (t423) and Lemma 100 §3 (t430). Chapter k is Lemma k . The notes were not edited for the volume.

Facsimile inclusion. As in Volume IV, the chapters are not re-typeset. Each note’s compiled pages are reproduced at 93% scale, with its own title, numbering, figures and reference list. Consequences:

- Statement numbers restart in each chapter.
- Each chapter keeps its own bibliography; there is no consolidated bibliography.
- Cross-references between chapters are by lemma number (“L109” in a note is Chapter 109).

File size. Each note embeds its own subsets of the same Latin Modern and AMS fonts. When the notes are included, pdfTEX does not copy these subsets; it embeds each font once, from the TEX distribution’s font files, as a subset containing every glyph used anywhere in the volume. Identical objects were then merged and the streams recompressed. No figure was rasterized: all text, equations and figures remain vector. Every page was checked to render pixel-identically to a build that copies each note’s original font subsets.

Pending errata. Three groups of corrections proposed in later notes for earlier notes (to Lemma 88 from Lemma 89, to Lemma 89 from Lemma 91, to Lemma 106 from Lemma 109) await J.W.’s decision. The earlier notes are reproduced unchanged; Appendix B, §B.1, states exactly what each proposed correction would change.

Not included. The project’s working notes (STATE.md, NOTATION.md), the synthesis drafts, scripts and data are not part of the volume. Each note names its scripts and output files.

Place in the series

Volume	Title	Contents and identifier
I	Arithmetic Unitarity I: Where the Riemann Hypothesis Lives in the Scaling Flow	Lemmas 1–13; ai.viXra:2610.0045v2
II	Arithmetic Unitarity II: The Light Cone of the Weil Form	Lemmas 14–29; ai.viXra:2610.0054
III	Arithmetic Unitarity III: In Search of Positivity — G5, Polarization, and Hilbert versus Krein	Lemmas 30–47 (and a condensation of Lemma 50); ai.viXra:2610.0052
IVa	Arithmetic Unitarity IVa: Counting Ghosts — Spinor Vacua, Saddles, the Weil Positivity Ladder, and the Prime Side	Lemmas 48–65; ai.viXra:2610.0056
IVb	Arithmetic Unitarity IVb: Counting Ghosts — Eisenstein Anomalies, Hyperbolic Geometry, the Ample-Class Obstruction, and Mordell–Weil Lattices	Lemmas 66–85; ai.viXra:2610.0055
V	Arithmetic Unitarity V: Boosts, Rotations and the Missing Surface — Hodge Index, Ghost Detection, Krein and Toeplitz Signatures, Newton over Hodge, and Passivity	Lemmas 86–118 (this volume)

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Introduction

The thread

Volume IV ended at the ample-class obstruction: over $\mathbb{F}_q$, the positivity in Weil’s proof comes from the Hodge index theorem on the surface $C \times C$, and the Rosati involution transports it to Frobenius; for ζ no surface, no ample class and no Rosati involution are known. Volume V asks, in many forms, what that missing surface would have to supply.

A picture recurs throughout. A Frobenius eigenvalue on the circle $|\alpha| = \sqrt{q}$ (an on-line zero) is a *rotation*: it preserves a positive form, a compatible complex structure J (a “Hodge circle”). An eigenvalue off the circle (a ghost) is a *boost*: it preserves only an indefinite form and a split-complex structure. The Hodge index theorem says that the intersection form has one timelike direction, and Lorentz geometry then allows at most one boost (Ch. 98); Rosati positivity removes even that one. The chapters look for an RH-independent source of the rotation for ζ , and keep meeting the same *dichotomy*: a structure built from Weil’s form (or from the zeros) is RH-equivalent, and a structure built without it is blind to ghosts. Volume V follows this along nine lines, in roughly the order the notes were written:

- central signs and saddle cones (Chs. 86–87);
- the standard conjectures and divisors on $C \times C$ (Chs. 88–90);
- testing the Weil form with ghosts, with certified numerics (Chs. 91–93);
- ample classes, two scalings, renormalization and weights (Chs. 94–97);
- boosts and rotations (Chs. 98–99);
- Hodge circles, two-point forms and spinors (Chs. 100–104);
- Siegel space, Rosati for $\mathbb{Q}$, logic, and the cusp (Chs. 105–109);
- commuting structures, Krein and Toeplitz signatures, Newton over Hodge (Chs. 110–113);
- the cusp, BRST, thermodynamics and passivity (Chs. 114–118).

Standing conventions

$s = \sigma + it$; $\rho = \beta + i\gamma$ is a nontrivial zero, a *ghost* if $\beta \neq \frac{1}{2}$; $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. The Weil form is Q or W (truncations Q_L , $Q|_{S(a,n)}$). Over $\mathbb{F}_q$, F is Frobenius on H^1 and $F^\dagger = qF^{-1}$ its Rosati partner; a finite “model” is a q -palindromic polynomial, and a *fake* or ghost model is one with roots off $|\alpha| = \sqrt{q}$. Normalized $F \in \mathrm{Sp}(2g, \mathbb{R})$ has type (k_u, k_r, k_c) : unit pairs, real pairs, quartets. Synthetic ghosts are inserted into lists of computed zeros of ζ ; such tests concern the modified list, and the insertion conventions are stated in each chapter (the pending convention question is §B.1, E6–E7). Appendix C lists symbols that are used differently in different chapters.

Chapter by chapter

Central signs and saddle cones (Chs. 86–87). **Ch. 86** asks whether central-value positivity (Gross–Zagier, Waldspurger, Kohnen–Zagier) is evidence for RH. It is not: the sign of the leading coefficient at the centre is the parity of the number of real zeros to the right of it (Thm. 3.1), so it cannot see complex off-line quartets. The Davenport–Heilbronn function has off-line zeros and a certified positive central value. **Ch. 87** rewrites the saddle picture of Chs. 55–56 in light-cone

language: on-line saddles have exact $\pm 45^\circ$ cones, off-line ones are rotated by $-\frac{1}{2} \arg c$, and Kim's identity becomes a statement about spacelike and timelike cones.

The standard conjectures and divisors (Chs. 88–90). **Ch. 88** measures every TAA counter against a true Hodge index over $\mathbb{F}_q$, where the algebraicity and positivity halves of Weil's mechanism are both available: on the constant surface $E \times E \rightarrow E$ the Mordell–Weil lattice is the Rosati trace form (Theorem A), and Rosati index, saddle count and Néron–Severi index agree (Thm. 4.1). **Ch. 89** takes the divisor-first view: $\mathrm{NS}(C \times C) \cong \mathbb{Z}^2 \oplus \mathrm{End}(J)$, irreducible factors of the zeta polynomial are cut out by $\mathbb{Q}$ -divisors, ghosts localize per factor, and all positivity sits in the product part $\langle H, V \rangle$. It also finds that Theorem A of Ch. 88 is classical and proposes errata to Ch. 88 (pending; §B.1). **Ch. 90** takes J.W.'s Yoneda idea literally: the wish-list functor for $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$ would give uniqueness, not positivity, because positivity is an input; four obstructions are listed, starting with the collapse of the absolute square ($\mathrm{Spec} \mathbb{Z}$ is terminal).

Testing the Weil form with ghosts (Chs. 91–93). **Ch. 91** computes the Connes–Consani self-intersection $D \bullet D$ from the prime side and from 2500 zeros, with and without synthetic quartets. A ghost is detected only above a support threshold that grows like $\log \gamma$, and the detection is Weil's criterion restated. **Ch. 92** makes this rigorous where it can: certified positivity of the Weil form on spline spaces up to support $a = 8$ (Arb), certified detection of hypothetical quartets, a 4×4 inertia criterion, and a function-field calibration in which a weak ghost is detected exactly at $N^* = g$. **Ch. 93** answers “what does it mean for a zero to pair with a zero”: a zero has a duality partner and a conjugation partner, and RH says they coincide.

Ample classes, two scalings, renormalization, weights (Chs. 94–97). **Ch. 94** tests constructions of the two missing ingredients named in the synthesis draft (an ample class P4 and a rational structure P9). Both exist in weak forms over $\mathbb{Q}$, and in those forms they are ghost-blind; the RH content sits in a Riemann–Roch-type effectivity axiom. **Ch. 95** asks whether two independent scalings (and Hecke operators as a template) give more than the single-scaling Weil form: they cannot (a ceiling theorem), and the Hecke lattice is blind to Ramanujan while the Frobenius lattice sees it. **Ch. 96** separates three objects called Δ (diagonal, cohesion, discriminant) and tests renormalizations of Δ^2 : none is finite, of Hodge sign and ghost-sensitive. **Ch. 97** shows that grading by weight is circular and tests “dequantization” of Weil's form; it includes a Mellin section and the $1 \oplus L$ dictionary.

Boosts and rotations (Chs. 98–99). **Ch. 98** examines a sum-of-squares certificate for Weil positivity read through hyperbolic geometry. The SOS that exists is the zero side; Levinson's recursion is a sequence of boosts; and the one-boost lemma for $O(1, n)$, with Hodge index, gives Hasse's bound over $\mathbb{F}_q$. **Ch. 99** checks J.W.'s proposal that the needed positive involution is a Cartan involution: Rosati is the Cartan involution restricted to the commutant of $J = h(i)$, and “Wick rotation by the ample class” turns Hodge index into positivity, which over $\mathbb{Q}$ is again a restatement.

Hodge circles, two-point forms and spinors (Chs. 100–104). **Ch. 100** asks for a compact Hodge circle for $\mathbb{Q}$. The functional equation acts as a half-turn, not a mirror, and a compatible J commuting with Frobenius exists iff the duality partner σ equals conjugation c on the eigenvalues, i.e. iff the Frobenius torus is compact (Thm. 4.1); for ζ no RH-independent circle is available. **Ch. 101** asks where J comes from when it exists: over $\mathbb{F}_q$, from the degree form and the negative discriminant, $J = (2F - a)/\sqrt{4q - a^2}$. **Ch. 102** shows that one alternating form has two twists, by a rotation J (Hodge–Riemann) and by a mirror S (Minkowski), and that compatibility (positivity) is what singles out J . **Ch. 103** shows that Weil's functional is a two-point function only after a complex structure is added, and then its positivity is Weil positivity (a restatement); it also fixes the single-zero replacement convention for ghost experiments. **Ch. 104** asks whether a Euclidean

Clifford pair with $J = SR_x$ can be built on a space carrying the zeros: for finite models iff RH holds; for ζ a spinor J is RH-equivalent if built from Weil’s form and ghost-blind otherwise.

Siegel space, Rosati for $\mathbb{Q}$, logic, and the cusp (Chs. 105–109). **Ch. 105** identifies Hodge–Riemann states with Siegel space, decomposes the modular scattering density exactly into Lorentzians, and shows why Lax–Phillips closure cannot turn resonances into eigenvalues. **Ch. 106** asks what a Rosati-type positivity for $\mathbb{Q}$ would need, and finds that over $\mathbb{F}_q$ the positivity comes from geometry (a count of intersection points), not arithmetic; for an algebra containing the scaling flow, “Rosati positivity” is Weil positivity. Its ghost numerics use a double-zero convention that Ch. 109 proposes to correct (pending; §B.1). **Ch. 107** records the logical form of RH: it is Π_1^0 , so a proof of independence from ZFC would prove it. **Ch. 108** reads ghosts as S -objects: a ghost commutes with an anti-symplectic involution whose null eigenlines are geodesic endpoints, not cusps, and the geometric reflection of the cusp is ghost-blind. **Ch. 109** asks whether the split-complex structure can be built from closed geodesics: the detectors that work are Weil/Hasse positivity restated, and count-type detectors are blind.

Commuting structures, Krein and Toeplitz signatures, Newton over Hodge (Chs. 110–113). **Ch. 110** classifies all structures commuting with a normalized symplectic F by eigenvalue type, and computes the Rosati signature. **Ch. 111** relates Krein strong stability to Hasse (a restatement for distinct eigenvalues) and shows that along the de Bruijn–Newman flow the Krein signs of ζ ’s zeros alternate. **Ch. 112** reads the Weil form through Carathéodory–Toeplitz and Verblunsky: in finite models RH is positivity of one Toeplitz matrix, and for ζ a discrete Weil criterion holds for every step h outside a countable set. **Ch. 113** puts the same inequality at every place: Weyl’s inequality over $\mathbb{R}$, Mazur’s “Newton above Hodge” over $\mathbb{Q}_p$; Weil’s proof makes the Hodge polygon flat, and ghosts are invisible at every finite place.

The cusp, BRST, thermodynamics and passivity (Chs. 114–118). **Ch. 114** checks whether the discriminant Δ “diagonalizes” the Pythagorean and Platonic side: Klein’s icosahedral identity matches $E_6^2 = E_4^3 - 1728\Delta$, the tower $(2, 3, N)$ has the cusp as its limit, and the Lorentzian family built from ξ is unconditional. **Ch. 115** asks whether the physicists’ no-ghost theorems (Gupta–Bleuler, BRST, Goddard–Thorn) give Weil positivity: they give an exact dictionary and one subsidiary condition, but no new positivity. **Ch. 116** tests thermodynamic geometry (Bost–Connes, Fisher–Ruppeiner, contact geometry, Bisognano–Wichmann): the one natural stability statement that sees zeros is necessary but not sufficient. **Ch. 117** makes precise the reading of Weil positivity as an arithmetic Castelnuovo–Severi inequality: Szegő steps are boosts, and in the doubled Lax–Phillips variable the stability is unconditional. **Ch. 118** gives the rigorous version of “passivity and the arrow of time”: irreversibility needs infinitely many channels, the doubled modular system is passive unconditionally, a reversible passive system is lossless, and the Euler product is not a passive cascade.

Editorial remarks

- Chapters were not edited. Where a later chapter corrects an earlier one, the correction is recorded in the later chapter and listed in Appendix A. Corrections that await the author’s decision are listed in Appendix B, §B.1, and are not applied.
- Most notes begin from a question or a hunch of J.W.’s in the working conversation (tags such as t427 or pt472u), and check the claims made there before building on them.
- The order of chapters is numerical, which is the order in which the notes were written.

TAA LEMMA 86: CENTRAL POSITIVITY. GROSS–ZAGIER, WALDSPURGER, AND THE REAL-ZERO BLINDNESS OF CENTRAL SIGNS

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA; neither is GRH or BSD. Nothing in this note moves RH: its main theorem (§3) shows that central-value positivity is *blind* to the complex zeros RH is about. Tags as in Lemmas 80–85: [THEOREM] proved here or standard with proof sketched, [CITED]/[CITED-sec] checked source / checked secondary source, [BIB] bibliographic data only, [TAA-L] earlier TAA lemma, [RESTATEMENT] equivalent to RH, [ANALOGY] analogy, [CONJECTURE] conjecture, [OPEN] open, [NUMERICAL] computation, [HEURISTIC] heuristic. Every citation is listed in `lit/lemma86/verified_refs.md`. Every number is computed here (PARI/GP 2.17.2 via `cy pari2`; `mpmath`; Arb via `python-flint`); none is recalled.*

Task (J.W., approved [t370] widget and [t371u]; item I2 of Lemma 82 excluded per [t371u]). (A) The verified core: Gross–Zagier and Waldspurger/Kohnen–Zagier with exact hypotheses. (B) The central-sign theorem: what a positive central value says about zeros. (C) Numerics: quadratic twists of 11a1 against a computed weight-3/2 form; a Gross–Zagier consistency check for 37a1. (D) Davenport–Heilbronn. (E) Corrections of the pasted [t370] text and of the parent’s chat claims. (F) Landau–Siegel zeros and family statistics. (G) Bloch–Kato: where central algebraic positivity lives.

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GLOSSARY (RESTATED WHERE USED)

- *Normalizations.* For an elliptic curve $E/\mathbb{Q}$ of conductor N , $L(E, s) = \sum a_n n^{-s}$ is the *arithmetic* normalization: Euler product absolutely convergent for $\operatorname{Re} s > 3/2$, functional equation $s \leftrightarrow 2-s$, centre $s = 1$. The *analytic* normalization is $L(E, s + \frac{1}{2})$, with centre $\frac{1}{2}$ and strip $0 < \operatorname{Re} s < 1$. $\Lambda(E, s) = N^{s/2} (2\pi)^{-s} \Gamma(s) L(E, s) = \varepsilon \Lambda(E, 2-s)$, $\varepsilon = \pm 1$ the root number.
- *Quadratic twist.* For a fundamental discriminant D , $\chi_D = (\frac{D}{\cdot})$ and E_D (also E^D) is the twist, with $L(E_D, s) = \sum \chi_D(n) a_n n^{-s}$ when $(D, N) = 1$. For $(D, N) = 1$: $\varepsilon(E_D) = \varepsilon(E) \chi_D(-N)$.
- *Néron–Tate height* $\hat{h}$ on $E(K)$: a quadratic form, positive definite on $E(K) \otimes \mathbb{R}$ (it vanishes exactly on torsion). This is the only “positivity” in the Gross–Zagier formula.
- *Half-integral weight.* $S_{3/2}^+(4N)$ is Kohnen’s plus space: cusp forms $g = \sum c(n) q^n$ of weight $3/2$ on $\Gamma_0(4N)$ with $c(n) = 0$ unless $-n \equiv 0, 1 \pmod{4}$. For odd squarefree N it is Hecke-isomorphic to $S_2(\Gamma_0(N))$ (Shimura correspondence).
- *Ghost* (TAA): a zero off the critical line. A *real* ghost is a real zero in $(\frac{1}{2}, 1)$ (analytic normalization), the Landau–Siegel type; a *complex* ghost comes in a quadruple $\frac{1}{2} \pm \beta \pm i\gamma$, $\gamma \neq 0$. κ = number of off-line pairs (Lemma 84; a complex quadruple counts 2).

1. ANSWER IN BRIEF

- (1) **Verified core** [CITED]/[CITED-sec]. Gross–Zagier (1986): for $K = \mathbb{Q}(\sqrt{D})$ imaginary quadratic with D odd, prime to N , and $D \equiv \square \pmod{4N}$ (Heegner condition), $L(E/K, s)$ vanishes at $s = 1$ and $L'(E/K, 1) = \|\omega\|^2 \hat{h}(P_K) / (c^2 u^2 |D|^{1/2})$; since $\hat{h} \geq 0$ on $E(K) \otimes \mathbb{R}$, $L'(E/K, 1) \geq 0$. Kohnen–Zagier (level 1) and Kohnen (odd squarefree level), building on Waldspurger: $L(E_D, 1)$ is a positive multiple of $c(|D|)^2 / |D|^{1/2}$ for fundamental D in a sign and Atkin–Lehner class, hence ≥ 0 there.
- (2) **Central-sign theorem** [THEOREM]. If Λ is real on $\mathbb{R}$, $\Lambda(s) = \varepsilon \Lambda(1-s)$ and $\Lambda(\sigma) > 0$ for large σ , then the leading Taylor coefficient of Λ at $\frac{1}{2}$ has sign $(-1)^{N_{\mathbb{R}}}$, where $N_{\mathbb{R}}$ counts real zeros in $(\frac{1}{2}, \infty)$ with multiplicity. For $\varepsilon = +1$ and $\Lambda(\frac{1}{2}) \neq 0$: $\Lambda(\frac{1}{2}) > 0 \iff N_{\mathbb{R}}$ even; for $\varepsilon = -1$ and a simple central zero: $\Lambda'(\frac{1}{2}) > 0 \iff N_{\mathbb{R}}$ even. Complex ghosts contribute positive factors and are invisible. *Correction*: the parent’s “ $\Lambda(\frac{1}{2}) \geq 0 \iff \text{even}$ ” fails when $\Lambda(\frac{1}{2}) = 0$; use the leading coefficient.
- (3) **What positivity says** [THEOREM]. The Kohnen–Zagier and Gross–Zagier inequalities are therefore *parity statements about real zeros*: when the value is nonzero, the number of real zeros in $(\frac{1}{2}, 1)$ is even. They say nothing about complex zeros. Kohnen–Zagier note that a negative value “would contradict the Riemann hypothesis for $L(f, D, s)$ ”, which is the converse direction of the same theorem.
- (4) **Twists of 11a1** [NUMERICAL]. All 3040 fundamental D with $|D| \leq 5000$: $L(E_D, 1) \geq 0$ in every case (1512 with $\varepsilon = +1$, of which 182 vanish; minimum positive value 0.0421168 at $D =$

- −4799). Against the weight-3/2 Kohnen form of level 44, computed here, $|D|^{1/2}L(E_D, 1)/c(|D|)^2$ is constant ($= 2.9176332338769904586617792258$) for all 631 eligible D with $c \neq 0$ (relative spread 7×10^{-37}), and $c(|D|) = 0 \iff L(E_D, 1) = 0$ (66 cases).
- (5) **Gross–Zagier check** [NUMERICAL]. For 37a1 and the 38 Heegner discriminants $|D| < 400$: $L'(E/K, 1)u^2|D|^{1/2}/(\|\omega\|^2\hat{h}(P)) = 2n_D^2$ with $n_D \in \{1, \dots, 7\}$ an integer in all 32 nonzero cases, to 10^{-36} ($P = (0, 0)$). The other six D are exactly those with $L(E^D, 1) = 0$.
 - (6) **Davenport–Heilbronn** [NUMERICAL]. $\Lambda_{\text{DH}}(\frac{1}{2}) = 1.43319 > 0$, and $f_{\text{DH}} > 0$ on $[\frac{1}{2}, \infty)$ (Arb-certified on $[\frac{1}{2}, 2]$, elementary beyond), so $N_{\mathbb{R}} = 0$, consistent with the central-sign theorem, although f_{DH} has the off-line zero $0.8085171824566 + 85.6993484854i$. The central sign is RH-blind, like the detectors of Lemmas 80 and 85. (DH has no Euler product; this is a test case, not a counterexample to anything.)
 - (7) **Corrections to [t370]** [NUMERICAL]/[THEOREM]. The “twisted period matrix” (diagonal 1, off-diagonal uniform in $(-0.2, 0.2)$, no $c(|D|)$) is positive definite by Gershgorin only for $M \leq 6$; it becomes indefinite at the semicircle threshold $M = 1/(4\sigma^2) = 18.75$ asymptotically, with simulated $P(\text{indefinite}) = 1\%, 50\%, 95\%$ first at $M = 18, 25, 30$. A Gram matrix of real $c(|D|)$ is PSD trivially. “Sato–Tate dictates cross-terms” is unfounded; no matrix algebra over $\mathbb{Q}$ is constructed; “positivity forces GRH inequalities” holds only at the centre, and only for real zeros.
 - (8) **Landau–Siegel** [CITED]/[THEOREM]. Real zeros are exactly what central signs see. Iwaniec–Sarnak tie “more than 50% positive central values” to Landau–Siegel zeros; Conrey–Soundararajan prove $L(\sigma, \chi_{-8d}) > 0$ on all of $[0, 1]$ for $\geq 20\%$ of d ; Goldfeld plus Gross–Zagier give an effective class number bound. Family statistics (Iwaniec–Luo–Sarnak, Katz–Sarnak) are averages, not Weil positivity.
 - (9) **Bloch–Kato** [CITED]/[CONJECTURE]. The Mordell–Weil group sits in $H_f^1(\mathbb{Q}, V_\ell E)$ (Bloch–Kato Prop. 5.4); its dimension conjecturally equals the order of the central zero; Gross–Zagier–Kolyvagin prove this when that order is ≤ 1 . For ζ the relevant group $H_f^1(\mathbb{Q}, \mathbb{Q}_\ell(1)) = 0$, and the centre $\frac{1}{2}$ is not an integer, so there is no motivic object there at all. *Verdict*: central-point algebraic positivity lives where the two-dimensional “spinor” $V_\ell E$ lives, and by §3 it cannot see the complex zeros RH is about.

2. (A) THE VERIFIED CORE, WITH EXACT HYPOTHESES

2.1. Gross–Zagier.

Theorem 2.1 (Gross–Zagier 1986, Ch. I (6.3), (6.5); Ch. V (2.1)). *[CITED] Let $E/\mathbb{Q}$ be an elliptic curve of conductor N with newform f and modular parametrization $\pi : X_0(N) \rightarrow E$, $\pi^*\omega = c\omega_f$ (ω a Néron differential, $c > 0$ the Manin constant). Let $K = \mathbb{Q}(\sqrt{D})$ be imaginary quadratic with discriminant D prime to N and $D \equiv \square \pmod{4N}$ (so Heegner points of discriminant D exist; Ch. I §3), and, as assumed throughout that paper, D odd. Let $u = \#\mathcal{O}_K^\times/2$ ($u = 1$ unless $D = -3$) and $P_K = \sum_{\sigma \in \text{Gal}(H/K)} \pi(x^\sigma) \in E(K)$ for a Heegner point x . Then $L(E/K, s) = L(E, s)L(E^D, s)$ vanishes at $s = 1$ and*

$$L'(E/K, 1) = \frac{\|\omega\|^2 \hat{h}_K(P_K)}{c^2 u^2 |D|^{1/2}}, \quad \|\omega\|^2 = \int_{E(\mathbb{C})} |\omega \wedge \bar{\omega}|,$$

with $\hat{h}_K$ the canonical height over K ; heights over $H, K, \mathbb{Q}$ satisfy $\langle \cdot, \cdot \rangle_H = h\langle \cdot, \cdot \rangle_K = 2h\langle \cdot, \cdot \rangle_{\mathbb{Q}}$ (Ch. I (6.4)).

Positivity. The height is “the canonical Néron–Tate height associated to $2\Theta \dots$ Since this divisor is ample, $\hat{h}$ defines a positive definite quadratic form on the real vector space $J(H) \otimes \mathbb{R}$ ” (Ch. I §4). Hence $L'(E/K, 1) \geq 0$, with equality iff P_K is torsion [CITED]. The root number -1 of $L(E/K, s)$ is a consequence of the Heegner condition (Ch. I (5.5)), not an extra hypothesis. Later work removes “ D odd”; we use only the 1986 statement.

What Gross–Zagier do with it [CITED]. Thm. (7.3): if $L(E, 1) = 0$, $L'(E, 1) = \alpha \Omega \hat{h}(P)$ with $P \in E(\mathbb{Q})$, $\alpha \in \mathbb{Q}$. Prop. (7.4): a curve with $\text{ord}_{s=1} L(E, s) = \text{rank } E(\mathbb{Q}) = 3$. §8: with Goldfeld’s theorem this gives an *effective* bound $h(D) > \kappa(\epsilon)(\log |D|)^{1-\epsilon}$. Kolyvagin (with auxiliary nonvanishing, e.g. Bump–Friedberg–Hoffstein) then gives $\text{rank} = \text{analytic rank}$ and Sha finite when the analytic rank is ≤ 1 [BIB] (Lemma 85, correction (4)).

2.2. Waldspurger, Kohnen–Zagier, Kohnen.

Theorem 2.2 (Kohnen–Zagier 1981, Thm. 1 and Cor. 1). [CITED] Let $f \in S_{2k}(\text{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform and $g = \sum c(n)q^n \in S_{k+1/2}^+(4)$ a form corresponding to f . For every fundamental discriminant D with $(-1)^k D > 0$,

$$\frac{c(|D|)^2}{\langle g, g \rangle} = \frac{(k-1)!}{\pi^k} |D|^{k-1/2} \frac{L(f, D, k)}{\langle f, f \rangle}, \quad \text{hence } L(f, D, k) \geq 0.$$

Kohnen–Zagier add two remarks that matter here: Waldspurger’s proportionality “would imply only that the values $L(f, D, k)$ for a given f are either all ≥ 0 or all ≤ 0 ; of course, one would expect them to be ≥ 0 since the contrary would contradict the Riemann hypothesis for $L(f, D, s)$ ”; and there is no direct proof, “since the point $s = k$ lies half a unit to the left of the abscissa of absolute convergence” [CITED].

Theorem 2.3 (Kohnen 1985, odd squarefree level; as restated by Ono–Skinner, eq. (7)). [CITED-sec] Let $F \in S_{2k}(\Gamma_0(M))$ be a newform, M odd squarefree. There is $0 \neq g = \sum b(n)q^n \in S_{k+1/2}^+(4M)$ with

$$L(F_D, k) = 2^{-\nu(M)} |D|^{1/2-k} \frac{\pi^k}{(k-1)!} \frac{\langle F, F \rangle}{\langle g, g \rangle} |b(|D|)|^2$$

for every fundamental D with $(-1)^k D > 0$ and $\chi_D(\ell) = w_\ell$ for each prime $\ell \mid M$, where w_ℓ is the Atkin–Lehner eigenvalue of F at ℓ . In particular $L(F_D, k) \geq 0$ for these D .

The restrictions for 11a1 [THEOREM]. $k = 1$, $M = 11$, $\varepsilon(E) = +1$, and for weight 2, $\varepsilon = -w_N$, so $w_{11} = -1$. The class is: $D < 0$ fundamental, $(\frac{D}{11}) = -1$. For such D , $\varepsilon(E_D) = \chi_D(-11) = \chi_D(-1)\chi_D(11) = +1$, as it must be. For $D < 0$ with $(\frac{D}{11}) = +1$ the twist has $\varepsilon = -1$ and $L(E_D, 1) = 0$ trivially.

General nonnegativity (status). For D outside the Kohnen class (e.g. $D > 0$ here) nonnegativity of $L(E_D, 1)$ is expected and is often attributed to Waldspurger’s work; we did not verify a source for a general statement and do not cite one. What we can say: it follows from GRH for $L(E_D, s)$ by Theorem 3.1 below (this is Kohnen–Zagier’s remark) [THEOREM], and it holds in every case we computed (§4) [NUMERICAL].

2.3. Normalizations. In the arithmetic normalization the centre is $s = 1$ (Gross–Zagier, Kohnen–Zagier with $k = 1$); in the analytic one it is $\frac{1}{2}$ (Iwaniec–Sarnak write $L(\frac{1}{2}, f)$). The strip $0 < \text{Re } s < 1$ (analytic) is $\frac{1}{2} < \text{Re } s < \frac{3}{2}$ (arithmetic). “Real zeros in $(\frac{1}{2}, 1)$ ” below are analytic; arithmetically they lie in $(1, \frac{3}{2})$.

3. (B) THE CENTRAL-SIGN THEOREM

Theorem 3.1 (central sign). [THEOREM] Let Λ be entire (or merely continuous on $[\frac{1}{2}, \infty)$ and analytic near $\frac{1}{2}$), real on $\mathbb{R}$, not identically zero, with $\Lambda(s) = \varepsilon \Lambda(1-s)$, $\varepsilon = \pm 1$, and $\Lambda(\sigma) > 0$ for all $\sigma \geq \sigma_0$. Let $m = \text{ord}_{s=1/2} \Lambda$, $a_m = \Lambda^{(m)}(\frac{1}{2})/m!$, and let $N_{\mathbb{R}}$ be the number of zeros of Λ in $(\frac{1}{2}, \infty)$, counted with multiplicity. Then

$$(-1)^m = \varepsilon \quad \text{and} \quad \text{sgn } a_m = (-1)^{N_{\mathbb{R}}}.$$

In particular: if $\varepsilon = +1$ and $\Lambda(\frac{1}{2}) \neq 0$, then $\Lambda(\frac{1}{2}) > 0 \iff N_{\mathbb{R}}$ is even; if $\varepsilon = -1$ and $m = 1$, then $\Lambda'(\frac{1}{2}) > 0 \iff N_{\mathbb{R}}$ is even.

Proof. $t \mapsto \Lambda(\frac{1}{2} + t)$ is even ($\varepsilon = 1$) or odd ($\varepsilon = -1$), so m has the parity of ε . The real zeros in $(\frac{1}{2}, \infty)$ lie in $(\frac{1}{2}, \sigma_0)$ and are isolated, so $N_{\mathbb{R}} < \infty$. For small $t > 0$, $\text{sgn } \Lambda(\frac{1}{2} + t) = \text{sgn } a_m$. Moving along the real axis from $\frac{1}{2} + t$ to σ_0 , Λ changes sign exactly at zeros of odd order; the total number of sign changes has the parity of $N_{\mathbb{R}}$. Since $\Lambda(\sigma_0) > 0$, $\text{sgn } a_m = (-1)^{N_{\mathbb{R}}}$. $\square$

No Hadamard product, growth condition or Euler product is needed. Complex zeros play no role, because the argument never leaves the real axis.

Corollary 3.2 (product form). *[THEOREM] If Λ is entire of order 1 and $\varepsilon = +1$, write $z = s - \frac{1}{2}$; then $\Lambda(\frac{1}{2} + z) = a_m z^m \prod_w (1 - z^2/w^2)$, the product over the zeros $w = \rho - \frac{1}{2} \neq 0$ taken modulo $w \sim -w$ (it converges absolutely: as a function of z^2 , Λ has order $\frac{1}{2}$). For real $x \rightarrow +\infty$ the factors are: on-line zero $w = i\gamma$: $1 + x^2/\gamma^2 > 0$; complex ghost quadruple $w = \pm(\beta \pm i\gamma)$: $|1 - x^2/(\beta \pm i\gamma)^2|^2 \geq 0$; real ghost $w = b \in (0, \frac{1}{2})$: $1 - x^2/b^2 < 0$ for $x > b$. So only real ghosts change the sign.*

The parent's form [THEOREM]. Normalizing instead at $s = 0$, as for $\xi(\frac{1}{2})/\xi(0) = \prod_{\rho}(1 - \frac{1}{2\rho})$ (paired), a complex quadruple $\rho = \frac{1}{2} \pm \beta \pm i\gamma$ contributes $\prod w/\rho = (\beta^2 + \gamma^2)^2/(|\rho|^2|1 - \rho|^2)$. The parent's " $(\beta^2 + \gamma^2)^2 > 0$ " is the numerator; the denominator is positive, so the conclusion stands. A real pair $\frac{1}{2} \pm b$ contributes $-b^2/(\frac{1}{4} - b^2) < 0$. For the DH ghost the quadruple factor is 0.999931926841 = 53941291.193/53944963.397 (§5).

3.1. Consequences.

Corollary 3.3 (positivity theorems as real-zero parity). *[THEOREM](i) For 11a1 and every fundamental D in the Kohnen class with $L(E_D, 1) \neq 0$, the number of real zeros of $L(E_D, s)$ in $(1, \frac{3}{2})$ (arithmetic), with multiplicity, is even. (ii) Under the hypotheses of Theorem 2.1, if $L'(E/K, 1) \neq 0$ the number of real zeros of $L(E/K, s)$ in $(1, \frac{3}{2})$ is even. (iii) Conversely, GRH for $L(E_D, s)$ implies $L(E_D, 1) \geq 0$ for every D , inside or outside the Kohnen class.*

Proof. Apply Theorem 3.1 to $\Lambda(E_D, s + \frac{1}{2})$ (resp. $\Lambda(E/K, s + \frac{1}{2}) = \Lambda(E, s + \frac{1}{2})\Lambda(E^D, s + \frac{1}{2})$): it is real on $\mathbb{R}$ (real coefficients) and positive for large real s ($L \rightarrow 1$, $\Gamma > 0$). There are no zeros with $\text{Re } s \geq \frac{3}{2}$ (arithmetic): the Euler product converges absolutely for $\text{Re } s > \frac{3}{2}$, and $L \neq 0$ on $\text{Re } s = \frac{3}{2}$ (Jacquet–Shalika [BIB]). So $N_{\mathbb{R}}$ counts zeros in $(1, \frac{3}{2})$. Use Theorems 2.3 and 2.1 for (i), (ii). For (iii), GRH gives $N_{\mathbb{R}} = 0$. $\square$

So the algebraic positivity proved by Kohnen–Zagier and Gross–Zagier is, analytically, the statement that real zeros come in even numbers. Since GRH predicts zero, the inequality is a *consequence* of GRH that happens to be provable; it carries no information about complex zeros.

3.2. Function fields, κ , and Lemmas 80 and 85.

Remark 3.4 (function fields). *[THEOREM] For $L(T) = \prod_i (1 - \alpha_i T)$ with $\alpha \mapsto q/\alpha$ symmetry, the centre is $T = q^{-1/2}$; put $x_i = \alpha_i/\sqrt{q}$. An on-circle pair gives $|1 - x|^2 \geq 0$; an off-circle quadruple $\{x, 1/x, \bar{x}, 1/\bar{x}\}$ gives $|1 - x|^4/|x|^2 > 0$; a real pair $\{x, 1/x\}$ gives $(1 - x)(1 - 1/x) = -(1 - x)^2/x$, negative for $x > 0$, positive for $x < 0$. A real $x > 0$ off the circle is a real zero $s \in (\frac{1}{2}, 1)$ of $L(q^{-s})$; $x < 0$ corresponds to $\text{Im } s = \pi/\log q$, which is not on the real axis. This matches Theorem 3.1 exactly.*

κ [THEOREM]. Split $\kappa = \kappa_{\mathbb{R}} + \kappa_{\mathbb{C}}$ into real ghost pairs and complex ghosts (Lemma 84 counts a quadruple as 2). Theorem 3.1 says $\text{sgn } a_m = (-1)^{\kappa_{\mathbb{R}}}$ (each real ghost pair has one member in $(\frac{1}{2}, 1)$, and there are no real zeros ≥ 1 when $\Lambda \neq 0$ on $\text{Re } s \geq 1$). The central sign is a mod-2 invariant of $\kappa_{\mathbb{R}}$ alone; $\kappa_{\mathbb{C}}$, the part RH is mostly about, is invisible. Lemma 84's trace form sees all of κ ; the central sign sees one bit of one part.

RH-blind detectors [TAA-L]/[THEOREM]. Lemma 80: positivity from scaling-commuting multipliers is blind (or restates Weil, or needs the zeros). Lemma 85 §(I): the Fricke split/modularity detector

sees the functional equation and passes DH to 10^{-41} despite its ghost. Lemma 82: the cutoff Weil form does see the DH ghost. The central sign joins the first two: it is blind to every complex ghost, and §5 confirms this on DH. A third kind of blindness, distinct from both: it sees *some* ghosts (real ones), and only their parity.

4. (C) NUMERICS: TWISTS OF 11A1 AND A GROSS–ZAGIER CHECK FOR 37A1

4.1. The weight-3/2 form, computed. [NUMERICAL] With PARI, `mfini([44,3/2],1)` has dimension 2 and `mfkohnenbasis` gives a one-dimensional plus space $S_{3/2}^+(44)$, matching $\dim S_2(\Gamma_0(11)) = 1$. Its generator, normalized with $c(3) = 1$, is

$$g = q^3 - q^4 - q^{11} - q^{12} + q^{15} + 2q^{16} + q^{20} - q^{23} - q^{27} - q^{31} + q^{44} + q^{55} + 2q^{56} - q^{59} - 3q^{60} + \dots$$

(coefficients computed to $n = 5000$, range $-12 \leq c(n) \leq 20$), and its Shimura lift (`mfshimura`, $t = 3$) equals the 11a1 newform $q - 2q^2 - q^3 + 2q^4 + q^5 + \dots$ in the first 40 coefficients. We did not take g from the literature.

4.2. All fundamental D with $|D| \leq 5000$. [NUMERICAL] Script `num/lemma86_twists.py`: for each of the 3040 fundamental discriminants $D \neq 1$ with $|D| \leq 5000$, `ellrootno` and, if $\varepsilon(E_D) = +1$, `elll1(E_D, 0)` at 38 digits. Results (`num/186/twist_stats.json`):

class	# D	$\varepsilon(E_D)$	result
$D < 0, (\frac{D}{11}) = -1$ (Kohnen)	697	+1	$L \geq 0$; $c \neq 0$ for 631, ratio constant; $c = 0 \iff L = 0$ (66)
$D < 0, (\frac{D}{11}) = +1$	702	-1	$L = 0$ forced; $c(D) = 0$ in all cases
$D < 0, 11 \mid D$	125	+1	$L \geq 0$; ratio = $2 \times$ constant (109 with $c \neq 0$); $c = 0 \iff L = 0$ (16)
$D > 0, (\frac{D}{11}) = +1$	690	+1	$L \geq 0$; 100 zeros
$D > 0, (\frac{D}{11}) = -1$ or $11 \mid D$	698+128	-1	$L = 0$ forced

Totals: $\varepsilon = +1$ for 1512 D , $\varepsilon = -1$ for 1528. Among the 1512: no negative value (the smallest computed value is -1.4×10^{-38} , i.e. zero at working precision); 182 zeros ($|L| \leq 1.1 \times 10^{-38}$; even analytic rank ≥ 2); 1330 positive values, minimum 0.0421167957 ($D = -4799$), maximum 23.674.

Proportionality [NUMERICAL]. For the 631 Kohnen-class D with $c(|D|) \neq 0$,

$$\frac{|D|^{1/2} L(E_D, 1)}{c(|D|)^2} = 2.917633233876990458661779225807 \dots,$$

with relative spread 7.1×10^{-37} over all 631 (Fig. 1(b)). This is Theorem 2.3 with $k = 1$, checked to working precision. Observation [NUMERICAL]: the constant equals $2|\operatorname{Im} \omega_2|$ for PARI's period lattice $[\omega_1, \omega_2]$ of 11a1 ($\omega_2 = 0.6346 \dots - 1.45881661693849522933 \dots i$) to 30 digits; we record this without deriving it (it depends on the normalization $c(3) = 1$). For $11 \mid D$, outside Kohnen's hypotheses, the same ratio is exactly twice the constant (109 cases, to 1.3×10^{-37}); we did not find this stated and leave it as an observation.

4.3. Gross–Zagier consistency check for 37a1. [NUMERICAL] Script `num/lemma86_gz.py`. $E = 37a1$ ($y^2 + y = x^3 - x$, $\varepsilon = -1$, $E(\mathbb{Q}) = \mathbb{Z}P$, $P = (0, 0)$, trivial torsion). For the 38 discriminants D with $|D| < 400$, odd, fundamental, prime to 37 and $D \equiv \square \pmod{148}$, we computed $L'(E/K, 1) = L'(E, 1)L(E^D, 1)$ (here $\varepsilon(E^D) = +1$ automatically) and

$$R_D = \frac{L'(E/K, 1) u^2 |D|^{1/2}}{\|\omega\|^2 \hat{h}(P)}, \quad \|\omega\|^2 = 2 \cdot \operatorname{area}(\mathbb{C}/\Lambda),$$

with PARI's `E.area` and `ellheight`. We do *not* compute P_K . But when $L(E^D, 1) \neq 0$, the twist has rank 0 (Kolyvagin), so $P_K \in E(\mathbb{Q}) \otimes \mathbb{Q} = \mathbb{Q}P$; Theorem 2.1 then predicts $R_D = \lambda n_D^2$ with $n_D = [E(\mathbb{Q}) : \mathbb{Z}P_K]$ and a fixed normalization constant λ (from $\hat{h}_K = 2\hat{h}_\mathbb{Q}$ and PARI's height normalization; with $c = 1$). Found: $R_D = 2n_D^2$ for all 32 D with $L(E^D, 1) \neq 0$, $n_D \in \{1, 2, \dots, 7\}$,

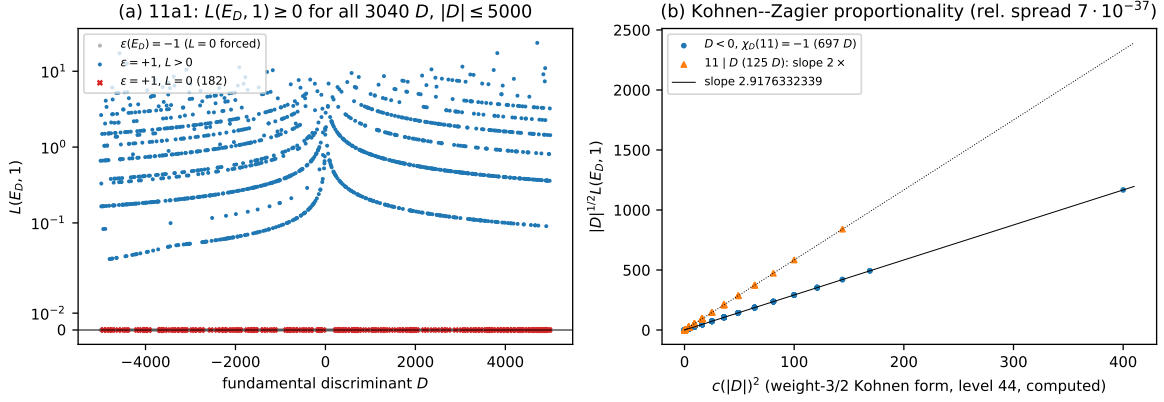


FIGURE 1. 11a1. (a) $L(E_D, 1)$ for all 3040 fundamental D , $|D| \leq 5000$ (symlog scale): no negative value; grey: $\varepsilon = -1$; red: $\varepsilon = +1$ zeros. The sheets are the values $c(|D|)^2 \cdot \text{const}/|D|^{1/2}$ for $c^2 = 1, 4, 9, \dots$ (for $D < 0$). (b) $|D|^{1/2} L(E_D, 1)$ against $c(|D|)^2$: a line through the origin (Kohnen) and a line of twice the slope ($11 \mid D$). *Diagnostic*: a negative value, or a point off the lines, would falsify the computation or the theorem.

maximum deviation 1.0×10^{-36} ; for $D = -3$, $u = 3$ and $n_D = 3$, consistent with the factor u in Gross–Zagier’s Conjecture (2.2). The six D with $L(E^D, 1) = 0$ ($-95, -107, -139, -215, -255, -391$) give $L'(E/K, 1) = 0$, so P_K is torsion there (Fig. 2). This checks the shape of the formula (a square times a constant), not the constant itself, and not Conjecture (2.2).

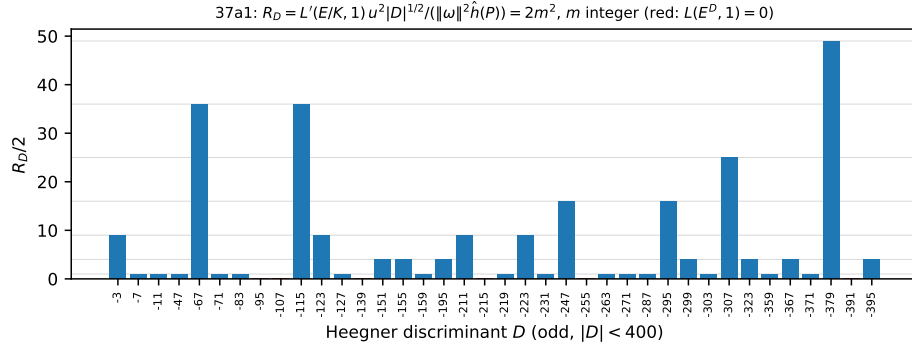


FIGURE 2. 37a1, Heegner discriminants $|D| < 400$: $R_D/2$ is a perfect square n_D^2 (written m^2 in the plot title; grid lines) in all 32 nonzero cases. *Diagnostic*: a non-square would contradict Theorem 2.1 (or the rank-0 input).

5. (D) DAVENPORT–HEILBRONN: THE CENTRAL SIGN IS RH-BLIND

[NUMERICAL] We use the Davenport–Heilbronn function of Lemma 85: $f(s) = \sum c(n)n^{-s}$, c periodic mod 5 with $(c(1), \dots, c(5)) = (1, \kappa, -\kappa, -1, 0)$, $\kappa = \frac{\sqrt{10-2\sqrt{5}-2}}{\sqrt{5}-1} = 0.28407904384041229603$, and $\Lambda(s) = (\pi/5)^{-(s+1)/2} \Gamma(\frac{s+1}{2}) f(s)$, which satisfies $\Lambda(s) = \Lambda(1-s)$ (checked: $|\Lambda(\sigma) - \Lambda(1-\sigma)| < 4 \times 10^{-31}$ at $\sigma = 0.2, 0.35, 0.45$). Script `num/lemma86_dh.py`, output `num/186/dh.json`.

- The off-line zero is $\rho = 0.8085171824566374 + 85.69934848537759i$ ($|f(\rho)| = 2 \times 10^{-30}$), the Lemma 85 value. By the functional equation and reality, $\rho, \bar{\rho}, 1 - \rho, 1 - \bar{\rho}$ are all zeros: a complex ghost quadruple.
- $f(\frac{1}{2}) = 0.82538302382110216695$, $\Lambda(\frac{1}{2}) = 1.4331921687750084417 > 0$.
- *Certified*: $f(\sigma) > 0$ on $[\frac{1}{2}, 2]$. Arb interval arithmetic was used on 3000 subintervals, writing $f = \frac{1-i\kappa}{2}L(s, \chi) + \frac{1+i\kappa}{2}L(s, \bar{\chi})$, $\chi \bmod 5$ with $\chi(2) = i$; near $s = 1$ we evaluated at $1 - s$ via the functional equation. The smallest certified lower bound was 0.4557. For $\sigma \geq 2$, $|f(\sigma) - 1| \leq \zeta(2) - 1 = 0.6449 < 1$. So $N_{\mathbb{R}} = 0$.

Thus $\Lambda_{\text{DH}}(\frac{1}{2}) > 0$, exactly as Theorem 3.1 predicts with $N_{\mathbb{R}} = 0$, *despite* the off-line zero. In the $s = 0$ normalization the quadruple contributes $(\beta^2 + \gamma^2)^2 / (|\rho|^2 |1 - \rho|^2) = 53941291.193 / 53944963.397 = 0.999931926841$; in the centre-normalized product it contributes 1 at the centre. So the central sign is blind to complex ghosts on an explicit example (Fig. 3).

Caveat [THEOREM]. DH has no Euler product. It is not an L -function of an automorphic form or motive, so no Waldspurger/Gross–Zagier theorem applies to it, and its positivity at $\frac{1}{2}$ is a numerical fact, not an instance of Theorem 2.2. DH shows that the *implication* “central positivity $\Rightarrow$ no ghosts” is false for functions with a functional equation. It cannot rule out that Euler products somehow tie the two together. But Theorem 3.1 shows that no such tie can come from the sign alone: the sign is determined by real zeros, for every Λ .

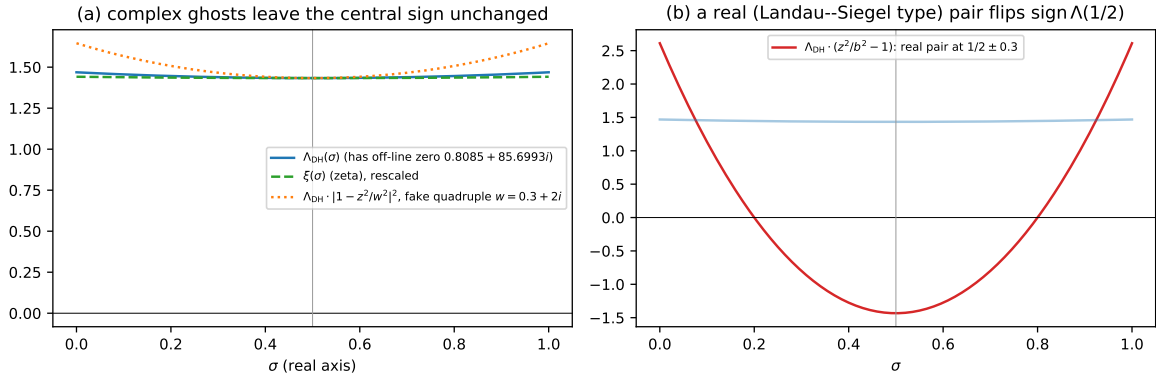


FIGURE 3. (a) $\Lambda_{\text{DH}}(\sigma)$ on $[0, 1]$ (it has the off-line quadruple at $0.8085 \pm 85.6993i$), $\xi(\sigma)$ rescaled, and Λ_{DH} multiplied by a fake complex quadruple factor: all positive at $\frac{1}{2}$. (b) Multiplying by a real pair at $\frac{1}{2} \pm 0.3$ flips the sign. *Diagnostic*: this is Theorem 3.1 made visible. Only real zeros change the sign on the real axis.

6. (E) THE PASTED “TWISTED PERIOD MATRIX” TEXT (T370)

Note. The verbatim t370 text was not available to the formalizer (the main chat is not readable from the build). The items below follow the parent’s description of it.

6.1. The random matrix. [NUMERICAL] The text proposes a “twisted period matrix” with unit diagonal and off-diagonal entries “bounded by 0.2” (modelled as random), which is positive definite by Gershgorin at $M = 5$. We computed this model: $A = I + B$, B symmetric, $B_{ij} \sim U(-0.2, 0.2)$ i.i.d. ($i < j$), 20000 trials per M (script `num/lemma86_rmt.py`, seed 86).

- *Gershgorin* [THEOREM]: with $|B_{ij}| < 0.2$ strictly, $\lambda_{\min} > 1 - 0.2(M - 1) \geq 0$ for $M \leq 6$, so PD is guaranteed up to $M = 6$, not just 5. At $M = 7$ the bound fails and is sharp: $B_{ij} = -a'$ with $a' \uparrow 0.2$ gives $\lambda_{\min} = 1 - 6a' \rightarrow -0.2$ (worst case $1 - 0.2(M - 1)$: 0.2, 0, -0.2, ... at $M = 5, 6, 7$).

- *Random* [NUMERICAL]: no indefinite matrix in 20000 trials for $M \leq 12$; $P(\text{indef}) = 0.0109$ at $M = 18$, 0.0472 at $M = 20$, 0.533 at $M = 25$, 0.981 at $M = 30$, and 1 from $M = 35$ on. The median $\lambda_{\min}$ first goes negative at $M = 25$.
- *Semicircle* [HEURISTIC]: $\sigma = 0.2/\sqrt{3}$, $\lambda_{\min} \approx 1 - 2\sigma\sqrt{M}$, which vanishes at $M = 3/(4 \cdot 0.04) = 18.75$. This is where the parent’s “ $M \approx 20$ ” comes from. Because of fluctuations it is a lower estimate: the actual 50% point is $M = 25$.

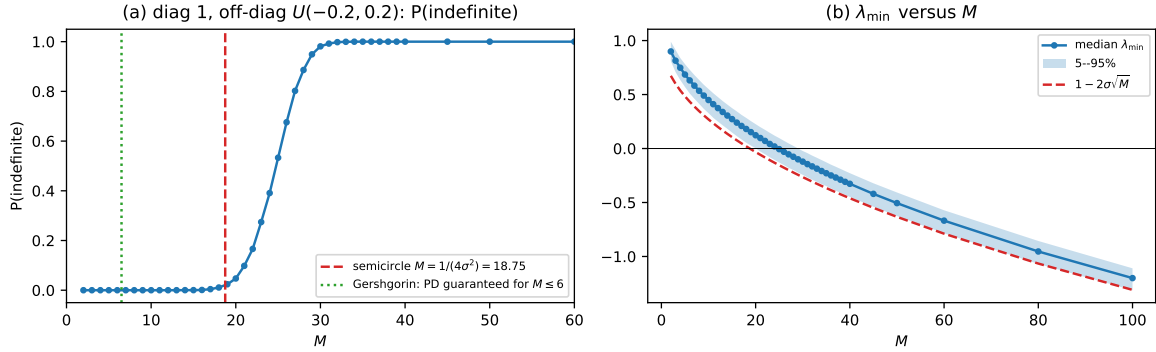


FIGURE 4. The t370 random model. (a) Probability of indefiniteness against M ; Gershgorin guarantees PD only for $M \leq 6$; the semicircle prediction 18.75. (b) $\lambda_{\min}$ (median, 5–95% band) against $1 - 2\sigma\sqrt{M}$. *Diagnostic*: a matrix with random entries is a model of nothing arithmetic; its definiteness is a property of the random model and has nothing to do with L -functions.

6.2. What is wrong with the text, item by item.

- T1 (random matrix) [RESTATEMENT]:** Nothing in Waldspurger or Kohnen–Zagier produces a matrix with random off-diagonal entries. Positive definiteness at $M = 5$ and loss of it near $M \approx 20$ are properties of $U(-0.2, 0.2)$, not of L -values. The computation is correct for the model, but the model is irrelevant.
- T2 (Gram matrix) [THEOREM]:** If one forms $G_{DD'} = c(|D|)c(|D'|)$, or any Gram matrix $\langle v_D, v_{D'} \rangle$, it is PSD trivially; $G = cc^T$ has rank 1. Kohnen–Zagier is a statement about each *diagonal* entry, $c(|D|)^2 \propto |D|^{1/2}L(E_D, 1)$; it says nothing about cross-terms, and there is no positivity content beyond the diagonal.
- T3 (“Sato–Tate dictates cross-terms”) [RESTATEMENT]:** Sato–Tate (not cited or used here) concerns $a_p(E)/2\sqrt{p}$ for a fixed E as p varies. It says nothing about products $c(|D|)c(|D'|)$ of half-integral weight coefficients across discriminants. The claim is unfounded.
- T4 (“infinite-dimensional matrix algebra over $\mathbb{Q}$ ”) [OPEN]:** No such algebra, positive involution or trace form is constructed. Naming one is not constructing one. (Lemma 84 built a finite-dimensional trace form whose signature counts ghosts; the text does not connect to it.)
- T5 (“positivity forces GRH inequalities”) [THEOREM]:** Only at the centre, and only about real zeros. By Corollary 3.3, a nonzero central value is positive if and only if the count is even, i.e. central positivity is equivalent to evenness of the real-zero count in $(\frac{1}{2}, 1)$ (analytic), and it is *implied* by GRH (Kohnen–Zagier’s own remark). It does not imply any inequality off the centre.

7. (F) LANDAU–SIEGEL ZEROS AND FAMILIES

7.1. Where central signs do see something. A Landau–Siegel zero is a real zero β of $L(s, \chi_D)$ close to 1, i.e. a *real ghost* (analytic normalization). Theorem 3.1 applies to $\Lambda(s, \chi_D)$ (real primitive χ_D , $\varepsilon = +1$, positive for large s , no zeros on $\operatorname{Re} s \geq 1$): a single simple real zero in $(\frac{1}{2}, 1)$ forces $L(\frac{1}{2}, \chi_D) < 0$. So the central sign is the one place where central positivity meets the ghost problem, and it does so only for real ghosts [THEOREM].

Corollary 7.1. [THEOREM] *For 11a1 and every fundamental $D < 0$ with $(\frac{D}{11}) = -1$ and $L(E_D, 1) \neq 0$ (631 of the 697 such D with $|D| \leq 5000$), $L(E_D, s)$ does not have an odd number of real zeros in $(1, \frac{3}{2})$ (arithmetic). In particular it does not have exactly one simple “Siegel-type” zero there. The same holds for $L(E/K, s)$ under Theorem 2.1 when $L'(E/K, 1) \neq 0$.*

This is Corollary 3.3(i), (ii) restated. It is unconditional and individual, but weak: it excludes odd counts and says nothing about pairs. We know of no analogous unconditional theorem for $L(\frac{1}{2}, \chi_D)$ and do not claim one.

7.2. What is known on average (cited, not reproved).

- [CITED-sec] Iwaniec–Sarnak (2000), from the abstract: for newforms of fixed even weight on $\Gamma_0(N)$ with even functional equation, at least 50% of the values $L(\frac{1}{2}, f)$ are positive as $N \rightarrow \infty$, and “any improvement of 50% is intimately connected to Landau–Siegel zeros”. This is an *average* statement over a family. It is the precise sense in which “central positivity bears on Siegel zeros”: through *proportions* of positive central values, and only for real zeros.
- [CITED-sec] Conrey–Soundararajan (2002), from the abstract: at least 20% of odd squarefree d have $L(\sigma, \chi_{-8d}) > 0$ for all $0 \leq \sigma \leq 1$, i.e. no real zeros at all. This is again a proportion, and again about real zeros only.
- [CITED-sec] Iwaniec–Luo–Sarnak (2000) and Katz–Sarnak (1999): low-lying zeros (near $\frac{1}{2}$) of families follow the symmetry type of the family (orthogonal for quadratic twists of a fixed E). These are statements about averages over a family, most results of ILS conditional on GRH. They do not locate any individual zero and say nothing about whether a given L -function has a ghost.
- [CITED-sec] Siegel (1935): $h(D) \gg_\epsilon |D|^{1/2-\epsilon}$, ineffective because of a possible real zero. Goldfeld (1976) with Gross–Zagier (§8, Thm. (8.1)) gives the effective $h(D) > \kappa(\epsilon)(\log |D|)^{1-\epsilon}$; the input is a curve of analytic rank 3 (Prop. (7.4)). This is the one place where Gross–Zagier positivity, through heights of Heegner points, has actually bought something about Siegel zeros. It does so by bypassing them, not by excluding them.

Verdict [THEOREM]/[HEURISTIC]. Central positivity (Waldspurger, KZ, GZ) bears on Landau–Siegel zeros in exactly two ways: individually, through parity (Corollary 7.1); and on average, through proportions of positive central values (Iwaniec–Sarnak). Neither touches complex zeros.

8. (G) BLOCH–KATO: WHERE THE POSITIVITY LIVES

8.1. What Bloch–Kato define.

- [CITED] Def. 5.1: for a p -adic representation V of G_K , the global Selmer group $H_f^1(K, V)$ consists of classes in $H^1(K, V)$ that lie in the local H_f^1 at every place. *Plainly*: Galois cohomology classes that “look geometric” everywhere.
- [CITED] Prop. 5.4: for an abelian variety A , Kummer theory gives $A(K) \otimes \hat{\mathbb{Z}} \hookrightarrow H_f^1(K, T)$. This is an isomorphism for tori; for abelian varieties $A(K) \otimes \mathbb{Z}_\ell \cong H_f^1$ if and only if $\operatorname{Sha}(A)\{\ell\}$ is finite. *Plainly*: if Sha is finite, the Selmer group *is* the Mordell–Weil group (tensored up).
- [THEOREM] For $V = \mathbb{Q}_\ell(1)$ over $\mathbb{Q}$: by Prop. 5.4 for $\mathbb{G}_m$, $H_f^1(\mathbb{Q}, \mathbb{Q}_\ell(1)) = \mathbb{Z}^\times \otimes \mathbb{Q}_\ell = 0$.

8.2. The conjectural dictionary. [CONJECTURE] In the Bloch–Kato framework (as commonly formulated; we do not quote a specific statement of BK 1990 for this), for a pure motive M of odd weight w with ℓ -adic realization V (self-dual up to twist), $\text{ord}_{s=(w+1)/2} L(M, s) = \dim H_f^1(\mathbb{Q}, V(\frac{w+1}{2})) - \dim H^0(\mathbb{Q}, V(\frac{w+1}{2}))$, the centre being $s = (w+1)/2$; for $M = h^1(E)$, $V(1) = V_\ell E$ and $H^0 = 0$. *Plainly*: the order of vanishing at the centre counts independent “rational points” of the motive. For $M = h^1(E)$ this is BSD’s rank statement.

Where the centre is an integer [THEOREM]. The centre $(w+1)/2$ is an integer exactly when w is odd. For E ($w = 1$) the centre is $s = 1$: there is a motivic object there ($E(K)$, Heegner points), and the Néron–Tate height gives a positive definite form on it. This is the positivity that Gross–Zagier expresses (Theorem 2.1) and that Kohnen–Zagier/Waldspurger realize as a square $c(|D|)^2$. In the language of Lemmas 50 and 85, it “lives where the spinor lives”: on a half-integral weight form (the metaplectic, spinor-like object) or on a Heegner point. For ζ , $w = 0$, the centre $\frac{1}{2}$ is not an integer, and the required twist $V(\frac{1}{2}) = \mathbb{Q}_\ell(\frac{1}{2})$ does not exist: there is no half Tate twist, so no motivic object at the centre on which a height could be positive. (A “half Tate twist” is the missing object of Lemma 50; direction M6.)

8.3. Verdict. [THEOREM] (mathematics) and [HEURISTIC] (interpretation). Central algebraic positivity, when it exists, is a positive form on a Selmer/Mordell–Weil space attached to the centre. Through Theorem 3.1 its only analytic shadow is the parity of real zeros. It cannot see complex zeros, which is to say it cannot see what RH is about. For ζ there is not even an object for it to live on.

RH is not proved here, nor GRH, nor BSD, nor any new case of them. Lemma 86 proves an elementary theorem (Theorem 3.1) and verifies known theorems numerically. Its conclusion is negative: central positivity cannot detect complex zeros.

9. CORRECTIONS

J.W.’s chat claims.

- (J1) “ $\Lambda(\frac{1}{2}) \geq 0 \iff \text{even number of real zeros in } (\frac{1}{2}, 1)$; a complex quadruple contributes $(\beta^2 + \gamma^2)^2 > 0$.” Correct in spirit, with three fixes. (a) The equivalence fails when $\Lambda(\frac{1}{2}) = 0$ (then $\Lambda(\frac{1}{2}) \geq 0$ holds regardless); the correct statement is about the *leading Taylor coefficient*: $\text{sgn } a_m = (-1)^{N_\mathbb{R}}$ (Theorem 3.1). (b) The count is over $(\frac{1}{2}, \infty)$; it reduces to $(\frac{1}{2}, 1)$ only when there are no real zeros ≥ 1 (true for L -functions with Euler product, by absolute convergence and Jacquet–Shalika). (c) In the $s = 0$ normalization the quadruple factor is $(\beta^2 + \gamma^2)^2 / (|\rho|^2 |1 - \rho|^2)$; the numerator alone is not the factor, but the sign conclusion is unaffected. In the centre normalization the factor is 1 at the centre. Also: no Hadamard product is needed; continuity on the real axis suffices.
- (J2) “*Gershgorin gives PD at $M = 5$.*” With entries strictly inside $(-0.2, 0.2)$, Gershgorin guarantees PD for $M \leq 6$. With entries allowed to equal ± 0.2 , $M \leq 5$. At $M = 7$ adversarial choices are indefinite (§6).
- (J3) “*Indefinite at $M \approx 20$.*” $M \approx 20$ is the semicircle estimate 18.75. In the simulation $P(\text{indef}) = 0.047$ at $M = 20$, 0.53 at $M = 25$ (median $\lambda_{\min} < 0$ from $M = 25$), 0.98 at $M = 30$.
- (J4) *Kohnen–Zagier restrictions on D .* Level 1 (KZ 1981): D fundamental with $(-1)^k D > 0$. Odd squarefree level M (Kohnen 1985): additionally $\chi_D(\ell) = w_\ell$ for every $\ell \mid M$. For 11a1: $D < 0$ and $(\frac{D}{11}) = -1$. Outside that class: for $(\frac{D}{11}) = +1$, $D < 0$, the root number is -1 and $c(|D|) = 0$ (observed in all 702 cases); for $11 \mid D$, the ratio is twice the Kohnen constant (observed, 109 cases); for $D > 0$ there is no formula with this g .

The pasted t370 text. (T1)–(T5) of §6: random matrix irrelevant [RESTATEMENT]; Gram matrix PSD of rank 1, no cross-term content [THEOREM]; Sato–Tate claim unfounded [RESTATEMENT];

algebra not constructed [OPEN]; positivity only at the centre, only about real zeros, implied by GRH [THEOREM].

Self-corrections during the build. The first version of the RMT script recorded Gershgorin PD up to $M = 5$; for the open interval the correct value is 6, and script, data and figure were corrected. The arithmetic/analytic normalizations of the centre (1 vs. $\frac{1}{2}$) are now stated explicitly throughout.

10. RANKED DIRECTIONS

Score = 0.6·significance+0.4·probability (subjective, [HEURISTIC]).

	direction	sig	prob	score
M1	Siegel-zero direction: certify (Arb) that $L(E_D, s)$ has no real zeros in $(1, \frac{3}{2})$ for the 1512 root-number-+1 twists of §4; compare with Corollary 7.1, Iwaniec–Sarnak and Conrey–Soundararajan proportions	.60	.80	.68
M3	Goldfeld–Gross–Zagier mechanism made explicit: 5077a1 (rank 3), $L'''(E, 1)$, and the effective class-number bound of GZ §8 numerically	.55	.75	.63
M2	unified table of RH-blind vs. RH-seeing detectors (L80, L82, L85, L86) run on DH and on an Epstein zeta function with real/complex ghosts	.45	.85	.61
M4	Gross–Zagier full index: compute P_K for 37a1 and test Conjecture (2.2) ($n_D = cmu Sha ^{1/2}$)	.40	.80	.56
M5	Waldspurger-type formula for $D > 0$ twists of 11a1 (a second half-integral weight form) and its positivity	.35	.75	.51
M6	a “half Tate twist” / spinor object for ζ at $s = \frac{1}{2}$ (Lemma 50)	.80	.05	.50

11. QUESTIONS FOR J.W.

- Q1. Do you accept Theorem 3.1 (leading coefficient, count over $(\frac{1}{2}, \infty)$) and the verdict that central positivity is blind to complex zeros?
- Q2. Are corrections (J1)–(J4) and (T1)–(T5) acceptable, in particular (J1)(a) ($\Lambda(\frac{1}{2}) = 0$ case) and (J2) ($M \leq 6$)?
- Q3. Should TAA adopt the phrase “KZ/GZ positivity is a parity theorem for real zeros” (Corollary 3.3) as standard language?
- Q4. Do you accept the Bloch–Kato verdict (§8): the positivity lives where the spinor lives, and ζ has no object at its centre? Should the half Tate twist (Lemma 50, M6) be revisited?
- Q5. Next lemma: M1 (Siegel-zero certification, .68), M3 (Goldfeld–GZ, .63) or M2 (detector table, .61)?

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CODE AND DATA

num/lemma86_twists.py (num/186/twists_5000.json), lemma86_twist_stats.py (twist_stats.json), lemma86_gz.py (gz37.json), lemma86_dh.py (dh.json), lemma86_rmt.py (rmt.json), lemma86_figs.py (four figures in build86/figs/). PARI/GP 2.17.2 (cypari2), mpmath, python-flint (Arb), matplotlib. Nothing was posted or submitted.

TAA LEMMA 87: SADDLE LIGHT CONES. SEPARATRIX CONES OF $|\xi|$, CAUSAL DIAMONDS, THE CASSINI WALL, LEGENDRE SLOPES, HEAT-FLOW LANDINGS, AND WHY THE LANDSCAPE HAS NO GENUS

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Every RH-equivalent statement below is labelled [RESTATEMENT]; Lemma 87 reorganizes the saddle picture of Lemmas 55–56 in light-cone language, proves the elementary facts that make the language exact, and tests it numerically on ζ and on Davenport–Heilbronn (DH). Tags as in Lemmas 80–86: [THEOREM] proved here (or standard, proof given), [CITED] checked source, [BIB] bibliographic data only, [TAA-L] earlier TAA lemma, [RESTATEMENT] equivalent to RH, [ANALOGY] analogy, [CONSTRUCTION] construction (a choice is made), [OPEN] open, [NUMERICAL] computation, [HEURISTIC] heuristic. Citations: `lit/lemma87/verified_refs.md`. Every number is computed here (python-flint/Arb evaluation at 96 bits, used as floating point and not certified; mpmath; NumPy/SciPy) or read from the data files of Lemmas 48, 55, 56 named in the text; none is recalled.

Task (J.W., approved [t375u]; addition [t376]). Check J.W.’s chat claims A–G rigorously: (A) local saddle geometry; (B) the light-cone theorem; (C) causal diamonds; (D) the trivial-to-Möbius wall; (E) the Legendre relation (Lagarias, Sondow–Dumitrescu); (F) envelope/heat flow, with the precise statement of “ κ counts landings”; (G) Morse topology and genus ([t374]). Built on Lemmas 43, 55, 56, 62, 81, whose content is cited, not redone.

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Date: October 24, 2026.

NOTATION (RESTATED WHERE USED)

- Λ is $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ or $\Lambda_{\text{DH}}(s) = (5/\pi)^{s/2}\Gamma(\frac{s+1}{2})f_{\text{DH}}(s)$ (DH as in Lemmas 55, 56, 86: $f_{\text{DH}} = \sum c(n)n^{-s}$, c 5-periodic $(1, \kappa_{\text{DH}}, -\kappa_{\text{DH}}, -1, 0)$). Both are entire, real on $\mathbb{R}$, with $\Lambda(1-s) = \overline{\Lambda(s)}$, hence $\Lambda(1-\bar{s}) = \Lambda(s)$. On the critical line $\Xi(t) := \Lambda(\frac{1}{2} + it)$ is real.
- Coordinates near a point $s_0 = \frac{1}{2} + it_0$: $x = \sigma - \frac{1}{2}$, $y = t - t_0$. “Time” is t ; the *light cone* of s_0 is $|y| = |x|$; *timelike* sectors $|y| > |x|$, *spacelike* $|y| < |x|$.
- A *saddle* is a zero of Λ' where $\Lambda \neq 0$ (Lemma 55 Def. 2.1); at a saddle $c := \Lambda''(s_0)/\Lambda(s_0)$, its *height* is $h = |\Lambda(s_0)|$. On-line saddle t_0 : $c = -\Xi''/\Xi(t_0) \in \mathbb{R}$; *longitudinal* if $c > 0$ (i.e. $\Xi''/\Xi < 0$), *transverse* if $c < 0$ (Lemma 55 Thm. 4.2; Lemma 56 Def. 2.1).
- Ghost: zero $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$; its *Jensen interval* on the line is $|t - \gamma| \leq \beta - \frac{1}{2}$. Counters (Lemma 56): τ transverse on-line saddles, S^+ off-line saddles with $\sigma > \frac{1}{2}$, K^+ ghosts; Kim’s identity $K^+ = \kappa := \tau + S^+$ on windows with admissible ends (Lemma 56 Thm. 4.3, = Kim 1996 Thm. 1). (H): every saddle on the line and longitudinal; $\text{RH} \Leftrightarrow (\text{H})$ (Lemma 56 Thm. 4.10, [RESTATEMENT]).
- Heat flow: $H_t(z) = \int_0^\infty e^{tu^2} \Phi(u) \cos(zu) du$, $\partial_t H_t = -\partial_{zz} H_t$, $H_0(z) = \frac{1}{8}\xi(\frac{1}{2} + \frac{iz}{2})$, so $z = 2t_{\text{height}}$ (Rodgers–Tao; Polymath). DH analogously (Lemma 56 v2/v3).

1. ANSWER IN BRIEF

- (1) **(A) Local geometry: correct [THEOREM]**. At a simple saddle the level set through s_0 is four rays at 90° , and the four steepest (gradient = phase) rays bisect them. If Λ' vanishes to order m at s_0 (and $\Lambda(s_0) \neq 0$), there are $2(m+1)$ level rays at angle $\pi/(m+1)$, with $2(m+1)$ gradient rays between them.
- (2) **(B) Light cones: correct, with two corrections [THEOREM]/[RESTATEMENT]**.
 - Correct: the reflection symmetry plus harmonicity of $\log |\Lambda|$ make the Hessian at every on-line point $\text{diag}(a, -a)$ in (σ, t) . So every on-line saddle has the separatrix tangents $t - t_0 = \pm(\sigma - \frac{1}{2})$ exactly. Longitudinal saddles descend into the timelike sectors, transverse ones into the spacelike sectors.
 - Correction 1: an off-line saddle’s cone is rotated by $-\frac{1}{2} \arg c$, not by $\arg c$.
 - Correction 2: “zeros inside the timelike cone” is trivially true for on-line zeros and so carries no content. The exact statement (Theorem 3.2) is about *spacelike* cones and needs the off-line saddles too. On a window with admissible ends: no ghost $\Leftrightarrow (S^+ = 0$ and no on-line saddle has a zero in its closed spacelike cone).
 - Checks: every DH ghost below height 1000 (28/28) lies in the spacelike cone of its transverse saddle; for the first 31 ζ saddles the computed c is real to relative 4×10^{-16} .
- (3) **(C) Diamonds: “same scale” is right, but the picture is not a diamond [NUMERICAL]**.
 - The true separatrices leave their tangents at the scale of the saddle gap g . At $r = g/4$ the turn is $\leq 7.2^\circ$ upward and 3.8° – 23.5° downward.
 - The downward lobes close at $r/g \in [0.28, 1.2]$; the asymmetry comes from the decay of $|\xi|$ in t .
 - Each zero’s basin occupies only 5.7%–55% of its diamond’s half-width and lies inside the diamond, up to $\leq 0.4\%$ of grid pixels.
 - For on-line zeros “inside the diamond” is automatic.
- (4) **(D) Cassini wall: correct; and it is orientation-blind [THEOREM]/[NUMERICAL]**.
 - The level circle encloses the critical value $\Leftrightarrow$ the lifted loop has a transposition as monodromy (one oval). At the boundary the level set is the lemniscate.

- Computed: the small-loop monodromy is a transposition at all 30 ζ saddles and all 75 DH saddles below 130, transverse ones included. The level-circle wall is verified at the ζ saddle 59.931 and at the DH transverse saddle 85.747.
 - The $\mathbb{Z}/2$ datum cannot tell longitudinal from transverse. “Möbius” should read “connected double cover of the circle” (its line bundle is the Möbius bundle).
- (5) **(E) Legendre: partly correct** [CITED]/[THEOREM]/[NUMERICAL].
- Lagarias Thm. 1.1 and Sondow–Dumitrescu Cor. 1 are as J.W. said (exact statements in §6).
 - The zero-slope intercept $b(0) = \inf_{\sigma \geq 1/2} \varphi$ equals the saddle height only when t is a saddle height *and* φ is monotone there. It fails at the DH transverse saddle: $b(0) = -67.541$ vs. $\log h = -66.386$.
 - New, and trivial: at a transverse saddle $\varphi''(\frac{1}{2}) = -\Xi''/\Xi < 0$, so SD monotonicity fails at its height.
 - Concavity sits in light cones: all 7514 concave grid points for ζ ($10 \leq t \leq 100$) lie in the spacelike cone of the nearest zero.
- (6) **(F) Heat flow: correct, made exact; “ κ counts landings” is a Restatement** [THEOREM]/[NUMERICAL]/[RESTATEMENT].
- The *signed* critical value crosses 0 linearly at a collision, with $H/H''(z^*) = t_c - t + O((t - t_c)^2)$, a fold.
 - The $\pm 45^\circ$ cone does *not* rotate. What rotates by 90° is the figure-eight lemniscate, i.e. which pair of sectors descends.
 - ζ backward: the clock gives $t_c = -1.28526$ (Lemma 56: -1.28527). DH forward at 85.747: 0.17652 (Lemma 56: 0.17651).
 - The static predictor $-4/c$ at $t = 0$ over-estimates all 28 DH landing times, by ratios 1.0008–1.71.
 - Under generic collisions and complete landing, κ = number of landings still ahead. This is an RH restatement, not a quadratic form with negative index κ .
- (7) **(G) Genus: none in the landscape** [THEOREM]/[CONSTRUCTION].
- Bounded sublevel components of $|\Lambda|$ are disks (max modulus), and each has zeros – saddles = 1. Verified on 34 components.
 - Genus 0 is automatic for planar domains; the content is “no holes”.
 - A genus appears only on a constructed cover. The canonical choice is $y^2 = \Lambda'(s)$, of infinite genus, with real structure $(s, y) \mapsto (1 - \bar{s}, iy)$. Its real ovals lie over alternate gaps between on-line saddles, so it is blind to τ and to DH’s ghosts. Its truncations are non-canonical (j drifts by a factor ~ 6).

2. (A) LOCAL SADDLE GEOMETRY

Proposition 2.1 (holomorphic Morse lemma). *[THEOREM] Let F be holomorphic near s_0 with $F(s_0) \neq 0$, and suppose F' vanishes to order exactly $m \geq 1$ at s_0 . Write $\log F(s) = \log F(s_0) + a(s - s_0)^{m+1} + O(|s - s_0|^{m+2})$ with $a = F^{(m+1)}(s_0)/((m+1)!F(s_0)) \neq 0$. Then:*

- (i) *There is a local coordinate ζ with $\log F = \log F(s_0) + \zeta^{m+1}$. So the level set $\{|F| = |F(s_0)|\}$ near s_0 consists of $2(m+1)$ analytic arcs leaving s_0 , with tangent directions $\arg(s - s_0) \in \frac{\pi/2 - \arg a}{m+1} + \frac{\pi}{m+1}\mathbb{Z}$, i.e. at mutual angle $\pi/(m+1)$.*
- (ii) *The steepest-ascent and steepest-descent curves of $|F|$ through s_0 are the level curves of $\arg F$ through s_0 : $2(m+1)$ arcs bisecting the previous ones (Cauchy–Riemann; Lemma 62 §7).*
- (iii) *For $m = 1$, $a = c/2$ with $c = F''/F(s_0)$: four level rays at 90° in the directions $\frac{1}{2}(\frac{\pi}{2} - \arg c) + \frac{\pi}{2}\mathbb{Z}$, and four gradient rays at 45° to them.*

Proof. $\log F - \log F(s_0)$ has a zero of order $m+1$ at s_0 , so it is ζ^{m+1} for a local biholomorphism $\zeta = (s - s_0)(a + O(s - s_0))^{1/(m+1)}$. $\log |F| = \operatorname{Re} \log F$ and $\arg F = \operatorname{Im} \log F$. Hence the level

set is $\operatorname{Re} \zeta^{m+1} = 0$ and the phase set is $\operatorname{Im} \zeta^{m+1} = 0$. Tangent directions follow from $\arg \zeta = \arg(s - s_0) + \frac{\arg a}{m+1}$ to first order. $\square$

J.W.'s claim (A): “a simple saddle gives 4 separatrix rays at 90° plus 4 steepest-path rays at 45° to them; order m gives $2(m+1)$ rays at $\pi/(m+1)$.” **Correct**, with the order m meaning the order of vanishing of Λ' (the monkey saddle is $m = 2$: 6 level rays at 60°). The hypothesis $\Lambda(s_0) \neq 0$ matters: at a zero of Λ there is no separatrix (level curves are small circles), and $|\Lambda|$ has no critical points other than zeros and saddles (maximum and minimum modulus principles). Multiple on-line saddles do occur, but only at saddle collisions in the heat flow (Lemma 56 Thm. 7.1(ii)).

3. (B) THE LIGHT-CONE THEOREM

Theorem 3.1 (on-line cones). *[THEOREM] Let Λ be as in the Notation and $u(\sigma, t) = \log |\Lambda(\sigma + it)|$.*

- (i) *At every point of the critical line where $\Lambda \neq 0$: $u_{\sigma t} = 0$ and $u_{\sigma\sigma} = -u_{tt}$. The Hessian is $\operatorname{diag}(a, -a)$ in (σ, t) .*
- (ii) *At an on-line saddle t_0 , $a = c = -\Xi''/\Xi(t_0) \in \mathbb{R}$, and the separatrix tangents are exactly $t - t_0 = \pm(\sigma - \frac{1}{2})$.*
- (iii) *If the saddle is longitudinal ($c > 0$), $u - u(s_0) \approx \frac{c}{2}(x^2 - y^2)$ is negative in the timelike sectors: the descent valleys, and hence the sublevel lobes, are timelike. If it is transverse ($c < 0$), they are spacelike.*
- (iv) *At an off-line saddle, $c \in \mathbb{C}$ and the cone (tangent pair) is the on-line cone rotated by $-\frac{1}{2} \arg c \pmod{\frac{\pi}{2}}$. The mirror saddle $1 - \bar{s}_0$ has $\bar{c}$ and the opposite rotation.*

Proof. (i) $\Lambda(1 - \bar{s}) = \overline{\Lambda(s)}$ gives $u(1 - \sigma, t) = u(\sigma, t)$. So u is even in $x = \sigma - \frac{1}{2}$, and u_σ and $u_{\sigma t}$ are odd in x , hence vanish at $x = 0$. u is harmonic where $\Lambda \neq 0$. (ii) $\Lambda(s) = \Xi(-i(s - \frac{1}{2}))$, so $\Lambda'' = -\Xi''$, and Proposition 2.1(iii) applies: the directions are $\frac{1}{2}(\frac{\pi}{2} - \arg c) + \frac{\pi}{2}\mathbb{Z}$, which are $\pm 45^\circ$ for c real. (iii) From $\log \Lambda = \log \Lambda(s_0) + \frac{c}{2}(x + iy)^2 + \dots$, Re gives $\frac{c}{2}(x^2 - y^2)$. (iv) Proposition 2.1(iii). For the mirror, apply the reflection, which conjugates c and reverses angles. $\square$

So the “light cone” of an on-line saddle is not a metaphor for the tangents: they really are the null lines of $dt^2 - d\sigma^2$, and this is forced by symmetry plus harmonicity. It is the same fact as the $\pi/4$ rotation between level and phase lines (Lemma 62 §7) and the vertical/horizontal figure-eight of Lemma 55 §4. The *orientation* (longitudinal or transverse) is one sign, $\operatorname{sgn} c$, and the cone itself carries no further information.

3.1. Where the zeros are: the exact cone statement.

Theorem 3.2 (light-cone form of Kim’s identity). *[THEOREM] ([RESTATEMENT] in the limit)*

- (i) *An on-line zero never lies in the closed spacelike cone $\{|t - t_0| \leq |\sigma - \frac{1}{2}|\}$ of an on-line saddle t_0 . Conversely, every on-line zero lies in the open timelike cone of every on-line saddle.*
- (ii) *(Lemma 56 Thm. 3.1, Jensen–Pólya via Kim.) Every transverse saddle t_0 lies in the Jensen interval of some ghost $\rho = \beta + i\gamma$, i.e. $|\gamma - t_0| \leq \beta - \frac{1}{2}$. Equivalently, ρ and $1 - \bar{\rho}$ lie in the closed spacelike cone of t_0 .*
- (iii) *Let $[T_1, T_2]$ have admissible ends (no endpoint in a closed Jensen interval; Lemma 56). Then Λ has no ghost with $T_1 < \gamma < T_2$ if and only if $S^+ = 0$ in the window and no on-line saddle in the window has a zero in its closed spacelike cone.*

Proof. (i) An on-line zero has $\sigma = \frac{1}{2}$; it lies in the closed spacelike cone of t_0 only if $t = t_0$, which is impossible since $\Lambda(s_0) \neq 0$. (iii) $(\Rightarrow)$ With no ghost, $K^+ = 0 = \tau + S^+$ by Kim’s identity (Lemma 56 Thm. 4.3), so $S^+ = 0$. A zero in a closed spacelike cone of an on-line saddle in the window would be a ghost by (i). It would also lie in the window: its Jensen interval contains t_0 , so if γ were outside, the interval would contain an end. $(\Leftarrow)$ If no on-line saddle has a zero in its closed spacelike cone, then by (ii) there is no transverse saddle, so $\tau = 0$. With $S^+ = 0$, Kim gives $K^+ = 0$. $\square$

Consequence [RESTATEMENT]. Combined with Lemma 56 Thm. 4.10 ($\text{RH} \Leftrightarrow (\text{H})$), RH holds iff there are no off-line saddles and no on-line saddle has a zero in its closed spacelike cone. This is the *correct* form of J.W.’s “every on-line saddle has its zeros inside its timelike cone”. That phrasing fails in two ways:

- It is trivially true for on-line zeros (part (i)), and a ghost far away also lies in the timelike cone of most saddles, so it carries no content.
- It omits the off-line saddles. By Kim’s identity, ghosts can be “paid for” by S^+ instead of τ ; the toy of Fig. 1(c) has ghosts and an off-line saddle pair.

The local (descent-sector) version of J.W.’s statement is exactly the Laguerre sign: (H) says every saddle is on the line and descends into its timelike sectors.

3.2. Numerics. [NUMERICAL] Script `num/lemma87_cones.py` (output `num/187/cones.json`).

- ζ . The 31 on-line saddles below $\gamma_{31} = 103.7255$ (the saddle at $t = 0$ and one in each gap) are all longitudinal ($c \geq 0.0462$), with $|\text{Im } c|/|c| \leq 4.3 \times 10^{-16}$. In particular the Hessian is diagonal, as Theorem 3.1(i) requires.
- DH , $0 < t < 130$. There are 75 on-line saddles, 2 of them transverse: 85.74720 and 114.16958. The two ghosts below 130, $0.80852 + 85.69935i$ and $0.65083 + 114.16334i$, each lie in the closed spacelike cone of their transverse saddle ($|\gamma - t_0| = 0.0479 < 0.3085$, resp. $0.0062 < 0.1508$) and of no other saddle.
- *All 28 DH ghosts below 1000* (Lemma 48 zero list; transverse saddles from Lemma 55, the 4 small- δ ones re-located here by a 1-D scan): 28/28 lie in the spacelike cone of a transverse saddle.

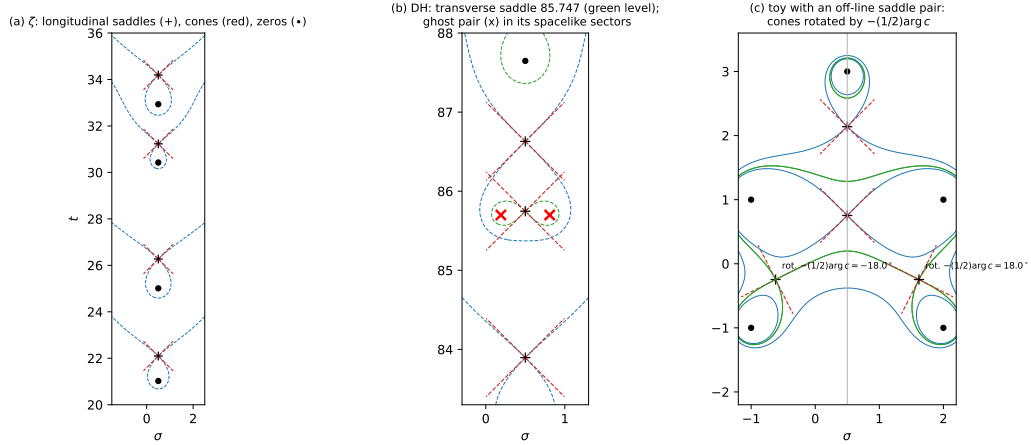


FIGURE 1. (a) ζ , $20 < t < 36$: the level curves through the saddles (blue), the tangent cones $t - t_0 = \pm(\sigma - \frac{1}{2})$ (red dashed) and the zeros. (b) DH near the ghost $0.8085 + 85.6993i$: the transverse saddle 85.747 (green level curve) has the ghost pair ($\times$) in its spacelike sectors; the longitudinal saddles 83.896, 86.629 have timelike lobes. (c) A polynomial toy, $\Lambda(s) = p(-i(s - \frac{1}{2}))$ with zeros $\pm 1 \pm 1.5i$ and 3 in the u -plane: an off-line saddle pair whose cones are rotated by $\mp 18.0^\circ = -\frac{1}{2} \arg c$. *Diagnostic*: green curves must follow the rotated tangents near the off-line saddles, and the on-line tangents must be exactly $\pm 45^\circ$.

Prior art: separatrices and sectors of the ξ flow are in Broughan (2005; Lemma 43). The light-cone wording and Theorem 3.2 were not found stated; we did not search systematically, and no novelty is claimed.

4. (C) CAUSAL DIAMONDS

Geometry [THEOREM]. For consecutive on-line saddles $t_k < t_{k+1}$, the forward cone of t_k and the backward cone of t_{k+1} meet where $t - t_k = t_{k+1} - t = |\sigma - \frac{1}{2}|$, i.e. at $\sigma - \frac{1}{2} = \pm \frac{1}{2}(t_{k+1} - t_k)$, $t = \frac{1}{2}(t_k + t_{k+1})$. J.W.'s formula is correct. The *diamond* is $D_k = \{|\sigma - \frac{1}{2}| < \min(t - t_k, t_{k+1} - t)\}$. An on-line zero in (t_k, t_{k+1}) lies in D_k automatically, so “each zero sits inside its diamond” has content only for ghosts, and Theorem 3.2 is the exact form of that content.

Scale of the deviation [HEURISTIC]. Expand $\log \Lambda = \log \Lambda(s_0) + \frac{c}{2}z^2 + \frac{d}{6}z^3 + \dots$ with $d = (\Lambda'/\Lambda)''(s_0)$. The cubic term bends the separatrices by $O(1)$ angles at $r \sim |c/d|$. By the Hadamard product, $c \sim \sum (s_0 - \rho)^{-2}$ and $d \sim 2 \sum (s_0 - \rho)^{-3}$ are dominated by the nearest zeros, so $|c/d|$ is of the order of the local zero spacing. “Same scale” is therefore expected; it is not a theorem.

Numerics [NUMERICAL] (`num/lemma87_cones.py`, `lemma87_basins.py`; ζ saddles 1–29, $15.59 \leq t_0 \leq 99.63$; g = smaller adjacent saddle gap). We continued each of the four separatrix branches from its tangent by solving $|\Lambda(s_0 + re^{i\theta})| = \log h$ for $\theta(r)$, $r \leq 1.2g$.

- *Upward branches* (towards larger t , where $|\xi|$ decays): the turn is $\leq 7.2^\circ$ at $r = g/4$ and $\leq 14.4^\circ$ (median 3.6°) at $r = g/2$. All 58 continue at least to $r = 0.74g$.
- *Downward branches*: the turn is 3.8° – 23.5° at $r = g/4$. They curl towards the line and end where the lobe closes, at $r/g \in [0.28, 0.68]$ for 52 of 58; the other 6 reach $1.2g$.
- *Basins*. The basin of each zero γ_k is the component of $\{|\xi| < \min(h_{k-1}, h_k)\}$ containing it, i.e. the level of its lower flanking saddle. For the first 30 zeros, its half-width is 0.057–0.551 of the diamond half-width $\frac{1}{2}(t_{k+1} - t_k)$. The fraction of basin pixels outside the diamond is 0 for 27 zeros and $\leq 0.43\%$ for 3, at the grid resolution of $\sim 0.02g$.

So: J.W.'s “same scale” is right, and the cones are faithful only out to $r \approx g/4$. The basins are narrow lenses well inside the diamonds. The up/down asymmetry is the t -decay of $|\xi|$: the lower flanking saddle is usually the upper one (27/30).

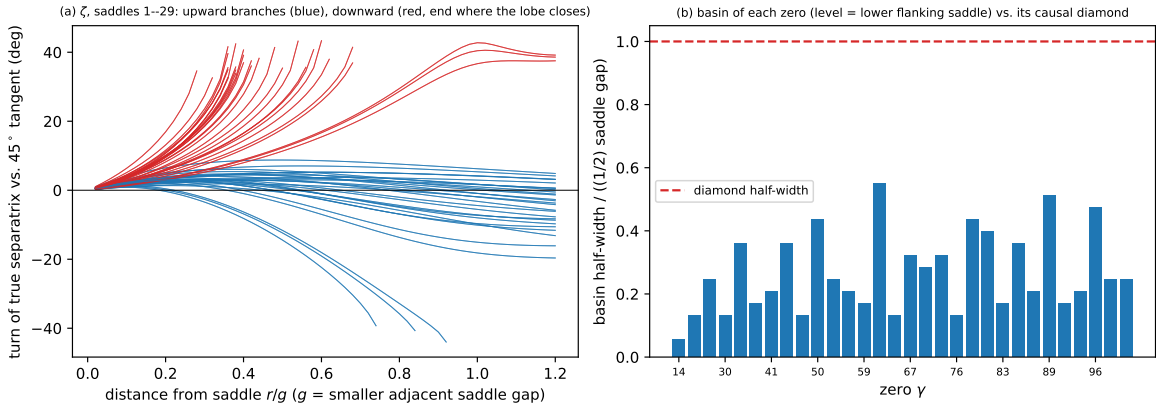


FIGURE 2. (a) Turn of the true separatrix relative to its 45° tangent against r/g , for 29 ζ saddles (blue: upward branches; red: downward branches, ending where the lobe closes). (b) Basin half-width of each zero as a fraction of the diamond half-width (red line = 1). *Diagnostic*: a ghost would appear as a basin that is not threaded by the line. A bar above 1 would mean a basin escaping its diamond (none does).

5. (D) THE TRIVIAL-TO-MÖBIUS WALL

Proposition 5.1 (Cassini model). [THEOREM] Let $w = z^2 - a^2$, $a > 0$, and $C_r = \{|w| = r\}$. The map is a double cover branched only over $w = -a^2$ (and ∞).

- (i) If $r < a^2$, C_r does not enclose $-a^2$; the monodromy of C_r is trivial and the preimage is two ovals around $\pm a$.
- (ii) If $r > a^2$, C_r encloses $-a^2$; the monodromy is the transposition and the preimage is one connected oval.
- (iii) If $r = a^2$, the preimage is the lemniscate $|z^2 - a^2| = a^2$ through the critical point 0.

Proof. A loop's preimage under a double cover has one component per cycle of its monodromy permutation. Around a simple branch point the monodromy is the transposition, and C_r winds once around $-a^2$ iff $r > a^2$. $\square$

J.W.'s claim is **correct** (Needham 1997, §2.II.3, for the curves; Lemma 55 Prop. 2.2 and Lemma 81 §7 for ξ). Terminology: the connected double cover of a circle is again a circle, and orientable. “Möbius” refers to the associated real line bundle, which is the Möbius bundle (Lemma 81, “Möbius strip and branched double covers”).

Proposition 5.2 (the wall for Λ). [THEOREM]

- (i) Near a simple saddle, $\Lambda = \Lambda(s_0) + \zeta^2$ in a local coordinate (Proposition 2.1). So Λ is locally the Cassini model, and the small-loop monodromy around $w_0 = \Lambda(s_0)$ is a transposition.
- (ii) Let U be a bounded component of $\{|\Lambda| < c\}$ (c not a critical value). Then $\Lambda : U \rightarrow \{|w| < c\}$ is a proper branched cover of degree $m = \text{number of zeros in } U$. The monodromy of $|w| = c'$ ($c' < c$) is the product of the local transpositions around the critical values of modulus $< c'$, and the number of components of $\{|\Lambda| = c'\} \cap U$ is the number of its cycles.
- (iii) The datum is orientation-blind: (i)–(ii) do not depend on $\text{sgn } c$.

Proof. (i) is Proposition 2.1 with $m = 1$. (ii) Properness follows because $|\Lambda| = c$ on ∂U . The degree is the number of preimages of 0. Monodromy factorizes over a generating set of loops around the branch values. (iii) The local model $w_0 + \zeta^2$ is the same for $c > 0$ and $c < 0$, since only $\arg c$ enters the coordinate change. $\square$

Orientation-blindness [THEOREM]/[TAA-L]. The Laguerre sign that distinguishes longitudinal from transverse (and hence (H), hence RH) is a second-order metric datum, $\text{sgn Re } c$ on the line, not a topological one. The $\mathbb{Z}/2$ monodromy cannot see it. This is consistent with Lemma 80's no-go, where detectors invariant under the relevant symmetries see the functional equation but not the zeros. The monodromy is RH-blind in the same way.

Numerics [NUMERICAL] (num/lemma87_monodromy.py, num/187/monodromy.json).

- *Small loops* $w_0 + 0.05|w_0|e^{i\phi}$, lifted by Newton continuation (240 steps): the two local preimages are exchanged at all 30 ζ saddles ($15.59 \leq t \leq 102.07$) and at all 75 DH saddles in $(0.3, 130)$, including the transverse 85.747 and 114.170. The maximal closure error is 4.3×10^{-14} .
- *Level-circle wall*, ζ : the saddle $t_0 = 59.931$ is a local minimum of the saddle-height sequence, $\log h = (-37.3577, -40.3684, -40.3524)$ at $t = (57.263, 59.931, 62.110)$. The component of $\{|\xi| < c\}$ containing the zeros 59.347 and 60.832 is therefore bounded for $\log c - \log h \in (0, 0.0159)$. Lifting $|w| = c$ from the base point ic : at $\log c - \log h = -0.00797$ the permutation is the identity (two ovals); at $+0.00797$ it is the transposition (one oval).
- *Level-circle wall*, DH : the transverse saddle 85.747, with the ghost ρ and its mirror $1 - \bar{\rho}$ as the two preimages of 0. At $\log c - \log h = \mp 0.520$ we get the identity and the transposition respectively. The wall is the same for a transverse saddle whose lobes are spacelike.

6. (E) THE LEGENDRE RELATION

Fix t and put $\varphi(\sigma) = \log |\Lambda(\sigma + it)|$. Then $\varphi'(\sigma) = \text{Re } \frac{\Lambda'}{\Lambda}(\sigma + it)$ and $\varphi''(\sigma) = \text{Re } \left(\frac{\Lambda'}{\Lambda}\right)'(\sigma + it)$. By the reflection symmetry, φ is even about $\frac{1}{2}$, so $\varphi'(\frac{1}{2}) = 0$ for every t .

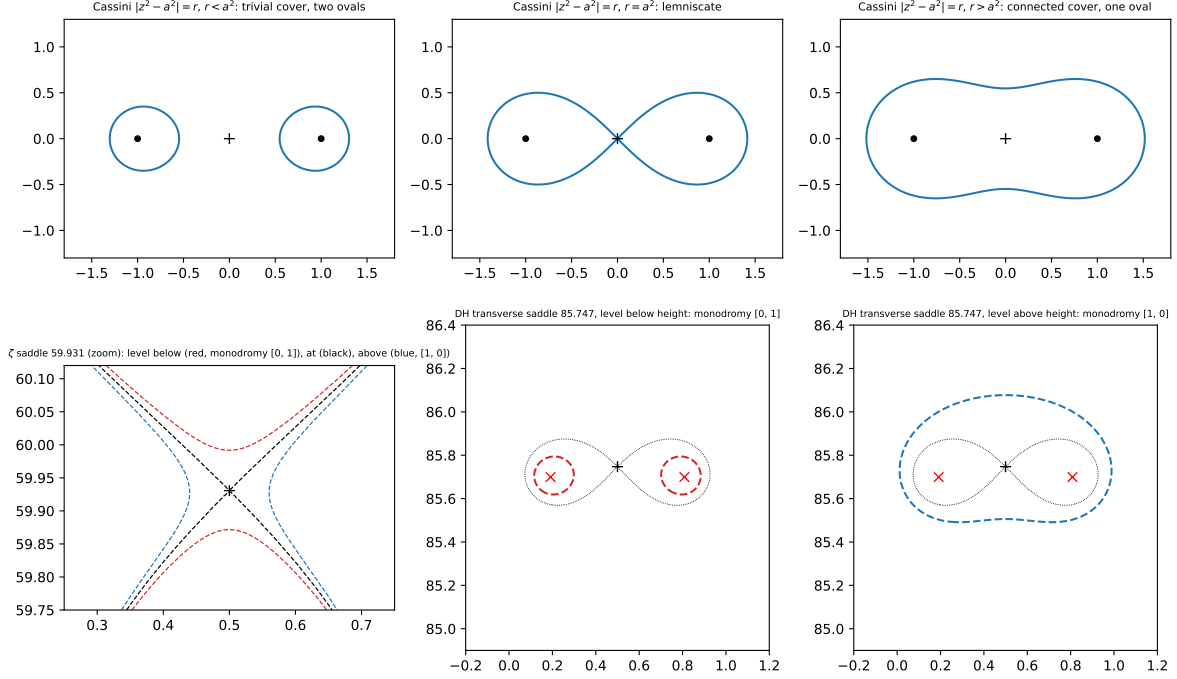


FIGURE 3. Top: the Cassini model $|z^2 - a^2| = r$ below, at and above the critical value. Bottom left: ζ saddle 59.931 (zoom), level curves just below (red), at (black) and just above (blue) the saddle height, with the computed monodromies. Bottom middle and right: DH transverse saddle 85.747, with the ghost pair ($\times$) and levels below/above the saddle height (dotted: the lemniscate). *Diagnostic*: the topology change is identical for the longitudinal (ζ) and the transverse (DH) saddle, so the wall cannot detect ghosts.

6.1. The two cited criteria (exact statements).

- [CITED] **Lagarias 1999, Thm. 1.1.** Let Ω be an admissible zero set (symmetric under conjugation, with $\sum(1 + |\operatorname{Re} \rho|)/(1 + |\rho|^2) < \infty$) and f_Ω its canonical product. Then all $\rho \in \Omega$ have $\operatorname{Re} \rho \leq \theta$ iff $\operatorname{Re} f'_\Omega / f_\Omega(s) > 0$ for $\operatorname{Re} s > \theta$. For ξ this gives $\text{RH} \iff \operatorname{Re} \xi' / \xi(s) > 0$ for $\operatorname{Re} s > \frac{1}{2}$, his (1.5), crediting Hinkkanen for the equivalence. J.W.'s statement is correct; the theorem is function-theoretic (it concerns zero sets, not Euler products).
- [CITED] **Sondow–Dumitrescu 2010.** Thm. 1: $|\xi|$ is strictly increasing along every horizontal half-line in any zero-free open right half-plane (and decreasing in left ones). Cor. 1: $\text{RH} \iff$ for every real t , $|\xi(\sigma + it)|$ is increasing for $\frac{1}{2} < \sigma < \infty$. J.W.'s statement is correct.

6.2. The intercept, and two exact consequences. Let $\varphi^*(p) = \sup_{\sigma \geq 1/2} (p\sigma - \varphi(\sigma))$. The supporting line of slope p has intercept $b(p) = -\varphi^*(p)$, and $b(0) = \inf_{\sigma \geq 1/2} \varphi$.

Proposition 6.1. [THEOREM]

- (i) If φ is non-decreasing on $[\frac{1}{2}, \infty)$ (SD at this t), then $b(0) = \varphi(\frac{1}{2}) = \log |\Xi(t)|$. As a function of t this is stationary exactly at the on-line saddles, where it equals $\log h$. Without monotonicity, $b(0) < \log |\Xi(t)|$ is possible.

- (ii) At an on-line saddle t_0 , $\varphi''(\frac{1}{2}) = -\Xi''/\Xi(t_0) = c$. So at every transverse saddle, φ has a strict local maximum at $\sigma = \frac{1}{2}$, and the Sondow–Dumitrescu monotonicity fails at the height t_0 .
- (iii) (light-cone law for convexity, one zero) A zero ρ_0 contributes $-\operatorname{Re}(s - \rho_0)^{-2} = (Y^2 - X^2)/(X^2 + Y^2)^2$ to φ'' , where $X + iY = s - \rho_0$. This is negative exactly in the spacelike cone $|X| > |Y|$ of ρ_0 .

Proof. (i) inf is at the left end. $\frac{d}{dt} \log |\Xi| = \Xi'/\Xi$ vanishes at saddles. (ii) $\varphi''(\frac{1}{2}) = u_{\sigma\sigma} = -u_{tt} = -(\log |\Xi|)'' = -\Xi''/\Xi + (\Xi'/\Xi)^2$, and $\Xi'(t_0) = 0$. (iii) Direct computation. $\square$

So J.W.’s “the zero-slope intercept equals the saddle height” is **correct at saddle heights under SD**, and fails exactly where a ghost bends φ down: by (ii), at every transverse saddle. “Convexity fails near zeros (piecewise Legendre)” is **correct**: φ^* only sees the convex hull φ^{**} , and the concave windows are bridged by affine pieces. By (iii) and the census below, those windows are the spacelike cones of the nearby zeros.

Numerics [NUMERICAL] (num/lemma87_legendre.py, num/187/legendre.json; $\sigma \in [\frac{1}{2}, 3]$, 501 points).

Λ	t	$\varphi(\frac{1}{2})$	$b(0)$ (at σ)	$\min_{\sigma > 1/2} \varphi'$	note
ζ	15.586 (saddle)	-7.12895	-7.12895 (0.5)	0.0028	convex on $[\frac{1}{2}, 3]$
ζ	18.0	-8.11647	-8.11647 (0.5)	0.0012	convex on $[\frac{1}{2}, 3]$
ζ	$\gamma_1 + 0.05$	-9.60850	-9.60850 (0.5)	0.562	first concave point $\sigma = 0.555$
DH	85.747 (transverse)	-66.38620	-67.54065 (0.805)	-8.48	SD and Lagarias fail
DH	85.699 + 0.01	-66.37237	-69.05525 (0.810)	-47.5	near the ghost
DH	86.629 (longitudinal)	-65.34580	-65.34580 (0.5)	0.0160	

Census on grids $\sigma \in (\frac{1}{2}, 2]$:

- ζ ($10 \leq t \leq 100$, 24000 points): no point with $\varphi' \leq 0$. All 7514 points with $\varphi'' < 0$ lie in the spacelike cone of the nearest zero.
- DH ($80 \leq t \leq 92$, 14400 points): 115 points with $\varphi' \leq 0$, all with $t \in [85.42, 85.97]$ and $\sigma \leq 0.806$. Of 5608 concave points, 5564 lie in the nearest zero’s spacelike cone.
- DH ($10 \leq t \leq 130$, 24000 points): 24 points with $\varphi' \leq 0$, all with $t \in [85.53, 114.17]$, i.e. at the two ghosts. Of 9394 concave points, 9384 lie in the nearest zero’s spacelike cone.

The light-cone law for convexity, “ $\varphi'' < 0$ only in the spacelike cone of some zero”, holds at every grid point for ζ . For DH the exceptions are where the nearest zero in t is not the dominant one. We state it as an observation [NUMERICAL], not a theorem.

7. (F) ENVELOPE AND HEAT FLOW

Proposition 7.1 (fold at a collision). *[THEOREM] Let H_t satisfy $\partial_t H = -\partial_{zz} H$ and have a double real zero at (z_c, t_c) with $a := H_{zz}(z_c, t_c) \neq 0$. Then near (z_c, t_c) :*

- $H = \frac{a}{2}w^2 - a(t - t_c) + O(|w|^3 + |w||t - t_c| + |t - t_c|^2)$, with $w = z - z_c$. The zero set is the parabola $w^2 = 2(t - t_c)$, a fold of the projection to t : real zeros for $t > t_c$, a conjugate (ghost) pair for $t < t_c$. This is consistent with Polymath Prop. 3.1(ii).
- The real critical point $z^*(t)$ of H_t is smooth through t_c . Its signed critical value is $H(z^*) = -a(t - t_c) + O((t - t_c)^2)$, which crosses 0 linearly, so $\log |H(z^*)| \rightarrow -\infty$.
- $H/H''(z^*) = t_c - t + O((t - t_c)^2)$. Hence $H''/H = 1/(t_c - t)$ to leading order (the universal pole of Lemma 56 Thm. 7.1), and the saddle is transverse for $t < t_c$ and longitudinal for $t > t_c$.
- The tangent cone of the saddle stays at $\pm 45^\circ$ for all t (Theorem 3.1). At t_c the descent sectors switch between timelike and spacelike, i.e. the figure-eight lemniscate turns by 90° .

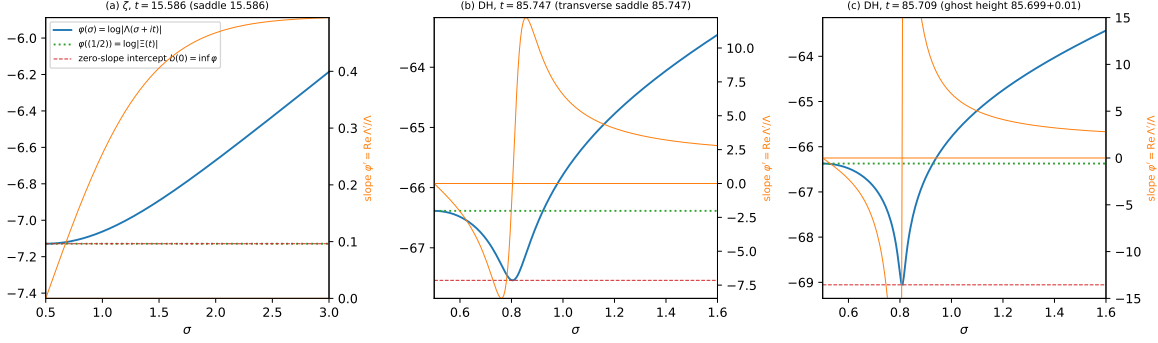


FIGURE 4. $\varphi(\sigma)$ (blue), $\varphi(\frac{1}{2}) = \log |\Xi(t)|$ (green dotted), the zero-slope intercept $b(0)$ (red dashed) and the slope φ' (orange, right axis). (a) ζ at the saddle 15.586: φ increasing and convex, intercept = saddle height. (b) DH at the transverse saddle 85.747: $\varphi''(\frac{1}{2}) < 0$, the slope goes negative, and the intercept drops below the saddle height. (c) DH just above the ghost height. *Diagnostic*: a negative orange curve to the right of $\frac{1}{2}$ is a Lagarias/SD violation, and therefore a ghost.

Proof. At (z_c, t_c) : $H = H_z = 0$, $H_t = -H_{zz} = -a$. Taylor expansion gives (i). $H_z = aw + \dots = 0$ gives $w^* = O(t - t_c)$, hence (ii), and $H_{zz}(z^*) = a + O(\cdot)$ gives (iii). (iv) is Theorem 3.1 applied to H_t , which is real on $\mathbb{R}$ and even. $\square$

J.W.'s claims (“height $\rightarrow -\infty$ at the collision; the saddle returns transverse; the cone rotates 90° ; fold/caustic; ghost birth = saddle height reaching zero”): **correct**, with two precisions.

- What reaches zero is the *signed* critical value; it changes sign.
 - What rotates is the lemniscate, i.e. the choice of descent sectors; the cone is fixed by symmetry.
- (iii) gives a *collision clock*: $\hat{t}_c := t + H/H''(z^*)$. At $t = 0$ it reads $\hat{t}_c = 4\Xi/\Xi''(t_0) = -4/c$ (since $H_0(z) = \frac{1}{8}\Xi(z/2)$), computable from the static saddle alone.

7.1. “ κ counts landings”: the precise statement (J.W. [t376]).

Proposition 7.2. [THEOREM] under hypotheses; [RESTATEMENT] as a criterion Consider the forward flow $t \geq 0$ of ξ (or Λ_{DH}) and a height window W with admissible ends at $t = 0$. Assume:

- (a) every ghost lands at some finite $t > 0$ (for ξ : all zeros of H_t are real for $t \geq \Lambda$, by Newman’s definition of Λ as stated in Rodgers–Tao §1, and $\Lambda \leq 0.2$ by Platt–Trudgian Cor. 2; for DH: no theorem is claimed here; below height 1000 it is verified by Lemma 56’s flow runs);
- (b) landings are generic, i.e. each is a simple double real zero as in Proposition 7.1;
- (c) no zero tracked from W leaves the real axis again in forward time (no forward births);
- (d) zeros are tracked along their trajectories, not by a fixed window.

Then the number of landing events still ahead for the ghosts in W equals $K^+(W) = \kappa(W) = \tau + S^+$ (Kim). Each landing removes one ghost pair: in the generic case it is a transverse saddle whose signed height goes through 0 (Proposition 7.1), after which the saddle is longitudinal and the double zero splits along the line. A saddle collision (Lemma 56 Thm. 7.1(ii)) moves one unit from τ to S^+ , leaves κ unchanged, and is not a landing; the off-line saddle pair must recombine on the line before that ghost lands.

Proof. Each ghost pair contributes one to K^+ and, by (a)–(d), lands exactly once. Kim’s identity at $t = 0$ gives $K^+ = \kappa$. The description of the event is Proposition 7.1. $\square$

Status [RESTATEMENT]. “ $\kappa \equiv 0$ on all windows” $\iff$ “no landings ahead at $t = 0$ ” $\iff$ RH. This is a *restatement* of RH through Kim’s identity and the flow. It is *not* a quadratic form whose

negative index is κ : no Hermitian or Krein form is constructed whose negative index is κ , and no positivity statement follows from it. Hypothesis (c) is assumed, not cited; it is observed in all of Lemma 56's runs.

Comparison with Lemma 56's DH landing data [NUMERICAL] (num/156/dhflow.json, num/156v3/tc_fft.json). Below height 1000, DH has 28 ghosts, 28 transverse saddles and $S^+ = 0$, so the static count is $\kappa = \tau = 28$. All 28 ghosts land in forward time, $t_c \in [0.000507, 0.2306]$. For each one, Lemma 56 v2's snapshot counts go from $(\tau, S^+, K^+) = (1, 0, 1)$ to $(0, 0, 0)$, with two new real zeros (28/28). So the dynamic count equals the static count, $28 = 28$.

New numerics [NUMERICAL] (num/lemma87_heat.py, lemma87_heat4.py; num/187/heat.json, heat_pred28.json).

- ζ backward, using the Lemma 56 engine at the saddle $z^* = 119.861$ ($t_0 = 59.931$, the local height minimum of §5). $H(z^*)$ changes sign between $t = -1.28$ and -1.29 . The clock reads $\hat{t}_c = -0.9064$ at $t = 0$, -1.2830 at $t = -1.2$, and -1.28526 at $t = -1.29$ (Lemma 56: $t_c = -1.28527$, $z \approx 121.0$).
- *DH forward* at the transverse saddle 85.747 ($z^* = 171.494$), using the Lemma 56 v3 FFT engine. The signed $e^{cz} H_t(z^*)$ changes sign between $t = 0.17$ and 0.18 , with clock minimum 0.17652 (Lemma 56 v3: 0.17651). At $t = 0$ the flow clock 0.21398 agrees with the static $-4/c = 0.21398$, which cross-checks the two engines.
- *Static predictor* $-4/c$ for all 28 DH ghosts: the ratio $\hat{t}_c/t_c$ lies in $[1.0008, 1.713]$ (median 1.219). It tends to 1 for early landings (e.g. 1.0008 at $t_c = 0.000507$, height 520.94), with correlation 0.69 between ratio and $\log t_c$. The first-order clock always over-estimates.
- ζ , the 14 saddles with $31 < z^* < 125$ at $t = 0$: $-4/c$ ranges from -0.906 (at $z^* = 119.86$, next to the first collision $t_c = -1.285$) to -7.14 (at $z^* = 31.17$). The ranking only roughly follows Lemma 56's collision times, since these lie far outside the linear regime.

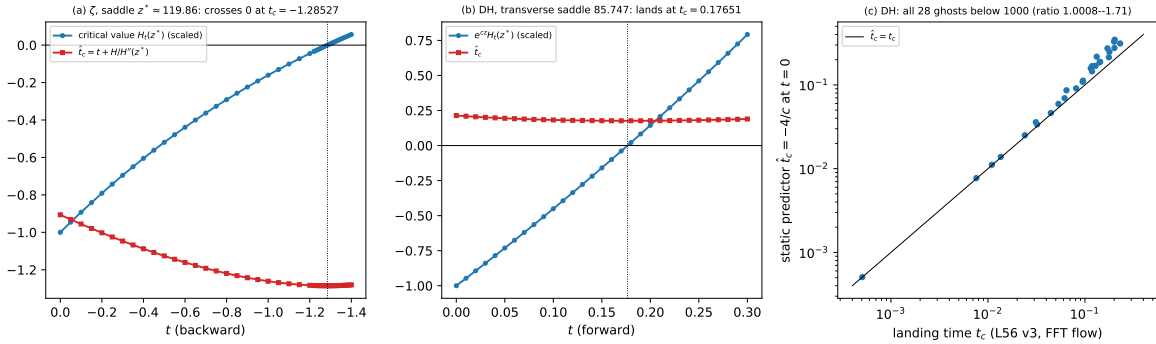


FIGURE 5. (a) ζ , backward flow: the signed critical value (scaled) and the clock $\hat{t}_c$ at $z^* \approx 119.86$; zero crossing at $t_c = -1.28527$ (dotted). (b) DH, forward flow at the transverse saddle 85.747: linear crossing at $t_c = 0.17651$. (c) The static predictor $-4/c$ at $t = 0$ against the computed landing times of all 28 DH ghosts below 1000. *Diagnostic:* a linear zero crossing of the signed height with a flat clock is the fold. Points on the diagonal in (c) mean the landing is visible in the static saddle alone.

8. (G) MORSE TOPOLOGY AND THE QUESTION OF GENUS

J.W.'s claims ([t374]): sublevel sets are disks; zeros – saddles = components; there is a hyperelliptic cover with an a -cycle per zero, a real structure, and ovals counting S^+/τ ; and the construction is canonical. We separate what is a theorem, a *Construction*, and an *Analogy*.

Proposition 8.1 (sublevel topology). *[THEOREM] Let ℓ be a regular value of $\log |\Lambda|$ and U a bounded component of $\{\log |\Lambda| < \ell\}$. Then U is simply connected. If U contains m zeros (with multiplicity),*

then $\Lambda : U \rightarrow \{|w| < e^\ell\}$ is a proper branched cover of degree m , and U contains exactly $m - 1$ critical points of Λ (with multiplicity). Summed over bounded components: $\#\text{zeros} - \#\text{saddles} = \#\text{bounded components}$.

Proof. If U had a hole V , then $\partial V \subset \partial U$, where $|\Lambda| = e^\ell$, while $|\Lambda| \geq e^\ell$ on V . Then the minimum principle applied to $1/\Lambda$, which is holomorphic on V since V has no zeros, forces $|\Lambda|$ to be constant. Properness holds because $|\Lambda| \rightarrow e^\ell$ at ∂U . Riemann–Hurwitz gives $1 = \chi(U) = m\chi(\mathbb{D}) - \sum(e_p - 1) = m - \#\text{crit}$. $\square$

Caveat. In the strip, $\xi(\sigma + it) \rightarrow 0$ as $|t| \rightarrow \infty$, so for every ℓ the sublevel set has *unbounded* components as well. There the count is replaced by the window identity of Lemma 56 Thm. 4.3 (Kim) and the global count Lemma 56 Thm. 4.8, which carry boundary and asymptotic terms. So “zeros – saddles = components” is **correct for bounded components** and needs Lemma 56’s boundary terms globally. “Sublevel sets are disks” is **correct for bounded components**; it is the maximum principle, not Morse theory. Disks are planar, so **no genus arises from the landscape itself**.

Census [NUMERICAL] (num/lemma87_morse.py, num/187/morse.json). Components were labelled on grids at levels between consecutive saddle heights.

- ζ : 13 bounded components over 12 levels, all of type (1 zero, 0 saddles).
- DH: 21 bounded components over 9 levels, including two of type (2 zeros, 1 saddle). These are the components containing a ghost pair and its transverse saddle 85.747, at the levels -66.370 and -65.850 .

Every component satisfies zeros – saddles = 1.

8.1. Hyperelliptic covers: what exists.

Construction 8.2 (the canonical cover). [CONSTRUCTION] Let $X = \{(s, y) : y^2 = \Lambda'(s)\}$, branched over the critical points of Λ (infinitely many, so X has infinite genus). The reflection gives $\Lambda'(1 - \bar{s}) = -\overline{\Lambda'(s)}$, so $\iota(s, y) = (1 - \bar{s}, i\bar{y})$ is an antiholomorphic involution of X (a real structure). Its fixed points lie over $s = \frac{1}{2} + it$ with $y = u(1 + i)$, $u \in \mathbb{R}$, i.e. where $\Lambda'(\frac{1}{2} + it) = -i\Xi'(t) = 2iu^2$. So the real locus consists of *ovals over the gaps between consecutive on-line saddles on which $\Xi' \leq 0$* : every other gap.

Proposition 8.3 (what the cover sees). [THEOREM]

- (i) X depends on Λ only through its critical points (up to an isomorphism class of the double cover). The transverse/longitudinal type of a saddle is the sign of Ξ''/Ξ , which involves the critical value. Hence the topology and real structure of X are blind to τ : they are the same whether or not a ghost sits in a saddle’s spacelike cone.
- (ii) Off-line saddles are non-real branch points, coming in pairs $\{s, 1 - \bar{s}\}$. S^+ is seen only as missing real branch points, i.e. two adjacent gaps merge.
- (iii) Suppose all saddles in a window are on the line. A gap between consecutive saddles contains no zero of Ξ iff one of its endpoints is transverse. So DH has 2τ zero-free gaps (when transverse saddles are not adjacent), and ζ , on its range, none.

Proof. (i), (ii): by construction. (iii): Ξ is monotone on a gap. If both endpoints are local maxima of $|\Xi|$ (longitudinal), monotonicity forces a sign change. If an endpoint a is a local minimum of $|\Xi|$ with $\Xi(a) \neq 0$, then $|\Xi|$ increases from a , and by monotonicity it does so all the way to the other endpoint, so there is no zero. $\square$

The a -cycle claim: incorrect as stated. A zero of Λ is not a branch point of X , so a loop around a zero does not lift to a distinguished cycle. The natural a -cycles encircle two consecutive branch points (saddles), i.e. a gap. For ζ each such gap carries one zero, so “one a -cycle per zero” holds numerically. For DH, 2τ of these cycles enclose no zero. So the correspondence is the zero/gap interlacing, which fails exactly at the transverse saddles. It does not come from the cover.

Numerics [NUMERICAL].

- ζ : 30 gaps between consecutive on-line saddles, each containing exactly 1 zero.
- DH: 74 gaps, of which $4 = 2\tau$ are zero-free: (83.896, 85.747), (85.747, 86.629), (112.915, 114.170), (114.170, 115.319), all with one transverse endpoint (Figure 6).

Construction 8.4 (truncations; *Analogy*). [ANALOGY] For $2g+2$ consecutive on-line saddles u_j , put $C_g : y^2 = \prod_j (u - u_j)$. With all branch points real, $C_g(\mathbb{R})$ has $g+1$ ovals (elementary: the product is positive on g bounded intervals and on the two unbounded ones, which join through infinity). Numerically [NUMERICAL], $g+1$ ovals for $g=1, 2, 3$ in every window of consecutive saddles, for ζ and DH alike. Hence the ovals carry *no* information on τ , and on S^+ only via non-real u_j . In the toy example of §3, $p(u) = ((u-1)^2 + d^2)((u+1)^2 + d^2)(u-3)$, the four critical points give 2 ovals at $d=0.5$ (all real) and 1 oval at $d=1.5$ (one off-line pair).

Not canonical: C_g depends on the window. For $g=1$, the j -invariant of four consecutive saddles ranges over [2064, 12125] (median 3951) for ζ and [1794, 11474] (median 3938) for DH, varying from window to window with no visible difference between the two.

Verdict on G .

- The landscape has no genus (planar disks).
- The only canonical cover is $y^2 = \Lambda'$. It has infinite genus and a real structure, but it is *blind* to τ , and with it to the ghosts in the $S^+ = 0$ regime that is the only one ever observed. That regime covers DH below 1000 and every pre-collision window of Lemma 56.
- The finite hyperelliptic curves are Analogies, and they are non-canonical. Their ovals are $g+1$ regardless of ghosts.
- The hyperelliptic/genus route therefore **cannot distinguish ζ from DH** as tested.

The Riemann Hypothesis is not proved here. Lemma 87 gives exact local geometry, restatements (Theorem 3.2, Proposition 7.2) and numerics. None of them implies RH.

9. CORRECTIONS

J.W.'s chat claims.

- (J1) (A) “4 separatrix rays at 90° , 4 steepest rays at 45° ; order m gives $2(m+1)$ rays.” Correct. The order is that of Λ' , and $\Lambda(s_0) \neq 0$ is needed (Proposition 2.1).
- (J2) (B) “Off-line saddles have their cone rotated by $\arg c$.” The rotation is $-\frac{1}{2} \arg c$ (Theorem 3.1). In the toy example it is $\mp 18.0^\circ$.
- (J3) (B) “ $RH \iff$ every on-line saddle has its zeros inside its timelike cone.” This is vacuous for on-line zeros, and it omits off-line saddles. The correct form is Theorem 3.2: no ghost $\iff S^+ = 0$ and no zero lies in the closed spacelike cone of an on-line saddle [RESTATEMENT].
- (J4) (C) “The deviation from the tangent cone is set by the zero spacing; each zero sits in its causal diamond.” The scale is right, [HEURISTIC] and [NUMERICAL]. Containment is automatic for on-line zeros and has content only for ghosts (Theorem 3.2). Basins fill only 5.7%–55% of the diamond half-width.
- (J5) (D) “Trivial-to-Möbius wall”. Correct. The connected double cover of a circle is a circle; “Möbius” is its line bundle. The wall is orientation-blind (all 105 saddles tested swap), consistent with Lemma 80’s multiplier no-go.
- (J6) (E) “The zero-slope Legendre intercept equals the saddle height.” Only at saddle heights where SD monotonicity holds. It fails at every transverse saddle (Proposition 6.1(ii)): at DH 85.747, $b(0) = -67.541 < \log h = -66.386$.
- (J7) (F) “The cone rotates 90° at the collision.” The cone is fixed at $\pm 45^\circ$. The lemniscate, i.e. which pair of sectors descends, turns by 90° (Proposition 7.1(iv); Lemma 56 Thm. 7.1). “Saddle height $\rightarrow -\infty$ ” is correct for the signed critical value, which passes through 0 linearly.

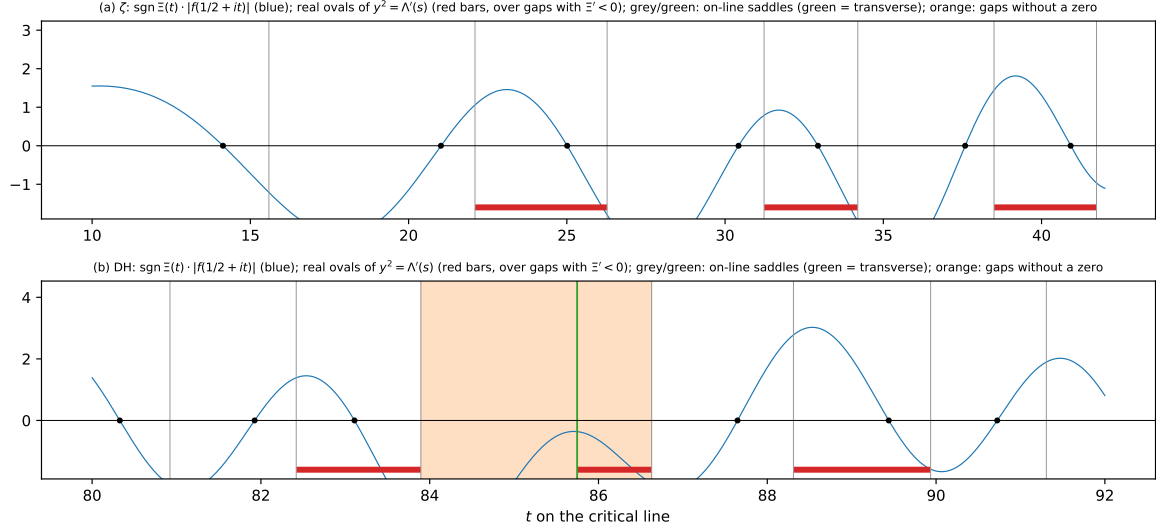


FIGURE 6. Real locus of the canonical cover $y^2 = \Lambda'(s)$ over the critical line. Blue: $\text{sgn } \Xi \cdot |f(\frac{1}{2} + it)|$, where f is Λ with the Gamma decay removed. Red bars: ovals over gaps with $\Xi' < 0$; only gaps entirely inside the window are marked. Grey and green lines: longitudinal and transverse saddles. Orange: zero-free gaps. (a) ζ : every gap has a zero. (b) DH: the transverse saddle 85.747 makes both adjacent gaps zero-free, yet the oval pattern (every other gap) is unchanged. *Diagnostic*: the cover's real structure is identical in (a) and (b); only the orange shading, which uses Ξ itself, sees the ghost.

(J8) (F , [t376]) “Ghost birth = saddle height hitting zero, and κ counts those events.” Precise form (Proposition 7.2). In a height window with admissible ends, $\kappa = \tau + S^+$ equals the number of forward-flow landing events still ahead. In each event a transverse saddle’s signed height goes to 0, giving a double real zero that splits along the line, after which the saddle is longitudinal. This assumes:

- generic pairwise collisions;
- all ghosts land;
- no forward births;
- zeros tracked along their trajectories.

Four further points:

- “Birth” is the backward-time reading of a landing.
- Saddle collisions move $\tau \rightarrow S^+$ without changing κ , and they are not landings.
- *Comparison*. Kim’s identity $K^+ = \tau + S^+$ is the static, saddle-side count at $t = 0$. Lemma 56’s DH data give the dynamic count: below height 1000 there are 28 ghosts, 28 transverse saddles and $S^+ = 0$, and 28 landings with $t_c \in [0.000507, 0.2306]$, each taking (τ, S^+, K^+) from $(1, 0, 1)$ to $(0, 0, 0)$. The two counts agree.
- *Status*. This is a **Restatement** [RESTATEMENT]. It is **not** a quadratic form whose negative index is κ ; no such form is constructed or implied.

(J9) (G) “Sublevel sets are disks; zeros – saddles = components; a hyperelliptic cover with an a -cycle per zero, a real structure, and ovals counting S^+/τ ; canonical.”

- Disks and the count: correct for bounded components; globally use Lemma 56 Thm. 4.3/4.8.
- a -cycle per zero: incorrect, since zeros are not branch points. The cycles are attached to gaps, and DH has 2τ zero-free gaps.

- Real structure: exists on $y^2 = \Lambda'$.
- Ovals: they do not count τ , and they count S^+ only via missing branch points.
- Canonical: only $y^2 = \Lambda'$ is; finite truncations are not.

Self-corrections during the build.

- Four DH transverse saddles below 1000 with very small ghost offset were missing from `155/dh_ghost_saddles.json`. They were located by a fine scan, giving 28/28.
- A draft of §5 had wrong ζ wall log-heights. They were replaced by the values in `num/187/monodromy.json`.
- Several internal theorem numbers (Lemma 55/56/81) were checked against the compiled `.aux` files and corrected.

10. RANKED DIRECTIONS

Score = 0.6·significance+0.4·probability (subjective, [HEURISTIC]).

	direction	sig	prob	score
N1	Static landing clock to higher order: add the $O((t - t_c)^2)$ terms (using H''''/H'' at the saddle) to $-4/c$, and test whether the 28 DH ratios 1.0008–1.71 collapse to 1	.55	.80	.65
N2	Lehmer pairs as near-walls: the backward clock $-4/c$ at the smallest ζ saddle heights, compared with Lemma 56's collision times and with known lower-bound methods for Λ (Rodgers–Tao)	.60	.70	.64
N3	Light-cone Kim census: Theorem 3.2 checked window by window for DH to height 1000, plus the function-field analogue (Lemma 56 §6)	.45	.85	.61
N6	Off-line cone rotation $-\frac{1}{2} \arg c$ tracked through saddle collisions in the backward ζ flow	.35	.75	.51
N5	Prove the convexity light-cone law of §6 (concavity of φ only in spacelike cones of zeros) under a spacing hypothesis	.45	.55	.49
N4	Periods of the canonical cover $y^2 = \xi'$ on the gap cycles: do they see τ where the topology does not?	.40	.60	.48

11. QUESTIONS FOR J.W.

- Q1. Do you accept Theorem 3.2 (spacelike cones plus $S^+ = 0$) as the official light-cone form of (H)?
- Q2. Are corrections (J1)–(J9) acceptable, in particular (J2) ($-\frac{1}{2} \arg c$), (J3) (spacelike, not timelike) and (J7) (the lemniscate rotates, not the cone)?
- Q3. Should TAA adopt the landing clock $\hat{t}_c = t + H/H''(z^*)$ and the κ -landing Restatement (Proposition 7.2) as standard language, labelled [RESTATEMENT]?
- Q4. Do you accept the verdict on (G): the landscape has no genus; the canonical cover $y^2 = \Lambda'$ is blind to τ ; finite hyperelliptic curves are non-canonical Analogies?
- Q5. Next lemma: N1 (higher-order clock, .65), N2 (Lehmer near-walls, .64) or N3 (light-cone Kim census, .61)?

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All entries are checked in `lit/lemma87/verified_refs.md`. Harnack's theorem is *not* used.

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Internal: TAA Lemmas 43, 55, 56, 62, 80, 81 (theorem and section numbers checked against their compiled `.aux` files).

CODE AND DATA

- `num/lemma87_cones.py` → `num/187/cones.json`: saddles, c , and cones for ζ and DH; ghost-cone tests; toy.
- `num/lemma87_basins.py` → `basins.json`: separatrix deviation, basins against diamonds.
- `num/lemma87_monodromy.py` → `monodromy.json`: small-loop and level-circle monodromy.
- `num/lemma87_legendre.py` → `legendre.json`: φ , intercepts, convexity census.
- `num/lemma87_heat.py`, `lemma87_heat4.py` → `heat.json`, `heat_pred28.json`: clocks and predictors (engines `num/lemma56_heatcore.py`, `num/lemma56v3_fft.py`; landing data `num/156v3/tc_fft.json`).
- `num/lemma87_morse.py` → `morse.json`: component census, gaps, ovals, j -invariants, toy.
- `num/lemma87_figs.py` → `build87/figs/lemma87_*.pdf`.

Evaluation uses `python-flint` (Arb) at 96 bits as floating point, not certified. Nothing was posted or submitted.

**TAA LEMMA 88: THE STANDARD CONJECTURES AS COMMON ROOT OF
HODGE, BSD AND RH.
ALGEBRAICITY VERSUS POSITIVITY, FROBENIUS AS A MORDELL–WEIL
SECTION,
THE ROSATI–NS–SADDLE IDENTITY, AND NÉRON–SEVERI NUMERICS
OVER $\mathbb{F}_q$**

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Lemma 88 studies the one setting, varieties over $\mathbb{F}_q$, where the algebraicity half and the positivity half of the Weil mechanism are both available. It measures every TAA counter (κ , k , τ , Z_{off} , the Rosati index, n_+ of the Néron–Severi Gram, ρ , the Mordell–Weil rank) against a true Hodge index. It then states precisely what is missing for ζ . Tags as in Lemmas 80–87: [THEOREM] proved here or standard with proof given; [CITED] source opened and checked; [CITED-sec] through an opened secondary source; [BIB] bibliographic data only; [TAA-L] earlier TAA lemma; [RESTATEMENT] equivalent to RH; [ANALOGY] analogy; [OPEN] open; [NUMERICAL] computation done here; [HEURISTIC] heuristic; [CONJECTURE] conjecture. Citations: `lit/lemma88/verified_refs.md`. All signatures, ranks, multiplicities and factorizations are computed in exact integer/rational arithmetic (SymPy); only the saddle classifier of Lemma 56 uses mpmath at 50 digits. Every number is computed here (`num/lemma88/*.py` $\rightarrow$ `num/188/*.json`) or read from the named data files of Lemmas 82 and 85; none is recalled.

Task (J.W. [t379u], plan of the t378u review, §7): “We havent really touched on the hodge conjecture at all...I have a feeling that if we can actually do this – it might actually require building a set of concepts that oneshot the BSD, hodge, and RH in the same go...I want to spend some dedicated time on this.”

The Riemann hypothesis is not proved, here or anywhere in TAA. Nor are the Hodge conjecture, BSD, or Grothendieck’s standard conjectures. Over $\mathbb{F}_q$, every positivity used below is a known theorem (Hodge index on surfaces, Rosati positivity, Néron–Tate/Shioda heights). For ζ , every positivity statement is [RESTATEMENT]. The “common root” found here is precise over $\mathbb{F}_q$ and an analogy over $\mathbb{Q}$.

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Date: October 24, 2026.

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1. SCOPE AND THE ALGEBRAICITY/POSITIVITY SPLIT

The Weil mechanism in two halves. Weil’s proof for a curve $C/\mathbb{F}_q$ (Lemmas 3, 14, 46 [TAA-L]; Milne [4, §1] [CITED]) has two independent inputs.

- *Algebraicity*: the Frobenius graph Γ_π , the diagonal Δ and H, V are divisors on $C \times C$, so they live in a finite-rank lattice $\text{NS}(C \times C)$ on which $\mathbb{Q}[F]$ acts.
- *Positivity*: the Hodge index theorem on that lattice; equivalently, Rosati positivity on $\text{End}^0(J_C)$ (Milne Thm. 1.27 [CITED]).

Grothendieck’s standard conjectures are exactly this split, raised to all dimensions. In his words [1, p. 193] [CITED]: “The first is an existence assertion for algebraic cycles. . . The second is a statement of positivity, generalising Weil’s well-known positivity theorem in the theory of abelian varieties. . . and is in fact a consequence of [the Hodge inequalities] in characteristic zero.”

J.W.’s hunch, made testable. If one set of concepts “onshots” Hodge, BSD and RH, it should be visible in a setting where all three have honest versions. That setting is $\mathbb{F}_q$:

- RH for curves is Weil’s theorem;
- BSD for elliptic surfaces is equivalent to Tate’s conjecture for their divisors (§2, S8);
- the Hodge-type standard conjecture for surfaces is the Hodge index theorem.

Lemma 88 checks that the same lattice, the same positivity and the same counters appear in all three, proves two linking theorems (A and B), and makes the ρ formula for $C \times C$ precise (Theorem C). It then lists what fails over $\mathbb{Q}$.

What is *not* attempted. No step toward RH for ζ , the Hodge conjecture, or BSD over $\mathbb{Q}$. No “proof sketch” of anything open.

2. THE STATEMENTS, READ AT THE SOURCE

- (S1) *Origin and Bombieri.* Grothendieck [1, p. 193] (scanned page viewed, [CITED]): the conjectures “are not really new, and they were worked out about three years ago independently by Bombieri and myself”. Milne [4, §3] [CITED] dates this “about 1964”; Grothendieck’s “three years” before the 1968 colloquium suggests c. 1965. This is a minor discrepancy between two secondary datings, not resolved here. **Bombieri’s own text was not found.** A MathOverflow answer names Bombieri’s 1968 “Varietà algebriche omeomorfe” as the only candidate publication; it was not opened and is not cited as a source.
- (S2) *Lefschetz type.* Kleiman [2, pp. 359–360] [CITED]: $B(X)$ says that the operator Λ is induced by an algebraic cycle on $X \times X$; $B(X) \Rightarrow C(X)$ (Künneth components algebraic). Kleiman’s introduction says $B(X)$ is satisfied “by curves, by surfaces and by abelian varieties”. Milne [4, Ex. 3.13] [CITED] restricts the surface case to surfaces with $\dim H^1 = 2 \dim \text{Pic}$ (citing Kleiman 1968, 2. Appendix). Both are recorded; $C \times C$ is covered either way, because J_C is an abelian variety. Prop. 3.11 of Milne (“Kleiman 1994, Thm. 4-1”) [CITED-sec].
- (S3) *Hodge type.* $I^p(X, L)$: $a, b \mapsto (-1)^p (L^{n-2p} a \cdot b)$ is positive definite on primitive algebraic classes [2, §3] [CITED]. Kleiman Rem. 3.10: a complex X satisfies $I(X, L)$ by the Hodge index theorem [CITED]. In characteristic p , it is known essentially only for surfaces, where it is the Hodge index theorem (Milne, text after Ex. 3.13 and Thm. 1.2) [CITED].
- (S4) *Standard conjectures $\Rightarrow$ RH over $\mathbb{F}_q$.* Kleiman Thm. 3.11 [CITED]: under strong Lefschetz, $B(X)$, $B(Y)$, the transpose u' of an algebraic correspondence is algebraic and $\text{Tr}(u' \circ u) \in \mathbb{Q}$; if also $I(X \times Y)$ holds, then $\text{Tr}(u' \circ u) > 0$ for $u \neq 0$. Lemma 4.6 [CITED]: $\text{Tr}(h' \circ h_1)$ is then an inner product on the $\mathbb{Q}(a)$ -algebra generated by g , and $g'g = \text{id}$ forces g to be semisimple with

- eigenvalues of absolute value 1. Thm. 4.7 [CITED]: the “Riemann hypothesis” follows under strong Lefschetz $+B(X) + I(X \times X)$. Milne Thms. 3.15/3.17 restate this [CITED].
- (S5) *Kleiman’s Lemma 4.6 is Lemma 84.* The L84 argument (Rosati trace form on $\mathbb{Q}[F]$; negative index $= \kappa$) is the curve case of Kleiman’s Lemma 4.6, with $g = q^{-1/2}F$ and $'$ the Rosati involution. [THEOREM] (identification by inspection of both statements). L84’s “one negative direction per ghost pair” is the quantitative failure of Kleiman’s inner product for a fake.
- (S6) *Hodge and Tate.* Milne Conj. 3.19, 3.20, 3.21 (Hodge $\Rightarrow$ Lefschetz over $\mathbb{C}$; Lefschetz over $\mathbb{C} \Rightarrow$ Hodge for abelian varieties, Abdulali, André), 3.22 (Tate $\Rightarrow$ Lefschetz; full Tate over $\mathbb{F}_p \Rightarrow$ Hodge type in characteristic p) [CITED-sec]. “Tate = ℓ -adic Hodge” is a slogan [ANALOGY].
- (S7) *Tate for surfaces over $\mathbb{F}_q$.* Tate [5, Thm. 5.2] [CITED]. For a surface $X/\mathbb{F}_q$ the following are equivalent: (i) $\text{Br}(X)(\ell)$ is finite; (ii) $\text{NS}(X) \otimes \mathbb{Z}_\ell \rightarrow H^2(\bar{X}, \mathbb{Z}_\ell(1))^G$ is bijective; (iv) $\rho(X)$ equals the multiplicity of q as a reciprocal root of $P_2(X, T)$. If they hold, Artin–Tate (C) holds up to a power of p . Tate’s conjecture is known for products of curves and for elliptic K3 surfaces (Schütt–Shioda [8, §14.6] [CITED]).
- (S8) *Tate $\iff$ BSD over function fields.* Tate [5, §4, conjecture (d)] [CITED]: for $f : X \rightarrow V$ with smooth generic fibre, (B) for the Jacobian of the generic fibre holds iff Artin–Tate (C) holds for X . It was “checked” when f is smooth with a section, and Tate was “99 % sure” for elliptic generic fibres with a rational point. Gordon [7, Thm. 6.1] [CITED] proves $AT(X, k) \iff BSD(\text{Jac } X_K, K)$ for proper flat cohomologically flat f whose generic fibre is smooth and geometrically irreducible with a K -rational cycle of degree 1. So, over $\mathbb{F}_q(t)$, *BSD for E is a statement about the Néron–Severi lattice of the elliptic surface*, which is where the Hodge index lives (Lemma 85).

3. THEOREM A: FROBENIUS AS A MORDELL–WEIL SECTION

Theorem 3.1 (Rosati trace form = Mordell–Weil lattice on the constant surface). *Let $E/\mathbb{F}_q$ be an elliptic curve and $S = E \times E \xrightarrow{pr_2} E$ the constant elliptic surface. Then:*

- (a) *the sections of S are the $\mathbb{F}_q$ -morphisms $E \rightarrow E$; modulo the torsion translations by $E(\mathbb{F}_q)$ they form $\text{End}_{\mathbb{F}_q}(E)$;*
- (b) *for $\varphi \in \text{End}_{\mathbb{F}_q}(E)$, Shioda’s height is $h(\varphi) = 2 \deg \varphi$, and the height pairing is $\langle \varphi, \psi \rangle = \deg(\varphi + \psi) - \deg \varphi - \deg \psi = \text{Tr}(\varphi \hat{\psi})$;*
- (c) *on $\mathbb{Z}[\pi]$ the Gram matrix in the basis $\{1, \pi\}$ is $\begin{pmatrix} 2 & a \\ a & 2q \end{pmatrix}$, $a = q + 1 - \#E(\mathbb{F}_q)$. This is exactly the Lemma 84 trace form on $\mathbb{Q}[F]$;*
- (d) *Shioda–Tate: $\rho(E \times E/\mathbb{F}_q) = 2 + \text{rank } \text{End}_{\mathbb{F}_q}(E)$.*

Proof. (a) A section is a morphism $s : E \rightarrow E \times E$ with $pr_2 \circ s = \text{id}$, i.e. the graph of $pr_1 \circ s$. Every morphism of curves $E \rightarrow E$ is a translation composed with an endomorphism; translations by $E(\mathbb{F}_q)$ are torsion sections.

(b) Schütt–Shioda [8, Thm. 11.5, §11.6] [CITED]: $h(P) = 2\chi(\mathcal{O}_S) + 2(P \cdot \mathcal{O}) - \sum_v \text{contr}_v(P)$. For $S = E \times E$: $\chi(\mathcal{O}_S) = 0$, all fibres are smooth (no contributions), and $\Gamma_\varphi \cdot (E \times \{0\}) = \# \ker \varphi = \deg \varphi$ for separable φ , counted with intersection multiplicity in general. Polarizing h gives the pairing; $\deg(\varphi + \psi) - \deg \varphi - \deg \psi = \text{Tr}(\varphi \hat{\psi})$ is the standard degree identity (Milne [4, §1, Thm. 1.27 context]).

(c) $h(1) = 2$, $h(\pi) = 2q$, and $\langle 1, \pi \rangle = \text{Tr}(\hat{\pi}) = a$.

(d) [8, Cor. 6.13] [CITED] with trivial lattice U (no reducible fibres). [THEOREM]. $\square$

Check [NUMERICAL] (num/review88/mw_constant.py, 15 curves). $h(1 \mp \pi) = 2\#E^{(\text{tw})}(\mathbb{F}_p)$ reproduces $\langle 1, \pi \rangle = a$ in 15/15 cases. This is a sign-convention check only.

Remark 3.2 (What Theorem A says about the gap). Over $\mathbb{F}_q$, the finite $\mathbb{Q}$ -algebra $\mathbb{Q}[F]$ (gap item (i)) is a sublattice of a Mordell–Weil group. Its positivity (item (ii)) is Néron–Tate/Shioda positivity,

which is the Hodge index theorem on the orthogonal complement of the trivial lattice. BSD's lattice and RH's lattice coincide here. This is the simplest honest instance of the t379 hunch. [THEOREM].

Over $\mathbb{Q}$, the positivity half exists: Néron–Tate heights on $A(\mathbb{Q})$ are positive definite, which is the arithmetic Hodge index of Faltings–Hriljac ([BIB], L84/L85). The missing object is a “Frobenius section”, a point whose height pairing would be the Weil form of ζ . This is Lemma 85's obstruction O3 restated, **not** a way around it. [OPEN].

Proposition 3.3 (genus g , [HEURISTIC]). *For a curve $C/\mathbb{F}_q$ of genus g and the constant abelian scheme $J_C \times C \rightarrow C$ over $K = \mathbb{F}_q(C)$, one expects $J_C(K) \supseteq \text{End}_{\mathbb{F}_q}(J_C)$ via the Albanese map $C \rightarrow J_C$, with Néron–Tate height a positive multiple of the Rosati trace form $\text{Tr}(\varphi\varphi')$.*

Status. [HEURISTIC]: Lang–Néron/Shioda-type folklore. No source was opened for the height normalization in genus g ; the genus-1 proof above does not transfer verbatim, because $J_C \times C$ is not an elliptic surface. Direction D4 (§7) is to source it.

4. THEOREM B: THE ROSATI–NS–SADDLE IDENTITY, AND THEOREM C: $\rho(C \times C)$

Setting. $P(T) = 1 + c_1T + \dots + c_{2g}T^{2g} \in \mathbb{Z}[T]$ is *Weil-shaped* if $c_{2g-i} = q^{g-i}c_i$ (the reciprocal roots α are closed under $\alpha \mapsto q/\alpha$ and under conjugation). Put $L(u) = P(u/\sqrt{q})$ and $S(u) = uL'(u) - gL(u)$ (Lemma 56). Let $k = \#\{\alpha : |\alpha| > \sqrt{q}\}$ over distinct roots: a real off-circle pair counts 1, a quartet 2. Let τ be the number of transverse simple saddles of $h(\varphi) = e^{-ig\varphi}L(e^{i\varphi})$ on the circle, and Z_{off} the number of roots of S off the circle. $G_n(P)$ is the Lemma 46 Gram of $\{H, V, \Gamma_0, \dots, \Gamma_{n-1}\}$, with entries $\Gamma_m \cdot \Gamma_n = q^{\min} N_{|m-n|}$.

Theorem 4.1 (Four-way identity). *If P is squarefree and S has only simple roots on $|u| = 1$, then for $n \geq 2g$*

$$n_-(\text{Tr}(xx^\dagger)|_{\mathbb{Q}[F]}) = k = \tau + \frac{1}{2}Z_{\text{off}} = n_+(G_n(P)) - 1.$$

Proof. The first equality is Lemma 84 (spectral Albert step, Lemma 30), i.e. the quantitative form of Kleiman's Lemma 4.6 (S5). The second is Lemma 56's circle identity, built on Cohn 1922 Fall IV (read in Lemma 56). The third is Lemma 46 ($n_+ = 1 + k$). [THEOREM] by composition of [TAA-L] theorems. $\square$

Proposition 4.2 (Where the saddles degenerate; genus 1 and 2). *(a) $g = 1$: $S(u) = L_2u^2 - L_0 = u^2 - 1$ always has simple roots ± 1 , which are degenerate only if $h = 0$ there, i.e. $a^2 = 4q$. (b) $g = 2$: with $L = (1, L_1, L_2, L_1, 1)$ one has $S(u) = (u^2 - 1)(2u^2 + L_1u + 2)$, independent of L_2 . Hence S has a multiple root on $|u| = 1$ iff $|L_1| = 4$, i.e. $|c_1| = 4\sqrt{q}$. Then $u = -\text{sgn}(c_1)$ is a triple root. For integer c_1 this needs $16q = c_1^2$, so q must be a square.*

Proof. Expand $S = \sum_j (j - g)L_j u^j$. For $g = 2$: $S = 2u^4 + L_1u^3 - L_1u - 2 = (u^2 - 1)(2u^2 + L_1u + 2)$. The quadratic has a double root iff $L_1^2 = 16$, at $u = -L_1/4 = \mp 1$, which is also a root of $u^2 - 1$. It shares a root with $u^2 - 1$ iff $4 \pm L_1 = 0$, the same condition. [THEOREM]. $\square$

This is exactly the boundary of the Lemma 56 genus-2 criterion ($\tau = k$ iff $|a_1| \leq 4\sqrt{q}$). The two Review-88 exceptions ($q = 4$, $c_1 = 8$) and all 10 degenerate cases of N4 are of this type.

Corollary 4.3 (Perturbation multiplicity). *Let P be squarefree. For every Weil-shaped perturbation P_ε small enough that S_ε has simple roots on the circle, $\tau(P_\varepsilon) + \frac{1}{2}Z_{\text{off}}(P_\varepsilon) = k(P)$. The “perturbation multiplicity” $k - \frac{1}{2}Z_{\text{off}}(P_\varepsilon)$ is therefore the correct transverse count of a degenerate saddle; it may depend on the side of the perturbation, but the identity holds on every side.*

Proof. A squarefree self-inversive real P keeps its simple circle roots on the circle under small self-inversive perturbation, and its off-circle roots stay off. So $k(P_\varepsilon) = k(P)$. Apply Theorem 4.1 to P_ε . [THEOREM]. $\square$

Theorem 4.4 (ρ of $C \times C$ over $\mathbb{F}_q$; sourced form). *Let $C/\mathbb{F}_q$ be a curve with Frobenius characteristic polynomial $Q(T) = \prod_i (T - \alpha_i)$, and let $m(\lambda)$ be the multiplicity of λ among the α_i . Then*

$$\rho(C \times C/\mathbb{F}_q) = 2 + \#\{(i, j) : \alpha_i \alpha_j = q\} = 2 + \sum_{\lambda} m(\lambda)^2.$$

In particular the Frobenius span $\langle H, V, \Gamma_{F^m} \rangle$, of rank $2 + D$ with D the number of distinct roots (Lemma 46), is all of $\text{NS}(C \times C)_{\mathbb{Q}}$ iff Q is squarefree.

Proof. By Künneth, $P_2(C \times C, T) = (1 - qT)^2 \prod_{i,j} (1 - \alpha_i \alpha_j T)$. Tate’s conjecture holds for products of curves ([8, §14.6] [CITED]), and by [5, Thm. 5.2 (ii) $\Leftrightarrow$ (iv)] [CITED] this means ρ equals the multiplicity of q as a reciprocal root of P_2 . That gives the first equality. For the second, $\alpha_j = q/\alpha_i$, and the multiset of roots is invariant under $\lambda \mapsto q/\lambda$ (functional equation), so $\#\{(i, j)\} = \sum_{\lambda} m(\lambda)m(q/\lambda) = \sum_{\lambda} m(\lambda)^2$. Finally $\sum m^2 \geq \sum m \cdot [m \geq 1]$ forces $2 + \sum_{\lambda} m(\lambda)^2 = 2 + D$ iff all $m = 1$. [THEOREM] from [CITED] inputs. $\square$

Remark 4.5 (The “ $2 + \text{rank End}(J)$ ” form). $\sum_{\lambda} m(\lambda)^2 = \sum_i m_i^2 \deg P_i$ over the $\mathbb{Q}$ -factorization $Q = \prod P_i^{m_i}$ is the dimension of the commutant of F on V_{ℓ} (Frobenius is semisimple on abelian varieties: Weil, Milne Aside 3.18 [CITED-sec]). By Tate’s theorem [6] (main theorem via Milne §1.15, [CITED-sec]; the paper itself was not opened) this commutant has dimension $\text{rank End}_{\mathbb{F}_q}(J_C)$. So $\rho = 2 + \text{rank End}_{\mathbb{F}_q}(J_C)$ holds [CITED-sec]. The direct geometric isomorphism $\text{NS}(C \times C) \cong \mathbb{Z}^2 \oplus \text{End}(J_C)$ (Weil’s correspondences) was not opened at a source and is labelled [HEURISTIC]. The numerics use only the sourced form $2 + \text{mult}_q$.

5. NUMERICS N1–N5

All signatures are exact. Each is computed by Descartes’ rule on the characteristic polynomial of an integer symmetric matrix, which is exact because the polynomial is real-rooted. $\rho_{\text{Tate}} = 2 + \text{mult}_q R$, where $R(T) = \text{Res}_X(Q(X), X^{2g}Q(T/X)) = \prod_{i,j} (T - \alpha_i \alpha_j)$ and the multiplicity is found by exact division. Point counts are brute force over $\mathbb{F}_{p^k}$ (polynomial-basis arithmetic). Library: `num/lemma88_lib.py`.

N1. $E \times E$ over $\mathbb{F}_p$, $5 \leq p \leq 47$. `lemma88_n1.py` $\rightarrow$ `188/n1.json`. For each prime p , every curve $y^2 = x^3 + Ax + B$ was counted. Every integer a with $a^2 < 4p$ occurred: 249 traces in total, $2\lfloor 2\sqrt{p} \rfloor + 1$ per p (observed, not used).

- *Curves (249/249).* The Gram of $\{H, V, \Delta, \Gamma_{\pi}\}$ has signature exactly $(1, 3)$; $\rho_{\text{Tate}} = 4$, equal to the Gram rank; the Rosati form is positive definite; $\tau = 0$.
- *Supersingular, base-changed to $\mathbb{F}_{p^2}$ (13/13).* $Q = (T + p)^2$. The Frobenius span has signature $(1, 2, 1)$, rank 3, while $\rho_{\text{Tate}} = 6$ and $\sum m^2 = 4$ (the full quaternion endomorphism ring). **Here the Frobenius classes see only 3 of the 6 dimensions of NS**, as Theorem 4.4 predicts.
- *Fakes $|a| > 2\sqrt{p}$, three on each side, all p (78/78).* Signature $(2, 2)$, Rosati $(1, 1)$, $\tau = 1$, $Z_{\text{off}} = 0$: the Hodge index fails by exactly $k = 1$.

N2. $C \times C$ for genus-2 curves. `lemma88_n2.py` $\rightarrow$ `188/n2.json`. Curves $y^2 = f(x)$ with f monic squarefree of degree 5, counted over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$:

p	curves	Weil polys	checks ok	Q non-squarefree	ρ_{Tate}	histogram
3	162 (all f)	32	32	0	6^{32}	
5	2500 (all f)	81	81	4	$6^{77} 10^4$	
7	504 (600 draws)	120	120	2	$6^{118} 10^2$	
11	232 (250 draws)	136	136	0	6^{136}	

Checks, all exact, on all 369 Weil polynomials:

- $n_+(G_6) = 1$ (Hodge index);

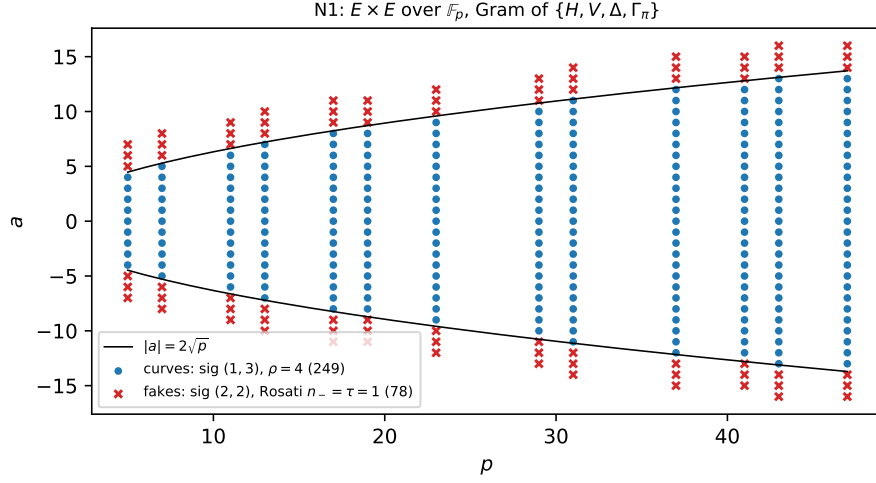


FIGURE 1. N1. Traces a of all elliptic curves over $\mathbb{F}_p$, $5 \leq p \leq 47$ (dots), and Weil-shaped fakes (crosses). Every dot has Gram signature $(1, 3)$; every cross has $(2, 2)$.

- Gram rank $r_F = 2 + D$;
- $\rho_{\text{Tate}} = 2 + \sum m_i^2 \deg P_i$;
- $\rho_{\text{Tate}} = r_F$ iff Q is squarefree;
- the Rosati form is positive semidefinite.

The six non-squarefree cases are $Q = (T^2 - bT + p)^2$ with $b \in \{0, \pm 2\}$, plus $(T^2 - 5)^2$ at $p = 5$. They have $r_F = 4 < \rho = 10$, so most of $\text{NS}(C \times C)$ is invisible to Frobenius. In the cases $b \in \{0, \pm 2\}$ the Jacobian is isogenous to E^2 . The case $(T^2 - 5)^2$ is the Weil number $\sqrt{5}$, which by Honda–Tate theory (Milne 1.16, [CITED-sec]) corresponds to a simple isogeny class. That this class is a surface is [HEURISTIC]; it is not used.

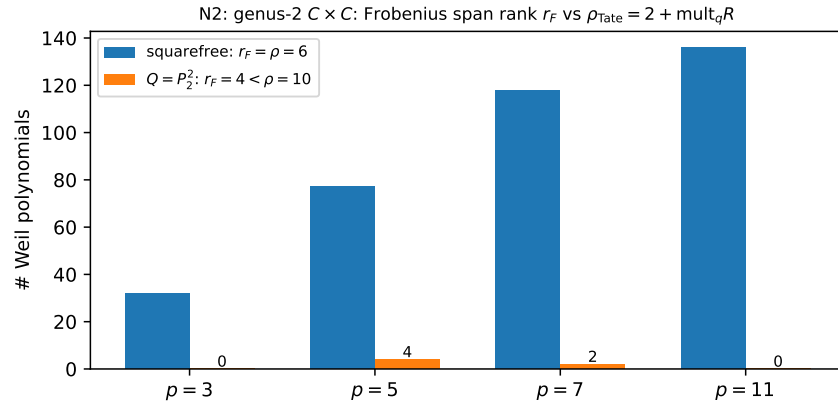


FIGURE 2. N2. Genus-2 Weil polynomials: Frobenius-span rank versus ρ_{Tate} .

N3. Elliptic surfaces (Lemma 85 data, exact re-verification). lemma88_n3.py $\rightarrow$ 188/n3.json.

- *Rational surface over $\mathbb{F}_5$* (12 fibres of type I_1). $L(E/K, T) = (5T - 1)^4(5T + 1)^2(25T^2 + 1)$, so $\text{ord}_{T=1/5} L = 4$ and $\rho(\mathbb{F}_5) = 2 + 4 = 6$; geometric $\rho = 2 + 8 = 10 = b_2$. The Mordell–Weil

Gram is positive definite with $\det = 8$ ($A_3 \oplus A_1$). The NS Gram $\langle O, F \rangle \oplus (-\text{MW})$ has signature exactly $(1, 5)$ and $\det = -8$. This is Tate's Thm. 5.2(iv) with equality, and BSD's rank equals the analytic order 4.

- *Trace forms* on $\mathbb{Q}[\gamma]$, $\gamma \mapsto q^2/\gamma$. The true L gives $(4, 0, 4)$: positive semidefinite, with a degenerate part from the repeated eigenvalue 5. Lemma 85's two ghosts (each a real off-circle pair, e.g. the factor $25T^2 - 12T + 1$) give $n_- = 1 = \kappa$.
- *33 elliptic K3s over $\mathbb{F}_7$* ($2 \times II^*$, $4 \times I_1$, trivial rank 18). All 33 agree with Lemma 85: $\rho(\mathbb{F}_7) \in \{18^{12}, 19^{18}, 20^3\}$ and $\rho_{\text{geom}} \in \{18^9, 20^{24}\} \leq 22$; every trace form has $n_- = 0$.

N4. The four-way identity on real curves and fakes. lemma88_n4.py $\rightarrow$ 188/n4.json.

- *Real curves: 740 Weil polynomials* (249 of genus 1 from N1, 369 of genus 2 from N2, 122 of genus 3: hyperelliptic curves of degree 7 over $\mathbb{F}_3$ and $\mathbb{F}_5$, counted over $\mathbb{F}_{p^k}$, $k \leq 3$). All have $k = 0$. Rosati $n_- = 0$ and $n_+(G_{2g+2}) = 1$ hold in 740/740. The saddle identity holds in 739/740. The exception is $Q = (T^2 - 2T + 5)^2(T^2 + 4T + 5)$, $q = 5$: P is not squarefree and has a *double root on the circle*, which lies outside Theorem 4.1. There the critical point is a zero of h and the classifier h''/h is meaningless. No real curve had a degenerate saddle.
- *1000 random integer fakes* ($g \in \{1, 2, 3\}$, $q \in \{2, 3, 4, 5, 7\}$, seed 8888; 10 non-squarefree draws skipped). Distribution $k \in \{0^{147}, 1^{433}, 2^{411}, 3^9\}$. Rosati $= k$ in 1000/1000 and $n_+ - 1 = k$ in 1000/1000; the saddle identity holds in 990/990 non-degenerate cases. The 10 degenerate cases are all $q = 4$, $|c_1| = 8$ (Prop. 4.2). After the perturbations $c_1 \mapsto c_1 \pm \varepsilon$, $c_3 \mapsto c_3 \pm q\varepsilon$ with $\varepsilon \in \{10^{-12}, 10^{-9}\}$, the identity holds for all perturbations in 10/10 cases, and also in the 2/2 Review-88 cases. Side dependence: $\varepsilon < 0$ gives $(\tau, Z_{\text{off}}) = (1, 0)$, $\varepsilon > 0$ gives $(0, 2)$. Perturbing c_g does not change S and leaves the degeneracy in place.

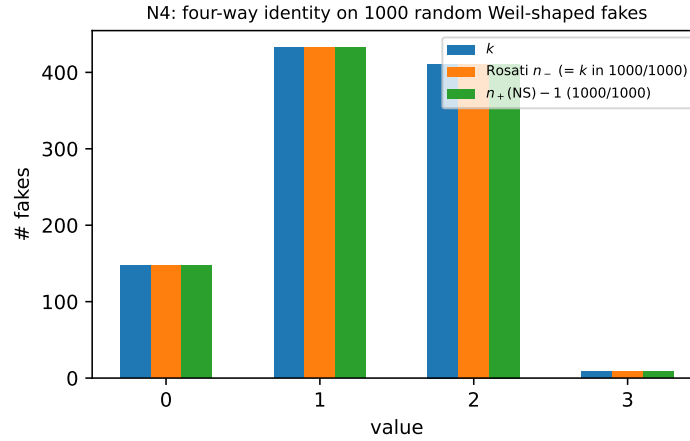


FIGURE 3. N4. Distribution of k on 1000 fakes; the Rosati index and $n_+(\text{NS}) - 1$ coincide with k case by case.

N5. Cutoff Weil forms as a finite Hodge-index diagnostic (no new ζ claim). lemma88_n5.py $\rightarrow$ 188/n5.json, read from num/182/cutoff.json (Lemma 82, band [70, 102]). By Lemma 15 the Hodge-index excess of $\mathcal{L}_L = P - Q_L$ is $n_-(Q_L)$.

- ζ : excess 0 at every $L = 2, \dots, 12$ ($N = 20, \dots, 122$). The smallest eigenvalue is $\geq -9.67 \cdot 10^{-10}$, at the Lemma 82 numerical floor.
- DH: excess 0 at $L = 2, 3$ and 2 for $L = 4, \dots, 12$ ($\lambda_{\min}$ from -1.34 down to -315). Lemma 82 locates the onset at $L \approx 3.73$.

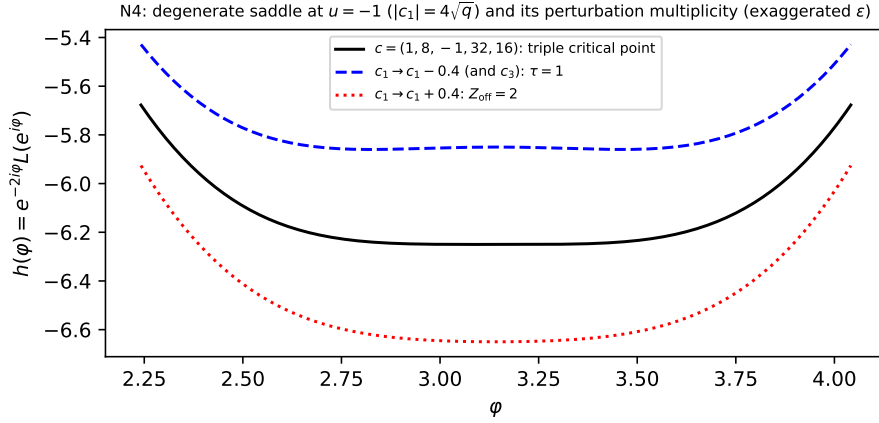


FIGURE 4. N4. The degenerate saddle of $c = (1, 8, -1, 32, 16)$, $q = 4$, at $\varphi = \pi$ (triple critical point, solid), and the two sides of its perturbation, exaggerated to $\varepsilon = \mp 0.4$ for visibility. Dashed: $\tau = 1$. Dotted: $Z_{\text{off}} = 2$. At $\varepsilon = \mp 0.4$ the counts were recomputed and agree with the infinitesimal ones.

This is the same table as Lemma 82, read as “index above 1”. It is [RESTATEMENT]-shaped and is recorded only so that the $\mathbb{F}_q$ and ζ counters sit side by side.

Counter	$\mathbb{F}_q$ curves	$\mathbb{F}_q$ fakes	ζ (cutoff)	DH (cutoff)
Hodge-index excess	0 (N1, N2, N4)	k (N1, N4)	0 ($L \leq 12$)	2 ($L \geq 4$)
$n_+ - 1$				
Rosati n_-	0	k	no finite $\mathbb{Q}[F]$ (L84)	no finite $\mathbb{Q}[F]$
$\tau + \frac{1}{2}Z_{\text{off}} / \text{Kim } \tau + S^+$	0	k	0 (L56, windows)	28 ghosts < 1000 (L56/L87)
ρ vs Frobenius span	equal iff Q squarefree	—	—	—

6. THE COMMON ROOT: WHAT HOLDS OVER $\mathbb{F}_q$, $\mathbb{C}$ AND $\mathbb{Q}$

Problem	algebraicity half	positivity half	TAA lemmas / this lemma
RH, curves over $\mathbb{F}_q$	known: Γ_π a divisor; Tate for $C \times C$ (Thm. 4.4)	known: Hodge index; Rosati = Kleiman 4.6 (S5)	L3, 14, 15, 46, 84; Thms. B, C; N1, N2, N4
Hodge conjecture over $\mathbb{C}$	open	known: Hodge–Riemann; Kleiman Rem. 3.10	L81 (HR form)
BSD over $\mathbb{F}_q(C)$	$\iff$ Tate/Artin–Tate for the surface (Tate (d), Gordon Thm. 6.1)	known: Shioda height = $-$ intersection	L85; Thm. A; N3
BSD over $\mathbb{Q}$	rank = order and finite Sha: open	known: Néron–Tate (Faltings–Hriljac [BIB])	L84, 85, 86
RH for ζ	no object to be algebraic in (M1–M3)	open ([RESTATEMENT])	L1, 15, 64, 82; N5

Verdict on the hunch ([HEURISTIC], with the $\mathbb{F}_q$ entries [CITED]/[THEOREM]). Over $\mathbb{F}_q$ the hunch is correct in a precise sense. The same two conjectures (Lefschetz + Hodge type) give RH

(S4); Tate’s conjecture for the elliptic surface *is* BSD (S8); Theorem A makes the BSD lattice and the Rosati lattice literally equal on $E \times E \rightarrow E$; Theorems B and C tie every TAA counter to the NS signature and rank. Over $\mathbb{Q}$ the three problems are open on *different halves*. So a proof of the Hodge conjecture (algebraicity over $\mathbb{C}$) would not, by any mechanism known here, supply positivity for ζ , and Néron–Tate positivity (known) has not supplied it either. **The common root is real over $\mathbb{F}_q$ and an analogy over $\mathbb{Q}$ ([ANALOGY]).**

What is missing for ζ , precisely.

(M1) *No surface*: $\text{Spec } \mathbb{Z} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z} = \text{Spec } \mathbb{Z}$ (L65 [THEOREM]).

(M2) *No finite cohomology*: no finite $\mathbb{Q}[F]$ (L84 [THEOREM]). A substitute H^1 is infinite-dimensional; its “Hodge type” is a regularized positivity whose natural candidate is Weil’s form ([RESTATEMENT], L1).

(M3) *No algebraic Frobenius / Frobenius section* (L85 O3; Remark 3.2).

(M4) *No ample class*: the pole plane plays $\langle H, V \rangle$, and index ≤ 1 on its complement is [RESTATEMENT] (L15, L64; N5).

Candidate substitutes, citing only what was opened: Moriwaki’s arithmetic Hodge index (alg-geom/9403011, [CITED]) gives Néron–Tate positivity on “curves over $\text{Spec } \mathbb{Z}$ ”, not a ζ -surface. Deninger (ICM 1998, §2, [CITED]) makes RH conditional on a positive Hodge *, a [RESTATEMENT]-shaped hypothesis. For Connes–Consani (scaling site, Riemann–Roch for $\overline{\text{Spec } \mathbb{Z}}$, archimedean Weil positivity) only abstracts and titles were seen; nothing is claimed.

7. RANKED DIRECTIONS, OPEN PROBLEMS, QUESTIONS

Score = 0.6 sig + 0.4 prob, judgement calls.

	Direction	sig	prob	score	First step
D1	Formulate the “Frobenius point over $\mathbb{Q}$ ” (Rem. 3.2) without presupposing RH, or prove that every formulation is [RESTATEMENT]	.70	.50	.620	List candidate groups (Arakelov Chow, Connes–Consani cokernel); test each against L84 (no finite $\mathbb{Q}[F]$)
D2	Full NS($C \times C$) for split Jacobians: Hodge index on the part invisible to Frobenius (N2’s $r_F = 4 < \rho = 10$), with an explicit $\text{End}(J)$ basis	.45	.85	.610	Genus-2 curves with $J \sim E^2$ from N2; compute $M_2(\text{End } E)$ Rosati Gram exactly
D3	Read Connes–Consani, “Weil positivity and trace formula: the archimedean place” (Selecta 2021) at the source	.80	.30	.600	Obtain the text; map its statement onto M4
D4	Source Prop. 3.3 (genus- g Theorem A)	.45	.70	.550	Lang–Néron / Shioda on constant abelian schemes
D5	Artin–Tate numerics on the L85 surface (Brauer group, regulator) as a Gordon Thm. 6.1 check	.40	.70	.520	Exact NS discriminant vs P_2 special value
D6	Arakelov ζ -surface substitutes	.80	.10	.520	None actionable in one lemma

Top three: D1 (.620), D2 (.610), D3 (.600).

Open problems. (O1) Prop. 3.3 at a source. (O2) Bombieri’s 1964–65 text (S1). (O3) A non-[RESTATEMENT] formulation of M3. (O4) Is the side-dependence of the perturbation multiplicity (Cor. 4.3) intrinsic, i.e. can a canonical half-count be assigned to the triple saddle?

Questions for J.W.

Q1. Adopt the algebraicity/positivity split (§6) as the organizing frame of the Hodge/BSD/RH thread?

- Q2. Lemma 89: D1 (Frobenius point, conceptual), D2 (split Jacobians, computational), or D3 (Connes–Consani at the source)?
- Q3. Queue Theorem B with Prop. 4.2/Cor. 4.3 as a Volume 3 addendum in `pending_replacements.md` (drafted, not applied, not submitted)?
- Q4. Do you accept that “Hodge $\Rightarrow$ RH” has no known mechanism, since Hodge is open on the algebraicity half and ζ on the positivity half? If so, should the thread pursue positivity substitutes (M4) rather than algebraicity?
- Q5. Your chat claim “saddle cones sum to (n, n) ” is recorded below as true but empty. Is the four-way identity (Thm. 4.1) the version of that idea you had in mind?

8. CORRECTIONS

- (K1) *J.W.’s chat claim “the saddle cones are $(1, 1)$ locally and sum to (n, n) ”: **true but empty.** Every non-degenerate saddle of a harmonic function has Hessian signature $(1, 1)$, longitudinal or transverse alike (Lemma 87). The direct sum of n of them is (n, n) on a space carrying no arithmetic, and no global form glues them. The ghost information lies in each cone’s *orientation* relative to ∂_σ . Restricting to the σ -lines gives $n_- = \tau$, which is [RESTATEMENT] via Kim’s identity (Lemma 56). The rigorous version of the intuition is Theorem 4.1, which needs the global Hodge index of $\mathbb{F}_q$.*
- (K2) *“Rosati $\kappa = \text{ghost pairs}$ ” needs the quartet convention.* The Rosati negative index counts *reciprocal* off-circle pairs $\{\alpha, q/\alpha\}$: a real pair counts 1, a complex quartet $\{\alpha, \bar{\alpha}, q/\alpha, q/\bar{\alpha}\}$ counts 2. This equals $k = \#\{|\alpha| > \sqrt{q}\}$ (Theorem 4.1; N4: $k \in \{0, 1, 2, 3\}$, 1000/1000).
- (K3) *Lemma 56 circle identity: simple critical points must be assumed.* The proof counts maxima and minima of $|h|$ per gap and tacitly assumes non-degenerate critical points. Fix: add “ P squarefree and S with simple roots on $|u| = 1$ ”, or count degenerate saddles by perturbation multiplicity (Cor. 4.3). In genus 2 degeneracy happens exactly at $|c_1| = 4\sqrt{q}$ (Prop. 4.2), the boundary of Lemma 56’s genus-2 criterion. Lemma 56’s random tests (real random roots) could not see it; integer polynomials with square q do (10/1000 here, 2/600 in the review). Non-squarefree P with a double root on the circle is a separate exclusion (one real genus-3 case in N4).
- (K4) *Review 88* said “ $\rho = 2 + \text{rank End}(J)$, formula to be checked at a source”. Now $\rho = 2 + \sum m(\lambda)^2$ is sourced (Thm. 4.4: Tate Thm. 5.2 + Schütt–Shioda §14.6, both [CITED]). The $\text{End}(J)$ form is [CITED-sec] (Tate 1966b via Milne), and $\text{NS}(C \times C) \cong \mathbb{Z}^2 \oplus \text{End}(J)$ is [HEURISTIC].
- (K5) *Review 88 / Milne date the Bombieri–Grothendieck work “about 1964”;* Grothendieck’s own page says “about three years ago” in 1968 (S1). Recorded, not resolved.
- (K6) *Review 88 S8* gave Tate $\iff$ BSD as [BIB]. It is now [CITED]: Tate’s conjecture (d) and Gordon’s Thm. 6.1 were read, with the hypotheses stated in S8.
- (K7) *Review 88 C2 table entry* “Gram on $\{1, \pi\} = [[2, a], [a, 2q]]$ ” is unchanged and is now part of Theorem 3.1(c).

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Also used through earlier TAA records: Faltings 1984, Hriljac 1985, Connes–Consani (abstracts), Artin–Swinnerton-Dyer 1973 (*lit/lemma84*, *lit/lemma85* verified-reference files).

CODE AND DATA

`num/lemma88_lib.py` (exact tools); `lemma88_n1.py–lemma88_n5.py` → `num/188/n1.json–n5.json`; `lemma88_figs.py` → `build88/figs/`. Earlier checks reused: `num/review88/`. Sources: `lit/lemma88/` (tate306, gordon79, kleiman68, groth69, kleiman94). No earlier lemma file, build or never-touch file was modified. Nothing was posted or submitted.

TAA LEMMA 89: DIVISORS FIRST. CORRESPONDENCE RINGS, IRREDUCIBLE FACTORS, AND THE MISSING FROBENIUS

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. J.W.’s hunch (t380) is that the $C \times C$ approach, Weil’s proof, $\mathbb{F}_1$ and $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ all lean too hard on direct-sum and product structure. He proposes factoring the structure of $C \times C$ into irreducible pieces (“least common divisors”) and letting the surface emerge from them. This lemma makes that precise for curves over $\mathbb{F}_q$ and tests what survives over $\mathbb{Q}$. **(A)** Weil’s proof is divisor-first. Its content is the axes H, V , the diagonal and the Frobenius graphs, and point counts are intersection numbers. $\text{NS}(C \times C) \cong \mathbb{Z}H \oplus \mathbb{Z}V \oplus \text{End}(J)$, with intersection form $-\text{Tr}(\alpha\beta^\dagger)$ on the third summand. **(B)** The correspondence ring $\text{End}^0(J)$ is semisimple. Its Wedderburn factors are the isotypic pieces of J , each cut out by an explicit $\mathbb{Q}$ -divisor built from Frobenius graphs, with Albert type read off from Honda–Tate. **(C)** κ localizes: the Rosati trace form splits orthogonally over $\dagger$ -orbits of $\mathbb{Q}$ -irreducible factors, with an exact negative count per orbit. **(D)** Over $\mathbb{Q}$ there is no Frobenius, no finite $\mathbb{Q}[F]$ and no $\mathbb{Q}$ -structure on the zeros. The reformulation “infinitely many simple factors, one per zero” is inaccurate as stated, and once corrected it is a restatement. **(E)** What divisors-first buys and what it cannot buy: the ample class comes from the product part. Numerics N1–N3 are exact except N3, which uses `mpmath` (30 digits) and is labelled. The corrections section contains the prior-art note for Lemma 88 Theorem A (J.W. t383) and items C89.1–C89.8.

Tags as in Lemmas 80–88: [THEOREM] proved here or standard with proof; [CITED] source opened and checked; [CITED-sec] through an opened secondary source; [BIB] bibliographic only; [TAA-L] earlier TAA lemma; [RESTATEMENT] equivalent to RH; [ANALOGY] analogy; [OPEN] open; [NUMERICAL] computation done here; [HEURISTIC] heuristic; [CONJECTURE] conjecture. References: `lit/lemma89/verified_refs.md`. Scripts: `num/lemma89_{lib,n1,n2,n3,n3_eval,figs}.py`, writing to `num/189/n{1,2,3}.json`.

Task (J.W. t380 hunch, t381, approval t382u; scope as recorded in STATE/memory, since the verbatim t380/t381 text was not available to the formalizer, see Q1): “Divisors First”: the correspondence ring $\text{NS} = \mathbb{Z}^2 \oplus \text{End}(J)$, Wedderburn/Albert factors, whether κ localizes per factor, and why $\mathbb{Q}$ has no Frobenius.

The Riemann hypothesis is not proved, here or anywhere in TAA. Over $\mathbb{F}_q$, every positivity used below is a known theorem (Hodge index, Rosati positivity, Weil). For ζ , every positivity statement is [RESTATEMENT]. The per-zero block decomposition in §5 is formal: it holds whether or not RH is true and proves nothing about where the zeros lie.

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Date: October 24, 2026.

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1. SETTING AND SOURCES

$C/\mathbb{F}_q$ is a smooth projective geometrically connected curve of genus g with Jacobian J and q -Frobenius $\pi = F$. $P(T) = \prod(T - \alpha_i)$ is the characteristic polynomial of F on H^1 , of degree $2g$. $V = qF^{-1}$ is Verschiebung, and $\dagger$ is the Rosati involution, which swaps F and V . NS is taken over $\bar{k}$ unless stated otherwise. The main new source is Milne’s *Abelian Varieties* notes, v2.0 (2008), opened at `lit/lemma89/AV.pdf`; printed page numbers are cited. The items used are: I 14.2 (NS(A) $\otimes \mathbb{Q}$ = Rosati-symmetric elements, p. 61), I 14.3 (Rosati positivity, p. 62), I 15.3 (direct factors $\leftrightarrow$ idempotents, p. 64), II Lemma 1.3 (pp. 77–78), II 2.1 (Honda–Tate bijection, p. 79), II 2.9 (structure of End^0 of a simple abelian variety over $\mathbb{F}_q$, p. 81), III Cor. 6.3 ($\text{Hom}(J_1, J_2) \leftrightarrow$ divisorial correspondences, p. 104), III Prop. 11.2 and Thm. 11.1 (Lefschetz formula and RH via $C \times C$, pp. 117–118), and the semisimplicity/Wedderburn paragraph on p. 137. All are [CITED].

2. A. WEIL’S PROOF IS DIVISOR-FIRST

Theorem 2.1 (the divisor lattice of $C \times C$). *Let $H = \{pt\} \times C$ and $V_0 = C \times \{pt\}$, so $H^2 = V_0^2 = 0$ and $H \cdot V_0 = 1$. For a correspondence $\alpha \in \text{End}(J)$ write D_α for the associated divisor class orthogonal to H and V_0 .*

- (a) $\text{NS}(C \times C) \cong \mathbb{Z}H \oplus \mathbb{Z}V_0 \oplus \text{End}(J)$ over $\bar{k}$, with $D_\alpha \cdot D_\beta = -\text{Tr}(\alpha\beta^\dagger \mid H^1)$ on the third summand.
- (b) For the Frobenius graphs $\Gamma_k = \Gamma_{F^k}$ ($\Gamma_0 = \Delta$): $\Gamma_k \cdot H = q^k$, $\Gamma_k \cdot V_0 = 1$, and $\Gamma_m \cdot \Gamma_n = q^{\min(m,n)} N_{|m-n|}$ with $N_j = \#C(\mathbb{F}_{q^j})$, where $N_0 := 2 - 2g$, so that $\Delta^2 = 2 - 2g$. The projections $D_k = \Gamma_k - H - q^k V_0$ satisfy $D_m \cdot D_n = -q^{\min(m,n)} s_{|m-n|} = -\text{Tr}(F^m V^n)$ with $s_j = \sum \alpha_i^j$.
- (c) Point counts are intersection numbers: $N_k = \Gamma_k \cdot \Delta = 1 + q^k - \text{Tr}(F^k)$.
- (d) Hodge index [2, Thm. 1.2] on $\langle H, V_0 \rangle^\perp$ gives $\text{Tr}(\alpha\alpha^\dagger) \geq 0$. Applied to $\alpha = m + nF$ this gives $|N_1 - q - 1| \leq 2g\sqrt{q}$, and applied to $\mathbb{Q}[F]$ it gives RH.

Proof. (a) Milne [1, III Cor. 6.3] [CITED]: for curves with rational points P_1, P_2 , $\text{Hom}(J_1, J_2)$ is in bijection with isomorphism classes of divisorial correspondences normalized at P_1, P_2 . A divisorial correspondence is trivial when it is of the form $p_1^* D_1 + p_2^* D_2$ [1, I §6, printed p. 27]. Since $\text{NS}(C) = \mathbb{Z}$, passing from Pic to NS gives (a) for $C_1 = C_2 = C$ over k . This passage is our deduction ([THEOREM]). The intersection formula is the L84 computation ([TAA-L]) and agrees with (b). (b) $\Gamma_k \cdot V_0$ counts the points of Γ_k over one point of the second factor (q^k or 1, depending on which factor carries pt ; the convention fixed here matches the L46 Gram, [TAA-L]). $\Gamma_m \cap \Gamma_n$ consists of the points x with $F^m x = F^n x$, counted with multiplicity $q^{\min}$. Expanding $D_m \cdot D_n$ and using $N_j = q^j + 1 - s_j$ gives $q^m(q^{n-m} + 1 - s_{n-m}) - q^n - q^m = -q^m s_{n-m}$ for $m \leq n$. This equals $-\text{Tr}(F^n V^m)$, since $F^n V^m = q^m F^{n-m}$. (c) is Milne [1, III Prop. 11.2] [CITED] with $\alpha = F^k$: $(\Gamma_\alpha \cdot \Delta) = 1 - \text{Tr}(\alpha') + \deg \alpha$. (d) is Weil’s argument, as in [1, III Thm. 11.1] and [2, §1] [CITED]. The genus-1 Gram form is the prior-art paragraph (1) in §8. $\square$

What “divisor-first” means precisely. The finite list $H, V_0, \Gamma_0, \dots, \Gamma_{2g-1}$ already spans $\mathbb{Q}H \oplus \mathbb{Q}V_0 \oplus \mathbb{Q}[F]$ inside $\text{NS} \otimes \mathbb{Q}$. Its Gram matrix is a function of q and $N_1, \dots, N_{2g}$ alone, and the signature of that matrix is RH (L46, L84, [TAA-L]). Nothing about the surface beyond these divisors enters

Weil's proof. This is the rigorous form of the first half of J.W.'s hunch. The second half, letting the surface *emerge* from the pieces, is tested in §6.

Remark 2.2 (L88 heuristic upgraded). Lemma 88 (Thm. C) labelled $\text{NS}(C \times C) \cong \mathbb{Z}^2 \oplus \text{End}(J)$ as [HEURISTIC]. Theorem 2.1(a) gives it at [CITED] level, through Milne III 6.3 plus the routine $\text{Pic} \rightarrow \text{NS}$ step (see C89.5). Note the contrast with $\text{NS}(J) \otimes \mathbb{Q}$, which is only the *Rosati-symmetric* part of $\text{End}^0(J)$ [1, I Prop. 14.2] [CITED]. The surface $C \times C$ sees the whole correspondence ring, and the Jacobian sees half of it.

3. B. THE CORRESPONDENCE RING: WEDDERBURN FACTORS, IDEMPOTENT DIVISORS, ALBERT TYPES

Theorem 3.1 (irreducible factors are cut out by $\mathbb{Q}$ -divisors). *Let $P = \prod_i f_i^{m_i}$ be the $\mathbb{Q}$ -factorization (the f_i distinct, monic, irreducible). Let $\mu = \prod_i f_i$, the minimal polynomial of F on H^1 ; F acts semisimply, [1, II §2, p. 81, after 2.9] [CITED].*

- (a) $\mathbb{Q}[F] \cong \prod_i \mathbb{Q}[T]/(f_i)$. The CRT idempotents $e_i = \sum_{k < \deg \mu} c_{ik} F^k$ ($c_{ik} \in \mathbb{Q}$) satisfy $e_i^\dagger = e_i$. The $\mathbb{Q}$ -divisors $D_{e_i} = \sum_k c_{ik} D_k$ are pairwise orthogonal, and $D_{e_i}^2 = -\text{Tr}(e_i) < 0$.
- (b) $\text{Tr}(e_i | H^1) = m_i \deg f_i$, and this trace is determined by the intersection numbers $\Gamma_k \cdot \Delta = N_k$: $\text{Tr}(e_i) = \sum_k c_{ik}(1 + q^k - N_k)$.
- (c) (Over $\mathbb{F}_q$.) e_i is the projector onto the f_i -isotypic part $J_i \sim B_i^{n_i}$ of J , with B_i simple. By Honda–Tate [1, II 2.1, 2.9] [CITED], $\text{End}_{\mathbb{F}_q}^0(J_i) \cong M_{n_i}(D_i)$, where D_i is a central division algebra over $\mathbb{Q}(\pi_i) = \mathbb{Q}[T]/(f_i)$ with $[D_i : \mathbb{Q}(\pi_i)] = e_i^2$, $2 \dim B_i = e_i \deg f_i$ and $m_i = n_i e_i$. Hence $\text{End}_{\mathbb{F}_q}^0(J) = \prod_i M_{n_i}(D_i)$ is the Wedderburn decomposition [1, p. 137] [CITED], and $\text{rank End}_{\mathbb{F}_q}(J) = \sum_i m_i^2 \deg f_i$.
- (d) (Albert type per factor.) If f_i has no real root, $\mathbb{Q}(\pi_i)$ is a CM field and the factor is of type IV. In the commutative case $e_i = 1$, $D_i = \mathbb{Q}(\pi_i)$. If f_i has a real root ($\pi_i = \pm\sqrt{q}$), then $\text{inv}_v(D_i) = \frac{1}{2}$ at the real places [1, II 2.9(b)], so D_i is a totally definite quaternion algebra: type III. The table of Albert types is [CITED-sec] (Mumford §21 via L84).

Proof. (a) CRT gives $e_i \in \mathbb{Q}[T]/(\mu)$. Since $\dagger$ maps $\mathbb{Q}[F]$ to itself and permutes the primitive idempotents, and every f_i is self-reciprocal for a genuine curve (RH, Weil), $\dagger$ fixes each e_i . The first statement of Theorem 4.1 below covers the general case. Orthogonality follows because $D_{e_i} \cdot D_{e_j} = -\text{Tr}(e_i e_j^\dagger) = -\text{Tr}(e_i e_j) = 0$ for $i \neq j$. Then $D_{e_i}^2 = -\text{Tr}(e_i^2) = -\text{Tr}(e_i)$. (b) e_i acts as 1 on the $m_i \deg f_i$ -dimensional generalized eigenspace and as 0 elsewhere. The second formula is Theorem 2.1(c). (c) Idempotents of $\text{End}^0(J)$ correspond to direct factors up to isogeny [1, I 15.3]. $\mathbb{Q}[\pi]$ is central in $\text{End}^0(J)$, and for simple B it is the centre of $\text{End}^0(B)$ [1, II 2.9(a)]. Tate's theorem, as stated in [1, II §2, after 2.9] [CITED], gives $\text{Hom}(B_i, B_j) = 0$ for non-conjugate Weil numbers, hence the isotypic decomposition. Finally $\dim M_{n_i}(D_i) = n_i^2 e_i^2 \deg f_i = m_i^2 \deg f_i$. (d) is read off from [1, II 2.9(b)]. $\square$

Remark 3.2 (the “least common divisors”, concretely). For the genus-2 curve $y^2 = x^5 + x$ over $\mathbb{F}_3$ (N2), $N_1 = 4$, $N_2 = 14$, $P = (T^2 - 2T + 3)(T^2 + 2T + 3)$, so $J \sim E_1 \times E_2$. The two idempotents are

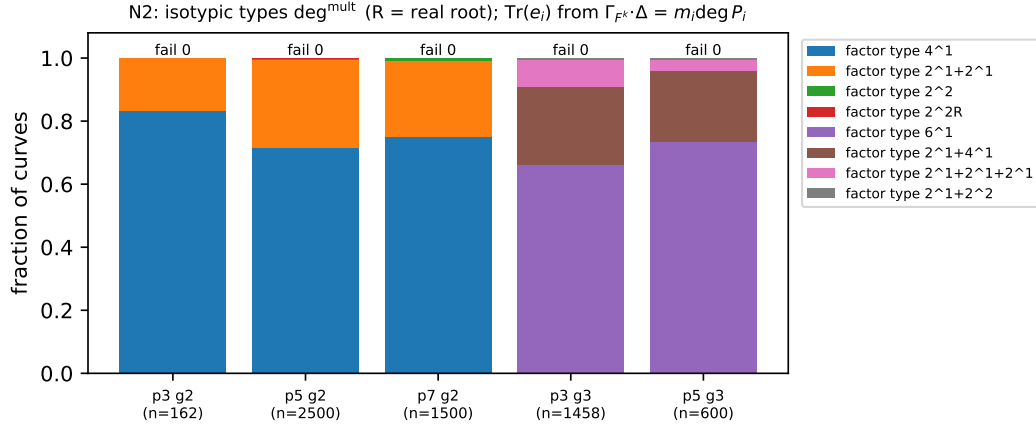
$$e_{\pm} = \frac{1}{2} \pm \frac{1}{12} F \mp \frac{1}{12} F^3, \quad D_{e_{\pm}} = \frac{1}{2} D_0 \pm \frac{1}{12} D_1 \mp \frac{1}{12} D_3.$$

That is, explicit $\mathbb{Q}$ -combinations of the diagonal and the Frobenius graphs, each with $D_{e_{\pm}}^2 = -2$ and $D_{e_+} \cdot D_{e_-} = 0$ ([NUMERICAL], exact). For $y^2 = x^5 + 4x$ over $\mathbb{F}_5$, $P = (T^2 - 5)^2$. There J is simple (since $e = 2$) and $\text{End}_{\mathbb{F}_5}^0(J)$ is a totally definite quaternion algebra over $\mathbb{Q}(\sqrt{5})$ of rank 8 (type III), so $\rho(C \times C/\mathbb{F}_5) = 10$ while $\mathbb{Q}[F]$ has dimension only 2. Here the divisors Γ_k do *not* generate the correspondence ring: the non-commutative part is invisible to Frobenius (L88 direction D2).

N2. Genuine curves: idempotent divisors and per-factor positivity (exact). We take all hyperelliptic curves $y^2 = f(x)$ with f monic and squarefree of degree $2g + 1$ over $\mathbb{F}_p$: all of them for $(p, g) = (3, 2), (5, 2), (3, 3)$, and seeded random samples of 1500 for $(7, 2)$ and 600 for $(5, 3)$. These are counted as equations, not up to isomorphism. For each curve we computed $N_1, \dots, N_g$ by brute force over $\mathbb{F}_{p^k}$, then P , the $\mathbb{Q}$ -factorization, the CRT idempotents, $\text{Tr}(e_i)$ from the intersection numbers $N_k = \Gamma_k \cdot \Delta$ alone (Theorem 3.1(b)), the local Rosati signature, and $\kappa(f_i)$. We also checked $\Delta^2 = N_0 = 2 - 2g$.

(p, g)	curves	isotypic types $\deg f_i^{m_i}$ (R = real root)
(3, 2)	162	$4^1:135, 2^1+2^1:27$
(5, 2)	2500	$4^1:1794, 2^1+2^1:690, 2^2:15, 2^2\text{R}:1$ ($y^2 = x^5 + 4x$, type III)
(7, 2)	1500	$4^1:1127, 2^1+2^1:362, 2^2:11$
(3, 3)	1458	$6^1:966, 2^1+4^1:360, 2^1+2^1+2^1:126, 2^1+2^2:6$
(5, 3)	600	$6^1:440, 2^1+4^1:136, 2^1+2^1+2^1:21, 2^1+2^2:3$

In every case (6220 curves), $\text{Tr}(e_i)$ computed from the intersection numbers equals $m_i \deg f_i$, every local Rosati form is positive definite, and $\kappa(f_i) = 0$, with **0 failures**. This is Theorem 4.1(e) checked factor by factor ([NUMERICAL], n2.json).



4. C. κ LOCALIZES PER $\dagger$ -ORBIT OF IRREDUCIBLE FACTORS

Let $P \in \mathbb{Z}[T]$ be any q -Weil-shaped polynomial of degree $2g$: monic, $P(0) = q^g$, and with coefficients satisfying $c_{2g-j} = q^{g-j}c_j$, so that the roots are closed under $\alpha \mapsto q/\alpha$. It may be a genuine L -polynomial or a fake. Assume P squarefree. Put $A = \mathbb{Q}[T]/(P)$ and let $\dagger$ be the ring involution $T \mapsto qT^{-1}$. L84 ([TAA-L]) proved that $\langle x, y \rangle = \text{Tr}_{A/\mathbb{Q}}(xy^\dagger)$ is positive definite iff all $|\alpha| = \sqrt{q}$, and that its negative index is $\kappa(P)$, the number of reciprocal pairs $\{\alpha, q/\alpha\}$ with $|\alpha| \neq \sqrt{q}$ (a complex quartet counts 2).

Theorem 4.1 (localization). *Write $P = \prod_i f_i$ over $\mathbb{Q}$ and $f^*(T) = T^{\deg f} f(q/T)/f(0)$. Then:*

- $f \mapsto f^*$ permutes the factors. Its orbits are self ($f = f^*$) or pairs $\{f, f^*\}$ with $f \neq f^*$.
- $\langle, \rangle$ is the orthogonal sum of its restrictions to the orbit blocks $A_O = \prod_{f \in O} \mathbb{Q}[T]/(f)$.
- On a pair block, $\langle, \rangle$ is hyperbolic of signature $(\deg f, \deg f)$. On a self block it has signature $(\deg f - \kappa(f), \kappa(f))$.
- Hence $\kappa(P) = \sum_{\text{self}} \kappa(f_i) + \sum_{\text{pairs}} \deg f_j$, and RH for P holds iff there are no pairs and $\kappa(f_i) = 0$ for every self factor.
- For a genuine curve every factor is self-reciprocal with $\kappa(f_i) = 0$. So Weil's positivity holds factor by factor: the Rosati form restricted to each isotypic piece J_i is positive.

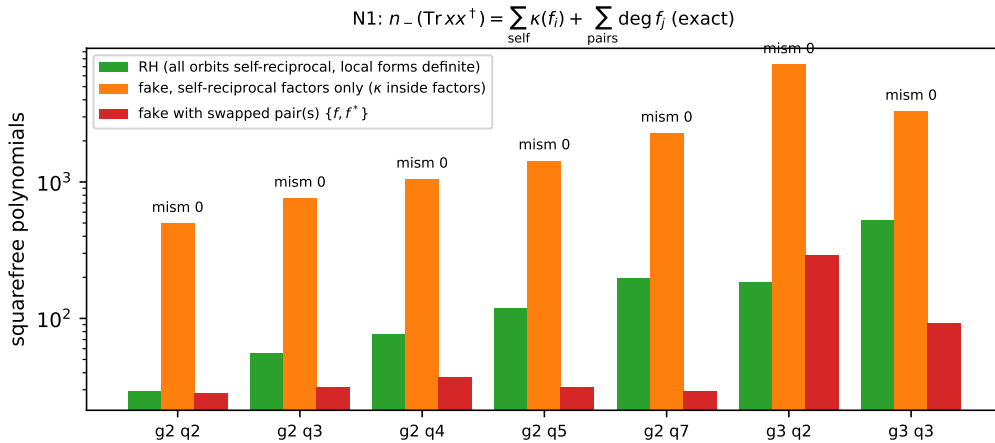
Proof. (a) The roots of f^* are the q/α for α a root of f ; it is irreducible, it divides P by the shape condition, and $f^{**} = f$. (b) $\dagger$ maps the ideal A_f to A_{f^*} . For distinct orbits $O \neq O'$, $A_O \cdot A_{O'}^\dagger = A_O \cdot A_{O'} = 0$. (c) Pair: $A_O = K \times K^*$ with $\dagger(a, b) = (\tau^{-1}b, \tau a)$ for a field isomorphism $\tau : K \rightarrow K^*$. Then $\langle (a, b), (a, b) \rangle = 2 \operatorname{Tr}_{K/\mathbb{Q}}(a \tau^{-1}b)$, which is the hyperbolic form on $K \oplus K$. Self: $\operatorname{Tr}_{K/\mathbb{Q}}(x x^\dagger) = \sum_i \iota(x) \iota(x^\dagger)$. Group the embeddings into pairs $\{\alpha, q/\alpha\}$ and quartets. A root on the circle has $q/\alpha = \bar{\alpha}$ and contributes $|\iota x|^2$. An off-circle reciprocal pair contributes a hyperbolic plane (L84's count, now done inside K). (d) Sum (c) over (b). (e) Weil (RH for curves) puts every root on the circle, so $\bar{\alpha} = q/\alpha$ and complex conjugation preserves each factor. The positivity statement is Theorem 3.1(a) together with Rosati positivity [1, I 14.3]. $\square$

Prior art (honest). Milne [1, II Lemma 1.3] [CITED] already contains the localization mechanism. If $\dagger$ is positive on $\mathbb{Q}[\alpha]$, then $\dagger$ “preserves the factors of $\mathbb{Q}[\alpha]$, and is a positive-definite involution on each of them”. Otherwise $\operatorname{Tr}(a^\dagger a)$ fails to be positive on $(a_1, 0, 0, \dots)$. Theorem 4.1 is the quantitative converse for fakes, an exact negative count per orbit, and that count is the only new item here ([THEOREM]). Verdict: **yes, κ localizes**, in the precise form “per $\dagger$ -orbit of $\mathbb{Q}$ -irreducible factors”. A swapped pair $\{f, f^*\}$ contributes $\deg f$ negative directions by itself, so “per irreducible factor” without the orbit qualifier is false for fakes.

N1. Exhaustive exact check of Theorem 4.1. All q -Weil-shaped squarefree polynomials in a box. Genus 2: $q \in \{2, 3, 4, 5, 7\}$, $|c_1| \leq \lceil 4\sqrt{q} \rceil + 2$, $|c_2| \leq 6q + 4$. Genus 3: $q = 2$ with $(|c_1|, |c_2|, |c_3|) \leq (4, 10, 20)$, and $q = 3$ with $\leq (3, 8, 16)$. For each polynomial we computed the global signature of $\langle, \rangle$, the local signatures per orbit, the predicted κ from (d), and κ from the roots (mpmath, 50 digits). Signatures are exact, by Descartes' rule on the real-rooted characteristic polynomial of the integer Gram matrix.

case	squarefree	RH	fakes w. pair	mism.
$g = 2, q = 2, 3, 4, 5, 7$	553, 845, 1161, 1577, 2499	29, 55, 77, 119, 195	28, 31, 37, 31, 29	0
$g = 3, q = 2$	7703	183	289	0
$g = 3, q = 3$	3906	523	92	0
total	18244	1181	537	0

No RH polynomial has a pair (0 of 1181). Examples ($q = 3$): $T^4 - 4T^3 - 22T^2 - 12T + 9 = (f)(f^*)$ is a single pair with signature (2, 2); $T^4 - 7T^3 + 18T^2 - 21T + 9 = (T^2 - 4T + 3)(T^2 - 3T + 3)$ is a pair $\{T - 1, T - 3\}$ (1, 1) plus a positive self factor (2, 0), total (3, 1); $(T^2 + T + 3)(T^2 - 5T + 3)$ has two self factors (2, 0) $\oplus$ (1, 1) ([NUMERICAL], n1.json).



5. D. WHY $\mathbb{Q}$ HAS NO FROBENIUS, AND “ONE SIMPLE FACTOR PER ZERO”

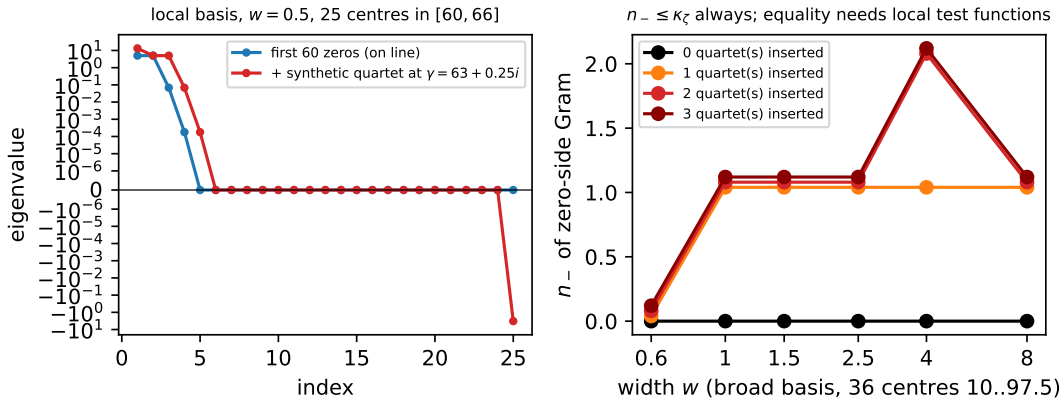
D1. What the Frobenius is made of. Over $\mathbb{F}_q$, F exists because $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ is topologically generated by $x \mapsto x^q$. It acts on a *finite* H^1 of dimension $2g$, its characteristic polynomial has *rational* coefficients, $Z(C, t) = P(t)/((1-t)(1-qt))$ [1, III Cor. 11.4] [CITED], and it is a *correspondence*, the divisor Γ_1 (Theorem 2.1). Three things fail over $\mathbb{Q}$: (i) $\text{Spec } \mathbb{Z}$ has no constant field and $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is not procyclic, so there is no canonical generator [ANALOGY]. (ii) Deninger’s expected $H^1(\text{Spec } \mathbb{Z})$ is *infinite-dimensional*, with the spectrum of an infinitesimal generator Θ (a flow, not a Frobenius) equal to the non-trivial zeros. The regularized-product formula $\prod_\rho \frac{s-\rho}{2\pi}$ for $\hat{\zeta}$ holds as a theorem (Deninger ICM 1998, §3, p. 168, quoting [D2], [SchS]; opened, [CITED] for the quoted statements). A Hodge $*$ commuting with the flow would give $\Theta = \frac{1}{2} + A$ with A skew, hence RH. That step is conjectural ([CONJECTURE], same source). (iii) There is no finite-dimensional $\mathbb{Q}[F]$ (L84, [THEOREM]/[TAA-L]), and Connes’s scaling action realizes only the critical zeros spectrally, with non-critical ones appearing as resonances (Connes 1999, Thm. 1, read in L79, [TAA-L]).

D2. The zero side of Weil’s form is block-diagonal per zero orbit (formal).

Proposition 5.1. *Take even test functions φ whose transforms $\hat{\varphi}$ are real on $\mathbb{R}$, and write $\rho = \frac{1}{2} + i\gamma$. The zero side $Q_Z(\varphi) = \sum_\rho \hat{\varphi}(\gamma_\rho)^2$ of Weil’s explicit formula splits over orbits. An on-line orbit $\{\pm\gamma\}$, $\gamma \in \mathbb{R}$, contributes $2vv^\top$ with $v_j = \hat{\varphi}_j(\gamma)$, which is positive semidefinite of rank ≤ 1 . An off-line quartet $\{\pm\gamma, \pm\bar{\gamma}\}$ contributes $2(vv^\top + \bar{v}\bar{v}^\top) = 4\text{Re}(vv^\top)$, of signature $(1, 1)$ when $\text{Re } v$ and $\text{Im } v$ are independent. Consequently, on any finite span of test functions, $n_-(Q_Z) \leq \kappa_\zeta$, the number of off-line quartets. $\text{RH} \iff \kappa_\zeta = 0 \iff \text{Weil positivity (Weil 1952, Bombieri 2000; L1/L18, [TAA-L], [RESTATEMENT])}$.*

Proof. $4\text{Re}(vv^\top) = 4(uu^\top - ww^\top)$ with $v = u + iw$. The bound $n_-(A+B) \leq n_-(A) + n_-(B)$ (Weyl) is applied block by block. [THEOREM]. $\square$

N3 ([NUMERICAL], mpmath 30 digits; n3.json). We use the first 60 zeros of ζ (mpmath; all on the line numerically) and *insert synthetic* off-line quartets at $\gamma = 40 + 0.35i$, $63 + 0.25i$, $87 + 0.40i$ by hand. This makes no claim about ζ . Every inserted block has inertia exactly $(1, 1)$. With a broad basis (36 Gaussian bumps, centres $10, \dots, 97.5$), the total n_- ranges from 0 to 2 when 3 quartets are inserted, depending on the width w . With a dense basis local to each ghost (25 centres in $[\gamma_0 - 3, \gamma_0 + 3]$, $w \in \{0.3, 0.5, 1\}$), $n_- = 1$ per ghost and $n_- = 3$ with all three, in 9/9 and 3/3 runs, with relative minimal eigenvalue between -0.05 and -0.49 . Without ghosts, $n_- = 0$ in all 15 runs (6 broad, 9 local). In the broad runs the detected negative eigenvalues are tiny, down to 10^{-16} relative at $w = 8$ against a threshold of 10^{-18} , so broad-basis detection is numerically fragile. So the negative directions live *at the ordinate of the off-line zero*, and a finite cutoff sees $n_- \leq \kappa_\zeta$, with equality only if it resolves that ordinate. Over $\mathbb{F}_q$, by contrast, $\mathbb{Q}[F]$ is finite and $n_- = \kappa$ exactly (N1).



D3. Verdict on “infinitely many simple factors, one per zero”.

- (i) *Inaccurate as stated.* Over $\mathbb{F}_q$, the “simple factors” are the $\mathbb{Q}$ -irreducible factors f_i , that is, Galois orbits of Frobenius eigenvalues. An orbit always has at least 2 elements, apart from $\pm\sqrt{q}$ when q is a square, so there is never one factor per eigenvalue. Their existence rests on a $\mathbb{Q}$ -structure, namely rational coefficients and rational correspondences e_i (Theorem 3.1). For ζ no $\mathbb{Q}$ -structure on the zeros is known: there is no known $\mathbb{Q}$ -algebra containing Θ , and no algebraicity of the ordinates is known. So “simple factor” has no defined meaning there. If such a structure existed, its factors would be orbits of zeros, not single zeros ([HEURISTIC]).
- (ii) *After correction it is a restatement.* The defensible version is “a real decomposition of the zero side into orbits $\{\pm\gamma\}$, positive on on-line orbits and $(1, 1)$ on off-line quartets” (Prop. 5.1). That is the regularized product together with the explicit formula. It carries no arithmetic, and positivity block by block is RH ([RESTATEMENT]).
- (iii) *What would be new* is the $\mathbb{F}_q$ feature with no ζ counterpart: block projectors that are *rational correspondences*, as with $e_{\pm} = \frac{1}{2} \pm \frac{1}{12}F \mp \frac{1}{12}F^3$ in Remark 3.2. Producing them would amount to constructing the missing H^1 with its $\mathbb{Q}$ -structure ([OPEN]).

6. E. WHAT DIVISORS-FIRST BUYS, AND WHAT IT CANNOT

Theorem 6.1 (positivity sits in the product part). *In $\text{NS}(C \times C) \otimes \mathbb{R}$, the summand $\text{End}^0(J) \otimes \mathbb{R}$ (all correspondences, including every idempotent D_{e_i}) is negative definite. The summand $\langle H, V_0 \rangle$ has signature $(1, 1)$, and the total signature is $(1, \rho - 1)$. The unique positive direction, the ample class $H + V_0$, therefore comes from the product structure. The irreducible factors carry only negative-definite blocks, and κ would live there.*

Proof. $D_{\alpha}^2 = -\text{Tr}(\alpha\alpha^{\dagger}) < 0$ for $\alpha \neq 0$ by Theorem 2.1(a) and [1, I 14.3]. $(H + V_0)^2 = 2$. Hodge index [2, Thm. 1.2]. [THEOREM]. $\square$

Verdict on the hunch. J.W.’s hunch splits into two parts.

- *Right:* factoring into irreducible pieces is how the structure is organized over $\mathbb{F}_q$. The pieces are $\mathbb{Q}$ -divisors (Theorem 3.1), κ localizes on them (Theorem 4.1), and Weil positivity holds piece by piece (N2).
- *Not right as stated:* the surface cannot emerge from the pieces alone. The pieces are negative definite, and the one positive direction that Hodge index needs is the ample class of the product (Theorem 6.1). Divisors-first reorganizes the product; it does not remove it.

Over $\mathbb{Q}$ the roles reverse ([ANALOGY]). The product part has a counterpart: Deninger’s $H^0 = \mathbb{R}$ and $H^2 = \mathbb{R}(-1)$, the poles of $\hat{\zeta}$ at $s = 0, 1$ (the L84/L88 pole plane, signature $(1, 1)$). What is missing is the factor ring, a $\mathbb{Q}$ -algebra of correspondences on H^1 with a positive involution (§5).

Bridge to Lemma 90 (J.W. t385, proposed “Yoneda specification”, [HEURISTIC]). Over $\mathbb{F}_q$ the data assembled here are exactly Hom-data: $\text{Hom}^0(h^1(C), h^1(C')) = \text{Hom}^0(J, J')$, the divisorial correspondences on $C \times C'$ modulo trivial ones [1, III 6.3], with composition, $\dagger$ and Tr . A Yoneda-style specification of the missing $h^1(\text{Spec } \mathbb{Z})$ would have to supply these as *functors* (how they transform under maps), not just as groups. Lemma 89 gives the $\mathbb{F}_q$ template for that.

7. RANKED DIRECTIONS, OPEN PROBLEMS, QUESTIONS

Score = 0.6 sig + 0.4 prob, judgement calls.

	Direction	sig	prob	score	First step
E1	Lemma 90: Yoneda/Tannaka specification of the missing h^1 using the $\mathbb{F}_q$ Hom-template (§6; J.W. t385)	.65	.55	.610	State the functor $T \mapsto \text{Hom}(h^1(T), -)$ on curves over $\mathbb{F}_q$ and list which parts have a $\mathbb{Q}$ analogue
E2	Read Milne 1968a/b at the source: the genus- g Theorem A of L88 and the trace-pairing regulator (C89.3)	.35	.95	.590	Obtain Invent. Math. 5, 6 (1968)
E3	Infinite-dimensional Albert algebra: a positive involution on an infinite product of orbit blocks, and which topology/state makes Weil’s form a trace form $\text{Tr}(xx^\dagger)$ (L84 K1 + Thm. 4.1)	.70	.35	.560	Formalize the blocks of Prop. 5.1 as a $*$ -algebra; check whether positivity of the state is [RESTATEMENT]
E4	Split Jacobians with non-commutative End^0 (type III, $y^2 = x^5 + 4x$ over $\mathbb{F}_5$): Hodge index on the part invisible to $\mathbb{Q}[F]$ (L88 D2)	.40	.80	.560	Explicit quaternion-order basis of $\text{End}(J)$, exact Gram
E5	Fake detection by factor type: classify all fakes in N1 by orbit pattern; asymptotic share of swapped pairs	.30	.90	.540	Extend N1 to $g = 4$ with a smaller box

Top three: E1 (.610), E2 (.590), E3 (.560; tied with E4).

Open problems. (O1) A $\mathbb{Q}$ -structure on any candidate H^1 for ζ whose idempotents are “rational correspondences” (§5, D3(iii)). (O2) Does a Hodge-index-type statement survive when the product part is replaced by the pole plane (§6)? (O3) The genus- g Theorem A at a source (Milne 1968a/b).

Questions for J.W.

- Q1. The formalizer did not have the verbatim t380/t381 text. Sections A–E follow the recorded scope (correspondence ring, Wedderburn/Albert, κ per factor, no Frobenius over $\mathbb{Q}$). Is anything from your original wording missing?
- Q2. By “least common divisors”, did you mean the irreducible factors of P (Theorems 3.1 and 4.1), or minimal generating divisors of NS? The second fails for type III Jacobians (Remark 3.2).
- Q3. Do you accept the verdict that the surface cannot emerge from the pieces alone because the ample class is product structure (Theorem 6.1)?
- Q4. Start Lemma 90 (Yoneda specification, t385) next, using §6 as its $\mathbb{F}_q$ template?
- Q5. C89.4 (the reported Hodge-for-CM-abelian-varieties proof): update the L88 statements now, flagged as unrefereed, or wait for a refereed version?

8. CORRECTIONS

Prior art for Lemma 88 Theorem A (J.W. t383: “Is this a known concept?”) Answer: yes, in every mathematical part. Theorem A puts together three classical facts. The only new thing in it is the TAA framing: Lemma 84’s trace form on $\mathbb{Q}[F]$ and Lemma 85’s Mordell–Weil/Hodge-index positivity are the same lattice in this case. (This note was first delivered as a 3-page draft, `build89/189_priorart_thmA.pdf`, and is merged here. `lemma88.pdf` is unchanged.) (1) Theorem A(c) is the textbook proof of Hasse. Let π be the q -Frobenius of $E/\mathbb{F}_q$ and put $a = \pi + \hat{\pi} = q + 1 - \#E(\mathbb{F}_q)$, Lemma 88’s sign convention. For $m, n \in \mathbb{Z}$,

$$\deg(m + n\pi) = (m + n\pi)(m + n\hat{\pi}) = m^2 + amn + qn^2,$$

so $2\deg$ has Gram matrix $\begin{pmatrix} 2 & a \\ a & 2q \end{pmatrix}$ in the basis $\{1, \pi\}$ (identity re-checked symbolically here, [NUMERICAL]). This is the matrix of Theorem A(c). Since $\deg$ is a positive-definite quadratic form on $\text{End}(E)$, this binary form is positive semidefinite, so $a^2 \leq 4q$. That is Hasse’s theorem, and the

argument is the standard one. In Silverman, *The Arithmetic of Elliptic Curves* (2nd ed., GTM 106, 2009) it reads:

- Cor. III.6.3: $\deg$ is a positive-definite quadratic form on $\text{Hom}(E_1, E_2)$;
- Lemma V.1.2: $|\deg(\psi - \phi) - \deg \psi - \deg \phi| \leq 2\sqrt{\deg \psi \deg \phi}$, i.e. Cauchy–Schwarz for that form;
- Thm. V.1.1 (Hasse, p. 138): proved by applying V.1.2 to $\psi = 1$, $\phi = \pi$, using that $1 - \pi$ is separable (Cor. III.5.5), so $\#E(\mathbb{F}_q) = \deg(1 - \pi)$.

Tag: [CITED-sec]. The locations were checked in two opened secondary sources, an REU exposition (S. Li, UMich REU 2014, quoting “Lemma V.1.2 in [Sil09]”, Cor. III.5.5, Thm. III.6.2) and Elkies’ Math 223 notes (Thm. V.1.1, p. 138; Cor. III.6.3), recorded in Lemmas 19 and 64. We did not open the book itself; we found only an unauthorized copy and did not use it. Lemma 88 Theorem A(c) and Lemma 84’s statement “trace form PD $\Leftrightarrow$ RH” in genus 1 are therefore Silverman V.1.1–V.1.2 in Gram-matrix form.

(2) Theorem A(a),(b): $\text{Hom}(E, E')$ as an even lattice with norm $2 \deg$. The constant surface $E' \times E \rightarrow E$ has generic fibre E'_K , $K = \mathbb{F}_q(E)$. Its sections are the morphisms $E \rightarrow E'$, so $E'(K) = \text{Mor}(E, E') = \text{Hom}(E, E') \oplus E'(\mathbb{F}_q)$, the second summand being the translations. The height is $2 \deg$. This is classical, and in the Mordell–Weil-lattice literature $\text{Hom}(E, E')$ is routinely given this lattice structure:

- Shioda, *K3 surfaces and sphere packings*, J. Math. Soc. Japan 60 (2008) 1083–1105, §2, p. 1085: “ $\text{Hom}(C_1, C_2)$ is viewed as a positive-definite even integral lattice by defining the norm of $\varphi : C_1 \rightarrow C_2$ to be $2 \deg(\varphi)$ ”. It enters $\det \text{NS}$ and $\det \text{MW}$ of the Inose K3 surfaces $F_{\alpha, \beta}^{(n)}$ (Thms. 2.2–2.3). [CITED] (opened). Note that this is a non-constant K3 fibration, not $E' \times E \rightarrow E$.
- Schütt–Shioda, *Elliptic surfaces* (arXiv:0907.0298): $\rho(\text{Km}(E \times E')) = 18 + \text{rank } \text{Hom}(E, E')$. [CITED] (opened). Again a K3 statement.
- Schütt–Shioda, *Mordell–Weil Lattices*, Ergebnisse 70, Springer 2019: [BIB] only. We did not open it, so we cannot point to where (or whether) it treats the constant surface.

For $E' \times E$ itself the cleanest source is the elementary intersection computation, already in TAA as Lemma 84 (NS complement = $-\text{Tr}$ form, [TAA-L]). On $E \times E'$ take $H = \{pt\} \times E'$, $V = E \times \{pt\}$ and the graph Γ_φ of $\varphi : E \rightarrow E'$. Then $H^2 = V^2 = 0$, $H \cdot V = 1$, $\Gamma_\varphi \cdot H = 1$, $\Gamma_\varphi \cdot V = \deg \varphi$, and $\Gamma_\varphi^2 = 0$ by adjunction ($\Gamma_\varphi \cong E$, $K = 0$). Hence $D_\varphi = \Gamma_\varphi - (\deg \varphi)H - V \perp \langle H, V \rangle$ and $D_\varphi^2 = -2 \deg \varphi$ ([THEOREM]; exact symbolic check [NUMERICAL]). Theorem A(b) follows from this computation with no Mordell–Weil-lattice input. The “height” is $-D_\varphi^2$, and Hodge index makes it positive.

(3) BSD for constant curves is a theorem (Milne 1967/68). BSD is known for constant abelian varieties over global function fields: Milne, *The Tate–Šafarevič group of a constant abelian variety*, Invent. Math. 6 (1968) 91–105 (thesis, Harvard 1967, under Tate). The regulator there is a trace pairing, and Milne, *Extensions of abelian varieties defined over a finite field*, Invent. Math. 5 (1968) 63–84, uses “the discriminant of the pairing $\text{Hom}(A, B) \times \text{Hom}(B, A) \rightarrow \mathbb{Z}$ sending two homomorphisms to the trace of their composite”. That is Theorem A(b)’s pairing $\text{Tr}(\varphi\hat{\psi})$, appearing in a BSD-type formula in 1968. Tag: [CITED-sec]. The content is taken from Milne’s own annotated list of abstracts (jmilne.org, opened); the papers themselves were not opened. Tate’s Bourbaki 306 (opened, lit/lemma88) supplies the general frame (conjecture (d), Artin–Tate $\Leftrightarrow$ BSD, and ρ versus Frobenius multiplicity, Thm. 5.2). We found no explicit constant-curve statement in it.

Verdict. Theorem A(a)–(d) consists of known results: Hasse via the $\deg$ form (Silverman V.1.1–V.1.2), Hom as an even lattice with norm $2 \deg$ (Shioda; elementary on $E \times E'$), $\rho(E \times E) = 2 + \text{rank End}$ (classical, [TAA-L] L71/L84), and BSD for constant curves (Milne 1968). **Ours is only the framing:** the BSD regulator lattice and the RH positivity lattice are literally the same Gram matrix in this case, and that matrix is Lemma 84’s trace form and Lemma 85’s MW lattice. That this is the “simplest honest instance” of the t379 hunch (L88 Remark 3.2) still stands, but as an observation about known mathematics, not a new theorem. Proposed score change for direction

D4 of L88 (“source Prop. genus- g Theorem A”): its significance should drop, because the genus- g version is very likely in Milne 1968a/b (trace pairing on $\text{Hom}(A, B)$). It should become “read Milne 1968a/b” rather than “prove”. [HEURISTIC].

Correction items.

C89.1 L88 Theorem A: the proof cites a source outside its stated scope. The proofs of (b) and (d) cite Schütt–Shioda [9], Thm. 11.5/§11.6 (height formula) and Cor. 6.13 (Shioda–Tate), for $S = E \times E \rightarrow E$. But [9] §1, Conventions (2), reads: “Every elliptic surface S has a singular fibre. In particular, S is not isomorphic to a product $E \times C$ ” (opened, arXiv:0907.0298v3). Shioda’s original MWL setup (Proc. Japan Acad. 65 (1989)) likewise assumes f is not smooth ([BIB], search snippet only). The *statements* of Theorem A remain true. *Repair*: replace (b) by the intersection computation $D_\varphi = \Gamma_\varphi - (\deg \varphi)H - V$, $D_\varphi^2 = -2 \deg \varphi$ ([THEOREM], which is L84 [TAA-L] and Theorem 2.1(a) here). Replace (d) by $\text{NS}(E \times E) \cong \mathbb{Z}^2 \oplus \text{End}(E)$ (Theorem 2.1(a), [1, III 6.3]). Read the “Shioda height” in L88 as $-D_\varphi^2$, which coincides with Shioda’s height wherever his definition applies. The `mw_constant.py` check is unaffected.

C89.2 L88 §3 should say Theorem A is classical. It should carry the attribution “(classical; Silverman V.1.1–V.1.2, Shioda 2008 §2, Milne 1968)”. The L84 genus-1 statement “trace form $\text{PD} \Leftrightarrow \text{RH}$ ” is likewise Silverman V.1.2 in Gram form.

C89.3 L88 Prop. 3.3 ([HEURISTIC]) and direction D4. The genus- g trace pairing on $\text{Hom}(A, B)$ is in Milne 1968a ([CITED-sec]). Prop. 3.3 stays [HEURISTIC] until Milne 1968a/b is opened, but the direction is now “locate”, not “discover” (E2 above).

C89.4 A reported change of status, unrefereed; we have not read the proofs. L88 §2 (the Hodge-type item) says, following Milne 2015, that the Hodge-type standard conjecture in characteristic p is “known essentially only for surfaces”. That was accurate for that source. We record only what the following sources say:

- OpenAI, *The rational Hodge conjecture for CM abelian varieties* (preprint dated 6 Oct 2026). Its abstract claims a proof that “every rational Hodge class on such a variety is a rational linear combination of algebraic cycle classes”. As consequences it lists “the generalized Hodge conjecture for CM abelian varieties, the Tate conjecture for abelian varieties over finite fields, and the Hodge standard conjecture for abelian varieties in arbitrary characteristic”, the last two via two theorems of Milne.
- P. van Hoften, J. Sempliner, *The Hodge conjecture for CM abelian varieties*, Hexagon preprint 2610.00259v2 (expository). Abstract: “Following the recent work [Ope26], we give a proof of the Hodge conjecture for CM abelian varieties.” Its introduction derives the Hodge standard conjecture for abelian varieties from [Mil02, Thm. 4.6(b)]. Its provenance statement says the authors made extensive use of ChatGPT Pro.
- Milne’s annotated list of abstracts (jmilne.org, opened 24 Oct 2026) adds notes dated 7 and 14 Oct 2026 to his 1999b and 2002a entries. They say that OpenAI “has announced a proof” and that van Hoften–Sempliner’s exposition is “confirming that it has in fact been proved”.

Status: [BIB]/[CITED-sec]. We saw abstracts and introductions only. The proofs were not read here, and neither preprint is refereed. Scope: the claims concern CM abelian varieties and, through Milne, abelian varieties over $\mathbb{F}_q$. They do not touch the general Hodge conjecture, ζ , or RH, and they leave L88 §6 and every result of this lemma unchanged (Tate for divisors on $C \times C$ was already known). Proposed L88 erratum, not applied: append “abelian varieties: claimed Oct. 2026, unrefereed preprint” to the §2 Hodge-type item.

C89.5 L88 Theorem C / correction K4: $\text{NS}(C \times C) \cong \mathbb{Z}^2 \oplus \text{End}(J)$ is upgraded from [HEURISTIC] to [CITED], via Milne [1, III Cor. 6.3] plus the routine $\text{Pic} \rightarrow \text{NS}$ step, which is our deduction (Theorem 2.1(a)). Over $\mathbb{F}_q$ itself, take Galois invariants.

- C89.6 L88 cited Tate 1966b only via Milne 2015.** The statement used (Tate’s theorem; the structure of End^0 over $\mathbb{F}_q$) is now also read in Milne’s AV notes [1, II §2, 2.1, 2.9] [CITED]. Tate’s paper itself is still unopened.
- C89.7 Scope of L84’s “#neg = κ ”.** It was proved for squarefree P . For non-squarefree P the algebra is $\mathbb{Q}[T]/(\mu)$ with μ the radical, and κ must be counted on distinct roots, which is the convention used in N2 and Theorem 3.1. This is a clarification, not an error.
- C89.8 Provenance of this lemma’s brief.** The verbatim t380/t381 messages were not available to the formalizer. Sections A–E were rebuilt from the recorded scope (STATE/memory: “correspondence ring $\text{NS} = \mathbb{Z}^2 \oplus \text{End}(J)$, Wedderburn/Albert factors, κ localizing per factor, and why $\mathbb{Q}$ has no Frobenius”). See Q1.

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TAA LEMMA 90: YONEDA DONE PROPERLY — THE WISH-LIST FUNCTOR FOR $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$, BASES BELOW $\mathbb{Z}$, EULER PRODUCTS, AND WHY THE PAIRING IS HARD

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis. Every claim carries one label: [THEOREM], [RESTATEMENT], [ANALOGY], [CONJECTURE], [OPEN], [NUMERICAL], [HEURISTIC]; sources are marked [CITED] (opened), [CITED-sec] (via an opened secondary source), [BIB] (bibliographic only), [TAA-L] (earlier TAA lemma). All references are listed in `lit/lemma90/verified_refs.md`.

ABSTRACT. We take J.W.’s Yoneda idea (t385) at face value. An object is pinned down by its functor of points $\mathrm{Hom}(-, X)$ *together with naturality*, not by the bare Hom sets (Theorem 2.1). We write the axioms that L88/L89 want of the missing surface “ $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$ ” as a functor F (axioms W1–W5) and ask whether F is representable. Representability would give uniqueness and functoriality, but it would give no positivity: the positivity is an input to F , not an output. There are four obstructions: (i) $\mathrm{Spec} \mathbb{Z}$ is terminal, so the absolute square collapses; (ii) Serre’s quaternion argument; (iii) mixed Tate motives over $\mathbb{Z}$ carry the ζ -values but not the zeros; (iv) a Tannakian fibre functor would have to be infinite-dimensional. Going below $\mathbb{Z}$ (t386): $\mathbb{N} \rightarrow \mathbb{Z}$ is an epimorphism of semirings, so $\mathbb{Z} \otimes_{\mathbb{N}} \mathbb{Z} = \mathbb{Z}$ (Theorem 4.1). In characteristic one, $x \mapsto x^\lambda$ is a Frobenius-type endomorphism (Theorem 4.2). Of the $\mathbb{F}_1$ frameworks we surveyed, only Connes–Consani’s has a non-collapsing square with Frobenius correspondences, and nobody has a positive intersection theory on it. The added sections treat the Euler product as a sum over the cone $\mathbb{N}^\times \subset \mathbb{Q}_+^\times$, the idele-class “bundle” and its splitting (for *every* number field), and why a Hodge-index pairing on the square is hard. In short: assuming it is circular (Weil), proving it over $\mathbb{F}_q$ uses Riemann–Roch together with a finite $h^1 = 2g$, and over $\mathbb{Q}$ the corresponding h^1 is infinite-dimensional. Verdict: F as wished is *not* representable in any surveyed category. A weakened F' (a continuous family of correspondences, real-valued dimensions, positivity as the goal and not an axiom) is exactly the open Connes–Consani programme.

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1. CATEGORICAL GLOSSARY FOR J.W.

All definitions are the standard ones from Riehl [1] [CITED].

- **Category, functor, natural transformation.** A category has objects and arrows. A functor F sends objects to objects and arrows to arrows, preserving composition. A natural transformation $\alpha : F \Rightarrow G$ is a family $\alpha_T : F(T) \rightarrow G(T)$ compatible with every arrow $T' \rightarrow T$. That compatibility is what *naturality* means.
- **Representable functor** (Riehl Def. 2.1.4). A contravariant $F : \mathcal{C}^{op} \rightarrow \mathbf{Set}$ is *represented* by X if there is a natural isomorphism $\mathrm{Hom}(-, X) \cong F$. Example: a scheme is represented by its functor of points $T \mapsto \mathrm{Hom}(T, X)$.
- **Yoneda lemma** (Riehl Thm. 2.2.4, Cor. 2.2.8). $\mathrm{Nat}(\mathrm{Hom}(-, X), F) \cong F(X)$, naturally. Consequently $X \mapsto \mathrm{Hom}(-, X)$ is fully faithful, and $X \cong Y$ if and only if $\mathrm{Hom}(-, X) \cong \mathrm{Hom}(-, Y)$ *naturally*.
- **Terminal / initial object** (Riehl Def. 1.6.14). t is terminal if every object has exactly one arrow to t ; dually for initial. $\mathbb{Z}$ is initial in commutative rings (Riehl Ex. 1.6.15(v)), so $\mathrm{Spec} \mathbb{Z}$ is terminal (final) in schemes (Stacks 01I4) [2].
- **Epimorphism** (Riehl Def. 1.2.7). $f : x \rightarrow y$ is epi if $hf = kf$ implies $h = k$. An epi need not be surjective: $\mathbb{Z} \rightarrow \mathbb{Q}$ in rings and $\mathbb{N} \rightarrow \mathbb{Z}$ in semirings (Theorem 4.1).
- **Tannakian category and fibre functor** (Deligne–Milne [3] Def. 2.19, Thm. 2.11 [CITED]). A rigid abelian tensor category $\mathcal{C}$ with $\mathrm{End}(\mathbb{K}) = k$, together with an exact faithful k -linear tensor functor $\omega : \mathcal{C} \rightarrow \mathrm{Vec}_k$ into *finite-dimensional* vector spaces. Then $G = \mathrm{Aut}^{\otimes}(\omega)$ is an affine group scheme and $\mathcal{C} \simeq \mathrm{Rep}_k(G)$. “Tannaka duality” is this reconstruction of a group from its representations. In spirit it is Yoneda for groups: G is recovered from the functor ω and its natural automorphisms.

2. YONEDA DONE PROPERLY AND THE WISH-LIST FUNCTOR

2.1. Bare Hom sets do not determine an object.

Theorem 2.1 ([THEOREM]). *In the category of $\mathbb{Q}$ -vector spaces, let $X = \mathbb{Q}$ and $Y = \mathbb{Q}^{(\mathbb{N})}$ (countable dimension). For every object T there is a bijection of sets $\mathrm{Hom}(T, X) \cong \mathrm{Hom}(T, Y)$ (maps in have equal cardinalities), yet $X \not\cong Y$.*

Proof. Let $\kappa = \dim T$. Then $\mathrm{Hom}(T, V) \cong V^\kappa$ as sets, which has cardinality 1 if $\kappa = 0$, $\aleph_0$ if $0 < \kappa < \infty$ (since $|V| = \aleph_0$), and $\aleph_0^\kappa = 2^\kappa$ if κ is infinite. These depend only on $|V| = \aleph_0$, so they agree for $V = X, Y$. Dually, $\mathrm{Hom}(V, T) \cong T^{\dim V}$: for $T = 0$ both are points, and for $T \neq 0$ we get $|T|$ versus $|T|^{\aleph_0}$. These differ in general, for example $T = \mathbb{Q}$ gives $\aleph_0$ versus $2^{\aleph_0}$. So the *maps-out* sets already distinguish X from Y , but the *maps-in* sets do not. In any case $\dim X = 1 \neq \aleph_0 = \dim Y$. $\square$

Remark 2.2 ([RESTATEMENT]). Yoneda needs the bijections $\mathrm{Hom}(T, X) \cong \mathrm{Hom}(T, Y)$ to commute with every $T' \rightarrow T$. Equivalently, it needs one universal element in $\mathrm{Hom}(X, Y)$ that induces all of them. With naturality, maps *in* suffice (Yoneda) and so do maps *out* (co-Yoneda). Without naturality, neither suffices in general (Theorem 2.1 for maps in). This sharpens the t385 chat answer (correction L90.C1).

2.2. The wish-list functor F . From L88 Thm. C, L89 §§A–E and the t378u review [TAA-L], the axioms wanted of the missing surface S are:

W1 (factor/correspondence ring):

A group $\text{NS}(S) \supseteq \mathbb{Z}H \oplus \mathbb{Z}V \oplus R$, where R is a ring of correspondences playing the role of $\text{End}(J)$.

Over $\mathbb{F}_q$ this is $\text{NS}(C \times C) \cong \mathbb{Z}H \oplus \mathbb{Z}V \oplus \text{End}(J)$ (L89 A, [THEOREM]/[CITED]).

W2 (Hodge index):

A symmetric intersection form on $\text{NS}(S)$ of signature $(1, \rho - 1)$ with $(H + V)^2 > 0$, and $H + V$ ample.

W3 (Frobenius graph):

An endomorphism Φ with graph $\Gamma_\Phi \in \text{NS}(S)$.

W4 (counts as intersections):

$\Gamma_{\Phi^k} \cdot \Delta =$ the k -th “point count”, which over $\mathbb{Q}$ is the prime side of the explicit formula.

W5 (κ -compatibility):

The inertia of the Rosati/trace form on R equals κ (L84/L89 C).

To ask about representability one must fix a base category $\mathcal{C}$ and set $F(T) = \{\text{W1–W5 data on “}T \times T\text{” over }T\}$, with pull-back along $T' \rightarrow T$. Candidate bases are schemes, $\mathbb{N}$ -schemes, $\mathbb{F}_1$ -schemes (Deitmar/Toën–Vaquié), blueprints, Λ -schemes, Durov’s generalized schemes, and characteristic-one semiringed topoi. The choice of $\mathcal{C}$ is part of the question (Q1).

Proposition 2.3 ([THEOREM], formal). *Suppose F is represented by S . Then: (a) S is unique up to unique isomorphism (Yoneda). (b) Every W -structure is induced by a universal one on S , so correspondences, Frobenius and pairing are functorial in T . (c) If W2 is part of the definition of F , then “ F representable” implies the existence of an object carrying a positive pairing. Representability therefore cannot prove positivity: it presupposes it. (d) Conversely, if one drops W2 from F and S represents the rest, the positivity of the pairing on S is a separate theorem to be proved on S .*

Proof. (a),(b) are Riehl Cor. 2.2.8 applied to $\text{Hom}(-, S) \cong F$: the universal element is the image of id_S . (c),(d) are immediate from the definitions. $\square$

Remark 2.4 ([HEURISTIC]). By Weil’s criterion (§7), W2+W4 over $\mathbb{Q}$ with the explicit-formula pairing is equivalent to RH. So “ F (with W2) is representable” is at least as strong as RH. The useful question is (d): construct S representing W1, W3, W4, W5, and *then* prove W2.

3. OBSTRUCTIONS

Theorem 3.1 ([THEOREM]/[CITED], terminal collapse). *$\text{Spec } \mathbb{Z}$ is terminal in schemes, and $\text{Spec } A \times \text{Spec } B = \text{Spec}(A \otimes_{\mathbb{Z}} B)$ (Stacks Lemma 26.6.7, tag 01I4, and its proof [2]). Hence $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z} = \text{Spec}(\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) = \text{Spec } \mathbb{Z}$, the diagonal is an isomorphism, and $\text{Pic}(\text{Spec } \mathbb{Z}) = 0$. The absolute square carries no W1–W5 structure beyond the trivial one.*

Proof. Morphisms $X \rightarrow \text{Spec } \mathbb{Z}$ correspond to ring maps $\mathbb{Z} \rightarrow \mathcal{O}_X(X)$, and there is exactly one (Riehl Ex. 1.6.15(v)). The product formula is the cited lemma. $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$. Finally $\mathbb{Z}$ is a PID, so $\text{Pic} = 0$. $\square$

Theorem 3.2 ([CITED-sec], Serre via Milne [6] §2.2 pp. 27–28). *Let E be supersingular over a field containing $\mathbb{F}_{p^2}$. Then $\text{End}^0(E)$ is a quaternion division algebra over $\mathbb{Q}$, nonsplit exactly at p and ∞ . It cannot act faithfully on a 2-dimensional $\mathbb{Q}_p$ - or $\mathbb{R}$ -vector space, so there is no Weil cohomology in characteristic p with $\mathbb{Q}_p$, $\mathbb{R}$, or (a fortiori) $\mathbb{Q}$ coefficients. Milne’s footnote 25: “It was Serre who explained this to Grothendieck, sometime in the 1950s.”*

Remark 3.3 ([ANALOGY]). For W1: the correspondence ring R will in general be non-commutative and non-split at some places. So the coefficients of any “ $H^1(\text{Spec } \mathbb{Z})$ ” cannot be fixed naively. In particular, a $\mathbb{Q}$ -linear Weil cohomology is ruled out even over $\mathbb{F}_p$. Connes–Consani use idempotent semimodules with real-valued dimensions instead (§4).

Theorem 3.4 ([CITED], Deligne–Goncharov [4], Brown [5]). $\mathrm{MT}(\mathbb{Z})$ is a Tannakian category with fibre functor ω , motivic Galois group $G_\omega = \mathbb{G}_m \ltimes U_\omega$, U_ω free pro-unipotent, $\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Q}$ (equal to $\mathbb{Q}$ for odd $n \geq 3$), and $\mathrm{Ext}^2 = 0$. Brown: $\mathrm{MT}(\mathbb{Z})$ is generated by the motivic π_1 of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and its periods are $\mathbb{Q}[(2\pi i)^{-1}]$ -combinations of multiple zeta values.

Remark 3.5 ([RESTATEMENT]+[HEURISTIC]). Every object of $\mathrm{MT}(\mathbb{Z})$ has a finite-dimensional realization, filtered by weight with Tate graded pieces $\mathbb{Q}(n)$. The values $\zeta(n)$, $n \geq 2$, occur as periods. Nothing in the category has a “Frobenius” whose eigenvalues are the zeros ρ . So $\mathrm{MT}(\mathbb{Z})$ contains ζ -values, not ζ -zeros. (The second sentence is Heuristic: it describes the absence of such a structure in the sources, not a theorem that none can exist.)

Proposition 3.6 ([THEOREM]given [CITED] inputs, Tannakian requirements). Suppose a neutral Tannakian $\mathcal{C}$ over k contained an object h^1 whose “Frobenius flow” had the zeros of ζ as spectrum, counted with multiplicity. Then $\omega(h^1)$ would be infinite-dimensional, since ζ has infinitely many zeros (Fig. 3). This contradicts Def. 2.19 of [3], where ω takes values in Vec_k (finite-dimensional). So such an h^1 needs an ind-/pro-Tannakian or topological variant. That is consistent with Deninger’s infinite-dimensional H^1 [21] [TAA-L].

Remark 3.7 ([ANALOGY]). In Tannakian language the positivity W2 is a *polarization*. Deligne–Milne §4 (Thms. 4.16–4.17 [CITED]) relate polarizations to compact real forms of $\mathrm{Aut}^\otimes(\omega)$. A “polarized Tannakian category of zeros” would encode W2. None is known.

4. A BASE BELOW $\mathbb{Z}$ (T386)

Theorem 4.1 ([THEOREM]; cf. Toën–Vaquié Prop. 3.4 [CITED]). (a) $\mathbb{N}$ is initial among commutative semirings. (b) $\iota : \mathbb{N} \rightarrow \mathbb{Z}$ is an epimorphism of commutative semirings. (c) $\mathbb{Z} \otimes_{\mathbb{N}} \mathbb{Z} \cong \mathbb{Z}$ via multiplication, so the square of $\mathrm{Spec} \mathbb{Z}$ over $\mathrm{Spec} \mathbb{N}$ is $\mathrm{Spec} \mathbb{Z}$ again.

Proof. (a) A semiring map $\mathbb{N} \rightarrow R$ must send $1 \mapsto 1$ and $n = 1 + \cdots + 1 \mapsto n \cdot 1_R$, so it is unique. (b) Let $f, g : \mathbb{Z} \rightarrow R$ be semiring maps agreeing on $\mathbb{N}$. Then

$$f(-1) = f(-1) + g(1) + g(-1) = f(-1) + f(1) + g(-1) = f(0) + g(-1) = g(-1),$$

using $g(1) + g(-1) = g(0) = 0$ and $f(1) = g(1)$. Additivity then gives $f = g$. (c) In commutative monoids in $(\mathrm{CMon}, \otimes)$, pushouts are tensor products (Toën–Vaquié §2.2: “ $A' \sqcup_A B \cong A' \otimes_A B$ ” [CITED]). A map $A \rightarrow B$ is epi iff the codiagonal $B \sqcup_A B \rightarrow B$ is an isomorphism (a standard categorical fact, proved directly: epi $\iff$ the two coprojections $B \rightrightarrows B \sqcup_A B$ coincide $\iff$ the codiagonal is an iso). Apply this with (b). $\square$

Theorem 4.2 ([THEOREM], characteristic-one Frobenius). Let $(S, \vee, \cdot)$ be a commutative semiring with idempotent addition ($x \vee x = x$), totally ordered by $x \leq y \iff x \vee y = y$, with monotone multiplication. Then for every integer $n \geq 1$, $x \mapsto x^n$ is a semiring endomorphism. On $\mathbb{R}_+^{\max} = ([0, \infty), \max, \times)$, $\mathrm{Fr}_\lambda(x) = x^\lambda$ is an automorphism for every real $\lambda > 0$. This is the characteristic-one analogue of $(a + b)^p = a^p + b^p$.

Proof. Multiplicativity is clear. If $x \leq y$, monotonicity gives $x^n \leq y^n$, so $(x \vee y)^n = y^n = x^n \vee y^n$. On $\mathbb{R}_+^{\max}$, $t \mapsto t^\lambda$ is a strictly increasing bijection of $[0, \infty)$ for $\lambda > 0$, so it preserves $\max$, and $(xy)^\lambda = x^\lambda y^\lambda$. $\square$

Remark 4.3 ([CITED]/[RESTATEMENT]). Connes–Consani: $\mathrm{Aut}_{\mathbb{B}}(\mathbb{R}_+^{\max}) = \mathbb{R}_+^\times$, Fr_λ is “the analogue of the arithmetic Frobenius”, and on points over $\mathbb{R}_+^{\max}$ its action is the idele class group action [8] Thm. 1.1. On $\mathbb{Z}_{\max}$ only integer Fr_n exist ($x \mapsto nx$ additively). In [10] the general statement is attributed to Golan: in a semifield of characteristic 1, $x \mapsto x^n$ is an injective endomorphism. **N1** [NUMERICAL] (exact rationals, num/190/n123.json): 20000 tests of $\lambda \max(a, b) = \max(\lambda a, \lambda b)$ and 20000 of $\lambda(a + b) = \lambda a + \lambda b$ in log coordinates gave 0 failures. $\lambda = -1$ breaks additivity in 2000/2000 cases. For ordinary $+$, $(a + b)^k = a^k + b^k$ failed in 2000/2000 cases. Fr_λ preserves $\mathbb{Z}_{\max}$ only for

$\lambda \in \{1, 2, 3\}$ among the tested $\{1, 2, 3, \frac{1}{2}, \frac{3}{2}, \frac{5}{3}\}$. Freshman’s dream mod p holds for $p \leq 13$ as a control. So t381’s “no Frobenius on $\mathbb{Q}$ ” is answered *only in this setting*: the Frobenius acts on the *coefficient semifield*, not on $\mathbb{Q}$ (correction L90.C3).

4.1. Survey: what each $\mathbb{F}_1$ framework supplies. Table 1 gives what each source says (all [CITED], opened; the computations in the last column are ours, [THEOREM] where marked).

TABLE 1. Frameworks below $\mathbb{Z}$. “Above $\mathbb{F}_1$ ” means $\text{Spec } \mathbb{Z}$ has a non-trivial $\mathbb{F}_1$ -model.

Framework	Supplies	$\text{Spec } \mathbb{Z}$ above $\mathbb{F}_1$?	$\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$ (checked)
Deitmar [15]	monoid schemes; $\mathbb{Z}[-] \dashv (R, \cdot)$ (Thm. 1.1); toric varieties	only as $\text{Spec } \mathbb{F}_1 \otimes \mathbb{Z}$	$\mathbb{Z}[\{1\} \times \{1\}] = \mathbb{Z}$ ([THEOREM], Prop. 4.4)
Toën–Vaquié [11]	relative schemes over $(\mathcal{C}, \otimes)$; $\mathbb{N}$ -, $\mathbb{F}_1$ -schemes	$\text{Spec } \mathbb{Z} \rightarrow \text{Spec } \mathbb{N}$ mono (Prop. 3.4)	over $\mathbb{N}$: $\mathbb{Z}$ (Thm. 4.1); over $\mathbb{F}_1$: as Deitmar
Lorscheid [14]	blueprints; tensor products (Prop. 1.12)	yes, via the blueprint $\mathbb{Z}$	associated ring $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$; the finer blueprint structure was not checked in the source (Q5)
Borger [13]	$\mathbb{F}_1$ -algebra = Λ -ring; descent of Artin–Tate motives (Thm. 0.1)	$\mathbb{Z}$ is $\mathbb{F}_1$ (initial, §2.1)	$\mathbb{Z}$ (colimits as in rings, §2.3); the square “must be defined to be” $W^*(\text{Spec } \mathbb{Z})$ (§1.2)
Durov [12]	generalized rings (mon-ads); $\mathbb{Z}_\infty$; $\widehat{\text{Spec } \mathbb{Z}}$; $\deg : \text{Pic} \cong \log \mathbb{Q}_+^\times$	$\mathbb{F}_1 = \mathbb{F}_0 \langle 0^{[0]} \rangle$	“ $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$, and similarly $\mathbb{Z} \otimes \mathbb{Z}_\infty = \mathbb{R}$ ” (0.5.19)
Connes–Consani [7, 8, 9, 10]	arithmetic site $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$; scaling site; $\text{Fr}_\lambda; \Psi(\lambda)$	via $\Theta : \text{Spec } \mathbb{Z} \rightarrow \widehat{\mathbb{N}_0^\times}$ ([8] Thm. 5.3)	$\overline{\mathbb{N}} \otimes_{\mathbb{B}} \overline{\mathbb{N}} \cong \text{Sub}_{\geq}(\mathbb{N} \times \mathbb{N})$, reduced = $\text{Conv}_{\geq}$ ([7] Prop. 3.1): non-trivial

Proposition 4.4 ([THEOREM]). *In Deitmar’s setting (with $\mathbb{F}_1 = \{1\}$, $\mathbb{F}_1$ -algebras = commutative monoids, base change $A \mapsto \mathbb{Z}[A]$), $\mathbb{Z}[A] \cong \mathbb{Z}$ as rings iff A is trivial. Hence $\text{Spec } \mathbb{Z}$ has the unique $\mathbb{F}_1$ -model $\text{Spec } \mathbb{F}_1$, and $\text{Spec } \mathbb{Z} \times_{\text{Spec } \mathbb{F}_1} \text{Spec } \mathbb{Z}$ base-changes to $\text{Spec } \mathbb{Z}$. For pointed monoids (with 0, the convention of Lorscheid and Connes–Consani) the same holds with $A = \{0, 1\}$.*

Proof. $\mathbb{Z}[A]$ is free abelian of rank $|A|$ (or $|A| - 1$ for pointed A after setting $[0] = 0$). Rank 1 forces $|A| = 1$ (resp. 2). $\mathbb{Z}[-]$ is a left adjoint ([15] Thm. 1.1), so it preserves pushouts: $\mathbb{Z}[A \otimes B] = \mathbb{Z}[A] \otimes \mathbb{Z}[B]$. $\square$

Remark 4.5 ([OPEN], the open core). No source we opened has an intersection pairing on a non-collapsing square of $\text{Spec } \mathbb{Z}$ over any $\mathbb{F}_1$ with a *provably* positive ample class. Connes–Consani [8]: “The main open question is at this point the definition of a suitable Weil cohomology.” See §7.

5. CONSISTENCY: WHICH AXIOMS MUST BE WEAKENED

Theorem 5.1 ([THEOREM], verdict). *The functor F with $W0$ – $W5$ as stated (finite NS, $\mathbb{Q}$ -coefficients, square in schemes or over $\mathbb{N}$) is not representable: in schemes and over $\mathbb{N}$ the only candidate is $\text{Spec } \mathbb{Z}$ itself (Thms. 3.1, 4.1), which has NS = 0 and fails $W1$. In Deitmar/Toën–Vaquié/Borger/Durov the square base-changes to $\text{Spec } \mathbb{Z}$ (Table 1).*

Conjecture 5.2 ([CONJECTURE]/[OPEN]). The weakened functor F' (characteristic-one square, continuous correspondences, real-valued dimensions, Riemann–Roch existence in place of $W2$) is representable by the square of the Scaling Site. With a Riemann–Roch theorem on it, F' would yield RH by the Mattuck–Tate–Grothendieck route ([10] §3.1, steps 1–5). This is exactly the open Connes–Consani programme. It is not a new conjecture of ours.

Wish-list axioms vs frameworks (assessment; sources in Table 1)

Schemes (abs.)	--	--	--	--	--	--	--
Deitmar	--	--	--	--	--	--	--
Toen--Vaquie	--	--	--	--	--	--	--
Lorscheid	partial	--	--	--	--	--	--
Borger	partial	--	--	--	partial	--	--
Durov	--	--	--	--	--	--	--
CC arith./scaling site	yes	--	partial	--	yes	partial	--
Arakelov (fibred)	partial	partial	yes	yes	--	--	--
	nontrivial square	NS-type ring	intersection form	Hodge index / ample	Frobenius graph	points = intersections	kappa-compatible

FIGURE 1. Our assessment ([HEURISTIC]) of axioms W1–W5 against frameworks; “partial” = analogue present without the surface property.

TABLE 2. W-axioms against §§3–4.

Axiom	Obstruction	Weakening that survives
W0: square in schemes	terminal collapse (Thm. 3.1), also over $\mathbb{N}$ (Thm. 4.1)	go to characteristic one ($\overline{\mathbb{N}} \otimes_{\mathbb{B}} \overline{\mathbb{N}}$) or define the square (Borger’s W^*)
W1: finite NS coefficients	infinitely many zeros; finite-dimensional ω impossible (Prop. 3.6) Serre (Thm. 3.2)	infinite-rank / continuous: $D = \int f(\lambda) \Psi_{\lambda} d^* \lambda$ ([10] (14)) idempotent semimodules, real dimensions ([9] Thm. 5.17)
W2: Hodge index	equivalent to RH (§7); circular as an axiom	keep it as the <i>goal</i> ; replace by a Riemann–Roch existence axiom ([10] §3.1)
W3: Frobenius graph	no Frobenius on $\mathbb{Q}$; Fr_{λ} is on coefficients	correspondences $\Psi(\lambda)$, $\Psi(\lambda)\Psi(\lambda') = \Psi(\lambda\lambda')$ up to Id_{ϵ} ([7] Thm. 4.3)
W4: counts = intersections	$N(1) = -\infty$ ([10] §2.3)	distributional $N(u)$; intersection via $N(u)$ ([10] (14)); needs renormalization
W5: κ	only finite spans (L89 Prop. D [TAA-L])	on finite-rank truncations only

6. HOPF-LIKE MAPS AND THE EULER PRODUCT (T388U/T389U)

Notation: we write $\mathbb{N}^{\times}$ for the multiplicative monoid of positive integers, following Connes–Consani. (In other contexts $\mathbb{N}^{\times}$ would mean the unit group $\{1\}$.)

(a) The Euler product as a cone generating function.

Theorem 6.1 ([THEOREM]). (i) $\mathbb{N}^{\times}$ is the free commutative monoid on the primes, $\mathbb{N}^{\times} \cong \bigoplus_p \mathbb{N}$ via exponent vectors. Its group completion is $\mathbb{Q}_+^{\times} \cong \bigoplus_p \mathbb{Z}$, and $\mathbb{N}^{\times} \rightarrow \mathbb{Q}_+^{\times}$ is “ $\mathbb{N} \rightarrow \mathbb{Z}$ prime by prime”. (ii) For $\Re s > 1$, $\sum_{n \geq 1} n^{-s} = \prod_p \sum_{k \in \mathbb{N}} p^{-ks} = \prod_p (1 - p^{-s})^{-1}$, and each Euler factor sums over the $\mathbb{N}$ -half of the exponent lattice. (iii) For no $s \in \mathbb{C}$ does $\sum_{k \in \mathbb{Z}} p^{-ks}$ converge. Hence ζ is the generating function of the positive cone $\mathbb{N}^{\times}$ inside the lattice $\mathbb{Q}_+^{\times}$, and it has no lattice-wide analogue. (iv) Tate’s local factor: with $d^{\times}x$ normalized by $\text{vol}(\mathbb{Z}_p^{\times}) = 1$, $\int_{\mathbb{Q}_p^{\times}} \mathbf{1}_{\mathbb{Z}_p}(x) |x|_p^s d^{\times}x = \sum_{k \geq 0} p^{-ks} = (1 - p^{-s})^{-1}$. The indicator of $\mathbb{Z}_p$ is exactly what cuts $\mathbb{Z}$ down to $\mathbb{N}$.

Proof. (i) is unique factorization. (ii) Absolute convergence for $\Re s > 1$ allows the product to be rearranged as the sum over exponent vectors. (iii) $|p^{-ks}| = p^{-k\sigma}$ with $\sigma = \Re s$. If $\sigma \geq 0$ the terms with $k \rightarrow -\infty$ do not tend to 0, and if $\sigma \leq 0$ the terms with $k \rightarrow +\infty$ do not. (iv) $\mathbb{Q}_p^\times = \bigsqcup_k p^k \mathbb{Z}_p^\times$, and $p^k \mathbb{Z}_p^\times \subset \mathbb{Z}_p$ iff $k \geq 0$. $\square$

N2 [NUMERICAL]: the factorization round trip holds for $10^5/10^5$ integers. The partial Euler products converge to $\zeta(2), \zeta(3), \zeta(3/2)$ with errors $1.3 \cdot 10^{-6}, 5.0 \cdot 10^{-12}, 1.2 \cdot 10^{-3}$ at $X = 10^5$. The two-sided sum $\sum_{|k| \leq K} 2^{-2k}$ grows as 5.25, 21.3, 341, $8.7 \cdot 10^4$, $5.7 \cdot 10^9$ for $K = 1, 2, 4, 8, 16$, while the Euler factor is $4/3$ (Fig. 2). This is consistent with Connes–Consani’s remark ([10] §2.1, [CITED]) that replacing $\mathbb{N}^\times$ by $\mathbb{Q}_+^\times$ forces one to cross with a non-archimedean component, which leads to the adèle class space.

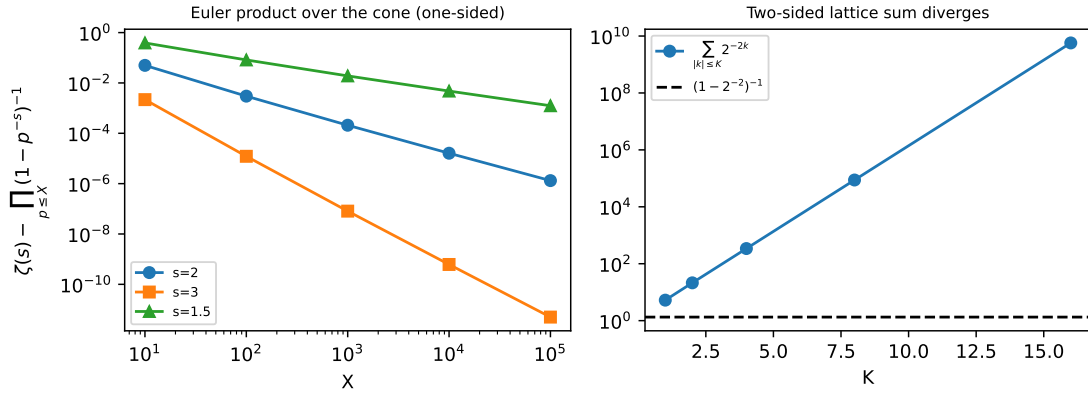


FIGURE 2. Left: one-sided Euler products converge ([NUMERICAL]). Right: the two-sided lattice sum diverges.

(b) Order of maps and the sign fibre.

Theorem 6.2 ([THEOREM]). *There are ring maps $\mathbb{N} \rightarrow \mathbb{Z} \rightarrow \mathbb{Q}$ but no ring map $\mathbb{Q} \rightarrow \mathbb{Z}$: such a map would send $2 \cdot \frac{1}{2} = 1$ to 1, so it would need an inverse of 2 in $\mathbb{Z}$. As monoids, $\mathbb{Z} \setminus \{0\} \cong \{\pm 1\} \times \mathbb{N}^\times$, a product.*

Remark 6.3 ([ANALOGY], weak). The fibre $\{\pm 1\} = S^0$ of $\mathbb{Z} \setminus \{0\} \rightarrow \mathbb{N}^\times$ is the fibre of the real Hopf map $S^1 \rightarrow \mathbb{R}P^1$, the Möbius double cover. But the arithmetic double cover is *trivial*: it is a product, and any double cover of a discrete base is trivial. The Möbius twist comes from the circle base, which $\mathbb{N}^\times$ lacks. At the archimedean place $\mathbb{R}^\times = \{\pm 1\} \times \mathbb{R}_{>0}$ is likewise split. As a source of a global constraint the analogy is weak (correction L90.C5).

(c) The idele class “bundle” splits, for every number field.

Theorem 6.4 ([THEOREM] with [CITED] inputs). *Let K be a number field of degree $n = r_1 + 2r_2$, with ideles $\mathbb{I}_K$, $C_K = \mathbb{I}_K / K^\times$, and content $c : C_K \rightarrow \mathbb{R}_{>0}$ (Milne CFT V.4.4 [16] [CITED]; c is well defined by the product formula and surjective). (i) $C_K^1 = \ker c$ is compact (Milne CFT V.4.4 [CITED]). (ii) The map $s(t) = [(t^{1/n})_{v|\infty}, (1)_{v|\infty}]$ is a continuous homomorphic section of c . Hence $C_K \cong C_K^1 \times \mathbb{R}_{>0}$ as topological groups, for every number field K . (iii) For $K = \mathbb{Q}$, $C_{\mathbb{Q}}^1 \cong \widehat{\mathbb{Z}}^\times$ and $C_{\mathbb{Q}} \cong \widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$. (iv) For general K there is an exact sequence $1 \rightarrow (\prod_{v|\infty} \mathcal{O}_v^\times \times K_{\infty,1}^\times) / \mathcal{O}_K^\times \rightarrow C_K^1 \rightarrow \text{Cl}_K \rightarrow 1$, so C_K^1 involves the class group and the compact unit torus. In particular C_K^1 is profinite (totally disconnected) iff $K = \mathbb{Q}$; for every other K it has a non-trivial connected torus part.*

Proof. (ii) $c(s(t)) = \prod_{v \text{ real}} t^{1/n} \prod_{v \text{ complex}} (t^{1/n})^2 = t^{(r_1+2r_2)/n} = t$ (normalized absolute values: $|x|^2$ at complex places). s is a continuous homomorphism, so the extension splits. (iii) Since $h_{\mathbb{Q}} = 1$, $\mathbb{I}_{\mathbb{Q}} = \mathbb{Q}^{\times} \cdot (\widehat{\mathbb{Z}}^{\times} \times \mathbb{R}^{\times})$ and $\mathbb{Q}^{\times} \cap (\widehat{\mathbb{Z}}^{\times} \times \mathbb{R}^{\times}) = \{\pm 1\}$ (Milne CFT V.4, Notes, [CITED]). So $C_{\mathbb{Q}} = (\widehat{\mathbb{Z}}^{\times} \times \mathbb{R}^{\times})/\{\pm 1\} \cong \widehat{\mathbb{Z}}^{\times} \times \mathbb{R}_{>0}$, and Connes–Consani [8] (proof of Thm. 3.8) write the same: “ $GL_1(\mathbb{A})/GL_1(\mathbb{Q}) = \widehat{\mathbb{Z}}^{\times} \times \mathbb{R}_{>0}^{\times}$ ”. (iv) The kernel of $C_K \rightarrow \text{Cl}_K$, $[x] \mapsto \text{ideal of } x$, is $K^{\times}(\prod \mathcal{O}_v^{\times} \times K_{\infty}^{\times})/K^{\times}$. Intersect with $\ker c$. For the last sentence: $K_{\infty}^{\times,1}$ is finite iff $r_2 = 0$ and $r_1 = 1$, i.e. $K = \mathbb{Q}$. Otherwise the identity component of $K_{\infty}^{\times,1}$ has positive dimension $r_1 + 2r_2 - 1$, and since $\mathcal{O}_K^{\times}$ is discrete in it (Dirichlet), the quotient keeps a positive-dimensional torus. $\square$

N3 [NUMERICAL]: $c(s(t))/t - 1$ has $|\cdot| \leq 8 \cdot 10^{-31}$ for 11 signatures (r_1, r_2) with $n \leq 5$ and $t \in \{0.37, 2, 10^7\}$ (30 digits). This is a sanity check only.

Remark 6.5 ([CITED]/[BIB]). Characters of $\widehat{\mathbb{Z}}^{\times}$ are Dirichlet characters, because they factor through $(\mathbb{Z}/N)^{\times}$. Mellin transform along $\mathbb{R}_{>0}$ of the resulting adelic integrals gives $L(s, \chi)$ with its gamma factor: Tate’s thesis [17] [BIB], with the local computation in Thm. 6.1(iv).

Remark 6.6 ([THEOREM], no twist means no constraint). For a global function field $K/\mathbb{F}_q$, the content lands in $q^{\mathbb{Z}}$. Since $q^{\mathbb{Z}}$ is free, $C_K \rightarrow q^{\mathbb{Z}}$ also splits (any surjection onto a free abelian group splits). By F.K. Schmidt’s theorem [BIB] the image is all of $q^{\mathbb{Z}}$. RH *holds* there (Weil). So the splitting $C_K \cong C_K^1 \times (\text{base})$ cannot be what blocks RH over $\mathbb{Q}$, and the absence of a Hopf-type twist carries no RH content (correction L90.C4).

(d) Heuristic: the cone $\mathbb{N}^{\times} \subset \mathbb{Q}_+^{\times}$ versus an ample cone.

Remark 6.7 ([HEURISTIC]). Write $\text{Div}(\text{Spec } \mathbb{Z}) = \bigoplus_p \mathbb{Z} \cong \mathbb{Q}_+^{\times}$ and $\text{Div}^+ = \mathbb{N}^{\times}$ (the effective divisors). Durov: $\deg : \widehat{\text{Pic}}(\text{Spec } \mathbb{Z}) \rightarrow \log \mathbb{Q}_+^{\times}$ is an isomorphism ([12] [CITED]). On a curve, “ample” means $\deg > 0$ (Hartshorne IV [BIB]).

Theorem 6.8 ([THEOREM], what survives). In $\mathbb{Q}_+^{\times} \otimes \mathbb{R} = \bigoplus_p \mathbb{R} \log p$: (i) the cone generated by $\mathbb{N}^{\times}$ is the closed positive orthant, and the Euler product is a sum over its lattice points (Thm. 6.1). (ii) The degree-positive set $\{\sum e_p \log p > 0\}$ is an open half-space, which strictly contains the orthant minus 0. For example $3/2$ has positive degree but is not in $\mathbb{N}^{\times}$. (iii) Hence $\mathbb{N}^{\times}$ is the effective cone, not the ample cone. An ample-type statement on a curve is a degree inequality, not a Hodge-index (signature) statement, which needs a surface.

So the precise statement that survives is (i): $\zeta = \text{cone generating function}$. The “ample cone” reading does not survive (correction L90.C6).

7. WHY THE PAIRING IS HARD (T390U)

(1) No $\mathbb{F}_1$ square with surface-like intersection theory.

Theorem 7.1 ([CITED], what Connes–Consani proved about the square). (i) [7] Prop. 3.1: $\overline{\mathbb{N}} \otimes_{\mathbb{B}} \overline{\mathbb{N}} \cong \text{Sub}_{\geq}(\mathbb{N} \times \mathbb{N})$; its reduction is $\text{Conv}_{\geq}(\mathbb{N} \times \mathbb{N})$ (Newton polygons). (ii) [7] Thm. 4.3: Frobenius correspondences $\Psi(\lambda)$, $\lambda \in \mathbb{R}_+^{\times}$, on the square, with $\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda\lambda')$, except that the product is Id_{ϵ} when λ, λ' are irrational and $\lambda\lambda' \in \mathbb{Q}$. (iii) [8] Prop. 6.11: the points of the square are the products of points. (iv) Scaling site [9]: points are $\mathbb{Q}^{\times} \backslash \mathbb{A}_{\mathbb{Q}} / \widehat{\mathbb{Z}}^*$ (Thm. 3.9). On each periodic orbit $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$ (a curve) one has $\text{Div}(C_p)/\mathcal{P} \cong \mathbb{R} \times \mathbb{Z}/(p-1)$ (Thm. 5.6) and Riemann–Roch $\dim_{\mathbb{R}} H^0(D) - \dim_{\mathbb{R}} H^0(-D) = \deg D$ with real-valued dimensions (Thm. 5.17). (v) [10] (1805.10501): “in the process to formulate a Riemann–Roch theorem on the square of the Scaling Site one faces a substantial difficulty. The problem, which is still open at this time, has to do with an appropriate definition of the sheaf cohomology ... H^1 .” The self-intersection $\mathfrak{s}(f, f) = D \bullet D$, $D = \int f(\lambda) \Psi_{\lambda} d^* \lambda$ (their (14)), is computed using the distribution $N(u)$ of the explicit formula. It is not derived from a cohomology theory or a Riemann–Roch theorem on the square. Their step 3 (Riemann–Roch on the square of the complex lift) is called “the most problematic”.

Remark 7.2 ([RESTATEMENT]). So the characteristic-one square has divisors, correspondences and a *defined* intersection number, but no Riemann–Roch, no H^1 , and no ampleness theorem. The other frameworks have no non-trivial square at all (Table 1). Statement (1) of t390 is accurate as of the sources opened, with the nuance that an intersection number *is* defined in [10] (correction L90.C7).

(2) **Weil positivity is RH; over $\mathbb{F}_q$ it is earned.**

Theorem 7.3 ([CITED]/[RESTATEMENT], Weil [18], Bombieri [19], as stated in [10] (12)–(13)). $\text{RH} \iff \mathfrak{s}(f, f) \leq 0$ for all real compactly supported f on $\mathbb{R}_+^*$ with $\int f d^*u = \int f du = 0$, where $\mathfrak{s}(f, g) = N(f \star \tilde{g})$ and $N(h) = \sum_n \Lambda(n)h(n) + \int_1^\infty \frac{u^2 h(u) - h(1)}{u^2 - 1} d^*u + c h(1)$, $c = \frac{1}{2}(\log \pi + \gamma)$.

Corollary 7.4 ([THEOREM], circularity). Any axiom asserting a Hodge-index inequality for the pairing $D \bullet D'$ of [10] (14) on the degree-0/codegree-0 subspace is equivalent to RH. Postulating W2 therefore assumes the conclusion.

Theorem 7.5 ([CITED-sec], the $\mathbb{F}_q$ mechanism; Milne [6] §1 Thm. 1.2, Cor. 1.3, Thm. 1.5, p. 13). On a smooth projective surface V over a field, the intersection form on $N(V)$ has index 1 (Hodge index). Grothendieck (1958a) showed this is “a simple consequence of the Riemann–Roch theorem for surfaces” (via Mattuck–Tate). The Castelnuovo–Severi inequality $D^2 \leq 2d_1 d_2$ on $C_1 \times C_2$ follows, and with it Weil’s RH for curves. The ingredients are: (a) Riemann–Roch on V ; (b) an ample class ($H + V$ on $C \times C$, $(H + V)^2 = 2 > 0$); (c) finiteness. $P(C, T)$ has degree $2g$, so H^1 is $2g$ -dimensional and NS has finite rank.

Remark 7.6 ([THEOREM]/[NUMERICAL]/[CITED], what fails over $\mathbb{Q}$). (a) No Riemann–Roch on any square (Thm. 7.1(v)). (b) No ample class, since the square collapses or has no positivity theorem. (c) ζ has infinitely many zeros: the mpmath count gives $N(T) = 10, 29, 79, 202, 341, 491, 649$ for $T = 50, \dots, 1000$ (Fig. 3). So any H^1 whose spectrum is the zeros is infinite-dimensional (Deninger [21] [TAA-L]). (d) The archimedean place contributes the continuous term $\int_1^\infty \frac{u^2 h(u) - h(1)}{u^2 - 1} d^*u + c h(1)$ to N . This is not a sum over closed points, so there is no “Frobenius graph meets diagonal at points” picture at ∞ .

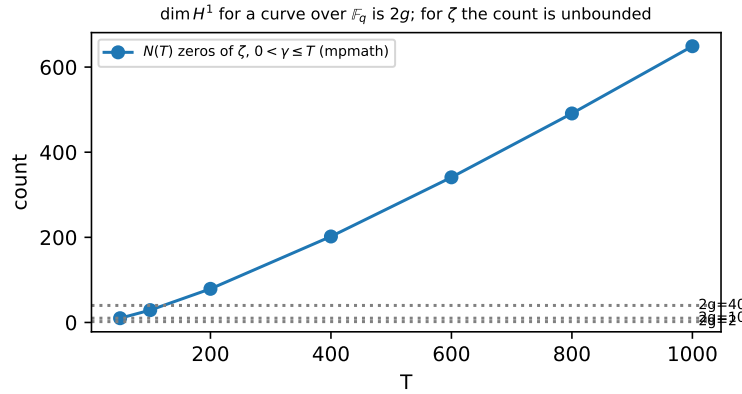


FIGURE 3. $\dim H^1 = 2g$ for a curve over $\mathbb{F}_q$, against the number of zeros $0 < \gamma \leq T$ of ζ ([NUMERICAL], num/190/n4_nzeros.json).

(3) **Heuristic: $\Delta^2 = 2 - 2g$ at “infinite genus”.**

Remark 7.7 ([HEURISTIC]). For a curve $C/\mathbb{F}_q$, $\Delta^2 = 2 - 2g = \sum (-1)^i \dim H^i$, the Lefschetz number of the identity (adjunction, Hartshorne V [BIB]). Formally $g = \infty$ gives $\Delta^2 = -\infty$. Connes–Consani find exactly this divergence: in Soulé’s $q \rightarrow 1$ limit the exponent $N(1)$ “is equal to $-\infty$ due to the infinite number of its zeros” ([10] §2.3 [CITED]). In Connes’ cut-off trace formula [20] [TAA-L] a divergent $\log \Lambda$ term appears, and [10] §3.4 speculate it is “due to the lack of good definition of self-intersection of the diagonal”. A positive pairing must therefore come with a *renormalized* Δ^2 , and its sign must survive the renormalization. No scheme is known that does this while keeping a provable inequality. This is the renormalization problem ([OPEN]).

(4) Contrast: Arakelov surfaces have Hodge index but are fibred.

Theorem 7.8 ([CITED-sec], Faltings 1984 [22]; Hriljac 1985 [23]). *For an arithmetic surface $X \rightarrow \text{Spec } \mathbb{Z}$ (fibres are curves; Arakelov intersection theory with Green’s functions at ∞ [24] [BIB]), Faltings and Hriljac independently proved a Hodge index theorem. The Arakelov intersection pairing, restricted to suitably normalized degree-zero classes, is the negative of the Néron–Tate height pairing on the Jacobian of the generic fibre. (Sources: Hriljac’s introduction, “an analogue of the Hodge Index theorem for arithmetic surfaces, a result due independently to G. Faltings”, and the Pacific J. Math. 309 (2020) abstract. Faltings’ paper was not opened.)*

Remark 7.9 ([ANALOGY]/[THEOREM]). $X \rightarrow \text{Spec } \mathbb{Z}$ is fibred over $\text{Spec } \mathbb{Z}$. The fibre class F has $F^2 = 0$ and plays the role of V only. There is no “horizontal H ” from a second factor, because $\text{Spec } \mathbb{Z}$ is not the self-product (Thm. 3.1). Its Hodge index is the arithmetic counterpart of the fibred surface $C \rightarrow B$ over $\mathbb{F}_q$ (Shioda height = $2 \deg$, L88 Thm. A [TAA-L]). It controls heights, not the zeros of ζ . So Arakelov theory proves the *fibred* half and leaves the self-product half (the one Weil’s argument needs) untouched (correction L90.C8).

8. RANKED DIRECTIONS

Score = 0.6·significance+0.4·probability (both subjective, [HEURISTIC]).

	Direction	sig	prob	score
Y1	Test the Connes–Consani pairing $D \bullet D$ ((14) of [10]) on finite spans of Ψ_λ with synthetic ghost zeros, linking to L89 N3: does Hodge-index inertia detect ghosts?	.65	.70	.67
Y2	Function-field calibration: realize F for $C \times C/\mathbb{F}_q$ (representable, W1–W5 hold), then track which axioms survive Soulé’s $q \rightarrow 1$ limit	.70	.55	.64
Y3	Arakelov “V-half”: Faltings–Hriljac as fibred Hodge index; can Durov’s $\mathbb{Z} \otimes \mathbb{Z}_\infty = \mathbb{R}$ supply a horizontal H ?	.75	.30	.57
Y4	Positivity as a Tannakian polarization (Deligne–Milne §4) on a pro-Tannakian category of zeros	.80	.20	.56
Y5	Borger’s $W^*(\text{Spec } \mathbb{Z})$ as the defined square: divisors on truncations	.60	.35	.50

9. QUESTIONS FOR J.W.

- Q1. Which base category should F live on: schemes, $\mathbb{N}$ -schemes, characteristic-one topoi, Λ -spaces, or Durov’s generalized schemes? (The verdict depends on it; only characteristic one gives a non-trivial square.)
- Q2. Do you accept weakening W1 from a finite NS to a continuous family $\int f(\lambda) \Psi_\lambda d^* \lambda$ (infinite rank)?
- Q3. Should W2 be removed as an axiom (it is circular) and replaced by a Riemann–Roch existence axiom, as in Connes–Consani’s strategy?
- Q4. Hopf: was S^0 meant as the archimedean sign $\mathbb{R}^\times \rightarrow \mathbb{R}_{>0}$? That cover is also split, and in $C_\mathbb{Q}$ the sign is quotiented out.
- Q5. Could you paste the verbatim chat text of t385, t386 and t390? The corrections below use recorded paraphrases. Also: should the finer blueprint structure of $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z}$ in Lorscheid’s category be checked against his text (we only checked the associated ring)?

10. CORRECTIONS TO CHAT CLAIMS

Paraphrased (verbatim text unavailable; Q5).

L90.C1 (t385 “knowing all maps in and out determines the object”).:

Correct only *with naturality*: maps in (Yoneda) or maps out (co-Yoneda) each suffice, and one direction is enough. Bare Hom sets do not suffice (Thm. 2.1). “Tannaka is a cousin” is fair: it reconstructs $G = \text{Aut}^\otimes(\omega)$ from a fibre functor ([3] Thm. 2.11). The hard part is representability, which presupposes positivity if W2 is an axiom (Prop. 2.3).

L90.C2 (t386 “ $\mathbb{N} \rightarrow \mathbb{Z}$ epi, so $\mathbb{Z} \otimes_{\mathbb{N}} \mathbb{Z} = \mathbb{Z}$ ”).:

Correct; proof in Thm. 4.1; cf. Toën–Vaquié Prop. 3.4.

L90.C3 (t386 “ $\mathbb{N}^\times$ with max-plus, $x \mapsto x^\lambda$ is Frobenius; answers no Frobenius on $\mathbb{Q}$ ”).:

Notation: “ $\mathbb{N}^\times$ ” is Connes–Consani’s multiplicative monoid, not the unit group. Real $\lambda > 0$ requires $\mathbb{R}_+^{\max}$; on $\mathbb{Z}_{\max}$ only integer λ works (N1). It is a Frobenius on the *coefficients* ([ANALOGY]), not on $\mathbb{Q}$.

L90.C4 (steering: “over $\mathbb{Q}$ the bundle splits, so there is no Hopf twist to carry a global constraint”).:

The splitting holds for *every* number field and for function fields (Thm. 6.4 and the remark after it). RH is true for function fields, so “no twist” carries no RH content. What is special to $\mathbb{Q}$ is $C_{\mathbb{Q}}^1 = \hat{\mathbb{Z}}^\times$ (profinite).

L90.C5 (steering: “ $\{\pm\} = S^0$ is the real Hopf / Möbius cover”).:

The arithmetic cover is a trivial product; the analogy is weak.

L90.C6 (steering: “ $\mathbb{N}^\times$ as an ample cone”).:

$\mathbb{N}^\times$ is the effective cone; degree-positive classes form a strictly larger half-space (3/2). What survives is “ ζ is the cone generating function”.

L90.C7 (t390 (1) “no $\mathbb{F}_1$ square with surface-like intersection theory”).:

Accurate for the sources opened. Nuance: Connes–Consani *define* $D \bullet D$ via $N(u)$, but without Riemann–Roch, H^1 , or ampleness (Thm. 7.1).

L90.C8 (t390 (2)–(4)).:

(2) Correct. Weil’s criterion is for the codimension-2 test space ($\int f d^*u = \int f du = 0$). Over $\mathbb{F}_q$ the Hodge index follows from Riemann–Roch on surfaces (Grothendieck 1958a via Milne). The *finite* $h^1 = 2g$ is what makes the relevant space finite-dimensional, but it is not itself an input to the Hodge index theorem. (3) Heuristic, matching Connes–Consani’s $N(1) = -\infty$ and the log Λ remark. (4) Faltings 1984 and Hriljac 1985 confirmed (independent; Arakelov pairing = $-\text{Néron–Tate}$), [CITED-sec]. “Fibred, not self-product” is correct.

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TAA LEMMA 91: TESTING THE CONNES–CONSANI SELF-INTERSECTION $D \bullet D$ ON FINITE Ψ_λ -SPANS WITH TRUE AND GHOST ZEROS; THE POINT AT INFINITY; SQUARES IN THE EULER PRODUCT

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis. Every claim carries one label: [THEOREM], [RESTATEMENT], [ANALOGY], [CONJECTURE], [OPEN], [NUMERICAL], [HEURISTIC]; sources are marked [CITED] (opened), [CITED-sec] (via an opened secondary source), [BIB] (bibliographic only), [TAA-L] (earlier TAA lemma). All references are listed in `lit/lemma91/verified_refs.md`.

ABSTRACT. This chapter carries out direction Y1 of L90. Connes–Consani define the self-intersection of a “divisor” $D = \int f(\lambda) \Psi_\lambda d^* \lambda$ on the square of the scaling site through the explicit-formula distribution $N(u)$: $D \bullet D = s(f, f)$, and RH is equivalent to $s(f, f) \leq 0$ on the codimension-2 space $\widehat{f}(0) = \widehat{f}(1) = 0$ ([2] (12), (14)). We compute s on finite spans of test functions and of smoothed $\Psi(\lambda)$, from the prime side (that is, from $N(u)$) and from the zero side, using 2500 zeros of ζ . We repeat the computation after inserting synthetic off-line quartets $\rho = \frac{1}{2} \pm \delta \pm i\gamma$, including the Davenport–Heilbronn point $0.8085 + 85.6993i$. Results. (a) With the true zeros, no positive direction appears beyond the discretization error, and the prime-side and zero-side matrices agree to $6 \cdot 10^{-5}$ relative. (b) A ghost quartet produces *exactly two* positive directions (four for two quartets, six for three). It does so only once the support half-width a (test functions on $[e^{-a}, e^a]$) exceeds a threshold $a_{\min}(\gamma) \approx 0.56 \log \gamma + c$, which barely depends on δ ; the size of the violation scales as δ^2 . Below the threshold the ghost is invisible. This is consistent with the theorem of Yoshida and Connes–Consani that positivity holds unconditionally for supports inside $[\frac{1}{2}, 2]$. (c) Smoothed $\Psi(\lambda)$ with *rational* $\lambda = m/n$ still detect ghosts; integers alone detect only strong ghosts; primes alone fail. Verdict: on these spans $D \bullet D$ is Weil’s functional restated ([RESTATEMENT]). The Ψ_λ structure adds a support filtration and an arithmetic sampling of λ , but no new source of positivity. A short section answers t391u: $\Gamma_{\mathbb{R}}$ is Tate’s local integral at ∞ ; adding ∞ completes a *curve*, and the surface still needs the square. A remark answers t393u: the square/squarefree split $\zeta(s) = \zeta(2s) \cdot \zeta(s)/\zeta(2s)$ is an Adams-type operation $\psi^2 = \text{Sym}^2 - \Lambda^2$. It sees only the diagonal eigenvalue products α_i^2 ; the off-diagonal products $\alpha_i \alpha_j$ of $H^1 \otimes H^1$, where Weil positivity lives, cancel out of it.

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Date: 24 October 2026 (t392u: “Yes do Y1”; t391u; addition t393u).

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GLOSSARY (BRIEF RESTATEMENTS FOR J.W.)

Test function, support a :

A smooth f on $\mathbb{R}_+^*$ supported in $[e^{-a}, e^a]$; in the variable $t = \log u$, it is a function on $[-a, a]$.

Mellin transform $\hat{f}(s) = \int f(u) u^s d^*u$.

Ψ_λ , $D = \int f(\lambda) \Psi_\lambda d^*\lambda$:

The Frobenius correspondences of the scaling site, with $\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda\lambda')$ [1]. A “divisor” D on the square is a superposition of them with weight f .

$N(u)$, $s(f, f)$:

The explicit-formula distribution: primes plus the archimedean term. Then $s(f, f) = N(f * \tilde{f})$ with $\tilde{f}(u) = f(u^{-1})$, up to normalization.

Ghost quartet:

A hypothetical off-line zero set $\{\frac{1}{2} \pm \delta \pm i\gamma\}$ inserted by hand into the zero side.

n_+ :

The number of positive eigenvalues of s on a finite span after imposing $\hat{f}(0) = \hat{f}(1) = 0$. These are the “wrong-sign” directions. Hodge index $\Leftrightarrow n_+ = 0$.

1. THE OBJECT UNDER TEST

1.1. What Connes–Consani define. Two facts are [CITED] from the opened sources. (i) [2] (12): RH holds iff $s(f, f) \leq 0$ for all test functions with $\int f d^*u = \int f du = 0$, i.e. $\hat{f}(0) = \hat{f}(1) = 0$ (citing Bombieri). (ii) [2] (14): $s(f, f) = D \bullet D$ for $D = \int f(\lambda) \Psi_\lambda d^*\lambda$, where the intersection is computed “using the distribution $N(u)$ ”. In [3] (1)–(2) the same criterion is written as $\sum_v \mathcal{W}_v(g * \bar{g}^\sharp) \leq 0$ for $\tilde{g}$ vanishing on a finite set $F \supset \{0, 1\}$. The local terms involve only finitely many primes when g has compact support. The vanishing at $0, 1$ (equivalently at $\pm \frac{i}{2}$ on the critical line) “corresponds to focus on the key contribution of H^1 in Lefschetz’s trace formula” in the function-field case [3], p. 3. So the class “orthogonal to the analogue of $H + V$ ” of the brief is $V_0 := \{\hat{f}(0) = \hat{f}(1) = 0\}$ ([ANALOGY], with the dictionary [CITED]).

1.2. Our normalization. Write $t = \log u$, $\varphi(t) = f(e^t)$ and $\hat{f}(s) = \int \varphi(t) e^{st} dt$. Let $h = f * \tilde{f}$ be the function with

$$\hat{h}(s) = \hat{f}(s) \hat{f}(1 - s).$$

Theorem 1.1 ([CITED] for the formula, [NUMERICAL] for the constant). *Let*

$$N(h) := \sum_{n \geq 2} \Lambda(n) h(\log n) + W_\infty(h),$$

where the archimedean principal-value term W_∞ is implemented in `num/lemma91_lib.py`. Then

$$N(h) = \frac{1}{2} \left(\hat{h}(0) + \hat{h}(1) - \sum_\rho \hat{h}(\rho) \right).$$

Numerically, for 16 test functions (supports $a \in \{0.5, 1, 1.5, 2\}$, four shapes) and 2500 zeros, the two sides agree to $\leq 7.5 \cdot 10^{-8}$ absolutely.

The factor $\frac{1}{2}$ appears because our prime sum is one-sided (h is symmetric). This positive constant does not affect any sign or count below. We define $s(f, f) := N(f * \tilde{f})$ and its polarization $s(f, g)$.

Proposition 1.2 ([RESTATEMENT]). *On V_0 one has $s(f, f) = -\frac{1}{2} \sum_{\rho} \hat{f}(\rho) \hat{f}(1 - \rho)$. If every zero is on the line, this equals $-\sum_{\gamma > 0} |\hat{f}(\frac{1}{2} + i\gamma)|^2 \leq 0$. Hence “Hodge index for $D \bullet D$ on V_0 ” $\iff$ Weil positivity $\iff$ RH [2, 5]. Consequently, with the true zeros, any test computed from the zero side is negative semidefinite by construction. The only non-tautological check is the prime side, i.e. $N(u)$ itself (§2, X1).*

1.3. Ghost quartets: exact algebra. Insert $Q = \{\frac{1}{2} \pm \delta \pm i\gamma\}$ ($0 < \delta < \frac{1}{2}$) into the zero side. Put $A(\sigma) := \hat{f}(\frac{1}{2} + \sigma + i\gamma)$ and use real f throughout.

Proposition 1.3 ([THEOREM]). (i) *The quartet adds $g_Q(f) = -\frac{1}{2} \sum_{\rho \in Q} \hat{f}(\rho) \hat{f}(1 - \rho) = -2 \operatorname{Re} [A(\delta) \overline{A(-\delta)}]$.* (ii) *As a Hermitian form in $(A(\delta), A(-\delta)) \in \mathbb{C}^2$, g_Q has signature $(1, 1)$; as a real form on $\mathbb{R}^4$ it has signature $(2, 2)$. Hence, on any finite span, with k quartets added to on-line zeros, $n_+ \leq 2k$.* (iii) *If $A(0) = 0$, then $g_Q(f) = 2\delta^2 |A'(0)|^2 + O(\delta^4)$.* (iv) *Let $\psi = e^{t/2} \varphi$ (the unitary normalization of [3]) and $R\psi(t) = \psi(-t)$. Then R preserves s and every g_Q . Hence s splits into R -even and R -odd blocks, and each block has at most one wrong-sign direction per quartet.*

Proof. (i) The pairs $\rho \leftrightarrow 1 - \rho$ in Q are $\frac{1}{2} + \delta \pm i\gamma \leftrightarrow \frac{1}{2} - \delta \mp i\gamma$. Since $\hat{f}(\bar{s}) = \overline{\hat{f}(s)}$, the sum is $4 \operatorname{Re} [A(\delta) \overline{A(-\delta)}]$. (ii) The matrix is $-(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})$. The true zeros contribute a negative semidefinite form, and the positive index is subadditive. (iii) Write $A(\pm\delta) = \pm\delta A_1 + \frac{\delta^2}{2} A_2 + O(\delta^3)$. The cubic term of $A(\delta) \overline{A(-\delta)}$ is purely imaginary, so the real part is $-\delta^2 |A_1|^2 + O(\delta^4)$. (iv) Put $\Psi(z) := \hat{f}(\frac{1}{2} + z) = \int \psi e^{zt} dt$. Then $\hat{h}(\frac{1}{2} + z) = \Psi(z) \Psi(-z)$, and R sends $\Psi(z) \mapsto \Psi(-z)$. Both the zero multiset (true or with Q) and the pole pair $\{\pm \frac{1}{2}\}$ are symmetric under $z \mapsto -z$. So R is an s -isometry with $R^2 = 1$, and its eigenspaces are s -orthogonal. On the even block Ψ is even, and Q contributes $-2 \operatorname{Re} \Psi(\delta + i\gamma)^2$, a real form in one complex variable of signature $(1, 1)$. The odd block is the same up to sign. $\square$

Remark 1.4 ([HEURISTIC]: what the ghost model is). A genuine off-line zero would also change $N(u)$, i.e. the primes. Our model changes only the zero side. It therefore tests the *quadratic form*, not a consistent alternative arithmetic. In the real world, the archimedean positivity theorem ([3], Theorem 1 and the statement proved by Yoshida [4]) gives $D \bullet D \leq 0$ unconditionally for f supported in $[2^{-1/2}, 2^{1/2}]$ (with the vanishing conditions), i.e. $a \leq \frac{1}{2} \log 2 \approx 0.347$. No real ghost can be seen at those supports.

2. NUMERICS

All runs use 2500 zeros $\gamma_n \leq 3031.29$ (mpmath, `zeta_zeros_2500.npy`, monotone). Test spaces are spans of $\operatorname{bump}(t/a) \cos(\omega t)$ and $\operatorname{bump}(t/a) \sin(\omega t)$, with ω on a grid of step π/a in a window ± 15 around the ghost ordinate (or $[0, 60]$ for X1). We impose $\hat{f}(0) = \hat{f}(1) = 0$, orthonormalize in $L^2(dt)$, dropping Gram eigenvalues below 10^{-11} relative, and count n_+ above 10^{-8} of the largest $|\text{eigenvalue}|$. The quadrature grid has Nyquist frequency above the largest zero used. *Self-correction*: a first scan with a coarse grid aliased high zeros onto low frequencies and produced a spurious law $a_{\min} \propto 1/\delta$. It was discarded; the zero-cutoff checks at 120 vs. 250 above the window now agree to 10^{-11} . Scripts: `num/lemma91_{lib,prime,weil,scan,curve,x345,even,t393,figs}.py`; data `num/191/*.json`. X1 ([NUMERICAL], true zeros, prime side vs zero side). We build the matrix of s from $N(u)$ directly (FFT correlations, exact prime powers, archimedean integral) and compare it with the zero-side matrix on the same span:

a	dim	$\lambda_{\max}$ (prime side)	$\max \Delta \text{eig} $	rel. matrix error
0.3	13	-0.391	$2.4 \cdot 10^{-6}$	$1.6 \cdot 10^{-6}$
0.6	25	$-3.7 \cdot 10^{-4}$	$1.2 \cdot 10^{-5}$	$3.6 \cdot 10^{-6}$
1.0	41	$+3.2 \cdot 10^{-7}$	$6.4 \cdot 10^{-5}$	$1.4 \cdot 10^{-5}$
1.5	59	$+2.0 \cdot 10^{-6}$	$2.3 \cdot 10^{-4}$	$2.3 \cdot 10^{-5}$
2.0	79	$+1.4 \cdot 10^{-5}$	$6.1 \cdot 10^{-4}$	$3.5 \cdot 10^{-5}$
2.5	97	$+7.0 \cdot 10^{-5}$	$1.5 \cdot 10^{-3}$	$5.8 \cdot 10^{-5}$

The small positive $\lambda_{\max}$ for $a \geq 1$ are always below the eigenvalue discretization error (column 4). This is consistent with $D \bullet D \leq 0$, and of course it is no proof. At $a = 0.3 < \frac{1}{2} \log 2$ the form is strictly negative, as the archimedean theorem predicts.

X2 ([NUMERICAL], minimal violating support). With one ghost quartet we find, by bisection on a , the smallest support with $n_+ > 0$. Detection persists at every larger a tested (monotone in all 15 cases).

$\gamma \backslash \delta$	0.3085	0.2	0.15	0.1	0.07	$\frac{1}{2} \log \frac{\gamma}{2\pi}$
39.3	1.321	1.348	1.370	1.391	1.407	0.917
85.699 (DH)	1.762	1.795	1.810	1.832	1.885	1.311
200.3	2.243	2.304	2.304	2.304	2.316	1.731

Observations ([NUMERICAL]): $a_{\min}$ grows with $\log \gamma$ (slope 0.566 at $\delta = 0.3085$ and 0.558 at $\delta = 0.07$), and it barely depends on δ (by $\leq 7\%$ over a factor 4.4 in δ). Fig. 1 shows the mechanism at the DH height. Without a ghost, $\lambda_{\max}$ rises from -0.54 ($a = 1$) to ~ 0 (-10^{-9} at $a = 2.2$, then rounding level). That is, at $a \approx 1.8$ – 2.2 the span acquires directions that almost annihilate all true zeros. A ghost then shows up at once, for every δ tested, with $\lambda_{\max} \approx C(a) \delta^2$ ($C(3) \approx 2.0$ to 2.2 for $\delta \leq 0.1$), as Prop. 1.3(iii) predicts.

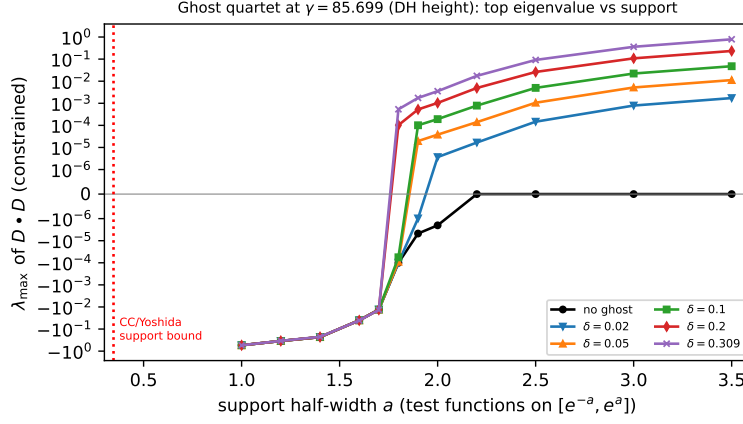


FIGURE 1. Top eigenvalue of $D \bullet D$ on V_0 vs. support, ghost at the DH height $\gamma = 85.699$. Diagnostic: the ghost becomes visible exactly where the no-ghost curve (black) reaches 0; the red line is the unconditional archimedean positivity range.

Remark 2.1 ([HEURISTIC]: why $\frac{1}{2} \log(\gamma/2\pi)$). Functions of exponential type a have zero density a/π per unit height, while the zeros of ζ near γ have density $\frac{1}{2\pi} \log \frac{\gamma}{2\pi}$. “Room to vanish on all nearby true zeros” therefore needs $a \gtrsim \frac{1}{2} \log \frac{\gamma}{2\pi}$. Our $a_{\min}$ exceeds this by 0.40–0.58, which is plausibly the effective-support loss of the bump envelope. The measured slope 0.56 is close to $\frac{1}{2}$. Our $a_{\min}$ are basis-dependent *upper bounds* on the true infimum over all $f \in C_c^\infty([e^{-a}, e^a])$.

X3 ([NUMERICAL], counting). Ghosts at $(0.3085, 85.699)$, $(0.2, 120.3)$, $(0.25, 160.3)$; $a \in \{2.5, 3.5, 5\}$; $n_+ = 0, 2, 4, 6$ for $k = 0, 1, 2, 3$ quartets in all 12 runs, with the zero-ghost runs at rounding level ($\leq 5 \cdot 10^{-10}$). In the unitary normalization (Prop. 1.3(iv)), the R -even block gives $n_+ = k$, the R -odd block $n_+ = k$, and the full span $2k$, in all 27 runs ($a \in \{3.5, 5, 7\}$). The bound of Prop. 1.3(ii) is attained.

X4 ([NUMERICAL], restricted $\Psi(\lambda)$ classes). Here $D = \sum_i c_i \Psi(\lambda_i)$, smoothed by a bump of width $\varepsilon = 0.02$ in $\log \lambda$, with $\lambda_i \in [e^{-a+\varepsilon}, e^{a-\varepsilon}]$; ghost at $\gamma = 85.699$.

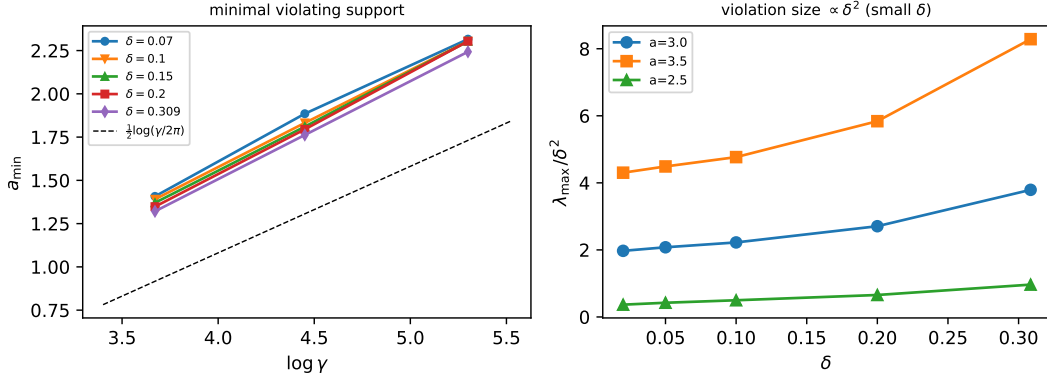


FIGURE 2. Left: $a_{\min}$ vs. $\log \gamma$ for five δ , against the density heuristic $\frac{1}{2} \log(\gamma/2\pi)$ (dashed). Right: $\lambda_{\max}/\delta^2$ is nearly constant for small δ .

class ($a = 3.5$)	dim	no ghost	$\delta = 0.1$	$\delta = 0.3085$
log-grid $h = 0.01$	697	$6 \cdot 10^{-15}$	+0.073	+1.19
log-grid $h = 0.03$	233	$-2.8 \cdot 10^{-4}$	$-2.8 \cdot 10^{-4}$	+0.130
log-grid $h = 0.05$	140	$-1.7 \cdot 10^{-3}$	$+2.5 \cdot 10^{-3}$	+0.243
rational m/n , $m, n \leq 40$	963	$5 \cdot 10^{-9}$ (10^{-10} rel.)	+0.059	+0.922
integers n and $1/n$	63	$-9.0 \cdot 10^{-3}$	$-9.3 \cdot 10^{-3}$	+0.029
primes p , $1/p$, and 1	23	$-3.2 \cdot 10^{-2}$	$-3.2 \cdot 10^{-2}$	$-3.1 \cdot 10^{-2}$

At $a = 2.5$ only the dense grid and the rationals detect ($\lambda_{\max} = 0.21$ and 0.17 at $\delta = 0.3085$); the integer and prime classes stay negative. The rational class is the arithmetic one: $\Psi(m/n) = \Psi(m) \circ \Psi(n)^{-1}$ are compositions of the integer Frobenius correspondences [1]. It detects as well as the dense grid. Integers alone detect only the strong ghost at the larger support; primes alone never do.

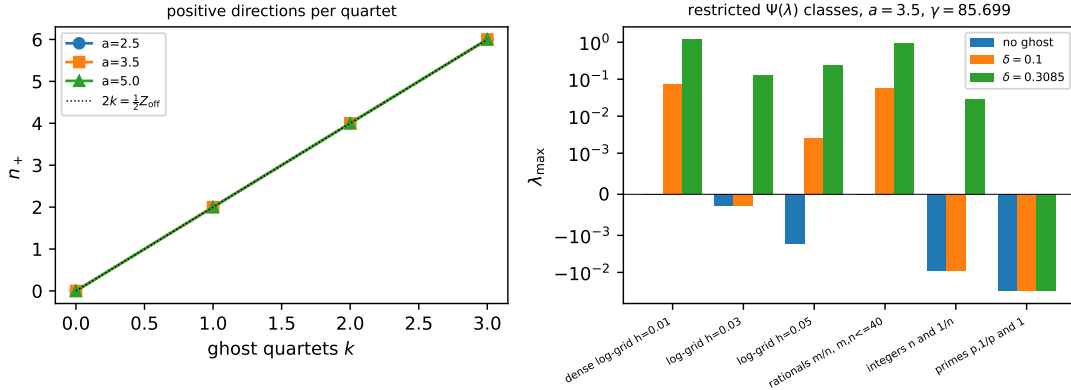


FIGURE 3. Left: $n_+ = 2k$ exactly (X3). Right: restricted $\Psi(\lambda)$ classes (X4, $a = 3.5$; symlog scale).

X5 ([NUMERICAL], the diagonal). Let $\Delta_\varepsilon = \int \eta_\varepsilon(\lambda) \Psi_\lambda d^* \lambda$ with η_ε a bump of width ε and $\hat{\eta}(0) = 1$ (a smoothed diagonal; no vanishing conditions). From the prime side:

ε	0.5	0.3	0.2	0.1	0.05	0.025	0.0125
$\Delta_\varepsilon \bullet \Delta_\varepsilon$	1.010	0.971	0.708	-1.049	-6.90	-23.28	-65.40
$\sum_\rho \hat{\eta}(\rho) ^2$ (from N)	0.020	0.072	0.591	4.099	15.80	48.56	132.81
same, direct from zeros	0.020	0.072	0.591	4.099	—	—	—

Here $\Delta_\varepsilon^2 = \hat{\eta}(0)\hat{\eta}(1) - \frac{1}{2} \sum_\rho |\hat{\eta}(\rho)|^2$ (Thm. 1.1; the direct values agree to $\leq 10^{-7}$), and the smoothed Riemann–von Mangoldt main term tracks the zero sum to within 0.25 for $\varepsilon \leq 0.1$. This is the numerical form of the L90 heuristic “ $\Delta^2 = 2 - 2g$ at $g = \infty$ ”: in our normalization $\Delta^2 = \frac{1}{2}(2 - 2g_\varepsilon)$ with $2g_\varepsilon := \sum_\rho |\hat{\eta}_\varepsilon(\rho)|^2 \sim \frac{1}{\varepsilon} \log \frac{1}{\varepsilon}$. It matches Connes–Consani’s $N(1) = -\infty$ [2] ([NUMERICAL], [HEURISTIC]).

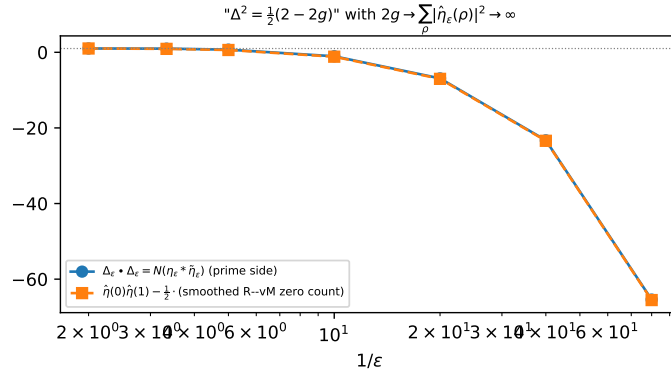


FIGURE 4. Smoothed self-intersection of the diagonal diverges like the smoothed zero count.

3. VERDICT ON THE Y1 QUESTIONS

Does the Hodge-index inequality fail exactly when a ghost is present?:

Yes, given enough support ([NUMERICAL], with [THEOREM] for the “only if”). With the true zeros, the prime-side form shows no positive direction beyond the discretization error up to $a = 2.5$ (X1). By Prop. 1.2 the zero-side form is ≤ 0 identically. With a ghost quartet, $n_+ = 2$ for $a > a_{\min}(\gamma)$ and $n_+ = 0$ below. “Exactly when” therefore needs the qualifier *at supports* $a \geq a_{\min}(\gamma)$. Small supports are blind: provably so for $a \leq \frac{1}{2} \log 2$ in the real world [3], and numerically up to $a_{\min}$ in the model.

How does the minimal violating test function scale with δ and γ ?:

The support threshold is governed by γ , not δ : $a_{\min} \approx 0.56 \log \gamma + c$, close to the density heuristic $\frac{1}{2} \log(\gamma/2\pi)$ ([NUMERICAL]+[HEURISTIC]). The *size* of the violation is governed by δ : $\lambda_{\max} \approx C(a)\delta^2$ ([THEOREM] at leading order, Prop. 1.3(iii); [NUMERICAL]). The violating vector vanishes at the ghost ordinate on the line ($A(0) = 0$), almost vanishes at all nearby true zeros, and has a nonzero σ -derivative there, i.e. it detects the displacement off the line. In λ -space it is a wave packet at frequency γ spread over all of $[e^{-a}, e^a]$.

Is this Weil’s criterion restated?:

Yes ([RESTATEMENT]). On every span tested, $D \bullet D = s(f, f) = N(f * \tilde{f})$ is Weil’s functional. This is Connes–Consani’s own definition (14) of [2], and the inequality is equivalent to RH by (12). The Ψ_λ structure contributes two things that are not mere notation: (a) a *support filtration* $C_c^\infty([e^{-a}, e^a])$, which is semi-local (finitely many primes, [3] (2)). It explains the blind zone and the $\log \gamma$ threshold. This is the setting of Connes–Consani’s archimedean theorem, which is the one place where positivity is *proved* ([CITED]). (b) an *arithmetic sampling*: rational $\lambda = m/n$, i.e. compositions of integer Frobenius correspondences, already detect ghosts (X4). So a restricted,

arithmetically defined test class still detects ghosts. But detection here is a sampling/Nyquist statement (spacing $\lesssim \pi/\gamma$ near the relevant λ), not a new positivity mechanism ([HEURISTIC]). Neither (a) nor (b) supplies what L90 identified as missing: a Riemann–Roch/ampleness argument that would *prove* $n_+ = 0$ ([OPEN]).

3.1. Links to L88 Theorem B, L89 zero blocks, and κ .

Proposition 3.1 ([THEOREM] + [NUMERICAL]). (i) *L89 Prop. D ([TAA-L]) works with R -even test functions (real transforms on the line). There each quartet contributes signature $(1, 1)$, so $n_- \leq \kappa_\zeta$ = the number of quartets. This is the even block of Prop. 1.3(iv). (ii) On the full real test space, which is Connes–Consani’s class, each quartet contributes $(2, 2)$ and $n_+ \leq 2\kappa_\zeta = \frac{1}{2}Z_{\text{off}}$, where Z_{off} is the number of off-line zeros. The bound is attained once $a > a_{\min}$ (X3). (iii) This reproduces the off-line part $\frac{1}{2}Z_{\text{off}}$ of L88 Theorem B’s count $k = \tau + \frac{1}{2}Z_{\text{off}}$ ([TAA-L]). The τ term has no counterpart in the ghost model. So the TAA counts line up: the number of wrong-sign directions of the would-be Hodge pairing equals half the number of off-line zeros ([ANALOGY] across $\mathbb{F}_q/\mathbb{Q}$, [THEOREM] within each side).*

The L89 observation “ $n_- = 1$ per ghost with a local basis” is therefore correct for its (even) class. The full Connes–Consani class sees two directions per ghost, one even and one odd (correction L91.C9). The L89 figure `n3_zero_blocks` shows the even block only.

4. THE POINT AT INFINITY (J.W. T391U)

- (1) **$\Gamma_{\mathbb{R}}$ is Tate’s local zeta integral at ∞** ([THEOREM]; [CITED] [9, 10]; [NUMERICAL]). $Z_{\infty}(e^{-\pi x^2}, s) = \int_{\mathbb{R}^{\times}} e^{-\pi x^2} |x|^s d^{\times}x = \pi^{-s/2} \Gamma(s/2) = \Gamma_{\mathbb{R}}(s)$ (substitute $u = \pi x^2$). The product with the local factors $(1 - p^{-s})^{-1}$ (from $1_{\mathbb{Z}_p}$) is $\xi(s) = \Gamma_{\mathbb{R}}(s)\zeta(s)$. So the completed zeta function *already contains* the place at infinity. Quadrature check: relative error $\leq 6.5 \cdot 10^{-14}$ at $s \in \{0.3, 0.7, 2, 5\}$. In this chapter, the archimedean term W_{∞} in $N(u)$ (Thm. 1.1) *is* that place. It is also the only regime where positivity is proved [3].
- (2) **Compactified $\text{Spec } \mathbb{Z}$ adds ∞** ([CITED]). Arakelov divisors are $\sum_P x_P P + \sum_{\sigma} x_{\sigma} \sigma$ with $x_{\sigma} \in \mathbb{R}$. The degree is $\sum x_P \log NP + \sum x_{\sigma}$, and $\text{Pic}^{(0)}(F)$ is compact (its volume is hR) [6] §2. For $F = \mathbb{Q}$, degree gives $\text{Pic}(\mathbb{Q}) \cong \mathbb{R}$. Durov’s $\widehat{\text{Spec } \mathbb{Z}}$ also adds ∞ : $\mathbb{Z} \otimes \mathbb{Z}_{\infty} = \mathbb{R}$ and $\deg : \text{Pic}(\widehat{\text{Spec } \mathbb{Z}}) \cong \log \mathbb{Q}_+^{\times}$ [7] 0.5.19 (opened in L90).
- (3) **Tate’s functional equation as Riemann–Roch** ([CITED] [6] Prop. 1; [NUMERICAL]). With $h^0(D) = \log \sum_{f \in I} e^{-\pi \|f\|_D^2}$, one has $h^0(D) - h^0(\kappa - D) = \deg D - \frac{1}{2} \log |\Delta|$. The proof is Poisson summation, and the identity is “a special case of Tate’s Riemann–Roch formula”. For $\mathbb{Q}$: $h^0(D_x) - h^0(D_{-x}) = x$, checked to 10^{-41} at $x \in \{0.1, 0.5, 1, 2, 3\}$. Caveat from the source: “we do not have a definition of $h^1(D)$... without recourse to duality”. So the arithmetic H^1 that a surface theory would need is missing already on the curve.
- (4) **Hilbert reciprocity; Hasse–Minkowski for conics** ([THEOREM], [CITED] [8]; [NUMERICAL]). $\prod_v (a, b)_v = 1$ over all places including ∞ ([8] III Aside 4.2 and V Ex. 5.4), and $(a, b)_v$ is the class of $H(a, b) \otimes \mathbb{Q}_v$ in $\text{Br}(\mathbb{Q}_v)$ (III Rem. 4.7–4.8). Hence the quaternion algebra $H(a, b)$, equivalently the conic $X^2 = aY^2 + bZ^2$, is ramified at an *even* number of places, ∞ counted like any prime. The conic has a rational point iff it has one at every v (Hasse–Minkowski, [8] VIII Thm. 3.5(b)). Check: all 6400 pairs $0 < |a|, |b| \leq 40$ give an even ramification set; 1600 of them are ramified at ∞ (e.g. $H(-1, -1)$ at $\{2, \infty\}$, $H(-1, 3)$ at $\{2, 3\}$). Local symbol formulas from Serre’s *Cours* ([BIB]).
- (5) **Adding ∞ completes a curve, not a surface** ([ANALOGY], with (2)–(3) [CITED]). Everything in (1)–(4) is one-dimensional: divisors are points (finite and infinite), and $\text{Pic}^{(0)}$ plays the Jacobian [6]. Compactification turns the affine curve $\text{Spec } \mathbb{Z}$ into the analogue of a *complete* curve C . Weil’s positivity lives on $C \times C$, and the arithmetic surfaces that do have a Hodge index (Faltings,

Hriljac) are fibred over $\text{Spec } \mathcal{O}_K$, not self-products (L90 §7, [TAA-L]). The square is still required. Connes–Consani’s construction is the candidate, and this chapter tests exactly its pairing.

5. REMARK: SQUARES IN THE EULER PRODUCT ARE NOT THE SQUARE (J.W. t393u)

Theorem 5.1 ([THEOREM]). *Every $n \geq 1$ is uniquely $n = a^2b$ with b squarefree. Hence $\zeta(s) = \zeta(2s) \cdot \sum_n |\mu(n)| n^{-s} = \zeta(2s) \cdot \zeta(s) / \zeta(2s)$, and locally $(1-p^{-s})^{-1} = (1+p^{-s})(1-p^{-2s})^{-1}$. Equivalently, $\zeta(2s) = \zeta(s) \cdot \sum_n \lambda(n) n^{-s}$ with Liouville’s λ .*

Check ([NUMERICAL]): uniqueness holds for $10^5/10^5$ integers. Partial sums to 10^5 match $\zeta(s)/\zeta(2s)$ to within the truncation tail ($6 \cdot 10^{-6}$, $3 \cdot 10^{-11}$, $2 \cdot 10^{-16}$ at $s = 2, 3, 4$), and the Liouville sums match $\zeta(2s)/\zeta(s)$ to $7 \cdot 10^{-10}$ at $s = 2$.

Proposition 5.2 ([THEOREM]; [NUMERICAL] on 14 curves). (i) For a curve $C/\mathbb{F}_q$: $Z_{C/\mathbb{F}_{q^2}}(T^2) = Z_C(T) Z_C(-T)$. Proof: $\log Z_C(T) + \log Z_C(-T) = \sum_r (1 + (-1)^r) N_r T^r / r = \sum_k N_{2k} (T^2)^k / k$, and $N_k(C/\mathbb{F}_{q^2}) = N_{2k}(C/\mathbb{F}_q)$. On eigenvalues, base change is $\alpha \mapsto \alpha^2$, i.e. Frobenius squared, the Adams operation ψ^2 . (ii) Over $\text{Spec } \mathbb{Z}$, Thm. 5.1 is the same identity place by place: $\zeta(2s) = \prod_p Z_{\text{pt}}(T) Z_{\text{pt}}(-T)|_{T=p^{-s}}$, with Liouville’s series playing $Z(-T)$. Over $\text{Spec } \mathbb{Z}$ the substitution $s \mapsto 2s$ (i.e. $T \mapsto T^2$, $n \mapsto n^2$) and base change coincide, because every local eigenvalue is 1. For curves they differ: $Z_C(T^2) \neq Z_{C/\mathbb{F}_{q^2}}(T^2)$ in all 14 elliptic curves tested, already because the H^2 eigenvalue q becomes q resp. q^2 . (iii) Brute-force point counts over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ ($p \in \{5, 7, 11, 13\}$, 14 curves $y^2 = x^3 + Ax + B$) confirm $N_2 = p^2 + 1 - (a_p^2 - 2p)$ and identity (i) exactly to T^8 , 14/14.

Proposition 5.3 ([THEOREM]; [NUMERICAL]). (i) $\zeta(s)^2 = \sum d(n) n^{-s}$ is the zeta function of $\mathbb{Z} \times \mathbb{Z}$, i.e. of $\text{Spec } \mathbb{Z} \sqcup \text{Spec } \mathbb{Z}$: ideals are $I_1 \times I_2$, and the norm is multiplicative. Over $\mathbb{F}_q$, $Z_C(T)^2 = Z_{C \sqcup C}(T)$ has coefficients $2N_r$. Pairs of integers form a disjoint union, not a product. (ii) Künneth for $C \times C$, with $P_1(T) = \prod_{i=1}^{2g} (1 - \alpha_i T)$:

$$Z_{C \times C}(T) = \frac{P_1(T)^2 P_1(qT)^2}{(1-T)(1-qT)^2 \prod_{i,j} (1 - \alpha_i \alpha_j T) (1 - q^2 T)},$$

where $\prod_{i,j}$ over $4g^2$ pairs is the $H^1 \otimes H^1 \subset H^2$ factor. It contains $\alpha_i \bar{\alpha}_i = q$, i.e. the eigenvalue- q (Tate) classes in $\text{NS}(C \times C)$ beyond H, V , where the Hodge index and hence Weil’s positivity act (L89 §A, [TAA-L]). For $E/\mathbb{F}_p$ the $H^1 \otimes H^1$ eigenvalues are $\{\alpha^2, \bar{\alpha}^2, p, p\}$. The formula matches $\exp \sum_r N_r(E)^2 T^r / r$ exactly to T^8 in 14/14 curves; at T^1 , $Z(E \times E)$ has N_1^2 (e.g. 81) while $Z(E)^2$ has $2N_1$ (e.g. 18). (iii) What the square/squarefree split misses. In K -theory $\psi^2 = \text{Sym}^2 - \Lambda^2$, so $\text{Tr}(F | \psi^2 H^1) = \sum \alpha_i^2$, whereas $\text{Tr}(F | H^1 \otimes H^1) = (\sum \alpha_i)^2$. The difference $2 \sum_{i < j} \alpha_i \alpha_j$, the off-diagonal products, is invisible to every one-variable operation ($s \mapsto 2s$, base change, Liouville twist). It is exactly the part that contains the algebraic classes $\alpha_i \bar{\alpha}_i = q$. (iv) Over $\text{Spec } \mathbb{Z}$, the Hadamard square of the local counts ($N_p = 1 \mapsto 1$) returns ζ itself. This is the collapse $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z} = \text{Spec } \mathbb{Z}$ (L90 Thm. 3.1, [TAA-L]). A “ ζ of zeros paired with zeros” would need the γ ’s as Frobenius eigenvalues, which we do not have ([HEURISTIC], [OPEN]).

Answer to t393u ([THEOREM] for (i)–(iii), [ANALOGY] for the reading). The squared and squarefree factors come from $n \mapsto n^2$ on one copy of $\mathbb{N}^\times$, i.e. Frobenius-squared/Adams ψ^2 . That is an operation on a single curve. It never produces the mixed eigenvalue products $\alpha_i \alpha_j$ ($i \neq j$) that define $H^1 \otimes H^1$, which is where $C \times C$ carries its Hodge-index positivity.

6. RANKED DIRECTIONS

Score = 0.6·significance+0.4·probability (both subjective, [HEURISTIC]).

	Direction	sig	prob	score
Z1	Certified semi-local positivity: interval-arithmetic bound on the largest a for which the <i>prime-side</i> $D \bullet D \leq 0$ on $V_0 \cap C_c^\infty([e^{-a}, e^a])$, beyond the proved $a = \frac{1}{2} \log 2$ (primes 2, 3, ... enter one at a time; Connes–Consani step)	.75	.45	.63
Z2	Threshold theorem: show $a^*(\gamma) = \frac{1}{2} \log \frac{\gamma}{2\pi} + O(1)$ for detectability of a ghost at height γ (Paley–Wiener/sampling; X2 gives slope 0.56)	.70	.50	.62
Z3	Function-field calibration of this chapter: same finite-span $D \bullet D$ for $C/\mathbb{F}_q$ and L88 fakes; check $n_+ = \frac{1}{2} Z_{\text{off}}$ and the support threshold vs. genus	.60	.65	.62
Z4	Arithmetic sampling: the least height M such that $\Psi(m/n)$, $m, n \leq M$, detects every ghost below height γ	.55	.60	.57
Z5	$H^1 \otimes H^1$ for ζ : compare with GL_2 , where Sym^2/Λ^2 L -functions exist, and isolate what ζ lacks	.50	.50	.50

7. QUESTIONS FOR J.W.

- Q1. Next step: certified semi-local positivity (Z1, numerical analysis) or the threshold theorem (Z2, harmonic analysis)?
- Q2. Is “the inequality fails exactly when a ghost is present, *provided the support exceeds* $a_{\min}(\gamma)$ ” the form of answer you wanted, or did you expect δ -dependence of the support?
- Q3. Should the L89 erratum (C9 below: $n_- = 1$ per ghost is the even class; the full class has 2) be written into `lemma89.pdf`, or kept here only? We have not touched L89.
- Q4. t393u: by “Frobenius-squared” did you mean base change (geometric, $\alpha \mapsto \alpha^2$) or the substitution $T \mapsto T^2$? They agree for $\text{Spec } \mathbb{Z}$ but not for curves (Prop. 5.2(ii)).
- Q5. Please paste the verbatim t391u and t393u texts; the corrections below use the brief’s paraphrase.

8. CORRECTIONS TO CHAT CLAIMS AND TO EARLIER LEMMAS

Paraphrased from the brief (Q5).

L91.C1 (t391u: “ $\Gamma_{\mathbb{R}}$ is Tate’s local integral at ∞ , so ξ already includes ∞)”):

Correct (§4(1)).

L91.C2 (t391u: “compactified $\text{Spec } \mathbb{Z}$ adds ∞ ; Durov’s $\mathbb{Z}_{\infty}$ too”):

Correct (§4(2)). Durov: $\mathbb{Z} \otimes \mathbb{Z}_{\infty} = \mathbb{R}$ and $\deg$ on $\widehat{\text{Pic}(\text{Spec } \mathbb{Z})}$.

L91.C3 (t391u: “Tate’s functional equation is Riemann–Roch on compactified $\text{Spec } \mathbb{Z}$; vdG–S 2000”):

Correct, via Poisson summation (vdG–S Prop. 1, “special case of Tate’s Riemann–Roch”). Nuance: there is no intrinsic h^1 (defined only by duality). Journal data (Selecta Math. 2000) [BIB]; arXiv text opened.

L91.C4 (t391u: “a conic/quaternion algebra is ramified at an even number of places including ∞)”):

Correct, reading “including” as “ ∞ counts as a place”. ∞ need not be ramified: $H(-1, 3)$ is ramified at $\{2, 3\}$. Hasse–Minkowski for conics: correct (Milne CFT VIII 3.5).

L91.C5 (t391u: “adding ∞ gives a complete curve, not a surface”):

Correct ([ANALOGY]; §4(5)).

L91.C6 (t393u (1)):

Correct (Thm. 5.1).

L91.C7 (t393u (2): “ $Z(T) \mapsto Z(T^2)$ relates to base change; Frobenius-squared/Adams, not $C \times C$)”):

Correct with one precision: base change is $Z_{C/\mathbb{F}_{q^2}}(T^2) = Z(T)Z(-T)$, which is *not* $Z(T^2)$ for curves. The two coincide for $\text{Spec } \mathbb{Z}$ (all local eigenvalues 1). “Not $C \times C$ ”: correct.

L91.C8 (t393u (3): “ ζ^2 counts pairs; the Künneth $H^1 \otimes H^1$ part is missing; that is where positivity lives”).:

Correct. Precisely: ζ^2 is the zeta of a disjoint union, and $\psi^2 = \text{Sym}^2 - \Lambda^2$ keeps only the diagonal α_i^2 . The missing piece is the off-diagonal $\alpha_i \alpha_j$, which contains the algebraic classes $\alpha_i \bar{\alpha}_i = q$ (Prop. 5.3).

L91.C9 (L89 N3 / Prop. D, [TAA-L])::

“ $n_- = 1$ per ghost” and “ $n_- \leq \kappa_\zeta$ ” hold for R -even test functions (L89’s class). On the full real class (Connes–Consani’s) the count is 2 per quartet $= \frac{1}{2}Z_{\text{off}}$, attained (X3; Prop. 3.1), matching L88 Thm. B. L89 is not wrong for its class; its broad-basis “0–2” fragility is the support threshold of X2.

L91.C10 (L90 Y1 wording “does Hodge-index inertia detect ghosts?”)::

Yes, but only above a support threshold that grows like $\log \gamma$, and the detection is Weil’s criterion restated (§3).

L91.C11 (self-correction, this chapter)::

A preliminary scan aliased zeros above the grid’s Nyquist frequency and suggested $a_{\min} \propto 1/\delta$. It was replaced (§2); no number from it is used.

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TAA LEMMA 92: CERTIFIED SEMI-LOCAL POSITIVITY ON SPLINE SPACES, THE GHOST-DETECTION THRESHOLD $a^*(\gamma)$, AND CALIBRATION ON CURVES OVER $\mathbb{F}_q$

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis. A positivity certificate on a finite-dimensional space of test functions is *not* RH, and not even Weil positivity at a single support size. Every claim carries one label: [THEOREM], [RESTATEMENT], [ANALOGY], [CONJECTURE], [OPEN], [NUMERICAL], [HEURISTIC]; sources are marked [CITED] (opened), [CITED-sec] (via an opened secondary source), [BIB] (bibliographic only), [TAA-L] (earlier TAA lemma). All references are listed in `lit/lemma92/verified_refs.md`.

ABSTRACT. This chapter carries out the three directions Z1–Z3 of L91.

Z1. We compute Weil’s quadratic form Q from the prime side, with no zeros used, in rigorous ball arithmetic (Arb). The test functions are cubic B-splines supported in $[e^{-a}, e^a]$. For support size a the prime sum is finite ($p^k \leq e^{2a}$), so there is no truncation tail; every other term is a one-dimensional integral over $[0, 2a]$ that is evaluated rigorously. On spline spaces of spacing $h \approx 0.1$ (dimension up to $\approx 2a/h$) we certify $Q > 0$, with a rigorous two-sided bracket on the smallest eigenvalue, for every a in a grid up to $a = 8$. Refinement at fixed a shows the smallest eigenvalue falling without saturation. This is consistent with the doubly exponentially small infimum predicted by Connes–Consani’s prolate analysis. We explain why such certificates cannot extend to all test functions without a tail (closure) argument. The published rigorous result for *all* test functions is still Yoshida’s $a = \frac{1}{2} \log 2$; an unreferenced 2026 computer-assisted claim reaches $a = \frac{1}{2} \log 19$.

Z2. Insert a hypothetical extra zero quartet (“ghost”) $\frac{1}{2} \pm \delta \pm i\gamma$. We prove that the detection set in a is upward closed. We also give an exact 4×4 inertia criterion (Haynsworth) on any finite test space and a small- δ law: the violation is $-4\delta^2 |F'(\gamma)|^2$ on $\{F(\gamma) = 0\}$, and the threshold solves $4\delta^2 \kappa(a, \gamma) = 1$. We give *rigorous* certificates that the actual Weil form plus a ghost is negative at explicit (γ, δ, a) , e.g. the Davenport–Heilbronn point $\gamma = 85.699$, $\delta = 0.3085$ at $a = 7/5$. Numerically, the L91 offset $a_{\min} - \frac{1}{2} \log(\gamma/2\pi) \approx 0.4$ – 0.6 was mostly an artefact of the basis. With finer test spaces it drops to 0.05 – 0.11 for $\delta = 0.1$ at the finest spacing tested, and it is still decreasing. The statement $a^*(\gamma) = \frac{1}{2} \log(\gamma/2\pi) + O(1)$ is *not* proved; we give the density heuristic (Landau) and the evidence.

Z3. Over $\mathbb{F}_q$, where RH is a theorem, the same finite-support test never fails on 50 true curves. For a weak ghost we *prove* that the threshold is exactly $N^* = g$ Frobenius steps (support $a^* = g \log q$, i.e. $\pi \times$ the zero density), independent of q ; the numerics agree in 35/35 cases. Stronger ghosts are seen earlier. For large support the number of negative directions equals $k = \#\{|\alpha|^2 > q\} = \frac{1}{2} Z_{\text{off}}(P)$ for all 982 non-degenerate L88 fakes. The answer to “log q or g ” is: N^* scales with g ; a^* scales with $g \log q$, and both are the same density law as $\frac{1}{2} \log(\gamma/2\pi) = \pi D(\gamma)$ for ζ .

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Date: 24 October 2026 (t394u: “Lets work on z1 through z3.”)

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GLOSSARY (BRIEF RESTATEMENTS FOR J.W.)

Test function, support a ::

A function ψ on the line $t = \log u$ vanishing outside $[-a, a]$ (so $u \in [e^{-a}, e^a]$; CC's $\lambda = e^a$). Fourier/Mellin transform $F(r) = \int \psi(t) e^{irt} dt$. A zero $\rho = \frac{1}{2} + ir$ contributes $F(r)F(-r)$, i.e. $|F(\gamma)|^2$ when ρ is on the line.

Weil form Q ::

$Q(\psi) = \sum_{\rho} F(r_{\rho})F(-r_{\rho})$ on $V_0 = \{F(\pm i/2) = 0\}$ (the codimension-2 condition that removes the poles at $s = 0, 1$). RH $\Leftrightarrow Q \geq 0$ for all a and all ψ (Weil). In L91 notation $Q = -2s(f, f) = -2D \bullet D$.

Prime side::

The explicit formula rewrites Q as primes plus an archimedean integral, with no zeros (§1). This is what we compute rigorously.

Ball arithmetic (Arb)::

Every number is an interval $[m \pm r]$ that provably contains the true value; a “certified” inequality holds for every point of the interval.

B-spline space $S(a, n)$::

Span of n shifted copies of a cubic B-spline (four boxes of width h convolved) inside $[-a, a]$; $h = 2a/(n + 3)$.

Ghost quartet, $a^*(\gamma, \delta)$::

A hypothetical extra zero set $\{\frac{1}{2} \pm \delta \pm i\gamma\}$ added to the zero side. a^* is the smallest support at which the form with the ghost becomes negative somewhere: the “detection threshold”.

$D(\gamma)$, πD ::

Mean density of zeros near height γ : $D = \frac{1}{2\pi} \log \frac{\gamma}{2\pi}$, so $\pi D = \frac{1}{2} \log \frac{\gamma}{2\pi}$.

Curves over $\mathbb{F}_q$; N , N^* ::

Test functions are finite sequences $\varphi(n)$, $|n| \leq N$ (support $a = N \log q$). N^* is the smallest N at which a ghost is detected.

SUMMARY OF VERDICTS

	What is established	What is not
Z1	[NUMERICAL] (rigorous, computer-assisted): $Q > 0$ on explicit spline spaces for all grid $a \leq 8$, with rigorous eigenvalue brackets; prime side matches the zero side to $\leq 3 \cdot 10^{-9}$.	Positivity for <i>all</i> test functions at any $a > \frac{1}{2} \log 2$ (needs a tail/closure bound). Not RH.
Z2	[THEOREM]: monotonicity, 4×4 inertia criterion, small- δ law. [NUMERICAL] rigorous: ghost-detection certificates (upper bounds on a^*). [NUMERICAL]: offset $\rightarrow$ small under refinement.	$a^*(\gamma) = \frac{1}{2} \log \frac{\gamma}{2\pi} + O(1)$ ([CONJECTURE]; heuristic via Landau density).
Z3	[THEOREM]: weak-ghost threshold $N^* = g$ (replacement) and $g + 2$ (added); kernel formula; [NUMERICAL]: 50 true curves no violation; $n_- = k$ on 982/982 non-degenerate fakes.	A formula for N^* for strong ghosts ([OPEN]).

1. SETUP: THE WEIL FORM FROM THE PRIME SIDE

Write $\psi(t) = e^{t/2} \phi(e^t)$ (unitary normalization), $k = \psi \star \psi$ ($k(x) = \int \psi(t) \psi(t+x) dt$, even, supported in $[-2a, 2a]$), and $F(r) = \int \psi e^{irt} dt$, so that $\widehat{k}(r) = |F(r)|^2$ for real ψ .

Proposition 1.1 (explicit formula, finite prime side; [THEOREM], classical, normalization checked in L91 Thm 1.1 [TAA-L]). *For $\psi \in V_0(a) = \{\psi \text{ supported in } [-a, a], F(\pm i/2) = 0\}$,*

$$Q(\psi) := \sum_{\rho} F(r_{\rho}) F(-r_{\rho}) = -2N(k),$$

$$N(k) = \sum_{n \leq e^{2a}} \frac{\Lambda(n)}{\sqrt{n}} k(\log n) + \int_0^{2a} \frac{e^{3x/2} k(x) - k(0)}{e^{2x} - 1} dx + k(0) \left(\frac{1}{2} \log(1 - e^{-4a}) + C \right),$$

$C = \frac{1}{2}(\log \pi + \gamma_E)$, where $\rho = \frac{1}{2} + ir_{\rho}$ runs over the nontrivial zeros with multiplicity. The prime sum is finite because $k(\log n) = 0$ for $n > e^{2a}$; there is no tail to bound. Equivalently, in frequency form, $Q = \frac{1}{2\pi} \int |F(r)|^2 (\operatorname{Re} \psi_{\Gamma}(\frac{1}{4} + \frac{ir}{2}) - \log \pi) dr - 2 \sum_n \Lambda(n) n^{-1/2} k(\log n)$ (checked numerically, [NUMERICAL]).

RH $\Leftrightarrow Q \geq 0$ on $V_0(a)$ for every a (Weil; [RESTATEMENT], CC [1] (12), (14)). In L91's notation $Q = -2s(f, f)$.

Basis and exact arithmetic. For rational a and $n \geq 1$, put $h = 2a/(n+3)$, $c_i = -a + 2h + ih$ ($0 \leq i < n$), $\psi_i = \mathbf{1}_{[0,h]}^4(\cdot - c_i + 2h)$ (centered cubic B-spline of width $4h$), so $F_i(r) = e^{ir c_i} (2 \sin(rh/2)/r)^4$. The matrix of Q is Toeplitz, $Q_{ij} = Q_{|i-j|}$, because $k_{ij}(x) = h^7 p_{|i-j|}(x/h)$ with p_L a symmetrized cardinal B-spline of order 8. That spline is piecewise polynomial with *exact rational* coefficients (fmpq). Thus:

- prime terms: $p_L(\log n/h)$ evaluated in Arb for every prime power $n \leq e^{2a}$ (15040 of them at $a = 6$);
- archimedean integral: on each cell p_L is a polynomial, so $\int \frac{e^{3x/2} x^j}{e^{2x} - 1} dx$ -type moments are needed. We compute them once per cell (shared across lags L) by rigorous series. The removable singularity at $x = 0$ in cell 0 is handled through ${}_1F_1(1; 2; z)$;
- V_0 : an exact (Arb) basis Z of $\{x : \sum_i x_i e^{\pm c_i/2} = 0\}$; reduced matrices $M = Z^T Q Z$, $G_0 = Z^T G Z$ ($G = L^2$ Gram).

Certification of positivity. A naive Cholesky in Arb fails at $a \geq 3.5$ through interval wrapping, even when M is very positive. We therefore use a preconditioned congruence. Take $X = L^{-1}$ from a high-precision floating Cholesky $M \approx LL^T$ (any matrix X is allowed). Form $B = XMX^T$ in Arb and check that B is strictly diagonally dominant with positive diagonal (Gershgorin). Since B is congruent to M , this proves $M \succ 0$ ([THEOREM], Sylvester). For the smallest generalized eigenvalue $\mu_{\min}$ of (M, G_0) :

- rigorous lower bound: certify $M - \mu_{\text{lo}}G_0 \succ 0$;
- rigorous upper bound: an exact Rayleigh quotient $x^T M x / x^T G_0 x$ with a binary-rational x .

Cross-check against the zero side (2500 zeros, mpmath) on the certified minimizers ([NUMERICAL]):

a	n	μ prime side	μ zero side (2500 zeros)	rel. diff.
1	17	$1.254311 \cdot 10^{-7}$	$1.254311 \cdot 10^{-7}$	$4.3 \cdot 10^{-13}$
2	37	$1.940169 \cdot 10^{-10}$	$1.940169 \cdot 10^{-10}$	$9.6 \cdot 10^{-12}$
3	57	$9.641082 \cdot 10^{-12}$	$9.641082 \cdot 10^{-12}$	$1.7 \cdot 10^{-11}$
1	40	$3.008759 \cdot 10^{-10}$	$3.008759 \cdot 10^{-10}$	$1.0 \cdot 10^{-10}$
6/5	80	$2.041037 \cdot 10^{-12}$	$2.041037 \cdot 10^{-12}$	$2.7 \cdot 10^{-9}$

The differences are what one expects from omitting zeros beyond $\gamma_{2500} = 3031.3$.

2. Z1: CERTIFIED SEMI-LOCAL POSITIVITY BEYOND $a = \frac{1}{2} \log 2$

Theorem 2.1 (computer-assisted, rigorous; [NUMERICAL]). *Let a run over the grid of Table 1, with $S(a, n)$ the cubic-spline space described there ($h \approx 0.1$). Then the Weil form Q is positive definite on $S(a, n) \cap V_0$ (dimension $n - 2$). Its smallest eigenvalue relative to the L^2 norm satisfies $\mu_{\text{lo}} \leq \mu_{\min} \leq \mu_{\text{up}}$ with the certified values listed. The largest support certified is $a = 8$ ($\lambda = e^a$, $\lambda^2 = e^{2a}$; all prime powers up to e^{2a} enter).*

What is certified, exactly. Space: the span of the n splines ψ_i of §1 with the exact rational c_i, h , intersected with V_0 through the Arb basis Z . *Errors:* every entry of M is an Arb ball; the largest radius over the whole grid is $\leq 2 \cdot 10^{-70}$ (256–320-bit precision). There is no truncation error, because the prime sum is finite and the moments are enclosed rigorously. *Inference:* congruence + Gershgorin as in §1. Timings are a few seconds to a few minutes on one core. Scripts: `num/lemma92_z1lib.py`, `lemma92_z1run.py`; data `num/192/z1_scanA.json`, `z1_scanC.json`. $\square$

Refinement ([NUMERICAL], rigorous at each entry). At fixed a the certified μ_{lo} keeps falling as the spline spacing shrinks. Each entry is again a certified positive lower bound:

$a \backslash n$	10	20	40	80	160
1/2	$1.14 \cdot 10^{-1}$	$5.06 \cdot 10^{-2}$	$2.79 \cdot 10^{-2}$	$1.91 \cdot 10^{-2}$	$1.53 \cdot 10^{-2}$
1	$1.06 \cdot 10^{-5}$	$4.00 \cdot 10^{-8}$	$3.01 \cdot 10^{-10}$	$3.01 \cdot 10^{-12}$	$2.32 \cdot 10^{-14}$
6/5	$3.25 \cdot 10^{-7}$	$5.80 \cdot 10^{-8}$	$3.89 \cdot 10^{-10}$	$2.04 \cdot 10^{-12}$	$1.33 \cdot 10^{-14}$
3/2	$1.13 \cdot 10^{-7}$	$3.06 \cdot 10^{-8}$	$2.98 \cdot 10^{-10}$	$1.95 \cdot 10^{-12}$	$7.51 \cdot 10^{-15}$

At $a = 1$ it falls by roughly two orders of magnitude per doubling of n , with no saturation by $n = 160$. Connes–Consani explain why: for $\lambda = e^a$ the form has many eigenvalues that are extremely small, coming from prolate spheroidal functions. In their numerics the smallest is $2.389 \cdot 10^{-48}$ at $\mu = \lambda^2 = 11$ ($a = \frac{1}{2} \log 11 \approx 1.20$). The number of tiny eigenvalues grows roughly like μ , and the eigenvectors are prolate vectors (Slepian–Pollak), of which there are $\approx 2\lambda^2$ ([3] §§2.5, 3, [CITED]). Kastoris’s bound [4] [CITED] has the scale $d_c \propto c^{9/2} e^{-4\pi c}$ with $c = \lambda^2$. A cubic-spline space of spacing 0.1 does not contain these near-radical vectors, which is why our $\mu_{\min}$ is so much larger. The normalizations of the three computations differ, so only the qualitative comparison is meaningful.

TABLE 1. Z1: certified positivity on $S(a, n) \cap V_0$ (cubic splines, $h \approx 0.1$). μ = smallest eigenvalue of Q relative to $\|\psi\|_2^2$; both bounds are rigorous.

a	n	h	$\#\{p^k \leq e^{2a}\}$	PD cert.	μ_{lo} (rig.)	μ_{up} (rig.)	time (s)
1/4	6	0.0556	1	yes	$1.113 \cdot 10^0$	$1.114 \cdot 10^0$	0.0
1/3	6	0.0741	1	yes	$8.251 \cdot 10^{-1}$	$8.259 \cdot 10^{-1}$	0.0
1/2	7	0.1000	2	yes	$2.505 \cdot 10^{-1}$	$2.507 \cdot 10^{-1}$	0.0
3/4	12	0.1000	4	yes	$2.021 \cdot 10^{-6}$	$2.023 \cdot 10^{-6}$	0.1
1	17	0.1000	6	yes	$1.253 \cdot 10^{-7}$	$1.254 \cdot 10^{-7}$	0.1
5/4	22	0.1000	9	yes	$2.366 \cdot 10^{-8}$	$2.368 \cdot 10^{-8}$	0.2
3/2	27	0.1000	12	yes	$1.131 \cdot 10^{-9}$	$1.132 \cdot 10^{-9}$	0.4
7/4	32	0.1000	18	yes	$6.458 \cdot 10^{-10}$	$6.465 \cdot 10^{-10}$	0.7
2	37	0.1000	24	yes	$1.938 \cdot 10^{-10}$	$1.940 \cdot 10^{-10}$	1.0
5/2	47	0.1000	48	yes	$3.522 \cdot 10^{-11}$	$3.525 \cdot 10^{-11}$	1.9
3	57	0.1000	98	yes	$9.631 \cdot 10^{-12}$	$9.641 \cdot 10^{-12}$	3.3
7/2	67	0.1000	210	yes	$1.017 \cdot 10^{-12}$	$1.018 \cdot 10^{-12}$	5.3
4	77	0.1000	465	yes	$3.031 \cdot 10^{-13}$	$3.034 \cdot 10^{-13}$	8.1
9/2	87	0.1000	1068	yes	$2.029 \cdot 10^{-14}$	$2.031 \cdot 10^{-14}$	12.0
5	97	0.1000	2533	yes	$2.211 \cdot 10^{-15}$	$2.214 \cdot 10^{-15}$	16.6
11/2	107	0.1000	6139	yes	$4.733 \cdot 10^{-16}$	$4.738 \cdot 10^{-16}$	24.2
6	117	0.1000	15040	yes	$2.510 \cdot 10^{-16}$	$2.512 \cdot 10^{-16}$	31.2
13/2	127	0.1000	37308	yes	$6.781 \cdot 10^{-17}$	$6.788 \cdot 10^{-17}$	67.8
7	137	0.1000	93371	yes	$9.349 \cdot 10^{-18}$	$9.358 \cdot 10^{-18}$	98.0
15/2	147	0.1000	235220	yes	$3.419 \cdot 10^{-18}$	$3.422 \cdot 10^{-18}$	117.4
8	157	0.1000	595877	yes	$1.324 \cdot 10^{-18}$	$1.326 \cdot 10^{-18}$	245.6

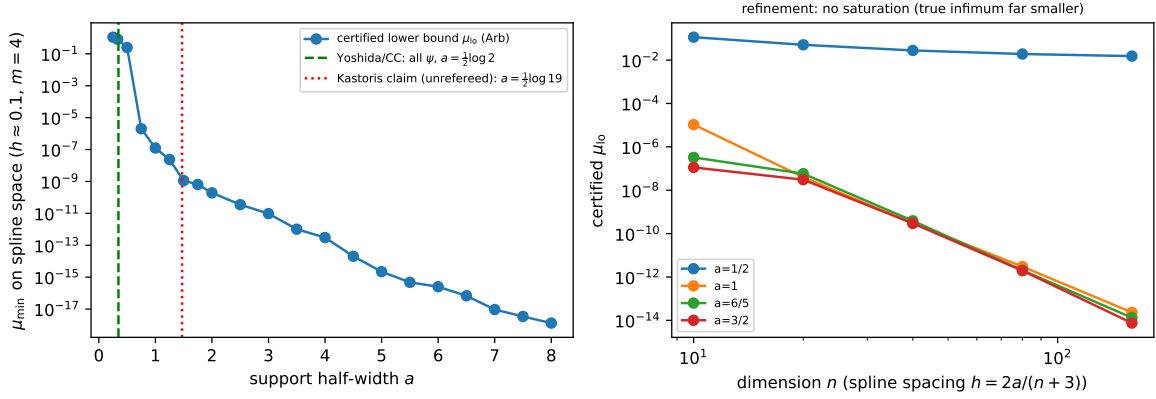


FIGURE 1. Left: certified μ_{lo} vs a on the $h \approx 0.1$ spaces, with Yoshida's $a = \frac{1}{2} \log 2$ and Kastoris's claimed $a = \frac{1}{2} \log 19$. Right: refinement at fixed a .

Why this does not extend to all test functions ([THEOREM]).. (i) Each certificate concerns a finite-dimensional subspace. Positivity of Q on all of $V_0(a)$ is a statement about an infinite-dimensional space. (ii) It cannot be reached by refinement alone. The infimum over $V_0(a)$ is attained (Bombieri [5] [CITED-sec]) but is tiny (previous paragraph). Proving that it is positive requires a bound on the *complement* of the finite space, i.e. a lower bound for Q on high-frequency/fine-scale functions plus control of the coupling. This is what Kastoris does with a multiscale Feshbach–Schur reduction [4]. (iii) Even positivity for all ψ at one a is not RH: RH is positivity for *all* a . By nesting, a positivity result at a covers all smaller a and nothing larger.

Why finite certificates were bound to succeed ([THEOREM]conditional / [HEURISTIC]).. If RH holds, $Q(\psi) = \sum_{\gamma} |F(\gamma)|^2 > 0$ for every $\psi \neq 0$. Equality would force the entire function F of exponential type a to vanish at all zeros, which is impossible because $N(T) \sim \frac{T}{2\pi} \log T$ grows faster than the zero count allowed for type a . This is CC's “no radical” argument [3] [CITED]. Unconditionally, a finite certificate is a *consistency check*: a single certified negative value would disprove RH, and no positive certificate proves it.

Comparison with the literature (honest).

- *Yoshida* [6] ([CITED-sec], as reported in [3]): positivity for *all* test functions with support in $[\lambda^{-1}, \lambda]$, $\lambda = \sqrt{2}$ ($a = \frac{1}{2} \log 2 \approx 0.347$), by numerical analysis. Bombieri [5] ([CITED-sec], abstract) reproves positivity for small support and shows the minimum is attained.
- *Connes–Consani* [3] ([CITED]): numerics (§2.2–2.5) show that the archimedean part alone loses positivity just beyond $L = 2 \log \lambda = \log 2$ (i.e. $a = \frac{1}{2} \log 2$), that adding the prime 2 restores it, and that replacing 2 by a real p outside an interval of size $< 10^{-3}$ destroys it. They find many tiny eigenvalues (prolate). They do not claim a rigorous result beyond Yoshida's.
- *Kastoris* (Zenodo, 12 Aug 2026, **unrefereed**) [4] ([CITED], pdf read): Thm 5.1 claims, computer-assisted with Arb/python-flint and multiscale Feshbach–Schur, $\inf \text{spec}(A_{\sqrt{c}}) > 5d_c$ for $c = 14, 17, 19$ on the continuum semilocal operator, both parities. By nesting, $c = 19$ would give all $a \leq \frac{1}{2} \log 19 \approx 1.47$. We have not checked the proof.
- *Connes–Consani, archimedean place* [2] ([CITED], opened in L91): Theorem 1 gives a trace-formula lower bound for the archimedean term for supports in $[2^{-1/2}, 2^{1/2}]$ (again $a = \frac{1}{2} \log 2$), a conceptual reason for positivity in Yoshida's window.
- *This chapter*: certified positivity on explicit finite spline spaces up to $a = 8$. This is *weaker* than all of the above in kind (finite spaces only) and further out in a . It is new only as a rigorous, reproducible prime-side computation at large a with exact control of every term. We do not claim novelty for the method.

3. Z2: THE GHOST-DETECTION THRESHOLD $a^*(\gamma, \delta)$

3.1. Definitions. A *ghost* is the quartet $r \in \{\pm\gamma \pm i\delta\}$, i.e. $\rho = \frac{1}{2} \pm \delta \pm i\gamma$. It is *added* to the actual zeros (we never assume anything about the actual zeros except through the prime side). Its contribution to the zero side is

$$G_{\gamma, \delta}(\psi) = \sum_{r \in \{\pm\gamma \pm i\delta\}} F(r)F(-r) = 4 \operatorname{Re}(u\bar{v}), \quad u = F(\gamma + i\delta), \quad v = F(\gamma - i\delta)$$

(real ψ , so $F(-\bar{z}) = \overline{F(z)}$). For a space $E \subset V_0$ of test functions, the ghost is *detected on E* if $Q + G_{\gamma, \delta}$ takes a negative value on E . Set $a^*(\gamma, \delta) = \inf\{a : \text{detected on } V_0(a)\}$.

3.2. Exact statements.

Theorem 3.1 ([THEOREM]). (i) If $E \subset E'$, detection on E implies detection on E' . Since $V_0(a) \subset V_0(a')$ for $a < a'$, the detection set $\{a\}$ is an up-ray $[a^*, \infty)$ or (a^*, ∞) . (ii) (Inertia criterion.) Let E be finite-dimensional with $Q|_E \succ 0$; Theorem 2.1 certifies this for our spline spaces. Write $u = \langle a_1, x \rangle + i\langle a_2, x \rangle$, $v = \langle b_1, x \rangle + i\langle b_2, x \rangle$, $U = [a_1 \ a_2 \ b_1 \ b_2]$ and $K = U^T Q^{-1} U$ (a 4×4 positive semidefinite Gram matrix). Then

$$n_-(Q + G_{\gamma, \delta})|_E = n_+ \left(\frac{1}{2} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} + K \right) - 2 \in \{0, 1, 2\}.$$

So detection is decided by a 4×4 matrix, and one quartet produces at most two negative directions. (iii) (Small- δ law.) Uniformly on bounded subsets of a finite-dimensional E ,

$$G_{\gamma, \delta} = 4|F(\gamma)|^2 - 4\delta^2(|F'(\gamma)|^2 + \operatorname{Re} \overline{F(\gamma)} F''(\gamma)) + O(\delta^4),$$

and on $\{F(\gamma) = 0\}$ simply $G = -4\delta^2|F'(\gamma)|^2 + O(\delta^4)$. Hence, with $B = Q + 4|F(\gamma)|^2$ and G_2 the δ^2 -coefficient,

$$\text{detected on } E \iff \delta^2 \lambda_{\max}(B^{-1}G_2) > 1 + O(\delta^2), \quad \kappa(E, \gamma) := \max_{x \in E} \frac{4|F'(\gamma)|^2}{Q(x) + 4|F(\gamma)|^2}.$$

To leading order, the threshold on E solves $4\delta^2\kappa = 1$.

Proof. (i) is immediate. (ii) Since $G = 4(\langle a_1, x \rangle \langle b_1, x \rangle + \langle a_2, x \rangle \langle b_2, x \rangle) = x^T U S U^T x$ with $S = 2 \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, apply Haynsworth’s inertia additivity [8] to $H = \begin{pmatrix} Q & U \\ U^T & -S^{-1} \end{pmatrix}$ in both orders: $\text{In}(Q) + \text{In}(-S^{-1} - K) = \text{In}(-S^{-1}) + \text{In}(Q + U S U^T)$. With $Q \succ 0$ and $n_-(-S^{-1}) = 2$ this gives the formula. Moreover $S^{-1} + K \succeq S^{-1}$ has $n_+ \geq 2$, which gives the range. (iii) Taylor-expand $u = F + i\delta F' - \frac{\delta^2}{2}F'' - \frac{i\delta^3}{6}F''' + \dots$ and v likewise with $\delta \rightarrow -\delta$. The δ and δ^3 terms of $u\bar{v}$ are purely imaginary. The criterion is the Rayleigh quotient for $B - \delta^2 G_2$ with $B \succ 0$. $\square$

Part (ii) is the finite-dimensional, two-sided version of L91 Prop. 1.3 ([TAA-L]; $n_+ \leq 2$ per quartet there, with L91’s opposite sign). Part (iii) fixes the normalization of L91’s “gain $2\delta^2|A'|^2$ ”: L91’s s is $-\frac{1}{2}Q$.

Theorem 3.2 (rigorous certificates, upper bounds for a^* ; [NUMERICAL] computer-assisted). *For each row of Table 2 there is an explicit ψ in the cubic spline space $S(a, n) \cap V_0$ with $Q(\psi) + G_{\gamma, \delta}(\psi) < 0$. Here Q is the actual Weil form computed from the prime side in Arb (§1), G is evaluated in Arb (complex balls), and ψ has binary-rational coefficients mapped exactly into V_0 by Z . Consequently $a^*(\gamma, \delta) \leq a$: if ζ had an extra zero quartet at (γ, δ) , the support- a Weil test would detect it.*

TABLE 2. Certified ghost detections (first a on a 0.05-grid at which the floating search found a candidate that Arb then certified). $Q_{\text{true}}(\psi) > 0$ is also certified in every row. The search is limited by double precision, so these are upper bounds, not thresholds.

γ	δ	$\frac{1}{2} \log \frac{\gamma}{2\pi}$	certified a	n	h	$Q_{\text{true}}(\psi) + G_{\gamma, \delta}(\psi)$ (Arb ball)
39.3000	0.3085	0.917	11/10	52	0.0400	$\leq -1.052 \cdot 10^{-15}$
39.3000	0.1	0.917	5/4	60	0.0397	$\leq -3.472 \cdot 10^{-16}$
85.6993	0.3085	1.306	31/20	156	0.0195	$\leq -6.991 \cdot 10^{-16}$
85.6993	0.1	1.306	37/20	186	0.0196	$\leq -1.178 \cdot 10^{-15}$
85.6993	0.3085	1.306	7/5	283	0.0098	$\leq -6.362 \cdot 10^{-21}$
85.6993	0.1	1.306	29/20	293	0.0098	$\leq -2.004 \cdot 10^{-20}$
85.6993	0.02	1.306	3/2	304	0.0098	$\leq -2.720 \cdot 10^{-22}$
120.0000	0.3085	1.475	33/20	233	0.0140	$\leq -3.599 \cdot 10^{-18}$
120.0000	0.1	1.475	37/20	262	0.0140	$\leq -9.047 \cdot 10^{-17}$
200.3000	0.3085	1.731	37/20	438	0.0084	$\leq -2.819 \cdot 10^{-17}$
200.3000	0.1	1.731	2	474	0.0084	$\leq -1.828 \cdot 10^{-17}$

The Davenport–Heilbronn point ($\gamma = 85.699$, $\delta = 0.3085$) is certified detectable at $a = 7/5$. The weak ghost $\delta = 0.02$ at the same height is certified at $a = 3/2$. These are rigorous statements about the true Weil form. With the coarser search ($h \approx 0.02$) the same height needs $a = 31/20$ and $37/20$ for $\delta = 0.3085, 0.1$; the gap shows that the certificates are search-limited, not threshold-limited.

3.3. Numerics: the threshold under refinement.

Method ([NUMERICAL]). We use modulated cubic splines $B_h(t - c_k) \cos \gamma t$, $B_h(t - c_k) \sin \gamma t$ (spacing h , centres filling $[-a, a]$). These span a frequency band of width $\sim 2\pi/h$ around $\pm\gamma$; we add low-frequency splines of spacing 0.25. F is in closed form, so the zero-side matrix $\sum_{j \leq 2500} |F(\gamma_j)|^2$ has no quadrature or aliasing error; the only error is the omission of zeros beyond 3031.3. Detection means $\lambda_{\min}(Q + G) < -10^{-13} \lambda_{\max}(Q)$ on V_0 . We bisect in a (11 steps,

resolution $\approx 10^{-3}$), assuming the up-ray structure of Theorem 3.1(i). Spline spaces at different a are not exactly nested, so this is checked only through the agreement with the certificates. Scripts `num/lemma92_z2{lib,run,run2,kappa,cert,certscan}.py`; data `num/192/z2_*.json`.

TABLE 3. $a_{\min}(\gamma, \delta)$ on modulated-spline spaces of spacing h ; n_5 = number of true zeros within ± 5 of γ .

γ	$\frac{1}{2} \log \frac{\gamma}{2\pi}$	n_5	$\delta = 0.02$			$\delta = 0.1$			$\delta = 0.3085$		
			$h=.025$	$.0125$	$.00625$	$.025$	$.0125$	$.00625$	$.025$	$.0125$	$.00625$
39.30	0.917	3	1.149	1.082	1.044	1.062	1.021	0.984	1.038	0.981	0.935
60.00	1.128	3	1.387	1.330	1.284	1.338	1.280	1.236	1.296	1.241	1.192
85.70	1.306	4	1.523	1.473	1.424	1.486	1.434	1.391	1.452	1.400	1.358
120.00	1.475	5	1.672	1.626	1.585	1.637	1.591	1.551	1.608	1.563	1.524
200.30	1.731	5	1.898	1.862	1.829	1.871	1.835	1.803	1.844	1.802	1.768
300.00	1.933	6	2.146	2.077	2.048	2.025	1.999	1.984	1.955	1.920	1.893
500.00	2.188	7	2.382	2.310	2.285	2.331	2.288	2.265	2.225	2.193	2.172
800.00	2.423	8	2.616	2.539	2.508	2.565	2.512	2.493	2.440	2.414	2.395

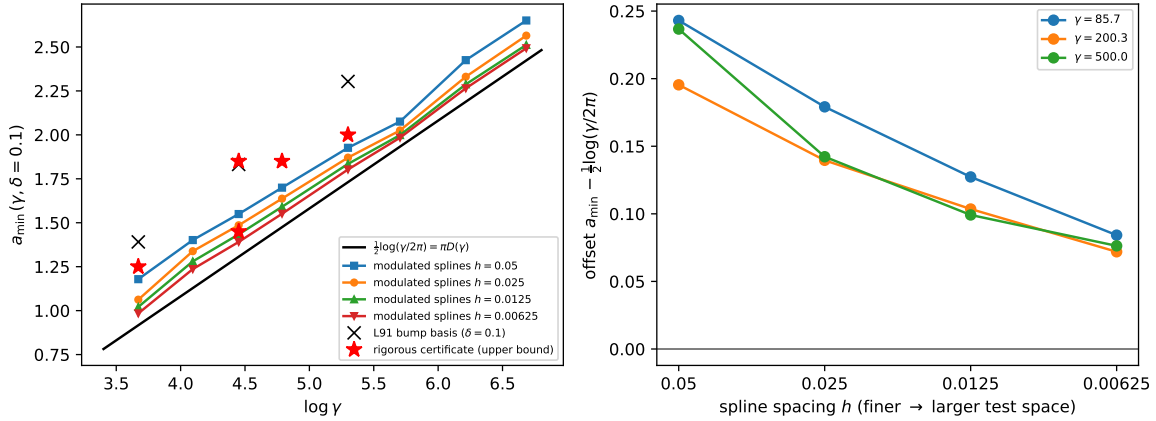


FIGURE 2. Left: $a_{\min}$ vs $\log \gamma$ ($\delta = 0.1$) for four spacings, the L91 values (bump basis) and the rigorous certificates; the line is $\frac{1}{2} \log(\gamma/2\pi)$. Right: offset vs h .

Findings ([NUMERICAL]).

- (1) *L91's offset was mostly a basis artefact.* With the L91 bump basis, $a_{\min} - \frac{1}{2} \log \frac{\gamma}{2\pi} \approx 0.4$ – 0.6 . With modulated splines the offset for $\delta = 0.1$ is 0.09 – 0.21 at $h = 0.025$, 0.07 – 0.15 at $h = 0.0125$ and 0.05 – 0.11 at $h = 0.00625$ ($\gamma \leq 300$). It shrinks by roughly 30% per halving of h (e.g. $0.179 \rightarrow 0.127 \rightarrow 0.084$ at the DH height), with no sign yet of a limit. For $\delta = 0.3085$ at $\gamma \geq 300$ it is already ≤ 0.04 , and slightly negative (-0.04 at $\gamma = 300$, $h = 0.00625$).
- (2) *Slope.* Across $\gamma \in [39, 800]$, $a_{\min}$ tracks $\frac{1}{2} \log \frac{\gamma}{2\pi}$ with least-squares slope 0.45 – 0.49 in $\log \gamma$ at every h and both δ (Fig. 2). L91's slope 0.56 came from the shrinking effective support of the bump envelope.
- (3) *δ -dependence is weak and explained.* From $\delta = 0.3085$ to 0.02 , $a_{\min}$ rises by only 0.06 – 0.16 . Theorem 3.1(iii) explains why: $\kappa(a, \gamma)$ stays of order 0.1 – 10 up to $a \approx \frac{1}{2} \log \frac{\gamma}{2\pi} + 0.1$ and then rises by four to six orders of magnitude within the next 0.15 – 0.3 in a ($h = 0.025$), so the level $1/(4\delta^2)$ is crossed at almost the same a for all δ . The measured thresholds fall on the κ -curves (Fig. 3), a quantitative check of the small- δ law even at $\delta = 0.3$.
- (4) The certificates (Table 2) lie 0.04 – 0.30 above the finest numerical thresholds, consistent with search-limited upper bounds; none contradicts them.

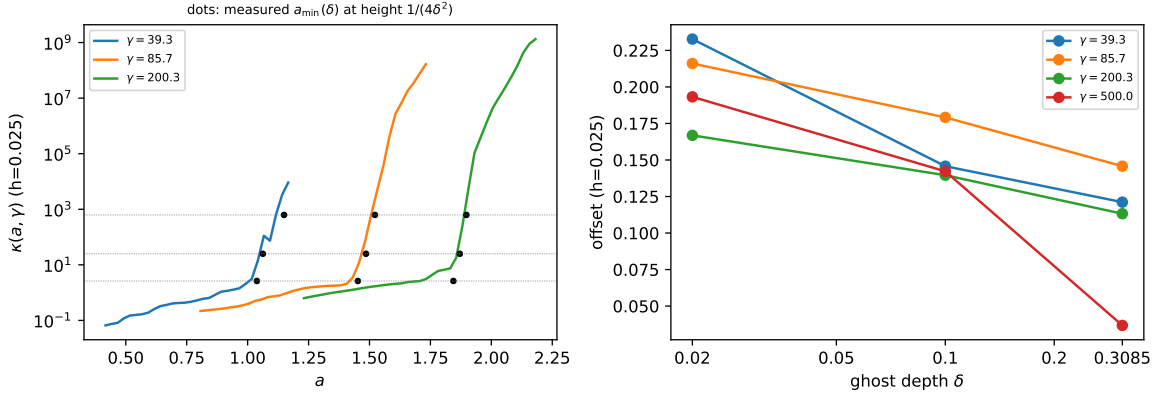


FIGURE 3. Left: $\kappa(a, \gamma)$ (Thm 3.1(iii), cross term dropped, $h = 0.025$), shown only where $Q + 4|F(\gamma)|^2$ is resolved above 10^{-13} relative. The dots are the measured $a_{\min}(\delta)$ placed at height $1/(4\delta^2)$; they lie on the curves. Right: offset vs δ .

3.4. What can be proved about $a^*(\gamma) = \frac{1}{2} \log \frac{\gamma}{2\pi} + O(1)$.

Proposition 3.3 (density heuristic; [HEURISTIC], ingredients [CITED]). *Detecting a weak ghost at γ requires some $F \in PW_a$ (F the transform of a ψ supported in $[-a, a]$) with $F(\gamma) = 0$, $F'(\gamma) \neq 0$, and $\sum_j |F(\gamma_j)|^2$ small compared with $\delta^2 |F'(\gamma)|^2$. That is an approximate interpolation problem on the set Λ of zeros near γ together with γ itself. For PW_a (spectrum in an interval of length $2a$), Landau [7] ([CITED], Thms 3–4) proves that a set of interpolation has $D^+(\Lambda) \leq 2a/2\pi = a/\pi$ and a set of sampling has $D^-(\Lambda) \geq a/\pi$. For a single interval the interpolation condition is also essentially sufficient (the theorem quoted there as Thm B). Near height γ the zeros have density $D(\gamma) = \frac{1}{2\pi} \log \frac{\gamma}{2\pi}$. Hence the heuristic threshold is $a^* \approx \pi D(\gamma) = \frac{1}{2} \log \frac{\gamma}{2\pi}$. This is exactly CC’s band edge: for support $\lambda = e^a$, CC Prop. 4.2 [3] ([CITED]) shows that the spectrum count of their prolate-perturbed Dirac operator matches $\langle N(E) \rangle$ at $E = 2\pi\lambda^2$, i.e. $a = \frac{1}{2} \log \frac{E}{2\pi}$. Ghosts at heights below the band edge $2\pi\lambda^2$ are the visible ones.*

Why this is not a proof (honest). Landau’s theorems are about *infinite*, uniformly discrete sets of *fixed* density and exact interpolation/sampling with constants. Our set has slowly drifting density ($\propto \log \gamma$) and no uniform separation. “Small” must be quantified against δ^2 , and the remaining zeros (all heights, and all zeros off the window, which are possibly off-line) enter Q with weights we cannot localize without RH-type input. We tried two routes:

- *Paley–Wiener/uncertainty (upper bound on a^*)*. Take $F = P_\Lambda(r) \cdot K(r)$, with P_Λ a polynomial vanishing on the zeros in $[\gamma - W, \gamma + W]$ and at γ , and K the transform of a smooth bump. Bounding the far-zero cost requires K to beat $|P_\Lambda| \sim W^{2WD}$. With Gevrey bumps this needs $a \gtrsim 2D \log W$, a *logarithmic* loss, so it gives only $a^* \leq C \log \gamma \cdot \log \log \gamma$ -type bounds ([HEURISTIC]; we did not write out the constants).
- *Beurling–Selberg majorants (lower bound on a^*)*. A lower bound needs $Q(\psi) \geq |G(\psi)|$ for all $\psi \in V_0(a)$. Minorizing Q by $\sum_{\text{zeros near } \gamma} |F(\gamma_j)|^2$ already uses RH for those zeros (verified numerically at these heights, but that is still an input). A Plancherel–Pólya/Bernstein local sampling bound $|F(\gamma \pm i\delta)|^2 \leq C(a, \delta) \sum_j |F(\gamma_j)|^2$ with $C < \frac{1}{4}$ would then be needed. Such bounds exist for uniformly dense separated sets (Beurling); making them local and explicit for zeta zeros is [OPEN].

Conjecture 3.4 ([CONJECTURE]). For each fixed $\delta \in (0, \frac{1}{2})$, $a^*(\gamma, \delta) - \frac{1}{2} \log \frac{\gamma}{2\pi} = O(1)$ as $\gamma \rightarrow \infty$. More strongly, the offset tends to 0 as $\gamma \rightarrow \infty$ ([OPEN]; the data cannot decide). If local zero statistics are GUE, fluctuations of the zero count in a window of L mean spacings are $O(\sqrt{\log L})$.

Heuristically this makes the density fluctuation over the effective window $o(D)$, so it supports $O(1)$ ([HEURISTIC]; conditional on GUE, not proved).

4. Z3: CALIBRATION ON CURVES OVER $\mathbb{F}_q$

4.1. Setting. Let $C/\mathbb{F}_q$ have genus g , with Frobenius eigenvalues $\alpha_1, \dots, \alpha_{2g}$ ($|\alpha_j| = \sqrt{q}$ by Weil's theorem). Write $\alpha_j = \sqrt{q} e^{i\theta_j}$. A test function is $\varphi : \mathbb{Z} \rightarrow \mathbb{R}$ supported in $[-N, N]$ (support $a = N \log q$), with $F(\theta) = \sum \varphi(n) e^{in\theta}$. The explicit formula gives ([THEOREM]; checked to 10^{-49} for $y^2 = x^5 + x + 1/\mathbb{F}_5$)

$$Q(\varphi) := \sum_j F(\theta_j) F(-\theta_j) = (2g - 2)k(0) - 2 \sum_{n \geq 1} k(n) q^{-n/2} N_n \quad \text{on } V_0 = \left\{ \sum \varphi(n) q^{\pm n/2} = 0 \right\},$$

with $k = \varphi \star \varphi$ and $N_n = \#C(\mathbb{F}_{q^n})$. For true curves $Q \geq 0$ (Weil). A *fake* is any palindromic set of $2g$ numbers α with $\alpha \mapsto q/\alpha$ symmetry, some of which are off the circle; then $\theta_j \notin \mathbb{R}$. The finite-support test is the inertia of Q on $V_0(N)$ ($2N - 1$ dimensions), computed in mpmath at 50 digits. Scripts `num/lemma92_z3.py`, `lemma92_z3run.py`; data `num/192/z3.json`.

4.2. Theorems.

Theorem 4.1 ([THEOREM]). *Let $z_1, \dots, z_r$ be the distinct values $e^{i\theta_j}$ and assume they are distinct from $q^{\pm 1/2}$.*

- (a) (True curves: kernel.) *If all α lie on the circle, $Q \geq 0$ on $V_0(N)$ and $\dim \ker Q|_{V_0(N)} = \max(0, 2N - 1 - r)$.*
- (b) (Weak ghost, replacement model.) *Replace two conjugate pairs of on-circle eigenvalues by the quartet $\theta_0 \pm i\varepsilon, -\theta_0 \pm i\varepsilon$ ($\varepsilon = \delta \log q$), keeping the other $2g - 4$ on the circle and distinct. Then there is $\varepsilon_1 > 0$ such that for $0 < \varepsilon < \varepsilon_1$ the ghost is detected iff $N \geq g$. Thus $N^* = g$, i.e. $a^* = g \log q$.*
- (c) (Weak ghost, added model.) *If the quartet is added to all $2g$ true eigenvalues, the same argument gives $N^* = g + 2$.*
- (d) (Large support.) *For N large enough that the evaluation functionals at all distinct z 's and at $q^{\pm 1/2}$ are independent ($2N + 1 \geq r + 2$), $n_-(Q|_{V_0(N)})$ equals the number of hyperbolic pairs $\{\alpha, q/\bar{\alpha}\}$ with $|\alpha| \neq \sqrt{q}$, i.e. $k = \#\{|\alpha|^2 > q\} = \frac{1}{2} Z_{\text{off}}(P)$, where $Z_{\text{off}}(P)$ is the number of roots of $P(T) = \prod (T - \alpha_j)$ off $|T| = \sqrt{q}$.*

Proof. $z^N F(\theta)$ is a polynomial of degree $\leq 2N$ in $z = e^{i\theta}$. Evaluation at $m \leq 2N + 1$ distinct points is therefore surjective onto $\mathbb{C}^m$ (Vandermonde), and its kernel has dimension $2N + 1 - m$. (a) $Q = \sum_j |F(\theta_j)|^2 \geq 0$. Its kernel on V_0 is the set of polynomials vanishing at the r points and at $q^{\pm 1/2}$; count $2N + 1 - (r + 2)$ (or 0). (b) Let $P_{\text{on}} = \sum_{\text{on}} |F(\theta_j)|^2$. As $\varepsilon \rightarrow 0$, $G_\varepsilon \rightarrow 4|F(\theta_0)|^2$ uniformly on the unit sphere of $V_0(N)$. If $N \leq g - 1$, then $2N + 1 \leq 2g - 1$, which is fewer than the $2g$ distinct points ($2g - 4$ on-circle, $e^{\pm i\theta_0}$, $q^{\pm 1/2}$). So $P_{\text{on}} + 4|F(\theta_0)|^2$ has trivial kernel on V_0 , is positive definite, and stays positive for small ε by compactness. If $N \geq g$, there is $x \neq 0$ in V_0 with F vanishing at all $2g$ points. For $N = g$ this F is, up to a constant, $z^{-g} \prod (z - z_i)$, which has *simple* zeros, so $F'(\theta_0) \neq 0$; for $N > g$ choose such an x . Then $Q(x) = G_\varepsilon(x) = -4\varepsilon^2 |F'(\theta_0)|^2 + O(\varepsilon^4) < 0$ (Theorem 3.1(iii), same expansion). (c) is identical with $2g + 4$ points. (d) The zero side is block-diagonal over orbits. On-circle orbits give rank-one positive terms. An off-circle pair $\{\theta, -\theta\}$ with $\theta \notin \mathbb{R}$ gives $F(\theta)F(-\theta) = \text{Re}$ of a product of two independent functionals, hence a hyperbolic block of signature $(1, 1)$ (a real- α pair) or $(2, 2)$ (a quartet, two pairs). With independent functionals the inertia adds (L89 Prop. D, L91 Prop. 1.3 [TAA-L]). $\square$

Remark 4.2 ([THEOREM], scaling answer). In the variable $r = \theta/\log q$, the curve has $2g$ zeros per period $2\pi/\log q$, i.e. density $D_C = g \log q/\pi$. Theorem 4.1(b) reads $a^* = g \log q = \pi D_C$, *exactly* the law $a^* \approx \pi D(\gamma) = \frac{1}{2} \log \frac{\gamma}{2\pi}$ conjectured for ζ . In Frobenius steps the threshold scales with g alone,

independent of q . In support units it scales with $g \log q$. The function-field case has no $O(1)$ offset in the weak-ghost limit: the Vandermonde count is exact. For ζ the analogue of the Vandermonde count is Landau’s density theorem, which is only asymptotic; this is where the $O(1)$ question lives.

4.3. Numerics ([NUMERICAL]).

- *True curves*: 50 hyperelliptic curves ($g = 1\text{--}4$, $p \in \{3, 5, 7, 11, 13\}$, seed 9292, char. polys from PARI `hyperellcharpoly`). All α are on the circle, there are 0 violations for $N \leq g + 3$, and the kernel formula (a) holds in 50/50.
- *Controlled ghosts*: for the 35 curves with $g \geq 2$, replace the two closest conjugate pairs by a quartet at their mean angle with $\delta \in \{0.02, 0.05, 0.1, 0.2, 0.3, 0.45\}$ (210 cases per model). At $\delta = 0.02$: $N^* = g$ in 35/35 (replacement) and $N^* = g + 2$ in 35/35 (added), as Theorem 4.1(b),(c) predict, for all q . Stronger ghosts are detected earlier ($N^* < g$; Table 4). In all 420 runs the largest N gives exactly $n_- = 2$ (one quartet = two hyperbolic pairs).
- *L88 integer fakes*: 1000 fakes regenerated with L88’s generator and seed 8888 ($g \in \{1, 2, 3\}$, $q \in \{2, 3, 4, 5, 7\}$; 10 non-squarefree skipped); k -histogram $\{0:147, 1:433, 2:411, 3:9\}$. n_- at $N = g + 3$ equals $k = \frac{1}{2}Z_{\text{off}}(P)$ in 982/1000. All 18 exceptions are *pole-degenerate*: they have eigenvalues with $|\alpha| \in \{1, q\}$, i.e. $e^{i\theta} = q^{\mp 1/2}$. These are exactly the points that V_0 annihilates, so the ghost is invisible by construction; in the case examined the form vanishes identically on V_0 (largest eigenvalue $1.1 \cdot 10^{-50}$) and our relative tolerance counted rounding noise. There are 52 pole-degenerate fakes in total; the other 34 agree. L88 Thm B’s identity $k = \tau + \frac{1}{2}Z_{\text{off}}(S)$ holds in 990/1000. The 10 failures are the degenerate case $q = 4$, $|c_1| = 8$, $\deg S = 3$ already flagged in L88 (N4: 10 degenerate, fixed by perturbation); the reproduction is exact.

TABLE 4. Threshold N^* for controlled ghosts (histogram “ N^* :count” over curves; $g = 2$: 15 curves, $g = 3, 4$: 10 each). Theory for small δ : $N^* = g$ (replace), $g + 2$ (add).

g	replacement model: N^* histogram						added model: N^* histogram					
	$\delta=0.02$	$\delta=0.05$	$\delta=0.1$	$\delta=0.2$	$\delta=0.3$	$\delta=0.45$	0.02	0.05	0.1	0.2	0.3	0.45
2	2:15	2:15	2:15	2:15	2:15	1:1,2:14	4:15	4:15	2:1,3:1,4:13	2:4,3:5,4:6	2:7,3:8	2:11,3:4
3	3:10	3:10	2:1,3:9	2:4,3:6	2:5,3:5	2:8,3:2	5:10	5:10	3:3,5:7	2:1,3:6,4:2,5:1	2:4,3:5,4:1	2:6,3:4
4	4:10	3:2,4:8	3:3,4:7	3:6,4:4	2:2,3:7,4:1	2:5,3:5	6:10	3:1,6:9	3:1,4:4,5:1,6:4	3:4,4:6	2:2,3:5,4:3	2:5,3:5

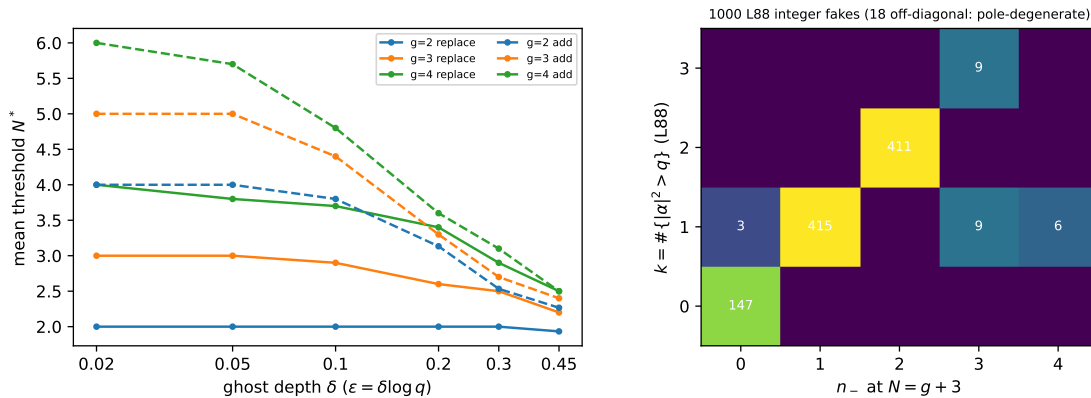


FIGURE 4. Left: mean N^* vs ghost depth for the two models (solid: replacement, dashed: added). Right: n_- at $N = g + 3$ vs L88’s k for 1000 fakes.

Calibration verdict. Over $\mathbb{F}_q$ the finite-support test behaves exactly as the density law predicts. There are no false alarms on true curves, the weak-ghost threshold equals $\pi \times \text{density}$ with zero offset, strong ghosts are seen earlier, and the large-support count is $\frac{1}{2}Z_{\text{off}}(P)$. This supports reading the ζ numerics of §3 as “ $a^* = \pi D(\gamma) + (\text{small, basis-dependent})$ ”. It does not prove it ([ANALOGY] for the transfer).

5. CORRECTIONS

L92.C1 (L91 X2 offset; chat claim “ $a_{\min} \approx \frac{1}{2} \log(\gamma/2\pi) + 0.4\text{--}0.58$ ”):.

Mostly a basis artefact. With refined test spaces the offset for $\delta = 0.1$ is 0.05–0.11 and still falling (§3). L91’s numbers are correct *for the bump basis* but are not properties of ζ . L93’s density heuristic (offsets 0.40–0.59, quoted from L91) inherits the same caveat; lemma93.pdf is not modified.

L92.C2 (L91 “slope 0.56 per $\log \gamma$ ”):.

On refined spaces the slope is 0.45–0.49, consistent with $\frac{1}{2}$. The excess came from the bump envelope shrinking the effective support.

L92.C3 (L91 “nearly δ -independent”; chat):.

Confirmed and explained. The weak dependence follows from the steep rise of $\kappa(a, \gamma)$ (Thm 3.1(iii)); measured spread 0.06–0.16 for $\delta \in [0.02, 0.3085]$.

L92.C4 (L91 “blind below threshold; real-world blind for $a \leq \frac{1}{2} \log 2$ by CC/Yoshida”):.

Imprecise. Yoshida/CC prove $Q_{\text{true}} \geq 0$ for $a \leq \frac{1}{2} \log 2$, which is a statement about the true zeros. Ghost visibility is a different question, governed to leading order by $4\delta^2 \kappa(a, \gamma) > 1$ (Theorem 3.1(iii)). Since $\delta < \frac{1}{2}$, this needs $\kappa > 1$. On our spaces $\kappa \lesssim 1$ until $a \approx \frac{1}{2} \log \frac{\gamma}{2\pi} - 0.1$ (§3). The corrected statement: ghosts at height γ are invisible for a somewhat below $\frac{1}{2} \log \frac{\gamma}{2\pi}$ ([NUMERICAL], leading order in δ). This is unrelated to Yoshida’s $\frac{1}{2} \log 2$, except that both are small-support phenomena.

L92.C5 (notation clash in the Z3 brief: “L88 Thm B count $n_+ = 2k = \frac{1}{2}Z_{\text{off}}$ ”):.

Two different k ’s and two different Z_{off} ’s. L91’s k = number of ghost *quartets*, with $n_+ = 2k_{\text{L91}}$. L88’s $k = \#\{|\alpha|^2 > q\}$ (= 2 per quartet, 1 per real pair). L88 Thm B is $n_-^{\text{Rosati}} = k_{\text{L88}} = \tau + \frac{1}{2}Z_{\text{off}}(S)$, where $S = uL' - gL$ is the *saddle* polynomial, not P . The correct chain is $n_- (\text{Weil, large } N) = k_{\text{L88}} = \frac{1}{2}Z_{\text{off}}(P) = 2k_{\text{L91}}$ (for quartets) = $\tau + \frac{1}{2}Z_{\text{off}}(S)$ (L88, non-degenerate). All of these were verified in §4.

L92.C6 (L91 sign convention):.

L91 works with $s(f, f) = D \bullet D \leq 0$; here $Q = -2s \geq 0$. L91’s “gain $2\delta^2|A'|^2$ ” equals our $-4\delta^2|F'|^2$ after this rescaling (consistent).

L92.C7 (own working note, Z1):.

An early draft said finite-space positivity is “essentially forced”. That is true *under RH* (CC’s no-radical argument), not unconditionally; corrected in §2.

L92.C8 (own, Z1 method):.

A first Arb run used a naive interval Cholesky, which failed at $a \geq 3.5$ through wrapping although the matrices are positive definite. All reported certificates use the preconditioned congruence test; $a = 7/2\text{--}6$ were re-certified with it.

L92.C9 (own, Z3 count):.

The raw count “ $n_- = k$ in 982/1000” must be read with the 18 pole-degenerate fakes excluded (§4); it is not a counterexample to Theorem 4.1(d), whose hypothesis excludes $z = q^{\pm 1/2}$.

6. RANKED DIRECTIONS

Scores: feasibility $\times$ value, averaged as in L88–L91 (0–1).

A1 Independent certified closure at one $a > \frac{1}{2} \log 2$ (score .66). Combine the Z1 Arb machinery with a high-frequency lower bound ($Q \geq c \int |F|^2 \log |r| - \text{primes}$) on the orthogonal

complement of a spline space, via a Schur/Feshbach step. Target $a = \frac{1}{2} \log 3$ first, then check Kastoris’s $c = 14$ claim independently. This is the only route from Z1 to a theorem about all test functions.

- A2 Function-field-to- ζ transfer of the threshold** (score .62). Prove $a^*(\gamma, \delta) \leq \frac{1}{2} \log \frac{\gamma}{2\pi} + C$ by a local Vandermonde/Lagrange construction. Use explicit zero counts (Riemann–von Mangoldt with explicit $S(T)$ bounds) and a sharper envelope than the Gevrey bump, to remove the log log loss of §3.
- A3 Prolate test spaces** (score .60). Replace splines by CC’s prolate vectors (band edge $2\pi\lambda^2$). This should push the numerical offset to its limit and connect a^* directly with CC Prop. 4.2.
- A4 Certified threshold table** (score .56). Use a higher-precision search (mpmath eigenvectors), so that certificates sit within 0.02 of the numerics for $\gamma \leq 10^3$.
- A5 Strong-ghost formula over $\mathbb{F}_q$** (score .52). Find N^* as a function of ghost depth (Table 4); conjecturally governed by κ as in Thm 3.1(iii).

Top three: A1 (.66), A2 (.62), A3 (.60).

7. QUESTIONS FOR J.W.

- Q1** The L89 erratum: should it go into lemma89.pdf, or stay as a separate note? (Still pending; lemma89–91 PDFs untouched.)
- Q2** Z1: is an all-test-function certificate at one modest a (A1) worth a long computation? It would be an independent check of Kastoris’s unrefereed claim, but it is not RH.
- Q3** Z2: which ghost model do you want as the reference? “Added” (extra quartet on top of the true zeros, used here for ζ) or “replacement” (same zero count, used over $\mathbb{F}_q$)? The thresholds differ by one zero pair’s worth of density.
- Q4** Should thresholds be reported in a (support half-width in $\log u$), in CC’s $\lambda = e^a$, or in band edge $E = 2\pi\lambda^2$? In the last, the conjecture reads “a ghost at height γ is visible iff $\gamma \lesssim E$ ”.
- Q5** Priority among A1–A5?

FILES AND PROVENANCE

Build `build92/{p0-p5,refs,lemma92.tex,tab_*.tex}`, figures `build92/figs/f1-f4`.
 Scripts: `num/lemma92_{z1float, z1lib, z1run, z1zero, z2lib, z2run, z2run2, z2kappa, z2cert, z2certscan, z3, z3run, figs, tables}.py`. Data: `num/192/*.json`. Zeros: `num/191/zeta_zeros_2500.npy` (L91, mpmath). Nothing was posted or submitted. lemma89–91 PDFs and the never-touch files are unchanged.

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- [10] TAA Lemmas 88, 89, 91, 93 (this project): `lemma88.pdf` Thm B; `lemma89.pdf` Prop. D; `lemma91.pdf` Prop. 1.3, X2; `lemma93.pdf` density heuristic. [[TAA-L]]

TAA LEMMA 93: ZEROS PAIRED WITH ZEROS. $H^1 \otimes H^1$, THE TWO PAIRINGS OF A ZERO, AND THE PLACE AT INFINITY

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis. Every claim carries one label: [THEOREM], [RESTATEMENT], [ANALOGY], [CONJECTURE], [OPEN], [NUMERICAL], [HEURISTIC]. Sources are marked [CITED] (opened), [CITED-sec] (via an opened secondary source), [BIB] (bibliographic only) or [TAA-L] (earlier TAA lemma). All references are listed in `lit/lemma93/verified_refs.md`.

ABSTRACT. J.W. asked what it means for a zero to “pair with” a zero. Our answer: a zero has *two* partners. Its *duality partner* comes from the functional equation ($\rho \leftrightarrow 1 - \rho$; over $\mathbb{F}_q$, $\alpha \leftrightarrow q/\alpha$). Its *conjugation partner* comes from complex conjugation ($\rho \leftrightarrow \bar{\rho}$; $\alpha \leftrightarrow \bar{\alpha}$). The duality pairing always exists: it is Poincaré duality, the cup product $H^1 \times H^1 \rightarrow H^2$. In $H^1 \otimes H^1 \subset H^2(C \times C)$ it produces the eigenvalue- q (Tate, algebraic) classes. The conjugation pairing is what makes a form Hermitian. RH is exactly the statement that the two partners coincide, and Weil positivity (Hodge index, Rosati) is the statement that the duality pairing, read through conjugation, is definite. We prove this over $\mathbb{F}_q$ and verify it on 2923 Weil-type polynomials. The *number* of Tate pairs is ghost-blind ($= 2g$ for every squarefree polynomial, fake or not), but the *signature* counts ghosts exactly ($n_- = k$, 2923/2923). Over $\mathbb{Q}$ the Weil quadratic form is literally a sum over duality pairs, $\sum_{\rho} \hat{f}(\rho) \hat{f}(1-\rho)$. Each on-line pair contributes a Hermitian square and each ghost quartet a block of inertia $(2, 2)$. We state precisely what Deninger’s programme says: the duality pairing $H_{\Theta \sim \alpha}^1 \times H_{\Theta \sim 1-\alpha}^1 \rightarrow H^2$ and the Hodge- $*$ argument for $\Theta = \frac{1}{2} + A$. We also say what it does not say: a spectrum $\{\rho + \rho'\}$ is our extrapolation. Rankin–Selberg $L(f \times g)$ tensors the *local* (fibre) data; its zeros are not sums of zeros, by counting, and numerically 5 zeros of $L(\chi_{12})$ lie below the smallest sum of zeros of $L(\chi_{-3})$ and $L(\chi_{-4})$. Montgomery’s pair correlation sees a ghost quartet only as an L^1 change of 0.007, against sampling noise of 0.34. At the place at infinity, the mean density of zeros is archimedean ($N(T) = \theta(T)/\pi + 1 + S(T)$). The L91 ghost-detection threshold behaves like a density threshold: $a_{\min} \approx \pi D(\gamma) + 0.5$, with $\pi D(\gamma) = \frac{1}{2} \log(\gamma/2\pi)$. On synthetic lattices of spacing Δ the fitted slope against π/Δ is 1.01–1.07 and the intercept 0.46–0.60, matching L91’s ζ offsets 0.40–0.59 ([NUMERICAL], [HEURISTIC]).

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Date: 24 October 2026 (t395u: “What does it mean for a zero to pair with a zero?”; continues L91 §§5–6).

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GLOSSARY (BRIEF RESTATEMENTS FOR J.W.)

Cohomology, h^1 ..:

For a curve C of genus g over $\overline{\mathbb{F}}_q$, H^0, H^1, H^2 are vector spaces (ℓ -adic) of dimensions $1, 2g, 1$ on which Frobenius F acts with eigenvalues $1; \alpha_1, \dots, \alpha_{2g}; q$. The α_i are the reciprocal zeros of $Z_C(T)$; $h^1 = \dim H^1 = 2g$ is the number of zeros.

Poincaré duality / cup product..:

A perfect pairing $H^1 \times H^1 \rightarrow H^2 \cong \mathbb{Q}_\ell(-1)$, compatible with F : it pairs the α -eigenline with the q/α -eigenline. This is the functional equation in cohomological form.

Künneth..:

$H^2(C \times C) = H^0 \otimes H^2 \oplus H^1 \otimes H^1 \oplus H^2 \otimes H^0$, with F -eigenvalues $q; \alpha_i \alpha_j$ ($4g^2$ of them); q .

Tate classes..:

Classes in $H^2(C \times C)(1)$ fixed by Frobenius, i.e. eigenvalue- q vectors in H^2 . Tate's theorem (for $C \times C$) says they are exactly the classes of divisors, $\text{NS} \otimes \mathbb{Q}_\ell$.

Hodge index..:

On a surface, the intersection form on $\text{NS} \otimes \mathbb{R}$ has exactly one positive direction (the ample class); everything orthogonal to $H + V$ pairs negatively. For $C \times C$ this is equivalent to RH for C (Weil; [7] Thm. 1.2; L84, L88).

Ghost..:

A hypothetical zero off the line: $\rho = \frac{1}{2} + \delta + i\gamma$ with $\delta \neq 0$ (over $\mathbb{Q}$), or $|\alpha|^2 \neq q$ for a “fake” q -palindromic polynomial (over $\mathbb{F}_q$, L88).

1. THE ANSWER IN ONE PAGE

Theorem 1.1 (Two partners of a zero; [THEOREM] over $\mathbb{F}_q$, [RESTATEMENT] over $\mathbb{Q}$). *Let α be a Frobenius eigenvalue on $H^1(\bar{C})$ ($C/\mathbb{F}_q$ a curve, or a root of a q -palindromic “fake” P), or let ρ be a nontrivial zero of ζ .*

- (i) Duality partner. q/α is again an eigenvalue, and $1 - \rho$ is again a zero. This is the functional equation. Cohomologically it is Poincaré duality: the cup product $H^1 \times H^1 \rightarrow H^2$ pairs the α -line with the q/α -line, because F acts on H^2 by q . This holds for every zero, including ghosts and fakes.
- (ii) Conjugation partner. $\bar{\alpha}$ is again an eigenvalue and $\bar{\rho}$ is again a zero, because the coefficients are real.
- (iii) RH $\iff$ the two partners coincide for every zero: $q/\alpha = \bar{\alpha} \iff |\alpha|^2 = q$, and $1 - \rho = \bar{\rho} \iff \Re \rho = \frac{1}{2}$.
- (iv) Positivity. The forms that prove RH over $\mathbb{F}_q$ have the shape “value at a zero times value at its duality partner”:

$$\text{Rosati (L84): } \text{Tr}(p(F)p(F)^\dagger) = \sum_i p(\alpha_i)p(q/\alpha_i), \quad \text{Weil (L91): } \sum_\rho \hat{f}(\rho)\hat{f}(1-\rho).$$

When the duality partner equals the conjugation partner, each term is a Hermitian square $|p(\alpha)|^2$ or $|\hat{f}(\rho)|^2$. When it does not, the terms of one ghost orbit form a hyperbolic block: inertia $(1, 1)$ for a real pair $\{\alpha, q/\alpha\}$, and $(2, 2)$ for a complex quartet.

Proof. (i)–(iii) are standard (Glossary; for ζ , $\xi(s) = \xi(1-s)$ and $\overline{\xi(\bar{s})} = \xi(s)$). (iv) is Thm. 2.4 below over $\mathbb{F}_q$, and Prop. 3.1 over $\mathbb{Q}$. $\square$

In words, for J.W. “A zero pairs with a zero” has a precise meaning: the cup product pairs each zero with its *functional-equation partner*. On $C \times C$ these pairs are the classes with eigenvalue $\alpha \cdot (q/\alpha) = q$ in $H^1 \otimes H^1$, which are exactly the divisor classes (Tate) where Hodge index acts. A ghost does *not* lose its partner, so the *count* of pairs cannot see it (Prop. 2.3). What a ghost changes is the *sign* of the pairing. Its partner is no longer its conjugate, so the pairing stops being a sum of squares. Every Weil-type positivity proof is a proof that “duality partner = conjugate”. Deninger’s programme makes exactly this argument in its conjectural cohomology (§3.2).

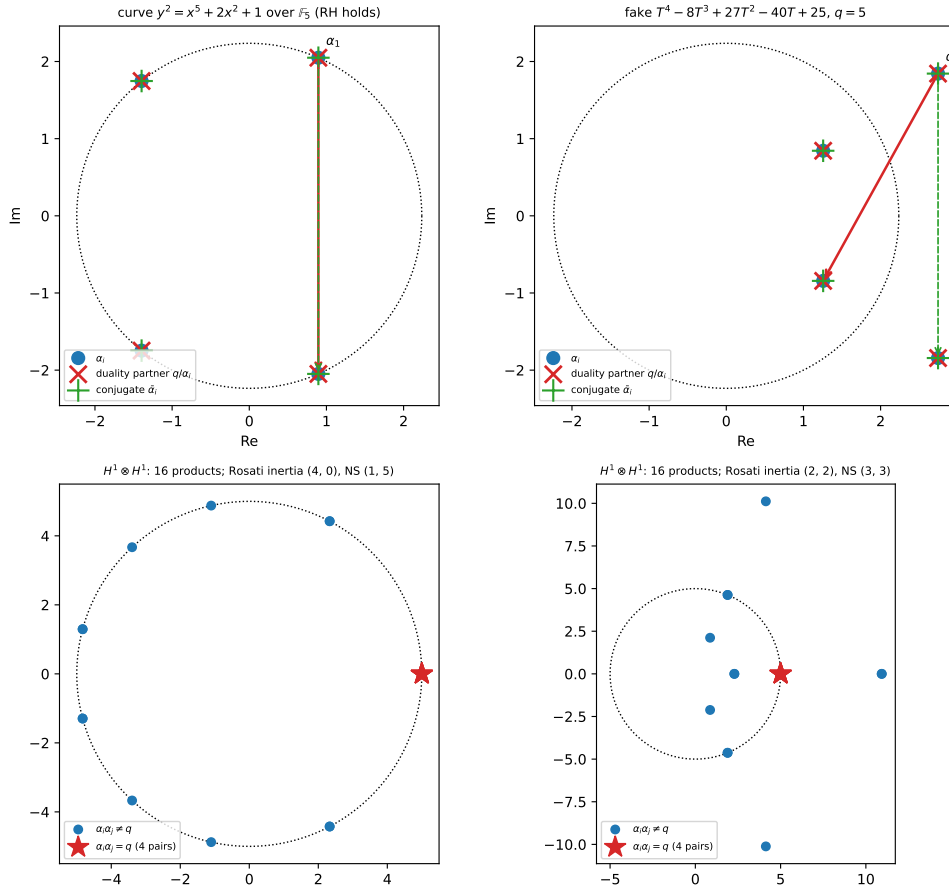


FIGURE 1. Diagnostic for Thm. 1.1. Top: Frobenius eigenvalues α_i (dots), duality partners q/α_i (crosses) and conjugates $\bar{\alpha}_i$ (plus signs); the arrows go from α_1 to q/α_1 (solid) and to $\bar{\alpha}_1$ (dashed). Left: a genus-2 curve over $\mathbb{F}_5$, where the arrows coincide. Right: a fake ($k = 2$), where they do not. Bottom: the 16 products $\alpha_i \alpha_j$ spanning $H^1 \otimes H^1$, with the circle $|z| = q$. In both cases *exactly four* products equal q (stars). The Tate count is the same, but the Rosati inertia changes from $(4, 0)$ to $(2, 2)$ and the NS signature from $(1, 5)$ to $(3, 3)$.

2. OVER $\mathbb{F}_q$: $H^1 \otimes H^1$, TATE CLASSES, AND THE SIGN OF THE PAIRING

Theorem 2.1 (Künneth and the additive dictionary; [THEOREM]). *Let $C/\mathbb{F}_q$ have genus g , with $P_1(T) = \prod_{i=1}^{2g} (1 - \alpha_i T)$. Then $H^1 \otimes H^1 \subset H^2(\overline{C} \times \overline{C})$ has Frobenius eigenvalues $\{\alpha_i \alpha_j\}_{i,j}$, and $Z_{C \times C}(T)$ contains the factor $\prod_{i,j} (1 - \alpha_i \alpha_j T)^{-1}$ (L91 Prop. 6.3(ii), checked there on 14 curves, [TAA-L]). Write $\alpha_i = q^{s_i}$ with $s_i \in \mathbb{C}/\frac{2\pi i}{\log q} \mathbb{Z}$, so the s_i are the zeros of $\zeta_C(s) = Z_C(q^{-s})$. Then:*

- (a) *the $H^1 \otimes H^1$ spectrum in the s -variable is the pair-sum set $\{s_i + s_j\}$. Over $\mathbb{F}_q$, “zeros paired with zeros” is literally a sum of zeros, modulo the period $2\pi i/\log q$;*
- (b) *$\alpha_i \alpha_j = q \iff s_i + s_j \equiv 1$: the eigenvalue- q (Tate) pairs are the special pairs $s + s' = 1$, i.e. duality pairs;*
- (c) *$q/\alpha_i = \bar{\alpha}_i \iff \Re s_i = \frac{1}{2}$.*

Proof. Künneth plus multiplicativity of eigenvalues on tensor products; (b) and (c) follow by taking $\log_q$. $\square$

Theorem 2.2 (Tate classes = divisors; [THEOREM], [CITED] [5] Thm. 5.2, [6] §14.6; L88 Thm. C, [TAA-L]). *For $C \times C$ over $\mathbb{F}_q$, $\text{NS}(C \times C) \otimes \mathbb{Q}_\ell$ is the eigenvalue- q part of H^2 (Tate’s conjecture, known for products of curves). Hence*

$$\rho(C \times C) = 2 + \#\{(i, j) : \alpha_i \alpha_j = q\} = 2 + \sum_{\lambda} m(\lambda) m(q/\lambda) = 2 + \sum_{\lambda} m(\lambda)^2,$$

where λ runs over the distinct eigenvalues on H^1 and m is the multiplicity ($m(q/\lambda) = m(\lambda)$ by the functional equation). The “2” is H, V (from $H^0 \otimes H^2$ and $H^2 \otimes H^0$). The rest is $H^1 \otimes H^1$, i.e. $\text{End}(J) \otimes \mathbb{Q}_\ell$ (L89 §A, [TAA-L]).

Proposition 2.3 (The Tate count is ghost-blind; [THEOREM], [NUMERICAL]). *For any q -palindromic P of degree $2g$ (a Weil polynomial or a fake) with distinct roots, $\#\{(i, j) : \alpha_i \alpha_j = q\} = 2g$: each root pairs with its duality partner and with nothing else. The formal “Picard number” $2 + 2g$ is therefore the same with or without ghosts. Check: the exact multiplicity of q in the resultant $\prod_{i,j} (T - \alpha_i \alpha_j)$ (L88 `mult_q_R`) equals $2g$ for 153/153 genus-1 and 2770/2770 genus-2 squarefree polynomials, of which 64 and 2490 are fakes (Fig. 2, left).*

Theorem 2.4 (The sign of the duality pairing; [THEOREM], with L84/L88/L89 [TAA-L]). *Let P be squarefree, $A = \mathbb{Q}[F] = \mathbb{Q}[T]/(P)$, and $x^\dagger = x(q/F)$ (Rosati: $F^\dagger = V = q/F$). For $x = p(F)$,*

$$\text{Tr}_{H^1}(x x^\dagger) = \sum_{i=1}^{2g} p(\alpha_i) p(q/\alpha_i).$$

(a) *If $|\alpha_i|^2 = q$ for all i , every term is $|p(\alpha_i)|^2$ and the form is positive definite on $A \otimes \mathbb{R}$. (b) Decompose $A \otimes \mathbb{R}$ by the real orbits of $\{\alpha \mapsto q/\alpha, \alpha \mapsto \bar{\alpha}\}$. An on-circle orbit contributes a positive definite block. A real off-circle pair $\{\alpha, q/\alpha\}$ contributes $2xy$ in the coordinates $x = p(\alpha), y = p(q/\alpha)$, of inertia (1, 1). A complex off-circle quartet contributes $4\Re(zw)$ with $z = p(\alpha), w = p(q/\alpha)$, of inertia (2, 2). Hence $n_- = k := \#\{\alpha : |\alpha|^2 > q\}$. (c) On $C \times C$, $D_x \cdot D_x = -\text{Tr}(x x^\dagger)$ for the part of NS orthogonal to H, V (L89 Thm. A). So Hodge index holds $\iff$ (a) $\iff$ RH, and the number of extra positive directions of NS is $n_+(\text{NS}) - 1 = k$. This is L88 Thm. B, $k = \tau + \frac{1}{2} Z_{\text{off}}$.*

Proof. (a), (b): Tr_{H^1} of $p(F)p(q/F)$ is the sum of its eigenvalues; then apply CRT, with coordinates $p(\alpha)$ independent across distinct roots. In the quartet, $z\bar{w}$ -type terms do not occur because the partner of α is q/α , not $\bar{\alpha}$; $\Re(zw)$ on $\mathbb{C}^2 \cong \mathbb{R}^4$ has inertia (2, 2). (c) is L89 Thm. A combined with (b). $\square$

Where a ghost’s “conjugate pair” goes. For a ghost, $\alpha \bar{\alpha} = |\alpha|^2 \neq q$, so the conjugate pair is *not* a Tate eigenvalue. It sits off the circle $|z| = q$ in $H^1 \otimes H^1$: the two real points 1.91 and 13.09 in the genus-1 fake below, and the points at 2.29 and 10.93 in Fig. 1, bottom right. The Tate eigenvalue q still comes from $(\alpha, q/\alpha)$. The Hodge-index violation arises because complex conjugation sends

the eigenline $e_\alpha \otimes e_{q/\alpha}$ to the *different* eigenline $e_{\bar{\alpha}} \otimes e_{q/\bar{\alpha}}$, so the real structure glues them into a hyperbolic block.

Example ($Q(T) = \prod(T - \alpha_i)$)	$ \alpha_i ^2$	$\#\{\alpha_i \alpha_j = q\}$	Rosati (n_+, n_-, n_0)	NS (n_+, n_-, n_0)	k
$E : y^2 = x^3 + x + 1/\mathbb{F}_5, T^2 + 3T + 5$	5, 5	2	(2, 0, 0)	(1, 3, 1)	0
$E : y^2 = x^3 + x/\mathbb{F}_9$ (supersingular), $(T + 3)^2$	9, 9	4 ($\rho = 6$)	(1, 0, 1)*	(1, 2, 2)*	0
fake $g = 1: T^2 - 5T + 5, q = 5$	1.910, 13.090	2	(1, 1, 0)	(2, 2, 1)	1
$C : y^2 = x^5 + x/\mathbb{F}_3, T^4 + 2T^2 + 9$	$3^{\times 4}$	4	(4, 0, 0)	(1, 5, 1)	0
$C : y^2 = x^5 + 2x^2 + 1/\mathbb{F}_5, T^4 + T^3 +$ $5T^2 + 5T + 25$	$5^{\times 4}$	4	(4, 0, 0)	(1, 5, 1)	0
$C : y^2 = x^5 + 3x^3 + x + 2/\mathbb{F}_7, T^4 +$ $4T^3 + 14T^2 + 28T + 49$	$7^{\times 4}$	4	(4, 0, 0)	(1, 5, 1)	0
fake $g = 2: T^4 - 8T^3 + 27T^2 - 40T +$ 25	$2.287^{\times 2}, 10.930^{\times 2}$	4	(2, 2, 0)	(3, 3, 1)	2
fake $g = 2: T^4 - 8T^3 - 12T^2 - 40T +$ 25	0.269, 93.05, 5, 5	4	(3, 1, 0)	(2, 4, 1)	1

TABLE 1. Explicit examples ([NUMERICAL], exact signatures via L88 **sig**; curves by brute-force point counts over $\mathbb{F}_p, \mathbb{F}_{p^2}$). The NS Gram matrix is that of $\{H, V, \Gamma_0, \dots, \Gamma_{2g}\}$ (L46), which has one null direction. *Non-squarefree ($Q[F] = \mathbb{Q}$): Thm. 2.4 does not apply, and $\rho = 2 + m^2 = 6$ exceeds the rank of the Frobenius-graph span (L88 N1, [TAA-L]). Script `num/lemma93_fq.py`, data `num/193/fq.json`.

Family check ([NUMERICAL]). We tested all squarefree q -palindromic integer polynomials with $g = 1$ ($q \in \{2, 3, 4, 5, 7, 8, 9, 11, 13\}$, $|a| \leq 2\lfloor\sqrt{q}\rfloor + 4$; 153 polynomials, 64 fakes) and $g = 2$ ($q \leq 5$, $|c_1| \leq 4\sqrt{q} + 2$, $-2q - 4 \leq c_2 \leq 6q + 4$; 2770 polynomials, 2490 fakes, with $k = 1$: 1626 and $k = 2$: 864). In all 2923/2923 cases: Tate count = $2g$; Rosati $n_- = k$; NS $n_+ - 1 = k$. “RH-type” here means all $|\alpha|^2 = q$; such polynomials need not come from curves.

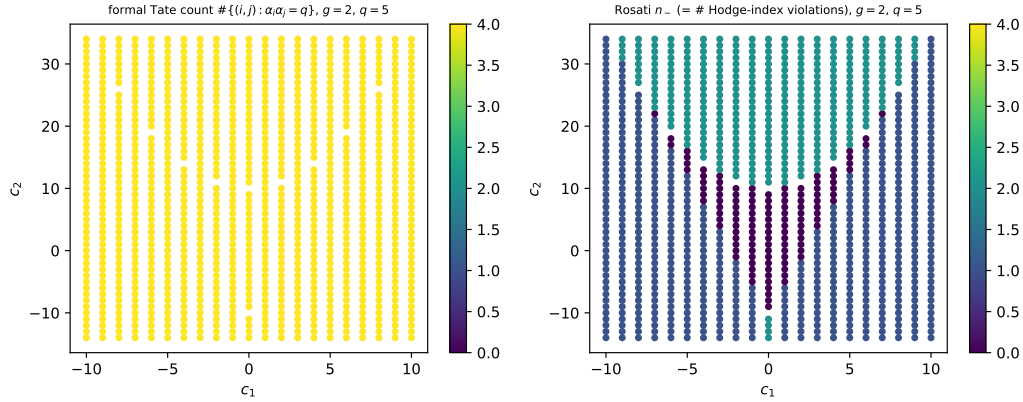


FIGURE 2. Diagnostic for Prop. 2.3 versus Thm. 2.4 ($g = 2, q = 5$, all squarefree (c_1, c_2) in range). Left: the formal Tate count is 4 everywhere and cannot see ghosts. Right: the Rosati n_- is 0 exactly on the RH region and equals $k = 1, 2$ outside it.

Link to L91 ([TAA-L], [RESTATEMENT]). L91 found $n_+ = 2$ for $D \bullet D$ per ghost quartet, with 12/12 runs exact. This is the $\mathbb{Q}$ -version of Thm. 2.4(b): one complex quartet gives inertia (2, 2), hence 2 wrong-sign directions. L89’s “one per ghost” is the even half (L91 C9).

3. OVER $\mathbb{Q}$: THE ADDITIVE PICTURE, AND WHAT ACTUALLY EXISTS

3.1. The formal dictionary.

Pair sums ([ANALOGY]).:

Thm. 2.1(a) suggests attaching to ζ the “pair spectrum” $\{\rho + \rho'\}$. On the line, $\rho + \rho' = 1 + i(\gamma + \gamma')$.

Special pairs ([RESTATEMENT]).:

$\rho + \rho' = 1 \iff \rho' = 1 - \rho$, the duality partner. By the functional equation *every* zero has one, ghost or not. They are the analogue of Tate pairs, and like those they are ghost-blind.

Conjugate pairs ([RESTATEMENT]).:

$\rho + \bar{\rho} = 2\Re\rho$, which equals 1 for every $\rho \iff$ RH. On the line, “special pair” and “conjugate pair” are the same pair. For a ghost $\frac{1}{2} + \delta + i\gamma$, the conjugate sum is $1 + 2\delta$.

What breaks ([THEOREM]).:

Over $\mathbb{Q}$ there is no period $2\pi i/\log q$ and no Frobenius (L89 §D, L90). The pair set is not the spectrum of any operator we can construct. Its counting function up to height T grows like $N(T)^2 \asymp (T \log T)^2$.

Proposition 3.1 (The Weil form is a sum over duality pairs; [RESTATEMENT], [NUMERICAL]). *For real test functions f in the Connes–Consani normalization (L91 Thm. 1.1), $D \bullet D = s(f, f) = \frac{1}{2}(2\hat{f}(0)\hat{f}(1) - \sum_{\rho} \hat{f}(\rho)\hat{f}(1-\rho))$. The zero side is the duality-pair sum of Thm. 1.1(iv). On-line zeros contribute $|\hat{f}(\rho)|^2 \geq 0$. A quartet $\{\rho, \bar{\rho}, 1-\rho, 1-\bar{\rho}\}$ contributes $4\Re(zw)$ with $z = \hat{f}(\rho)$, $w = \hat{f}(1-\rho)$, which has real inertia $(2, 2)$. Numerical check (support $a = 2.5$, 50 basis functions around height 85.7, orthonormalized): each on-line orbit gives an inertia $(2, 0)$ block (orbits at 84.735, 87.425). The ghost quartet gives $(2, 2)$ for $\delta = 0.3085, 0.05$ and 0.01 alike. So the inertia does not depend on δ ; only the size, $O(\delta^2)$ by L91 Prop. 1.3, does.*

3.2. What Deninger actually states. We opened two sources: Deninger’s ICM 1998 lecture [1], pp. 168–169 (page images), and his 2002 survey [2] (arXiv text).

- (D1) **Conjectural cohomology of $\overline{\text{Spec } \mathbb{Z}}$** ([CITED] [1] p. 168; [CONJECTURE] in the source). $H^0 = \mathbb{R}$ with $\Theta = 0$; H^1 infinite-dimensional, with the spectrum of Θ equal to the nontrivial zeros; $H^2 \cong \mathbb{R}$ with $\Theta = \text{id}$; and ζ is recovered as a regularized determinant (his (3)).
- (D2) **Duality pairs a zero with a zero** ([CITED] [1] p. 168). The cup product induces, for every $\alpha \in \mathbb{C}$,

$$\cup : H^i(\overline{\text{Spec } \mathbb{Z}}, \mathbb{C})^{\Theta \sim \alpha} \times H^{2-i}(\overline{\text{Spec } \mathbb{Z}}, \mathbb{C})^{\Theta \sim 1-\alpha} \longrightarrow H^2(\overline{\text{Spec } \mathbb{Z}}, \mathbb{C})^{\Theta \sim 1} \cong \mathbb{C},$$

and Deninger expects these pairings to be nondegenerate, “compatible with the functional equation”. This is exactly the special pairs $\rho + \rho' = 1$ of Thm. 2.1(b), on the curve itself.

- (D3) **The Hodge- $*$ argument** ([CITED] [1] p. 169; conditional). *If there is a Hodge $*$ on H^1 commuting with Θ such that $\langle f, f' \rangle = \text{tr}(f \cup *f')$ is positive definite, then the Leibniz rule $f_1 \cup f_2 = \Theta(f_1 \cup f_2) = \Theta f_1 \cup f_2 + f_1 \cup \Theta f_2$ gives $\Theta = \frac{1}{2} + A$ with A skew, “hence the Riemann conjecture would follow”. In our language: duality pairing + a positive conjugation structure compatible with it $\Rightarrow$ partners coincide (Thm. 1.1(iii)).*
- (D4) **Proven pieces in the foliation setting** ([CITED] [2]). Θ is a derivation of the cup product ((1)). There is a Künneth isomorphism for reduced leafwise cohomology of Riemannian foliations ((6), attributed to a thesis in preparation). Thm. 2.1: for a conformal flow (e^{at}) , $\Theta = \frac{\alpha}{2} + S$ on $\bar{H}^1$ with S skew. Cor. 3.5: all $\rho \in \text{Sp}^1(\Theta)$ have $\Re\rho = \alpha/2$. In the manifold setting the hypotheses force $\alpha = 0$ (Remark 2), whereas number theory needs $\alpha = 1$. For an ordinary CM elliptic curve this gives “a dynamical proof for the Riemann hypotheses for $\zeta_E(s)$ ”, which Deninger notes is essentially Hasse’s.
- (D5) **The infinite place** ([CITED] [2] Thm. 4.1 and table). The real place corresponds to a *fixed point* of the flow with $\kappa = -2$, and W_∞ “is determined by the Γ -factor”. Primes correspond to closed orbits of length $\log p$.

What is our extrapolation, not Deninger’s ([HEURISTIC]). From (D4)’s Leibniz rule and Künneth, the diagonal flow on a putative $\bar{H}^1 \hat{\otimes} \bar{H}^1$ would have generator eigenvalues $\rho + \rho'$, and only the pairs with $\rho + \rho' = 1$ would map nontrivially to H^2 . Neither opened source states a spectrum $\{\rho + \rho'\}$ or discusses $\overline{\text{Spec } \mathbb{Z}} \times \overline{\text{Spec } \mathbb{Z}}$. What Deninger does predict is the duality pairing (D2) and positivity via $*$ (D3) on the *curve’s* H^1 . Over $\mathbb{F}_q$, the $H^1 \otimes H^1$ component of the diagonal class on $C \times C$ is that cup-product pairing (Künneth decomposition of $[\Delta]$, [THEOREM]). So the surface and the curve formulations carry the same pairing.

3.3. Rankin–Selberg: the automorphic $H^1 \otimes H^1$ tensors the fibre, not the base.

Proposition 3.2 ([THEOREM] for (a)–(b), [ANALOGY] for (c)). (a) [CITED] [4]: for f, g of GL_2 with Satake parameters $\{\alpha_p, \alpha'_p\}$ and $\{\beta_p, \beta'_p\}$, $L(f \otimes g, s) = \prod_p \prod (1 - \alpha\beta p^{-s})^{-1}$ over the four products, a degree-4 Euler product. The tensor product acts on the local eigenvalues at each prime: the ℓ -adic H^1 of the motives $h^1(f) \otimes h^1(g)$, i.e. the fibre over $\text{Spec } \mathbb{Z}$. (b) Its zeros are not the sums of zeros of $L(f)$ and $L(g)$. A degree- d L -function has $O_d(T \log T)$ zeros up to height T ([BIB], standard Riemann–von Mangoldt), while the pair sums number $\asymp (T \log T)^2$. Numerically, for $\text{GL}_1 \times \text{GL}_1$, $L(\chi_{-3} \otimes \chi_{-4}, s) = L(s, \chi_{12})$ (product character checked for $n < 200$). Its 21 zeros on $0 < t < 40$ start at 3.8046, 6.6922, 8.8906, 11.1884, 12.9662, and these five lie below the smallest pair sum $8.0397 + 6.0209 = 14.0607$. Above it, the mean nearest-sum distance is 0.48, against 0.37 for random points (no attraction) (Fig. 3; `num/lemma93_rs.py`). (c) Over $\mathbb{F}_q$ the two tensors are distinct as well. $H^1(C) \otimes H^1(C)$ is the base tensor of Thm. 2.1, whose spectrum consists of products of zeros. For elliptic curves E, E' over $K = \mathbb{F}_q(C)$, $L(E \otimes E'/K)$ is a fibre tensor, and its zeros are eigenvalues on the cohomology of the base with coefficients in the tensor of the local systems, not sums of zeros. The multiplicative picture works over $\mathbb{F}_q$ because one Frobenius governs all places at once (the base is a point). Over $\mathbb{Q}$, Rankin–Selberg multiplies the local parameters prime by prime, and the zeros are global.

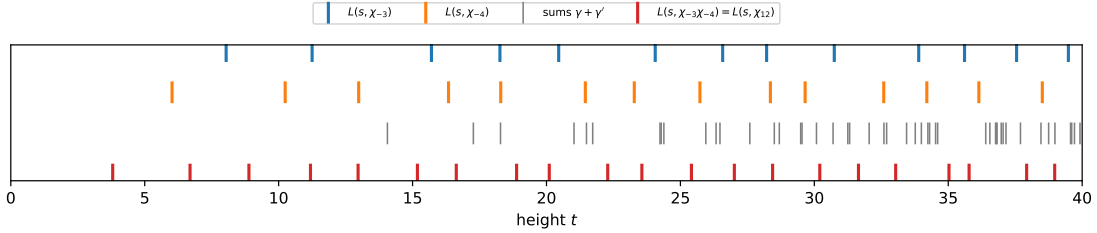


FIGURE 3. Diagnostic for Prop. 3.2(b). Rows, top to bottom: zeros of $L(s, \chi_{-3})$, of $L(s, \chi_{-4})$, all sums $\gamma + \gamma' < 40$, and zeros of the Rankin–Selberg product $L(s, \chi_{12})$. The bottom row is not the third.

3.4. Pair correlation as a statistical shadow; the formal pair spectrum. Montgomery ([CITED] [3]; [THEOREM] assuming RH; [CONJECTURE] in general). Montgomery studies the differences $\gamma - \gamma'$. Assuming RH, he proves the asymptotics of $F(\alpha)$ for $0 \leq \alpha < 1$, and he conjectures that $1 - (\sin \pi u / \pi u)^2$ is the pair correlation function. Dyson pointed out to him that this is the pair correlation of random Hermitian matrices. In our language, $\rho + \bar{\rho}' = 1 + i(\gamma - \gamma')$ on the line, so pair correlation is the *conjugation*-pair statistic. It is defined from the ordinates only, so it cannot see $\Re \rho$ directly.

Numerics ([NUMERICAL]; 2500 zeros ≤ 3031.29 ; `num/lemma93_pairs.py`). We compare the zeros with and without a synthetic quartet: $(\delta, \gamma) = (0.3085, 85.699)$ (DH-like) and $(0.1, 200.3)$.

- *Special-pair count*: every zero has its duality partner (5000/5000, resp. 5004/5004). This is blind, as Prop. 2.3 predicts.
- *Real parts of pair sums*: none off 1 without the ghost. With it, 10002 of the $6.26 \cdot 10^6$ upper sums are off 1 (fraction $1.6 \cdot 10^{-3}$), at $1 \pm \delta$ and $1 \pm 2\delta$. This detects the ghost only because the pair set is built from the zeros' real parts, i.e. it presupposes the answer.
- *Conjugation mismatch* $\#\{\rho : 1 - \rho \neq \bar{\rho}\}$: 0 versus 4. This is RH restated.
- *Pair correlation* (unfolded with $\theta(t)/\pi + 1$): adding the quartet changes the histogram by $L^1 = 0.0068$ (DH-like) or 0.0075. The first-half-versus-second-half sampling noise is 0.344, and the distance to Montgomery's curve is 0.197. The ghost is $\sim 50\times$ below the noise; it appears only as one extra coincidence at $u = 0$ (two zeros at the same height), indistinguishable from a double zero.

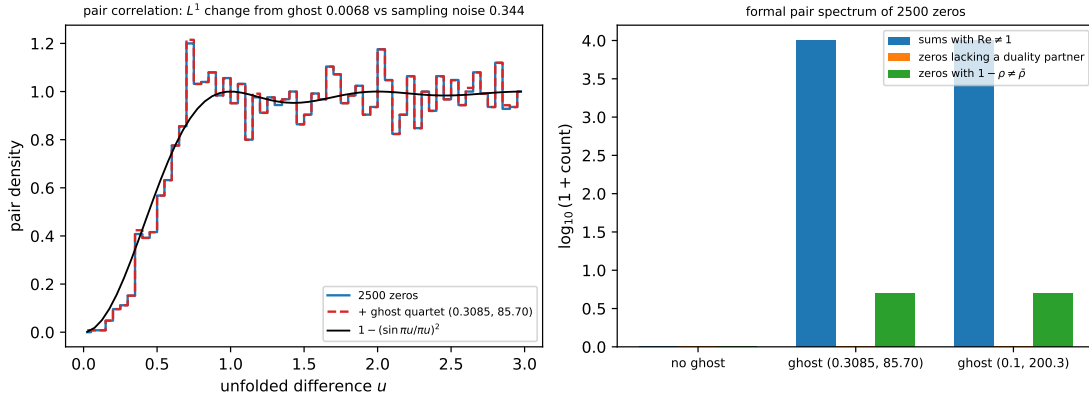


FIGURE 4. Diagnostics for the formal pair spectrum. Left: pair correlation with and without a ghost quartet (the two step curves overlap), with Montgomery's $1 - (\sin \pi u / \pi u)^2$. Right: three counts on a $\log_{10}(1 + \cdot)$ scale. Real parts $\neq 1$ and the conjugation mismatch detect the ghost; the duality-partner deficit (orange, always 0) does not.

Verdict on the object carrying $\{\rho + \rho'\}$ ([OPEN], [HEURISTIC]). No known object carries it. Automorphic tensor products (Rankin–Selberg, and ζ^2 trivially) tensor the fibre. Statistical objects (pair correlation) see differences of ordinates only. Deninger's framework, extrapolated, would put the pair sums on $H^1 \otimes H^1$ of a square, but the opened texts predict only the duality pairing on the curve. The information that matters for RH, namely whether duality partner equals conjugate, sits in the *sign* of the pairing, which the Weil form already measures (Prop. 3.1).

4. THE PLACE AT INFINITY: WHERE THE PAIRING'S DENSITY COMES FROM

4.1. Recap of L91 §5, with sources. ([TAA-L]; all items checked in L91.) (1) $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$ is Tate's local zeta integral at ∞ for $e^{-\pi x^2}$ ([THEOREM], [CITED] [10, 11]), so $\xi = \Gamma_{\mathbb{R}} \zeta$ is the product over *all* places. (2) Arakelov/van der Geer–Schoof: the compactified $\bar{\text{Spec}} \mathbb{Z}$ has divisors with real coefficients at ∞ , and Riemann–Roch $h^0(D) - h^0(\kappa - D) = \deg D - \frac{1}{2} \log |\Delta|$ follows by Poisson summation; there is no intrinsic h^1 ([CITED] [8] Prop. 1). (3) Hilbert reciprocity $\prod_v (a, b)_v = 1$ over all places including ∞ , so ramification sets are even; ∞ need not be ramified ([CITED] [9]; 6400/6400 in L91). (4) In Deninger's dictionary ∞ is a fixed point of the flow, and W_{∞} is determined by $\Gamma_{\mathbb{R}}$ (D5).

4.2. The archimedean term is the mean density of the zero side.

Theorem 4.1 ([THEOREM]; [CITED] [12], [BIB] [13]). $N(T) = \theta(T)/\pi + 1 + S(T)$, where $\theta(t) = \arg \Gamma(\frac{1}{4} + \frac{it}{2}) - \frac{t}{2} \log \pi$ is the phase of $\Gamma_{\mathbb{R}}(\frac{1}{2} + it)$ and $S(T) = O(\log T)$. Hence the mean density of zeros, $D(t) = \theta'(t)/\pi \sim \frac{1}{2\pi} \log \frac{t}{2\pi}$, is purely archimedean. Check ([NUMERICAL]): over the first 2500 zeros, $n - \frac{1}{2} - (\theta(\gamma_n)/\pi + 1)$ has mean $-2.6 \cdot 10^{-5}$ and maximum modulus 0.79.

Reading ([HEURISTIC], [ANALOGY]). Split the zero side of the pairing as $\sum_{\rho} \hat{f}(\rho) \hat{f}(1 - \rho) = \int |\hat{f}(\frac{1}{2} + it)|^2 D(t) dt + (\text{fluctuation})$, plus pole terms. The first term is controlled by $\Gamma_{\mathbb{R}}$ (the W_{∞} part of $N(u)$). The fluctuation is carried by the primes. This suggests treating the archimedean term as the *continuous part* of the pairing, which is the reading Deninger’s 2002 Remark after Cor. 3.5 allows: “ $\int \Phi(\lambda) m(\lambda) d\lambda$ where $m(\lambda) \geq 0$ is the spectral density function of the continuous part of the spectrum of Θ ” ([CITED] [2]). So “ $H^1 \otimes H^1$ at ∞ ” would be the smooth density $D(t) dt$ that the duality pairing integrates against. This is an Analogy: no theorem identifies W_{∞} with a continuous spectrum of an operator on a space attached to ζ .

Blind regime ([NUMERICAL]). For $f = \text{bump}(t/a) \cos(\gamma_0 t)$, the relative difference between $\sum_{\rho} |\hat{f}(\rho)|^2$ (2500 zeros) and $\int |\hat{f}|^2 D dt$ is $\leq 1.5\%$ for $a \leq 0.5$ at $\gamma_0 \in \{39.3, 85.7, 200.3\}$. For such small supports the zero side is the archimedean density and cannot see individual zeros, ghost or not. The point where the difference exceeds 30% (at $a \approx 1.75, 2.5, 1.5$ respectively) is *not* monotone in γ_0 . So this crude diagnostic does not locate the threshold (Fig. 5, right).

4.3. Is the L91 support threshold a density threshold? Heuristic (Paley–Wiener/density; [HEURISTIC]). If f is supported in $[-a, a]$ (log variable), then $t \mapsto \hat{f}(\frac{1}{2} + it)$ is entire of exponential type a . To expose a ghost at height γ , the form must find $f \in V_0$ whose transform is small at all true zeros near γ but not at the ghost. A function of type a can be forced small on a set of density D only if $a \gtrsim \pi D$, which is the Beurling–Landau sampling heuristic. For ζ , $\pi D(\gamma) = \frac{1}{2} \log(\gamma/2\pi)$, exactly the leading term L91 observed. Equivalently, $e^{a_{\min}} \approx e^c \sqrt{\gamma/2\pi}$: the threshold support is the length of the Riemann–Siegel main sum, itself a $\Gamma_{\mathbb{R}}$ -phenomenon. We do not prove this (direction W1).

Test on synthetic lattices ([NUMERICAL]; zero side only; num/lemma93_arch.py). Put on-line “zeros” at $100 + (n + o)\Delta$ out to ± 245 , with $o \in \{\frac{1}{2}, \frac{1}{4}\}$, and a ghost quartet $(\delta, 100)$. Use the L91 basis (bump envelope, frequencies $100 + k\pi/a$, $|k\pi/a| \leq 15$) on V_0 , and find $a_{\min}$ by bisection with tolerance 10^{-9} relative.

Δ	0.5	0.7	1.0	1.4	2.0	2.8
π/Δ	6.283	4.488	3.142	2.244	1.571	1.122
$a_{\min}, o = \frac{1}{2}, \delta = 0.1$	7.155	5.226	3.787	2.852	2.121	1.663
$a_{\min}, o = \frac{1}{4}, \delta = 0.1$	6.867	5.236	3.825	2.901	2.161	1.676
$a_{\min}, o = \frac{1}{2}, \delta = 0.2$	6.053	5.168	3.770	2.820	2.094	1.622

Least-squares fits $a_{\min} = s \cdot \pi/\Delta + c$: for $\delta = 0.1$, $(s, c) = (1.065, 0.456)$ at $o = \frac{1}{2}$ (max residual 0.014) and $(1.011, 0.602)$ at $o = \frac{1}{4}$. For $\delta = 0.2$, $(0.887, 0.802)$ and $(0.863, 0.811)$; the stronger ghost is seen slightly *before* the density limit at the densest lattice. For ζ (L91 X2, 15 points), the offsets $a_{\min} - \pi D(\gamma)$ are 0.40–0.49 ($\gamma = 39.3$), 0.46–0.58 (85.7) and 0.51–0.59 (200.3), with the free fit $(s, c) = (1.14, 0.33)$. **Conclusion** ([NUMERICAL], [HEURISTIC]): to leading order the threshold is the archimedean zero density, $a_{\min} \approx \pi D + 0.5$, with the same constant on lattices and on ζ . The constant presumably depends on the envelope; we did not vary it (direction W5). This answers the brief’s question “does $a_{\min}$ come from the archimedean density?” with *yes, numerically, to leading order*; it is not a theorem.

5. VERDICT

- (1) **Meaning ([THEOREM] over $\mathbb{F}_q$, [RESTATEMENT] over $\mathbb{Q}$).** A zero pairs with its *duality partner* (Poincaré duality, the functional equation) and is conjugate to its *conjugation partner*.

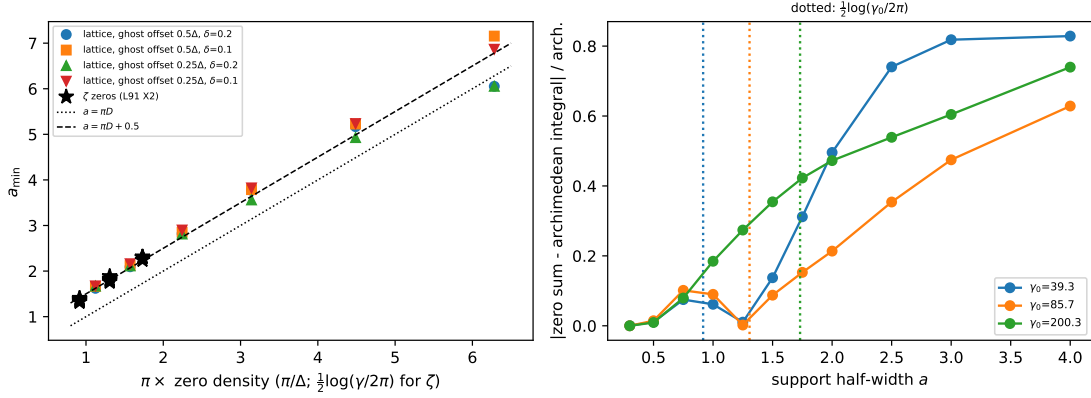


FIGURE 5. Diagnostic for the density heuristic. Left: $a_{\min}$ against $\pi \times \text{density}$ for lattices (four marker types) and for ζ (stars, L91 X2), with the lines $a = \pi D$ and $a = \pi D + 0.5$. Right: relative difference between the zero sum and the archimedean density integral, against support a . The dotted lines mark $\frac{1}{2}\log(\gamma_0/2\pi)$. Small supports see only the archimedean density.

On $C \times C$ the duality pairs are the eigenvalue- q classes in $H^1 \otimes H^1$, which are the divisors beyond H, V (Tate). RH means the two partners coincide; positivity (Hodge index, Rosati, Weil) means the duality pairing is a sum of Hermitian squares, which is the same thing.

- (2) **Count versus sign ([THEOREM], [NUMERICAL]).** The number of pairs is ghost-blind ($2g$; 2923/2923). The signature counts ghosts ($n_- = k$; 2923/2923). Over $\mathbb{Q}$, a ghost quartet is a $(2, 2)$ block of the Weil form (L91's 2 directions).
- (3) **The additive spectrum $\{\rho + \rho'\}$ ([ANALOGY], [OPEN]).** Over $\mathbb{F}_q$ it is exact (mod $2\pi i / \log q$). Over $\mathbb{Q}$ no object carrying it is known. Rankin–Selberg tensors the fibre, and its zeros are not pair sums ([THEOREM] by counting; [NUMERICAL]). Pair correlation is a shadow that does not see a ghost ([NUMERICAL]). Deninger predicts the duality pairing and the $*$ -positivity on the curve, not a pair spectrum ([CITED]).
- (4) **Infinity ([THEOREM] for the density, [HEURISTIC] for the threshold).** The mean density of zeros is archimedean, and the L91 detection threshold follows it: $a_{\min} \approx \pi D(\gamma) + 0.5 = \frac{1}{2}\log(\gamma/2\pi) + 0.5$, on ζ and on synthetic lattices alike.
- (5) Nothing here creates positivity. All of it restates Weil's criterion in the language of pairs. **RH is not proved.**

6. RANKED DIRECTIONS

Score = 0.6·significance+0.4·probability (both subjective, [HEURISTIC]).

	Direction	sig	prob	score
W2	Function-field calibration of the threshold: the same finite-span test for $C/\mathbb{F}_q$ and L88 fakes, where the “density” is g zeros per period $2\pi/\log q$. Check $a_{\min} \approx \pi D + c$ with the same c , and $n_+ = k$ (subsumes L91 Z3)	.60	.70	.64
W1	Density-threshold theorem: show that ghost detection on $V_0 \cap \{\text{supp} \subset [-a, a]\}$ needs $a \geq \pi D(\gamma) - O(1)$ and is achieved by $a \leq \pi D(\gamma) + O(1)$ (Paley–Wiener, Beurling–Landau, de Branges spaces; sharpened L91 Z2)	.70	.50	.62
W3	Finite Deninger toys: build $(\bar{H}^1, \Theta, \cup, *)$ from L88 fakes and real curves; show the axioms fail exactly at $*$ -positivity, and determine which explicit-formula data force a Θ with a duality pairing but no $*$	.55	.60	.57
W5	Optimal envelope: replace the bump by prolate spheroidal windows and minimize the intercept $c \approx 0.5$ in $a_{\min} = \pi D + c$	.45	.75	.57
W4	Fibre versus base over $\mathbb{F}_q(t)$: for elliptic surfaces compute $L(E \otimes E'/K)$ zeros and the $H^1 \otimes H^1$ of the base curve side by side (makes Prop. 3.2(c) numerical)	.50	.55	.52

7. QUESTIONS FOR J.W.

- Q1. Is “a zero pairs with its functional-equation partner; RH says that partner is also its conjugate” the meaning you were after? Or did you have in mind sums $\rho + \rho'$ as a *new* spectrum (§3)?
- Q2. Next: W2 (function-field calibration, high probability) or W1 (the density-threshold theorem, higher significance)?
- Q3. The pair set $\{\rho + \rho'\}$ grows like $N(T)^2$ and no known L -function carries it. Should we still look for an object with that spectrum (a “zeta of the square”), or set it aside in favour of the sign of the duality pairing?
- Q4. Do you want a lemma treating W_∞ as a *continuous spectrum* (Deninger 2002’s spectral density $m(\lambda)$), making “ $H^1 \otimes H^1$ at ∞ ” precise as far as it goes?
- Q5. The L89 erratum (L91 C9: “one per ghost” is the even half; the full class has 2 per quartet) is still recorded only in L91 and here. Should it be written into lemma89.pdf?

8. CORRECTIONS TO CHAT CLAIMS

L93.C1 (chat t393 answer: “ $\zeta(s)^2 \dots$ only gives the ‘pairs of points’ part, the H^0 and H^2 pieces”):

Wrong as stated. $Z_C(T)^2 = Z_{C \sqcup C}(T)$ contains $H^1 \oplus H^1$ (each zero twice) as well as H^0, H^2 twice. What it lacks is $H^1 \otimes H^1$ (the products). Over $\mathbb{Q}$, ζ^2 has the zeros of ζ with doubled multiplicity.

L93.C2 (“ ζ^2 zeros are just zeros of ζ counted twice, never $\rho + \rho'$ ”):

Correct ([THEOREM]). The same holds for Rankin–Selberg products, whose zeros are not pair sums either (Prop. 3.2).

L93.C3 (over- $\mathbb{F}_q$ account: “classes with $\alpha_i \alpha_j = q$ are Tate classes, equal to NS beyond H, V ”):

Correct for actual curves (Thm. 2.2; Tate’s theorem for $C \times C$), with $\rho = 2 + \sum_\lambda m(\lambda)m(q/\lambda)$. *Precision:* for fakes the same formal count holds, $2g$ for squarefree P , so the count of Tate pairs never detects a ghost (Prop. 2.3).

L93.C4 (“functional-equation partner q/α versus conjugate $\bar{\alpha}$: RH $\iff$ they coincide”):

Correct ([THEOREM]; Thm. 1.1(iii)).

L93.C5 (“for a ghost, the pair $(\alpha, \bar{\alpha})$ gives $|\alpha|^2 \neq q$ and a $(1, 1)$ block violating Hodge index”):

Half right. $\alpha \bar{\alpha} = |\alpha|^2 \neq q$ is correct, but that product is *not* where the violating class lives. The Tate eigenvalue still comes from $(\alpha, q/\alpha)$. The violation arises because conjugation glues $e_\alpha \otimes e_{q/\alpha}$ to $e_{\bar{\alpha}} \otimes e_{q/\bar{\alpha}}$, giving inertia $(1, 1)$ for a real off-circle pair and $(2, 2)$ for a complex quartet. The total count is k (Thm. 2.4).

L93.C6 (the $\mathbb{Q}$ additive analogy: “special pairs $\rho + \rho' = 1$ are the Tate classes; on the line $1 - \rho = \bar{\rho}$)”):

An Analogy, exact over $\mathbb{F}_q$ modulo $2\pi i/\log q$ (Thm. 2.1). Over $\mathbb{Q}$ there is no Frobenius and no period. Special pairs exist for every zero, ghosts included, so they do not detect RH; “ $1 - \rho = \bar{\rho}$ for all ρ ” is RH restated.

L93.C7 (“Deninger’s programme predicts such a spectrum heuristically”):

Partly. Deninger (ICM 1998, pp. 168–169) predicts the duality pairing $H_{\Theta \sim \alpha}^1 \times H_{\Theta \sim 1-\alpha}^1 \rightarrow H^2$, i.e. the special pairs, and the conditional $*$ -argument $\Theta = \frac{1}{2} + A$. His 2002 survey proves the Leibniz rule and Künneth in the foliation setting, and RH-type conclusions only where $\alpha = 0$ is forced or in the ordinary CM elliptic case. A spectrum $\{\rho + \rho'\}$ on $H^1 \otimes H^1$ of a square is *our* extrapolation; it is not stated in the opened texts.

L93.C8 (L91 chat: “the $H^1 \otimes H^1$ part, zeros paired with zeros, is exactly what the squaring map misses”):

Correct, with one precision: ψ^2 keeps the diagonal α_i^2 . The missed off-diagonal part contains the duality pairs $\alpha_i \cdot q/\alpha_i = q$, the algebraic classes.

L93.C9 (L91 X2 reading “ $a_{\min} \approx \frac{1}{2} \log(\gamma/2\pi) + 0.4\text{--}0.58$ ”):

Confirmed, and now interpreted. The leading term is $\pi \times$ the archimedean zero density, and lattices reproduce the constant ($c = 0.46\text{--}0.60$ for weak ghosts). Still [NUMERICAL]/[HEURISTIC], not a theorem.

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TAA LEMMA 94: THE AMPLE CLASS AND THE RATIONAL STRUCTURE (P4, P9)

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis. Every claim carries one label: [THEOREM] (proved here or classical with proof indicated), [RESTATEMENT] (a reformulation, no new content about ζ), [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (computed; scripts and data in `num/lemma94_*.py`, `num/194/`). Sources: [CITED] (opened), [CITED-sec] (secondary), [BIB] (bibliographic), [TAA-L] (earlier TAA lemma).
References: `lit/lemma94/verified_refs.md`.

ABSTRACT. The synthesis draft (Ch. 16) names two missing ingredients for a Weil-type argument on the Connes–Consani (CC) square: **P4**, an ample class whose hyperbolic plane is the pole plane, and **P9**, a rational structure on the correspondences Ψ_λ . We test new constructions for both, over $\mathbb{F}_q$ (where everything is known) and over $\mathbb{Q}$ (on smoothed Ψ_λ , prime side computed directly). The verdict is that *both P4 and P9 exist in their weak forms over $\mathbb{Q}$, and in those forms they are ghost-blind*. The RH content sits elsewhere: in a Riemann–Roch-type effectivity axiom, or, equivalently over $\mathbb{F}_q$, in the *additive* (ring) structure of the correspondences.

Results. (1) Over $\mathbb{F}_q$, effectivity of the Frobenius graphs ($\Gamma_m \cdot \Gamma_n \geq 0$ and nonnegative Euler exponents) does not imply RH. In genus 1, $N_k = (1 - \alpha^k)(1 - \beta^k)$, so every $2\sqrt{q} < |a| \leq q + 1$ is an “effective fake” (Thm 2.2). In the 1000 L88 fakes, 81 violate RH yet are effective on the Frobenius cone ([NUMERICAL]). (2) Effectivity of graphs of *all ring elements* $m + n\pi$ is RH (Hasse; classical). A ghost is an endomorphism of negative degree, and the Frobenius powers approach the null cone from the effective side without crossing it (Fig. 2). (3) Over $\mathbb{Q}$, intersection numbers of separated effective (nonnegative) classes are ≥ 0 *unconditionally* (Thm 3.1). An extra *on-line* zero violates this just as a ghost does, while a ghost seen at coarse resolution can respect it. So effectivity and Hodge index are independent, over $\mathbb{Q}$ (Fig. 3) as over $\mathbb{F}_q$. (4) A formal Nakai–Moishezon class $h = H + V$ exists over $\mathbb{Q}$ (Prop. 4.1). At moderate smoothing a Perron class built from integer correspondences alone is positive on every generator. Hodge index then follows from Nakai only together with an RR-type axiom (Prop. 2.4). (5) An RH-equivalent statement phrased purely on the effective cone is the Lorentzian reverse Cauchy–Schwarz inequality $(D \bullet D')^2 \geq D^2 D'^2$ for D interior (Thm 4.2, [RESTATEMENT]). Numerically, ghost violations of it are confined to a thin sliver of the cone around D . (6) L91’s “integers barely detect, primes never” is a range statement: primes do detect ghosts once $N \approx 80\text{--}452$ ([NUMERICAL], Table 1). (7) J.W.’s additions (t399u, t400u): $\text{Spec } \mathbb{Z}^{\times 4}$ collapses, but CM $E \times E$ has a Minkowski (1, 3) NS whose ample cone *is* the future light cone (§5). The difference map $C \times C \rightarrow J$ contracts Δ with $\delta^* \theta = (g - 1)(H + V) + \Delta$ on the nef boundary. $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Q}_+^\times$ is the arithmetic analogue, and the CC form factors through it (rank $43 = \#\{m/n\}$ on 8×8 lattice points), so the square carries no data beyond its quotient (§6).

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Date: 24 October 2026 (t398u: “Lets do a bit more work. Now on P4 and P9 ... See if we can come up with anything new.”)

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GLOSSARY (BRIEF RESTATEMENTS FOR J.W.)

Effective divisor / correspondence.:

A nonnegative integer combination of curves on a surface S . On $S = C \times C$ the Frobenius graphs $\Gamma_n = \Gamma_{F^n}$, the diagonal $\Delta = \Gamma_0$, $H = C \times \text{pt}$ and $V = \text{pt} \times C$ are effective. Over $\mathbb{Q}$ (§3) “effective” means a nonnegative test function $\psi \geq 0$, i.e. a positive superposition of Ψ_λ ’s, together with nonnegative multiples of H, V .

Nef.:

D is nef if $D \cdot C \geq 0$ for every irreducible curve C .

Ample.:

Some multiple of D embeds S in projective space. Numerically (Nakai–Moishezon, below) ample means positive square and positive on all curves. The ample cone is the interior of the nef cone (Kleiman).

Nakai–Moishezon.:

On a smooth projective surface, D is ample iff $D^2 > 0$ and $D \cdot C > 0$ for every irreducible curve C [1] V §1 [CITED]. The proof first uses Riemann–Roch to make a multiple of D effective. That is the step that has no $\mathbb{Q}$ -analogue (§4).

Hodge index.:

On $\text{NS}(S)_\mathbb{R}$ the intersection form has signature $(1, \rho - 1)$, so it is negative definite on $h^\perp$ whenever $h^2 > 0$ [1] V Thm 1.9 [CITED].

Positive cone, Lorentzian.:

$\{D : D^2 > 0, D \cdot h > 0\}$. A symmetric form is *Lorentzian* if it has exactly one positive eigenvalue. For timelike D such a form satisfies reverse Cauchy–Schwarz, $(D \cdot D')^2 \geq D^2 D'^2$.

Metzler matrix.:

A symmetric matrix with nonnegative off-diagonal entries. Perron–Frobenius: its top eigenvector can be chosen ≥ 0 .

Frobenius cone.:

The cone spanned by H, V, Γ_n ($n \geq 0$). It is the $\mathbb{F}_q$ analogue of the cone spanned by Ψ_n , $n \in \mathbb{N}$.

Pole plane, V_0 .:

Over $\mathbb{Q}$, $\langle H, V \rangle$ is represented by the functionals $F(\pm i/2)$, and $V_0 = \{F(i/2) = F(-i/2) = 0\}$ is its orthogonal complement (codimension 2).

SUMMARY OF VERDICTS

Construction	Status	RH content?
(a) Frobenius-cone effectivity over $\mathbb{F}_q$	[THEOREM] $g = 1$ band theorem; [NUMERICAL] 81/853 ghost fakes effective	none: ghost-blind
(a) ring effectivity (deg ≥ 0 on $\mathbb{Z}[\pi]$)	[THEOREM] (Hasse/Rosati)	classical =RH, but needs addition of maps
(b) $\mathbb{Q}_+^\times$ -span, off-diagonal ≥ 0	[THEOREM] unconditional	none: violated by on-line extra zeros too
(b) $\mathbb{Q}$ -span as a lattice	[THEOREM] fails: $\Psi^2 = -\infty$, prime-power pairs $+\infty$ as $R \rightarrow 0$	obstruction to P9 in finite rank
(b) integer/prime detection	[NUMERICAL] thresholds (Table 1)	range effect, not effectivity
(c) Nakai class $h = H + V$ / Perron class	[THEOREM] formal; [NUMERICAL] Perron at $R \in [0.01, 0.05]$	none by itself
(c) Nakai + (RR) $\Rightarrow$ Hodge index	[THEOREM] (formal)	(RR) is the content ([OPEN])
(c) reverse C–S on effective cone	[RESTATEMENT]	=RH; violations thin ([NUMERICAL])
(d) Hecke over $\mathbb{C}$, Bost–Connes, Borger	[THEOREM]/[CITED]	no new handle (§7)

1. WHAT P4 AND P9 ASK FOR, STATED PRECISELY

On $S = C \times C$ over $\mathbb{F}_q$, [2] and [TAA-L] L89 (A) give $\text{NS}(S) \cong \mathbb{Z}H \oplus \mathbb{Z}V \oplus \text{End}(J)$. A graph Γ_f of a morphism of degree d has class $H + dV + D_f$, where $D_f \perp \langle H, V \rangle$ and $D_f \cdot D_g = -\text{Tr}(fg^\dagger)$ [CITED]. Hence

$$(1) \quad \Gamma_m \cdot \Gamma_n = q^m + q^n - q^n \text{Tr}(F^{m-n}) = q^n N_{m-n} \quad (m > n), \quad \Gamma_n^2 = q^n(2 - 2g),$$

which is [THEOREM] (a two-line computation from L89 (A)). The ample class $h = H + V$ satisfies $h^2 = 2$ and $h \cdot \Gamma_n = q^n + 1$.

P4 (synthesis Table 16.1) asks for h with the pole plane $\langle H, V \rangle$ as its hyperbolic plane and V_0 as complement. Precision (C94.3 below): V_0 is the complement of the *plane*, while $h^\perp = V_0 \oplus \mathbb{R}(H - V)$. The extra direction $H - V$ has $(H - V)^2 = -2 < 0$, so Hodge index on $h^\perp$ is equivalent to negativity on V_0 [THEOREM].

P9 asks for a $\mathbb{Q}$ - or $\mathbb{Z}$ -structure on the Ψ_λ such that positivity comes from effectivity. Over $\mathbb{F}_q$ this is $\text{NS} \otimes \mathbb{Q}$ with the Frobenius graphs effective. The remainder of this lemma asks *which* effectivity, and shows that the obvious candidates carry no RH content.

2. (A) OVER $\mathbb{F}_q$: WHICH EFFECTIVITY CARRIES RH?

We test three conditions, each stated using only data that has a $\mathbb{Q}$ -analogue:

(N): Nakai for $h = H + V$ on the Frobenius cone: $h^2 > 0$ and $h \cdot \Gamma_n > 0$.

(E1): Distinct Frobenius graphs meet nonnegatively: $\Gamma_m \cdot \Gamma_n \geq 0$ for $m \neq n$. By (1) this is $N_k \geq 0$ for all $k \geq 1$.

(E3): Euler effectivity: the exponents a_d in $Z(T) = \prod_d (1 - T^d)^{-a_d}$ (the numbers of closed points of degree d) are nonnegative integers. Here $da_d = \sum_{e|d} \mu(d/e)N_e$, and (E3) $\Rightarrow$ (E1).

(N) holds for *every* q -Weil-type polynomial, true or fake, since $h \cdot \Gamma_n = q^n + 1$ [THEOREM].

Proposition 2.1 (Landau-type bound). *For any real q -palindromic P with roots α_i : (E1) $\Rightarrow \max |\alpha_i| \leq q$. [THEOREM]*

Proof. Let $M = \max |\alpha_i| > q$ and let $\theta_1, \dots, \theta_m$ be the arguments of the roots of modulus M . By Dirichlet's simultaneous approximation there are infinitely many n with $\|n\theta_j/2\pi\| < 1/8$ for all j . For those n , $\Re \sum_{|\alpha_i|=M} \alpha_i^n \geq (m/\sqrt{2})M^n$. The other roots contribute $o(M^n)$, and $\sum_i \alpha_i^n$ is real. Hence $N_n = q^n + 1 - \sum \alpha_i^n < 0$ for large such n . $\square$

So Frobenius-cone effectivity confines the spectrum only to the closed “critical strip” $|\alpha| \leq q$. This is the function-field form of the fact that the Euler product with $\Lambda \geq 0$ gives only $\Re s \leq 1$ (Landau's theorem on Dirichlet series with nonnegative coefficients) [ANALOGY].

Theorem 2.2 (effective fakes in genus 1). *Let $P = T^2 - aT + q$ with $a \in \mathbb{Z}$ and $a^2 \neq 4q$, and let α, β be its roots. Then $N_k = (1 - \alpha^k)(1 - \beta^k)$, and*

$$(E1) \iff |a| \leq q + 1, \quad RH \iff |a| < 2\sqrt{q} \iff \text{Gram}\{H, V, \Delta, \Gamma\} \text{ has signature } (1, 3).$$

Hence every a with $2\sqrt{q} < |a| \leq q + 1$ is a fake that satisfies (N) and (E1) but violates Hodge index. Its Gram has signature (2, 2) and $\det = a^2 - 4q$. [THEOREM]

Proof. $N_k = q^k + 1 - \alpha^k - \beta^k = (1 - \alpha^k)(1 - \beta^k)$ because $\alpha\beta = q$. For on-circle roots $N_k = |1 - \alpha^k|^2 \geq 0$. For real roots of equal sign and $|\alpha| \geq |\beta|$: if $|\beta| \geq 1$, then both factors are ≤ 0 when $a > 0$ or k is even, and both are ≥ 2 when $a < 0$ and k is odd, so $N_k \geq 0$. If $|\beta| < 1$, then for even k we have $\alpha^k > 1 > \beta^k$, so $N_k < 0$. Finally $|\beta| \geq 1 \iff |a| \leq q + 1$, and the determinant is a direct expansion. $\square$

(E3) numerics [NUMERICAL] (num/lemma94_fq.py, $K = 60$). We took all a with $|a| \leq q + 4$, $a^2 \neq 4q$, for the 22 prime powers $q \leq 47$: 1088 polynomials.

- 368 satisfy RH.
- 566 are fakes satisfying (E1) and (E3).
- 22 are fakes satisfying (E1) but not (E3). These are exactly $a = -(q + 1)$, the pole-degenerate root $\beta = -1$, with $a_2 < 0$.
- 132 are fakes failing (E1).

“(E1) $\iff |a| \leq q + 1$ ” held in 1088/1088 cases, and “RH $\iff n_+ = 1$ ” in 1088/1088. Whether (E3) holds for *all* d in the band is Q2.

L88 fakes [NUMERICAL] (the 1000 squarefree polynomials of L88/L92, seed 8888, $g \leq 3$, $q \leq 7$):

- 853 have $k > 0$ (off-circle roots).
- 223 satisfy (E1) $\wedge$ (E3). Of these, **81 have** $k > 0$: 47 with $g = 1$, 24 with $g = 2$, 10 with $g = 3$. One example is $g = 2$, $q = 7$, $T^4 + 4T^3 + 25T^2 + 28T + 49$ with $k = 2$.
- Of the 147 members with $k = 0$, 5 fail (E1) or (E3). An example is $T^4 - 7T^3 + 22T^2 - 35T + 25$ over $\mathbb{F}_5$, which has $N_1 = -1$. These are RH-satisfying but “non-effective”, so they are not curves.
- $n_+(\text{Gram}\{H, V, \Gamma_0, \dots, \Gamma_{2g+2}\}) = 1 + k$ in 1000/1000 (L88 Thm B again).
- (E1) $\implies \max |\alpha| \leq q$ held in all cases (Prop. 2.1).

All 50 true curves of L92 satisfy (E1) $\wedge$ (E3). So *effectivity and RH are independent*: both off-diagonal cells of the 2×2 table are populated (Fig. 1).

2.1. Ring effectivity: where RH actually sits over $\mathbb{F}_q$. For $g = 1$, every $\varphi = m + n\pi \in \mathbb{Z}[\pi]$ is an endomorphism (an isogeny when nonzero). Its graph is an effective curve of class $\Gamma_\varphi = H + \deg(\varphi)V + D_\varphi$ with $\Gamma_\varphi^2 = 0$. So $D_\varphi^2 = -2\deg \varphi$ and $\Gamma_\varphi \cdot H = \deg \varphi = m^2 + amn + qn^2$ ([3] III.6, V.1; [TAA-L] L88 Thm A) [CITED]. The lattice class $H + \deg(\varphi)V + D_\varphi$ makes sense in a *fake* lattice too.

Proposition 2.3 (Hasse, in effectivity form). *For $g = 1$: $RH \iff$ every graph class Γ_φ ($\varphi \in \mathbb{Z}[\pi]$) meets H nonnegatively $\iff \deg$ is positive definite on $\mathbb{Z}[\pi]$. For a fake, the classes with $\Gamma_\varphi \cdot H < 0$ are “endomorphisms of negative degree”; D_φ is then a positive class in $h^\perp$. For general g the same holds with $\deg$ replaced by $\text{Tr}(xx^\dagger)/2g$ on $\mathbb{Z}[F]$ (Rosati; [TAA-L] L84, L88 Thm B). [THEOREM] (classical)*

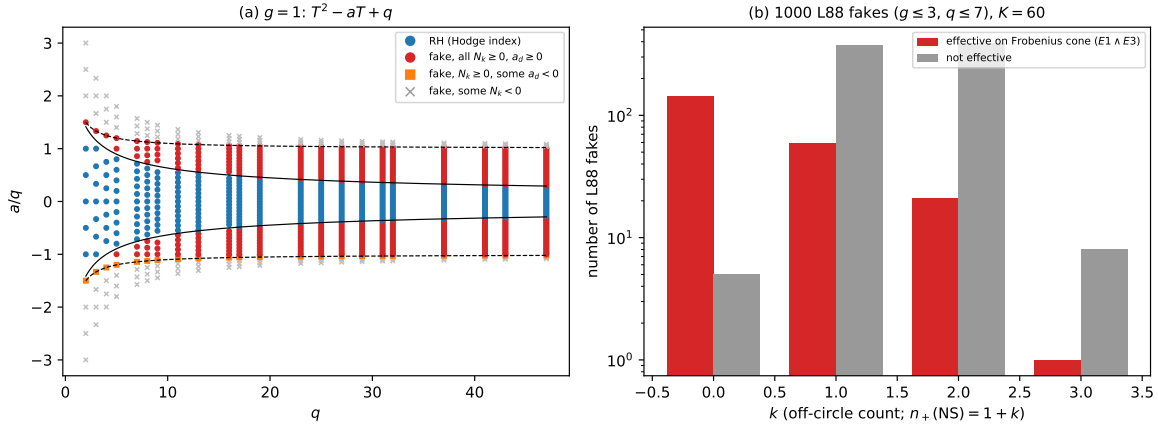


FIGURE 1. (a) Genus 1: traces a/q by type. Red marks the effective fakes, filling the band $2\sqrt{q} < |a| \leq q + 1$ between the solid $(\pm 2/\sqrt{q})$ and dashed $(\pm(q+1)/q)$ curves. (b) The L88 fakes by off-circle count k , effective ($E1 \wedge E3$) or not. *Diagnostic*: if Frobenius-cone effectivity carried RH, the red bars at $k \geq 1$ would be empty. They are not.

The monoid $\{\pi^n\}$ never enters the negative region, since $\deg \pi^n = q^n > 0$. As $n \rightarrow \infty$ the direction of $\pi^n = u_n + v_n \pi$ tends to an eigen-direction of multiplication by π . For a fake this direction is a null direction of $\deg$, so the powers approach the light cone from inside and never cross it. The negative region is reached only by mixed elements $m + n\pi$, and forming those requires *adding maps* via the group law (Fig. 2). [THEOREM]/[NUMERICAL]

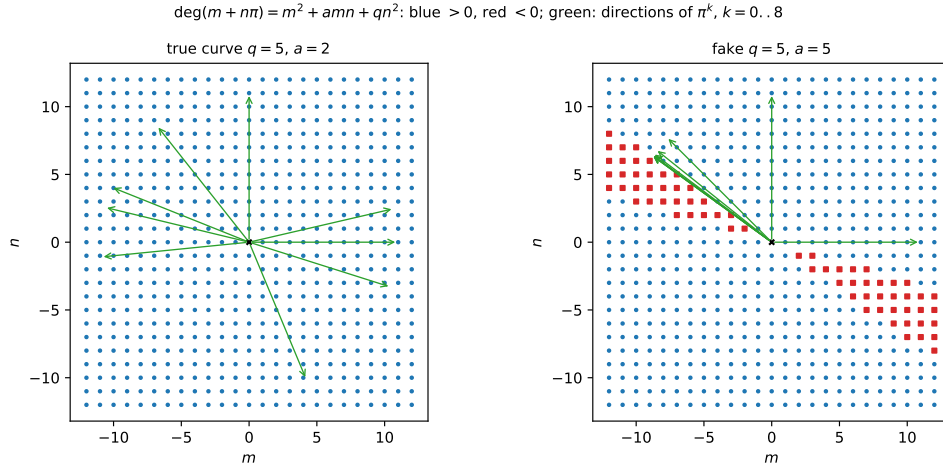


FIGURE 2. $\deg(m + n\pi)$ on $\mathbb{Z}[\pi]$. Left: true $E/\mathbb{F}_5$, $a = 2$. Right: fake $a = 5$, an effective fake by Thm 2.2. Green arrows: directions of π^k , $k \leq 8$. *Diagnostic*: in the fake, the Frobenius monoid hugs the boundary of the red (negative-degree) sector without entering it. Detecting the ghost needs ring elements, not effective Frobenius divisors.

2.2. Cones of $E \times E$ and the RR step. On an abelian surface every irreducible curve C satisfies $C^2 \geq 0$: a general translate $C + t$ shares no component with C , so $C^2 = C \cdot (C + t) \geq 0$. Hence the

closed effective cone, the nef cone and the closure of the positive cone coincide, and the ample cone is $\{D^2 > 0, D \cdot h > 0\}$. This is [THEOREM] (classical; Nakai–Moishezon [1] V §1, Riemann–Roch for the inclusion of the positive cone in the effective cone). For ordinary $E/\mathbb{F}_q$ these cones sit in $\text{NS}_{\mathbb{R}} = \langle H, V, \Delta, \Gamma \rangle$ with Gram determinant $a^2 - 4q < 0$. The Frobenius graphs are null ($\Gamma_n^2 = 0$ for $g = 1$), so they lie on the boundary of the cone.

In a fake “lattice” (signature $(2, 2)$) the set $\{D^2 > 0\}$ is not a convex double cone, so *no ample cone exists*, although (N) and (E1) hold. What fails is the Riemann–Roch step in the proofs of Hodge index and Nakai–Moishezon:

Proposition 2.4 (Nakai + RR $\Rightarrow$ Hodge index). *Let $(L, \cdot)$ be a finite-dimensional real quadratic space, $K \subset L$ a cone, and $h \in L$ with $h^2 > 0$. Assume (N) $h \cdot E > 0$ for all $0 \neq E \in K$, and (RR) every D with $D^2 > 0$ lies in $K \cup (-K)$. Then the form is negative semidefinite on $h^\perp$, and negative definite if it is nondegenerate there. [THEOREM] (the argument of [1] V.1.9)*

Proof. If $D \perp h$ and $D^2 > 0$, then $\pm D \in K$ by (RR), so $h \cdot (\pm D) > 0$ by (N), contradicting $D \cdot h = 0$. So $D^2 \leq 0$ on $h^\perp$, and a nondegenerate semidefinite form is definite. $\square$

For fakes, (N) holds on the Frobenius cone and on any cone generated by graph classes of positive degree. (RR) fails exactly at the ghost classes D_φ of Prop. 2.3.

3. (B) OVER $\mathbb{Q}$: THE $\mathbb{Q}_+^\times$ -SPAN AND ITS EFFECTIVE CONE

Setup (`num/lemma94_qlib.py`). Let $\psi_x(t) = M_m((t-x)/h)/h$ be the nonnegative B-spline bump of order $m = 4$ and radius $R = mh$ at $x = \log \lambda$. It is the smoothed Ψ_λ , in L91/L92’s unitary normalization. The intersection number is CC’s $s(f, g) = D \bullet D'$, computed from the prime side [TAA-L] L92 Prop. 1.1:

$$(2) \quad I(\psi_x, \psi_y) = N(k_d), \quad N(k) = \sum_n \frac{\Lambda(n)}{\sqrt{n}} k(\log n) + \int_0^\infty \frac{e^{3t/2} k(t) - k(0)}{e^{2t} - 1} dt + C k(0),$$

where k_d is the symmetrized cross-correlation, $d = |x - y|$, and $C = \frac{1}{2}(\log \pi + \gamma_E)$. We checked numerically that $I = F(\frac{i}{2})F(-\frac{i}{2}) - \frac{1}{2} \sum_\rho F(r_\rho)F(-r_\rho)$ (pole plane minus zero form): with 2500 zeros and $R = 0.15$, the ratio is 1 to within $1.2 \cdot 10^{-12}$ on random vectors [NUMERICAL]. So I is the full Lorentzian form, and I depends only on d : it is translation invariant on $\log \mathbb{Q}_+^\times$.

Theorem 3.1 (unconditional off-diagonal positivity). *If $\psi, \varphi \geq 0$ and $\text{supp } \psi - \text{supp } \varphi$ avoids a neighbourhood of 0, then $I(\psi, \varphi) \geq 0$. In particular the Gram matrix of $\{\psi_{\log n}\}$ with pairwise separated supports is Metzler. This holds for ζ with no hypothesis on its zeros. [THEOREM]*

Proof. In (2) we have $k \geq 0$ and $k(0) = 0$. Both remaining terms are integrals of k against nonnegative measures: $\Lambda(n)n^{-1/2}\delta_{\log n}$ and $e^{3t/2}/(e^{2t} - 1) dt$. $\square$

This is the $\mathbb{Q}$ -analogue of (E1). Its proof uses only $\Lambda \geq 0$ and the positivity of the archimedean density, i.e. the Euler product (P3), and not P5. In the numerics ($N = 60$) the minimum off-diagonal entry is 0.0646, which equals half the archimedean density at $d = \log 60$.

Theorem 3.2 (effectivity and Hodge index are independent over $\mathbb{Q}$). [NUMERICAL] (`num/lemma94_metzler.py`, integers $n \leq 60$, Fig. 3). *Write $I_{\text{fake}} = I - \frac{1}{2}G$, where G is the zero form of the added zeros.*

- An extra on-line pair at $\gamma = 40$ (RH-compatible) makes 24.6% of the separated pairs negative at $R = 0.004$, and 15.4% at $R = 0.1$. It keeps $n_+(I) = 1$: effectivity fails, Hodge index holds.
- A ghost quartet at $\gamma = 40$, $\delta = 0.3$ at $R = 0.4$ leaves every separated pair positive (minimum 0.509) but gives $n_+(I) = 3$: effectivity holds, Hodge index fails.

So, as over $\mathbb{F}_q$, the two properties are logically independent. Off-diagonal effectivity tests whether the zero set is *exactly* that of an Euler product, i.e. rigidity. It does not test where the zeros lie.

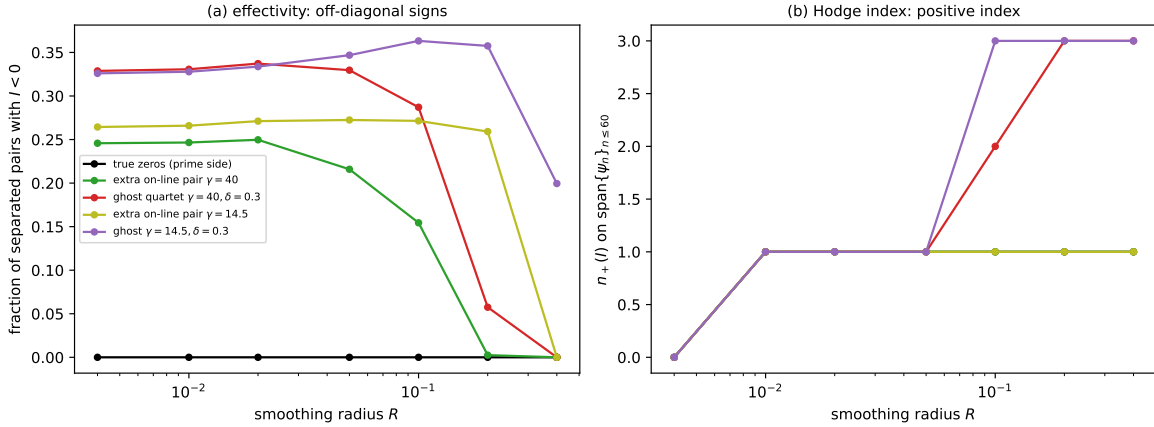


FIGURE 3. Integer correspondences $\psi_{\log n}$, $n \leq 60$, against the smoothing radius R . (a) Fraction of separated pairs with negative intersection: zero for ζ (Thm 3.1), positive for on-line extra zeros and for ghosts. (b) $n_+(I)$: equal to 1 for ζ and for on-line extra zeros (0 at $R = 0.004$, see Prop. 3.3), larger for ghosts at $R \geq 0.1$. *Diagnostic:* the curves in (a) and (b) are not ordered the same way, so effectivity and signature measure different things.

Proposition 3.3 (the sharp $\mathbb{Q}$ -span is not a lattice). *As $R \rightarrow 0$ at fixed points, $I(\psi_x, \psi_x) \sim -\frac{1}{2}c(0)\log(1/R) \rightarrow -\infty$, with $c(0) = M_{2m}(0)/h$. If $d = \log(p^k)$, then $I(\psi_x, \psi_{x+d}) \sim \frac{1}{2}\Lambda(p^k)p^{-k/2}c(0) \rightarrow +\infty$; the other off-diagonal entries stay finite. Hence the $\mathbb{Q}$ -span of the unsmoothed $\Psi_{m/n}$ carries no finite intersection form. Moreover, for fixed finite S , $n_+(I_S) = 0$ once R is small: no positive class lies in the span of sharp correspondences. [THEOREM] (leading-order asymptotics of (2); $\Delta^2 = 2 - 2g \rightarrow -\infty$, [TAA-L] L90 §7(3), [14] §2.3 “ $N(1) = -\infty$ ”)*

[NUMERICAL] ($N = 60$): the mean diagonal is $-1105, -470, -354, -144, -40$ for $R = 0.004, 0.008, 0.01, 0.02, 0.05$. The top eigenvalue λ_1 is -219.6 and -10.6 at $R = 0.004$ and 0.008 , then $+19.7, +62.0, +74.2$ at $R = 0.01, 0.02, 0.05$.

Perron class (a constructive P4 candidate). For $R \in \{0.01, 0.02, 0.05\}$ the integer Gram is Metzler (all pairs) with $\lambda_1 > 0$. Its Perron vector v is > 0 , so v is an effective class with $v^2 > 0$ and $I(v, \psi_n) > 0$ for every generator. It is “ample relative to the cone” [NUMERICAL] (num/194/perron_vsR.json). It exists for fakes too whenever the matrix is Metzler (Perron–Frobenius), so it is ghost-blind by construction. At $R \leq 0.008$ it fails ($\lambda_1 < 0$), as Prop. 3.3 predicts.

3.1. Integer and prime blindness revisited.

Proposition 3.4. $V_0 \cap \{\psi \geq 0\} = \{0\}$: a nonnegative ψ has $F(\pm i/2) = \int \psi e^{\mp t/2} > 0$. Hence every ghost-detecting vector is a signed combination, and the effective cone itself never detects anything. [THEOREM]

So the blindness L91 found is a property of the signed $\mathbb{Z}$ -span on a sparse sampling set, not of effectivity. Table 1 gives the first N in the grid $\{5, 7, 10, \dots, 452, 640\}$ at which $\lambda_{\min}(Q+G)|_{V_0 \cap \text{span}} < -10^{-8}\lambda_{\max}$. Here $Q = -2I$ is computed from the prime side, and round-off is $\geq -3 \cdot 10^{-14}$ relative (num/lemma94_qscan.py). **Reading.** Primes *do* detect ghosts once $N \approx 80$ –640 (67–116 primes). L91’s “primes never” was a statement about its range ($N \lesssim e^{3.5} \approx 33$), and is consistent with this table. Rationals with denominator ≤ 3 detect at about half the integer N . The uniform control sometimes detects *later* than the integers at large γ . That is an artefact of our relative tolerance on dense, near-degenerate grids, so we do not claim a sampling law (Q4).

TABLE 1. First detecting N (support $[0, \log N]$). $\text{int} = \{\log n\}_{n \leq N}$; $\text{prime} = \{0\} \cup \{\log p\}_{p \leq N}$; $\text{rat3} = \{\log(m/k)\}$, $k \leq 3$, $m/k \in [1, N]$; $\text{unif} = \text{grid of step } R/4$. Entries $\text{i/p/r/u} = \text{int/prime/rat3/unif}$. “–” means not detected for $N \leq 640$. [NUMERICAL]

R, δ	sets	$\gamma = 20$	30	40	60	85.7
0.1, 0.3	i/p/r/u	40/320/20/5	40/320/20/5	56/320/28/10	80/452/40/14	80/452/40/40
0.2, 0.3	i/p/r/u	28/80/10/5	20/113/10/5	28/113/14/10	56/320/20/80	56/320/28/80
0.1, 0.1	i/p/r/u	80/640/28/5		80/640/40/10		80/–/40/80

4. (C) NAKAI–MOISHEZON IN THE CC/WEIL SETUP

Proposition 4.1 (formal P4). *Adjoin H, V to any finite span S by $H \cdot \psi = F(i/2)/\sqrt{2}$, $V \cdot \psi = F(-i/2)/\sqrt{2}$, $H^2 = V^2 = 0$, $H \cdot V = 1$. Then $h = H + V$ has $h^2 = 2$, and $h \cdot E > 0$ for every nonzero E in $K_S = \{aH + bV + \psi : a, b \geq 0, \psi \geq 0\}$. The extended Gram G_{ext} satisfies $n_+(G_{\text{ext}}) = 1 + n_-(Z|_S)$, by Haynsworth with the Schur complement $I - P = -\frac{1}{2}Z$. [THEOREM]*

Thus Nakai’s numerical conditions hold over $\mathbb{Q}$ for free. By Prop. 2.4, Hodge index would follow from (RR): *every class of positive square is $\pm$ effective*. Over $\mathbb{Q}$ we know no source for (RR). CC’s square lacks the Riemann–Roch theorem and the H^1 needed ([TAA-L] L90 Thm 7.1(v), [14] [CITED]). Whether RH implies (RR) is Q1.

Theorem 4.2 (effective-cone form of RH). *Let L be a finite-dimensional quadratic space and K a cone with nonempty interior, and let $D \in \text{int } K$ with $D^2 > 0$. Then $(D \cdot D')^2 \geq D^2 D'^2$ for all $D' \in K \iff$ the form is ≤ 0 on $D^\perp \iff n_+(L) = 1$. Applied to L_S, K_S (B-spline bases, so K_S has interior) and dense families of S , this gives: $\text{RH} \iff$ the reverse Cauchy–Schwarz inequality holds for all pairs of effective correspondences with D interior and timelike. [THEOREM] (linear algebra) / [RESTATEMENT] (as a statement about ζ)*

Proof. For $D_0 \perp D$ and t large, $D' = D_0 + tD \in K$, and $(D \cdot D')^2 - D^2 D'^2 = -D^2 D_0^2$. The converse is the usual Lorentzian reverse Cauchy–Schwarz inequality. $\square$

[NUMERICAL] (`num/lemma94_revcs.py`, $D = H + V + \sum \psi_n$, $N = 60$, $R = 0.2$). Under the ghost ($\gamma = 40$, $\delta = 0.3$; $n_+ = 3$) an explicit violator $D' \in K$ (minimum coefficient $3.8 \cdot 10^{-4} > 0$) has $(D \cdot D')^2 = 3.346 \cdot 10^6 < D^2 D'^2 = 3.348 \cdot 10^6$. But $2 \cdot 10^4$ random effective D' (sparse exponential coefficients) found *no* violation for true zeros, on-line extra zeros *or* ghosts. The ghost-violating region of the cone is a thin sliver around the ray of D . This is the cone-geometric picture of why effectivity tests are weak.

5. FOUR FACTORS VERSUS A LORENTZIAN NS (J.W. T399U)

J.W.: “What if $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ is the real object? It would be like a 4-vector, maybe?”

Theorem 5.1 (collapse). *$\text{Spec } \mathbb{Z}$ is the terminal object of schemes, so $\text{Spec } \mathbb{Z}^{\times n} = \text{Spec}(\mathbb{Z} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}) = \text{Spec } \mathbb{Z}$ for every n . The absolute fourfold product is a point over itself and carries no intersection theory. [THEOREM] (standard)*

Non-trivial “squares” exist only over a base below $\mathbb{Z}$: CC’s $\bar{\mathbb{N}} \otimes_{\mathbb{B}} \bar{\mathbb{N}} \cong \text{Sub}_{\geq}(\mathbb{N} \times \mathbb{N})$ ([TAA-L] L90 Thm 7.1(i)), or Borger’s $\mathbb{Z} \otimes_{\mathbb{F}_1} \mathbb{Z} = W^*(\text{Spec } \mathbb{Z})$ ([13] §1.2, via [TAA-L] L46). We did not find a fourfold version in the opened sources, and we claim none.

Higher self-products over $\mathbb{F}_q$. On C^n ($n \geq 3$) the positivity that would replace Hodge index is Grothendieck’s *Hodge standard conjecture* [4]: on primitive algebraic classes of codimension $r \leq \dim / 2$, the form $(-1)^r L^{\dim - 2r} x \cdot y$ is positive definite. Status, as stated in [5] (Introduction), [6] (Introduction, Thm. 1.1) and [7] (Introduction) [CITED-sec]/[CITED]:

- It is known in characteristic 0 (Hodge theory).
- It is known in characteristic p for surfaces, where it is the Hodge index theorem (Segre 1937, Grothendieck 1958).
- It is known for abelian fourfolds in characteristic p (Ancona, Invent. Math. 223 (2021), for numerical equivalence). This covers E^4 .
- It is known for squares of K3 surfaces and for all powers of certain K3 surfaces (Ito–Ito–Koshikawa).
- For abelian varieties it follows conditionally from the Hodge conjecture for CM abelian varieties (Milne 2002).
- Otherwise, and in particular for C^n with $g \geq 2$ and $n \geq 3$, it is open.

RH for a curve needs only the surface $C \times C$, where Hodge index is proved [1]. Higher self-products add nothing to Weil’s proof. Over $\mathbb{Z}$ they collapse (Thm 5.1).

Theorem 5.2 (where the 4-vector actually lives). $\mathrm{NS}(E \times E) \cong \mathbb{Z}H \oplus \mathbb{Z}V \oplus \mathrm{End}(E)$, so $\rho = 2 + \mathrm{rk} \mathrm{End}(E)$. This gives $\rho = 3$ without CM (char 0), $\rho = 4$ with CM, and $\rho = 6$ for supersingular E over $\overline{\mathbb{F}}_p$, where End is a maximal order in a quaternion algebra. By Hodge index the signature is $(1, \rho - 1)$. [THEOREM]/[CITED] ([2] via [TAA-L] L89 (A); [1] V.1.9)

Example (CM by $\mathbb{Z}[i]$). $E : y^2 = x^3 - x$ over $\mathbb{C}$, with $[i](x, y) = (-x, iy)$. In the basis H, V, D_1, D_i , using $D_\alpha \cdot D_\beta = -\mathrm{Tr}(\alpha\bar{\beta}) = -2\Re(\alpha\bar{\beta})$,

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \quad D = xH + yV + D_{u+iv} : D^2 = 2xy - 2u^2 - 2v^2 = t^2 - z^2 - 2u^2 - 2v^2,$$

with $t = (x + y)/\sqrt{2}$ and $z = (x - y)/\sqrt{2}$. This is a Minkowski $(1, 3)$ form with $\det G = -4$ [THEOREM] (num/194/extra.json).

- $h = H + V$ has $h^2 = 2 > 0$, so it is timelike.
- Each graph $\Gamma_\alpha = H + |\alpha|^2 V + D_\alpha$ ($\alpha \in \mathbb{Z}[i]$) has $\Gamma_\alpha^2 = 0$ and $h \cdot \Gamma_\alpha = 1 + |\alpha|^2 > 0$. It is a *future null vector*; checked for $\alpha = 1, i, 1 + i, 2 + i, 1 - 2i$.
- On an abelian surface every curve has $C^2 \geq 0$ (translation argument, §2). So the effective curves lie in the closed future cone $\bar{P}$.
- Riemann–Roch puts the positive cone P inside the effective cone [1] V.
- Since $\bar{P}$ is self-dual, $\mathrm{Nef} = \bar{\mathrm{Eff}}^\vee = \bar{P}$. Hence $\mathrm{ample} = P = \{D^2 > 0, D \cdot h > 0\}$ and $\mathrm{nef} = \bar{P}$, for every abelian surface, ordinary or not [THEOREM].

The ample cone is exactly the future light cone (Fig. 4a).

Tie to P4. Ample means future-timelike *and* positive on every curve. For abelian surfaces the second condition is automatic. For $C \times C$ with $g \geq 2$ it is not: $\Delta^2 = 2 - 2g < 0$, so the hyperplane $\Delta^\perp$ cuts through P , and the ample cone is strictly smaller than P . Its boundary contains the nef, big, non-ample class $h_J = \delta^* \theta$ of §6 (Fig. 4b) [THEOREM].

A $\mathbb{Q}$ -analogue of P4 must therefore supply three things:

- a timelike class;
- the list of effective classes of negative square (over $\mathbb{Q}$ the diagonal Ψ_1 has square $-\infty$, Prop. 3.3), which fixes the walls of the ample cone;
- the RR inclusion “positive $\Rightarrow$ effective” (Prop. 2.4).

Only (i) is available (Prop. 4.1). (ii) is available only after renormalization ([TAA-L] L90 §7(3)), and (iii) is the open content.

6. $\mathbb{N} \times \mathbb{N}$, THE DIFFERENCE MAP AND THE DIAGONAL (J.W. T400U)

J.W.: “How about the natural numbers then? Squaring the natural numbers would get you to $\mathrm{Spec} \mathbb{Z}$.”

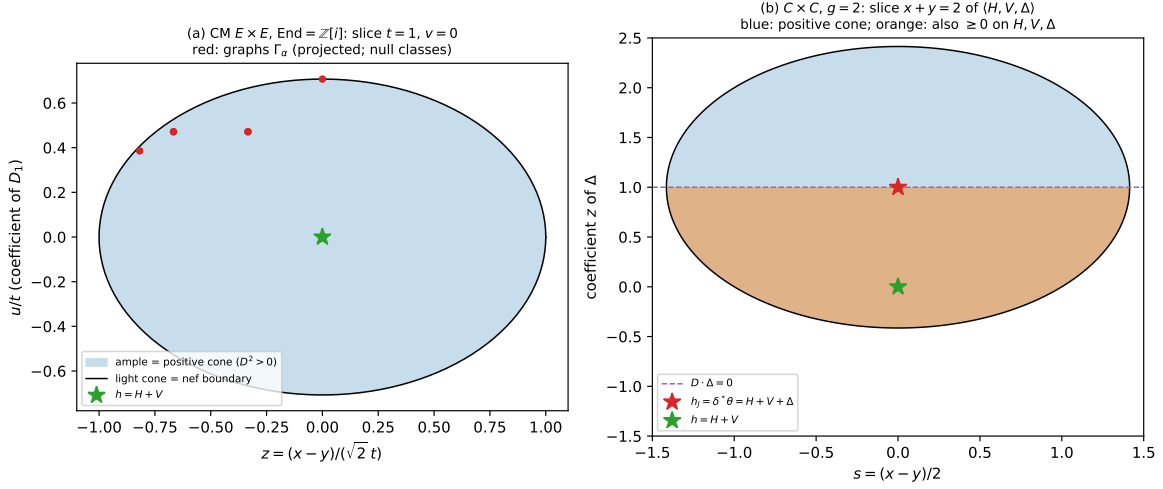


FIGURE 4. (a) $CM E \times E$: slice $t = 1$, $v = 0$ of the $(1, 3)$ space. Ample = positive cone (blue), with the light cone as its boundary. Graph classes are null; those with $\Im \alpha \neq 0$ appear inside because the D_i coordinate is projected out. (b) $C \times C$, $g = 2$, in $\langle H, V, \Delta \rangle$. The positive cone (blue) is cut by $D \cdot \Delta = 0$, and the region also nonnegative on H, V, Δ (orange) is an upper bound for the nef cone. $h_J = H + V + \Delta$ lies on the wall. *Diagnostic*: in (a) the ample cone fills the light cone; in (b) a negative curve removes part of it.

Theorem 6.1 (group completion collapses the diagonal). $\mathbb{Z} = (\mathbb{N} \times \mathbb{N}) / \sim$ with $(a, b) \sim (a + c, b + c)$, via $(a, b) \mapsto a - b$. Each class is a translate of the diagonal, and the diagonal itself goes to 0. The ring structure descends, because $\mathbb{N}$ is a semiring, so this gives the ring $\mathbb{Z}$ and hence $\text{Spec } \mathbb{Z}$. Multiplicatively, $\mathbb{Q}_+^\times = (\mathbb{N}^\times \times \mathbb{N}^\times) / \sim$ via $(m, n) \mapsto m/n$, with the diagonal going to 1. [THEOREM] (standard)

Theorem 6.2 (the difference map). Let C have genus $g \geq 2$, and let $\delta : C \times C \rightarrow J$, $(x, y) \mapsto [x - y]$.

- (a) $\delta(\Delta) = 0$, and the image is the difference surface $C - C$.
- (b) δ is injective off Δ , hence birational onto $C - C$, iff C is non-hyperelliptic. If $x - y \sim x' - y'$ with $(x, y) \neq (x', y')$, then $x + y' \sim x' + y$ is a g_2^1 . If C is hyperelliptic, δ has degree 2 off Δ , since (x, y) and (y, x) have the same image. This holds in every genus, including $g = 2$, where $C - C = J$. [8] §2.1 [CITED] (opened; it cites ACGH p. 223 Ex. D [BIB]). So “ $g \geq 3$ ” is necessary but not sufficient.
- (c) $\delta^* \theta \equiv (g - 1)(H + V) + \Delta$. *Proof*: on H^1 , $\delta^* = p_1^* - p_2^*$, so $\delta^* \theta$ has Künneth components $g[\text{pt}] \otimes 1 + 1 \otimes g[\text{pt}]$ plus $\pm$ the $H^1 \otimes H^1$ part of Δ . The sign is fixed by $\delta^* \theta \cdot \Delta = 0$, because Δ is contracted. Then $(\delta^* \theta)^2 = 2g(g - 1)$, which equals $\deg \delta \cdot \theta^2 = 2 \cdot 2$ for $g = 2$ [THEOREM].
- (d) Any contracted curve has negative square (Hodge index applied to δ^* of an ample class); here $\Delta^2 = 2 - 2g < 0$. Over $\mathbb{C}$, Grauert’s criterion gives analytic contractibility of a curve with negative square [8] §2.1, citing BHPV p. 91 Thm. 2.1 ([9]) [CITED-sec]. Artin’s algebraic criterion [10] also requires $p_a(Z) \leq 0$, which fails for Δ ($p_a = g$). Here the algebraic contraction is supplied by δ itself, via Stein factorization [THEOREM]/[CITED-sec].

Genus-2 example [NUMERICAL] (num/lemma94_extra.py). Take $C/\mathbb{F}_3$ with $P = T^4 - 2T^3 + 2T^2 - 6T + 9$ (an L92 curve), so $N_1 = 2$. The Gram of $\{H, V, \Delta, \Gamma\}$ is $\begin{pmatrix} 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & 2 \\ 3 & 1 & 2 & -6 \end{pmatrix}$, with signature $(1, 3)$. For $h_J = H + V + \Delta$:

- $h_J^2 = 4 = \deg \delta \cdot \theta^2$;
- $h_J \cdot \Delta = 0$, so Δ is contracted;
- $h_J \cdot H = 2 = g$, since $\delta|_{C \times \text{pt}}$ is Abel–Jacobi;
- $h_J \cdot \Gamma = q + 1 + N_1 = 6 > 0$.

So h_J is nef and big, not ample, and $\Delta \subset h_J^\perp$ is negative, as Hodge index requires. For an L88 fake ($q = 4$, $k = 1$) the same numbers $h_J^2 = 4$ and $h_J \cdot \Delta = 0$ come out: *the difference-map class is ghost-blind*. The ghost appears only with $\Gamma_0, \dots, \Gamma_{2g+2}$ (Thm B).

Analogy and P9 test. [ANALOGY] CC’s square is built on $\mathbb{N} \times \mathbb{N}$ ([TAA-L] L90 Thm 7.1(i)), and the correspondences are indexed by the quotient: $\Psi_{m/n}$. The point (m, n) plays the role of (x, y) , and $(m, n) \mapsto m/n$ the role of δ . [THEOREM]: the Weil form on lattice-point classes $\psi_{(m,n)} := \psi_{\log(m/n)}$ factors through m/n . For $\{(m, n) : m, n \leq 8\}$ (64 points, $R = 0.05$) the Gram has rank 43 = $\#\{m/n\}$, and $I(e_{(1,2)} - e_{(2,4)}) = 0$ exactly. The form is a pullback along the quotient, and $n_+ = 1$ (true zeros).

What matches: the diagonal goes to the identity, and the diagonal class is negative ($2 - 2g$; $-\infty$ over $\mathbb{Q}$, Prop. 3.3). What does not match [THEOREM]:

- On $C \times C$, $\delta^* \text{NS}(J) \subset \Delta^\perp$ misses the contracted class Δ , and H, V are not pullbacks either. NS of the square is strictly bigger than what comes from J .
- In the $\mathbb{N} \times \mathbb{N}$ picture the CC form is defined only downstairs, on $\mathbb{Q}_+^\times$. The diagonal *points* (n, n) are identified with one another, and the diagonal *class* Ψ_1 survives.

So $\mathbb{N} \times \mathbb{N}$ carries no intersection information beyond its quotient. The extra classes that make $\text{NS}(C \times C)$ Lorentzian with a finite negative Δ^2 have no counterpart. This is P9 again, now phrased as “the square must not factor through the difference map”.

7. (D) RATIONAL STRUCTURES FROM ELSEWHERE

Hecke correspondences. On $X = X_0(N)$ over $\mathbb{C}$ the T_n are effective, symmetric correspondences of bidegree $(\sigma(n), \sigma(n))$ for $(n, N) = 1$, with a genuine $\mathbb{Q}$ -structure (the Hecke algebra) and Hodge index on $X \times X$ [11, 1]. For a self-adjoint correspondence T , Hodge index applied to D_T, D_Δ gives $\text{Tr}(T|H^1)^2 \leq 2g \text{Tr}(T^2|H^1)$. That is Cauchy–Schwarz in H^1 and carries no arithmetic content [THEOREM].

The Ramanujan bound for weight 2 needs the *asymmetric* pieces from Eichler–Shimura in characteristic p : $T_p = F + F^t$, with F of bidegree $(p, 1)$ and $\text{Tr}(FF^\dagger) = 2gp$, giving $|\text{Tr } F| \leq 2g\sqrt{p}$ ([TAA-L] L20 Thm 2.1, [11] Ch. 8) [CITED]. CC’s Ψ_λ have bidegree $(\lambda, 1)$, so they are of Frobenius type. The “level $\rightarrow \infty$ ” or $q \rightarrow 1$ direction keeps only the symmetric T_n and therefore loses exactly the needed piece. Over $\mathbb{Q}$, moreover, $g = \infty$ makes Weil’s single-correspondence inequality $|\text{Tr } \Psi| \leq 2g\sqrt{\lambda}$ vacuous; only V_0 -combinations can carry content [HEURISTIC].

Bost–Connes. The $\mathbb{Q}$ -structure is the arithmetic subalgebra, with isometries μ_n , $\mu_n \mu_m = \mu_{nm}$ (the $\Psi_n/\sqrt{n}$), $\sigma_t(\mu_n) = n^{it} \mu_n$, partition function $\zeta(\beta)$ [12] [CITED]. The KMS data live in $\beta > 1$, the Euler-product half-plane. That is exactly the region Prop. 2.1 and Thm 3.1 show to be ghost-blind [ANALOGY].

Borger. Λ -structures are commuting Frobenius lifts. There is none on curves of genus ≥ 1 ([13] §2.9, via [TAA-L] L46) [CITED]. So there is no Λ -ring analogue of $\text{End}(J)$, the ring P9 actually needs.

8. CORRECTIONS

C94.1 (chat/synthesis: “the realistic place to build is P4 together with P9”).:

Partly right. In their weak forms both exist over $\mathbb{Q}$: Nakai for $h = H + V$ (Prop. 4.1), a Perron class at moderate smoothing, and the $\mathbb{Q}$ -span of $\Psi_{m/n}$. Both are ghost-blind. Corrected form: “the gap is an RR-type effectivity axiom (Prop. 2.4), or equivalently a finite-rank *ring* of correspondences with a degree form (Prop. 2.3). P4 and P9 in their weak forms are not the bottleneck.”

C94.2 (synthesis: “Over $\mathbb{F}_q$, positivity follows from these by Riemann–Roch. This is J.W.’s ‘divisors first’ instinct (L89), stated in the form that survived the corrections”).:

Needs restating. Positivity over $\mathbb{F}_q$ comes from RR applied to *all* divisors, or from Rosati positivity, which uses an ample class on J . It does not come from effectivity of the Frobenius divisors: 81 L88 ghost fakes and the whole band $2\sqrt{q} < |a| \leq q+1$ satisfy every Frobenius-cone effectivity condition (Thm 2.2). “Divisors first” survives only as “ring first” or “RR first”.

C94.3 (synthesis P4 row: “ $V_0 = h^\perp$ -type complement”):

Precision: $V_0 = \langle H, V \rangle^\perp$ has codimension 2, and $h^\perp = V_0 \oplus \mathbb{R}(H - V)$ with $(H - V)^2 < 0$. The two formulations are equivalent for Hodge index (§1).

C94.4 (synthesis/L91 reading: “integers only a strong ghost at $a = 3.5$, primes never”):

Range statement. Integers detect at $N = 20$ –80 and primes at $N = 80$ –640 for the ghosts tested (Table 1). L91’s numbers stand; “never” should read “not for $N \lesssim 33$ ”.

C94.5 (t399u: “Spec $\mathbb{Z}^{\times 4}$... like a 4-vector”):

The product collapses (Thm 5.1). A Minkowski 4-space does occur, but as $\text{NS}(E \times E)$ of a CM elliptic curve (Thm 5.2). The “4” there is ρ , not the number of factors; for $C \times C$ the Lorentzian space has dimension $\rho = 2 + \text{rk End}(J)$.

C94.6 (steering: “ $\rho = 3$ without CM, 4 with CM in char 0, 6 if supersingular”):

Correct (6 over $\overline{\mathbb{F}}_p$). Precision: over a finite field every ordinary E has $\text{End} \otimes \mathbb{Q} = \mathbb{Q}(\pi)$ imaginary quadratic, so “without CM” does not occur there.

C94.7 (steering: “over ordinary abelian surfaces the ample cone equals the positive cone”):

True for *all* abelian surfaces; “ordinary” is unnecessary (proof in §5).

C94.8 (steering: “Hodge standard conjecture open in char p in dimension ≥ 3 ”):

Correct in general, with the verified exceptions: abelian fourfolds (Ancona 2021) and squares and certain powers of K3 surfaces (Ito–Ito–Koshikawa). For surfaces it is Hodge index.

C94.9 (t400u: “squaring the natural numbers would get you to Spec $\mathbb{Z}$ ”):

Correct as group or ring completion, $\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\sim$. Geometrically it is a *quotient that collapses the diagonal*, the analogue of δ . The CC form lives on the quotient, so “the square” $\mathbb{N} \times \mathbb{N}$ adds no intersection data (§6).

C94.10 (steering: “birational onto its image for $g \geq 3$ ”):

Imprecise. Birational iff non-hyperelliptic. Hyperelliptic curves of every genus give degree 2.

C94.11 (steering: “ $\Delta^2 < 0$ is what makes Δ contractible (Grauert/Artin)”):

Half right. $\Delta^2 < 0$ is necessary, and sufficient analytically over $\mathbb{C}$ (Grauert). Artin’s algebraic criterion needs $p_a \leq 0$, which fails for Δ . The algebraic contraction here is the difference map itself.

9. RANKED DIRECTIONS

Scores 0–1 (importance $\times$ tractability):

R1 (.62) (RR) axiom. Formulate and test “positive $\Rightarrow$ $\pm$ effective” on L_S numerically. For fixed D with $D^2 > 0$, ask whether D is numerically equivalent to a nonnegative ψ plus $aH + bV$; this is an LP feasibility problem per span. Compare true zeros with ghosts. This is the one condition left standing (Prop. 2.4, Q1).

R2 (.55) Ring before cone. Look for a finite-rank additive structure in which $\deg$ is a quadratic form compatible with $\Psi_\lambda \Psi_\mu = \Psi_{\lambda\mu}$, as $\mathbb{Z}[\pi]$ is for $g = 1$ (Prop. 2.3, Q3). Test candidates on the Hasse-lattice diagnostic of Fig. 2.

R3 (.48) Negative curves over $\mathbb{Q}$. Renormalize $\Psi_1^2 = -\infty$ and determine the walls of a would-be ample cone ((ii) of §5). Compare with the $g \rightarrow \infty$ behaviour of $\Delta^\perp \cap P$ on $C \times C$.

R4 (.40) Sampling law for integer and prime detection thresholds (Table 1; Q4).

R5 (.30) Perron class as $N \rightarrow \infty$ (Q5).

10. QUESTIONS FOR J.W.

- Q1. Over $\mathbb{Q}$, does RH imply (RR), i.e. is every class of positive square in L_S numerically $\pm$ effective? Or is (RR) strictly stronger?
- Q2. Genus 1: does the band $2\sqrt{q} < |a| \leq q + 1$, $a \neq -(q + 1)$, satisfy $a_d \geq 0$ for *all* d ? Verified for $d \leq 60$ and $q \leq 47$.
- Q3. Is there an additive structure on the $\mathbb{Q}$ -span of Ψ_λ making $\deg$ a quadratic form, as on $\mathbb{Z}[\pi]$? This is the concrete form of P9.
- Q4. What is the scaling law of the first detecting N for integers and primes, and is the uniform-grid anomaly at large γ purely a matter of tolerance?
- Q5. Does the Perron class (fixed R , $N \rightarrow \infty$) converge to the pole class h ? Where does $\lambda_1(R)$ change sign as a function of N ?

WHAT IS NEW AND WHAT IS CLASSICAL (HONEST LIST)

- *Classical*: Hasse/Rosati ring positivity; Nakai–Moishezon; Hodge index; the cones of abelian surfaces; the difference map; Landau-type bounds; Perron–Frobenius.
- *Elementary, new to TAA, probably folklore*: Thm 2.2 (the $g = 1$ effective band via $N_k = (1 - \alpha^k)(1 - \beta^k)$); Thm 3.1 (unconditional off-diagonal positivity); Prop. 3.3; Thm 4.2 (a reformulation).
- *New numerics*: the independence of effectivity and Hodge index on both sides (Figs. 1, 3); the Perron class; the thin-sliver phenomenon; prime detection thresholds; the lattice-point factorization.

No claim here bears on the truth of RH.

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TAA LEMMA 95: TWO SCALINGS: $\mathbb{N}^3$, HECKE OPERATORS, AND THE RIEMANN–ROCH AXIOM

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis, or GRH for $L(\Delta, s)$. Every claim carries one label: [THEOREM] (proved here, or classical with the proof indicated), [RESTATEMENT] (reformulation), [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (computed; scripts `num/lemma95_*.py`, data `num/195/`). Sources: [CITED] (opened), [CITED-sec] (secondary), [BIB] (bibliographic), [TAA-L] (earlier TAA lemma). References: `lit/lemma95/verified_refs.md`.

ABSTRACT. J.W.’s $8 \times 8 \times 8$ computation (t401u) and the Ramanujan τ discussion (t402u) raise one question: is there a pairing with *two* independent scalings that carries more than the single-scaling Weil form, using the Hecke gcd correction as a template? And does the L94 Riemann–Roch axiom separate RH from ghosts? We verify the chat claims, build the candidate pairings, test them on true zeros, synthetic ghosts and the Davenport–Heilbronn (DH) quartet, and work out the Hecke model over $\mathbb{F}_p$. The verdict is mostly negative, and it is sharp.

Results. (1) The chat claims are correct except one: the Eichler–Shimura relation has $FV = p$ in weight 2 but $FV = p^{11}$ for Δ (Deligne 1969, Thm. 4.9). The Hecke relations hold for 12275 pairs with $mn \leq 3000$, including 3794 with $\gcd > 1$, and at operator level on S_{24} ([NUMERICAL]). (2) *Dichotomy* (Thms. 2.1, 2.2). Every pairing obtained by pulling back the one-variable Weil form, whether by composition or by Hecke multiplication, has rank and inertia bounded by the single-scaling span. The Hecke-corrected pairing on 8×8 has rank 30, *below* the 43 of the composition pairing, because $mn/d^2 = (m/d)(n/d)$. Genuinely two-variable pairings are restrictions of the Künneth product $I \otimes I$. Their inertia is fixed by the one-variable inertia, and their signature moves even for an extra *on-line* zero. No natural choice is both new and ghost-sensitive in a new way. (3) Over $\mathbb{F}_p$ the Hecke correspondence $T_p = \Gamma_F + \Gamma_F^t$ spans, with H, V, Δ , a lattice of signature $(1, 2)$ for *every* trace a_p , true or fake (Thm. 3.1). The Ramanujan bound is invisible to T_p . It appears only after the Eichler–Shimura splitting $T_p = F + V$, as Hodge index on $\langle H, V, \Delta, \Gamma_F \rangle$ (Thm. 3.2). A fake Satake parameter gives signature $(2, 2)$ and remains Lefschetz-effective: for Δ at $p = 2$, 3918 of the 4099 integer traces in the effective band are fakes. (4) Local versus global (§3.3): the Weil form of $L(\Delta, s)$, validated against 28 computed zeros (absolute agreement $\leq 10^{-9}$), is positive definite on all tested spans. Random Satake angles, which satisfy Ramanujan at every p , already break positivity at support width $a = 1$ in 57% (Sato–Tate) and 63% (uniform) of samples, and in all samples for $a \geq 3$. (5) The RR axiom (L94 R1): with Nakai and a free choice of effective cone, it is *equivalent* to $n_+ = 1$ (Thm. 4.2). It therefore separates 720/720 genus-1 fakes, where effectivity let 566 through, but only because it *is* Hodge index. With a fixed cone (the LP version) it fails for true curves too: 43% of sampled positive classes lie in the cone for true curves and 44% for fakes. This answers L94 Q1: at the lattice level RR is RH-equivalent, neither stronger nor weaker. Its real content is an effectivity notion (h^0) independent of the form, which is exactly what P9 lacks over $\mathbb{Q}$.

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GLOSSARY (BRIEF RESTATEMENTS FOR J.W.)

Hecke operators.:

On weight- k forms $f = \sum a(n)q^n$ for $\mathrm{SL}_2(\mathbb{Z})$, T_n acts on coefficients by $(T_n f)_m = \sum_{d|(m,n)} d^{k-1} a(mn/d^2)$. The T_n commute and satisfy $T_m T_n = \sum_{d|(m,n)} d^{k-1} T_{mn/d^2}$; in particular $T_p T_{p^r} = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}$. Serre writes the weight as $2k$, so his exponent is $2k-1$ [4] VII §5 [CITED-sec], [5] Ch. 5 [BIB]. For Δ , $T_n \Delta = \tau(n) \Delta$. For level N the sum runs over d prime to N .

Satake parameters.:

For a Hecke eigenform of weight k and $p \nmid N$, write $1 - a(p)X + p^{k-1}X^2 = (1 - \alpha_p X)(1 - \beta_p X)$. Then $\alpha_p + \beta_p = a(p)$ and $\alpha_p \beta_p = p^{k-1}$. The Ramanujan–Petersson bound is $|\alpha_p| = |\beta_p| = p^{(k-1)/2}$, i.e. $|a(p)| \leq 2p^{(k-1)/2}$. For Δ it was reduced to the Weil conjectures in [1] and proved by [2] [CITED].

Eichler–Shimura.:

For $p \nmid N$ the Hecke correspondence on $X_0(N)$ reduces mod p to Frobenius plus its transpose: $T_p = F + V$ in $\mathrm{End}(J_0(N)_{\mathbb{F}_p})$, with $FV = p$ [5] Ch. 8, [6, 7] [CITED-sec].

Kuga–Sato.:

For weight $k \geq 3$ one uses the $(k-2)$ -fold fibre power of the universal elliptic curve, suitably compactified and desingularized. For Δ this is a variety of dimension 11 whose H^{11} contains the rank-2 ℓ -adic representation W_ℓ of Δ . On it, $T_p = F + V$ and $FV = p^{k+1}$, where Deligne’s k is the weight minus 2, so $FV = p^{11}$ for Δ , and $1 - T_p X + p^{11} X^2 = (1 - FX)(1 - VX)$ ([1] Thm. 4.9 and §5 [CITED]). Scholl cuts out the corresponding Chow motive [3] [CITED].

Weil form of an entire L -function.:

For $\Lambda(s) = \Gamma_{\mathbb{C}}(s + \frac{11}{2}) L(\Delta, s)$ (normalized, centre $\frac{1}{2}$, entire, $\varepsilon = +1$), $W(f) = \sum_{\gamma} |\hat{f}(\gamma)|^2$ over the zeros $\frac{1}{2} + i\gamma$. GRH for $L(\Delta) \iff W \geq 0$ on all test functions (Weil’s criterion [8] [BIB], form for entire L). There is *no pole plane*, so “Hodge index” here means plain semidefiniteness.

RR axiom (L94).:

“A class of positive square is $\pm$ effective.” Nakai: h is positive on every nonzero effective class. L94 Prop. NRR: Nakai + RR $\Rightarrow$ Hodge index.

SUMMARY OF VERDICTS

Construction	Status	RH content?
Composition pairing on $\mathbb{N}^3/\text{scaling}$	[THEOREM] collapses to a/c : rank 43	that of the single form
Hecke-corrected pairing	[THEOREM] rank $\leq \#\{mn\}$ (= 30 on 8×8)	that of the single form
Künneth pairing $I(a/b)I(b/c)$	[THEOREM] rank 439, inertia fixed by I	none new; moves with on-line zeros
Hecke lattice $\langle H, V, \Delta, T_p \rangle$ over $\mathbb{F}_p$	[THEOREM] signature (1, 2) for all a_p	none: blind to Ramanujan
Frobenius lattice $(T_p = F + V \text{ split})$	[THEOREM] $(1, 3) \iff a_p^2 < 4p$	= Ramanujan (local)
Ramanujan at all p	[NUMERICAL] random models break Weil positivity at $a \geq 1$	local $\nrightarrow$ global
RR + Nakai, free cone	[THEOREM] $\iff n_+ = 1$	=RH (via Hodge index)
RR with a fixed (LP) cone	[THEOREM]/[NUMERICAL] fails for true curves	stronger than RH, false

1. THE CHAT CLAIMS AND THE HECKE RELATIONS

1.1. The grids.

Proposition 1.1 (counts). (i) On the 8×8 grid the pairing $I(\Psi_{m/n}, \Psi_{m'/n'})$ has rank 43 = $\#\{m/n\}$ (L94 §6, [NUMERICAL]). (ii) The $8^3 = 512$ triples (a, b, c) fall into 439 classes under $(a, b, c) \sim (ka, kb, kc)$. The map $(a, b, c) \mapsto (a/b, b/c) \in \mathbb{Q}_+^\times \times \mathbb{Q}_+^\times$ is injective on classes (439 distinct pairs), while a/c takes only 43 values. (iii) The products mn ($m, n \leq 8$) take 30 values, and the triple products abc take 80. [THEOREM] (the injectivity) and [NUMERICAL] (the counts; `num/lemma95_grid.py`).

The class set is the positive part of $\mathbb{P}^2(\mathbb{Q})$, i.e. the $\mathbb{Q}_{>0}$ -points of the split 2-torus $\mathbb{G}_m^3/\mathbb{G}_m$. It is a group, $(\mathbb{Q}_+^\times)^2$, rather than a projective plane in the incidence sense (C95.2).

Proposition 1.2 (composition collapses). If a pairing on classes is defined through the composite $\Psi_{a/b} \circ \Psi_{b/c} = \Psi_{a/c}$, i.e. $B((a, b, c), (a', b', c')) = I(\Psi_{a/c}, \Psi_{a'/c'})$, then $B = S I_{43} S^T$ with S the 439×43 incidence matrix of a/c . Hence $\text{rank } B \leq 43$ and $n_\pm(B) = n_\pm(I_{43})$. [THEOREM] (Sylvester; S has full column rank).

1.2. **Hecke relations for τ .** $\Delta = q \prod (1 - q^n)^{24} = \sum \tau(n) q^n$ was expanded exactly (python-flint) to $n \leq 3000$: $\tau(2..6) = -24, 252, -1472, 4830, -6048$ [NUMERICAL].

- $\tau(m)\tau(n) = \sum_{d|(m,n)} d^{11} \tau(mn/d^2)$: all 12275 pairs $m \leq n$ with $mn \leq 3000$, of which 3794 have $\text{gcd} > 1$. Zero failures. $\tau(p)^2 = \tau(p^2) + p^{11}$ for the 16 primes with $p^2 \leq 3000$ [NUMERICAL].
- Operator level: on $S_{24} = \langle \Delta E_4^3, \Delta^2 \rangle$ the 2×2 matrices satisfy $T_m T_n = \sum_{d|(m,n)} d^{23} T_{mn/d^2}$ for all 576 pairs $m, n \leq 24$, and each T_n ($n \leq 24$) preserves the space with residual exactly 0 on 50 coefficients. The characteristic polynomial of T_2 is $x^2 - 1080x - 20468736$, with discriminant $2^6 3^2 144169$ [NUMERICAL].
- Deligne's bound $|\tau(p)| < 2p^{11/2}$ holds for all 430 primes $p \leq 3000$. The maximum of $|\tau(p)|/2p^{11/2}$ is 0.9594 (at $p = 103$) and the mean of $\cos^2 \theta_p$ is 0.238 (Sato–Tate predicts $\frac{1}{4}$; Fig. 1) [NUMERICAL].

Non-factorization through mn . Equal products, different values: $\tau(2)^2 = 576 \neq \tau(1)\tau(4) = -1472$; $\tau(2)\tau(6) = 145152 \neq \tau(3)\tau(4) = -370944$; $\tau(3)^2 = 63504 \neq \tau(9) = -113643$ [NUMERICAL]. The difference is the d^{11} term. Locally $\tau(p)^2 - \tau(p^2) = p^{11} = \alpha_p \beta_p$ [THEOREM].

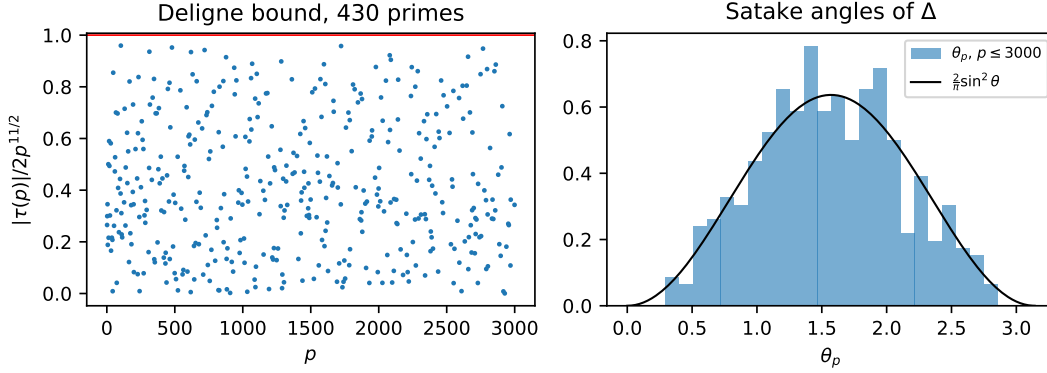


FIGURE 1. Left: $|\tau(p)|/2p^{11/2} < 1$ for $p \leq 3000$ (Deligne). Right: Satake angles against the Sato–Tate density, a theorem for Δ [9]. [NUMERICAL]

1.3. Partners, Eichler–Shimura, and what is open.

Proposition 1.3 (partner = conjugate, locally). *Let $\alpha\beta = p^{11}$ with $\alpha + \beta = \tau(p) \in \mathbb{R}$. Then $|\alpha| = p^{11/2} \iff \beta = \bar{\alpha} \iff \tau(p)^2 \leq 4p^{11}$. The duality partner $\beta = p^{11}/\alpha$ (the local shadow of $s \leftrightarrow 1-s$; L93 Thm. 1.1) coincides with the conjugation partner exactly under Ramanujan. [THEOREM] (elementary)*

Eichler–Shimura, precisely. In weight 2, $T_p = F + V$ on $J_0(N)_{\mathbb{F}_p}$ with $FV = p$, and no Kuga–Sato variety is needed [5, 6, 7] [CITED-sec]. For Δ , Deligne’s congruence formula on the Kuga–Sato cohomology reads $T_p = F + V$ with $FV = p^{k+1} = p^{11}$ ($k = 10$), and $1 - \tau(p)X + p^{11}X^2 = \det(1 - FX; W_\ell)$. The Weil conjectures then give $|\alpha_p| = p^{11/2}$ ([1] Thm. 4.9, Thm. 5.6, (5.7); [2]) [CITED]. So the chat’s “ $FV = p$ ” is correct only in weight 2 (C95.7). **Open.** GRH for $L(\Delta, s)$ is open [OPEN]. Ramanujan is a statement about each Euler factor; GRH is about the zeros of the whole product (§3.3).

2. TWO-SCALING PAIRINGS

Let I be a one-variable Weil form on test functions on $\mathbb{R}_+^\times$: the ζ form of L91–L94, with its pole plane (Lorentzian under RH), or the $L(\Delta)$ form of §3.3 (no poles, semidefinite under GRH). Evaluate I on smoothed Ψ_λ , i.e. bumps $\psi_{\log \lambda}$ ($m = 4$, $h = 0.05$, as in L94). For a finite set $\Lambda \subset \mathbb{Q}_+^\times$ write I_Λ for the Gram matrix.

Theorem 2.1 (ceiling for pulled-back pairings). *Let X be any finite index set (pairs, triples, classes) and $\Phi : \mathbb{R}^X \rightarrow \mathbb{R}^\Lambda$ linear, sending x to a combination of Ψ_λ ’s. Then $B = \Phi^T I_\Lambda \Phi$ satisfies $\text{rank } B \leq \text{rank } I_\Lambda \leq |\Lambda|$ and $n_\pm(B) \leq n_\pm(I_\Lambda)$, with equality when Φ is onto. [THEOREM] (Sylvester’s law of inertia)*

Composition (Prop. 1.2) is the case $\Phi(a, b, c) = \Psi_{a/c}$. The Hecke-corrected pairing, our template from $T_m T_n = \sum d^{k-1} T_{mn/d^2}$, is the case

$$\Phi_H(m, n) = \sum_{d|(m, n)} d^{-w} \psi_{\log(mn/d^2)} \quad (w = \tfrac{1}{2}; \text{ the triple version } T_a T_b T_c \text{ uses } w = 1).$$

It does not factor through mn , as (2,6) and (3,4) show. But $mn/d^2 = (m/d)(n/d)$ is again a product, so the image lies in the span of the 30 product points and Φ_H is onto them (rank 30, [NUMERICAL]). The Hecke correction adds no new direction. It re-expresses the same span, and on 8×8 its rank (30) is below the composition rank (43) [THEOREM]. The same holds for triples: rank 80 = $\#\{abc\}$.

Theorem 2.2 (Künneth pairings). *A pairing that is genuinely bilinear in two independent scalings is a restriction of $I \otimes I$ on $\mathbb{R}^\Lambda \otimes \mathbb{R}^\Lambda$. For classes, $B_\otimes((a, b, c), (a', b', c')) = I(\Psi_{a/b}, \Psi_{a'/b'}) I(\Psi_{b/c}, \Psi_{b'/c'})$, the Hadamard product of two pulled-back Gram matrices, which is a principal submatrix of $I_{43} \otimes I_{43}$. If I_Λ has inertia (p, n) , then $I \otimes I$ has inertia $(p^2 + n^2, 2pn)$, and $n_\pm(B_\otimes)$ are bounded by these. If $I_\Lambda \succeq 0$ (the $L(\Delta)$ form under GRH), then $B_\otimes \succeq 0$ (Schur product theorem). [THEOREM]*

So the Künneth pairing *does* exceed the 43-dimensional collapse (rank 439 on the classes, [NUMERICAL]). But it is not Lorentzian: for the true ζ form $n_+(B_\otimes) = 375$. Its inertia is bounded by the one-variable inertia and carries no Hodge-index reading. This is the scaling-side version of L93's finding that the pair spectrum is the tensor square, not a new spectrum. Rankin–Selberg $L(\Delta \times \Delta, s)$ is again a *one-variable* Dirichlet series, so its Weil form falls under Thm. 2.1. The tensor structure sits in the local Satake parameters $\{\alpha_p, \beta_p\}^{\otimes 2}$, not in the scalings [THEOREM].

TABLE 1. Inertia of the pairings (num/195/grid.json). ζ : entries are n_+ (true value 1); P2 shows n_+/n_- . Ghosts are quartets $\pm\gamma \pm i\delta$ added to the zero side; DH = (85.699, 0.3085); “on-line” adds the real pair ± 30 . [NUMERICAL]

pairing	rank	true	(20, .3)	(20, .1)	DH	on-line
single $\Psi_{m/n}$ (43 pts)	43	1	3	3	1	1
P1 composition, 439 classes	43	1	3	3	1	1
P3 Hecke $T_m T_n$, 36 pairs	30	1	3	1	1	1
P3' product Ψ_{mn}	30	1	3	1	1	1
single Ψ_r , $r \in \{abc\}$ (80)	80	1	3	3	3	1
P4 Hecke $T_a T_b T_c$, 120	80	1	3	3	3	1
P2 Künneth, 439 classes	439	375/64	368/71	376/63	375/64	376/63

$L(\Delta)$: entries are n_- (GRH value 0); P2 shows n_-/n_+ ; ghosts (15, .3), (15, .1), DH, on-line ± 30 .

pairing	rank	true	(15, .3)	(15, .1)	DH	on-line
single $\Psi_{m/n}$ (43 pts)	43	0	1	0	0	0
P1 composition, 439 classes	43	0	1	0	0	0
P3 Hecke $T_m T_n$, 36 pairs	30	0	0	0	0	0
P3' product Ψ_{mn}	30	0	0	0	0	0
single Ψ_r , $r \in \{abc\}$ (80)	80	0	2	2	0	0
P4 Hecke $T_a T_b T_c$, 120	80	0	2	2	0	0
P2 Künneth, 439 classes	439	0/439	2/437	0/439	0/439	0/439

Reading the table [NUMERICAL]. (a) Every pulled-back pairing (P1, P3, P3', P4) has exactly the inertia of its single-scaling span, as Thm. 2.1 predicts. Whether a ghost is seen depends only on which scalings are spanned. The weak ghost (20, 0.1) is seen on the 43 ratios but not on the 30 products. The DH quartet is seen only on the 80-point triple span, whose range is $\log 512$. (b) The Künneth pairing P2 changes under ghosts ($64 \rightarrow 71$ negative directions for (20, 0.3)), but also under an extra on-line zero ($64 \rightarrow 63$). Its changes are therefore not a ghost criterion. For $L(\Delta)$ it is PSD in every model where the single form is PSD (Schur), and it picks up $n_- = 2$ only where the single form already has $n_- = 1$. (c) *Honest verdict*: every natural two-scaling pairing we could build either collapses (Thm. 2.1) or is Künneth (Thm. 2.2). In both cases its ghost sensitivity is inherited from the one-variable form. We found no pairing on $\mathbb{N}^3/\text{scaling}$ whose positivity is a new condition [THEOREM]/[NUMERICAL].

3. THE HECKE MODEL OVER $\mathbb{F}_p$

3.1. $X_0(11)$: $T_p = F + V$ **checked**. Take $E = X_0(11)$: $y^2 + y = x^3 - x^2 - 10x - 20$, with newform $\eta(z)^2 \eta(11z)^2$ [BIB]. For all 303 primes $p \leq 2000$, the count $a_p = p + 1 - \#E(\mathbb{F}_p)$ equals the q -expansion coefficient (0 mismatches; $a_{11} = 1$). The level-11 Hecke relations hold for 7781 pairs.

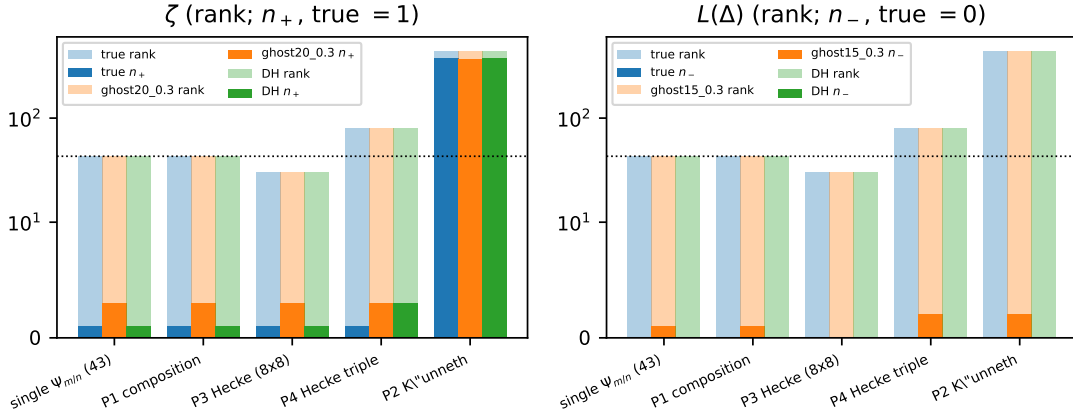


FIGURE 2. Ranks (light) and the RH/GRH-relevant inertia index (dark) of the pairings, with 43 dotted. Only K\"unneth exceeds 43, and its index is not Lorentzian. [NUMERICAL]

$\#E(\mathbb{F}_{p^2}) = p^2 + 1 - (a_p^2 - 2p)$ for $p = 3, 5, 7, 13$ (direct count over $\mathbb{F}_{p^2}$), so F satisfies $F^2 - a_p F + p = 0$. Hasse holds strictly, with $\max |a_p|/2\sqrt{p} = 0.974$ [NUMERICAL] (num/lemma95_fp.py). On $E_{\mathbb{F}_p}$, $V = \hat{F} = a_p - F$, so $T_p = F + V = a_p$ and $FV = \deg F = p$. This is Eichler–Shimura in its simplest case [THEOREM].

3.2. The Hecke correspondence is blind to Ramanujan. On $S = E \times E$ over $\mathbb{F}_p$ use the basis H, V, D_1, D_F with $D_1 = \Delta - H - V$ and $D_F = \Gamma_F - H - pV$. Then $D_f \cdot D_g = -(f\hat{g} + g\hat{f})$, the degree form ([TAA-L] L94 §1, [10] III.6 [CITED]). The Gram matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -2 & -a \\ -a & -2p \end{pmatrix}$ with $\det = a^2 - 4p$.

Theorem 3.1 (Hecke lattice is ghost-blind). *Reduced mod p , the Hecke correspondence is $T_p = \Gamma_F + \Gamma_F^t$, and its class is $(p+1-a)(H+V) + a\Delta$. Hence $T_p \in \langle H, V, \Delta \rangle$, with*

$$T_p \cdot \Delta = 2(p+1-a) = 2\#E(\mathbb{F}_p), \quad T_p^2 = 2((p+1)^2 - a^2),$$

and $\langle H, V, \Delta, T_p \rangle$ has rank 3 and signature $(1, 2)$ for every $a \in \mathbb{Z}$, whether or not $a^2 < 4p$. More generally, on $J_0(N)$ the Hecke algebra $\mathbb{T}$ is stable under the Rosati involution, and T_p ($p \nmid N$) is self-adjoint (Petersson; [5] Ch. 5 [BIB]). So $D_t^2 = -\text{Tr}(tt^\dagger) = -\text{Tr}(t^2) = -2\sum_i \lambda_i(t)^2 < 0$ for $0 \neq t \in \mathbb{T} \otimes \mathbb{R}$, because the eigenvalues $\lambda_i = \alpha_i + \beta_i$ are real. That holds for fake Satake pairs with real trace as well. [THEOREM]

Proof. $\Gamma_F^t = \Gamma_V$ has class $pH + V + D_{\hat{F}}$, and $D_F + D_{\hat{F}} = D_{F+\hat{F}} = aD_1$. Then $(p+1)(H+V) + aD_1 = (p+1-a)(H+V) + a\Delta$. The products follow from $H \cdot V = 1$, $\Delta \cdot H = \Delta \cdot V = 1$, $\Delta^2 = 0$. In the basis H, V, Δ the Gram matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has eigenvalues $2, -1, -1$. For the general statement, $\text{Tr}(t^2) = \sum$ of squares of the real eigenvalues, each counted twice on H^1 . $\square$

Theorem 3.2 (the Frobenius splitting sees Ramanujan). $\langle H, V, \Delta, \Gamma_F \rangle$ has signature $(1, 3)$ iff $a^2 < 4p$, and $(2, 2)$ iff $a^2 > 4p$ ([TAA-L] L94 Thm. g1). A fake Satake parameter (real $\alpha \neq \beta$, $\alpha\beta = p$) therefore violates Hodge index. If moreover $|a| \leq p+1$, it still satisfies the Lefschetz effectivity $N_k = (1 - \alpha^k)(1 - \beta^k) \geq 0$. [THEOREM]

[NUMERICAL] (exact signatures, fp.json). For $p \in \{2, 3, 5, 7, 13, 101\}$, true a_p gives $(1, 3)$ and fakes ($a = \lfloor 2\sqrt{p} \rfloor + 1$ and $a = p+1$) give $(2, 2)$. The Hecke sublattice has rank 3 and signature $(1, 2)$ in all 18 cases, and $T_p \cdot \Delta = 2\#E(\mathbb{F}_p)$ in each (Fig. 3).

Δ (weight 12). The ring $\mathbb{Z}[F]$ with $F^2 - \tau(p)F + p^{11} = 0$ and $F^\dagger = V = p^{11}/F$ (Deligne's congruence formula) carries the Rosati form $\text{Tr}((m+nF)(m+nV)) = 2(m^2 + \tau(p)mn + p^{11}n^2)$. It is positive definite iff $\tau(p)^2 < 4p^{11}$ [THEOREM]. The model lattice $\langle H, V \rangle \oplus \mathbb{Z}[F](-1)$ with $q = p^{11}$ has signature $(1, 3)$ for $p = 2, 3, 5, 7, 11, 13, 103$. The fakes $t = \lfloor 2p^{11/2} \rfloor + 1$ give $(2, 2)$ with $N_1, \dots, N_5 \geq 0$. At $p = 2$, $q = 2048$: of the 4099 integer traces with $|t| \leq q + 1$ (Lefschetz-effective), only 181 satisfy Ramanujan [NUMERICAL]. **Caveat [ANALOGY].** For weight 2 this lattice is the Néron–Severi lattice of an actual surface, and positivity is Weil's Hodge index (a theorem). For Δ , $\mathbb{Z}[F]$ acts on the rank-2 piece of H^{11} of an 11-dimensional Kuga–Sato variety. Positivity of the intersection form on *algebraic cycles* of $W \times W$ in middle dimension over $\mathbb{F}_p$ is a Hodge-standard-conjecture statement, open in that dimension ([TAA-L] L94 §5). Deligne obtains $|\alpha_p| = p^{11/2}$ from Weil I [2], not from cycle positivity. The Rosati statement above is *equivalent* to Ramanujan, not a proof of it.

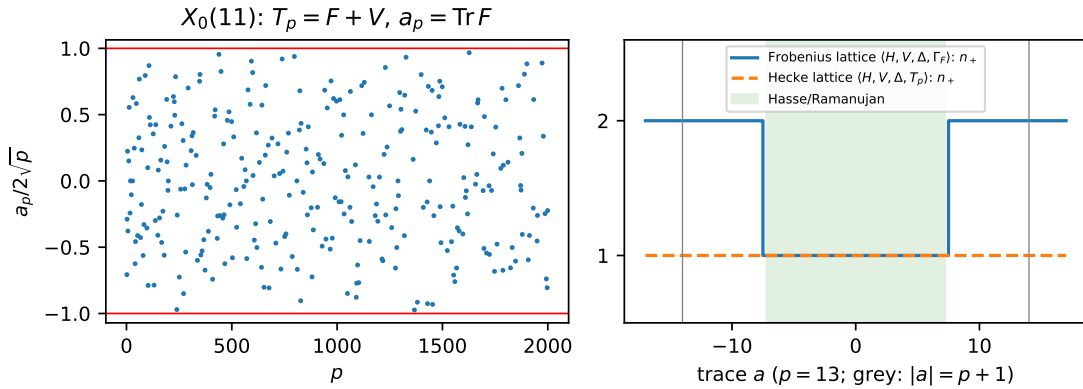


FIGURE 3. Left: $X_0(11)$, $a_p/2\sqrt{p}$ for $p \leq 2000$. Right ($p = 13$): n_+ of the Frobenius lattice (2 for fakes) versus the Hecke lattice (always 1) as the trace a varies. Shaded: Hasse/Ramanujan. Grey: the Lefschetz-effective band edge $|a| = p + 1$. [NUMERICAL]/[THEOREM]

3.3. Local versus global for $L(\Delta, s)$. Weil form [THEOREM] (standard contour argument, cf. [8]). For $\Lambda(s) = \Gamma_{\mathbb{C}}(s + \frac{11}{2})L(\Delta, s)$ and even g with $h(r) = \int g(u)e^{iru} du$,

$$\sum_{\gamma} h(\gamma) = \frac{1}{2\pi} \int_{\mathbb{R}} h(r) \left[-2 \log 2\pi + 2 \Re \psi(6 + ir) \right] dr - 2 \sum_n \frac{\Lambda_{\Delta}(n)}{\sqrt{n}} g(\log n),$$

where $\Lambda_{\Delta}(p^k) = (\alpha^k + \beta^k) \log p$ and $\alpha + \beta = \tau(p)/p^{11/2}$, $\alpha\beta = 1$. There are no pole terms, since Λ is entire. **Validation [NUMERICAL].** The zeros of $\Lambda(6 + it)$ (raw normalization) were computed from $\Lambda(s) = \sum \tau(n)[\Gamma(s, 2\pi n)(2\pi n)^{-s} + \Gamma(12 - s, 2\pi n)(2\pi n)^{s-12}]$, 90 digits, $n \leq 45$. This gives 28 zeros with $0 < \gamma < 62$, the first at 9.2223793999. The two sides of the formula agree to $\leq 10^{-9}$ absolutely (relative 4×10^{-6} at $W \approx 2 \times 10^{-4}$) for $h \in \{0.5, 0.8\}$ and five shifts (`dcheck.json`).

Experiment [NUMERICAL] (`num/lemma95_local.py`). The test space is n bumps on a 0.1-grid of width a . We compare: the true τ ; 30 random angle sets with Sato–Tate (ST) or uniform θ_p (each satisfies Ramanujan at every p , but there is no modularity and no functional equation); a non-Ramanujan factor at $p = 2$ ($t_2 = 2.2$); and the true form with a ghost quartet $(15, 0.3)$.

a	n	true $\lambda_{\min}/\lambda_{\max}$	ghost	ST: PSD	unif: PSD	$t_2 = 2.2$
0.5	6	0.1447	0.1291	30/30	30/30	0.145
1	11	0.0888	0.0649	13/30	11/30	-0.186
1.5	16	0.0770	0.0527	2/30	0/30	-0.436
2	21	0.0683	0.0443	1/30	0/30	-0.690
3	31	0.0544	0.0295	0/30	0/30	-0.892
4	41	0.0265	-0.0051	0/30	0/30	-1.11
5	51	0.0085	-0.0762	0/30	0/30	-1.19
6	61	0.0022	-0.171	0/30	0/30	-1.33

Reading [NUMERICAL]/[HEURISTIC]. (i) The true form stays positive definite on every span tested, as GRH predicts, although its margin shrinks with a . (ii) Ramanujan at every prime does *not* give Weil positivity. Random local data break it already at $a = 1$, where only $p = 2, 3$ enter: 17/30 ST and 19/30 uniform samples fail. (iii) Conversely, at small support Weil positivity constrains the *actual values* of the first few Satake angles, not just their size. (iv) The ghost is detected from $a \approx 4$. That is a range effect, as for ζ (L92, L94). This is L85 (H) in quantitative form: there is no local-to-global route; the gap is cross-prime correlation (modularity and the functional equation), which random models lack [TAA-L].

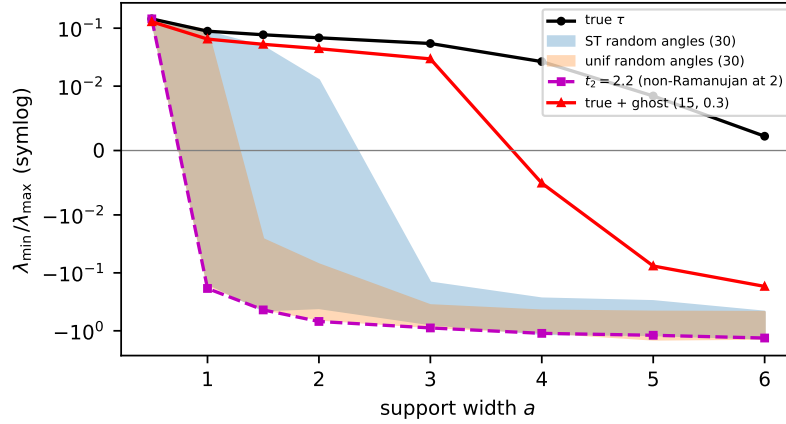


FIGURE 4. $\lambda_{\min}/\lambda_{\max}$ of the $L(\Delta)$ Weil form versus support width. Bands: range over 30 random-angle models. [NUMERICAL]

4. R1 FROM L94: THE RIEMANN-ROCH AXIOM AS A FINITE TEST

Let L_S be a finite lattice spanned by generators g_i (over $\mathbb{F}_q$: $H, V, \Gamma_0, \dots, \Gamma_n$; over $\mathbb{Q}$: H, V and bumps $\psi_i \geq 0$), with Gram G_S and $h = H + V$. Work modulo the numerical kernel.

Definition 4.1. RR&N(S): there is a convex cone $E \supseteq \{g_i\}$ such that (Nakai) $h \cdot x > 0$ for all $0 \neq x \in E$, and (RR) every D with $D^2 > 0$ lies in $E \cup -E$.

Theorem 4.2 (RR + Nakai = Hodge index). Suppose $h^2 > 0$ and $h \cdot g_i > 0$ for all i . Then $\text{RR\&N}(S) \iff n_+(G_S) = 1$. [THEOREM]

Proof. ($\Leftarrow$) If $n_+ = 1$, the positive cone $P = \{D^2 > 0, D \cdot h > 0\}$ is a round convex sheet. Reverse Cauchy-Schwarz gives $h \cdot x > 0$ on $\bar{P} \setminus 0$. Put $E = \text{cone}(\bar{P} \cup \{g_i\})$. Nakai holds termwise, and RR holds because $\{D^2 > 0\} = P \cup -P$. ($\Rightarrow$) If $n_+ \geq 2$, then $h^\perp$ contains D with $D^2 > 0$. RR puts $\pm D \in E$, but $h \cdot (\pm D) = 0$ contradicts Nakai. $\square$

Corollary 4.3. *Over $\mathbb{F}_q$, RR&N for the Frobenius lattices $\iff RH$, because $n_+ = 1 + k$ ([TAA-L] L88 Thm. B, L94 Thm. g1). Over $\mathbb{Q}$, RR&N for every finite extended span $\iff n_-(Z|S) = 0$ for all spans (L94, Haynsworth) $\iff$ Weil positivity $\iff RH$. [RESTATEMENT]*

Numerics [NUMERICAL] (num/lemma95_rr.py; exact rational arithmetic over $\mathbb{F}_q$).

- *Genus 1, 1088 polynomials ($q \leq 47$, as L94):* RR&N holds for 368/368 RH cases and fails for 720/720 fakes. Every fake has an exact rational violator D with $D \cdot h = 0$ and $D^2 > 0$ (0 failures). Effectivity (E1 $\wedge$ E3) had passed 566 of these fakes (L94).
- *L88 set (1000, seed 8888) and 50 true curves:* 853 fakes with $k > 0$ are all infeasible, each with a violator. The 147 cases with $k = 0$ (all roots on the circle) and the 50 true curves are feasible.
- $\mathbb{Q}$, *extended Gram (N bumps at $\log n$):* the true zeros are feasible for $(N, R) = (30, .2), (60, .2), (60, .1)$, and so is an extra on-line pair ± 40 . The ghost $(40, 0.3)$ is infeasible ($n_+ = 2, 3, 3$). DH is infeasible at $N = 60$ ($n_+ = 2$; relative D^2 of the violator 2.4×10^{-6} at $R = .2$ and 4.3×10^{-4} at $R = .1$) and not seen at $N = 30$.

The fixed-cone (LP) version fails for true curves [THEOREM]/[NUMERICAL]. Take $E = \text{cone}(g_i)$ fixed and ask whether $P_S \subseteq \pm E$. This is an LP per class. Over $\mathbb{F}_q$, a fraction of sampled positive classes lie in $\pm E$, but it is the same for true curves (mean 0.43, range 0.32–0.57, 40 lattices) and for band fakes (0.44, 0.33–0.57, 36). Over $\mathbb{Q}$, with $E = \{aH + bV + \psi : a, b \geq 0, \psi \geq 0\}$, the fraction is 0/400 for every model. The reason is that $h - \varepsilon\psi_1 \in P$ has a negative coordinate in a basis, while $h \cdot (\varepsilon\psi_1 - h) < 0$. On an actual surface $n(H + V) - \Delta$ is effective for $n \gg 0$, but only by Riemann–Roch: the true effective cone is larger than the generated cone (Fig. 5).

Answer to L94 Q1 [THEOREM]. At the lattice level, RR (with Nakai) is *equivalent* to RH, neither stronger nor weaker. It “separates” 720/720 where effectivity failed, but only because it is Hodge index restated. The fixed-cone version is strictly stronger than RH and is false. The geometric content of RR on a surface is an effectivity notion ($h^0(nD) > 0$) defined *independently* of the intersection form. Over $\mathbb{Q}$ no such notion is available, and that is the precise form of P9’s gap.

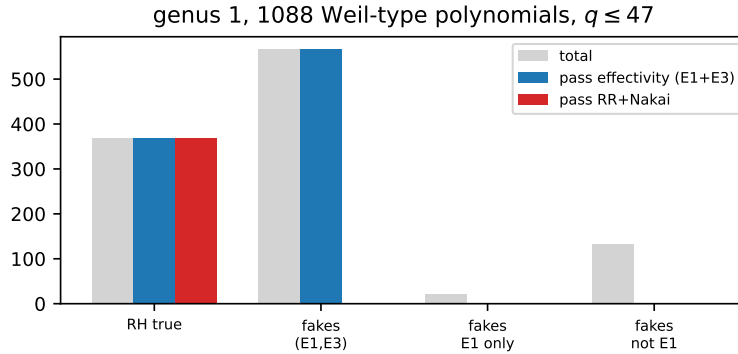


FIGURE 5. Genus 1: effectivity passes 566 fakes; RR+Nakai passes none, and passes all 368 RH cases. [NUMERICAL]

5. WHAT IS NEW AND WHAT IS CLASSICAL (HONEST LIST)

- *Classical:* the Hecke relations; Deligne’s congruence formula and Weil $\Rightarrow$ Ramanujan; Eichler–Shimura; $T_p \cdot \Delta$ as a point count; self-adjointness of Hecke operators; Sylvester and Schur; Weil’s criterion.
- *Elementary, probably folklore, new to TAA:*

- Thm. 2.1 applied to Hecke-corrected pairings (rank = $\#\{mn\}$, below the composition rank);
- the ceiling/Künneth dichotomy (Thm. 2.2);
- Thm. 3.1, that the Hecke lattice is blind to Ramanujan and only the Frobenius splitting sees it;
- Thm. 4.2, that RR+Nakai is equivalent to $n_+ = 1$, which answers L94 Q1.
- *New numerics:*
 - the inertia table for two-scaling pairings (true, ghosts, DH, on-line);
 - the validated $L(\Delta)$ Weil form and its positive definiteness on spans up to width 6;
 - random Ramanujan-satisfying models failing Weil positivity from $a = 1$;
 - the RR tallies (720/720 fakes, with exact violators).

No claim here bears on the truth of RH or of GRH for $L(\Delta, s)$.

6. CORRECTIONS

C95.1 (t401u: “the $\Psi_{m/n}$ pairing on 8×8 has rank 43 = $\#m/n$, so it factors through $\mathbb{Q}_+$ ”).:

Correct (L94 §6; Prop. 1.1).

C95.2 (t401u: “512 points, 439 classes, two ratios $(a/b, b/c)$, an arithmetic projective plane”).:

Counts correct; the injectivity of the ratio map is proved. Precision: the class set is the positive part of $\mathbb{P}^2(\mathbb{Q})$, the group $(\mathbb{Q}_+^\times)^2$ of positive points of a split 2-torus. It is not a projective plane in the incidence sense.

C95.3 (t401u: “any pairing built from single scalings collapses ... back to the a/c values”).:

True for pairings built by composition (Prop. 1.2). It is not true for all pairings built from single scalings: the Künneth product $I(\Psi_{a/b})I(\Psi_{b/c})$ has rank 439. That product is, however, not new in content (Thm. 2.2).

C95.4 (t402u: “ $\tau(m)\tau(n) = \sum d^{11}\tau(mn/d^2)$ does not factor through mn when $\gcd > 1$ ”).:

Correct (12275 pairs; examples in §1). Precision: the span of the Hecke products equals the span of the single T_r with r a product, so no new dimensions appear (rank 30 on 8×8).

C95.5 (t402u: “ $\tau(p)^2 = \tau(p^2) + p^{11}$; $p^{11} = \alpha_p\beta_p$, the functional-equation-partner product”).:

Correct. $\beta_p = p^{11}/\alpha_p$ is the local duality partner (L93).

C95.6 (t402u: “Ramanujan $|\alpha_p| = p^{11/2}$ (Deligne 1974) is ‘partner = conjugate’ locally”).:

Correct (Prop. 1.3). Attribution: the reduction to Weil is Deligne 1969, the proof of Weil I Deligne 1974.

C95.7 (t402u: “Eichler–Shimura $T_p = F + V \bmod p$ with $FV = p$, via Kuga–Sato and the Weil conjectures”).:

Partly right. In weight 2 ($J_0(N)$), $T_p = F + V$ with $FV = p$, and no Kuga–Sato variety is involved. For Δ , the congruence on Kuga–Sato cohomology has $FV = p^{11}$ ([1] Thm. 4.9). The two statements were merged in chat.

C95.8 (t402u: “global RH for $L(\Delta, s)$ is open”).:

Correct.

C95.9 (brief: “a fake Satake parameter should give a Hodge-index violation”).:

Correct for the *Frobenius* lattice (Thm. 3.2). False for the *Hecke* lattice: T_p alone never violates Hodge index (Thm. 3.1).

C95.10 (L94 R1: “an LP feasibility problem per span”).:

Self-correction. With a fixed cone the axiom is false for true curves. With a free cone it is an eigenvalue condition, namely Hodge index (Thm. 4.2). LP is the wrong format.

C95.11 (L94 Q1):

Answered: $\text{RR+Nakai} \iff \text{RH at the lattice level (§4)}.$

7. RANKED DIRECTIONS

Scores 0–1 (importance $\times$ tractability):

- D1 (.58) Frobenius splitting over $\mathbb{Q}$.** Thm. 3.1/3.2 show that lattice positivity needs F , not $T_p = F + V$. Look for a non-self-adjoint operator on the Ψ -span with $F^\dagger F = \text{scaling}$, i.e. the “missing Frobenius” of L89 in operator form. This is the precise target for P9.
- D2 (.52) Certified $L(\Delta)$ windows.** Certify positive definiteness of W_Δ for widths $a \leq a_*$ (as L92 Z1 did for ζ), and quantify how far the true $\tau(2), \tau(3), \dots$ sit inside the region that random angles leave at $a \approx 1$ –2.
- D3 (.47) An h^0 over $\mathbb{Q}$.** Test candidate effectivity notions independent of the form, such as Paley–Wiener/prolate dimensions with prescribed support or Connes–Consani’s RR on $\overline{\text{Spec } \mathbb{Z}}$. Ask whether any of them implies RR&N (Thm. 4.2).
- D4 (.38)** Search for a Künneth invariant that moves only for off-line zeros (it is not the signature; Table 1).
- D5 (.30)** Genus-2 Hecke field: $X_0(23)$ with $\mathbb{T} \otimes \mathbb{Q} = \mathbb{Q}(\sqrt{5})$, worked out explicitly.

8. QUESTIONS FOR J.W.

- Q1. Should P9 be retargeted from “a $\mathbb{Q}$ -structure on Ψ_λ ” to “a Frobenius F with $F + F^\dagger$ the Hecke/scaling operator” (D1)?
- Q2. Do you have a two-variable pairing on $\mathbb{N}^3/\text{scaling}$ in mind that is neither a pull-back nor Künneth? An incidence structure on $(\mathbb{Q}_+^\times)^2$ would be the place to look.
- Q3. Is a certified version of the random-model breakdown for $L(\Delta)$ worth the compute (D2)?
- Q4. For weight 12, should we stay with Deligne’s ring $\mathbb{Z}[F]$ (a theorem) or pursue the cycle-level positivity, which is a Hodge-standard-conjecture question?
- Q5. Which effectivity notion over $\mathbb{Q}$ would you accept as an “ h^0 ” for the RR axiom (D3)? The earlier errata decision (L89 C89.4 and others) is also still pending.

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TAA LEMMA 96: THE DIAGONAL, THE DISCRIMINANT, AND COHESION: RENORMALIZING Δ^2

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis. Every claim carries one label: [THEOREM] (proved here, or classical with the proof indicated), [RESTATEMENT] (reformulation), [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (computed; scripts `num/lemma96_*.py`, data `num/196/`). Sources: [CITED] (opened), [CITED-sec] (secondary), [BIB] (bibliographic), [TAA-L] (earlier TAA lemma). References: `lit/lemma96/verified_refs.md`.

ABSTRACT. In t406–t408, J.W. proposed that three objects called “ Δ ” are one object: the diagonal curve $\Delta_C \subset C \times C$, the cohesion/diagonal functor Disc , and the discriminant $\Delta_{12} = \eta^{24}$. On this view the anomaly of E_2 and $\log \eta$ would be the correct “renormalized Δ^2 ” for ζ , whose “genus” is infinite. He also asked (t408) whether the η -anomaly is a Rosati-like involution. We separate the three Δ ’s, check every chat constant, and test each candidate renormalization against three requirements: (a) finite, (b) the Hodge-index sign, (c) sensitive to ghosts, i.e. off-line zero quartets as in L91/L93.

Results. (1) Weil’s proof needs exactly one finite number: $-\Delta_C'^2 = 2g$, the square of the diagonal projected off $\langle H, V \rangle$. It enters as a Cauchy–Schwarz constant uniform in n (Thm. 1.2). Any finite uniform constant would do. Over $\mathbb{Q}$ none exists: the zero sum grows like $(\log T)^2$ ([NUMERICAL]) and $\pi(x) - \text{li}(x)$ is not $O(\sqrt{x}/\log x)$ (Littlewood). (2) The η -anomaly is *odd*. Rademacher’s integer $\Phi(\gamma)$ satisfies $\Phi(\gamma^{-1}) = -\Phi(\gamma)$ and $\Phi(J\gamma J) = -\Phi(\gamma)$ (Thm. 4.1). Checked: 400 random γ ; $\log \eta$ transformation to 8×10^{-30} in 58 tests; cocycle $-3 \text{sign}(c_{ACBCAB})$ in 296/296 tests. The involution $\gamma \mapsto S\gamma^T S^{-1} = \gamma^{-1}$ is the Rosati (dual-isogeny, adjugate) involution on H^1 of a torus. Its positive form $\text{Tr}(\gamma\gamma^\dagger) = 2 \det \gamma$ is identically 2 on $\text{SL}_2(\mathbb{Z})$. So J.W.’s t408 guess is half right: the anomaly sits in the *odd* part, where Rosati positivity carries no information. This matches Atiyah’s identification of the same invariant with signature defects and η -invariants, which measure spectral *asymmetry*. It is ghost-blind, since it depends only on $\text{SL}_2(\mathbb{Z})$. (3) The E_2 anomaly is an order-of-summation effect. Rectangular truncations $|c| \leq M$, $|d| \leq \lambda M$ converge to $G_2 - \frac{2\pi i}{\tau} \left(1 - \frac{2}{\pi} \arctan \frac{\lambda}{i\tau}\right)$, which interpolates between G_2 and $G_2^* = G_2 - 2\pi i/\tau$ ([NUMERICAL], agreement 10^{-6} , Fig. 2). (4) Every renormalized Δ_C^2 we built fails at least one requirement. $g_{\text{reg}} = Z(0) = 7/8$ (Voros; reproduced as 0.8748 from 2500 zeros) gives $\Delta_C^2_{\text{reg}} = +1/4$, the wrong sign for genus ≥ 2 . The Weil bound with $2g_{\text{reg}} = 7/4$ already fails at width $T = 20$ and is off by $660\times$ at $T = 1000$. The log-renormalized diagonal has slope $0.503 \approx \frac{1}{2}$ (L94), but its finite part depends on the scheme ($0.65 \rightarrow 0.36$ as the spline order goes $2 \rightarrow 6$), and its ghost part tends to 0 like R . Every candidate is ghost-blind. (5) Cohesion. In a presheaf topos, $\Pi \dashv \text{Disc} \dashv \Gamma$ is $\text{colim} \dashv \text{const} \dashv \text{lim}$, and Disc is the categorical diagonal functor. The discrete/continuous split of the explicit formula is canonical: prime atoms versus archimedean density. But it is RH-blind. On V_0 the prime-only and arch-only forms are both indefinite (e.g. (22, 14) and (5, 31)), and only their sum is negative definite ([NUMERICAL]). Net verdict: the three Δ ’s meet at one number, $\zeta(-1) = -\frac{1}{12}$, which appears as the exponent $\frac{1}{24}$ of η , the anomaly $\pi i/12$, and $\chi(\text{SL}_2(\mathbb{Z}))$. That number is a special value, not a zero statistic, and it carries no positivity.

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NOTATION: WHICH Δ

Symbol	Meaning	Where
Δ_C	diagonal curve $\{(x, x)\} \subset C \times C$	§1
Δ_C'	its projection to $\langle H, V \rangle^\perp$	§1
Disc	discrete-object functor of a cohesive topos (a “diagonal” functor)	§2
δ_X	diagonal morphism $X \rightarrow X \times X$	§2
Δ_{12}	the discriminant $\eta^{24} = q \prod (1 - q^n)^{24}$, weight 12	§3
Φ, Ψ	Rademacher’s integer function and class function on $\mathrm{SL}_2(\mathbb{Z})$	§4

GLOSSARY (BRIEF RESTATEMENTS FOR J.W.)

Cohomology, H^1 ..

For a curve C of genus g over $\mathbb{F}_q$, H^1 (ℓ -adic) is a $2g$ -dimensional space on which Frobenius acts with eigenvalues $\alpha_1, \dots, \alpha_{2g}$. Then $\#C(\mathbb{F}_{q^n}) = q^n + 1 - \sum_i \alpha_i^n$.

Intersection numbers, Lefschetz..

On the surface $C \times C$, two curves D, E have an intersection number $D \cdot E$. If $f : C \rightarrow C$ has graph Γ_f , then $\Gamma_f \cdot \Delta_C = \sum_i (-1)^i \mathrm{Tr}(f^*|H^i)$, which is the number of fixed points counted with multiplicity.

Hodge index..

On the Néron–Severi group of a surface, the intersection form has exactly one positive direction. Hence it is negative definite on the orthogonal complement of any class of positive square, e.g. $H + V$ with $H = \{pt\} \times C$, $V = C \times \{pt\}$ ([6] V Thm. 1.9 [BIB]).

Adjoint functors..

$L \dashv R$ means $\mathrm{Hom}(LX, Y) \cong \mathrm{Hom}(X, RY)$ naturally. A chain $\Pi \dashv \mathrm{Disc} \dashv \Gamma \dashv \mathrm{coDisc}$ is four functors, each left adjoint to the next.

Topos..

A category that behaves like the category of sheaves on a space, e.g. presheaves $\mathrm{Set}^{C^{\mathrm{op}}}$.

Dedekind η , Dedekind sums..

$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n)$ with $q = e^{2\pi i \tau}$. The Dedekind sum is $s(h, k) = \sum_{r=1}^{k-1} ((r/k))((hr/k))$, where $((x)) = x - \lfloor x \rfloor - \frac{1}{2}$ for $x \notin \mathbb{Z}$ and $((x)) = 0$ for $x \in \mathbb{Z}$.

Quasimodular form..

A function like E_2 that transforms like a weight-2 form up to an explicit additive term. Derivatives of modular forms are quasimodular.

 η -invariant (APS)..

For a self-adjoint elliptic operator with eigenvalues λ , $\eta(s) = \sum \mathrm{sign}(\lambda) |\lambda|^{-s}$ and $\eta(0)$ measures spectral asymmetry. Atiyah notes the unfortunate clash with the Dedekind η [1] [CITED].

Ghosts.:

As in L91/L93/L94: hypothetical off-line zero quartets $\pm\gamma \pm i\delta$ inserted into the Weil form, giving the fake form $I - \frac{1}{2}G$.

1. (A) THE DIAGONAL Δ_C AND WHAT WEIL'S PROOF NEEDS

Let $C/\mathbb{F}_q$ have genus g , with $H = \{pt\} \times C$, $V = C \times \{pt\}$, $H^2 = V^2 = 0$, $H \cdot V = 1$, and Frobenius graph Γ_F .

Proposition 1.1 ([RESTATEMENT]; [6] Ch. V [BIB], [14, 2] context). $(1) \Gamma_f \cdot \Delta_C = \sum_i (-1)^i \text{Tr}(f^*|H^i)$ (Lefschetz). In particular $\Gamma_{F^n} \cdot \Delta_C = q^n + 1 - \sum_i \alpha_i^n = \#C(\mathbb{F}_{q^n})$.
 $(2) \Delta_C^2 = \deg N_{\Delta_C} = \deg T_C = 2 - 2g$ (self-intersection is the degree of the normal bundle, which for the diagonal is the tangent bundle). Likewise $\Gamma_F^2 = q(2 - 2g)$, $\Gamma_F \cdot V = 1$, $\Gamma_F \cdot H = q$.
 (3) Riemann–Hurwitz: for a separable $f : C \rightarrow D$ of degree n with tame ramification, $2g_C - 2 = n(2g_D - 2) + \sum_P (e_P - 1)$ ([6] Ch. IV §2 [BIB]). Ramification is where Γ_f fails to meet the fibres transversally; it plays no role in Weil's argument.

Theorem 1.2 (What Weil's proof needs; [THEOREM]). For D on $C \times C$ put $D' = D - (D \cdot V)H - (D \cdot H)V$. Then $D' \perp H, V$ and $D'^2 = D^2 - 2(D \cdot H)(D \cdot V)$. Hence

$$\Delta_C'^2 = -2g, \quad \Gamma_F'^2 = -2gq, \quad \Delta_C' \cdot \Gamma_F' = -a, \quad a := q + 1 - N_1.$$

By Hodge index the Gram matrix $\begin{pmatrix} -2g & -a \\ -a & -2gq \end{pmatrix}$ is negative semidefinite, so $a^2 \leq 4g^2q$. This is Cauchy–Schwarz $|\Delta_C' \cdot \Gamma_F'|^2 \leq \Delta_C'^2 \Gamma_F'^2$. Applied to F^n it gives $|\sum_i \alpha_i^n| \leq 2gq^{n/2}$ for all n , hence $|\alpha_i| \leq \sqrt{q}$, since a finite sum of powers $\sum_i (\alpha_i/\sqrt{q})^n$ that stays bounded forces $|\alpha_i| \leq \sqrt{q}$. The input that must be **finite** is the single number $\|\Delta_C'\|^2 = 2g = \dim H^1$, uniformly in n . Any finite uniform C with $|\sum \alpha_i^n| \leq Cq^{n/2}$ would suffice. The precise value $2g$ matters only for the sharp Hasse–Weil constant.

Proof. $D' \cdot H = D \cdot H - (D \cdot H)(V \cdot H) = 0$ and similarly $D' \cdot V = 0$. Expanding gives $D'^2 = D^2 - 2(D \cdot V)(D \cdot H) - 2(D \cdot H)(D \cdot V) + 2(D \cdot V)(D \cdot H)$. Then insert Prop. 1.1. The determinant is $4g^2q - a^2$ (sympy, lemma96_misc.py). The bound $|a| \leq 2\sqrt{q}$ was checked for $y^2 = x^3 + x + 1$ at $p \leq 1009$ ([NUMERICAL]). The last step is the standard “bounded power sums” lemma. $\square$

Remark 1.3 (The ζ analogue; [THEOREM]/[NUMERICAL]/[BIB]). For ζ the role of $\{\alpha_i\}$ is played by the infinitely many zeros, so “ $g = \infty$ ” and $\Delta_C'^2 = -\infty$. The analogue of Theorem 1.2 would be a bound $|\sum_\rho h(\gamma)e^{i\gamma u}| \leq C \sup |h|$ uniform in the test function, and none exists. Even with RH the normalized zero sum diverges: $\sum_{0 < \gamma < T} 1/\gamma = \frac{1}{4\pi} \log^2 \frac{T}{2\pi} - 0.017 \pm 0.001$ for $T \in [100, 3000]$ ([NUMERICAL], misc.json). On the prime side, $\pi(x) - \text{li}(x) = \Omega_\pm(\sqrt{x} \log \log \log x / \log x)$ [9] [BIB], so no bound $C\sqrt{x}/\log x$ holds. “Make Δ_C^2 finite” therefore cannot mean “find the genus”. It must mean replacing the Cauchy–Schwarz constant by something else, which is what §5 tests.

2. (B) COHESION AND THE DIAGONAL FUNCTOR Disc

Definition 2.1 ([CITED] [12, 10], [8] [CITED]). A *cohesive topos* over Set is a topos $\mathcal{H}$ with an adjoint chain $\Pi \dashv \text{Disc} \dashv \Gamma \dashv \text{coDisc} : \mathcal{H} \rightleftarrows \text{Set}$. Here Disc and coDisc are fully faithful and Π preserves finite products. The induced modalities are shape $\int = \text{Disc}\Pi$, flat $\flat = \text{Disc}\Gamma$ and sharp $\sharp = \text{coDisc}\Gamma$, with $\int \dashv \flat \dashv \sharp$. Read: ΓX = points of X ; $\text{Disc}S = S$ with discrete cohesion; ΠX = connected components (shape); $\flat X = X$ with its cohesion forgotten.

Proposition 2.2 ([RESTATEMENT]; standard category theory). Let $\mathcal{C}$ have a terminal object $*$, and let $\mathcal{H} = \text{Set}^{\mathcal{C}^{\text{op}}}$. Then $\text{Disc} = \text{const}$ is the categorical diagonal functor $\text{Set} \rightarrow \text{Set}^{\mathcal{C}^{\text{op}}}$. Its left adjoint is colim ($= \Pi$) and its right adjoint is lim , which equals evaluation at $*$ ($= \Gamma$). For any X , the diagonal morphism $\delta_X : X \rightarrow X \times X$ is an isomorphism iff X is subterminal. In particular $\delta_1 : 1 \rightarrow 1 \times 1$ is an isomorphism, so any “ Δ^2 ” formed from the terminal object is trivial.

So J.W.’s identification “Disc is a diagonal” is correct in the categorical sense ([RESTATEMENT]). But it is a third Δ , unrelated to Δ_C^2 . The functor Disc has no self-intersection, and δ_X carries no number unless one adds a linear structure, e.g. a trace $\text{Tr}(\delta^*\delta_*)$, which is the Euler characteristic.

Theorem 2.3 (The discrete/continuous split of the explicit formula is RH-blind; [THEOREM]+[NUMERICAL]). *With the L92 normalization, $I(\psi_x, \psi_y) = N(k_{|x-y|})$ splits canonically as*

$$N(k) = \underbrace{\sum_{n \geq 2} \Lambda(n) n^{-1/2} k(\log n)}_{\text{atomic on } (0, \infty): \text{“discrete”}} + \underbrace{\int_0^\infty \frac{e^{3t/2} k(t) - k(0)}{e^{2t} - 1} dt + Ck(0)}_{\text{density + finite part at 0: “continuous”}}.$$

The split is unconditional and does not involve the zeros. A ghost changes only the total, by $-\frac{1}{2}G$, and G is a smooth function of $|x - y|$, i.e. it modifies the “continuous” part. On V_0 (orthogonal to the pole vectors), Table 1 and Fig. 1 show that neither part has a sign, while the total is negative definite. RH is a property of the sum, invisible to the modality-type split.

TABLE 1. Inertia (n_+, n_-) on bump spans $x \in [0.3, 4]$, step $R/2$ (`renorm.json`).

R	$\dim V_0$	prime only	arch only	total	fake (ghost $14.13 \pm 0.1i$)	max eig total
0.4	17	(7,10)	(5,12)	(0,17)	(2,15)	$-7.4 \cdot 10^{-5}$
0.2	36	(22,14)	(5,31)	(0,36)	(0,36)	$-4.6 \cdot 10^{-4}$

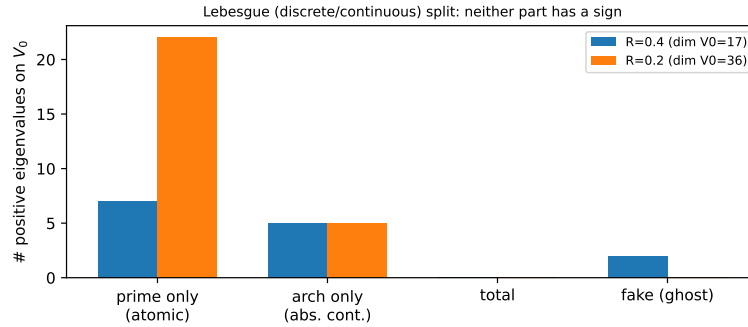


FIGURE 1. Positive inertia on V_0 of the prime (atomic) part, the arch (continuous) part, the total, and the ghost-modified total. The ghost is detected at $R = 0.4$ only by the total, and at $R = 0.2$ it is below resolution (cf. L93).

Remark 2.4 (Arithmetic site, condensed sets; [CITED]/[BIB], no claims). Connes–Consani’s arithmetic site [3] [CITED] (abstract) is the topos $\widehat{\mathbb{N}^\times}$ of $\mathbb{N}^\times$ -sets with a structure sheaf from the tropical semiring $\mathbb{Z}_{\max}$. Frobenius correspondences are realized in its square, which is the closest existing object to “ Δ_C for $\mathbb{Q}$ ”. Clausen–Scholze condensed sets [4] [CITED] (Lecture I) are sheaves on profinite sets and give a framework for topological algebra. We make *no* claim that either is a cohesive topos in the sense above, or that $\int, \flat$ act on the explicit formula there. This is [OPEN].

3. (C) E_2 , η , RADEMACHER AND THE EISENSTEIN SIDE

Throughout, $\vartheta = q d/dq = \frac{1}{2\pi i} \frac{d}{d\tau}$, $E_2 = 1 - 24 \sum \sigma_1(n) q^n$, and $G_2 = 2\zeta(2)E_2$.

Proposition 3.1 ([RESTATEMENT], [CITED], [NUMERICAL]). *Sources: [15] §2.3 Prop. 6 and §5.1 Prop. 15 (Zagier normalizes G_2 differently).*

- (1) $E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{6}{\pi i} c(c\tau+d)$. In particular $E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}$ (residual $\leq 1.6 \cdot 10^{-31}$ at 4 points; the variant with $\frac{12\tau}{\pi i}$ is off by 1.4–2.4).
- (2) $E_2 = \vartheta \log \Delta_{12}$, i.e. $\vartheta \Delta_{12} = E_2 \Delta_{12}$. Ramanujan: $\vartheta E_2 = \frac{E_2^2 - E_4}{12}$, $\vartheta E_4 = \frac{E_2 E_4 - E_6}{3}$, $\vartheta E_6 = \frac{E_2 E_6 - E_4^2}{2}$, and $1728 \Delta_{12} = E_4^3 - E_6^2$. All hold exactly for the coefficients of $q^0, \dots, q^{400}$ ([NUMERICAL]).
- (3) Serre derivative $\partial_k = \vartheta - \frac{k}{12} E_2$ maps $M_k \rightarrow M_{k+2}$, and $\partial_{12} \Delta_{12} = 0$. Interpretation ([RE-STATEMENT], ours, not stated in [15]): on the weight- k line bundle, ∂_k is a holomorphic connection and $-\frac{k}{12} E_2(2\pi i d\tau)$ is its connection form. The affine law in (1) is exactly the gauge transformation of a connection form, and $E_2^* = E_2 - \frac{3}{\pi y}$ is the non-holomorphic modular correction.

Proposition 3.2 (The anomaly is an order of summation; [THEOREM] for (1) (cf. [15] §2.3 (18)), [NUMERICAL]+[HEURISTIC] for (2)). (1) With d summed inside, $G_2(\tau) = \sum_c \sum'_d (c\tau + d)^{-2} = 2\zeta(2) + 2 \sum_{c \geq 1} \pi^2 / \sin^2(\pi c\tau)$. With c summed inside, $G_2^*(\tau) = \tau^{-2} G_2(-1/\tau)$, and $G_2^* - G_2 = -2\pi i / \tau$ (to 15 digits at two points). (2) Rectangular truncations $|c| \leq M$, $|d| \leq \lambda M$ converge to

$$G_2(\tau) - \frac{2\pi i}{\tau} \left(1 - \frac{2}{\pi} \arctan \frac{\lambda}{-i\tau}\right),$$

which interpolates from G_2^* ($\lambda \rightarrow 0$) to G_2 ($\lambda \rightarrow \infty$). It agrees with the $M = 400$ extrapolations to $\leq 3 \cdot 10^{-6}$ for $\lambda \in [\frac{1}{8}, 16]$ (Fig. 2). The formula comes from the continuum integral $\iint (x\tau + y)^{-2}$ over the boundary region ([HEURISTIC]).

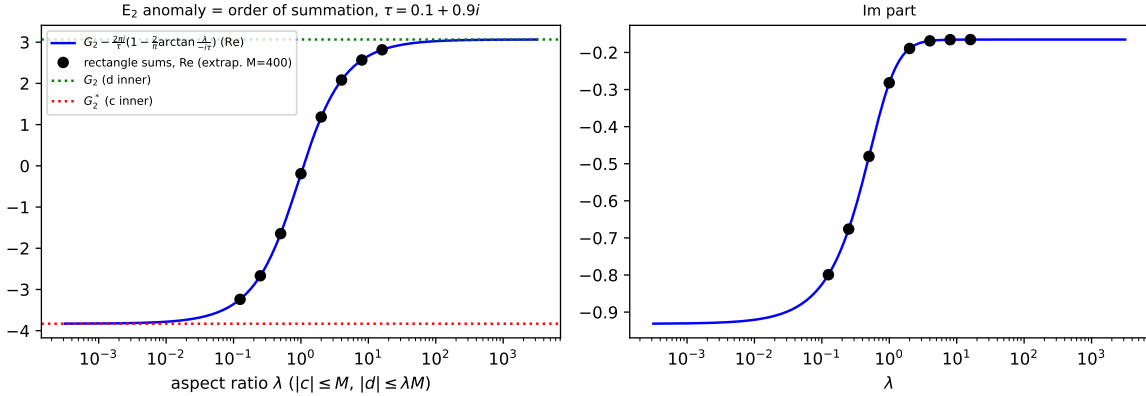


FIGURE 2. Conditionally convergent G_2 . The “anomaly” $-2\pi i / \tau$ is the difference between two orders of summation of one lattice sum.

Theorem 3.3 (Dedekind/Rademacher; [CITED] [1] §1, [11] [BIB]; conventions verified [NUMERICAL]). For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ define $\Phi(\gamma) = \frac{a+d}{c} - 12 \text{sign}(c) s(d, |c|)$ for $c \neq 0$ and $\Phi(\gamma) = b/d$ for $c = 0$. Then $\Phi \in \mathbb{Z}$ and, for $c > 0$,

$$\log \eta(\gamma\tau) = \log \eta(\tau) + \frac{1}{2} \log \frac{c\tau+d}{i} + \frac{\pi i}{12} \Phi(\gamma) \quad (\text{principal branches}).$$

Rademacher’s class function is $\Psi(\gamma) = \Phi(\gamma) - 3 \text{sign}(c(a+d))$ for $c \neq 0$. Atiyah writes it with $s(a, c)$, which equals $s(d, c)$ because $ad \equiv 1 \pmod{c}$.

[NUMERICAL]: integrality held for all 400 random γ with $|c| \leq 300$. The displayed formula held in 58 tests with maximal error $7.9 \cdot 10^{-30}$, and the opposite sign of Φ fails whenever $\Phi \neq 0$. Also $\Phi(S) = 0$, $\Phi(T) = 1$, $\Phi(-\gamma) = \Phi(\gamma)$, and Ψ is conjugation-invariant in 448/448 tests, while Φ is so in only 318/448.

Theorem 3.4 (Atiyah 1987, Thm. (5.60); [CITED]). *For hyperbolic $A \in \mathrm{SL}_2(\mathbb{Z})$, the following coincide: Meyer’s invariant $\phi(A)$ (the signature defect of the torus bundle $W(A) \rightarrow S^1$); Hirzebruch’s signature defect $\delta(A)$; the invariant describing the transformation of $\log \eta$ under A ; the logarithmic monodromy of Quillen’s determinant line bundle; the value $L_A(0)$ of the Shimizu L -function; the Atiyah–Patodi–Singer invariant $\eta(A)$; and its adiabatic limit $\eta_0(A)$. In Atiyah’s introduction these are tied to Rademacher’s Ψ , and Meyer’s signature cocycle on $\mathrm{SL}_2(\mathbb{Z})$ is a coboundary that yields ϕ .*

Proposition 3.5 (The Eisenstein side; [THEOREM]/[RESTATEMENT]). (1) $E_2 - 1 = -24 \sum_{m,n \geq 1} n q^{mn}$ groups the pairs $(m, n) \in \mathbb{N} \times \mathbb{N}$ by the product mn . Mellin: $\sum \sigma_1(n) n^{-s} = \zeta(s) \zeta(s-1)$ (partial sum at $s = 3$, $N = 2 \cdot 10^5$: 1.977296 vs 1.977304). The associated motive is $\mathbb{Q}(0) \oplus \mathbb{Q}(-1)$: a direct sum, with L a product. It is not a tensor product.

- (2) For E_k the Satake parameters at p are $(1, p^{k-1})$. They are non-tempered, because their absolute values differ. The Hasse form $m^2 + amn + p^{k-1}n^2$ with $a = 1 + p^{k-1}$ has discriminant $(p^{k-1} - 1)^2 > 0$, so it is indefinite. For Δ_{12} , $\tau(p)^2 - 4p^{11} < 0$ (checked at $p = 2, 3, 5, 7$). In the L95 Frobenius-lattice test (L95 Thm. frobsees [TAA-L]), an Eisenstein “Frobenius” reads as signature $(2, 2)$, exactly like a ghost. The reason is that $\mathbb{Q}(0) \oplus \mathbb{Q}(1-k)$ is mixed: it has two weights. Each pure summand has rank 1 and is trivially polarized.
- (3) ζ itself is “on the Eisenstein side”: $L(E_k, s) = \zeta(s) \zeta(s-k+1)$. The non-temperedness of E_k concerns the weight split, not the zeros of ζ .
- (4) The diagonal $m = n$ gives $\theta(\tau) = \sum_n q^{n^2}$. Then $\theta(i/t) = \sqrt{t} \theta(it)$ (residual 0 at 30 digits), and its Mellin transform gives Riemann’s functional equation (classical).

4. (D) IS THE ANOMALY A ROSATI-LIKE INVOLUTION? (T408)

Theorem 4.1 (Φ is odd; [THEOREM], [NUMERICAL]). *Let $J = \mathrm{diag}(1, -1)$ (orientation reversal), let $\gamma^\dagger := S\gamma^T S^{-1}$ with $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Then*

- (1) $\gamma^\dagger = \gamma^{-1}$. This is the adjugate, i.e. the adjoint for the symplectic form. On H^1 of an elliptic curve it is the Rosati involution $\varphi \mapsto \hat{\varphi}$ (dual isogeny), and $\varphi \hat{\varphi} = \deg \varphi$.
- (2) $\Phi(\gamma^{-1}) = -\Phi(\gamma)$, $\Phi(J\gamma J) = -\Phi(\gamma)$, $\Phi(-\gamma) = \Phi(\gamma)$. The class function Ψ satisfies $\Psi(\gamma^{-1}) = \Psi(\gamma^T) = -\Psi(\gamma)$, whereas $\Phi(\gamma^T) = -\Phi(\gamma)$ holds in only 214/389 tests.
- (3) $\Phi(AB) = \Phi(A) + \Phi(B) - 3 \mathrm{sign}(c_{AC} c_{BC} c_{AB})$ when $c_A, c_B, c_{AB} \neq 0$. Checked in 296/296 tests; the values are ± 3 only (Fig. 3).
- (4) The Rosati positive form $\mathrm{Tr}(\gamma \gamma^\dagger) = 2 \det \gamma = 2$ is constant on $\mathrm{SL}_2(\mathbb{Z})$, and $\Phi(\gamma \gamma^\dagger) = \Phi(1) = 0$.

Proof. (1) For a 2×2 matrix, $SM^T S^{-1} = \mathrm{adj}(M)$. (2) For $c \neq 0$, $\gamma^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so $\Phi(\gamma^{-1}) = -\frac{a+d}{c} + 12 \mathrm{sign}(c) s(a, |c|)$, and $s(a, |c|) = s(d, |c|)$ since $ad \equiv 1$. Next, $J\gamma J = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ flips the signs of c and $\mathrm{sign}(c)$. For $c = 0$, $\gamma = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ and both operations send $b/d \mapsto -b/d$. For Ψ : $\gamma^T = S^{-1} \gamma^{-1} S$, and Ψ is a class function. (3) This is Rademacher’s cocycle relation, verified numerically here. (4) is immediate. $\square$

Corollary 4.2 (Answer to t408; [THEOREM] + [ANALOGY]). *The involution $J.W.$ had in mind exists. $\gamma \mapsto S\gamma^T S^{-1}$ is literally the Rosati involution on H^1 of a torus, and S is the level-1 Fricke involution that gives $\Lambda(\Delta_{12}, s) = \Lambda(\Delta_{12}, 12-s)$. But the anomaly is not that involution. It is a function that is odd under it. Rosati positivity is an even, quadratic statement ($\mathrm{Tr} \varphi \hat{\varphi} = 2 \deg \varphi > 0$), and on $\mathrm{SL}_2(\mathbb{Z})$ it degenerates to the constant 2. By Theorem 3.4, Φ and Ψ are signature defects and η -invariants, i.e. measures of spectral asymmetry. The “partner” under $\dagger$ is γ^{-1} , and positivity needs $\gamma \gamma^\dagger$, which is trivial. **Ghost sensitivity:** Φ is defined by $\mathrm{SL}_2(\mathbb{Z})$ alone, so it is identical for the true ζ and for any fake ([THEOREM]). The L -values it equals are $L_A(0)$ for real quadratic fields: special values, not zero locations.*

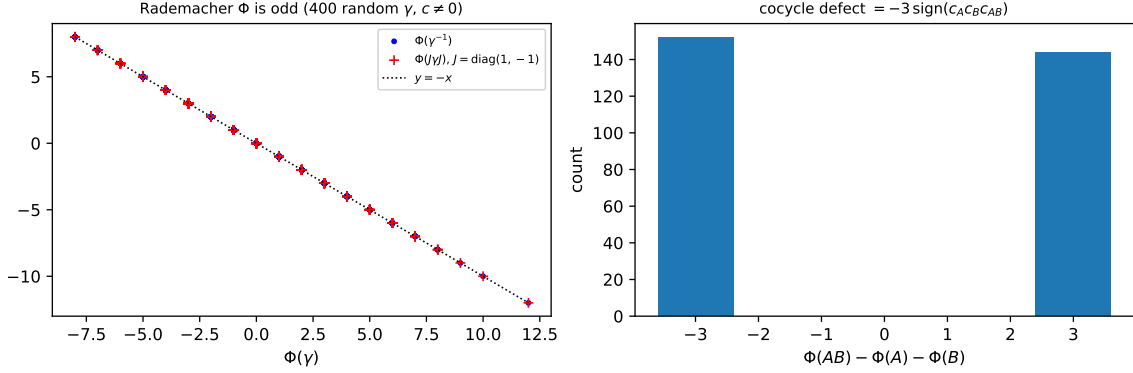


FIGURE 3. Left: $\Phi(\gamma^{-1})$ and $\Phi(J\gamma J)$ against $\Phi(\gamma)$ lie on $y = -x$. Right: the coboundary takes only the values ± 3 .

5. (E) RENORMALIZING Δ_C^2

5.1. Three appearances of $\zeta(-1)$.

Proposition 5.1 ([RESTATEMENT]; [7] [CITED]). (1) $\eta = q^{1/24} \prod (1 - q^n)$ and $\frac{1}{24} = -\frac{1}{2}\zeta(-1)$. Thus $\eta = q^{-\frac{1}{2} \sum_{\text{reg}} n} \prod (1 - q^n)$, and it is $1/\eta$ that carries $q^{+\frac{1}{2} \sum_{\text{reg}} n}$. (2) Harer–Zagier: $\chi(\Gamma_g^1) = \zeta(1 - 2g)$ for the mapping class group with one marked point (i.e. $\mathcal{M}_{g,1}$), and $\chi(\Gamma_g) = \zeta(1 - 2g)/(2 - 2g)$ for $g \geq 2$. For $g = 1$, $\Gamma_1^1 \cong \Gamma_1 \cong \text{SL}_2(\mathbb{Z})$ and $\chi = \zeta(-1) = -\frac{1}{12}$, an orbifold Euler characteristic with generic stabilizer ± 1 ; for $\text{PSL}_2(\mathbb{Z})$ it is $-\frac{1}{6}$. (3) For $g \geq 2$, $\Delta_C^2 = 2 - 2g = \chi(C) = \chi(\mathcal{M}_{g,1})/\chi(\mathcal{M}_g)$ (fibre = curve), checked exactly for $g = 2, \dots, 7$. At $g = 1$ the identity breaks: $\Delta_C^2 = 0$ but $\chi(\mathcal{M}_{1,1}) = -\frac{1}{12}$.

Remark 5.2 ([THEOREM]/[HEURISTIC]). The numbers $\frac{1}{24}$ (in η), $\frac{\pi i}{12}$ (in Theorem 3.3), and $\chi(\text{SL}_2(\mathbb{Z})) = -\frac{1}{12}$ are all $\zeta(-1)$. This is the precise sense in which J.W.’s “three Δ ’s” meet. But $\zeta(-1)$ is a special value; it does not depend on the location of any zero. A “genus ∞ ” limit of (3) does not exist: $|\zeta(1 - 2g)| \sim 2(2g - 1)!/(2\pi)^{2g} \rightarrow \infty$ ([HEURISTIC], Stirling, from $\zeta(1 - 2g) = -B_{2g}/2g$).

5.2. Spectral renormalization of the genus.

Proposition 5.3 ([13] [CITED]; [NUMERICAL]). Let $Z(\sigma) = \sum_k \tau_k^{-2\sigma}$ over the zeros $\frac{1}{2} + i\tau_k$ with $\tau_k > 0$. Then $Z(0) = \frac{7}{8}$. This is the constant in $N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + S(T) + o(1)$. Numerically, the mean of $N(T)$ minus the smooth term on $[100, 3031]$ is 0.8748. The heat-trace constant $\sum e^{-t\tau_k} - \frac{1}{2\pi t}(\log \frac{1}{2\pi t} - \gamma_E)$ is 0.800, 0.860, 0.871 at $t = 0.1, 0.02, 0.005$ (Fig. 4).

Theorem 5.4 (g_{reg} fails; [THEOREM]+[NUMERICAL]). Set $g_{\text{reg}} = Z(0) = \frac{7}{8}$, which counts each pair $\pm\tau_k$ once, like $g = \frac{1}{2} \dim H^1$. Then: (a) $\Delta_{C_{\text{reg}}}^2 = 2 - 2g_{\text{reg}} = +\frac{1}{4}$. This is finite but has the sign of genus < 1 , whereas the actual ζ analogue sits beyond every genus ≥ 2 (where $\Delta_C^2 \leq -2$). The projected value $\Delta_{C_{\text{reg}}}^2 = -\frac{7}{4}$ is negative, trivially. (b) The Weil-type bound $\sum_\rho h(\gamma) \leq 2g_{\text{reg}} \sup h$ for the Gaussian $h = e^{-(\gamma/T)^2}$ holds for $T \leq 14$ but fails for $T \geq 20$: the ratio is 1.57, 3.8, 9.8, 29.8, 140, 660 at $T = 20, 30, 50, 100, 300, 1000$. (c) Ghosts: adding a quartet changes $Z(0)$ by exactly $+2$, and replacing two on-line zeros by a quartet changes it by 0. Each zero contributes $\tau^{-2\sigma}|_{\sigma=0} = 1$ wherever it lies. The heat-trace constant shift from an added pair $100 \pm 0.3i$ is $2e^{-100t} \cos(0.3t) \rightarrow 2$. Hence g_{reg} is position-blind.

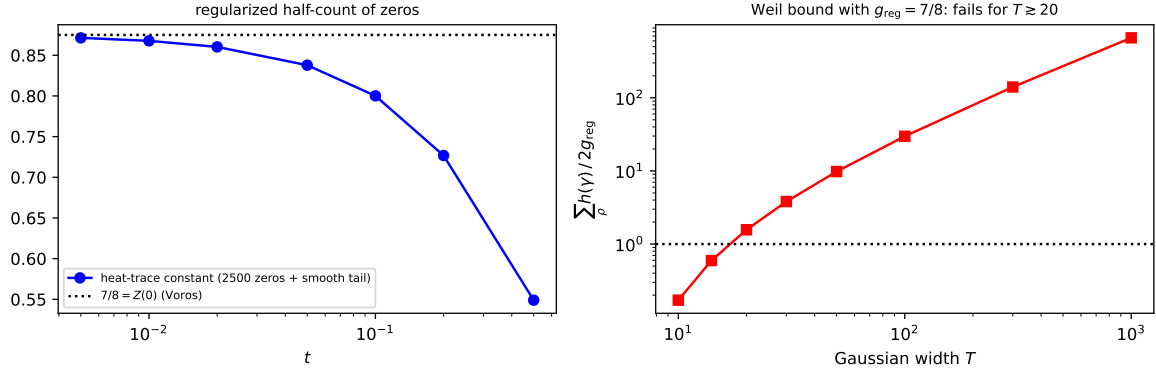


FIGURE 4. Left: regularized half-count of zeros $\rightarrow 7/8$. Right: the Weil bound with the finite constant $2g_{\text{reg}}$ is violated once the test function sees more than a few zeros.

5.3. Log-renormalized self-intersection of the diagonal. L94 (sharp proposition, [TAA-L]) gives $I(\psi, \psi) \sim -\frac{1}{2}c(0) \log \frac{1}{R}$ for a bump of support radius R . Define $\Delta_{C_{\text{ren}}}^2 = \lim_{R \rightarrow 0} [I(\psi, \psi)/c(0) - \kappa \log R]$.

Proposition 5.5 ([NUMERICAL]; ghost limit [THEOREM]). *For cardinal B-spline bumps of order $m = 4$ and $R = 0.4, \dots, 0.0125$: the fitted slope is $\kappa = 0.5027$ (L94 predicts $\frac{1}{2}$), and the finite part is 0.467. The finite part is **scheme dependent**: 0.655, 0.545, 0.467, 0.359 for $m = 2, 3, 4, 6$ (slopes 0.504–0.502). With two ghost quartets ($20 \pm 0.2i$, $50 \pm 0.4i$) the shift is $-\frac{1}{2}G/c(0) = -0.10, -0.16, -0.16, -0.10, -0.05, -0.026$ along the R grid. It tends to 0 because $G_{xx} = \sum_r \text{sinc}^{2m}(rh/2)$ stays bounded while $c(0) \propto 1/R$. So $\Delta_{C_{\text{ren}}}^2$ is ghost-blind.*

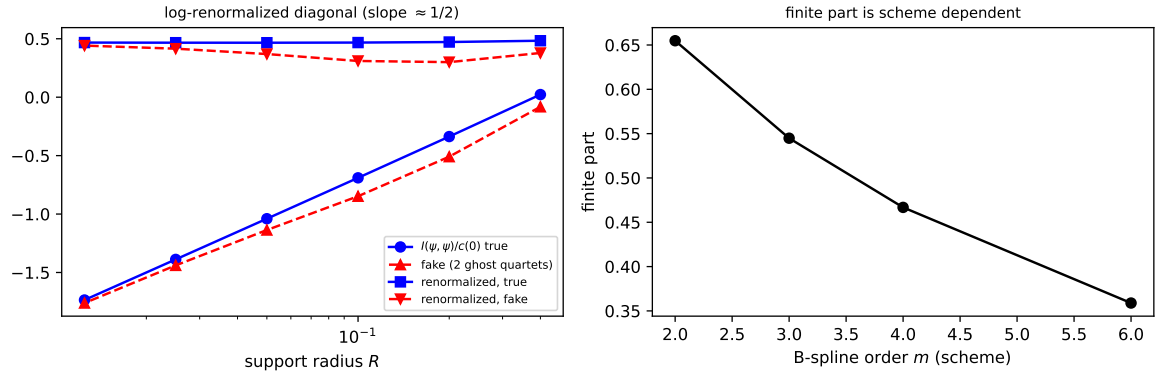


FIGURE 5. Left: raw and renormalized diagonal, true vs fake. The renormalized curves converge together as $R \rightarrow 0$. Right: the finite part depends on the regularization scheme.

Theorem 5.6 (Honest negative; [THEOREM] for the listed schemes, [HEURISTIC] in general). *No candidate tested here meets all of (a) finite, (b) Hodge-index sign, (c) ghost-sensitive. Spectral constants ($Z(0)$, heat-trace constants) fail (c) because they count zeros. Local diagonal renormalizations fail (c) because the divergence and the finite part come from $d = 0$, where ghosts contribute $O(R)$. In addition, the finite part fails universality. $\zeta(1-2g)$ and $\zeta(-1)$ are special values and fail (c) trivially. Heuristic reason: Weil's finite $2g$ enters as a Cauchy–Schwarz constant for the whole pairing with*

Γ_{F^n} , for all n . A renormalization subtracts a local divergence and therefore cannot supply a global constant. Regularized determinants ([5] [BIB], [13]: $Z'(0) = \frac{1}{4} \log 8\pi$) encode Γ - and L -factors, and again are not positivity.

6. CORRECTIONS TO THE CHAT CLAIMS (T406–T408)

The verbatim chat was not available to this worker. The claims below are as paraphrased in the brief (t409u).

C96.1 (E_2 constant):

Correct: $E_2(-1/\tau) = \tau^2 E_2 + \frac{6\tau}{\pi i} = \tau^2 E_2 + \frac{12\tau}{2\pi i}$. Any version with $\frac{12\tau}{\pi i}$, or without the factor τ , is wrong ([NUMERICAL]). For G_2 the shift is $-2\pi i\tau$, i.e. $G_2^* - G_2 = -2\pi i/\tau$.

C96.2 (Harer–Zagier):

$\zeta(1-2g)$ is the orbifold Euler characteristic of Γ_g^1 ($\mathcal{M}_{g,1}$, one marked point), not of $\mathcal{M}_g$. The latter is $\zeta(1-2g)/(2-2g)$. $-\frac{1}{12}$ is for $\mathrm{SL}_2(\mathbb{Z}) / \mathcal{M}_{1,1}$ as a stack; PSL_2 gives $-\frac{1}{6}$, and the coarse space $\mathbb{C}$ has $\chi = 1$.

C96.3 (Atiyah):

Atiyah 1987 proves that seven invariants coincide for *hyperbolic* A (Thm. (5.60)). They are invariants of the matrix A (torus bundles, real quadratic $L_A(0)$), not of zeros of ζ . The APS η is a different object from the Dedekind η , as Atiyah himself flags.

C96.4 (Φ oddness):

Confirmed for $\gamma \mapsto \gamma^{-1}$ and orientation reversal. Φ is even under $\gamma \mapsto -\gamma$. Φ is *not* odd under the transpose and is not a class function; Ψ is both.

C96.5 (“ E_2 is a connection”):

The Serre derivative ∂_k is the connection; $-\frac{k}{12}E_2$ is its connection form. The anomaly is the gauge term.

C96.6 (“the Eisenstein side has no positivity”):

Refined: the Eisenstein motive $\mathbb{Q}(0) \oplus \mathbb{Q}(1-k)$ is mixed. Each pure piece is polarized, and the cross-weight Hasse form is indefinite, which mimics a ghost in the L95 test.

C96.7 ($q^{1/24}$ sign):

$q^{1/24} = q^{-\frac{1}{2}\zeta(-1)}$. It is $1/\eta$ that carries $q^{+\frac{1}{2}\sum_{\mathrm{reg}} n}$.

7. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	Weight-graded positivity: formalize “positivity per pure weight” over $\mathbb{Q}$. The pole plane $\langle H, V \rangle$ is the Eisenstein-like mixed part, and V_0 is its pure complement (L91). Make the Eisenstein/ghost $(2, 2)$ coincidence precise.	0.5	0.7	0.58
2	Spectral asymmetry: look for an RH-equivalent statement about an <i>odd</i> (η -type) invariant attached to the explicit formula, since Rosati-even data on $\mathrm{SL}_2(\mathbb{Z})$ is trivial.	0.8	0.15	0.54
3	Position-sensitive regularized constants, e.g. $Z'(0)$ or $\sum \log(\rho - \frac{1}{2})^2$, which do see δ but have no sign.	0.6	0.25	0.46
4	Replace Weil’s uniform constant by a growth-rate constant. This is equivalent to RH-strength prime bounds, so the risk of circularity is high.	0.7	0.1	0.46
5	Realize the explicit formula in a genuine cohesive topos (condensed / arithmetic site) and compute $\int, \flat$.	0.3	0.5	0.38

8. QUESTIONS FOR J.W.

- Q1. By “the anomaly”, do you mean the integer $\Phi(\gamma)$, Atiyah’s class function Ψ/ϕ , or the E_2 term $\frac{6}{\pi i}c(c\tau + d)$? They differ by coboundaries and by parity under the transpose.
- Q2. Given Theorem 4.1(4), should the Rosati idea move to the odd part (direction 2), or to End of a non-unimodular lattice, where $\text{Tr } \varphi\hat{\varphi}$ is non-constant?
- Q3. Do you accept the three-part test (finite, Hodge sign, ghost-sensitive) as the criterion for a “renormalized Δ_C^2 ”? Every candidate here fails ghost-sensitivity.
- Q4. Which cohesive topos do you intend: condensed, the arithmetic site, or smooth sets? Without one, Π , Disc are formal and Prop. 2.2 is all that can be said.
- Q5. Should L97 take direction 1 (weight-graded positivity, most likely to give a theorem) or direction 2 (spectral asymmetry, higher risk)?

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TAA LEMMA 97: WEIGHT, DEQUANTIZATION, AND THE MISSING H^1

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis. Every claim carries one label: [THEOREM] (proved here, or classical with the proof indicated), [RESTATEMENT] (reformulation), [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (computed; scripts `num/lemma97_*.py`, data `num/197/`). Sources: [CITED] (opened), [CITED-sec] (secondary), [BIB] (bibliographic), [TAA-L] (earlier TAA lemma). References: `lit/lemma97/verified_refs.md`.

ABSTRACT. This chapter pursues L96 direction 1, *weight-graded positivity*. It also tests J.W.’s proposal (t414–t417) to “dequantize” Weil’s form in the way that $\hbar \log |f|$ turns q -series into orders of vanishing, and it collects the $\mathbf{1} \oplus \mathbb{L}$ dictionary. The Mellin section requested in t419 is included as §5.

Results. (1) *Grading by weight is circular* (Thm. 2.2). For any finite Frobenius-type multiset closed under $\mu \mapsto 1/\bar{\mu}$, the explicit-formula (duality) form is congruent to $S\Pi$. Each weight-1 piece is definite. Each pair $\mathrm{Gr}_w \oplus \mathrm{Gr}_{2-w}$ with $w \neq 1$ is a hyperbolic block, and a single Gr_w with $w \neq 1$ is isotropic. A “positive form on each graded piece” does exist, the graded polarization $\sum |\hat{\varphi}(\mu)|^2$, but it is positive for *every* multiset, true or ghost. Exact inertias: curve $(2g, 0)$; ghost $(2, 2)$ versus graded $(4, 0)$; E_4 at $p = 2$ $(1, 1)$ versus $(2, 0)$. On ζ the graded form stays $(0, 17)/(0, 36)$ on V_0 with ghosts present, while the duality form shows $n_+ = 2$. To split by weight one must know $\mathrm{Re} \rho$, which is RH. The non-circular grading is cohomological degree: poles $(H^0 \oplus H^2)$ versus zeros (H^1) . Over $\mathbb{Q}$ that grading is exactly V_0 , and $\mathrm{RH} \iff$ purity of the “ H^1 ” part. (2) *Dequantization keeps signs but truncates* (Thm. 3.1). With Gaussian test functions and $\hbar = 1/s^2 \rightarrow 0$, the eigenvalues on V_0 tropicalize to $-\mathrm{Re}(r^2) = -(\gamma^2 - \delta^2)$ ([NUMERICAL]: ghost $20 \pm 0.3i$ gives -399.89 , predicted -399.91). The ghost does *not* die the way the shadow $3/(\pi y) = 6\hbar$ does. It survives as two *positive* eigenvalues, i.e. in the signed tropical hyperfield. Its unsigned valuation equals that of an on-line zero at $\gamma' = \sqrt{\gamma^2 - \delta^2}$. With a fixed test space only the lowest zero levels survive: a ghost at $40 \pm 0.3i$ is invisible with $\dim V_0 = 6$. The tropical limit therefore depends on finitely many zeros and cannot carry RH. (3) *Chat claims*. $\xi_2(E_2^*) = 3/\pi$ in the Bruinier–Funke normalization ([NUMERICAL], 12 digits). The coefficient $3/\pi$ equals $1/\mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}) = \mathrm{Res}_{s=1} E(z, s)$ by Stokes (Thm. 1.3), so the attribution to ζ ’s pole is accurate up to the factor $\zeta(2)$. **Wrong:** “ $\partial - \frac{k}{12} E_2^*$ is the raising operator”. The raising operator is $\partial_k + \frac{k}{12} E_2^* = \partial - \frac{k}{4\pi y}$, and the claimed operator is not covariant (residual 1.25). **Unverified/corrected:** I found no statement in Lion–Vergne about the Rademacher function. The verified link is that the coboundary of Φ is $-3 \times$ the Kashiwara–Maslov index of $(\ell, A\ell, AB\ell)$ (Thm. 1.7, 1820/1820), consistent with the “area cocycle” of Kirby–Melvin and Szech. (4) Deninger’s formalism and the four Mellin corrections C97.m1–m4 (§5).

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Date: 24 October 2026 (approved t418u, “queue that for direction 1” pt418u; Mellin addendum t419/pt419u; chat t414–t417).

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GLOSSARY (BRIEF RESTATEMENTS FOR J.W.)

Motive.:

A “universal cohomology” piece of a variety. $h(X)$ decomposes as $h^0 \oplus h^1 \oplus \cdots$, and every cohomology theory (Betti, ℓ -adic, de Rham) realizes it. Its L -function is built from Frobenius eigenvalues.

Weight; purity.:

h^i of a smooth projective variety over $\mathbb{F}_q$ has *weight* i : every Frobenius eigenvalue has $|\alpha| = q^{i/2}$. This is *purity* (Deligne [5] [BIB]). A *mixed* motive is a sum or extension of pieces of different weights.

Lefschetz motive $\mathbb{L}$; Tate twist.:

$\mathbb{L} = h^2(\mathbb{P}^1)$, with Frobenius acting by q (weight 2). $\mathbb{Q}(n) = \mathbb{L}^{\otimes -n}$, and “twisting by $\mathbb{Q}(n)$ ” shifts the weight by $-2n$. $h(\mathbb{P}^1) = \mathbf{1} \oplus \mathbb{L}$.

Polarization; Hodge–Riemann.:

A pairing $h^i \otimes h^i \rightarrow \mathbb{L}^{-i}$ (it depends on i) that is definite on each primitive piece after the standard sign. Over $C \times C$ this is the Hodge index theorem. Note: one says “the first copy of C ” in $C \times C$, not “factor”.

Tropical / max-plus; Maslov dequantization.:

$\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with $x \oplus y = \max(x, y)$ and $x \odot y = x + y$. It is the $\hbar \rightarrow 0$ limit of $x \oplus_{\hbar} y = \hbar \log(e^{x/\hbar} + e^{y/\hbar})$ [13] [CITED]. Signs are lost. Viro’s *signed tropical hyperfield* $\{-1, 0, 1\}$ keeps them, with $1 + (-1) = \{-1, 0, 1\}$ [18] [CITED].

Harmonic Maass form; ξ_k .:

A real-analytic weight- k form annihilated by the weight- k Laplacian. $\xi_k f = v^{k-2} \overline{L_k f}$ with $L_k = -2iv^2 \partial_{\bar{\tau}}$, and the raising operator is $R_k = 2i\partial_{\tau} + k/v$ [1] [CITED].

1. THE CHAT CLAIMS T414–T417

1.1. (a) E_2^* , Hecke’s trick, ξ_2 .

Proposition 1.1 ([RESTATEMENT] [19] §2.3; [NUMERICAL]). E_2 is holomorphic and quasimodular. $E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi y}$ is modular of weight 2 and real-analytic: residual $\leq 5 \cdot 10^{-30}$ on 9 tests, whereas E_2 fails by ≥ 1.4 . Zagier obtains it by Hecke’s trick, regularizing $\sum' (m\tau + n)^{-2} |m\tau + n|^{-2\varepsilon}$ and letting $\varepsilon \rightarrow 0$ ([19] (20)–(21)).

Proposition 1.2 ([THEOREM] + [NUMERICAL]). In the normalization of [1], $\xi_2 E_2^* = \frac{3}{\pi}$ (finite differences at 3 points: 0.954929658551), and $\xi_2 E_4 = 0$. Hence $\Delta_2 E_2^* = -R_0 \xi_2 E_2^* = 0$ ($B\bar{F}$ (3.3)), so E_2^* is a harmonic Maass form of weight 2 lying in H_2 . It does not lie in H_2^+ , because its ξ -image is a nonzero constant, not a cusp form.

Proof. $\partial_{\bar{\tau}}(1/v) = -\frac{i}{2v^2}$, so $L_2(-\frac{3}{\pi v}) = -2iv^2 \cdot \frac{3i}{2\pi v^2} = \frac{3}{\pi}$, which is real. Then apply v^{0-} . $\square$

Theorem 1.3 (The constant $3/\pi$ is 1/vol; [THEOREM]). Let $E_2^* = E_2 - c/y$ be modular. Then $c \cdot \text{vol}(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}) = 1$ (the constant term of E_2), so $c = 3/\pi$. Also $\text{Res}_{s=1} E(z, s) = \sqrt{\pi} \Gamma(\frac{1}{2}) \cdot \frac{1}{2} / \zeta(2) =$

$3/\pi$ for $E(z, s) = \sum_{\Gamma_\infty \backslash \Gamma} \text{Im}(\gamma z)^s$ ([9] [CITED-sec]; [NUMERICAL] 0.95492965855). The pole comes from $\zeta(2s - 1)$ in the constant term.

Proof. The form $\omega = E_2^* d\tau$ is $\text{SL}_2(\mathbb{Z})$ -invariant. Since $\partial_{\bar{\tau}} E_2^* = \frac{ic}{2y^2}$ and $d\bar{\tau} \wedge d\tau = 2i dx \wedge dy$, we get $d\omega = -c d\mu$ with $d\mu = dx dy / y^2$. Integrate over the fundamental domain truncated at $y = T$. The side edges cancel, and the top edge gives $-\int_{-1/2}^{1/2} E_2^*(x + iT) dx = -(1 - c/T)$. Letting $T \rightarrow \infty$ gives $c \cdot \frac{\pi}{3} = 1$. $\square$

So J.W.'s attribution is accurate with one refinement. $3/\pi$ is $1/\text{vol}$, which is also the residue of $E(z, s)$. That residue is ζ 's pole (via $\zeta(2s - 1)$) divided by $\zeta(2)$.

1.2. (b) Serre derivative versus raising.

Proposition 1.4 ([RESTATEMENT] [19] §5.1 (55); [NUMERICAL]). *The Serre derivative $\vartheta_k f = \vartheta f - \frac{k}{12} E_2 f$ is holomorphic and covariant. The non-holomorphic covariant derivative is $\partial_k f = \vartheta f - \frac{k}{4\pi y} f = -\frac{1}{4\pi} R_k f$. They are related by*

$$\partial_k = \vartheta_k + \frac{k}{12} E_2^* \quad ([\text{NUMERICAL}]: 3 \cdot 10^{-32}).$$

The operator $\vartheta - \frac{k}{12} E_2^*$ is not covariant: for $f = E_4$ it fails by 1.25 under S and by 7.2 under $(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix})$, while ∂_4 and ϑ_4 hold to 10^{-18} .

1.3. (c) Mellin, Hecke, and the log-derivative.

Proposition 1.5 ([RESTATEMENT]/[ANALOGY]). $\sum \sigma_1(n) n^{-s} = \zeta(s) \zeta(s-1)$ (L96 [NUMERICAL]). For a normalized Hecke eigenform, multiplicative coefficients are equivalent to an Euler product ([19] §4, [17] VII [CITED-sec]). $E_2 = 1 - 24 \sum \frac{nq^n}{1-q^n} = \vartheta \log \Delta_{12}$ with $\Delta_{12} = q \prod (1 - q^n)^{24}$ (L96 [NUMERICAL], exact to q^{400}). The parallel with $-\zeta'/\zeta = \sum \Lambda(n) n^{-s}$ is structural (log-derivative of a product) but not literal. Δ_{12} is a product over all n , not over primes, and its log-derivative has divisor-sum coefficients $\sigma_1(n)$, not von Mangoldt weights ([ANALOGY]).

1.4. (d) Dequantization and the Maslov index.

Proposition 1.6 ([THEOREM] + [NUMERICAL]; valence formula [17] VII [CITED-sec]). With $\hbar = 1/(2\pi y)$, $\hbar \log |f(x + iy)| \rightarrow -\text{ord}_\infty f$ for $f = \sum_{n \geq n_0} a_n q^n$, since $|q^n| = e^{-n/\hbar}$. [NUMERICAL]: Δ_{12} gives $-1.0000032, -1 - 6 \cdot 10^{-12}, -1.0000000$ at $y = 2, 4, 8$, and $E_4, E_2^* \rightarrow 0$. The shadow is $3/(\pi y) = 6\hbar$ exactly, so it vanishes linearly in $\hbar$ (Fig. 1). Valence formula: $\text{ord}_\infty + \frac{1}{2} \text{ord}_i + \frac{1}{3} \text{ord}_\rho + \sum \text{ord}_p = \frac{k}{12}$. Checks: $E_4(\rho) = 0$ ($6 \cdot 10^{-28}, \frac{1}{3} = \frac{4}{12}$), $E_6(i) = 0$ ($\frac{1}{2} = \frac{6}{12}$), Δ_{12} : $\text{ord}_\infty = 1 = \frac{12}{12}$.

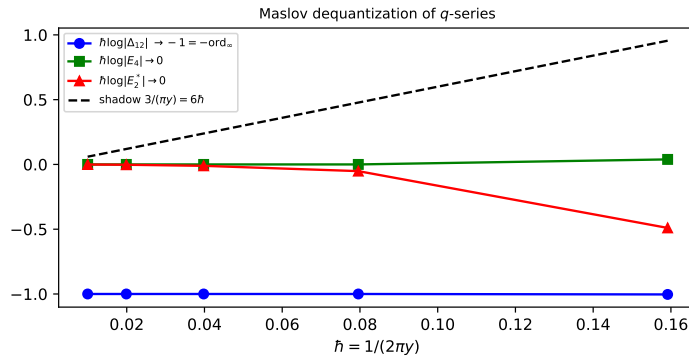


FIGURE 1. Dequantized q -series: $\hbar \log |f| \rightarrow -\text{ord}_\infty f$. The shadow of E_2^* is exactly $6\hbar$.

Theorem 1.7 (Rademacher coboundary = Maslov index; [THEOREM] + [NUMERICAL]). *Let $\ell = \mathbb{R}(1,0)^T \subset \mathbb{R}^2$ and let τ be the Kashiwara–Maslov index of three lines, i.e. the signature of $Q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$ on $\ell_1 \oplus \ell_2 \oplus \ell_3$. For $A, B \in \mathrm{SL}_2(\mathbb{Z})$ with $c_A, c_B, c_{AB} \neq 0$, $\tau(\ell, A\ell, AB\ell) = \mathrm{sign}(c_A c_B c_{AB})$. Hence Rademacher’s relation (L96 Thm. odd(3)) reads*

$$\Phi(AB) - \Phi(A) - \Phi(B) = -3\tau(\ell, A\ell, AB\ell).$$

Proof. $\omega(\ell, A\ell) = c_A$, $\omega(A\ell, AB\ell) = \omega(\ell, B\ell) = c_B$, and $\omega(AB\ell, \ell) = -c_{AB}$. The matrix $\frac{1}{2} \begin{pmatrix} 0 & a & c \\ a & 0 & b \\ c & b & 0 \end{pmatrix}$ has trace 0 and determinant $abc/4$, so its signature is $-\mathrm{sign}(abc)$, which here equals $\mathrm{sign}(c_A c_B c_{AB})$. [NUMERICAL]: 1820/1820 random pairs. $\square$

Szcech calls $\mathrm{sign}(c_1 c_2 c_3)$ the *area cocycle*, and Meyer’s function differs from $-\frac{1}{3}\Phi$ by an explicit correction whose coboundary is the signature cocycle ([16] [CITED] excerpt; [12] [BIB]: “geometric formulation of the Rademacher φ -function . . . signature cocycle”). Lion–Vergne [14] [BIB] (table of contents checked) treat the Maslov index, the Shale–Weil cocycle and the multiplier of θ . I could not verify that they state the Rademacher link, so it is attributed here to Thm. 1.7 and [12, 16].

1.5. (e) $\mathbf{1} \oplus \mathbb{L}$ and level N .

Proposition 1.8 ([THEOREM]/[RESTATEMENT]; [NUMERICAL]). $\#\mathbb{P}^1(\mathbb{F}_p) = 1 + p = \sigma_1(p)$ (checked for $p \leq 11$), and $\zeta_{\mathbb{P}^1/\mathbb{Z}}(s) = \prod_p (1 - p^{-s})^{-1} (1 - p^{1-s})^{-1} = \zeta(s)\zeta(s-1)$. $M_2(\mathrm{SL}_2(\mathbb{Z})) = 0$, but $E_2(\tau) - NE_2(N\tau) \in M_2(\Gamma_0(N))$, because the shadows cancel: $\frac{3}{\pi y} - N\frac{3}{\pi Ny} = 0$. [NUMERICAL]: residual $\leq 5 \cdot 10^{-12}$ for $N = 2, 3, 5$ ([8] Ch. 1 [BIB]).

2. (A) WEIGHT-GRADED POSITIVITY

Proposition 2.1 (Over $\mathbb{F}_q$; [RESTATEMENT] [5, 10] [BIB]). *For a curve C , H^0, H^1, H^2 have Frobenius absolute values $1, \sqrt{q}, q$ (weights 0, 1, 2). On $\mathrm{NS}(C \times C) \otimes \mathbb{R}$ the plane $\langle H, V \rangle$, built from H^0 of one copy of C times H^2 of the other, is hyperbolic of signature $(1, 1)$. The algebraic part of $H^1 \otimes H^1$ (correspondences orthogonal to H, V) is negative definite (Hodge index; L96 Thm. weilneeds).*

Theorem 2.2 (Graded decomposition and circularity; [THEOREM]). *Let $\mathcal{S}$ be a finite multiset of normalized eigenvalues $\mu = \lambda/\sqrt{q}$ with signs s_μ (+1 on “ H^1 ”, −1 on the poles $q^{\pm 1/2}$), closed under $\mu \mapsto \mu^* = 1/\bar{\mu}$ with $s_{\mu^*} = s_\mu$. On test sequences φ supported on $|n| \leq M$, with M large, put*

$$H(\varphi) = \sum_{\mu} s_{\mu} \hat{\varphi}(\mu) \overline{\hat{\varphi}(\mu^*)}, \quad P(\varphi) = \sum_{\mu} s_{\mu} |\hat{\varphi}(\mu)|^2, \quad \hat{\varphi}(\mu) = \sum_n \varphi_n \mu^{-n}.$$

H is the explicit-formula (duality) form; P is the graded polarization. Then:

- (1) *$H \cong S\Pi$ and $P \cong S$ (Sylvester), where Π is the permutation $\mu \mapsto \mu^*$.*
- (2) *The weight $w = 1 + 2 \log |\mu| / \log q$ satisfies $w(\mu^*) = 2 - w(\mu)$. Each $|\mu| = 1$ eigenvalue contributes a definite square of sign s_μ . Each pair $\{\mu, \mu^*\}$ with $|\mu| \neq 1$ contributes a hyperbolic plane $(1, 1)$, and a single Gr_w ($w \neq 1$) is isotropic for H .*
- (3) *P has the sign pattern of (s_μ) for every $\mathcal{S}$. It is ghost-blind.*
- (4) *The H^1 part of H is definite $\iff$ every H^1 eigenvalue has $|\mu| = 1$ (RH/purity). Computing Gr_w requires knowing $|\mu|$, i.e. $\mathrm{Re} \rho$: the weight grading presupposes the conclusion.*

Proof. $H_{nm} = \sum s_{\mu} \mu^{-n} \bar{\mu}^{*-m}$. Writing $V_{\mu n} = \mu^{-n}$ gives $H = V^T(S\Pi)\bar{V}$ and $P = V^T S \bar{V}$, and V has full row rank for $2M + 1 \geq |\mathcal{S}|$. The rest is reading off the blocks of $S\Pi$. $\square$

With the poles included, the full duality form is $(2g + 1, 1)$ for true curves. This is the Hodge-index pattern, with the $(1, 1)$ pole plane playing the role of $\langle H, V \rangle$. The Eisenstein rows make C96.6 precise. The mixed motive $\mathbf{1} \oplus \mathbb{L}^{k-1}$ is positive *per graded piece* under P , but indefinite under the one pairing compatible with the functional equation. That is exactly the ghost pattern.

TABLE 1. Exact inertia of the H^1 part (`graded.json`); normalized weights in brackets.

Model	weights	duality H	graded P
$E/\mathbb{F}_{101}$, $a = -3$	1, 1	(2, 0)	(2, 0)
genus 2 ($a = -3$, $a = 7$), $\mathbb{F}_{101}$	1^4	(4, 0)	(4, 0)
ghost quartet, $\delta = 0.15$	1.3, 0.7, 1.3, 0.7	(2, 2)	(4, 0)
ghost replaces a pair in genus 2	1.3, 0.7, 1.3, 0.7, 1, 1	(4, 2)	(6, 0)
E_4 at $p = 2$ ($\mu = 2^{\pm 3/2}$)	4, -2	(1, 1)	(2, 0)
E_{12} at $p = 2$	12, -10	(1, 1)	(2, 0)
$\Delta_{12} \oplus E_{12}$ at $p = 2$	1, 1, 12, -10	(3, 1)	(4, 0)

Proposition 2.3 (ζ on V_0 ; [NUMERICAL]). *On the L96 bump spans, using the L91 form I with pole vectors projected out (V_0), with ghost quartets $14.13 \pm 0.1i$ and $30 \pm 0.2i$:*

R , ghost	true	fake, duality ($I - \frac{1}{2}G$)	fake, graded ($I - \frac{1}{2}P_g$)	max eig. (duality)
0.4, $14.13 \pm 0.1i$	(0, 17)	(2, 15)	(0, 17)	+0.103
0.4, $30 \pm 0.2i$	(0, 17)	(2, 15)	(0, 17)	$+6.7 \cdot 10^{-4}$
0.2, $14.13 \pm 0.1i$	(0, 36)	(0, 36)	(0, 36)	$-4.5 \cdot 10^{-4}$
0.2, $30 \pm 0.2i$	(0, 36)	(2, 34)	(0, 36)	$+8.3 \cdot 10^{-3}$

The graded polarization $P_g = \sum_r F(r)\overline{F(r)}$ never detects a ghost; the duality form does when the resolution permits (Fig. 2).

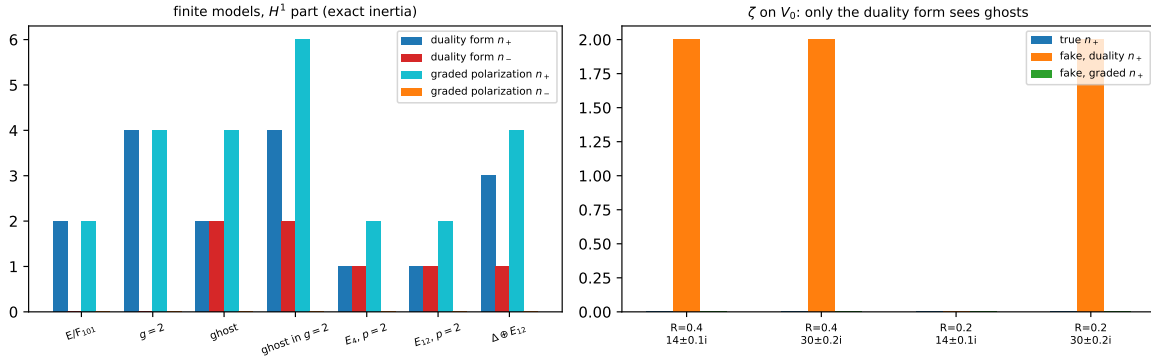


FIGURE 2. Left: duality form versus graded polarization on finite models. Right: ζ on V_0 ; only the duality form sees ghosts.

Remark 2.4 (What the “ H^1 piece” must be for ζ ; [RESTATEMENT]/[OPEN]). The available non-circular grading is cohomological *degree*: pole terms ($H^0 \oplus H^2$, the (1, 1) plane) versus zero terms. Over $\mathbb{Q}$ this is V_0 (L91/L96), and on V_0 the Weil form is negative semidefinite $\iff$ RH. “Weight-graded positivity” thus reduces to “degree = weight on H^1 ”, which is purity, which is RH. Over $\mathbb{F}_q$ the missing ingredient is supplied by geometry, not by the spectrum: H^1 of the curve, the cup product to $H^2 \cong \mathbb{L}$, and Hodge index on correspondences. For ζ it is Deninger’s conjectural H^1 with a Hodge $*$ (§5).

3. (B) MASLOV DEQUANTIZATION OF WEIL’S FORM

Take Gaussian test functions ψ_j centred at $x_j = j/2$ ($j = 0, \dots, 7$), with $F_j(r) = e^{irx_j} e^{-s^2 r^2/2}$, and project to V_0 (annihilator of $F(\pm i/2)$; $\dim V_0 = 6$). The zero form is $Q = -\frac{1}{2} \sum_r F_j(r) F_k(-r)$, using 200 zeros and, for the fake form, an added quartet. Let $\hbar = 1/s^2$.

Theorem 3.1 (Tropical limit; [THEOREM] (valuations), [NUMERICAL]). *Each zero $r = \gamma + i\delta$ contributes entries of modulus $e^{-\text{Re}(r^2)/\hbar} \times O(1)$, so its tropical valuation is $-(\gamma^2 - \delta^2)$. Consequently: (a) For a fixed finite test space, only the top valuations survive as $\hbar \rightarrow 0$, i.e. only finitely many lowest zeros. The limit cannot encode RH. (b) The unsigned valuation of a quartet $\pm\gamma \pm i\delta$ equals that of an on-line zero at $\gamma' = \sqrt{\gamma^2 - \delta^2}$. (c) Signs survive: the quartet contributes two positive eigenvalues at that valuation. In Viro’s signed tropical hyperfield the ghost is visible; in $\mathbb{R}_{\max}$ it is not.*

[NUMERICAL] (dequant.json):

- Ghost $20 \pm 0.3i$, which lies between γ_1 and γ_2 : $n_+ = 2$ for $s \geq 0.25$, with top positive valuation $\hbar \log \lambda_+ = -391.7, -390.5, -399.92, -399.89$ at $s = 0.35, 0.5, 0.7, 1.0$ (predicted -399.91).
- Low ghost $10 \pm 0.3i$: $\rightarrow -99.7$ (predicted -99.91).
- Ghost $40 \pm 0.3i$: $n_+ = 0$ at every s , masked by the three lowest zero levels in the 6-dimensional V_0 .
- True form: $n_+ = 0$ and top valuation $\rightarrow -198.4 \approx -\gamma_1^2 = -199.8$.
- At $s = 1.4$ the fake spectrum is $\{-199.1^-, -199.6^-, -399.4^-, -399.6^+, -399.6^+, -399.9^-\}$, while an on-line double pair at γ' gives $\{\dots, -398.8^-, -399.1^-, -441.4^-, -441.5^-\}$ (Fig. 3).

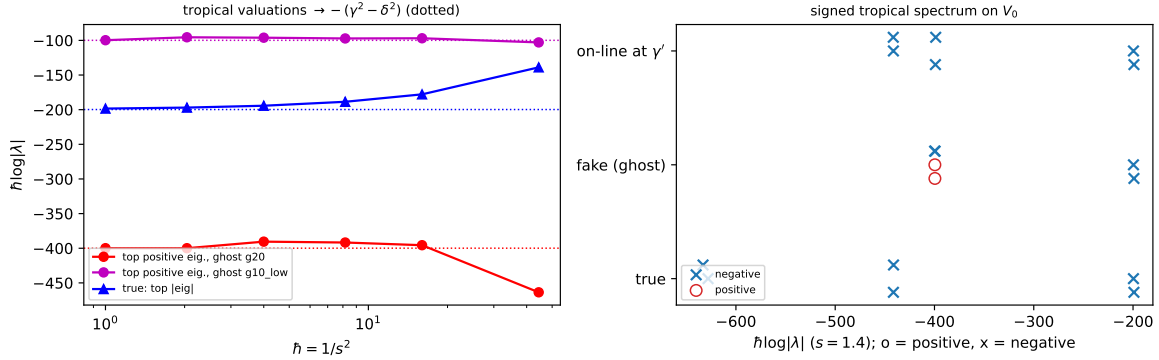


FIGURE 3. Left: tropical valuations of top eigenvalues versus $\hbar$; dotted lines are $-(\gamma^2 - \delta^2)$. Right: signed tropical spectrum at $s = 1.4$. The ghost is the pair of positive (red) eigenvalues.

Remark 3.2 (The shadow analogy fails; [THEOREM]/[HEURISTIC]). The shadow $3/(\pi y) = 6\hbar$ is an *additive* $O(\hbar)$ correction, so it disappears. A ghost changes *signs*, which are invariant under positive rescaling and survive any $\hbar$, and it shifts a valuation by $+\delta^2$, which is $O(1)$. What dequantization destroys is not the ghost but the *high-lying spectrum*. Max-plus has no subtraction, so “indefinite” only makes sense in a signed or hyperfield setting. There, “all signs negative on V_0 for every finite test space” is again exactly Weil’s criterion: nothing is simplified.

Remark 3.3 (Connes–Consani, tropical curves; [CITED]/[BIB], no claims). Connes–Consani develop “characteristic one” geometry, linking it to idempotent analysis and tropical geometry [2] [CITED] (abstract), and the arithmetic site over $\mathbb{Z}_{\max}$ [3] (L96). Tropical curves give enumerative geometry over $\mathbb{R}_{\max}$ [15] [BIB]. We make no claim that these frameworks contain a positivity statement equivalent to RH. [OPEN].

4. (C) THE $\mathbf{1} \oplus \mathbb{L}$ TABLE

Object	motive (weights)	L / zeta	positivity
$\mathbb{P}^1/\mathbb{Z}$	$\mathbf{1} \oplus \mathbb{L} (0, 2)$	$\zeta(s)\zeta(s-1)$; $\# = 1 + p = \sigma_1(p)$	each piece rank 1; no H^1
E_2 (level $N > 1$: $E_2 - NE_2(N\tau)$)	$\mathbf{1} \oplus \mathbb{L}$ at L -level	$\zeta(s)\zeta(s-1)$ up to factors at $p \mid N$	mixed: $(1, 1)$ in duality form
E_k	$\mathbf{1} \oplus \mathbb{L}^{k-1} (0, 2k-2)$	$\zeta(s)\zeta(s-k+1)$; Satake $(1, p^{k-1})$	mixed: $(1, 1)$; graded $(2, 0)$
Δ_{12}	pure, weight 11, rank 2 in H^{11} of Kuga–Sato	$L(\Delta_{12}, s)$; $ \alpha_p = p^{11/2}$	Ramanujan (Deligne [4, 5] [BIB]/L95)
$C/\mathbb{F}_q$	$\mathbf{1} \oplus h^1 \oplus \mathbb{L}$	$P(T)/((1-T)(1-qT))$	Hodge index $\Rightarrow$ RH (Weil)
ζ	$“\mathbf{1} \oplus ? \oplus \mathbb{L}”$	$\hat{\zeta} = \det(H^1)/\det(H^0)\det(H^2)$ (Deninger, conj.)	missing H^1 and its polarization

([RESTATEMENT]); the E_k “motive” is meant at the level of L -function and Galois data.)

5. WHAT THE MELLIN TRANSFORM CANNOT SUPPLY

Proposition 5.1 (Curves; [RESTATEMENT] [5, 10] [BIB]). *For a smooth projective curve $C/\mathbb{F}_q$ of genus g , the motive decomposes as $h(C) = \mathbf{1} \oplus h^1(C) \oplus \mathbb{L}$. Here $\mathbf{1} = h^0$, $\mathbb{L} = h^2$ is the Lefschetz motive (Frobenius acts by q), and h^1 has rank $2g$. Correspondingly*

$$Z(C, T) = \frac{P(T)}{(1-T)(1-qT)}, \quad P(T) = \det(1 - TF \mid H^1) = \prod_{i=1}^{2g} (1 - \alpha_i T).$$

Theorem 5.2 (Deninger’s formalism; [7] §3 [CITED]). (a) *Proved: for every place $p \leq \infty$, $\zeta_p(s) = \det_\infty(\frac{1}{2\pi}(s-\Theta) \mid \mathbb{R}_p)^{-1}$ ([7] Prop. 3.1). Here $\mathbb{R}_p$ is the space of real finite Fourier series on $\mathbb{R}/(\log p)\mathbb{Z}$ for $p < \infty$, $\mathbb{R}_\infty = \mathbb{R}[e^{-2y}]$, $\Theta = d/dy$, and $\zeta_\infty(s) = 2^{-1/2}\pi^{-s/2}\Gamma(s/2)$. (b) Conjectural ([7] (3)): $\hat{\zeta}(s) = \prod_{i=0}^2 \det_\infty(\frac{1}{2\pi}(s-\Theta) \mid H^i(\overline{\text{“Spec } \mathbb{Z}”}, \mathbb{R}))^{(-1)^{i+1}}$. That is, $\hat{\zeta}$ is the determinant on H^1 divided by the determinants on H^0 ($\Theta = 0$) and on H^2 ($\Theta = 1$). H^1 is infinite dimensional, with the spectrum of Θ equal to the nontrivial zeros. (c) Proved consequence ([7], citing Deninger and Schröter–Soulé): $\xi(s) := \frac{s}{2\pi} \frac{s-1}{2\pi} \hat{\zeta}(s) = \prod_\rho \frac{s-\rho}{2\pi}$ as a zeta-regularized product. (d) Heuristic ([7] §3): if there were a Hodge $*$ on H^1 with $\langle f, f' \rangle = \text{tr}(f \cup *f')$ positive definite, commuting with Θ , and if Θ were a derivation for $\cup$, then RH would follow.*

[NUMERICAL] (num/lemma97_mellin.py): the logarithmic derivative of (c), $\xi'/\xi(s) = \sum_\rho \frac{1}{s-\rho}$ (symmetric sum), agrees to $\leq 4 \cdot 10^{-6}$ at $s = 2, 3, \frac{1}{2} + 7i, 2 + 30i$ using 2500 zeros plus the smooth tail. This checks (c) only up to a multiplicative constant.

Proposition 5.3 ($\oplus$ versus $\otimes$; [THEOREM]/[RESTATEMENT]). (a) $L(h^1 \oplus h^1, s) = L(h^1, s)^2$. The zeros are the same, with doubled multiplicity, and the Weil form is doubled. Nothing new: positivity for $h^1 \oplus h^1$ is equivalent to positivity for h^1 . The same holds for every Mellin-type construction that produces products $\prod \zeta(s - a_j)$, e.g. $\sum \sigma_{k-1}(n)n^{-s} = \zeta(s)\zeta(s-k+1)$ (L96 Prop. eis [TAA-L]). (b) $h^1 \otimes h^1 \subset h^2(C \times C)$ (Künneth) carries the cup product $H^1 \otimes H^1 \rightarrow H^2 \cong \mathbb{L}$. Correspondences (in particular Γ_F) give classes in it. The Rosati involution is the adjoint for this pairing, and Weil’s positivity $\text{Tr}(\varphi\varphi^\dagger) > 0$, equivalently Hodge index on $C \times C$, is a statement about $h^1 \otimes h^1$, not about $L(h^1, s)$ alone. Frobenius acts on $h^1 \otimes h^1$ with eigenvalues $\alpha_i\alpha_j$.

Proposition 5.4 (Why Rankin–Selberg helps for automorphic L but not for ζ ; [THEOREM] + [11] [BIB]). *For a unitary cuspidal π on GL_n with Satake parameters $\{\alpha_{i,p}\}$, $L(\pi \times \bar{\pi}, s)$ has local factors $\prod_{i,j} (1 - \alpha_{i,p}\bar{\alpha}_{j,p}p^{-s})^{-1}$. Its logarithm has coefficients $|\sum_i \alpha_{i,p}^k|^2/k \geq 0$. Absolute convergence of*

$L(\pi \times \tilde{\pi}, s)$ for $\operatorname{Re} s > 1$ [11] forces $|\alpha_{i,p}| < p^{1/2}$ (elementary: the diagonal terms $|\alpha_{i,p}|^{2k} p^{-ks}$ must decay), which is a bound toward Ramanujan. ([NUMERICAL]: for Δ_{12} at $p = 2, 3, 5, 7$ the normalized coefficients $|\alpha^k + \beta^k|^2$, $k \leq 4$, lie in $[0.10, 3.46]$.) For ζ (GL_1 , trivial character), the local Satake set is $\{1\}$ and $1 \cdot \bar{1} = 1$, so $L(\mathbf{1} \times \mathbf{1}, s) = \zeta(s)$. Rankin–Selberg returns ζ itself and gives no new inequality.

Remark 5.5 (Local factors versus global zeros; [THEOREM]/[HEURISTIC]). For $p < \infty$, $\zeta_p(s) = (1 - p^{-s})^{-1}$ has no zeros. Motivically it is the pure weight-0 piece $h^0(\operatorname{Spec} \mathbb{F}_p)$, with one Frobenius eigenvalue 1. The archimedean factor $\Gamma(s/2)$ has no zeros either. Every zero of ζ is therefore a global phenomenon of analytic continuation. In Deninger’s Prop. 3.1 every local factor appears as an *inverse* determinant, i.e. in the position of H^0/H^2 , and only the conjectural global H^1 carries zeros.

Corrections to the t419 chat claims.

C97.m1.:

“Deninger’s conjectural $\xi(s)$ as a ratio of regularized determinants on H^1 over $H^0 \oplus H^2$ ”: essentially right, with two refinements. The conjecture ([7] (3)) is stated for $\hat{\zeta}(s)$ (with $\zeta_\infty = 2^{-1/2} \pi^{-s/2} \Gamma(s/2)$), and it is an alternating product: H^1 in the numerator, H^0 and H^2 each in the denominator. The formula for $\xi(s) = \frac{s(s-1)}{4\pi^2} \hat{\zeta}(s)$ as the regularized product over zeros is a *theorem*, not conjectural. What is conjectural is the cohomology H^i with its flow.

C97.m2.:

“ ζ ’s local factors are 1-dimensional”: true motivically for $p < \infty$ (one eigenvalue 1, rank 1). But in Deninger’s realization $\mathbb{R}_p$ is *infinite*-dimensional: its Θ -spectrum is $2\pi i k / \log p$, $k \in \mathbb{Z}$, the poles of ζ_p . At $p = \infty$ the factor $\Gamma(s/2)$ has infinitely many poles $s = -2\nu$. “Pure H^0 ” is accurate in the sense that local factors sit in the inverse-determinant position.

C97.m3.:

“ $h^1 \otimes h^1$ with cup product to $\mathbb{L}$ (Weil/Rosati)”: correct. Precisely, positivity is used on the algebraic (correspondence) part of $h^1 \otimes h^1(1)$, where it is Hodge index on $C \times C$, i.e. Rosati positivity.

C97.m4.:

“Rankin–Selberg works because local Satake pairs exist”: correct in spirit. The mechanism is nonnegativity of $\log L(\pi \times \tilde{\pi})$ plus the pole at $s = 1$. For GL_1 it degenerates to ζ itself, which is the precise sense of “nothing to pair”.

6. CORRECTIONS TO THE CHAT CLAIMS (T414–T417)

C97.1 ((a), constant and attribution).:

$E_2^* = E_2 - \frac{3}{\pi y}$ is correct. $\xi_2 E_2^* = 3/\pi$ (BF normalization), and E_2^* is harmonic, in H_2 but not H_2^+ . The coefficient is $3/\pi = 1/\operatorname{vol}$ (Stokes), which equals $\operatorname{Res}_{s=1} E(z, s) = \frac{\pi/2}{\zeta(2)}$. The attribution to “ ζ ’s pole” is right, provided the factor $\zeta(2)$ (the volume) is kept.

C97.2 ((b)).:

Wrong as stated. $\vartheta - \frac{k}{12} E_2^*$ is not covariant ([NUMERICAL]). The raising operator is $\partial_k = \vartheta - \frac{k}{4\pi y} = \vartheta_k + \frac{k}{12} E_2^* = -\frac{1}{4\pi} R_k$.

C97.3 ((c)).:

Correct, as an analogy. Δ_{12} ’s product runs over all n , and its log-derivative has σ_1 -coefficients, not Λ .

C97.4 ((d)).:

The dequantization limit is correct, and the shadow is exactly $6\hbar$. The Lion–Vergne attribution for Rademacher–Maslov is *not verified*. The verified statement is Thm. 1.7: coboundary of $\Phi = -3 \times$ Kashiwara–Maslov index (Kirby–Melvin, Sczech).

C97.5 ((e)).:

Correct. Refinement: the identification $E_k \leftrightarrow \mathbf{1} \oplus \mathbb{L}^{k-1}$ is at the level of L -functions and Satake data. In the duality form this mixed sum is indefinite, $(1, 1)$, like a ghost.

C97.6 (direction 1 itself).:

“Weight-graded positivity” as stated is circular (Thm. 2.2). The non-circular version is degree-graded, and there it is RH.

7. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	An intrinsic degree grading over $\mathbb{Q}$: an H^1 with a pairing (Deninger) whose polarization is defined without the spectrum.	0.9	0.05	0.56
2	Signed-tropical Weil criterion: certify the sign pattern on V_0 for finite test spaces, which amounts to finite certificates for low zeros (connects to L95 D2).	0.4	0.7	0.52
3	Weight filtration as extension data: treat the pole plane and H^1 as a mixed object, and study the Eisenstein/ ζ extension class.	0.6	0.3	0.48
4	A Bruinier–Funke-type pairing $\{g, f\} = (g, \xi f)$ via Stokes as a model for a positivity-producing pairing.	0.5	0.3	0.42
5	The odd part: Maslov/Rademacher as a signature cocycle (L96 D2) in the explicit-formula setting.	0.3	0.55	0.40

8. QUESTIONS FOR J.W.

- Q1. Do you accept Thm. 2.2(4), that weight grading presupposes $\text{Re } \rho$? If so, should direction 1 be renamed “degree-graded positivity”?
- Q2. In t419 you wrote “ $h^1 + h^1$ ”. Did you mean $h^1 \oplus h^1$ (L^2 , nothing new) or $h^1 \otimes h^1$ with the cup product (Weil)? See Prop. 5.3.
- Q3. For dequantization: is a *signed* tropical setting acceptable? Without signs, positivity cannot even be stated.
- Q4. Where did you see Lion–Vergne connected to Rademacher’s Φ ? Please send the page if you have it. Otherwise we cite Kirby–Melvin/Sczech.
- Q5. L98: signed-tropical certificates (direction 2), or the Eisenstein/ ζ extension class (direction 3)? Direction 1 has the highest score but very low probability.

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TAA LEMMA 98: SUMS OF SQUARES, RAPIDITY, AND THE ONE BOOST

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis. Every claim carries one label: [THEOREM] (proved here, or classical with the proof indicated), [RESTATEMENT] (reformulation), [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (computed; scripts `num/lemma98_*.py`, data `num/198/`). Sources: [CITED] (opened), [CITED-sec] (secondary/excerpt), [BIB] (bibliographic record only), [TAA-L] (earlier TAA lemma). References: `lit/lemma98/verified_refs.md`.

ABSTRACT. J.W.'s proposal (t420–t421): a sum-of-squares (SOS) certificate for Weil positivity, read through a hyperbolic picture in which “the entire system is hyperbolic orthogonal” and scaling is a geodesic translation measured by a rapidity. Section (D) (t422) adds the algebraic side: rapidity versus angle, positive involutions, polar decomposition.

Results. (1) *The SOS that exists is the zero side* (§2, §3). On spline spans of support $a \leq 12$ the prime+ $\Gamma_{\mathbb{R}}$ Gram matrix Q on V_0 is positive definite and Cholesky succeeds at every a ([NUMERICAL]), but $\lambda_{\min}$ falls from 1.7 to $5 \cdot 10^{-9}$ and the condition number reaches $6 \cdot 10^9$. Fitting the prime-side Gram by a nonnegative combination of zero-free frequency atoms (NNLS) *recovers the zeta zeros*: 14.135, 21.022, 25.011, 30.425, 32.935, 37.586, 40.918, 43.326, each with mass 2.000 ± 0.015 ([NUMERICAL], Fig. 2). By Bochner uniqueness this is forced (Prop. 3.1): any a -uniform SOS by frequency atoms *is* the zero side. J.W.'s “the content is uniformity in a ” is right but incomplete (C98.12). (2) *Levinson rapidities* (Thm. 3.2). The Toeplitz zero-side form is positive definite up to order k iff all reflection coefficients satisfy $|\kappa_j| < 1$, and each Levinson step is a $(1, 1)$ Lorentz boost of rapidity $\operatorname{artanh} \kappa_j$. True data: $|\kappa_j| < 1$ for all 60 orders, with closest approach $\kappa_8 = -0.986$. A ghost $14.13 \pm 0.1i$ forces $|\kappa_8| = 1.0009$, so the SOS fails at $a = 1.6$. One ghost quartet costs exactly ≤ 2 negative squares. First failures on V_0 : $a = 1.8, 3.2, 5.8, 6.0$ for ghosts at $\gamma = 14.1, 30, 50, 85.7$. (3) *One boost* (Thm. 4.1, Thm. 4.2). An element of $O(1, n)$ has at most one eigenvalue of modulus > 1 . Over $\mathbb{F}_q$, Hodge index makes $\operatorname{NS}(E \times E) \otimes \mathbb{R}$ Lorentzian, and the normalized Frobenius pullback is an isometry whose single boost is spent on the H – V plane, with rapidity $\frac{1}{2} \log q$. Hence $|\alpha| = \sqrt{q}$. Numerically: $q = 101$, true a gives one boost and $(1, 3)$; fake $a = 21, 25, 40$ give two boosts and $(2, 2)$. For ζ : translation $x \mapsto x + s$ ($\lambda = e^s$) is an isometry of the Weil form and a boost of rapidity $\frac{1}{2} \log \lambda$ on the pole plane (fit slope 0.4975). The form is Lorentzian with true zeros ($n_+ = 1$ up to $a = 8$) and has $n_+ = 3$ with one ghost quartet. This restates Weil's criterion, [RESTATEMENT]. (4) *Rapidity versus angle* (§5). The norm forms $x^2 + y^2$ vs $x^2 - y^2$, units as rapidities, closed geodesics of length $2 \log \varepsilon_d$ (8/8 checked). A positive involution is the same as a compact unitary group. Rosati trace form on $\mathbb{Q}(F)$: $(2, 0)$ and $(4, 0)$ for true curves, $(1, 1)$ and $(3, 1)$ for ghosts. $FF^\dagger = q$ holds for ghosts too, so the content of J.W.'s polar decomposition is the positivity of $\dagger$, not the identity $|F| = \sqrt{q}$ (C98.16). Over $\mathbb{Q}$ the target “algebra with positive involution whose scaling flow is the single boost” is a restatement of Weil positivity. No candidate algebra is known. (5) *Trig/hyperbolic systems and Yoneda* (§6, t423). Circle $= R_{\mathbb{Q}(i)/\mathbb{Q}}^1 \mathbb{G}_m$ and hyperbola $= \mathbb{G}_m$ are representable tori. Over $\operatorname{Spec} \mathbb{Z}$ the type changes from place to place: split iff $p \equiv 1 \pmod{4}$, 16/16 primes. The structure “one boost $\times$ compact” already exists, $C_{\mathbb{Q}} \cong \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^\times$ (Connes), and has no RH content. Positivity is free only on the compact part. For the explicit candidate functor $\mathfrak{F}$, non-emptiness implies RH (Prop. 6.1). Yoneda gives uniqueness, not existence. Rosati, SOS and Hodge index are linked by identities over $\mathbb{F}_q$, where Hodge index is proved first; over $\mathbb{Q}$ they are one inequality (C98.17–19).

Date: 24 October 2026 (approved t421u; owner pt421u “the entire system is hyperbolic orthogonal”; section (D) per t422, (E) per t423; chat t420–t423).

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GLOSSARY (BRIEF RESTATEMENTS FOR J.W.)

Positive semidefinite (PSD):

A real symmetric matrix M with $c^T M c \geq 0$ for all c ; positive definite (PD) if > 0 for $c \neq 0$.

Gram matrix:

$M_{ij} = B(f_i, f_j)$ for a bilinear form B and vectors f_i . B is PSD on their span iff M is PSD.

Cholesky; SOS:

A PD matrix factors as $M = LL^T$ with L lower triangular. Then $c^T M c = \sum_k (\sum_i L_{ik} c_i)^2$ is a *sum of squares* of linear forms. Any PSD matrix has such a factorization (eigen or Cholesky), so on a finite span “SOS” says nothing beyond “PSD”.

Signature (inertia):

(n_+, n_-, n_0) : the numbers of positive, negative and zero eigenvalues. These are invariant under congruence $M \mapsto P^T M P$ (Sylvester).

Lorentzian:

Signature $(1, n)$: one positive (“time-like”) direction and a negative-definite complement. Every vector then splits as $D = \frac{D \cdot H}{H^2} H + D'$ with $D'^2 \leq 0$, so $D^2 = (D \cdot H)^2 / H^2 - (\text{SOS})$: *one square minus a sum of squares*.

Hyperboloid model:

For a Lorentzian form, $\{D : D^2 = 1, D \cdot H > 0\}$ with the induced (negated) metric is hyperbolic n -space. Distance: $\cosh d(D, E) = D \cdot E$ [3] [CITED] (their Lemma in §10, with sign convention $a * b = -\cosh d$).

Rapidity:

The parameter η of a Lorentz boost $\begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix}$. On null coordinates it acts by $e^{\pm \eta}$. It equals the hyperbolic distance moved along the boost axis.

Geodesic:

A locally length-minimizing curve. In the upper half-plane $\mathbb{H}$: vertical lines and semicircles orthogonal to $\mathbb{R}$. In the hyperboloid model: intersections with planes through 0.

Product spaces:

For $C \times C$ we say “the first copy of C ”, not “the first factor”.

1. THE CHAT CLAIMS T420–T421

All symbolic checks are in `num/lemma98_claims.py` (198/claims.json).

- (a) $(2|\rho|)^2 = 1 + (2\gamma)^2$ for $\rho = \frac{1}{2} + i\gamma$: correct (sympy, [THEOREM]). It is the norm form $x^2 + y^2$ of $\mathbb{Q}(i)$ evaluated at $2\rho = 1 + 2i\gamma$, i.e. an *elliptic* (angle) quantity.

- (b) $\rho(1-\rho) = \frac{1}{4} + t^2$ with $\rho = \frac{1}{2} + it$: correct. For a ghost $\rho = \frac{1}{2} + \delta + i\gamma$ one has $t = \gamma - i\delta$ (checked), and $\rho(1-\rho) = \frac{1}{4} + \gamma^2 - \delta^2 - 2i\gamma\delta$. The sign is correct. For the mirror ghost $\frac{1}{2} - \delta + i\gamma$ the imaginary part flips to $+2i\gamma\delta$. Note $\frac{1}{4} + t^2$ is real exactly when $t \in \mathbb{R} \cup i\mathbb{R}$, so “ $\rho(1-\rho)$ real and $> \frac{1}{4}$ ” $\iff \rho$ on the line [THEOREM].
- (c) Li: $\lambda_n = \sum_{\rho} [1 - (1 - 1/\rho)^n]$ [10, 1] [CITED]. $|1 - 1/\rho|^2 - 1 = (1 - 2\sigma)/|\rho|^2$, so $|1 - 1/\rho| = 1 \iff \sigma = \frac{1}{2}$ [THEOREM]. The Cayley-type map $w = 1 - 1/\rho$ sends the critical line to the unit circle (angle). A ghost at distance δ sits at $\log|w| \approx \delta/\gamma^2$ (a *rapidity*). [NUMERICAL]: 2500 zeros plus a smooth tail give $\lambda_1 = 0.0230957$, against the exact $1 + \frac{\gamma_E}{2} - \frac{1}{2} \log 4\pi = 0.0230957$ (agreement $2 \cdot 10^{-8}$). Injected quartets first make $\lambda_n < 0$ at $n = 4442$ for $(\gamma, \delta) = (14.13, 0.40)$ and at $n = 3907$ for $(14.13, 0.45)$. For $\gamma \geq 21$ nothing turns negative up to $n = 9000$ (Fig. 5). Detection needs $n \gtrsim (\gamma^2/\delta) \log n$ [HEURISTIC].
- (d) Hasse: $\alpha = \frac{a}{2} + i\sqrt{q - a^2/4}$ gives $|\alpha|^2 = (a/2)^2 + b^2 = q$ for $|a| \leq 2\sqrt{q}$: correct, [THEOREM].
- (e) Hodge index as $D^2 = (D \cdot H)^2/H^2 - \text{SOS}$: correct when $H^2 > 0$, with $\text{SOS} = -D'^2$ written in an orthonormal basis of $H^\perp$ (residual $9 \cdot 10^{-16}$ on a random $(1, 4)$ lattice, [NUMERICAL]). This is the Lorentzian normal form.
- (f) Reverse Cauchy–Schwarz $(D \cdot E)^2 \geq D^2 E^2$ for time-like D, E : correct in signature $(1, n)$ [THEOREM]. It is false in signature $(2, n)$: two orthogonal time-like vectors exist (§4).
- (g) $z \mapsto \lambda z$ on $\mathbb{H}$ translates the imaginary axis by $\log \lambda$: correct ($d(i, \lambda i) - \log \lambda < 5 \cdot 10^{-16}$ for $\lambda = 1.5, 2, 10, 101$).
- (h) Hyperboloid model of a Lorentzian lattice, with $\cosh d = D \cdot E$: correct [3] [CITED]. Signature $(2, \cdot)$ has no hyperboloid model: $\{D^2 = 1\}$ then carries an indefinite induced metric and contains circles ([NUMERICAL]: $\cos \theta e_1 + \sin \theta e_2$ has norm 1.000000 for all θ). A ghost’s $(2, 2)$ block destroys the model: correct (§4).
- (i) Closed geodesics on $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ have lengths $2 \log \varepsilon_d$ [14, 12] [CITED], not $\log p$. Connes’s heuristic flow has periodic orbits of period $\log p$ with instability exponents $\pm \log p$, and Connes notes it *cannot* be a geodesic flow (time-reversal symmetry would double the orbit count) [5] [CITED]. “ $\log p$ as geodesic lengths” is therefore an [ANALOGY] (C98.11).
- (j) “Cholesky/SOS is trivial on any finite PSD matrix; the content is uniformity in a ”: essentially right; see C98.12 for three refinements.

2. (A) THE SOS FORMULATION

Set-up [TAA-L] (L91/L92/L94). For bumps $\psi_x(t) = M_m((t-x)/h)/h$ ($m = 4$, $R = mh = 0.4$) at centres x_j , Weil’s form is $I(\psi_x, \psi_y) = N(|x-y|)$ (primes + archimedean). We verified

$$I = P_t - \frac{1}{2}Z, \quad P_t = \text{sym}(F(\frac{i}{2})F(-\frac{i}{2})^T), \quad Z = \sum_{\rho} F(r_{\rho})F(-r_{\rho})^T$$

to $1.1 \cdot 10^{-12}$ against 2500 zeros ([NUMERICAL]). Here P_t has rank 2 and signature $(1, 1)$ (the pole plane), and Z is the zero side. On $V_0 = \ker F(\pm \frac{i}{2})$ we set $Q = Z|_{V_0} = -2I|_{V_0}$. Weil’s criterion [17, 2, 18]: $\text{RH} \iff Q \geq 0$ on V_0 for all test functions [CITED-sec].

Proposition 2.1 (What an SOS would and would not give). (i) [THEOREM] For each finite span, $Q \succ 0$ is equivalent to a factorization $Q = \sum_k \langle \cdot, \psi_k \rangle^2$ with ψ_k computed from primes and $\Gamma_{\mathbb{R}}$ only (Cholesky of the prime-side Gram). This is trivial once $Q \succ 0$ is known. (ii) [RESTATEMENT] A family $\{\psi_k(a)\}$ with $Q_a = \sum_k \langle \cdot, \psi_k(a) \rangle^2$ on $V_0(a)$ for all a (and all refinements) is equivalent to RH, by Weil/Yoshida. (iii) [THEOREM] Under RH such a family exists canonically: $\psi_{\gamma} = (F_{x_j}(\gamma))_j$, the zero side. So the question is not whether an SOS exists but whether there is one whose vectors are not the zeros. Prop. 3.1 shows that within frequency atoms there is none.

Proof. (i) is linear algebra. (iii): for real γ and real ψ , $F(-\gamma) = \overline{F(\gamma)}$, so $Z = \sum_{\gamma} \text{Re } F(\gamma)F(\gamma)^* \succeq 0$. $\square$

Prior attempts (verified only). *de Branges*. Conrey–Li [6] [CITED] show that the positivity condition required by de Branges’s approach fails for ζ . The value $-\operatorname{Re}\{\xi'(\rho)\xi(1+\rho)\} = -5.389100507182945 \cdot 10^{-69} < 0$ at the 34th zero is reproduced here to all 16 digits ([NUMERICAL], mpmath). So is their W -space value $\operatorname{Re}\xi(1+282i)/\xi(2+282i) = -0.000131957293$. *Yoshida* [18] [CITED-sec] (abstract): RH $\iff$ positive definiteness on $C(a)$ for all a , and positivity holds for small a . *Connes–Consani* [4] [CITED] (opened in L92): QW_λ is positive under RH, Yoshida’s positivity at $\lambda = \sqrt{2}$ is numerical, and the archimedean part alone loses positivity just beyond $L = \log 2$ while the prime 2 restores it. L92 [TAA-L]: certified PD on splines for $a \leq 8$ ($n = 157$). Burnol’s work is not cited (not verified this session).

3. (B) NUMERICAL SOS SEARCH

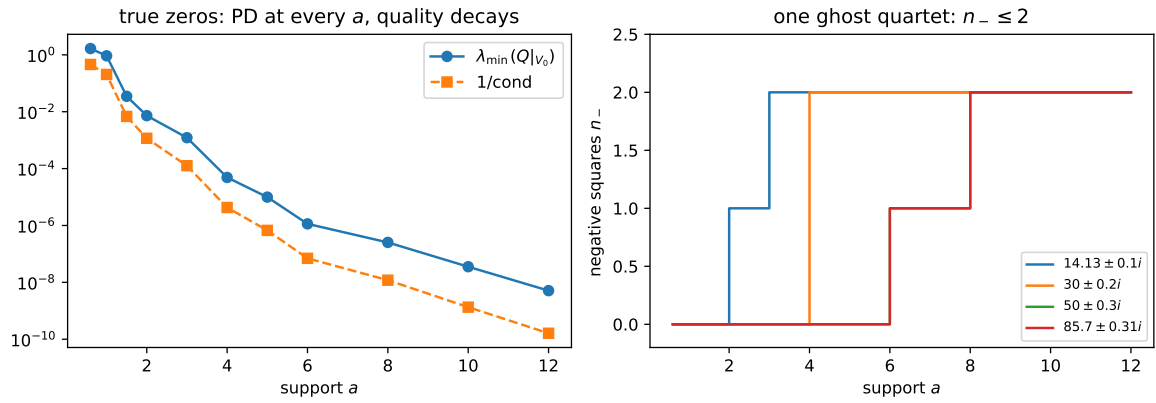


FIGURE 1. Left: $\lambda_{\min}(Q|_{V_0})$ and $1/\text{cond}$ vs support a (true zeros; PD throughout). Right: negative squares after injecting one ghost quartet.

B1. Cholesky/eigen vs a [NUMERICAL] (lattice $x_j = 0.2j$, $R = 0.4$).

a	$\dim V_0$	$\lambda_{\min}(Q)$	cond	prime-only on V_0	arch-only on V_0
0.6	2	1.66	2.2	(1, 1)	(2, 0)
1.0	4	$9.2 \cdot 10^{-1}$	4.9	(2, 2)	(4, 0)
2.0	9	$7.3 \cdot 10^{-3}$	$8.6 \cdot 10^2$	(5, 4)	(7, 2)
4.0	19	$4.9 \cdot 10^{-5}$	$2.3 \cdot 10^5$	(11, 8)	(14, 5)
8.0	39	$2.5 \cdot 10^{-7}$	$8.4 \cdot 10^7$	(18, 21)	(27, 12)
12.0	59	$5.1 \cdot 10^{-9}$	$6.2 \cdot 10^9$	(25, 34)	(40, 19)

Cholesky succeeds at every a . The decay of $\lambda_{\min}$ mostly reflects the bump spectrum ($\hat{\psi} \sim \text{sinc}^4$), not proximity to failure. Neither the prime part nor the archimedean part is PSD by itself (the arch part is PSD only for $a \leq 1$ in this family), so no *termwise* SOS exists (prime atoms enter Q with both signs). In float64, certification degrades past $a \approx 12$ (cond $\sim 10^{10}$). Interval certification (L92) is required beyond that.

B2. Bochner: the SOS is the zero side.

Proposition 3.1. [THEOREM] (classical; Bochner–Schwartz uniqueness). If $Z(f, f) = \int |\hat{f}(\tau)|^2 d\mu(\tau)$ for all test f with a positive measure μ , then μ is unique. By the explicit formula, under RH it is $\sum_\gamma \delta_\gamma$, and it is not positive if there is a ghost. [RESTATEMENT]: any a -uniform SOS by frequency atoms $\langle \cdot, e^{i\tau x} \rangle$ with nonnegative weights converges to the zero side.

[NUMERICAL] (Fig. 2). We used 61 jittered centres on $[0, 8]$; a lattice aliases τ modulo $2\pi/\Delta x$, and that was our first, failed attempt. The dictionary was $\tau \in [0, 120]$ with step 0.01, and the only input was the prime-side Gram. NNLS reaches relative residual $1.5 \cdot 10^{-4}$. Its clusters sit at 14.135, 21.022, 25.011, 30.425, 32.935, 37.586, 40.918, 43.326 (errors $< 10^{-3}$) with masses 2.000, 2.000, 2.000, 2.000, 2.000, 1.998, 1.999, 1.985 (each $\pm\gamma$ pair counts 2). Above $\gamma \approx 50$ the resolution set by $a = 8$ and the bump is exhausted. With the ghost $14.13 \pm 0.1i$ added, no nonnegative fit exists below residual $4.3 \cdot 10^{-2}$ ($1.8 \cdot 10^{-2}$ for $30 \pm 0.2i$).

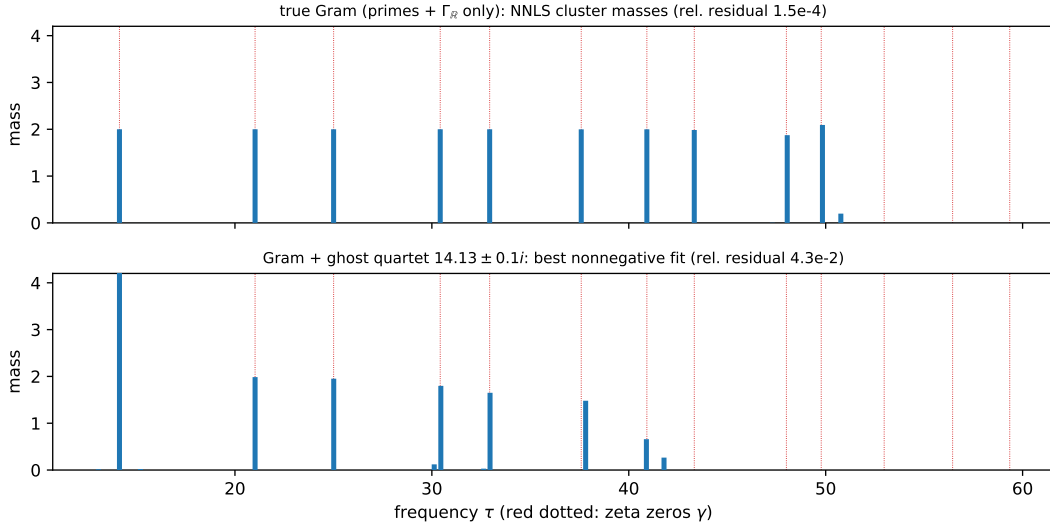


FIGURE 2. Nonnegative frequency-atom SOS of the prime-side Gram ($a = 8$): the atoms land on the zeta zeros (top). A ghost cannot be fitted (bottom).

B3. An a -uniform format: Levinson rapidities. The full-span form Z is Toeplitz on a lattice, because $Z_{ij} = z((i - j)\Delta x)$ with z computed from primes, $\Gamma_{\mathbb{R}}$ and the pole term.

Theorem 3.2. [THEOREM] (*Levinson–Durbin/Szegő recursion* [9, 15] [BIB]). Let $z_0 > 0$. The Toeplitz matrices $[z_{|i-j|}]_{i,j \leq k}$ are PD for all $k \leq K$ iff the reflection (Verblunsky) coefficients satisfy $|\kappa_j| < 1$ for $j \leq K$. Moreover $E_k = z_0 \prod_{j \leq k} (1 - \kappa_j^2)$ is the prediction error. Each step multiplies the pair (forward, backward predictor) by $\frac{1}{\sqrt{1-\kappa^2}} \begin{pmatrix} 1 & -\kappa \\ -\kappa & 1 \end{pmatrix}$, a $(1, 1)$ Lorentz boost of rapidity $\eta_j = \operatorname{artanh} \kappa_j$. PD to all orders means every step is a genuine boost; failure means a “superluminal” $|\kappa| \geq 1$.

[NUMERICAL] (Fig. 3). True data: $|\kappa_j| < 1$ for all $j \leq 60$ ($a \leq 12$). The closest approach is $\kappa_8 = -0.9861$ at $a = 1.6$, and the next margins are 0.112, 0.164. Ghost $14.13 \pm 0.1i$: $\kappa_8 = -1.0009$, failing at order 8 ($a = 1.6$). Other ghosts: order 13 ($30 \pm 0.2i$), 28 ($50 \pm 0.3i$), 30 ($85.7 \pm 0.31i$). The format is a -uniform, but each new κ_k needs the next Toeplitz entry. No prime-side mechanism bounding $|\kappa_k| < 1$ is known (direction 2). Since the spectral measure is purely atomic, Szegő’s condition fails and $\sum \kappa_j^2 = \infty$ [15] [BIB], so the κ_j do not decay.

B4. Ghost injection [NUMERICAL] ($Z \rightarrow Z + G$). First a with $n_- > 0$:

ghost	V_0	full span	Levinson	$\frac{1}{2} \log(\gamma/2\pi)$
$14.13 \pm 0.1i$	1.8	1.6	1.6	0.41
$30 \pm 0.2i$	3.2	2.6	2.6	0.78
$50 \pm 0.3i$	5.8	5.6	5.6	1.04
$85.70 \pm 0.31i$	6.0	6.0	6.0	1.31

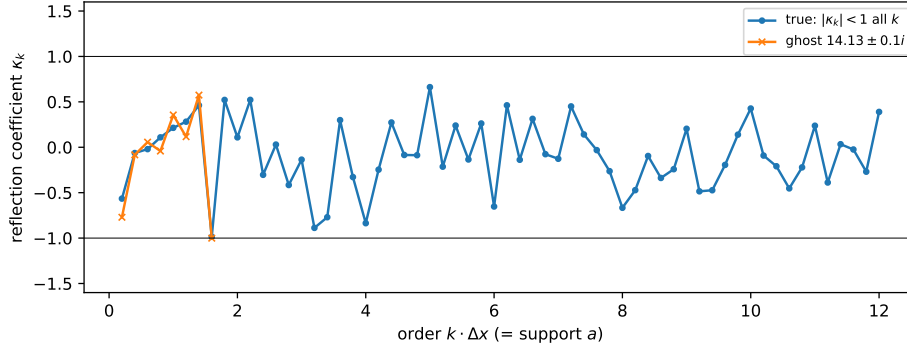


FIGURE 3. Reflection coefficients of the zero-side Toeplitz form (true) and with a ghost quartet; failure at $|\kappa| = 1$.

The count is always $n_- \leq 2$ per quartet (Fig. 1), confirming L92. The offset from $\frac{1}{2} \log(\gamma/2\pi)$ grows with γ and depends on δ and on the bump. The L92 “ $+O(1)$ ” is not uniform in this family [HEURISTIC].

4. (C) THE LORENTZIAN REFORMULATION

Theorem 4.1 (One boost). *[THEOREM] If $A \in O(1, n)$, then A has at most one eigenvalue of modulus > 1 , counted with algebraic multiplicity, and it is real. Its inverse is the only eigenvalue of modulus < 1 . All other eigenvalues lie on the unit circle.*

Proof. Extend B to the Hermitian form $h(v, w) = B(v, \bar{w})$, which has signature $(1, n)$. If $Av = \lambda v$ and $Aw = \mu w$, then $h(v, w) = \lambda \bar{\mu} h(v, w)$. Hence the sum W of generalized eigenspaces with $|\lambda| > 1$ is h -isotropic, since $\lambda \bar{\mu} \neq 1$ there (a Jordan-chain induction handles the generalized vectors). A Hermitian form of signature $(1, n)$ has no isotropic subspace of dimension > 1 , so $\dim W \leq 1$. If $\lambda \notin \mathbb{R}$, then $\bar{\lambda}$ would also occur, so $\lambda \in \mathbb{R}$. $\square$

[NUMERICAL]: 200 random elements of $SO(1, 4)$ have 0 (26 cases) or 1 (174 cases) eigenvalues outside the circle, never 2.

Theorem 4.2 ($\mathbb{F}_q$: Hodge index + one boost $\Rightarrow$ Hasse). *[THEOREM] Let $E/\mathbb{F}_q$ with $\mathbb{Z}[F]$ of rank 2, and let $L = \langle H, V, \Delta, \Gamma_F \rangle \otimes \mathbb{R} \subset \text{NS}(E \times E) \otimes \mathbb{R}$, where $H = \text{pt} \times E$ and $V = E \times \text{pt}$. The Gram matrix is*

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & q \\ 1 & 1 & 0 & N_1 \\ 1 & q & N_1 & 0 \end{pmatrix}, \quad N_1 = q + 1 - a,$$

using $\Gamma_\varphi \cdot \Gamma_\psi = \deg(\varphi - \psi)$. Let $f = F \times \text{id}$. Then $f^*H = qH$, $f^*V = V$, $f^*\Delta = \Gamma_F$, $f^*\Gamma_F = \Gamma_{F^2}$, and $f^*/\sqrt{q}$ is an isometry of L . By Hodge index [8] [BIB], L is Lorentzian. On the null pair (H, V) , $f^*/\sqrt{q}$ acts by $(q^{1/2}, q^{-1/2})$, a boost of rapidity $\frac{1}{2} \log q$ along the geodesic joining the ideal points $[H], [V]$. By Thm. 4.1 the remaining eigenvalues, which are $\alpha/\sqrt{q}$ and $\bar{\alpha}/\sqrt{q}$, have modulus 1. That is, $a^2 \leq 4q$.

[NUMERICAL] ($q = 101$, $E : y^2 = x^3 + x + 1$, $N_1 = 105$, $a = -3$): inertia $(1, 3)$, $\det = -395$, isometry error 10^{-13} , $|\text{eig}| = (10.05, 1, 1, 0.0995)$, translation length $2.3076 = \frac{1}{2} \log 101$. Fake $a = 21, 25, 40$: inertia $(2, 2)$ and two boosts, with second $|\text{eig}| = 1.347, 1.983, 3.711$ (Fig. 4 right). This is Weil’s argument in group language: [RESTATEMENT] of known material, but *non-circular* over $\mathbb{F}_q$, since Hodge index is proved geometrically.

Hyperboloid geometry [NUMERICAL]. $\cosh d(H+V, qH+V) = \frac{q+1}{2\sqrt{q}}$, so $d = \frac{1}{2} \log q$ exactly. For $g = 1$, $\Delta^2 = \Gamma_F^2 = 0$: the graphs are *ideal points* on the boundary sphere. For $g = 2$, $\Delta^2 = -2$

and $\Gamma_F^2 = -2q$: the graphs are space-like and define totally geodesic *walls*. The distance from $(H + V)/\sqrt{2}$ to $\Delta^\perp$ is $\operatorname{arsinh} 1 = 0.8814$, and to $\Gamma_F^\perp$ it is $\operatorname{arsinh} \cosh(\frac{1}{2} \log q)$ (2.327 for $q = 101$). For genus 2 ($y^2 = x^5 + x + 1$ over $\mathbb{F}_{101}, \mathbb{F}_{103}, \mathbb{F}_{107}$, $a = -2, 6, -2$) the lattice $\langle H, V, \Delta, \Gamma_F \rangle$ is $(1, 3)$; with fake $a > 4\sqrt{q}$ it is $(2, 2)$. CM $E \times E$ ($y^2 = x^3 - x$, L94 [TAA-L]): Minkowski $(1, 3)$, $\det -4$, with $H + V$ time-like (2) .

For ζ .

Proposition 4.3. (i) [THEOREM] Translation $U_s : \psi_x \mapsto \psi_{x+s}$ ($\lambda = e^s$) preserves I , because I depends only on $|x - y|$. It acts on the pole functionals by $F_{x+s}(\pm \frac{i}{2}) = e^{\mp s/2} F_x(\pm \frac{i}{2})$ ([NUMERICAL]: ratios 0.740818 and 1.349859 at $s = 0.6$, exact). The dual pole directions are null, and U_s is a boost of rapidity $\frac{1}{2} \log \lambda$ on the pole plane. (ii) [THEOREM] $I = \operatorname{sym}(uv^T) - \frac{1}{2}Z = \frac{1}{4}\langle u + v, \cdot \rangle^2 - (\frac{1}{4}\langle u - v, \cdot \rangle^2 + \frac{1}{2}Z)$: one square minus an SOS whenever $Z \succeq 0$. So $RH \Rightarrow n_+(I|_S) \leq 1$ on every finite span S . (iii) [NUMERICAL] True zeros: I has inertia $(1, n - 1)$ for all $a \leq 8$ (beyond that some eigenvalues fall below the tolerance). One ghost quartet gives $(3, \cdot)$ once $a \geq 2$ (14.13) or $a \geq 4$ (30). The exact mode model gives $(3, 23)$ (§5). (iv) [NUMERICAL] For a time-like D supported in $[0, 2]$, $d(D, U_s D) = 0.4975s + 0.047$ for $s \in [2, 9]$ (Fig. 4 left): the geodesic is translated by $\frac{1}{2} \log \lambda$, as $J.W.$ pictured. With a ghost the “rapidity” curve looks the same, since $\delta < \frac{1}{2}$ and the pole boost dominates. Ghosts are invisible in the asymptotic rapidity, an honest negative.

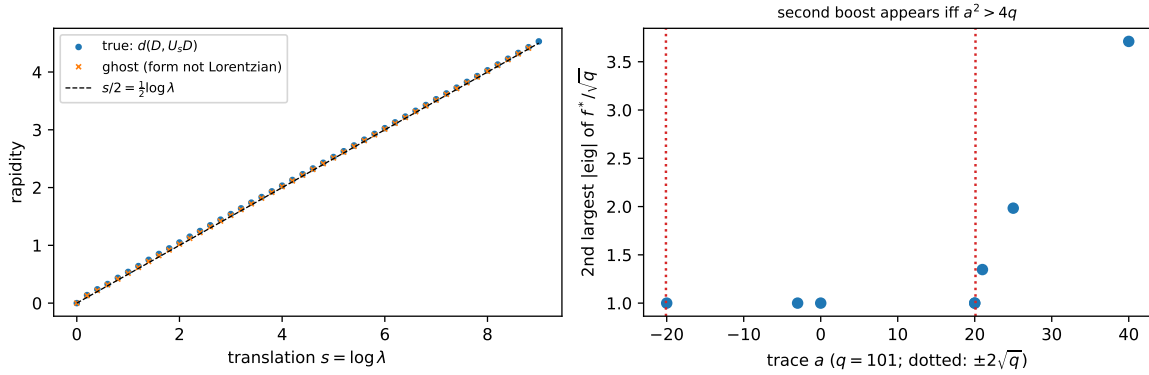


FIGURE 4. Left: rapidity of translation on the Weil form (slope $\frac{1}{2}$). Right: normalized Frobenius on $\langle H, V, \Delta, \Gamma_F \rangle$, $q = 101$: a second boost appears exactly when $a^2 > 4q$.

Verdict [RESTATEMENT]. “Lorentzian on all spans” is Weil positivity in the form one square minus an SOS. The one-boost lemma is free; the Lorentzian signature is the whole content. Over $\mathbb{F}_q$ it is supplied by Hodge index. Over $\mathbb{Q}$ nothing supplies it independently, so as a proof strategy the reformulation is circular. Its value is that it isolates the missing input exactly: an independently constructed time-like (ample) class. The SOS (space-like part) and the Lorentzian form are the same object, $I = (\text{one square}) - \frac{1}{2}Z$ on the complement of the pole plane.

5. (D) RAPIDITY VERSUS ANGLE: THE ALGEBRAIC FORM OF POSITIVITY

Script `num/lemma98_polar.py` (198/polar.json).

D1. Elliptic vs hyperbolic (claim (1)). $\mathbb{R}[i]$ has norm $x^2 + y^2$ and norm-one group $SO(2)$ (compact; $\cos, \sin$). The split-complex numbers $\mathbb{R}[j]$, $j^2 = 1$, have norm $x^2 - y^2$ and norm-one group $SO(1, 1)$ (non-compact; $\cosh, \sinh$, rapidity) [THEOREM]. Over $\mathbb{Q}$: an imaginary quadratic field has positive-definite norm form and finite unit group (elliptic). A real quadratic field has an indefinite norm form, and Dirichlet’s unit theorem gives units $\pm \varepsilon^n$ [BIB]. Writing $\varepsilon = e^\eta$, the regulator $\log \varepsilon$ is a rapidity: multiplication by ε is a boost with null coordinates $x \pm y\sqrt{d}$ scaled

by $\varepsilon^{\pm 1}$. Closed geodesics: a primitive hyperbolic $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ with trace t acts on $\mathbb{H}$ as $z \mapsto \varepsilon^2 z$, length $2 \log \varepsilon = 2 \operatorname{arccosh}(t/2)$ [14, 12] [CITED]. [NUMERICAL]: for $d = 5, 8, 12, 13, 17, 21, 24, 28$ the fundamental solution of $t^2 - du^2 = 4$ gives $2 \log \varepsilon_d = 2 \operatorname{arccosh}(t/2)$ to 10^{-15} (e.g. $d = 5$: 1.92485). Claim (1) is correct, including the factor 2.

D2. Positive involutions (claim (2)).

Theorem 5.1. *Let A be a finite-dimensional real semisimple algebra with involution $\dagger$, and $G_{\dagger} = \{x : xx^{\dagger} = 1\}$. (i) [THEOREM] If $\dagger$ is positive ($\operatorname{Tr}(xx^{\dagger}) > 0$ for $x \neq 0$), then $G_{\dagger}$ is compact. (ii) [CITED-sec] Conversely, if $G_{\dagger}$ is compact then $\dagger$ is positive. Equivalently, $\theta(x) = (x^{\dagger})^{-1}$ is a Cartan involution of $A^{\times}$.*

Proof. (i) $\|x\|^2 = \operatorname{Tr}(xx^{\dagger})$ is a Euclidean norm, and on $G_{\dagger}$ it equals $\operatorname{Tr}(1)$, so $G_{\dagger}$ is closed and bounded. (ii) By the classification of involutions on $M_n(\mathbb{R})$, $M_n(\mathbb{C})$, $M_n(\mathbb{H})$ and on swap-type $B \times B^{op}$, $G_{\dagger}$ is one of $O(p, q)$, $U(p, q)$, $Sp(p, q)$, $Sp_{2n}(\mathbb{R})$, $O^*(2n)$, GL_n . It is compact only for the definite forms, which are exactly the positive involutions. (Classification: Book of Involutions, cited via [16] [CITED-sec]; not re-derived here.) $\square$

Examples ([THEOREM]): on $\mathbb{R} \times \mathbb{R}$ the swap involution gives $G = \{(t, 1/t)\} \cong \mathbb{R}^{\times}$, which is a rapidity and not positive. On $M_2(\mathbb{R})$ the adjugate involution gives $G = \mathrm{SL}_2(\mathbb{R})$ and $\operatorname{Tr}(xx^{\dagger}) = 2 \det x$, which is indefinite. Albert’s classification of division algebras with positive involution [11, 16] [CITED-sec] (Mumford §21 via Totaro’s summary; Mumford not opened): Type I totally real field ($\dagger = \operatorname{id}$); II totally indefinite quaternion; III totally definite quaternion; IV central simple over a CM field ($\dagger|_{\text{centre}} = \text{complex conjugation}$). For a curve over $\mathbb{F}_q$, $\mathbb{Q}(F) \subset \operatorname{End}^0(\operatorname{Jac})$ is stable under the Rosati involution, and $F^{\dagger} = q/F$. Since Rosati is positive (ampleness), $\mathbb{Q}(F)$ is CM, or totally real when $F = \pm\sqrt{q}$ (then $\dagger = \operatorname{id}$ on it), and $\dagger = \text{complex conjugation}$. Hence $F\bar{F} = q$ in every embedding. This is Weil’s RH for curves, as in Mumford §21 [BIB]. Claim (2) is correct, with the caution that “ $\dagger = \text{complex conjugation}$ ” is the *conclusion* of positivity, not an independent input.

D3. $\alpha = \sqrt{q} e^{\eta + i\theta}$ (claim (3)). Correct [THEOREM]: RH $\iff \eta = 0$. For ζ , $p^{-s} = p^{-1/2} e^{-\delta \log p} e^{-i\gamma \log p}$ for $s = \frac{1}{2} + \delta + i\gamma$, so $\delta \log p$ is a rapidity and $\gamma \log p$ an angle. Refinement: the prefactor $p^{-1/2}$ is itself the pole boost (rapidity $\frac{1}{2} \log p$, Prop. 4.3(i)). The ghost rapidity $\delta \log p$ is a *second* boost on top of it. [NUMERICAL], $L(\Delta)$: for all 25 primes $p < 100$, $\tau(p)^2 < 4p^{11}$, so $\mathbb{Q}(\alpha_p)$ is imaginary quadratic (elliptic), with $\eta_p = 0$ and $\max |\tau(p)|/2p^{11/2} = 0.855$. A fake $\tau(2) = 2 \cdot 2^{11/2} \cosh 0.3$ gives a real quadratic α_2 with $\eta = 0.300$ and $|\alpha|/2^{11/2} = 1.3499, 0.7408$. Hecke operators are self-adjoint for Petersson, a positive involution of type I on the totally real Hecke field. That forces $a_p \in \mathbb{R}$, but it is *blind* to Ramanujan (compare L95’s “Hecke lattice blind” [TAA-L]). The positivity that matters is type IV on $\mathbb{Q}(\alpha_p)$.

D4. Polar decomposition (claim (4)) [NUMERICAL].

model	$ \alpha /\sqrt{q}$	Rosati trace form	$FF^{\dagger} - q$	sing. values (Rosati metric)
E , $q = 101$, $a = -3$	1, 1	(2, 0)	0	10.0499, 10.0499
E fake $a = 25$	1.983, 0.504	(1, 1)	0	— (no metric)
C : $y^2 = x^5 + x + 1$, $\mathbb{F}_{101}$	1, 1, 1, 1	(4, 0)	$2 \cdot 10^{-17}$	10.0499×4
C ghost ($e^{\pm 0.3}$ pair)	1, 1, 1.350, 0.741	(3, 1)	$3 \cdot 10^{-15}$	— (no metric)

For C : $N_1 = 104$, $N_2 = 10250$, $P(x) = x^4 + 2x^3 + 26x^2 + 202x + 10201$. The Rosati trace form is $T(y, z) = \operatorname{Tr}_{\mathbb{Q}(F)/\mathbb{Q}}(yz^{\dagger})$, with $\dagger : F \mapsto q/F$; multiplication by q/F is the T -adjoint of F (error $< 4 \cdot 10^{-8}$). Three findings: (a) $FF^{\dagger} = q$ holds identically, for ghosts too, because it is the definition of $\dagger$. (b) In a T -orthonormal basis (which exists only when $T \succ 0$), $F = \sqrt{q}U$ with U orthogonal: the positive part is scalar. In the naive coordinate metric the singular values are (101.04, 0.9996), so scalar-ness depends on the metric. (c) For ghosts, T is indefinite and no such polar decomposition exists. In a metric where F is normal, the positive part is $\sqrt{q} \operatorname{diag}(e^{\pm \eta})$, which is not scalar. On the

full cohomology of E the normalized positive part is $\text{diag}(q^{-1/2}, 1, 1, q^{1/2})$ ($\log = \mp 2.3076$): a single boost along the degree (ample) direction, and the identity on H^1 . So J.W.’s “scalar boost along the ample direction times unitary” is right after one correction: the boost is rank one on $H^0 \oplus H^2$, and the part that is scalar is the H^1 part (C98.16).

ζ mode model ([NUMERICAL], [RESTATEMENT]). Take the modes $e^{\pm x/2}$ (poles, paired hyperbolically), $e^{\pm i\gamma x}$ for the first 10 zeros (definite), and a ghost quartet with $r = \pm\gamma \mp i\delta$ paired with $\pm\gamma \pm i\delta$. Translation U_s is an exact isometry (error 10^{-16}). Polar decomposition with respect to the Cartan majorant gives $\log P/s$ eigenvalues: true $\{\pm \frac{1}{2}\}$ (one boost plane), inertia (1, 21); one ghost $\{\pm \frac{1}{2}, \pm 0.1, \pm 0.1\}$ (three boost planes), inertia (3, 23), matching the Weil-form window (3, 13); two ghosts: five boost planes, (5, 25). The real rank of the hyperbolic part equals n_+ , and $\text{RH} \iff$ rank one. This is a restatement.

D5. Is the proposed target new? J.W.’s target over $\mathbb{Q}$ is a $\mathbb{Q}$ -algebra with a positive involution (type-IV-like) in which the scaling flow is the single boost and everything else acts compactly. (i) [RESTATEMENT]: by Thm. 4.1, Thm. 5.1 and Prop. 4.3, “positive involution with one boost” for the scaling action on a space carrying the Weil form is equivalent to that form being Lorentzian, i.e. to Weil positivity. Positivity of $\dagger$ is RH, so as stated the target is circular. (ii) It is *not* circular over $\mathbb{F}_q$, because Rosati positivity there comes from an ample class (Riemann form/Hodge index), independently of Frobenius. (iii) So the non-circular target is the same missing piece as in (C): an ample class over $\mathbb{Q}$ whose Riemann form makes $\dagger$ positive. We know no candidate. Connes’s scaling action on adèle classes is the natural carrier of the flow [5] [CITED], but no positive involution on it is known to us [OPEN]. New in this chapter: the Levinson-rapidity format (Thm. 3.2) and the Bochner/NNLS recovery of zeros from prime-side Gram matrices. These are standard tools applied in a new place ([NUMERICAL]). Nothing in (C) or (D) is mathematically new; it is a dictionary.

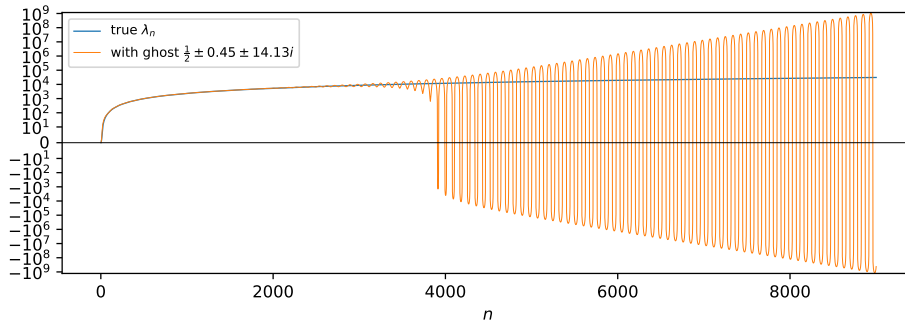


FIGURE 5. Li coefficients from 2500 zeros + tail (true), and with one injected ghost quartet $\frac{1}{2} \pm 0.45 \pm 14.13i$ (first negative at $n = 3907$).

6. (E) A TRIGONOMETRIC/HYPERBOLIC SYSTEM ON $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$, AND YONEDA

J.W. (t423): if we can define a trigonometric and a hyperbolic system on the missing base $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$, do Rosati positivity, positivity and SOS come for free? And does Yoneda then hand us the scaffolding of the object? Script `num/lemma98_tori.py` (198/tori.json).

E1. Circle and hyperbola as group schemes [THEOREM]. The circle $S = \text{Spec } \mathbb{Q}[x, y]/(x^2 + y^2 - 1)$ is the norm-one torus $R_{\mathbb{Q}(i)/\mathbb{Q}}^1 \mathbb{G}_m$. Its group law is $(x, y)(x', y') = (xx' - yy', xy' + x'y)$, i.e. multiplication of $x + iy$. The hyperbola $\text{Spec } \mathbb{Q}[x, y]/(x^2 - y^2 - 1)$ is isomorphic to the split torus $\mathbb{G}_m = \text{Spec } \mathbb{Q}[u, u^{-1}]$ via $u = x + y$, $u^{-1} = x - y$. Both are representable (affine) and become isomorphic over $\mathbb{Q}(i)$. So “trig vs hyperbolic” is the choice of quadratic algebra, $\mathbb{Q}(i)$ versus $\mathbb{Q} \times \mathbb{Q}$. In that sense J.W.’s “choice of number system” is right. Over $\mathbb{R}$: $S(\mathbb{R}) = S^1$ is compact and

$G_m(\mathbb{R}) = \mathbb{R}^\times$ is not. A connected reductive $\mathbb{R}$ -group is anisotropic iff its real points are compact [7] [CITED-sec]. *But the choice is place by place.* S splits over $\mathbb{F}_p$ and $\mathbb{Q}_p$ exactly when -1 is a square there. [NUMERICAL]: for all 16 primes $3 \leq p \leq 59$, $\#S(\mathbb{F}_p) = p-1$ (split, hyperbolic) iff $p \equiv 1 \pmod{4}$, and $p+1$ (elliptic) iff $p \equiv 3 \pmod{4}$, while the hyperbola always has $p-1$ points. Over $\text{Spec } \mathbb{Z}$ the same torus is a circle at some places and a hyperbola at others. Compactness, the property that yields an invariant positive inner product, is a statement about the *real* place. Rational points are dense on both (41 parametrized $\mathbb{Q}$ -points of each checked exactly), so $\mathbb{Q}$ -points do not tell the two apart.

E2. The existing “one boost $\times$ compact” structure. For $\mathbb{Q}$, the idele class group splits as $C_{\mathbb{Q}} \cong \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^\times$ [TAA-L] (L90 Thm 6.4). Connes [5] [CITED] (eqs. (20)–(23)) uses $C_k \simeq K \times N$ with $K = \{|g| = 1\}$ compact. He decomposes $\mathcal{H} = \bigoplus_{\chi} \mathcal{H}_{\chi}$ over characters of K , and on each $\mathcal{H}_{\chi}$ the $\mathbb{R}_+^*$ -action has generator D_{χ} with purely imaginary spectrum. This happens in a weighted space L_{δ}^2 in which only critical zeros appear; non-critical zeros would appear as resonances, and RH is reduced to a trace formula. What is free and what is missing: (i) [THEOREM] Positivity on the compact part is free: averaging over $\widehat{\mathbb{Z}}^\times$ (Weyl’s unitary trick) gives an invariant positive inner product, and this is the Dirichlet-character decomposition. (ii) [RESTATEMENT] The boost $\mathbb{R}_{>0}$ is non-compact, so no averaging exists. “Zeros are pure angles” means exactly that $\mathbb{R}_{>0}$ acts unitarily on an H^1 that sees *all* zeros with a positive inner product, which is RH. (iii) [TAA-L] The splitting itself carries no RH content: it exists for every number field and every function field (L90). Finite-field analogue ([NUMERICAL]): the powers $(F/\sqrt{q})^n$, $n \leq 200$, stay bounded (norm ≤ 10.17) for $E/\mathbb{F}_{101}$ and blow up to $2 \cdot 10^{60}$ for the fake $a = 25$. “Compact closure of the Frobenius group” is the same as “pure angles” (compare [7], Remark 1.4: $\gamma^{\mathbb{Z}}$ is bounded iff its eigenvalues are units).

E3. Yoneda: the candidate functor. Yoneda [13] [TAA-L] (L90) says $X \mapsto \text{Hom}(-, X)$ is fully faithful. A functor $\mathfrak{F}$ determines its representing object up to unique isomorphism, *if* one exists. Specifying how the object transforms is specifying $\mathfrak{F}$ on morphisms. The candidate functor reads:

$\mathfrak{F}(T) =$ isomorphism classes of tuples $(A, \dagger, \alpha, H^1, \langle, \rangle)$ over the test object T , where: A is a $\mathbb{Q}$ -algebra with a *positive* involution $\dagger$ (type IV-like); α is an action of $\mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^\times$ on $(A, \dagger)$; H^1 is an A -module with a *positive-definite* Hermitian form compatible with $\dagger$, on which $\widehat{\mathbb{Z}}^\times$ acts unitarily (characters) and $\mathbb{R}_{>0}$ acts *unitarily*, i.e. by pure angles; and the trace of $\alpha(f)$ on H^1 equals the zero side of the explicit formula for every test f (normalization). The pole plane $(H^0 \oplus H^2)$ carries the single boost (§4).

Proposition 6.1. (i) [THEOREM] If $\mathfrak{F}$ is representable by X , then X is unique up to unique isomorphism, and its symmetries are the natural automorphisms of $\mathfrak{F}$. That is the “scaffolding”. (ii) [THEOREM] If $\mathfrak{F}(\text{pt}) \neq \emptyset$, then RH holds. A unitary $\mathbb{R}_{>0}$ -representation has spectrum on $i\mathbb{R}$, and the trace normalization forces that spectrum to be the multiset of zeros (Weil/Bochner uniqueness, Prop. 3.1), so every γ is real. (iii) [RESTATEMENT] So representability needs: a category of test objects T ; $\mathfrak{F}$ to be a sheaf that preserves limits; and $\mathfrak{F}(\text{pt}) \neq \emptyset$, which by (ii) is RH. Dropping positivity gives a functor with known non-empty values (Connes’s L_{δ}^2 -type realizations), but then the uniqueness from (i) does not pin down a positive form.

Hence Yoneda gives uniqueness and functoriality, not existence. The transformation behaviour, an action of $\mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^\times$, is already known from $C_{\mathbb{Q}}$. Representability is the whole battle, not half of it (C98.19; compare L90 Prop. 2.3, “presupposes positivity if W2 is an axiom”). *Relation to Q3.* Over $\mathbb{F}_q$ the representing object $C \times C$ exists independently, and its ample class $H + V$ produces the positive Rosati involution through the Riemann form. Putting “positive $\dagger$ ” into $\mathfrak{F}$ as an axiom reverses that order and imports RH. The non-circular version of $\mathfrak{F}$ asks for an ample class first and derives $\dagger$.

E4. In what sense Rosati, SOS and Hodge index are “the same”. Over $\mathbb{F}_q$, for a curve C ([THEOREM], with sources): (a) Hodge index: $\text{NS}(C \times C) \otimes \mathbb{R}$ is Lorentzian. This is proved geometrically from Riemann–Roch on the surface, with no spectral input [8] [BIB], [TAA-L] (L90, Milne pRH).

(b) SOS: by the identity of §1(e), (a) is the same as $D^2 = (D \cdot H)^2 / H^2 - \sum_k (\text{linear form})_k^2$. This is a definition-level equivalence. (c) Rosati: for $\varphi \in \text{End}^0(\text{Jac})$, the trace $\text{Tr}(\varphi\varphi^\dagger)$ is, up to a positive factor, minus the self-intersection of the primitive part of the correspondence class (Castelnuovo–Severi/Weil, via Milne pRH Thm 1.5 [CITED-sec]). So (a) gives positivity of $\dagger$. (d) Positivity of $\dagger$ on $\mathbb{Q}(F)$ gives $|\alpha| = \sqrt{q}$ (Thm. 5.1, §5). Thus (a) $\Leftrightarrow$ (b) is an identity, (a) $\Rightarrow$ (c) is an intersection-number identity, and (c) $\Rightarrow$ (d) is algebra. The chain is non-circular because (a) is proved first. Over $\mathbb{Q}$ ([RESTATEMENT]): (a') I is Lorentzian on all spans; (b') $Q = Z|_{V_0} \succeq 0$, the SOS; (c') the Weil functional W is a positive trace for the involution $f^*(x) = \overline{f(x^{-1})}$ on the convolution algebra of $\mathbb{R}_{>0}$, $W(f * f^*) \geq 0$ [17, 2] [CITED-sec]; (d') RH. Here (b') $\Leftrightarrow$ (c') is the same inequality written two ways, (c') $\Leftrightarrow$ (d') is Weil's criterion, and (b') $\Rightarrow$ (a') is Prop. 4.3(ii). None is proved independently, so over $\mathbb{Q}$ all four are *one* inequality in different clothes. Defining a trig/hyperbolic system does not supply it. It supplies the group (known), not the positive form.

7. CORRECTIONS TO THE CHAT CLAIMS (T420–T423)

C98.1 $((2|\rho|)^2)$..:

Correct. It is the norm form of $\mathbb{Q}(i)$, i.e. elliptic.

C98.2 $(\rho(1 - \rho), \text{ghost})$..:

Correct, including the sign $-2i\gamma\delta$ for $\rho = \frac{1}{2} + \delta + i\gamma$ ($t = \gamma - i\delta$). The mirror zero has $+2i\gamma\delta$.

C98.3 (Li) ..:

Correct, with the references verified. Quantitatively, Li detects a ghost only at $n \gtrsim \gamma^2/\delta$: $n = 3907$ for (14.13, 0.45), and none up to 9000 for $\gamma \geq 21$.

C98.4 (Hasse) ..:

Correct for $|a| \leq 2\sqrt{q}$.

C98.5 (Hodge index as one square minus SOS) ..:

Correct when $H^2 > 0$.

C98.6 (reverse Cauchy–Schwarz) ..:

Correct in $(1, n)$, false in $(2, n)$.

C98.7 $(z \mapsto \lambda z, \log \lambda)$..:

Correct on $\mathbb{H}$. On the Weil form, scaling by λ is a boost of rapidity $\frac{1}{2} \log \lambda$ (pole vectors scale by $\lambda^{\mp 1/2}$). The two agree if the Weil boost is matched with $z \mapsto \sqrt{\lambda}z$. Over $\mathbb{F}_q$, Frobenius is a boost of rapidity $\frac{1}{2} \log q$.

C98.8–10 (hyperboloid; no model in $(2, \cdot)$; ghost $(2, 2)$ destroys it) ..:

Correct. With poles included, a ghost raises n_+ from 1 to 3.

C98.11 (geodesic lengths) ..:

Lengths are $2 \log \varepsilon_d$, not $\log p$. Connes's $\log p$ orbits belong to a hypothetical flow that is explicitly *not* a geodesic flow [5]. [ANALOGY] only.

C98.12 (“SOS trivial on finite spans; content is uniformity in a ”) ..:

Right, with three refinements. (i) Uniformity in the refinement at fixed a is also needed (L92: μ_{lo} falls with n). (ii) Under RH an a -uniform SOS exists canonically, the zero side, and by Bochner every frequency-atom SOS fitted to prime-side data converges to it (NNLS recovers the zeros). So the real requirement is an SOS whose vectors are *not* spectral. (iii) The Levinson format is a -uniform and equivalent, not easier.

C98.13 (D1: angle/rapidity, units, $2 \log \varepsilon_d$) ..:

Correct (8/8 numerical).

C98.14 (D2: positive $\iff$ compact) ..:

Correct: (i) is proved here and (ii) is via the classification. “ $\dagger$ = complex conjugation on $\mathbb{Q}(F)$ ” is a consequence of Rosati positivity, not an input.

C98.15 (D3: RH $\iff \eta = 0$; $\delta \log p$ rapidity):.

Correct. The $p^{-1/2}$ factor is the (allowed) pole boost, and a ghost adds a second boost.

C98.16 (D4: $|F| = \sqrt{FF^\dagger} = \sqrt{q}$):.

The identity $FF^\dagger = q$ is tautological and holds for ghosts too. The content is that $\dagger$ is positive, so that $T(y, z) = \text{Tr}(yz^\dagger)$ is a metric. Only then is $F = \sqrt{q}$ -unitary. On the total cohomology the positive part is a rank-one boost, not a scalar. The proposed $\mathbb{Q}$ -target is a restatement of Weil positivity (D5).

C98.17 (t423: “trig vs hyperbolic is a choice of number system”):.

Correct at one field: circle = $R_{\mathbb{Q}(i)/\mathbb{Q}}^1 \mathbb{G}_m$, hyperbola = $\mathbb{G}_m$. Over $\text{Spec } \mathbb{Z}$ it is not one global choice. The same torus is split at $p \equiv 1 \pmod{4}$ and non-split at $p \equiv 3 \pmod{4}$ (16/16 primes checked). Compactness, and hence positivity, lives at the real place.

C98.18 (t423: “Rosati, positivity, SOS for free”):.

Only for the compact part: averaging over $\widehat{\mathbb{Z}}^\times$ gives a positive invariant form for free. The zeros sit in the non-compact $\mathbb{R}_{>0}$ direction, where no averaging exists. Unitarity there, meaning pure angles on all of H^1 , is RH (Prop. 6.1(ii)). The “one boost $\times$ compact” group $C_{\mathbb{Q}} \cong \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^\times$ already exists and carries no RH content (L90).

C98.19 (t423: Yoneda gives “half the battle”):.

Yoneda gives uniqueness up to unique isomorphism, and functoriality, *if* the functor is representable. It needs the functor on all test objects (naturality, L90.C1), not just a transformation rule, and the rule is already known. With positivity in $\mathfrak{F}$, existence is RH; without it, uniqueness does not fix a positive form. Representability is the whole battle.

8. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	An ample class / Riemann form over $\mathbb{Q}$ making a $\dagger$ on the scaling algebra positive (J.W.’s type-IV target, D5). The only non-circular route, and no candidate is known.	0.95	0.03	0.58
2	Certified Lorentzian/Levinson certificates: interval-arithmetic $ \kappa_k < 1$ beyond L92’s $a \leq 8$, with the margin $1 - \kappa_8 = 0.014$ explained.	0.35	0.75	0.51
3	A prime-side mechanism for $ \kappa_k < 1$ (a Schur-function or Szegő-type argument). It would be an a -uniform non-spectral SOS.	0.7	0.10	0.46
4	Li vs Weil detection: rapidity $\log w \approx \delta/\gamma^2$ against the Weil threshold $a^*(\gamma)$, i.e. which criterion sees a ghost first.	0.3	0.7	0.46
5	Type I (Hecke/Petersson) vs type IV ($\mathbb{Q}(\alpha_p)$) positivity for $L(\Delta)$: why the Hecke side is blind (L95), via the Scholl motive.	0.5	0.3	0.42

9. QUESTIONS FOR J.W.

- Q1. Is the normalization acceptable, with rapidity $\frac{1}{2} \log \lambda$ on the Weil form (pole vectors scale by $\lambda^{\mp 1/2}$) and $\log \lambda$ on $\mathbb{H}$? Over $\mathbb{F}_q$ this is $\frac{1}{2} \log q$ per Frobenius.
- Q2. Bochner forces any frequency-atom SOS to be the zero side. Do you agree the aim must be a *non-spectral* SOS, e.g. Levinson/Schur (direction 3)?
- Q3. For the type-IV target: which comes first in your picture, the involution or the ample class? Over $\mathbb{F}_q$ the ample class produces the positive Rosati involution, not conversely. Section (E) shows that putting positive $\dagger$ into the Yoneda functor as an axiom makes non-emptiness equivalent to RH.
- Q4. Section (D) shows $FF^\dagger = q$ holds for ghosts too. Do you accept that “positive part scalar” must be read in the Rosati metric, whose existence is RH?

Q5. L99: direction 2 (certified Lorentzian/Levinson, high probability) or direction 1 (ample class over $\mathbb{Q}$, high significance, low probability)?

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TAA LEMMA 99: TWO BOOSTS MAKE A ROTATION — THE CARTAN INVOLUTION AS THE MODEL POSITIVE INVOLUTION

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing in this chapter proves or claims to prove the Riemann Hypothesis. Every claim carries one label: [THEOREM] (proved here, or classical with proof indicated), [RESTATEMENT] (reformulation), [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (computed; scripts `num/lemma99_*.py`, data `num/199/`). Sources: [CITED] (opened), [CITED-sec] (secondary/excerpt), [BIB] (record only), [TAA-L] (earlier TAA lemma). References: `lit/lemma99/verified_refs.md`.

ABSTRACT. J.W. (t424–t425) proposes that the positive involution we need is a *Cartan involution*: two non-commuting boosts generate a rotation (Thomas–Wigner), and the rotation part carries a positive form. We check the claims and make the link precise.

Results. (1) *Rosati is a restricted Cartan involution* (Thm. 2.1). On H^1 of a polarized abelian surface, the Rosati involution (the E -adjoint) has trace form $(8, 8)$ on all of $\text{End}(H_{\mathbb{R}}^1)$. On the centralizer of the Weil operator $J = h(i)$ it equals the adjoint for the positive form $E(x, Jy)$, with trace form $(8, 0)$. On Sp it is the Cartan involution $g \mapsto JgJ^{-1} = (g^\dagger)^{-1}$ ([NUMERICAL], errors 10^{-15}). (2) *Deligne’s criterion* (Milne [4] [CITED]): a rational Hodge structure is polarizable iff $\text{ad } h(i)$ is a Cartan involution of $G/w_h(\mathbb{G}_m)$, where G is its Mumford–Tate group. For E , $\text{ad } J$ is the Cartan involution of $\mathfrak{sl}_2(-B(X, \text{ad } JY))$ has signature $(3, 0)$, and the polarization is definite). A “hyperbolic J ” with $J^2 = +1$ gives $(1, 2)$ and an indefinite $(1, 1)$ polarization (Fig. 1). (3) *The Lorentzian-to-Euclidean step is a Cartan involution* (Thm. 3.2). The reflection θ_H in the ample class makes $Q(x, \theta_H y)$ positive definite exactly when NS is Lorentzian: $(4, 0)$ for $E/\mathbb{F}_{101}$, genus 2 and CM $E \times E$, and $(3, 1)$ for ghosts. No time-like reflection repairs a $(2, 2)$ form (1017 random H). The same holds for the Weil form on finite spans: $(n, 0)$ with true zeros, and $n_- = n_+(I) - 1 = 1, 2$ with a ghost quartet. (4) $\mathfrak{sl}_2(\mathbb{R})$: brackets and Killing form as claimed. Corrections: $R(W)$ acts on weight k by ik , and $\Delta = -\frac{1}{4}(H^2 + P^2 - W^2)$ with eigenvalue $s(1-s) = \frac{1}{4} + r^2$. Invariant Hermitian forms on K -types, computed numerically: positive for $s = \frac{1}{2} + ir$ and for $s \in (0, 1)$; indefinite for $s = 1.6$; nonexistent for $s = 0.8 + 3i$, which pairs only with $\pi_{1-\bar{s}}$ (hyperbolic, $(11, 11)$). Zeta zeros are scattering poles of $\varphi(s) = \xi(2s-1)/\xi(2s)$ at $\rho/2$, and $|\varphi| = 1$ on $\text{Re } s = \frac{1}{2}$ independently of RH. (5) (D) , *main question: honest negative*. The $\text{ax}+\text{b}$ group (Bost–Connes) is solvable, with degenerate Killing form $(1, 0, 1)$. It has no Cartan involution (Milne Thm. 1.16), and its brackets give translations, not rotations. The minimal completion that produces a rotation is $\mathfrak{sl}_2$, i.e. GL_2 , and there the zeros are resonances, not angles. The abelian scaling group has only an *outer* Cartan involution, with compact form $U(1)$. Asking the zeros to be angles for it is RH. In Hodge theory the angles come from the compact circle in the Deligne torus $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$; the archimedean place of $\mathbb{Q}$ is real ($\Gamma_{\mathbb{R}}$), and no such circle is available to us.

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Date: 24 October 2026 (approved t426u, pt426u “Yes make that lemma 99. Absolutely.”; chat t424–t425).

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GLOSSARY (BRIEF RESTATEMENTS FOR J.W.)

Lie algebra; bracket.:

The tangent space $\mathfrak{g}$ at the identity of a Lie group, with $[X, Y] = XY - YX$ for matrices. It measures failure to commute: $e^{sX}e^{tY}e^{-sX}e^{-tY} = e^{st[X, Y] + \dots}$.

Killing form.:

$B(X, Y) = \text{Tr}(\text{ad}X \text{ad}Y)$. For $\mathfrak{sl}_2$, $B = 4 \text{Tr}(XY)$. It is nondegenerate iff $\mathfrak{g}$ is semisimple.

Cartan involution / decomposition.:

An involution θ of $\mathfrak{g}$ with $-B(X, \theta Y)$ positive definite. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ ($\theta = +1, -1$) with $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$. Here $\mathfrak{k}$ = rotations (compact) and $\mathfrak{p}$ = boosts. For algebraic groups: θ is Cartan iff the twisted real form $\{g : \bar{g} = \theta(g)\}$ is compact [4] [CITED] (§1); one exists iff G is reductive (Thm. 1.16 there).

Casimir.:

The quadratic element of $U(\mathfrak{g})$ built from B^{-1} . It commutes with $\mathfrak{g}$ and acts as a scalar on irreducible representations. On $\mathbb{H}$ it is the Laplacian.

Principal series.:

Representations induced from characters y^s of the upper-triangular group. Unitary for $s = \frac{1}{2} + ir$, $r \in \mathbb{R}$.

Lefschetz $\mathfrak{sl}_2$; primitive class.:

For an ample class ω : $e = \omega \cup$, $h = (\text{degree} - \dim)$, $f =$ its adjoint. Primitive classes are those killed by the appropriate power of ω , i.e. the lowest-weight vectors [7] [CITED-sec].

Hodge–Riemann relations.:

The Lefschetz-twisted Poincaré form is definite, of a fixed sign, on each primitive Hodge piece $P^{p,q}$ [7] [CITED-sec].

Deligne torus; Weil operator.:

$\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$, $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times = \mathbb{R}_{>0} \times U(1)$. A Hodge structure is a representation $h : \mathbb{S} \rightarrow \text{GL}(V_{\mathbb{R}})$, and the Weil operator is $C = h(i)$ [4] [CITED] (§2).

Mumford–Tate group.:

The smallest $\mathbb{Q}$ -algebraic subgroup $G \subseteq \text{GL}(V)$ with $h(\mathbb{S}) \subseteq G_{\mathbb{R}}$ [4] [CITED].

1. THE CHAT CLAIMS T424–T425

Scripts: `lemma99_lie.py`, `lemma99_ps.py`.

- (a) *Thomas–Wigner*, $[K_x, K_y] = -J_z$: correct for $\eta = \text{diag}(1, -1, -1, -1)$, $K_x = E_{01} + E_{10}$, $K_y = E_{02} + E_{20}$, $J_z = E_{21} - E_{12}$ (counter-clockwise $x \rightarrow y$) [THEOREM]. The sign depends on conventions; with Hermitian generators one writes $[K_x, K_y] = -iJ_z$. Finite version [NUMERICAL]: two perpendicular boosts give boost $\times$ rotation with $\cos \theta = \frac{\gamma_1 + \gamma_2}{1 + \gamma_1 \gamma_2}$, verified by polar decomposition for 4 rapidity pairs to 10^{-15} (Fig. 2).
- (b) $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$, $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$: correct [THEOREM]. In $\mathfrak{sl}_2$: $\mathfrak{k} = \mathbb{R}W$, $\mathfrak{p} = \mathbb{R}H \oplus \mathbb{R}P$.
- (c) $-B(X, \theta Y)$ positive definite: correct. For $\theta X = -X^T$, $B = \text{diag}(8, 8, -8)$ in the basis H, P, W (signature $(2, 1)$), and $-B(X, \theta Y) = 8I$ ([NUMERICAL]).
- (d) $[H, P] = 2W$: correct. Also $[H, W] = 2P$ and $[P, W] = -2H$ [THEOREM]. *Two boosts make a rotation* is literally $[H, P] = 2W$.

- (e) “Weight k forms are W -eigenvectors with eigenvalue k ”: **normalization**. With $k_\theta = e^{\theta W}$ and $\phi_f(g) = f(g \cdot i) j(g, i)^{-k}$, one gets $j(k_\theta, i) = e^{-i\theta}$ ([NUMERICAL]), so $\phi_f(g k_\theta) = e^{ik_\theta} \phi_f(g)$. Thus $R(W)\phi_f = ik\phi_f$; the self-adjoint $-iR(W)$ has eigenvalue k (C99.5).
- (f) Casimir/Laplacian: **sign**. Numerically, $(H^2 + P^2 - W^2)y^s = 4s(s-1)y^s$ (right-invariant derivatives; $s = \frac{1}{2} + 3i, 0.7, 1.3, \frac{1}{2}$, relative error $< 10^{-5}$). Hence $\Delta = -y^2(\partial_x^2 + \partial_y^2) = -\frac{1}{4}(H^2 + P^2 - W^2)$ on K -invariant functions, with eigenvalue $s(1-s) = \frac{1}{4} + r^2$. The operator $(H^2 + P^2 - W^2)/4$ itself has eigenvalue $-(\frac{1}{4} + r^2)$ (C99.6).
- (g) Unitary dual of $\mathrm{SL}_2(\mathbb{R})$ (Bargmann [1] [CITED-sec]): trivial; principal series ($r \in \mathbb{R}, \lambda \geq \frac{1}{4}$); complementary series ($s \in (0, 1), 0 < \lambda < \frac{1}{4}$); discrete series ($k \geq 2$) and limits of discrete series. “Unitary $\iff r$ real” omits the complementary series and the discrete series (C99.7). The complementary series does not occur in $L^2(G)$ (Bargmann, via [1]). For $\mathrm{SL}_2(\mathbb{Z})$ all cuspidal eigenvalues are $> \frac{1}{4}$ [6] [CITED-sec]. Selberg’s conjecture $\lambda_1 \geq \frac{1}{4}$ for congruence groups is [OPEN].
- (h) $\mathbb{Q}_{>0}^\times$ is abelian, so commuting boosts give no rotation: correct [THEOREM] (bracket 0). Refinement (C99.8): compactness does not require brackets. The split torus has the *outer* Cartan involution $x \mapsto x^{-1}$ with compact form $U(1)$. What it lacks is an *inner* one, $\mathrm{ad} C$, because $\mathrm{ad} C = \mathrm{id}$ is Cartan only for compact groups [4] (Ex. 1.17(a)).
- (i) Zeta zeros as Eisenstein scattering resonances, not Casimir eigenvalues: correct [CITED] [3, 6, 5]. [NUMERICAL]: $\varphi(s) = \xi(2s-1)/\xi(2s)$ has $|\varphi(\frac{1}{2} + it)| = 1.000000$ for $t = 1, 5, 14.1, 30$, and $|\varphi(\rho_1/2 + \epsilon)| = 50.4, 5145, 5.1 \cdot 10^5$ for $\epsilon = 10^{-2}, 10^{-4}, 10^{-6}$ (a simple pole at $\rho_1/2 = \frac{1}{4} + 7.067i$; Fig. 4). Unitarity on $\mathrm{Re} s = \frac{1}{2}$ follows from the functional equation and holds with or without RH. Sarnak [5] [CITED]: “the spectral interpretation of the zeroes ... through poles of Eisenstein series (ie as resonances) has already proven to be very powerful” (nonvanishing on $\mathrm{Re} s = 1$).
- (j) (t425) Cartan’s 2-part split is not the 3-part $H^0 \oplus H^1 \oplus H^2$: correct. Künneth $H^2(C \times C) = H^0 \otimes H^2 \oplus H^1 \otimes H^1 \oplus H^2 \otimes H^0$, with $\oplus$ between degrees and $\otimes$ inside Künneth: correct. The 3-part structure is the Lefschetz $\mathfrak{sl}_2$. On a curve, $H^0 \oplus H^2$ is the 2-dimensional irreducible representation ($\mathbf{1} \oplus \mathbb{L}$) and H^1 is $2g$ copies of the trivial one (primitive). Verified for $g = 2$: $[h, e] = 2e, [h, f] = -2f, [e, f] = h$ ([NUMERICAL]).
- (k) Hodge structure = representation of $\mathbb{S}$, with $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times = \mathbb{R}_{>0} \times U(1) = \text{weight (boost)} \times \text{angle}$: correct [4] [CITED]. Ties to L98 (E) [TAA-L].
- (l) Hodge–Riemann $Q(x, Cx) > 0$: correct up to normalization. Milne’s definition: $\psi_C(u, v) = (2\pi i)^n \psi(u, Cv)$ is symmetric positive definite. On H^k with $k > 1$ it applies piecewise on primitive parts with Lefschetz signs $(-1)^{k(k-1)/2}$ [4, 7] (C99.12).
- (m) Deligne’s criterion: essentially correct; the precise statement is (C99.13): *a rational Hodge structure is polarizable iff its Mumford–Tate group (G, h) has $\mathrm{ad}(h(i))$ a Cartan involution on $G/w_h(\mathbb{G}_m)$* (Milne [4] Aside following 2.14, citing Milne 1994b 1.6, 1.27 and Milne 2013 6.2–6.3; real version Prop. 1.20). It is “modulo the weight”, and in Shimura data it is stated on G^{ad} (axiom SV2). Deligne 1979 is not opened here ([BIB] via Milne).

2. (A) THE CARTAN INVOLUTION AS THE MODEL POSITIVE INVOLUTION

Theorem 2.1 (Rosati = restricted Cartan). [THEOREM] *Let (V, E) be a real symplectic space of dimension $2g$, and J a complex structure with $E(Jx, Jy) = E(x, y)$ and $g_J(x, y) = E(x, Jy)$ positive definite (a Riemann form/polarization of (V, h) , $J = h(i)$). Let $X^\dagger$ be the E -adjoint (Rosati). (i) On $\mathrm{End}(V)$, the trace form $\mathrm{Tr}(XY^\dagger)$ has signature $(2g^2 + g, 2g^2 - g)$ [NUMERICAL] (for $g = 2$: $(8, 8)$). The unitary group of $\dagger$ is $\mathrm{Sp}_{2g}(\mathbb{R})$, which is non-compact, so $\dagger$ is not positive on $\mathrm{End}(V)$ (L98 Thm. 5.1). (ii) On the centralizer $\mathrm{End}_J(V) = \{X : XJ = JX\} \supseteq \mathrm{End}^0(A) \otimes \mathbb{R}$, $\dagger$ coincides with the g_J -adjoint, hence is positive (for $g = 2$: dimension 8, trace form $(8, 0)$; agreement $7 \cdot 10^{-15}$). (iii) On $\mathrm{Sp}(V, E)$, $\theta_J(S) = JSJ^{-1} = (S^*)^{-1}$ with S^* the g_J -adjoint. This is the Cartan involution whose compact form is $U(g) = \mathrm{Sp} \cap \mathrm{End}_J$ (error 10^{-15}).*

Proof. (ii) $E(Xx, Jy) = E(x, X^\dagger Jy) = E(x, JX^\dagger y)$ when $X^\dagger$ commutes with J , and $X^\dagger$ does because $J^\dagger = J^{-1} = -J$. So $X^\dagger$ is the g_J -adjoint, and $\text{Tr}(XX^\dagger) = \|X\|_{g_J}^2 > 0$. (iii) For $S \in \text{Sp}$, $S^\dagger = S^{-1}$, and $g_J(Sx, y) = E(Sx, Jy) = E(x, S^{-1}Jy) = g_J(x, J^{-1}S^{-1}Jy)$, so $S^* = J^{-1}S^{-1}J$ and $(S^*)^{-1} = J^{-1}SJ = JSJ^{-1}$ (as $J^{-1} = -J$). $\square$

Reading. Rosati positivity is the Cartan involution $\text{ad}(h(i))$ of the polarization group, restricted to the endomorphisms that commute with the Hodge structure. This is the precise sense of “Rosati = Cartan on the symplectic group preserving the polarization” (claim (A): correct, with the restriction to the commutant made explicit). Albert’s types (L98 D2) are the possible commutants: compact real forms intersected with a $\mathbb{Q}$ -algebra.

3. (B) LEFSCHETZ $\mathfrak{sl}_2$, HODGE–RIEMANN, DELIGNE’S CRITERION

Theorem 3.1 (Deligne/Milne criterion). *[CITED] [4] Prop. 1.20: if $\text{ad}(C)$ is a Cartan involution of G , every real representation of G carries a C -polarization. Conversely, if one faithful representation carries a C -polarization, then $\text{ad}(C)$ is Cartan. Aside following 2.14: a rational Hodge structure is polarizable iff $\text{ad}(h(i))$ is a Cartan involution on $G/w_h(\mathbb{G}_m)$.*

Elliptic curve ([NUMERICAL], `lie.json`). Here $G^{\text{der}} = \text{SL}_2$ and $h(i) = J = W$. Then $\text{ad}J = \theta$ exactly, $-B(X, \text{ad}JY)$ has signature $(3, 0)$, and $E(x, Jy)$ is definite. Replace J by the “hyperbolic” $J = H$ ($J^2 = +1$, the split torus of L98 E): $-B(X, \text{ad}JY)$ has signature $(1, 2)$ and $E(x, Jy)$ has signature $(1, 1)$. No polarization exists, which is the circle versus the hyperbola again.

Curve, total cohomology ([NUMERICAL], $g = 2$). The Poincaré pairing on $H^0 \oplus H^2$ is hyperbolic, $(1, 1)$. The Hodge-star-type operator ($1 \leftrightarrow \omega$ on $H^0 \oplus H^2$, J on H^1) gives $Q(x, \star y)$ of signature $(6, 0)$. The star is the Lefschetz–Hodge Cartan involution: it rotates the light cone of the pole plane into a circle.

Theorem 3.2 (Wick rotation by the ample class). *[THEOREM] Let (L, Q) be a real quadratic space with $Q(H, H) > 0$, and $\theta_H(x) = \frac{2Q(x, H)}{Q(H, H)}H - x$. Then θ_H is an isometric involution, and $Q(x, \theta_H y)$ is positive definite iff Q is Lorentzian. If Q has signature (p, q) with $p \geq 2$, then $Q(x, \theta_H y)$ has signature $(p + q - (p - 1), p - 1)$ for every time-like H . In $O(1, n)$, θ_H is the Cartan involution with compact form $O(n) = \text{Stab}(H)$.*

Proof. $\theta_H = +1$ on $\mathbb{R}H$ and -1 on $H^\perp$, and $Q|_{H^\perp}$ has signature $(p - 1, q)$. Flipping it gives $(q, p - 1)$, and adding $\mathbb{R}H$ gives $(q + 1, p - 1)$. $\square$

[NUMERICAL] (`lef.json`, Fig. 1). For $H + V$: $E/\mathbb{F}_{101}$ ($a = -3$), genus 2 $y^2 = x^5 + x + 1$ ($a = -2$) and CM $E \times E$ (L94) all give NS of signature $(1, 3)$ and $Q(x, \theta_H y)$ of $(4, 0)$. Fakes ($a = 25$; genus 2 $a = 42$) give $(2, 2)$ and $(3, 1)$. Over 1017 random time-like H the ghost never exceeds $n_+ = 3$. **The Weil form on finite spans** ([NUMERICAL], `zcartan.json`; $H = \text{top eigenvector}$, θ_H as above):

a	true	ghost $14.13 \pm 0.1i$	ghost $30 \pm 0.2i$
1	$I(1, 5), I_\theta(6, 0)$	$(1, 5), (6, 0)$	$(1, 5), (6, 0)$
2	$(1, 10), (11, 0)$	$(2, 9), (10, 1)$	$(1, 10), (11, 0)$
4	$(1, 20), (21, 0)$	$(3, 18), (19, 2)$	$(3, 18), (19, 2)$
6	$(1, 30), (31, 0)$	$(3, 28), (29, 2)$	$(3, 28), (29, 2)$

Verdict [RESTATEMENT]. The precise theorem linking “boost/rotation” to positivity on $H^0 \oplus H^1 \oplus H^2$ is Hodge–Riemann (definiteness on primitive pieces) plus Deligne’s criterion ($\text{ad } h(i)$ Cartan). In the NS/Weil-form setting it is Thm. 3.2: the ample (pole) class defines the Cartan involution, and its positivity is equivalent to the Lorentzian signature, i.e. to Hodge index, i.e. to Weil positivity. A ghost leaves $p - 1$ negative directions that no choice of H removes.

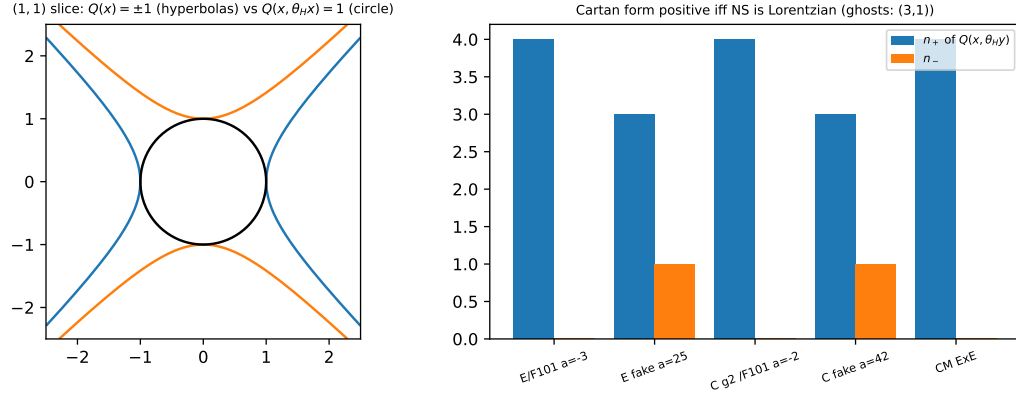


FIGURE 1. Left: the reflection in a time-like class turns the hyperbolas $Q = \pm 1$ into a circle. Right: inertia of $Q(x, \theta_{Hy})$ on NS lattices, true (definite) versus fake Frobenius (3,1).

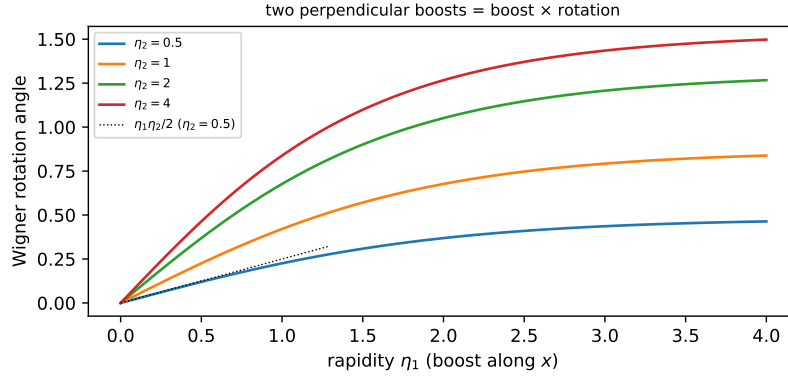


FIGURE 2. Thomas–Wigner: the rotation angle produced by two perpendicular boosts ($\approx \eta_1\eta_2/2$ for small rapidities).

4. (C) $\mathfrak{sl}_2(\mathbb{R})$ ON THE MODULAR SURFACE

Invariant Hermitian forms on K -types ([NUMERICAL], ps.json). We realize π_s on $f(n(x)a(y)k_\theta) = y^s e^{ik\theta}$ and compute the matrix of P on K -types $k \in \{-10, \dots, 10\}$ by numerical Iwasawa decomposition and Fourier projection. Invariance ($X^* = -X$ for real X) forces $c_{k+2} = -\bar{M}_{k,k+2} c_k / M_{k+2,k}$.

s	invariant Hermitian form	sign
$\frac{1}{2} + 3i, \frac{1}{2} + 0.7i$ (principal)	exists	positive ($c_k \equiv 1$)
0.8, 0.3 (complementary)	exists	positive
1.6 (real, $\lambda < 0$)	exists	indefinite (10 negative of 11)
$0.8 + 3i$ (ghost-like)	none (ratios non-real)	—

For $s = 0.8 + 3i$ the only invariant sesquilinear form pairs π_s with $\pi_{1-\bar{s}}$. On $\pi_s \oplus \pi_{1-\bar{s}}$ it is hyperbolic, of signature (11,11) on the truncation: the representation-theoretic version of the ghost (2,2) block (L97/L98) [RESTATEMENT]. See Fig. 3. **Selberg and Lax–Phillips** ([CITED-sec]). $L^2(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H})$ is unitary: cusp forms have $\lambda > \frac{1}{4}$ [6], and the continuous spectrum is the unitary principal series

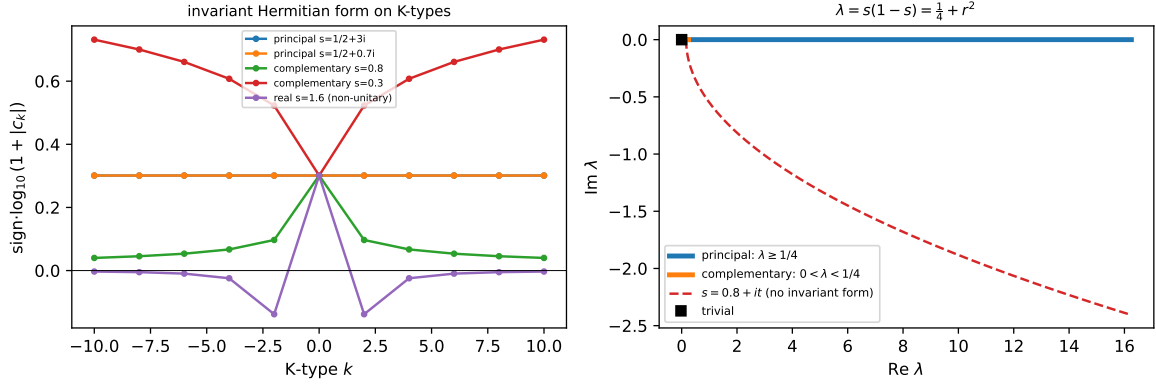


FIGURE 3. Left: invariant Hermitian forms on K -types. Right: unitary dual in the $\lambda = s(1-s)$ plane; off-line s gives complex λ .

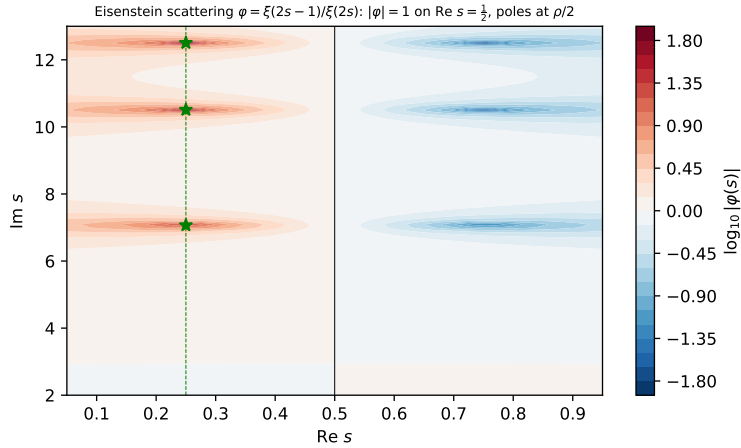


FIGURE 4. $\log_{10} |\varphi(s)|$: unitary on $\text{Re } s = \frac{1}{2}$; poles (stars) at $\rho/2$ on $\text{Re } s = \frac{1}{4}$.

for *all* real r , with or without RH. The zeros enter only through φ , as poles at $\rho/2$. Lax–Phillips [3] realize them as eigenvalues of the generator of a contraction semigroup on the scattering subspace, and Uetake [6] constructs an operator whose eigenvalues are exactly the zeros. That generator is *not* self-adjoint, so no Cartan-type positivity places its spectrum on a line. The $\mathfrak{sl}_2$ positivity is real but sits on the wrong spectral parameter.

5. (D) NON-ABELIAN BOOSTS OVER $\mathbb{Q}$: CAN BRACKETS PRODUCE THE ZEROS AS ANGLES?

Script `lemma99_candidates.py` (`cand.json`).

- (D1) *Abelian scaling* $\mathbb{Q}_{>0}^\times \times \mathbb{R}_{>0}$: brackets vanish [THEOREM]. Inner Cartan involutions are trivial, and id is Cartan only for compact groups [4] Ex. 1.17(a). The outer involution $x \mapsto x^{-1}$ has compact form $U(1)$. “Zeros are angles” for it means the characters of the representation are unitary, which is RH (L98 Prop. 6.1(ii)) [RESTATEMENT].
- (D2) $ax+b$ / Bost–Connes. Bost–Connes is the Hecke algebra of the $ax+b$ pair $(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$, with partition function $\zeta(\beta)$ [2] [CITED] (abstract). [NUMERICAL]: $[A, N] = N$, so dilation and translation bracket to a translation. The Killing form has signature $(1, 0, 1)$ (degenerate), the

algebra is solvable, and there is no Cartan involution (Milne Thm. 1.16: one exists iff the group is reductive). On $\ell^2(\mathbb{N})$: $[\mu_2, \mu_3] = 0$ (abelian scaling), $\|[\mu_2, e(\frac{1}{5})]\| = 1.90$, $\mu_2^* e(r) \mu_2 = e(2r)$, $\sigma_t(\mu_n) = n^{it} \mu_n$ (error 10^{-15}), and $\text{Tr } e^{-2H}$ approaches $\zeta(2)$. The non-commutativity is boost $\times$ phase, not boost $\times$ boost, so no rotation is generated. *Negative*.

- (D3) *Completion*. Adding the opposite translation $\bar{N}$ gives $[N, \bar{N}] = H$, $[H, N + \bar{N}] = 2(N - \bar{N})$, Killing signature $(2, 1)$: this is $\mathfrak{sl}_2$. So the *minimal* structure in which two boosts make a rotation is a rank-one semisimple group, SL_2/GL_2 [THEOREM]. Over $\mathbb{Q}$ that is the modular/adelic GL_2 setting of (C). There the Cartan positivity holds (unitarity of L^2) and the zeros are resonances, not angles. Hecke operators commute with each other and with Δ , which brings back abelian scaling at the finite places. *Negative*.
- (D4) *What would be needed* [RESTATEMENT]/[OPEN]: a unitary representation of a group containing the scaling boost, together with a Cartan-type involution, in which the zeros are spectral parameters of the boost (principal-series type: the boost acts by angles $e^{i\gamma \log \lambda}$). That is a Hilbert–Pólya realization, equivalent to RH; it is circular as a construction principle. The Hodge-theoretic template gets its angles from the compact circle $U(1) \subset \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ at a *complex* place, while the boost (weight) is central and only normalizes. The archimedean place of $\mathbb{Q}$ is real, with factor $\Gamma_{\mathbb{R}}$ and idele component $\mathbb{R}^\times = \mathbb{R}_{>0} \times \{\pm 1\}$: its only circle is $\{\pm 1\}$ [ANALOGY]. “Two boosts make a rotation” is one way to manufacture the compact part. Over $\mathbb{Q}$ the available ways either give no compact part (ax+b, abelian scaling) or give one whose spectrum is not the zeros (GL_2).

6. CORRECTIONS TO THE CHAT CLAIMS (T424–T425)

C99.1 (Thomas–Wigner):

Correct; the sign depends on conventions (stated in §1(a)).

C99.2–4 (Cartan decomposition, $-B(X, \theta Y)$, $[H, P] = 2W$):

Correct.

C99.5 (weight k):

$R(W)$ has eigenvalue ik (with $k_\theta = e^{\theta W}$); “eigenvalue k ” holds for $-iR(W)$.

C99.6 (Casimir):

$\Delta = -\frac{1}{4}(H^2 + P^2 - W^2)$; $(H^2 + P^2 - W^2)/4$ has eigenvalue $-(\frac{1}{4} + r^2)$.

C99.7 (unitary dual):

Add the complementary series ($0 < \lambda < \frac{1}{4}$), the trivial representation, and discrete series and limits. “Unitary iff r real” is the *principal* series only. Off-line complex s admits no invariant Hermitian form; it pairs with $\pi_{1-\bar{s}}$ hyperbolically.

C99.8 (abelian, no rotation):

Correct. Compactness can also come without brackets (outer Cartan involution, compact torus); the abelian scaling group lacks an *inner* Cartan involution.

C99.9 (resonances):

Correct, with sources (Lax–Phillips, Uetake, Sarnak).

C99.10–11 (t425: 2-part vs 3-part; Künneth; Lefschetz $\mathfrak{sl}_2$; Deligne torus):

Correct.

C99.12 (Hodge–Riemann):

Correct up to the factor $(2\pi i)^n$ and the Lefschetz signs on non-primitive pieces.

C99.13 (Deligne’s criterion):

Correct in substance. Precisely: $\text{ad } h(i)$ Cartan on $G/w_h(\mathbb{G}_m)$ for the Mumford–Tate group (G^{ad} in Shimura data). The source opened is Milne; Deligne 1979 is not opened.

7. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	A compact “Hodge circle” for $\mathbb{Q}$: an $\mathbb{S}$ -type torus whose $U(1)$ carries the zeros as angles, with the scaling boost central (Thm. 3.1 as template; D4).	0.9	0.05	0.56
2	Certified Cartan/Wick form $I(x, \theta_{Hy}) \succ 0$ on spans beyond $a = 6$ (equivalent to L98 direction 2, but a single PD check).	0.35	0.75	0.51
3	Lax–Phillips/Uetake generator: an indefinite (Krein-space) Cartan-type form on the scattering subspace, tested numerically against ghost resonances.	0.5	0.45	0.48
4	Maass–Selberg positivity (Sarnak’s nonvanishing method [5]): how far into $\operatorname{Re} s < 1$ the positivity of truncated Eisenstein norms constrains $\xi(2s)$.	0.45	0.4	0.43
5	Non-abelian scaling beyond $ax+b$: does a GL_2 -type Hecke system with a semisimple completion carry an inner Cartan involution compatible with the ζ time evolution? (No candidate verified.)	0.6	0.1	0.40

8. QUESTIONS FOR J.W.

- Q1. Do you accept that Rosati is the Cartan involution *restricted to the commutant of $h(i)$* (Thm. 2.1), not the Cartan involution of the whole symplectic group?
- Q2. In your picture, are the zeros meant to be the *rotation* spectrum (integer K -types) or the *boost* spectrum (principal-series r)? In unitary representations of $\operatorname{SL}_2(\mathbb{R})$ the rotation spectrum is discrete integers and the zeros cannot be K -types.
- Q3. Since $\mathbb{Q}$ ’s archimedean place is real, what plays the role of the Deligne-torus circle $U(1)$ for ζ ? (The sign group $\{\pm 1\}$ is the only compact piece of $\mathbb{R}^\times$.)
- Q4. Do you accept that Thm. 3.2 (ample-class reflection) is the exact form of “Lorentzian $\rightarrow$ positive via a Cartan involution”, and that over $\mathbb{Q}$ it restates Weil positivity?
- Q5. L100: direction 1 (Hodge circle, speculative) or direction 3 (Lax–Phillips Krein form, testable)?

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TAA LEMMA 100: A COMPACT HODGE CIRCLE FOR $\mathbb{Q}$? — HALF-TURNS, SADDLES AND THE MISSING VORTEX

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM] (proved here or classical), [RESTATEMENT] (reformulation), [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma100_*.py`, data `num/1100/`). Sources: [CITED] (opened), [CITED-sec] (excerpt/secondary), [BIB] (record only), [TAA-L] (earlier TAA lemma). References: `lit/lemma100/verified_refs.md`.

ABSTRACT. J.W. (t427) asks for a compact “Hodge circle” for $\mathbb{Q}$, and frames the problem as follows. Each zero has two partner involutions: $\sigma : \rho \mapsto 1 - \rho$ (functional equation) and $c : \rho \mapsto \bar{\rho}$ (conjugation). A ghost is a point where $\sigma \neq c$. A Hodge circle $C = h(i)$ is a device that intertwines the duality partner σ with conjugation c .

Results. (0) *Half-turn, not mirror* (§3, t430). $\sigma : \rho \mapsto 1 - \rho$ is the point reflection (half-turn) about $s = \frac{1}{2}$: $\sigma = R \circ c$ with R the mirror $\text{Re } s = \frac{1}{2}$, so $\sigma \circ c = R$. In Minkowski signature the analogue $-1 = PT$ swaps the light cones. By Cartan–Dieudonné, even products of reflections are rotations in $O(2)$ and $\pm$ boosts in $O(1,1)$; four reflections give a Thomas–Wigner rotation only with ≥ 2 space dimensions ([NUMERICAL], $O(1,2)$). No continuous $U(1)$ about $\frac{1}{2}$ preserves the zero set (Prop. 3.3): the discrete half-turn is the only symmetry about the present point, so a genuine vortex must act elsewhere. (1) *Two reflections make a boost* (Thm. 2.1). In $\mathbb{R}^{1,1}$, two reflections whose normals are both time-like (or both space-like) compose to a boost of rapidity twice the hyperbolic angle between the normals. Mixed normals give $-(\text{boost})$. In the s -plane, σ , c and σc form a Klein four-group: σc is the mirror $\text{Re } s = \frac{1}{2}$, and $\sigma = (\sigma c) \circ c$ is a rotation by π . σc is an involution, not a boost. The boost is the composite of two *parallel* mirrors $\text{Re } s = \frac{1}{2}$ and $\frac{1}{2} + \delta_0$: on the hyperbolic pair $(p^{-s}, p^{-(1-s)})$ it acts by (λ, λ^{-1}) with rapidity $2\delta_0 \log p$ ([NUMERICAL], 10^{-15}). (2) *The intertwining theorem* (Thm. 4.1). For $F \in \text{GSp}(V, Q)$ with multiplier q , semisimple and without real eigenvalues, the following are equivalent: a polarizing complex structure J ($J^2 = -1$, $J \in \text{Sp}$, $JF = FJ$, $Q(x, Jy) > 0$) exists; the Rosati partner $F^\dagger = qF^{-1}$ is the $Q(\cdot, J\cdot)$ -adjoint; $\sigma = c$ on the eigenvalues ($q/\alpha = \bar{\alpha}$); the Frobenius torus is compact. Numerics: $E/\mathbb{F}_{101}$, genus 2 $y^2 = x^5 + x + 1$ ($P = x^4 + 2x^3 + 26x^2 + 202x + 10201$) and CM $E \times E$ each have exactly one polarizing J (up to sign) and $\|F^\dagger - F^*\| \leq 8 \cdot 10^{-13}$. Ghosts: none. A real-root fake has no J at all. A ghost quartet $\sqrt{q}e^{\pm\delta \pm i\theta}$ has $Q(x, Jy)$ of signature $(2, 2)$ for *every* J , and a torus of Killing type $(1, 1)$ (boost $\times$ rotation, commuting) instead of $(0, 2)$. (3) *Candidates over $\mathbb{Q}$* (§5), all honest negatives: (i) The circle in $W_{\mathbb{R}}$ is exactly the commutator subgroup $(jzj^{-1}z^{-1} = \bar{z}/z)$; Tate (1.4.3)), so it dies in every character. The trivial motive (Hodge type $(0, 0)$) carries $J = 1$. A J with $J^2 = -1$ lives only in $\text{Ind } \chi_k$ with k odd. (ii) In Deninger’s picture the infinite prime is a *stationary point* of the flow, while finite primes are closed orbits of length $\log p$. His Hodge $\star$ on H^1 would imply RH, but it is assumed, not constructed. (iii) Base change to $\mathbb{Q}(i)$: $\zeta_{\mathbb{Q}(i)} = \zeta L(\chi_{-4})$, and the complex place’s $U(1)$ acts trivially on the $k = 0$ component. A ghost in ζ is still detected ($n_- = 2$ in 8/9 settings), so this is circular. (iv) Arakelov: arithmetic surfaces have $X(\mathbb{C})$ with its Hodge circle (Faltings–Hriljac). $\text{Spec } \mathbb{Z}$ has only $\text{Spec } \mathbb{C}$ with F_∞ , and there is no H^1 for the circle to act on. (v) Connes–Consani: at ∞ there is a genuine “two reflections make a rotation” structure (cutoff and Fourier cutoff; dihedral group with angles $\cos \alpha_n = |\lambda(n)|$, reproduced to 6 digits). But its angles are prolate eigenvalues, not zeros.

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Date: 25 October 2026 (approved t428u, pt428u “Lets do your 1. Next step – a compact hodge circle”; chat t427, t430; L99 direction 1).

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GLOSSARY

Weil group $W_{\mathbb{R}}.$:

$W_{\mathbb{R}} = \mathbb{C}^{\times} \cup j\mathbb{C}^{\times}$ with $j^2 = -1$ and $jzj^{-1} = \bar{z}$ [7] [CITED] (1.4.3). It is a non-split extension $1 \rightarrow \mathbb{C}^{\times} \rightarrow W_{\mathbb{R}} \rightarrow \text{Gal}(\mathbb{C}/\mathbb{R}) \rightarrow 1$. Its abelianization is $\mathbb{R}^{\times}$ ($z \mapsto z\bar{z}$, $j \mapsto -1$), and its commutator subgroup is $U(1)$ [6] [CITED-sec].

Frobenius at infinity $F_{\infty}.$:

Complex conjugation acting on the Betti realization M_B of a motive (and on $H^{p,q} \rightarrow H^{q,p}$). An $\mathbb{R}$ -Hodge structure with F_{∞} determines the archimedean Γ -factor [2] [CITED] (§2, citing Serre and Fontaine–Perrin-Riou).

Deligne torus; Weil operator.:

$\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$, $\mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times} = \mathbb{R}_{>0} \times U(1)$. A Hodge structure is $h : \mathbb{S} \rightarrow \text{GL}(V_{\mathbb{R}})$, and $C = h(i)$ [5] [CITED]. Note $\mathbb{S}(\mathbb{R}) = \mathbb{C}^{\times}$ is the identity component of $W_{\mathbb{R}}$.

Hodge–Riemann form.:

For a polarization Q of weight w , the form $H(x, y) = Q(x, C\bar{y})$ on $V_{\mathbb{C}}$ is Hermitian and positive definite (Milne’s normalization $(2\pi i)^w Q(x, Cy)$ is symmetric positive) [5] [CITED].

Arakelov geometry.:

Intersection theory on an arithmetic surface $\mathcal{X} \rightarrow \text{Spec } \mathbb{Z}$ completed by Green’s functions on the Riemann surface $\mathcal{X}(\mathbb{C})$ at ∞ . Its Hodge index theorem is due to Faltings and Hriljac [3, 4] [CITED-sec] (L90).

CM field.:

A totally imaginary quadratic extension K of a totally real field K^+ . Complex conjugation is then a well-defined automorphism, equal in every embedding. Positive involutions on number fields are exactly id on totally real fields and complex conjugation on CM fields [8] [CITED-sec] (Mumford §21 [BIB]).

1. THE CHAT CLAIMS (T427 AND THE FRAMING)

- (a) *Two partner involutions* σ, c . Correct [THEOREM]: $\xi(s) = \xi(1-s)$ and $\overline{\xi(\bar{s})} = \xi(s)$. They commute and generate a Klein four-group $V_4 = \{1, \sigma, c, \sigma c\}$ acting on the zero multiset (L93 [TAA-L]). A ghost $\rho = \frac{1}{2} + \delta + i\gamma$ has a free V_4 -orbit, the quartet.
- (b) σc is reflection across $\text{Re } s = \frac{1}{2}$. Correct [THEOREM]. Also c is the mirror $\text{Im } s = 0$, and $\sigma = (\sigma c) \circ c$ is the rotation by π about $\frac{1}{2}$: two perpendicular Euclidean mirrors make a half-turn ([NUMERICAL], error $4 \cdot 10^{-16}$).

- (c) “In p^{-s} terms σc is scaling by $p^{2\delta}$, a boost of rapidity $2\delta \log p$.” **Half right** (C100.3). On a given ghost, $|p^{-\sigma c \rho}|/|p^{-\rho}| = p^{2\delta}$ ([NUMERICAL], $p = 2, 3, 5, 7$, exact). But σc is the involution $u \mapsto -u$ in $u = \log(\sqrt{p} |p^{-s}|)$. Its “scaling factor” depends on the point, and it squares to the identity, so it is a *reflection*, not a boost. A boost is a composite of two reflections (Thm. 2.1).
- (d) $RH \iff \sigma c = \text{id on zeros} \iff \sigma = c \text{ on zeros}$. Correct and tautological [RESTATEMENT]: $1 - \bar{\rho} = \rho \iff \text{Re } \rho = \frac{1}{2}$.
- (e) Over $\mathbb{F}_q$, Rosati $\dagger$ equals c on $\mathbb{Q}(F)$ by positivity. Correct, with the logic made explicit (C100.5). Rosati positivity on $\text{End}^0(A)$ (proved first, via Castelnuovo–Severi/Hodge index, L98) makes $\mathbb{Q}(F)$ a commutative algebra with positive involution. Each factor is then totally real or CM with $\dagger$ = complex conjugation in every embedding [8] [CITED-sec]. Since $F^\dagger = qF^{-1}$, this gives $q/\alpha = \bar{\alpha}$, i.e. $|\alpha| = \sqrt{q}$ [THEOREM]. So $\dagger \circ c = \text{id on } \mathbb{Q}(F)$ is the *output*, and positivity is the input.
- (f) Ghost quartet = boost pair $e^{\pm\delta} \times$ angle pair $e^{\pm i\gamma}$; $(2, 2)$ vs $\mathfrak{so}(2, 2) \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$. The first half is correct ([THEOREM], [NUMERICAL]): $F/\sqrt{q} = BR$ with B a boost on the hyperbolic pair of Q -isotropic eigenplanes and R a rotation, $[B, R] = 0$. The centralizer torus of F in $\mathfrak{sp}_4$ has Killing signature $(1, 1)$ (one split, one compact direction), versus $(0, 2)$ (compact $U(1)^2$) for a true Frobenius. The second half is *only ambient* (C100.6, [ANALOGY]): $\mathfrak{so}(2, 2) \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ is the symmetry algebra of the split form $Q(x, Jy)$ of signature $(2, 2)$. The invariant content is the *type of the Frobenius torus*, not the $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ splitting.
- (g) Under abelian scaling the two partner boosts commute, so no Thomas rotation absorbs them. Correct [THEOREM] (L99 D1). Even inside GSp_4 the ghost’s boost and rotation commute (abelian torus). Absorbing a boost into a rotation needs a non-abelian bracket, and the centralizer of a regular F is abelian.
- (h) Key claim: $Q(x, C\bar{y})$ combines σ (the pairing Q) with c via $C = h(i)$, and positivity $\iff$ the combined form is definite; a compact Hodge circle is exactly a device intertwining σ with c . Correct after one correction (C100.8), made precise in Thm. 4.1. The sesquilinear form $iQ(x, \bar{y})$ (σ followed by c) is Hermitian but *indefinite*, of signature (g, g) ([NUMERICAL]: $(1, 1)$, $(2, 2)$). Inserting C makes it definite. “Exactly” needs the qualifier: J must commute with the dynamics F and be constructed *independently* of the spectrum. Otherwise Thm. 4.1(iii) $\Rightarrow$ (i) builds J from RH (circular).

2. TWO REFLECTIONS MAKE A BOOST

Theorem 2.1. [THEOREM] (i) In $\mathbb{R}^2$, the reflections in lines at angles a, b compose to the rotation by $2(b - a)$. (ii) In $\mathbb{R}^{1,1}$ with $\eta = \text{diag}(1, -1)$, let $r_n(x) = x - 2\frac{\eta(x, n)}{\eta(n, n)}n$. For time-like normals $n_k = (\cosh \phi_k, \sinh \phi_k)$, $r_{n_2}r_{n_1} = B(2(\phi_2 - \phi_1))$, the boost of rapidity $2(\phi_2 - \phi_1)$. The same holds for two space-like normals $(\sinh \phi_k, \cosh \phi_k)$. For one time-like and one space-like normal, $r_{n_2}r_{n_1} = -B(2(\phi_2 - \phi_1))$: determinant 1 but not orthochronous. Equivalently, in 2D a time-like normal is a space-like mirror. (iii) In the s -plane, two parallel mirrors $\text{Re } s = \frac{1}{2}$ and $\text{Re } s = \frac{1}{2} + \delta_0$ compose to $s \mapsto s + 2\delta_0$. On the pair $(u, v) = (p^{-s}, p^{-(1-s)})$ with invariant $uv = p^{-1}$ this acts as $(u, v) \mapsto (\lambda u, \lambda^{-1}v)$, $\lambda = p^{-2\delta_0}$: a Lorentz boost of the light-cone pair, of rapidity $2\delta_0 \log p$.

Proof. (ii) r_n fixes $n^\perp$ and negates n . With $n = (\cosh \phi, \sinh \phi)$ one computes $r_n = \begin{pmatrix} -\cosh 2\phi & \sinh 2\phi \\ -\sinh 2\phi & \cosh 2\phi \end{pmatrix}$, and the product of two such matrices is $B(2(\phi_2 - \phi_1))$. For a space-like normal, $r_n = -r_{n'}$ with n' time-like of the same rapidity, which gives the sign. (iii) Direct. $\square$

[NUMERICAL] (ref1.json): 200 random pairs per case, errors $\leq 3 \cdot 10^{-12}$. Orthochronous for tt and ss, not for ts. Translation error 10^{-16} ; $\lambda\mu = 1$ and rapidity $= -2\delta_0 \log p$ to 10^{-15} for $p = 2, 3, 5, 7$.

Reading. J.W.’s slogan is right in the precise form of (ii)–(iii). “Two reflections make a boost” needs *two* mirrors of the same causal type in a Lorentzian plane. The pair of zero-partners $\{\sigma, c\}$

lives in the Euclidean s -plane, with perpendicular mirrors, and makes the rotation σ . The ghost displacement $\rho \mapsto \sigma c \rho$ equals, on that one point, the boost $s \mapsto s - 2\delta$, which acts on $(p^{-s}, p^{-(1-s)})$ with rapidity $2\delta \log p$. This is exactly the L98 one-boost picture [TAA-L].

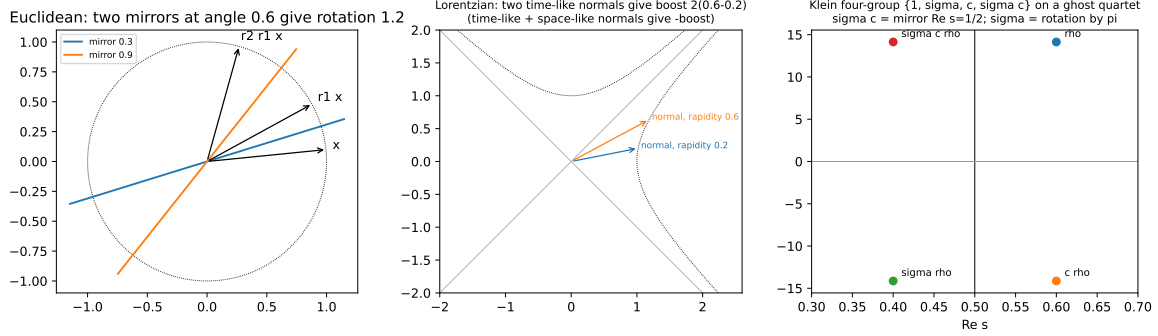


FIGURE 1. Left: Euclidean mirrors make a rotation. Middle: Lorentzian mirrors with time-like normals make a boost. Right: the Klein four-group $\{1, \sigma, c, \sigma c\}$ on a ghost quartet.

3. MIRRORS, HALF-TURNS, SADDLES AND VORTICES (T430)

Script `lemma100_halfturn.py` (`halfturn.json`), Fig. 2.

Proposition 3.1. [THEOREM] Let $c(s) = \bar{s}$ and $R(s) = 1 - \bar{s}$ (the mirror $\text{Re } s = \frac{1}{2}$). Then $\sigma(s) = 1 - s = R \circ c = c \circ R$ is the half-turn about $s = \frac{1}{2}$, a point reflection and not a line reflection, and $\sigma \circ c = R$ ([NUMERICAL]: exact). In the multiplicative variable p^{-s} , R becomes $u \mapsto -u$ with $u = \log(\sqrt{p} |p^{-s}|)$. On a point with $\text{Re } s = \frac{1}{2} + \delta$ it multiplies $|p^{-s}|$ by $p^{2\delta}$, but the factor depends on δ ($p = 3$: 1.246, 1.933, 1.116 for $\delta = 0.1, 0.3, 0.05$). So R is a boost only pointwise; a uniform boost needs two parallel mirrors (Thm. 2.1(iii)).

Proposition 3.2 (PT, Cartan–Dieudonné, Thomas–Wigner). [THEOREM] (i) In $\mathbb{R}^{1,1}$, $T = \text{diag}(-1, 1)$, $P = \text{diag}(1, -1)$, and $PT = -1$ has $\det = 1$. It is not in the identity component $SO^+(1, 1)$: it maps the future cone to the past $((1, 0.3) \mapsto (-1, -0.3))$. In $\mathbb{R}^2$, -1 is the half-turn, which is in $SO(2)$. (ii) (Cartan–Dieudonné) Every isometry of a nondegenerate n -dimensional real quadratic space is a product of at most n reflections. Even products have $\det = 1$: rotations in $O(2)$, $\pm$ boosts in $O(1, 1)$ ([NUMERICAL]: 400 random products of 2 and 4 reflections, all $\pm$ boosts). In $O(1, 1)$ no number of reflections gives a rotation. (iii) In $O(1, 2)$ take $B_x = r_{n_2} r_{n_1}$ and $B_y = r_{n_4} r_{n_3}$ (time-like normals in the tx - resp. ty -planes, rapidity $\eta = 1$). The 4-reflection product $B_y B_x$ has, in its polar decomposition, a rotation by 0.420784. This equals the Wigner angle $\arccos \frac{2\gamma}{1+\gamma^2}$ with $\gamma = \cosh 1$ ($2 \cdot 10^{-16}$; L99 [TAA-L]). Rotations from boosts need at least two space dimensions. (iv) Linear flows: $\dot{x} = Kx$ (K the boost generator, eigenvalues ± 1) is a saddle with hyperbolic orbits $t^2 - x^2 = \text{const}$. $\dot{x} = Wx$ (W the rotation generator, eigenvalues $\pm i$) is a centre/vortex with circles $t^2 + x^2 = \text{const}$. This is the hyperbolic-versus-trigonometric dichotomy of L98 [TAA-L].

Proposition 3.3 (no vortex at the present point). [THEOREM] Let Z be the multiset of nontrivial zeros. Its symmetries among the isometries of $\mathbb{C}$ include $V_4 = \{1, \sigma, c, R\}$. The rotations about $s = \frac{1}{2}$ that preserve Z form a finite group containing σ ; in particular no nontrivial continuous rotation group about $\frac{1}{2}$ preserves Z . Proof: Z is discrete, nonempty and avoids $\frac{1}{2}$ ($\zeta(\frac{1}{2}) = -1.46035$), so the closure of the rotation group would put a whole circle of zeros inside Z . [NUMERICAL]: rotating the first 20 zeros by $\theta = 10^{-3}, 10^{-2}, 0.1, \pi/2$ leaves them at distances 0.0141, 0.141, 1.41, 20.0 from the first 100 zeros and their conjugates; for $\theta = \pi$ the distance is $2 \cdot 10^{-15}$.

Reading ([RESTATEMENT]/[ANALOGY]). J.W.'s picture is consistent. $s = \frac{1}{2}$ is the “present point”. σ is a discrete half-turn about it (Euclidean -1), and its Lorentzian analogue $PT = -1$ swaps the cones instead of rotating. A genuine vortex, i.e. a continuous compact circle with closed orbits, cannot be a symmetry of the zero set about $\frac{1}{2}$ (Prop. 3.3). It must act *elsewhere*, on a space that carries the zeros as spectrum, as $h(e^{i\theta})$ acts on H^1 in Thm. 4.1. That is exactly the compact Hodge circle sought in §5.

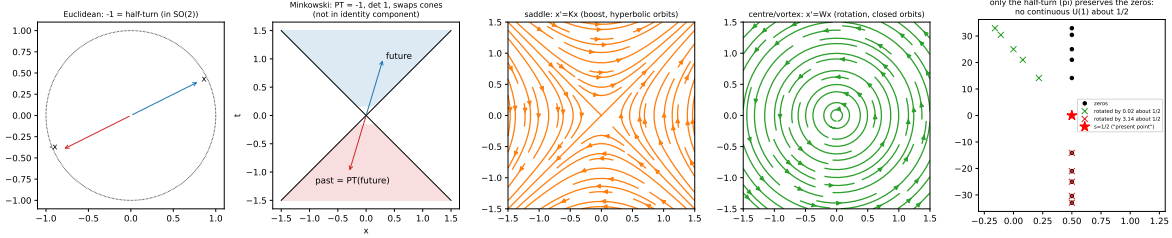


FIGURE 2. Euclidean half-turn; Minkowski $PT = -1$ swapping cones; saddle (boost) vs. centre (rotation) flows; rotations of zeros about $\frac{1}{2}$, where only $\theta = \pi$ preserves them.

4. (A) THE INTERTWINING THEOREM

Theorem 4.1 (Hodge circle $\iff \sigma = c$). [THEOREM] Let (V, Q) be a real symplectic space of dimension $2g$, and $F \in \mathrm{GSp}(V, Q)$ with $Q(Fx, Fy) = qQ(x, y)$, semisimple, with no real eigenvalue. Write $F^\dagger = qF^{-1}$ for the Q -adjoint (Rosati/duality partner, “ σ ”). The following are equivalent:

- (i) there is $J \in \mathrm{Sp}(V, Q)$ with $J^2 = -1$, $JF = FJ$, and $g_J(x, y) := Q(x, Jy)$ positive definite (a Hodge circle $h(e^{i\theta}) = \cos \theta + J \sin \theta$ polarized by Q and commuting with F);
- (ii) for some positive definite g with $F^\dagger$ the g -adjoint of F , one has $F^*F = q$;
- (iii) every eigenvalue satisfies $q/\alpha = \bar{\alpha}$ (“ $\sigma = c$ ”), i.e. $|\alpha| = \sqrt{q}$;
- (iv) the centralizer of F in $\mathrm{Sp}(V, Q)$ is compact (the Frobenius torus is anisotropic over $\mathbb{R}$).

In that case the Hermitian form $H(x, y) = Q(x, J\bar{y})$ on $V_{\mathbb{C}}$ is positive definite, while $iQ(x, \bar{y})$ has signature (g, g) .

Proof. (i) $\implies$ (ii): $g_J(Fx, y) = Q(Fx, Jy) = Q(x, F^\dagger Jy) = Q(x, JF^\dagger y) = g_J(x, F^\dagger y)$, since $F^\dagger = qF^{-1}$ commutes with J . So $F^* = F^\dagger$ and $F^*F = q$. (ii) $\implies$ (iii): $F/\sqrt{q}$ is g -orthogonal, so $|\alpha| = \sqrt{q}$ and $q/\alpha = \bar{\alpha}$. (iii) $\implies$ (i): the eigenvalues pair as $\{\alpha, \bar{\alpha}\} = \{\alpha, q/\alpha\}$, so $V = \bigoplus V_{\{\alpha, \bar{\alpha}\}}$ (real 2-planes, or larger isotypic blocks). Q pairs V_α only with $V_{q/\alpha}$, so each block is Q -nondegenerate and Q -orthogonal to the others. On a 2-plane, $J = \pm(2F - (\alpha + \bar{\alpha})/|\alpha - \bar{\alpha}|)$ satisfies $J^2 = -1$ and $J \in \mathrm{Sp}$, and one sign makes $Q(x, Jx) > 0$. Larger blocks are a direct sum of such planes after a Q -orthogonal splitting of F restricted to the block. (iii) $\iff$ (iv): the centralizer of a semisimple F is the group of units of the commutant intersected with Sp . On each pair $V_\alpha \oplus V_{q/\alpha}$ it is $U(1)$ -type if $q/\alpha = \bar{\alpha}$, and contains $\mathbb{R}_{>0}$ (scaling V_α by t and $V_{q/\alpha}$ by t^{-1}) otherwise. Hermitian statement: $iQ(x, \bar{x})$ is $\pm$ on the $\pm i$ eigenspaces of J , and $C = J$ corrects the sign. $\square$

Ghost mechanism [THEOREM]. If $q/\alpha \neq \bar{\alpha}$, the real plane $V_{\{\alpha, \bar{\alpha}\}}$ is Q -isotropic (its pairings need $\alpha\beta = q$). Every J commuting with F preserves it, so $Q(x, Jx) = 0$ there, and the best one can get is the hyperbolic signature $(2, 2)$ on $V_{\{\alpha, \bar{\alpha}\}} \oplus V_{\{q/\alpha, q/\bar{\alpha}\}}$.

Model (<code>intertwine.json</code>)	$\max q/\alpha - \bar{\alpha} $	J 's in Sp; best $Q(x, Jy)$	$\ F^\dagger - F_J^*\ $	torus Killing
$E/\mathbb{F}_{101}, a = -3$	$2 \cdot 10^{-15}$	2; (2, 0)	$3 \cdot 10^{-16}$	(0, 1)
fake $E, a = 25$	14.9	none ($\mathbb{R}[F] \cong \mathbb{R}^2$)	—	—
genus 2, $y^2 = x^5 + x + 1$	$7 \cdot 10^{-13}$	4; (4, 0) once, (2, 2), (2, 2), (0, 4)	$8 \cdot 10^{-13}$	(0, 2)
ghost quartet $\delta = 0.2$	4.05	2 in Sp; all (2, 2)	—	(1, 1)
CM $E \times E, y^2 = x^3 - x, a = -2$	$4 \cdot 10^{-15}$	$J_E \oplus J_E$: (4, 0)	$4 \cdot 10^{-17}$	—
CM ghost ($a = 25$ doubled)	—	J exists, $J \in \text{Sp}; (2, 2)$	—	—

The genus-2 Weil polynomial was computed by point counts: $N_1 = 104$ and $N_2 = 10250$ (one point at infinity). Models were conjugated by a random symplectic matrix before testing. For the true models, $iQ(x, \bar{y})$ has signature (1, 1) resp. (2, 2), and $H(x, y) = Q(x, J\bar{y})$ is definite (Fig. 3).

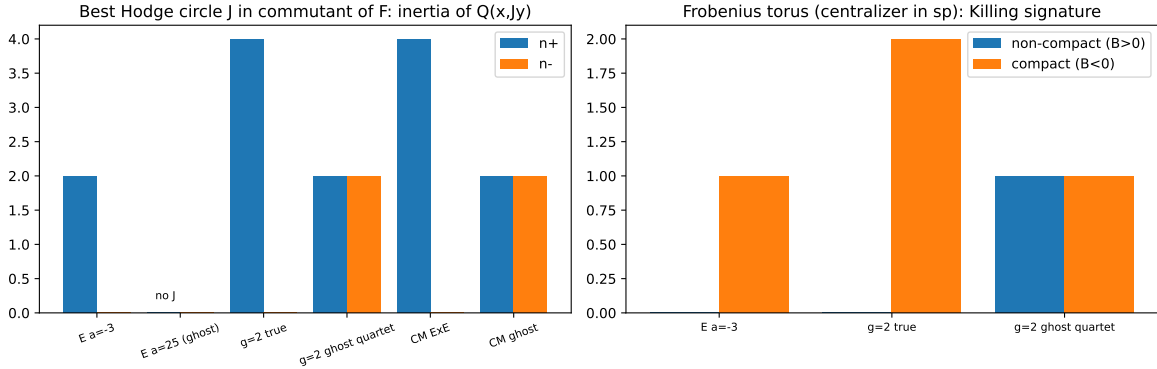


FIGURE 3. Left: best polarizing J in the commutant of F . Right: Killing signature of the Frobenius torus; a ghost has a split (boost) direction.

Where the circle comes from. Over $\mathbb{C}$ (Hodge theory), $J = h(i)$ is *input*: it comes from the complex structure of $X(\mathbb{C})$ and is independent of any Frobenius. Over $\mathbb{F}_q$ it is an *output*: Weil's argument proves Rosati positivity (Hodge index on $C \times C$) first, and (iii), hence (i), follow. Over $\mathbb{Q}$ we would need J as input, constructed without knowing the zeros [RESTATEMENT].

5. (B) CANDIDATES FOR THE CIRCLE OVER $\mathbb{Q}$

5.1. B1. The Weil group $W_{\mathbb{R}}$.

Proposition 5.1. [THEOREM] (from [7] (1.4.3), (3.1), (3.3.1) [CITED]). (i) $[W_{\mathbb{R}}, W_{\mathbb{R}}] = U(1)$, since $jzj^{-1}z^{-1} = \bar{z}/z$. Hence every character of $W_{\mathbb{R}}$ is trivial on $U(1)$, and characters factor through $\mathbb{R}^\times$: $L = \Gamma_{\mathbb{R}}(s + \epsilon)$ with $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. (ii) The irreducible 2-dimensional representations are $\text{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}} \chi$ with $\chi \neq \chi \circ c$, and $L = \Gamma_{\mathbb{C}}(s + \cdot)$ with $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s + 1)$. (iii) For $\chi_k(z) = (z/|z|)^k$, the element $i \in U(1)$ acts by $\text{diag}(i^k, (-i)^k)$. So a J with $J^2 = -1$ from the circle exists exactly for odd k ($k = |p - q|$, odd weight), and the circle acts nontrivially only for $k \neq 0$.

[NUMERICAL] (`weilgroup.json`): $j^2 = -1$ and $jzj^{-1} = \bar{z}$ exactly. The 400 commutators have $|\cdot| = 1$ and fill $U(1)$ (max angular gap 0.125). The abelianization is a homomorphism (error $7 \cdot 10^{-16}$). $\rho(i)$ is $\text{diag}(1, 1)$, $(i, -i)$, $(-1, -1)$, $(-i, i)$ for $k = 0, 1, 2, 3$. $\Gamma_{\mathbb{C}} = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s + 1)$ to 10^{-18} , and the local Tate integral $\int e^{-\pi x^2} |x|^s d^\times x = \Gamma_{\mathbb{R}}(s)$ holds at $s = 2.3$.

Verdict (B1). ζ is the L -function of the motive $\mathbb{Q}(0)$, of Hodge type (0, 0), with archimedean factor $\Gamma_{\mathbb{R}}(s)$ coming from a character of $W_{\mathbb{R}}$. Its circle acts by 1: $J = 1$, $J^2 = +1$. The $U(1)$ in $W_{\mathbb{R}}$ is “two reflections make a rotation” in its purest form (commutators of j with $\mathbb{C}^\times$), but it

acts on Hodge pieces with $p \neq q$, and ζ has none. Nothing carrying the zeros is acted on [THEOREM]/[RESTATEMENT]. A J with $J^2 = -1$ is available at ∞ for odd-weight motives (e.g. $H^1(E)$), but it acts on the 2-dimensional Betti space, not on the zeros of $L(E, s)$ [ANALOGY].

5.2. B2. Deninger: F_∞ , real Hodge structures, foliated dynamics. [CITED] [2].

- The archimedean factor “depends only on the real Hodge realization M_B over $\mathbb{R}$ ”, as $L_\infty(M, s) = \prod_n \Gamma(s - n)^{d_n}$ with $d_n = \dim \mathrm{Gr}_V^n M_B$ (§2). For $p = \infty$, integrality means that the real Hodge structure is split (§3).
- On the conjectural $H^1(\overline{\mathrm{Spec} \mathbb{Z}}, \mathbb{R})$, the generator Θ of the flow has spectrum equal to the zeros. If there were a Hodge $\star$ on H^1 commuting with Θ , with $\langle f, f' \rangle = \mathrm{tr}(f \cup \star f')$ positive definite, then $\Theta = \frac{1}{2} + A$ with A skew, “hence the Riemann conjecture would follow” (p. 169).
- In the dynamical picture the closed orbits correspond to primes with length $\log p$, and “there should be a stationary point x_∞ corresponding to the infinite prime” (p. 181). The local factor at finite p uses Fourier series on $\mathbb{R}/(\log p)\mathbb{Z}$ (a circle), while at ∞ it uses $\mathbb{R}[\exp(-2y)]$, with no circle (p. 167–168).

Verdict (B2). Deninger’s $\star$ is precisely the Hodge circle J.W. wants: it intertwines $\cup$ (duality, σ) with positivity, the analogue of Thm. 4.1. But it is a hypothesis of the formalism, and in his picture the infinite place is a *fixed point* of the flow, not a circle. The compact circles are the closed orbits at the finite primes (lengths $\log p$), which is where L98’s boosts live. For Riemannian foliations a leafwise Hodge $\star$ exists by Álvarez López–Kordyukov (p. 180, cited there [CITED-sec]). No foliated system realizing ζ is known [OPEN].

5.3. B3. Base change to $\mathbb{Q}(i)$. [NUMERICAL] (cm.json). $\zeta_{\mathbb{Q}(i)}(3)$ computed as $\sum_n (\sum_{d|n} \chi_{-4}(d)) n^{-3}$ ($n \leq 2 \cdot 10^5$) equals $\zeta(3)L(3, \chi_{-4}) = 1.1647284039$ (10^{-11}). We computed 25 zeros of $L(s, \chi_{-4})$ with $t < 60$ (first 6.020949; $|L| < 10^{-16}$ there), alongside 13 zeros of ζ . The complex place contributes $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$, i.e. the two real factors of ζ and of $L(\chi_{-4})$ (Fig. 4).

- The $U(1)$ at the complex place acts on the idele class group of $\mathbb{Q}(i)$. Its characters $(z/|z|)^{4k}$ index Hecke L -functions, and $\zeta_{\mathbb{Q}(i)}$ is the $k = 0$ member, on which the circle acts trivially ($J = 1$) [THEOREM]. The circle moves between L -functions; it does not act on the zeros of a fixed one.
- *Ghost sensitivity* (Toeplitz zero-side kernel $\sum_\gamma e^{-\gamma^2/2T^2} \cos \gamma \Delta$, 40-point grids). A ghost $14.13 + i\epsilon$ replacing ζ ’s first zero gives $n_- = 2$ for ζ in 9/9 settings. For the $\mathbb{Q}(i)$ zero set it gives $n_- = 2$ in 8/9 settings. The extra $L(\chi_{-4})$ zeros add positive mass, which masks the ghost only for the shortest span (4.7, $\epsilon = 0.5$). This is a diagnostic of finite spans, not the Weil functional.

Verdict (B3) [RESTATEMENT]. Positivity for $\zeta_{\mathbb{Q}(i)}$ is GRH for ζ and $L(\chi_{-4})$ together; it implies RH, and the complex place adds no structure on ζ ’s zeros. This is circular.

5.4. B4. Arakelov. For an arithmetic surface $\mathcal{X} \rightarrow \mathrm{Spec} \mathbb{Z}$, the fibre at ∞ is the compact Riemann surface $\mathcal{X}(\mathbb{C})$ with $H^1(\mathcal{X}(\mathbb{C}))$ and its Hodge circle. The Faltings–Hriljac Hodge index theorem uses it, relating the intersection pairing to minus the Néron–Tate height ([3, 4] [CITED-sec], L90 [TAA-L]). For $\mathrm{Spec} \mathbb{Z}$ itself the fibre at ∞ is $\mathrm{Spec} \mathbb{C}$ with F_∞ (Deninger: a stationary point). $H^1(\mathrm{Spec} \mathbb{C}) = 0$, so there is nothing for a circle to act on, and the zeros (Deninger’s H^1) do not live at ∞ [THEOREM]/[RESTATEMENT]. The positivity that uses the circle (Faltings–Hriljac) concerns H^1 of a curve over $\mathbb{Q}$, i.e. $L(\mathrm{Jac}, s)$ -type objects, not ζ [ANALOGY].

5.5. B5. Connes–Consani at the archimedean place. [CITED] [1] (abstract, introduction, §§1–4 opened). Weil’s positivity restricted to test functions supported in $(\frac{1}{2}, 2)$, where no prime enters, holds (proved earlier; their reference [34], not opened here). Connes–Consani give an operator reason: the scaling action compressed to Sonin’s space is a positive trace, and they control the difference from W_∞ . A pair of projections $\mathcal{P}_1$ (cutoff) and $\hat{\mathcal{P}}_1$ (Fourier cutoff) defines a unitary representation of the

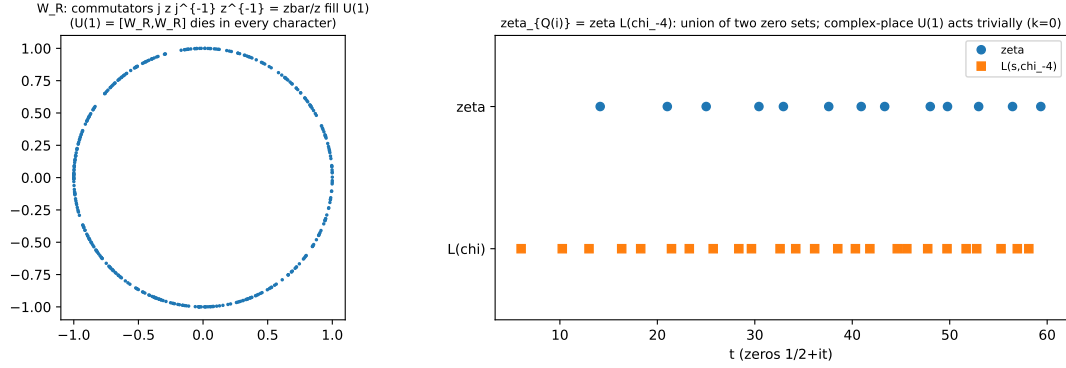


FIGURE 4. Left: the commutators $\bar{z}/z$ of $W_{\mathbb{R}}$ fill $U(1)$. Right: zeros of ζ and $L(s, \chi_{-4})$ (together, $\zeta_{\mathbb{Q}(i)}$).

infinite dihedral group generated by the two reflections $\mathcal{U} = 1 - 2\mathcal{P}_1$, $\mathcal{V} = 1 - 2\widehat{\mathcal{P}}_1$. Its irreducible pieces are 2-dimensional, labelled by an angle with $\cos \alpha(n) = |\lambda(n)|$ (even prolate eigenvalues, $c = 2\pi$). Sonin's space has angle 0 (Lemma 2, Prop. 5 there). [NUMERICAL] (prolate.json): $\lambda(n) = 0.999971, -0.979485, 0.524086, -0.0589766, 0.00273233, -7.6291 \cdot 10^{-5}$, matching their list. $\sum \lambda(n)^2 = 2.2374848349$ equals $2(\text{Si}(4\pi)/4\pi + 1)$ (10^{-15}). In finite dimensions, $\mathcal{UV}$ rotates by exactly 2α on each block (10^{-15}). The angles $2\alpha_n = 0.015, 0.406, 2.04, 3.02, \dots \rightarrow \pi$ are set by bandwidth, not by the zeros (Fig. 5).

Verdict (B5). This is the only candidate with a genuine compact “two reflections make a rotation” structure at ∞ acting on the space where Weil positivity lives. It yields archimedean positivity (support in $(\frac{1}{2}, 2)$), but its circle angles are prolate data, and extending it to supports containing primes is open (their programme) [OPEN]. It is not circular and not a Hodge circle on the zeros.

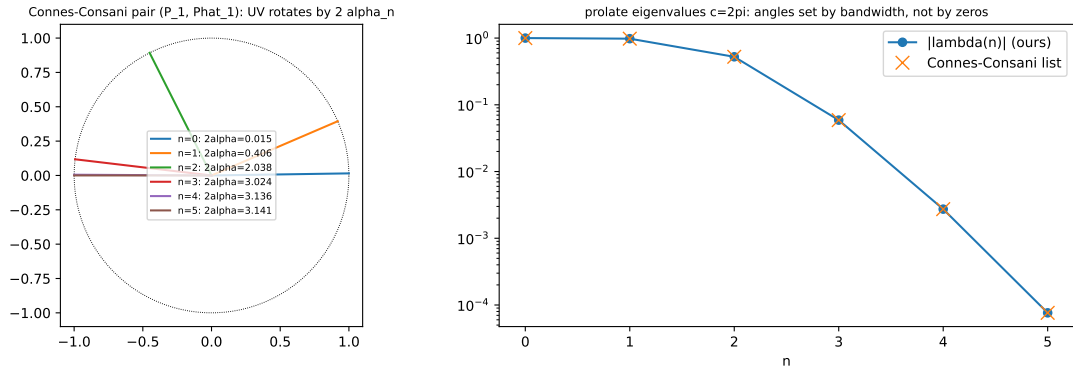


FIGURE 5. Connes–Consani dihedral pair: rotation angles $2\alpha_n$ (left), prolate eigenvalues ours vs. theirs (right).

6. (C) SCORECARD

Candidate	$J^2 = -1?$	acts on zeros?	polarization?	RH-free?	circular / why not
$U(1) \subset W_{\mathbb{R}}$	only k odd	no (ζ : $k = 0$, $J = 1$)	n/a	yes	dies in characters
Deninger $\star$	assumed	yes (on H^1)	assumed	no (implies RH)	hypothesis; $\infty =$ fixed point
$\mathbb{Q}(i)$ complex place	no on $k = 0$	no	n/a	yes	positivity for $\zeta_K \Rightarrow$ RH
Arakelov $X(\mathbb{C})$	yes	no ($H^1(\text{Spec } \mathbb{C}) = 0$)	yes	yes	wrong object (curves)
Connes–Consani pair	rotations, angles $2\alpha_n$	no (prolate spectrum)	positive trace	yes	only ∞ ; primes open
Thm. 4.1 build	yes	yes	yes	no	built from $\sigma = c$, i.e. RH

Conclusion [RESTATEMENT]. No candidate supplies a J with $J^2 = -1$ acting on a space carrying the zeros, commuting with a polarization, and constructed independently of RH. The obstruction is structural. The archimedean place of $\mathbb{Q}$ is real: its Weil group’s circle is a commutator and acts trivially on the $(0,0)$ motive, and in Deninger’s picture ∞ is a fixed point. The circles that do exist are either at finite primes (closed orbits, angles $\gamma \log p$, L98) or prolate (Connes–Consani).

7. CORRECTIONS TO THE CHAT CLAIMS

C100.1–2 (σ , c ; σc mirror):

Correct. Add: σ is a half-turn, and V_4 is abelian.

C100.3 (σc is a boost of rapidity $2\delta \log p$):

Only pointwise. σc is a reflection (an involution with a point-dependent factor). A boost is the composite of two parallel mirrors (Thm. 2.1(iii)).

C100.4 (RH $\iff \sigma = c$):

Correct, tautological.

C100.5 (Rosati = c over $\mathbb{F}_q$):

Correct, as a consequence of positivity (input) via the CM/totally-real classification. It is not an independent route.

C100.6 $((2,2)$ and $\mathfrak{so}(2,2) \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$):

Ambient only. The invariant is the Frobenius torus type: Killing $(1,1)$ for a ghost, $(0,2)$ when true.

C100.7 (commuting boosts, no Thomas rotation):

Correct.

C100.8 (Hodge circle = intertwiner of σ and c):

Correct as Thm. 4.1, provided J commutes with F and is built independently of the spectrum. $iQ(x, \bar{y})$ alone is indefinite (g, g) ; with C inserted, $Q(x, C\bar{y})$ is definite. (Sign: $Q(Cx, \bar{y})$ is then *negative* definite; normalizations vary.)

C100.9 ($W_{\mathbb{R}}$ ’s $U(1)$ for ζ):

It does not act: $U(1) = [W_{\mathbb{R}}, W_{\mathbb{R}}]$ is killed by every character, including the one giving $\Gamma_{\mathbb{R}}(s)$.

C100.10 (t430, J.W.’s self-correction: σ is a half-turn):

Confirmed: $\sigma = R \circ c$ is the point reflection about $\frac{1}{2}$, and $\sigma \circ c = R$ is the line reflection (Prop. 3.1). “Boost in p^{-s} ” holds only pointwise (C100.3).

C100.11 ($PT = -1$):

Correct: $\det = 1$, not in $SO^+(1,1)$, swaps the cones. In the Euclidean plane -1 is the half-turn.

C100.12 (Cartan–Dieudonné):

Correct as stated: even products give rotations in $O(2)$ and $\pm$ boosts in $O(1,1)$. Four reflections give a rotation only with ≥ 2 space dimensions (Thomas–Wigner, verified in $O(1,2)$).

C100.13 (saddle vs centre):

Correct: real versus imaginary generator eigenvalues, i.e. hyperbolic versus trigonometric.

C100.14 (no continuous $U(1)$ about $\frac{1}{2}$):

Correct and proved (Prop. 3.3). The vortex must live on a space carrying the zeros, not in the s -plane.

8. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	Deninger $\star$ in a foliated toy model: a Riemannian foliation with a stationary point “at ∞ ” and closed orbits of lengths $\log p$; test whether a leafwise $\star$ commuting with the flow exists (Álvarez López–Kordyukov), and what it forces on the spectrum.	0.9	0.05	0.56
2	Connes–Consani beyond $(\frac{1}{2}, 2)$: reproduce the Sonin-trace inequality numerically for support $(\frac{1}{3}, 3)$ (primes 2 enters) and track the dihedral angles; ghost-test the remainder.	0.6	0.35	0.50
3	Frobenius-torus type as a certified ghost detector on L98 finite models (Killing signature of the centralizer; Thm. 4.1(iv)).	0.3	0.8	0.50
4	Arakelov: a “ $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ ” surrogate inheriting a Hodge circle from Faltings–Hriljac (L90 Q3).	0.8	0.03	0.49
5	Odd-weight motives: for $L(E, s)$ the circle at ∞ exists ($\text{Ind } \chi_1$); test numerically whether it constrains anything beyond the Γ -factor (expected: no).	0.4	0.3	0.36

9. QUESTIONS FOR J.W.

- Q1. Do you accept Thm. 4.1 as the precise form of “a compact Hodge circle intertwines σ with c ”, including the requirement that J be built independently of the spectrum?
- Q2. By Thm. 2.1 and Prop. 3.1, $R = \sigma c$ and c are perpendicular Euclidean mirrors whose product is the half-turn σ , and a uniform boost needs R together with a *parallel* shifted mirror. Is that shifted mirror (a translate of the critical line) the second reflection in your picture?
- Q3. Since $U(1) = [W_{\mathbb{R}}, W_{\mathbb{R}}]$ acts trivially on the $(0,0)$ motive, would you accept a circle that acts on *families* of L -functions (Hecke characters) rather than on the zeros of one?
- Q4. Deninger places ∞ at a fixed point of the flow and puts the circles at the primes. Is your Hodge circle meant at ∞ or along the closed orbits?
- Q5. L101: direction 1 (Deninger $\star$ toy model) or direction 2 (Connes–Consani numerics with one prime)?

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TAA LEMMA 101: WHERE J COMES FROM — DEGREE, DISCRIMINANT, THE $\frac{1}{4}$ BOUND, AND THE SPACELIKE COMPLEMENT

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma101_*.py`, data `num/1101/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma101/verified_refs.md`.

ABSTRACT. L100 found that a Hodge circle J exists iff $\sigma = c$ on Frobenius eigenvalues. Here we ask where J comes from in the cases where it exists. Over $\mathbb{F}_q$ the order is fixed: object $\rightarrow$ endomorphisms $\rightarrow$ counting quadratic form (degree) $\rightarrow$ negative discriminant $\rightarrow J = (2F - a)/\sqrt{4q - a^2}$.

Results. (1) J is a polynomial in F (Prop. 2.1). In genus g it is a direct sum of blocks, with 2^g sign choices; positivity selects exactly one, the $h(i)$ of L100 Thm. 4.1. For the fake $a = 25$, $(2F - a)^2/(a^2 - 4q) = +1$: one gets a split j , not an i . (2) The degree form $m^2 + amn + qn^2$ counts points ($\deg(1 - F^k) = 104, 10400, 1029704$ for $y^2 = x^3 - x/\mathbb{F}_{101}$). Its positivity is Hasse (Thm. 3.1, Silverman V.1.1). For the fake it takes negative “point counts” ($\deg(-30 + 2F) = -196$). The Rosati trace form is $(2, 0)/(4, 0)$ true and $(1, 1)/(2, 2)$ fake. (3) Over $\mathbb{Q}$: $\lambda = \rho(1 - \rho)$ is real and $\geq \frac{1}{4}$ iff $\operatorname{Re} \rho = \frac{1}{2}$ (Prop. 4.1). Formally $J = (2D - 1)/\sqrt{4\lambda - 1}$ squares to -1 for every ρ , so the content is reality: a real D with $D^2 - D + \lambda = 0$, λ real, forces $\sigma = c$. Correction: $\langle \Delta f, f \rangle = \|df\|^2$ gives $\Delta \geq 0$, not $\geq \frac{1}{4}$. On the modular surface the zeros are resonances with complex λ ($50.14 + 3.53i$ for ρ_1). Positivity provable on short support ($< \log 2$) is ghost-blind (ghost perturbation $\leq 10^{-3}$). (4) Spacelike complement (Thm. 5.1). On $\langle H, V, \Delta, \Gamma_F \rangle \subset \operatorname{NS}(C \times C)$ the primitive block has Gram matrix $\begin{pmatrix} -2g & -a \\ -a & -2gq \end{pmatrix}$. It is negative definite iff $|a| < 2g\sqrt{q}$. Signature $(1, 3)$ true, $(2, 2)$ ghost, for genus 1, genus 2 and $E \times E$. (5) Reflections and orientations (§§6–7). The Klein four-group $\{1, R_x, R_y, -1\}$; $O(1, 1)$ has no $J^2 = -1$; the trace trichotomy (Hasse $\iff F/\sqrt{q}$ elliptic); $\langle J, R_x, R_y \rangle = D_4$. J inverts boosts, and $J = SR_x$ with S the mirror in a null line. Two orthogonalities: rotation J (oriented, $\omega(x, Jy)$ definite) and light-cone mirror S (unoriented, $\omega(x, Sy)$ of signature $(1, 1)$, $\langle x, Sx \rangle_M \equiv 0$, $S^T MS = -M$). Only J can carry positivity.

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GLOSSARY

Frobenius F ; trace a :

$F(x, y) = (x^q, y^q)$ on $E/\mathbb{F}_q$, with $\#E(\mathbb{F}_q) = q + 1 - a$ and $F^2 - aF + q = 0$ on H^1 .

Degree:

$\deg \psi = \# \ker \psi$ for separable ψ . It is a positive definite quadratic form on $\text{End}(E)$ [7] [CITED-sec].

Rosati involution $\dagger$:

The adjoint for the polarization: $F^\dagger = qF^{-1}$, with $\text{Tr}(\phi\phi^\dagger) > 0$ (Mumford §21 [BIB], [8] [CITED-sec]).

Néron–Severi / Hodge index:

$\text{NS}(S)$ is the divisor classes on a surface modulo numerical equivalence. If $H^2 > 0$, the intersection form is negative definite on $H^\perp$, i.e. signature $(1, n - 1)$ [6] [BIB] (V.1.9; Ex. 1.9–1.10 for Castelnuovo–Severi and Weil’s proof).

Laplacian; resonance:

$\Delta = -y^2(\partial_x^2 + \partial_y^2)$ on $\Gamma \backslash \mathbb{H}$, $\lambda = s(1 - s)$. A resonance is a pole of the scattering matrix off the spectrum (L99 [TAA-L]).

Orientation; split-complex j :

An orientation of $\mathbb{R}^2$ is a choice between J and $-J$ (axis $\pm z$). $\mathbb{R}[S]$ with $S^2 = +1$ is the split-complex numbers, with indefinite norm $x^2 - y^2$.

1. THE CHAT CLAIMS T431–T437 (AND T439)

Each claim is checked in the section indicated. Corrections are collected at the end.

- (a) $J = (2F - a)/\sqrt{4q - a^2}$, $J^2 = -1$, $JF = FJ$ for $a = -3$, $q = 101$: correct (§2).
- (b) Fake $a = 25$: $4q - a^2 = -221$, so there is no real J : correct. The same formula gives a split j with $j^2 = +1$ (§2).
- (c) $\deg(m + nF) = m^2 + amn + qn^2 \geq 0 \Rightarrow a^2 \leq 4q$: correct (§3). The citation is Silverman, Thm. V.1.1 (Hasse) with Lemma V.1.2 (the quadratic-form inequality).
- (d) The chain object $\rightarrow \text{End} \rightarrow \text{counting form} \rightarrow \text{discriminant} \rightarrow J$: correct (§3).
- (e) $\mathbb{Q}$: $(2\rho - 1)^2 = 1 - 4\lambda$; λ real and $\geq \frac{1}{4} \iff \text{Re } \rho = \frac{1}{2}$; $4\lambda - 1 = 799.16$ for ρ_1 and $799.00 - 22.62i$ for the ghost: correct (§4).
- (f) “ $\Delta \geq \frac{1}{4}$ proved by counting/squaring, $\langle \Delta f, f \rangle = \|df\|^2$ ”: **incomplete** (C101.5, §4).
- (g) Spacelike complement: Weil’s class lies in $H^\perp$ and is negative definite there $\iff$ Hasse/RH; a ghost is a spacelike direction turned timelike, giving $(2, 2)$. Correct, with the genus qualifier C101.7 (§5).
- (h) Four reflections R_x, R_y, R_x, R_y form a Klein four-group; $O(1, 1)$ has no $J^2 = -1$; trace trichotomy; the relations among J, S, R_x, R_y : correct, with one correction about S (C101.11, §6).
- (i) t439, orientation and the two orthogonalities: correct up to conventions (§7).

2. (A) J AS A POLYNOMIAL IN FROBENIUS

Proposition 2.1. [THEOREM] (i) If $F^2 - aF + q = 0$ on a real 2-dimensional V and $4q - a^2 > 0$, then $J = (2F - a)/\sqrt{4q - a^2}$ satisfies $J^2 = -1$ and $JF = FJ$. Indeed $(2F - a)^2 = 4(F^2 - aF) + a^2 = a^2 - 4q$. (ii) If $4q - a^2 < 0$, then $j = (2F - a)/\sqrt{a^2 - 4q}$ satisfies $j^2 = +1$. No real J commuting with F has $J^2 = -1$, because the commutant $\mathbb{R}[F] \cong \mathbb{R} \times \mathbb{R}$ contains none. (iii) Genus g : if F is semisimple with eigenvalue pairs $\alpha_k, \bar{\alpha}_k$, $|\alpha_k|^2 = q$, $t_k = 2\text{Re } \alpha_k$, then $J = \bigoplus_k \epsilon_k(2F_k - t_k)/\sqrt{4q - t_k^2}$ with $\epsilon_k = \pm 1$. These 2^g choices are all the complex structures in $\mathbb{R}[F]$. Exactly one makes $Q(x, Jy)$ positive definite, and it is the $h(i)$ of L100 Thm. 4.1 [TAA-L].

[NUMERICAL] (J. json). For $a = -3$: $\|J^2 + 1\| = 10^{-16}$, $\|[J, F]\| = 4 \cdot 10^{-15}$, and $Q(x, Jy)$ is $(2, 0)$ for $+J$, $(0, 2)$ for $-J$. For $a = 25$: $(2F - a)^2/(a^2 - 4q) = I$ exactly. For genus 2 ($P =$

$x^4 + 2x^3 + 26x^2 + 202x + 10201$; $t_k = 12.304, -14.304$; $4q - t_k^2 = 252.6, 199.4$), the four sign choices give $(4, 0)$, $(2, 2)$, $(2, 2)$, $(0, 4)$.

3. (B) DEGREE AS COUNTING POSITIVITY

Theorem 3.1 (Hasse via degree). *[CITED-sec] [7] (Thm. V.1.1, Lemma V.1.2; via [5]).* *deg is a positive definite quadratic form on $\text{End}(E)$, and $\#E(\mathbb{F}_q) = \deg(1 - F)$. Hence $\deg(m + nF) = m^2 + amn + qn^2 \geq 0$ for all $m, n \in \mathbb{Z}$, which forces $a^2 \leq 4q$ [THEOREM]. (A binary form $m^2 + amn + qn^2$ is ≥ 0 on $\mathbb{Z}^2$, equivalently on $\mathbb{Q}^2$, iff its discriminant is ≤ 0 .)*

Logical order [RESTATEMENT]: (1) the object E ; (2) $\text{End}(E) \ni F$; (3) the counting form $\deg$, positive because it counts; (4) discriminant $a^2 - 4q \leq 0$; (5) $J = (2F - a)/\sqrt{4q - a^2}$. J is the last step, an output. In higher genus, step (3) is the Rosati trace $\text{Tr}(\phi\phi^\dagger) > 0$, or equivalently Castelnuovo–Severi/Hodge index on $C \times C$ [6]. Step (4) becomes $|\alpha| = \sqrt{q}$ (L100 Thm. 4.1). [NUMERICAL]: $y^2 = x^3 - x/\mathbb{F}_{101}$ has $a = -2$, and $|1 - \alpha^k|^2 = 104, 10400, 1029704$ ($k = 1, 2, 3$) are the point counts. The fake $a = 25$ has $\deg$ of signature $(1, 1)$ with minimum -441 on the box $|m| \leq 30, |n| \leq 3$, e.g. $\deg(-30 + 2F) = -196$ (Fig. 1). The Rosati trace form $T(u, v) = \sum_\alpha u(\alpha)v(q/\alpha)$ on $\mathbb{Q}[F]$ is $(2, 0)$ for $a = -3$, $(1, 1)$ for $a = 25$, $(4, 0)$ for genus 2 and $(2, 2)$ for the ghost quartet.

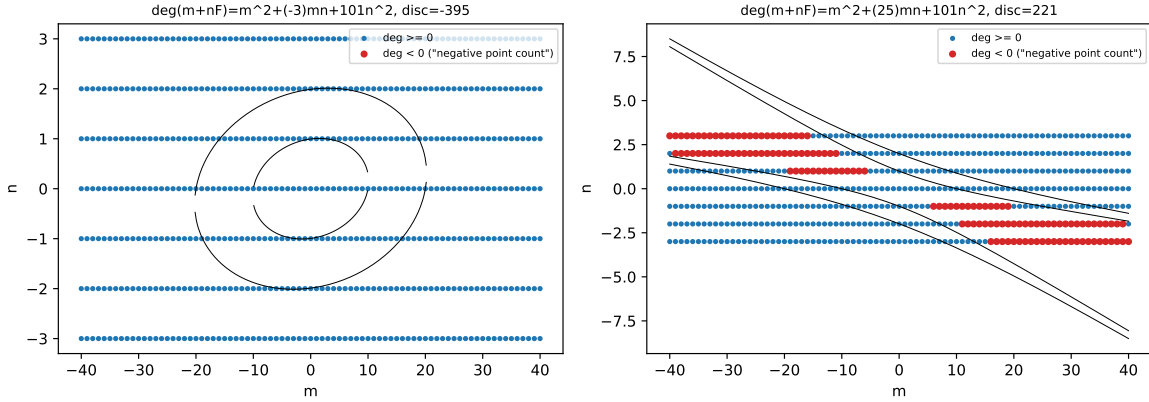


FIGURE 1. Degree form on $\mathbb{Z}^2$: ellipse (true, $a = -3$) vs. hyperbola (fake, $a = 25$) with negative “point counts” (red).

4. (C) THE $\mathbb{Q}$ ANALOGUE: $\lambda = \rho(1 - \rho)$ AND THE $\frac{1}{4}$ BOUND

Proposition 4.1. *[THEOREM] Write $\rho = \frac{1}{2} + \delta + i\gamma$ and $\lambda = \rho(1 - \rho) = \frac{1}{4} - (\delta + i\gamma)^2$, so that $(2\rho - 1)^2 = 1 - 4\lambda$ (the Pythagorean identity of L98/t420 [TAA-L]). Then $\lambda \in \mathbb{R}$ and $\lambda \geq \frac{1}{4}$ iff $\delta = 0$. Proof: $\text{Im } \lambda = -2\delta\gamma$. If $\gamma = 0$ then $\lambda = \frac{1}{4} - \delta^2 \leq \frac{1}{4}$, with equality only when $\delta = 0$. If $\delta = 0$ then $\lambda = \frac{1}{4} + \gamma^2$.*

[NUMERICAL] (qlambda.json): $4\lambda - 1 = 799.1618, 1767.70, 2502.17, \dots$ for the first zeros, and the identity holds exactly. The ghost $0.7 + 14.1347i$ gives $4\lambda - 1 = 799.00 - 22.62i$. A random test of 10^5 samples shows no counterexample. A real zero $\rho = 0.8$ would give $\lambda = 0.16 < \frac{1}{4}$, the analogue of an exceptional eigenvalue/complementary series (L99).

Proposition 4.2 (the content of J is reality). *[THEOREM] Formally $J = (2\rho - 1)/\sqrt{4\lambda - 1}$ satisfies $J^2 = (2\rho - 1)^2/(4\lambda - 1) = -1$ for every ρ , the ghost included ([NUMERICAL]: $J^2 = -1$ exactly). The real content: if D is a real operator on a 2-plane with $D^2 - D + \lambda = 0$ for a real scalar λ , its eigenvalues are $\{\rho, 1 - \rho\}$ and also closed under conjugation. So either $4\lambda - 1 > 0$ and $1 - \rho = \bar{\rho}$*

($\operatorname{Re} \rho = \frac{1}{2}$), or the eigenvalues are real ([NUMERICAL]: $\lambda = 50$ gives $0.5 \pm 7.053i$; $\lambda = 0.16$ gives $0.8, 0.2$). Hence “ $\Delta := D(1 - D)$ is a self-adjoint scalar on each block, $\geq \frac{1}{4}$ ” is $\sigma = c$ plus no real zeros, i.e. RH. It must be proved by an independent counting argument, as $\deg \geq 0$ is over $\mathbb{F}_q$, or it is circular [RESTATEMENT].

Why $\|df\|^2$ is not enough (C101.5, [THEOREM]). $\langle \Delta f, f \rangle = \|df\|^2$ gives $\Delta \geq 0$. On $\operatorname{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ the constants have $\lambda = 0 < \frac{1}{4}$. The continuous spectrum starts at $\frac{1}{4}$, and cusp forms have $\lambda > \frac{1}{4}$ [9] [CITED-sec] (L99). These are spectral facts about this surface, not consequences of squaring. Moreover the zeros are not eigenvalues there. They are scattering resonances $s = \rho/2$ with $\lambda = s(1-s) = 50.1351 + 3.5337i$ for ρ_1 , complex even though ρ_1 is on the line (Fig. 2). Selberg’s $\frac{1}{4}$ is therefore an analogy for the *shape* of the needed statement, not a route [ANALOGY]. **Known candidates** (only what is verified). Connes’ trace formula realizes only the critical zeros, with non-critical zeros as resonances ([4] [CITED], L30/L79). The semiclassical $H = xp$ count of Berry–Keating [1] [BIB] reproduces the smooth zero count and is identified with Connes’ absorption picture in the literature (L14 records). Neither supplies a self-adjoint $\Delta \geq \frac{1}{4}$ with the zeros as spectrum [OPEN]. **What “counting positivity over $\mathbb{Q}$ ” would have to be** [RESTATEMENT]. It would be a quadratic form on a lattice of “correspondences” of $\operatorname{Spec} \mathbb{Z}$, positive because it counts something, whose discriminant condition is $\lambda \geq \frac{1}{4}$. Weil’s functional is the candidate form, and its positivity is RH. On restricted test classes it is provable: for support $< \log 2$ no prime enters and positivity is archimedean (Yoshida, reproved by Bombieri [2] [CITED-sec]; Connes–Consani [3] [CITED]: the archimedean part alone loses positivity just beyond $\log 2$ and the prime 2 restores it). [NUMERICAL] (Fig. 3): a ghost $\delta = 0.2$ replacing γ_1 perturbs the zero-side kernel by $\|\Delta K\|/\|K\| = 1.2 \cdot 10^{-4}$ at $L = 0.3$ and $1.1 \cdot 10^{-3}$ at $L = \log 2$, growing to 0.62 at $L = 16$, where $n_- = 2$. Intermediate n_- values are numerically unstable on this truncated kernel. The bound is $\cosh(\delta L) - 1$. So short-support positivity is ghost-blind, and ghost-sensitivity needs long support, where the prime terms have no sign. That is the circularity [RESTATEMENT]. Bombieri: for large truncation the number of negative eigenvalues is half the number of off-line zeros [2] [CITED-sec].

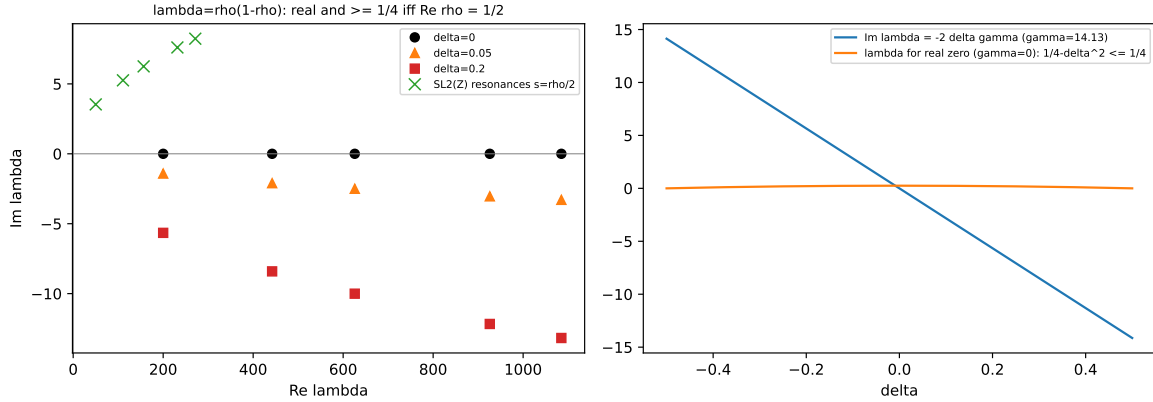
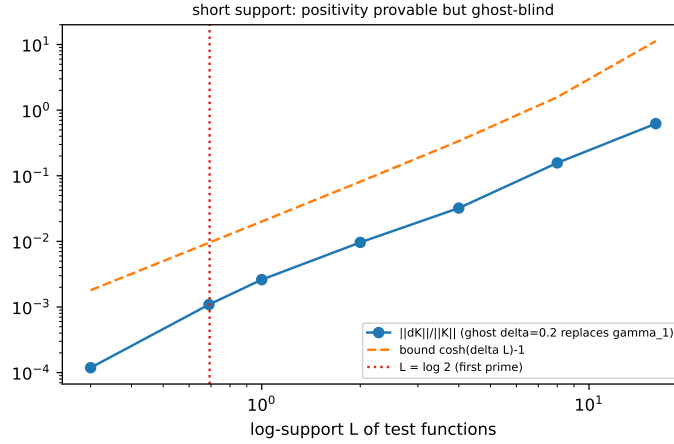


FIGURE 2. λ -plane: true zeros on $[\frac{1}{4}, \infty)$; ghosts leave the real axis ($\operatorname{Im} \lambda = -2\delta\gamma$); $\operatorname{SL}_2(\mathbb{Z})$ resonances are complex.

5. (D) THE SPACELIKE COMPLEMENT

Theorem 5.1. [THEOREM] *On a curve C of genus g over $\mathbb{F}_q$ with Frobenius trace a , let $H = C \times \text{pt}$, $V = \text{pt} \times C$, Δ the diagonal and $\Gamma = \Gamma_F$ the graph of F . Then $H^2 = V^2 = 0$, $H \cdot V = 1$, $\Delta \cdot H = \Delta \cdot V = 1$, $\Gamma \cdot H = q$, $\Gamma \cdot V = 1$, $\Delta^2 = 2 - 2g$, $\Gamma^2 = q(2 - 2g)$, $\Delta \cdot \Gamma = q + 1 - a$. The classes $\Delta_0 = \Delta - H - V$ and $\Gamma_0 = \Gamma - H - qV$ are orthogonal to H and V (the “spacelike complement” of*

FIGURE 3. Ghost perturbation of the zero-side form vs. log-support L .

the timelike $H + V$), with Gram matrix $\begin{pmatrix} -2g & -a \\ -a & -2gq \end{pmatrix}$. By Hodge index this block is negative definite, so $|a| < 2g\sqrt{q}$. For $g = 1$ this is Hasse. For $g \geq 2$ one needs Γ_{F^n} for all n to reach $|\alpha| = \sqrt{q}$, as in Weil's argument [6] (Ex. V.1.10).

[NUMERICAL] (J.json, Fig. 4):

Case	span $\langle H, V, \Delta, \Gamma \rangle$	primitive block
$g = 1, a = -3$	(1, 3)	(0, 2)
$g = 1, a = 25$	(2, 2)	(1, 1)
$g = 2, a = -2$	(1, 3)	(0, 2)
$g = 2, a = 42 (> 4\sqrt{101} = 40.2)$	(2, 2)	(1, 1)

For $\text{NS}(E \times E) \supseteq \mathbb{Z}e_1 + \mathbb{Z}e_2 + \{\Gamma_\psi\}$ with $\Gamma_\psi \cdot \Gamma_\chi = \deg(\psi - \chi)$: CM $j = 1728$ ($\deg(m + ni) = m^2 + n^2$) gives (1, 3); ordinary $a = -3$ gives (1, 3); a fake indefinite $\deg(m^2 + 5mn + n^2, \text{ or } a = 25)$ gives (2, 2). **Reading.** A ghost is a spacelike direction turned timelike: n_+ goes from 1 to 2, and the (2, 2) block appears. Positivity of the degree/Rosati form and negativity of the complement are one statement [RESTATEMENT]. **Logical vs causal order** ([ANALOGY] only). A circular argument (“assume the positivity to construct J ”) resembles a closed timelike loop, and the non-circular order of §3 resembles a causal chain. J.W.’s remark that “humans conflate logic with time” is recorded as [HEURISTIC] and is *not claimed* here.

6. (E) REFLECTIONS, BOOSTS AND THE 90-DEGREE TURN

Let $R_x = \text{diag}(1, -1)$, $R_y = \text{diag}(-1, 1)$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $M = \text{diag}(1, -1)$, and $B(r) = \cosh r + S \sinh r$.

Proposition 6.1. [THEOREM] ([NUMERICAL] exact, *reflect.json*). (i) The partial products of R_x, R_y, R_x, R_y are $R_x, -1, R_y, 1$, and $\langle R_x, R_y \rangle = \{1, R_x, R_y, -1\}$ is a Klein four-group: finite and abelian, not $U(1)$. (ii) $O(1, 1) = \{\pm B(r), \pm B(r)M\}$ contains no A with $A^2 = -1$: over 8000 sampled elements, $\min \|A^2 + 1\| = 2$. Indeed $B(r)^2 = B(2r)$ and $(B(r)M)^2 = 1$. (iii) Trace trichotomy in $SL_2(\mathbb{R})$: $|\text{tr}| > 2$ hyperbolic (two real eigenlines), $|\text{tr}| < 2$ elliptic (none), $|\text{tr}| = 2$ parabolic. For $F/\sqrt{q}$, $\text{tr} = a/\sqrt{q}$, so Hasse $\iff F/\sqrt{q}$ elliptic. Checked for $a = -3, \pm 20$ (elliptic) and ± 25 (hyperbolic, two real eigenlines). (iv) $JR_yR_xJ = 1$; $J^TMJ = -M$ (J is an anti-isometry of M , the Wick turn); $\langle J, R_x, R_y \rangle$ is dihedral of order 8; $JB(r)J^{-1} = B(r)^{-1}$, so J swaps the ghost pair $e^{\pm\delta}$. (v) $S^2 = 1$; its eigenlines $(1, \pm 1)$ are null; $[B(r), S] = 0$; $J = SR_x$ (mirrors 45° apart). But

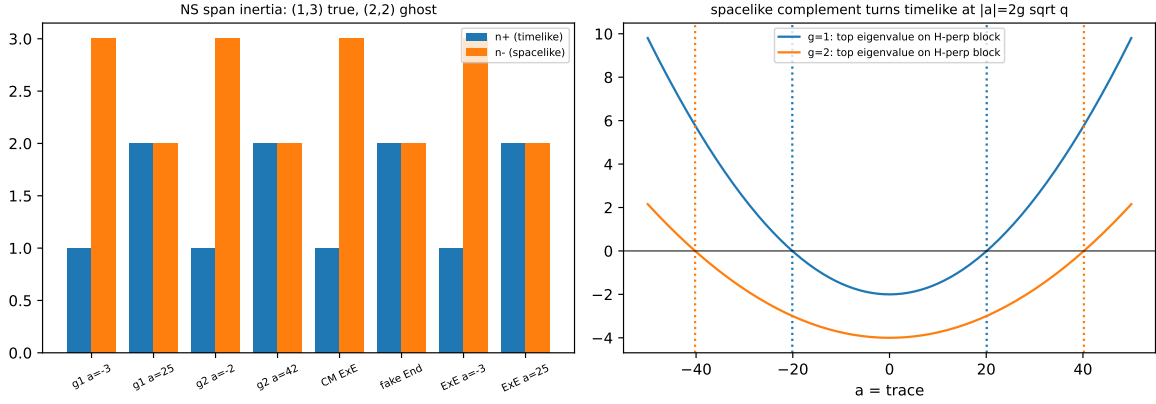


FIGURE 4. Left: NS inertia. Right: top eigenvalue of the $H^\perp$ block crosses 0 at $|a| = 2g\sqrt{q}$.

$S^T M S = -M$: S is not in $O(1,1)$. It is the Euclidean mirror in a null line, and an anti-isometry of M like J (C101.11).

Ghost-blindness of the j -world ([THEOREM], precise form). $\mathbb{R}[S] = \{x + yS\}$ has norm $x^2 - y^2$, which is indefinite, and its units include every boost $B(r)$. So the ghost pair $e^{\pm\delta}$ is an admissible element, and no inequality in the j -world excludes it. $\mathbb{R}[J] \cong \mathbb{C}$ has norm $x^2 + y^2 > 0$, and $F/\sqrt{q}$ lies in its norm-one group iff it is elliptic. Only an i -type J can exclude ghosts (L100 Thm. 4.1).

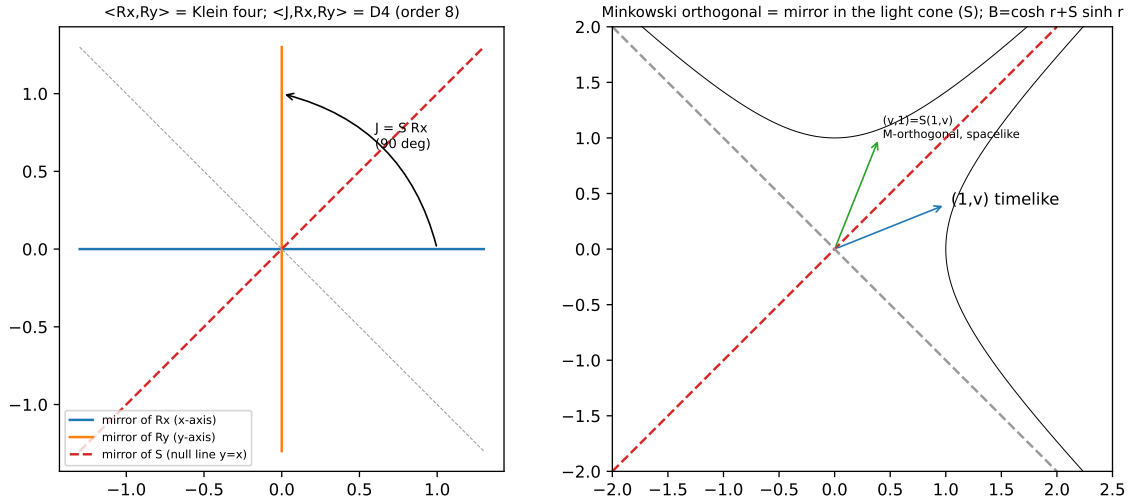


FIGURE 5. Left: mirrors of R_x , R_y , S ; $J = SR_x$. Right: the Minkowski orthogonal of $(1, v)$ is its mirror $(v, 1)$ in the light cone.

7. (F) ORTHOGONALITY THAT DEPENDS ON DIRECTION (T439)

Proposition 7.1. [THEOREM] ([NUMERICAL] reflect.json). (1) In $\mathbb{R}^2$ the vectors perpendicular to $x \neq 0$ with length $|x|$ are $\pm Jx$. Choosing one is choosing an orientation (rotation axis $+z$ or $-z$), i.e. J or $-J$, i.e. i or $-i$. Any reflection conjugates J to $-J$ ($R_x J R_x^{-1} = -J$). (2) $\omega(x, y) = (x \times y) \cdot \hat{z}$ gives $\omega(x, Jx) = |x|^2$ and $\omega(x, -Jx) = -|x|^2$ (error $2 \cdot 10^{-15}$). This is Hodge–Riemann $Q(x, Jx) > 0$,

with $Q = \omega$ playing the role of “ $+z$ ”. Replacing J by $-J$ swaps the $\pm i$ eigenlines $H^{1,0} \leftrightarrow H^{0,1}$, which is complex conjugation c : the conjugate of the $+i$ eigenvector is a $-i$ eigenvector. The sign of $i\omega(v, \bar{v})$ on each line is convention-dependent: -1 on the $+i$ line with this ω ; Milne inserts $(2\pi i)^n$. (3) Ghost quartet: for every $J \in \text{Sp}$ commuting with F , $Q(x, Jx)$ changes sign with direction. Over random $x \in \mathbb{R}^4$ the positive fraction is 0.51 and 0.49 for the two J ’s, consistent with signature $(2, 2)$ (L100). (4) Minkowski: $\langle (1, v), (v, 1) \rangle_M = 0$ and $(v, 1) = S(1, v)$, so the M -orthogonal is the mirror in the null diagonal. Null vectors are self-orthogonal. In fact $\langle x, Sx \rangle_M \equiv 0$ for all x , since MS is antisymmetric. So the “ S -form” vanishes identically, while $\omega(x, Sy)$ has signature $(1, 1)$ and $\omega(x, Jy)$ has $(2, 0)$. Also $\det J = +1$ (oriented) and $\det S = -1$ (unoriented).

Two orthogonalities ([RESTATEMENT]). Rotate $(J, i, \text{oriented})$: the paired form $\omega(x, Jy)$ can be definite and can carry positivity. Reflect in the light cone $(S, j, \text{unoriented})$: the paired form is indefinite or identically zero, and it commutes with boosts, so it is ghost-blind. A compact Hodge circle is a choice of the first kind, with a sign (orientation) fixed by positivity.

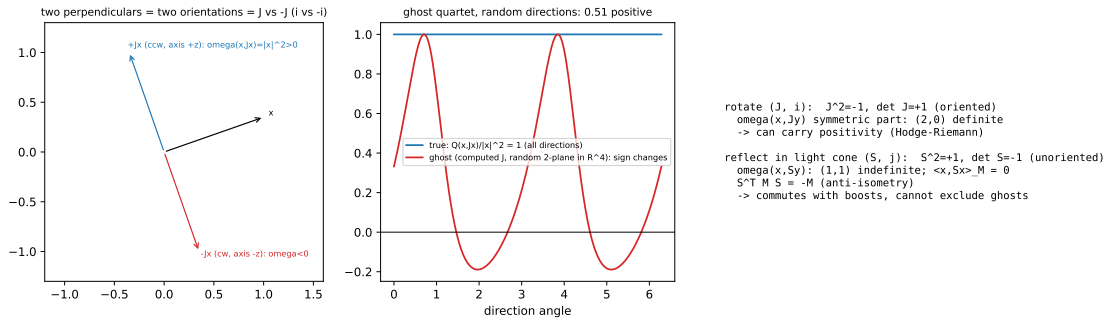


FIGURE 6. Left: the two perpendiculars $\pm Jx$ (axis $\pm z$). Middle: $Q(x, Jx)$ along a circle of directions, true (constant sign) vs. ghost (computed J , sign changes). Right: rotate vs. reflect.

8. CORRECTIONS TO THE CHAT CLAIMS

C101.1 (J polynomial in F):

Correct. Genus g : “unique up to sign per block” describes the complex structures in $\mathbb{R}[F]$ (2^g of them); positivity selects exactly one.

C101.2 (degree $\Rightarrow$ Hasse):

Correct. Citation detail: Silverman V.1.1 is Hasse’s theorem, and the inequality used is Lemma V.1.2.

C101.3 (logical chain):

Correct. J is the output, not the input.

C101.4 (λ , J over $\mathbb{Q}$):

The numbers and the Proposition are correct. Add: $J^2 = -1$ holds formally even for a ghost, and the content is the reality of λ (Prop. 4.2).

C101.5 ($\Delta \geq \frac{1}{4}$ by squaring):

Off. $\|df\|^2$ gives only $\Delta \geq 0$, and constants have $\lambda = 0$. On $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ the bound $\frac{1}{4}$ is a spectral fact about cusp forms and the continuous spectrum, and the zeros are resonances there with complex λ .

C101.6 (restricted test classes):

Positivity is provable for support $< \log 2$ but is ghost-blind there. Ghost sensitivity needs long support, where it is equivalent to RH.

C101.7 (spacelike complement $\iff$ Hasse/RH).:

Exact for $g = 1$. For $g \geq 2$ the $\langle \Delta, \Gamma \rangle$ block gives $|a| < 2g\sqrt{g}$, and RH needs all Γ_{F^n} .

C101.8 (ghost = spacelike turned timelike, (2,2)).:

Correct.

C101.9 (logic vs time).:

Analogy only. “Humans conflate logic with time” is a heuristic, not claimed.

C101.10 (Klein four, $O(1,1)$, trichotomy, D_4 , $JB J^{-1} = B^{-1}$).:

Correct. In 2D every line is isotropic for the symplectic form, so “null eigenlines” carries content only for the Minkowski M .

C101.11 (S “reflection across a null line”).:

Correct as a Euclidean mirror, with $S^2 = 1$, null eigenlines, $[B, S] = 0$ and $J = SR_x$. But $S \notin O(1,1)$: $S^T M S = -M$ (anti-isometry). “Reflection” here does not mean a Lorentz reflection, which a null normal does not define.

C101.12–14 (t439 (1)–(3)).:

Correct, with the convention note on the sign of $i\omega(v, \bar{v})$.

C101.15 (t439 (4)).:

Correct. Strengthened: $\langle x, Sx \rangle_M \equiv 0$ for all x , so the S -orthogonality carries no form at all.

9. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	A $\mathbb{Q}$ -analogue of the degree lattice: a lattice of “correspondences” of $\text{Spec } \mathbb{Z}$ with a quadratic form positive <i>because it counts</i> , whose discriminant condition is $\lambda \geq \frac{1}{4}$ (Connes–Consani scaling-site line).	0.9	0.05	0.56
2	Restricted-support Weil positivity with the true prime side: track the onset of ghost-sensitivity beyond $\log 2$ (Bombieri’s eigenvalue count), with an honest explicit formula rather than a truncated zero sum.	0.5	0.6	0.54
3	The 2×2 primitive-block test (Thm. 5.1) as a certified ghost detector on L98 finite models.	0.3	0.8	0.50
4	An i/j classification of ghost detection in finite models (which structures commute with a ghost F).	0.2	0.9	0.48
5	A Selberg-type $\frac{1}{4}$ mechanism for resonances (Lax–Phillips): any spectral-gap argument that moves resonances rather than eigenvalues.	0.5	0.3	0.42

10. QUESTIONS FOR J.W.

- Q1. Do you accept that over $\mathbb{Q}$ the formal $J = (2D - 1)/\sqrt{4\lambda - 1}$ carries no content until λ is known to be real (Prop. 4.2)?
- Q2. What would “counting” mean over $\mathbb{Q}$: what set is counted by the quadratic form whose discriminant should give $\lambda \geq \frac{1}{4}$?
- Q3. Given C101.5, which mechanism for $\frac{1}{4}$ do you have in mind: hyperbolic geometry (as on $\mathbb{H}$) or arithmetic counting?
- Q4. Is the spacelike complement for ζ meant to be the primitive part of Weil’s functional (positivity on $\hat{g}(\pm i/2) = 0$), as in Connes–Consani?
- Q5. L102: direction 1 (a lattice of correspondences) or direction 2 (onset of ghost-sensitivity beyond $\log 2$)?

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TAA LEMMA 102: COMMUTATORS, ANTICOMMUTATORS AND THE HODGE–RIEMANN FORM

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts num/lemma102_*.py, data num/l102/forms.json); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: lit/lemma102/verified_refs.md.

ABSTRACT. One alternating form $\omega(x, y) = x_1y_2 - x_2y_1$ has two twists. Twisted by a rotation J ($J^2 = -1$) it becomes the Euclidean Hodge–Riemann (HR) form $\omega(x, Jy)$. Twisted by the light-cone mirror S ($S^2 = +1$) it becomes the Minkowski form $\omega(x, Sy)$. Quantization sees both. The commutator of a CCR algebra is ω (Dirac 1925). The symmetric part of the two-point function in the vacuum of J is the HR metric $g = \omega(\cdot, J\cdot)$. The Clifford anticommutator of the pair (S, J) is the Lorentz metric η , since S, J are 1+1 gamma matrices.

Results. (1) $A^TWA = \det(A)W$ for every 2×2 A , so “ J preserves, S reverses ω ” is a determinant statement. What singles out J is compatibility: g positive definite. $\omega(x, Jy)$ has signature $(2g, 0)$ and $\omega(x, Sy)$ has (g, g) in genus g (Sylvester). Flipping the Hodge-index form on $H^\perp$ (L99 Wick) turns $(1, 3)$ into $(4, 0)$ for true models but $(2, 2)$ into $(3, 1)$ for ghosts (Prop. 2.1). (2) Bosons (Thm. 3.1): $\langle \xi_a \xi_b \rangle = \frac{1}{2}g_{ab} + \frac{i}{2}\omega_{ab}$. The uncertainty principle $g + i\omega \geq 0$ [7] is, for one degree of freedom, $\det g \geq 1$ (0 mismatches in $2 \cdot 10^4$ trials). A pure vacuum requires $g = \omega(\cdot, J\cdot)$ positive definite. For $-J$ the would-be vacuum $e^{+x^2/2}$ is not normalizable. Bogoliubov transformations change J . Squeezes are hyperbolic, the oscillator is elliptic and the free particle is parabolic (L101 trichotomy). Shale’s criterion is checked numerically (overlap 0.687 vs. $\rightarrow 0$). (3) Fermions and spinors (Thm. 4.1): $S^2 = 1$, $J^2 = -1$, $SJ + JS = 0$, so $\{1, S, J, SJ\}$ spans $M_2(\mathbb{R}) \cong \text{Cl}(1, 1)$, with chirality $SJ = R_x$ and $R_yS = J$. The Euclidean pair (S, R_x) generates $\text{Cl}(2, 0)$ with bivector J . Hence $\text{Spin}(2) = \{e^{\theta J}\}$ (elliptic, the Hodge circle acting on spinors, rotating vectors by 2θ) and $\text{Spin}(1, 1) = \{e^{rR_x/2}\}$ (hyperbolic squeezes). A Hilbert-space CAR forces its symmetric form to be ≥ 0 . (4) Flipping the page over the null diagonal is R_π about $(1, 1, 0)$. It restricts to S , sends $+z \mapsto -z$, and conjugates J to $-J = c$ (Grok Bot’s explanation of the hand-flip; J.W.’s t443 self-correction on null rays confirmed, C102.13). J cannot be joined to S in $GL_2(\mathbb{R})$ (det), nor to $-J$ within complex structures (two sheets) (t440). (5) Zeros (Thm. 5.1, [RESTATEMENT] of L100 Thm. 4.1): a finite model satisfies RH iff some J commuting with F makes $g + i\omega \geq 0$, i.e. unitary quantization. [NUMERICAL]: $\min \text{eig}(g + i\omega)$ is 0 to within $3 \cdot 10^{-16}$ for genus 1, genus 2 and CM $E \times E$, and -2 for both admissible J of the ghost quartet. A ghost is a negative-norm direction ([ANALOGY] Gupta–Bleuler).

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GLOSSARY

W, J, S, R_x, R_y, M :

$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with $\omega(x, y) = x^T W y$; $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $R_x = \text{diag}(1, -1) = M$; $R_y = -R_x$.

Compatible complex structure.:

$J^2 = -1$, $J^T W J = W$, and $g(x, y) = \omega(x, Jy)$ symmetric positive definite ([8] [CITED] eqs. (15)–(16) and the surrounding construction).

CCR / CAR.:

Canonical commutation relations $[\xi_a, \xi_b] = i\omega_{ab}$; canonical anticommutation (Clifford) relations $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$.

Two-point matrix.:

$T_{ab} = \langle \xi_a \xi_b \rangle$ in a state, $\hbar = 1$; $T = \frac{1}{2}(g + i\omega)$.

Bogoliubov transformation.:

A linear symplectic change of (q, p) ; equivalently a change of J .

1. THE CHAT CLAIMS T440–T443

- (a) One parent, two twists: $\omega(x, Jy)$ is Euclidean and $\omega(x, Sy)$ is Minkowski; $J^T W J = W$ and $S^T W S = -W$; Sylvester; Hodge index vs. Rosati via the flip; cup product on a curve. Correct, with the qualifier C102.1 (§2).
- (b) Dirac’s rule, the two-point split, the uncertainty matrix, Fock positivity, Bogoliubov = change of J , trichotomy. Correct. Dirac 1925, Simon–Mukunda–Dutta and Shale verified (§3).
- (c) Clifford relation, S, J as gamma matrices, chirality, $R_y S = J$, $S J S = -J$. Correct, but Dirac 1928 states the α -form, not the γ - η form (C102.10) (§4).
- (d) $\text{RH} \iff$ unitary quantization; ghost = negative norm; xp as a squeeze. Correct as restatement/analogy (§5). Correction: hyperbolic does not mean non-unitary (C102.16).
- (e) t440 path claim; the t443 self-correction (null rays end up opposite, CONFIRMED); the “commutation relations” answer. Logged as C102.12–14.

2. (A) ONE PARENT, TWO TWISTS

Proposition 2.1. [THEOREM] ([NUMERICAL] exact). (i) $WJ = I$ and $WS = M = \text{diag}(1, -1)$. So $\omega(x, Jy) = x \cdot y$ (HR, signature $(2, 0)$), $\omega(x, -Jy)$ has $(0, 2)$, and $\omega(x, Sy) = x_1 y_1 - x_2 y_2$ (Minkowski, $(1, 1)$). By Sylvester’s law of inertia the HR and Lorentz forms are not isometric. (ii) $A^T W A = \det(A) W$ for all $A \in M_2(\mathbb{R})$ (max error $2.5 \cdot 10^{-15}$ over 10^3 samples). Hence $J^T W J = W$ ($\det J = 1$) and $S^T W S = -W$ ($\det S = -1$). Preserving ω does not distinguish J from $-J$ or from any element of SL_2 ; compatibility ($g > 0$) does. (iii) Genus g : with $W_g = \oplus W$, $J_g = \oplus J$, $S_g = \oplus S$ and any $B \in \text{Sp}(2g, \mathbb{R})$, $\omega(x, B J_g B^{-1} y)$ has signature $(2g, 0)$ and $\omega(x, B S_g B^{-1} y)$ has (g, g) ([NUMERICAL], $g = 1, 2, 3, 5$). (iv) On $\langle H, V, \Delta, \Gamma_F \rangle \subset \text{NS}(C \times C)$ (L101 Thm. 6.1 [TAA-L]), let $h = H + V$ and $G' = 2(Gh)(Gh)^T / h^T G h - G$, i.e. flip the sign on $h^\perp$ (L99 Wick). The inertia goes from $(1, 3)$ to $(4, 0)$ for $g = 1$, $a = -3$ and for $g = 2$, $a = -2$ (Rosati-like), but from $(2, 2)$ to $(3, 1)$ for the fakes $a = 25$ and $a = 42$. (v) On a curve, ω is the cup product $Q(\alpha, \beta) = \int \alpha \cup \beta$ on H^1 . It is alternating and unimodular (Poincaré duality). HR reads $Q(x, Cx) > 0$ with Weil operator $C = h(i) = J$ (classical Riemann bilinear relations; L100 Thm. 4.1 [TAA-L]). In genus g this is (iii).

3. (B) THE BOSONIC SIDE: COMMUTATORS AND THE FOCK VACUUM

Dirac’s rule [CITED] [2]: “the difference between the Heisenberg products of two quantum quantities is equal to $i\hbar/2\pi$ times their Poisson bracket expression”, $xy - yx = i\hbar/2\pi \cdot [x, y]$ (eq. (11)), giving $q_r p_s - p_s q_r = \delta_{rs} i\hbar/2\pi$ (eq. (12)). With $\xi = (q, p)$ and $\hbar = 1$: $[\xi_a, \xi_b] = i\omega_{ab}$, so the commutator is ω .

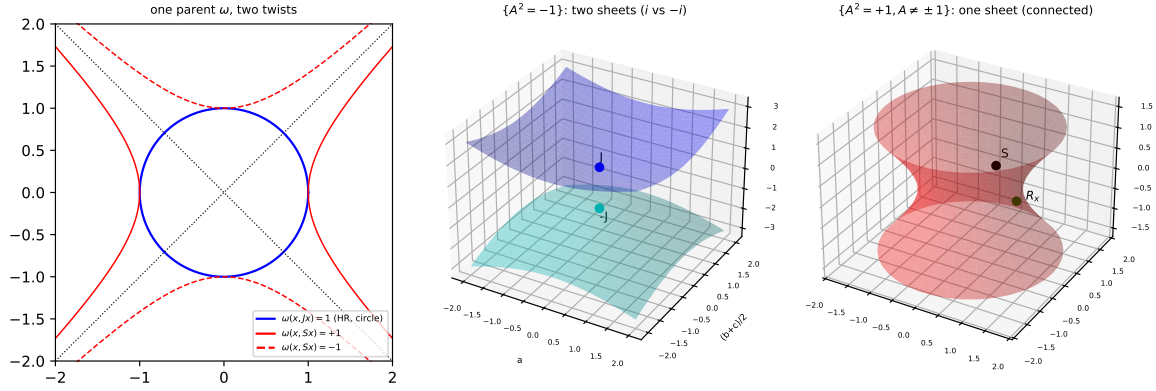


FIGURE 1. Left: the circle $\omega(x, Jx) = 1$ vs. the hyperbolas $\omega(x, Sx) = \pm 1$. Middle: complex structures $\{A^2 = -1\}$ form two sheets, J and $-J$ (i vs. $-i$). Right: split structures $\{A^2 = 1\} \setminus \{\pm 1\}$ form one connected sheet containing S and R_x . Coordinates $(a, \frac{b+c}{2}, \frac{c-b}{2})$ for $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$.

Theorem 3.1. [THEOREM] (one degree of freedom; the n -mode version is blockwise). (i) In any state $\langle \xi_a \xi_b \rangle = \frac{1}{2} \langle \{ \xi_a, \xi_b \} \rangle + \frac{1}{2} \langle [\xi_a, \xi_b] \rangle = \frac{1}{2} g_{ab} + \frac{i}{2} \omega_{ab}$, with $g = 2V$ (V the covariance, means 0). The antisymmetric part is state-independent; the symmetric part depends on the state. (ii) Since $T = \langle \xi \xi^T \rangle$ is a Gram matrix, $g + i\omega \geq 0$. This is the Robertson–Schrödinger uncertainty principle in matrix form [5] [BIB], [7] [CITED-sec] ($V + \frac{i}{2} \Omega \geq 0$). For $n = 1$ it is equivalent to $g > 0$ and $\det g \geq 1$ ($\det V \geq \frac{1}{4}$); [NUMERICAL]: 0 mismatches in $2 \cdot 10^4$ random trials. (iii) Equality $\det g = 1$ (a pure Gaussian state) holds iff $g = \omega(\cdot, J\cdot)$ for a compatible J , namely $J = W^{-1}g$ (so $WJ = g$; for $g = I$, $J = -W$ is the standard J). Then $\text{eig}(g + i\omega) = \{0, 2\}$ ([NUMERICAL]). This is the one-mode case of the Segal/Bargmann–Fock construction: the one-particle space is $(\mathbb{R}^2, J)$ with inner product $\frac{1}{2}(g + i\omega)$ ([8] [CITED]; book [9] [BIB]). (iv) For $-J$, $g = -I$ and $\text{eig}(g + i\omega) = \{-2, 0\}$. The annihilator for $-J$ is $(x - \partial_x)/\sqrt{2}$, whose kernel $e^{+x^2/2}$ is not normalizable: no positive-norm Fock space. Positivity of $g \iff$ unitarity. (v) Bogoliubov: for $T \in \text{Sp}(2, \mathbb{R}) = \text{SL}_2(\mathbb{R})$, $J \mapsto TJT^{-1}$ is again compatible. Squeezes $T = \text{diag}(e^r, e^{-r})$ are hyperbolic ($\text{tr} = 2 \cosh r$), with $\langle N \rangle = \sinh^2 r$ in the new vacuum (0.2715, 1.381 for $r = 0.5, 1$). The oscillator $H = \frac{1}{2}(q^2 + p^2)$ generates e^{-tJ} (elliptic), xp generates e^{tR_x} (hyperbolic) and $\frac{1}{2}p^2$ generates a shear (parabolic) (Fig. 5). This is the L101 trace trichotomy [TAA-L]. The Hessian of the Hamiltonian has signature (2,0), (1,1), (1,0) respectively. (vi) Shale [6] [CITED]: in the Fock–Cook quantization, the symplectic T that are unitarily implementable are exactly those with $(T^*T)^{1/2} - I$ Hilbert–Schmidt. [NUMERICAL] (Fig. 3): for $r_k = 1/k$ the HS sum converges (4.741) and the vacuum overlap $\prod (\cosh r_k)^{-1/2} \rightarrow 0.6874$. For $r_k = 1/\sqrt{k}$ the HS sum diverges (30.9 at $N = 10^6$) and the overlap goes to 0 (0.029).

4. (C) THE FERMIONIC SIDE: ANTICOMMUTATORS, GAMMA MATRICES, SPINORS

Dirac 1928 [CITED] [3]: the conditions are $\alpha_\mu^2 = 1$, $\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu = 0$ ($\mu \neq \nu$; $\mu, \nu = 1, \dots, 4$; eq. (6)), realized by 4×4 matrices built from Pauli matrices σ_r and ρ_r . The covariant form $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}$ is the later rewriting $\gamma \sim \beta\alpha$, with sign conventions varying (C102.10).

Theorem 4.1. [THEOREM] ([NUMERICAL] exact, forms. json:C). (i) $S^2 = 1$, $J^2 = -1$, $SJ + JS = 0$. So $\gamma_0 = S$, $\gamma_1 = J$ satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ with $\eta = \text{diag}(1, -1)$. Since $\{1, S, J, SJ\}$ has rank 4 in $M_2(\mathbb{R})$, $\text{Cl}(1, 1) \cong M_2(\mathbb{R})$, and $\mathbb{R}^2$ is its real (Majorana) spinor module. (ii) Chirality: $\gamma_0 \gamma_1 = SJ = R_x$, $(SJ)^2 = 1$, and SJ anticommutes with S and J . Also $R_y S = J$ and $SJS = -J$. (iii) Euclidean partner: $\{S, R_x\} = 0$ and $S^2 = R_x^2 = 1$, so (S, R_x) generates $\text{Cl}(2, 0) \cong M_2(\mathbb{R})$ with bivector

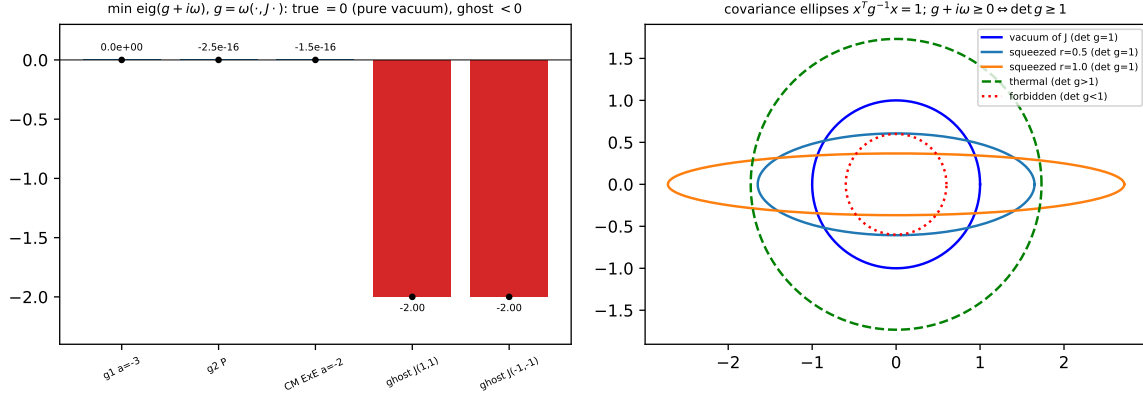


FIGURE 2. Left: $\min \text{eig}(g + i\omega)$ for true models ($J = h(i)$ from the L101 formula; 0, pure vacuum) and the ghost quartet (-2). Right: admissible covariance ellipses.

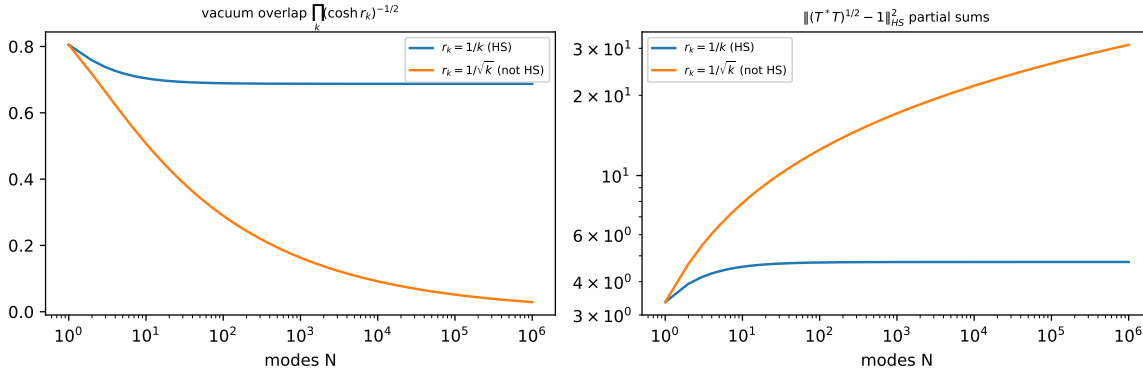


FIGURE 3. Shale's criterion numerically: Hilbert–Schmidt squeezes keep a common vacuum; non-HS ones do not.

$SR_x = J$. The spin groups in this one $M_2(\mathbb{R})$ are $\text{Spin}(2) = \{e^{\theta J}\} \cong U(1)$ and $\text{Spin}(1, 1)^0 = \{e^{rR_x/2}\}$. Conjugation by $e^{\theta J}$ rotates the vector plane $\langle S, R_x \rangle$ by 2θ : $S \mapsto \cos 2\theta S - \sin 2\theta R_x$. Conjugation by $e^{rR_x/2}$ boosts $\langle S, J \rangle$ with rapidity r : $S \mapsto \cosh r S - \sinh r J$. (iv) Elliptic vs. hyperbolic (L101): $F/\sqrt{q} \in SL_2(\mathbb{R})$ is elliptic iff it is conjugate into $\text{Spin}(2)$ (Hasse), and hyperbolic iff it is conjugate into $\pm \text{Spin}(1, 1)^0$ (ghost/fake). In spinor language: the Hodge circle is the Euclidean spin group, and a ghost Frobenius is a Lorentzian spin element. (v) CAR positivity: if operators $a(f)$ on a Hilbert space satisfy $a(f)a(f)^* + a(f)^*a(f) = h(f, f)1$, then $h(f, f)\langle \psi, \psi \rangle = \|a(f)^*\psi\|^2 + \|a(f)\psi\|^2 \geq 0$. So a *-CAR algebra with an indefinite h has no Hilbert representation; it needs a Krein (indefinite) space. CCR needs a positive $g = \omega(\cdot, J\cdot)$, and CAR needs its symmetric form to be positive. The Lorentz η of $\text{Cl}(1, 1)$ appears only as the algebraic relation among the generators, not as the inner product of a state space.

Reading ([THEOREM] for the computation, [HEURISTIC] for the interpretation). The Deligne-torus circle $h(e^{i\theta}) = e^{\theta J}$ acts on $H^1 \cong \mathbb{R}^2$ as the spin lift of a rotation by 2θ on $\langle S, R_x \rangle$. Its weights ± 1 on $H^{1,0}, H^{0,1}$ are “spin $\pm \frac{1}{2}$ ” relative to the weight ± 2 of the vector plane. A literature source for a spinor formulation of Hodge structures was *not* located or verified [OPEN]. **CCR vs. CAR** [THEOREM]: antisymmetric ω gives the Weyl/Heisenberg algebra, and symmetric forms give

Clifford algebras. In 2D, ω 's two twists produce one form of each type: $g = \omega(\cdot, J\cdot)$ (symmetric, PD, the CCR vacuum) and $\omega(\cdot, S\cdot)$ (symmetric, $(1, 1)$). The Gram matrix of the latter on spinors equals $\eta = \frac{1}{2}\text{tr}(\gamma_\mu\gamma_\nu)$ on vectors; both are $\text{diag}(1, -1)$, but on different spaces. **The flip** (Grok Bot's explanation accompanying t443; [THEOREM], [NUMERICAL]): the rotation R_π of $\mathbb{R}^3$ about $(1, 1, 0)/\sqrt{2}$ restricts to S on the plane ($3 \cdot 10^{-16}$) and sends $e_z \mapsto -e_z$. It conjugates the generator L_z to $-L_z$, i.e. $J \mapsto SJS = -J = c$ (Fig. 4). **Paths (t440)** [THEOREM]: $J(1, 1) = (-1, 1)$ and $J(1, -1) = (1, 1)$, so J swaps the null lines. S fixes $t = x$ pointwise and reverses $t = -x$. $\det J = 1$ and $\det S = -1$, so no path in $GL_2(\mathbb{R})$ joins them. The complex structures $\{a^2 + bc = -1\}$ form two sheets, with the sign of b matching the sign of $\omega(x, Ax)$ in all $2 \cdot 10^4$ samples. So J and $-J$ are not joined through complex structures, although they are joined in GL_2^+ . Split structures form one connected sheet (Fig. 1).

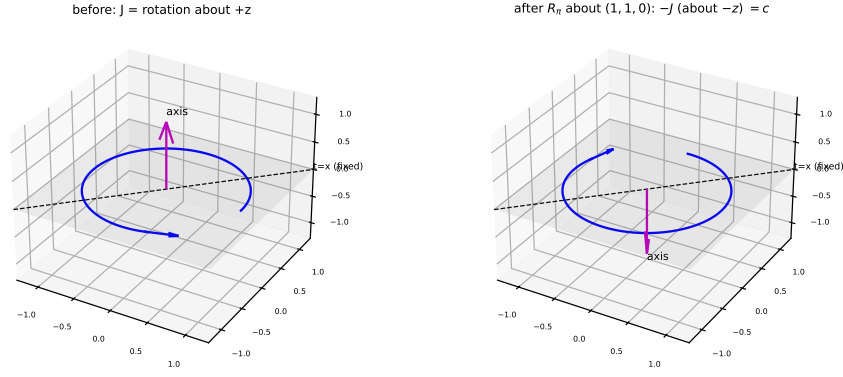


FIGURE 4. Flipping the page over the null diagonal $t = x$ (R_π about $(1, 1, 0)$) reverses the rotation axis: $J \mapsto -J = c$.

5. (D) ZEROS: RH AS UNITARY QUANTIZATION

Theorem 5.1. [RESTATEMENT] (L100 Thm. 4.1 [TAA-L], L101 Prop. 3.1 [TAA-L]). Let $F \in GL_{2g}(\mathbb{R})$ be semisimple with no real eigenvalues and $F^T W_g F = qW_g$. TFAE: (i) all eigenvalues satisfy $|\alpha| = \sqrt{q}$; (ii) some $J \in \text{Sp}(2g, \mathbb{R})$ commuting with F has $g = \omega(\cdot, J\cdot) > 0$; (iii) some such J has $g + i\omega \geq 0$, i.e. a pure Fock vacuum on which $F/\sqrt{q}$ acts unitarily. (ii) $\iff$ (iii): $g + i\omega \geq 0$ forces $g \geq 0$ on real vectors, and for compatible J with $g > 0$ the eigenvalues are $\{0, 2\}$ blockwise.

[NUMERICAL] (forms.json:D). J is built from F by the L101 formula $(2F_k - t_k)/\sqrt{4q - t_k^2}$ blockwise, then conjugated by a random $B \in \text{Sp}$.

model	$\ F^T W F - qW\ $	$\ J^2 + 1\ $	$\ [J, F]\ $	inertia g	min eig($g + i\omega$)
$g = 1, a = -3$	$1.4 \cdot 10^{-14}$	$2.5 \cdot 10^{-16}$	0	(2, 0)	0
$g = 2, P (t = 12.304, -14.304)$	$9.2 \cdot 10^{-14}$	$7.4 \cdot 10^{-16}$	$6.6 \cdot 10^{-15}$	(4, 0)	$-2.5 \cdot 10^{-16}$
CM $E \times E, a = -2$	$3.0 \cdot 10^{-14}$	$5.0 \cdot 10^{-16}$	$6.3 \cdot 10^{-15}$	(4, 0)	$-1.5 \cdot 10^{-16}$
ghost quartet $\delta = 0.2, J_{(+,+)}$	$2.8 \cdot 10^{-15}$	—	0	(2, 2)	-2
ghost quartet, $J_{(-,-)}$			0	(2, 2)	-2

For the ghost $F = \sqrt{q}(e^\delta R_\theta \oplus e^{-\delta} R_\theta)$ (eigenvalue moduli 0.819, 1.221 in units of $\sqrt{q}$), the four eigenvalues are distinct. So the commutant is $\mathbb{R}[A] \oplus \mathbb{R}[A^{-T}]$, its complex structures are $\text{diag}(\pm J, \pm J)$, and only the two with equal signs are symplectic. Both give (2, 2) [THEOREM]. The fake $a = 25$ has commutant $\mathbb{R} \times \mathbb{R}$ with no complex structure; its split j has $\omega(x, jy)$ of signature $(1, 1)$. **Ghost = negative norm** [ANALOGY]: the (2, 2) form is an indefinite-metric state space, as in

Gupta’s treatment of longitudinal photons [4] [BIB] (Gupta–Bleuler; analogy only). xp (Berry–Keating [1] [BIB]): $H = xp = \frac{1}{2}\xi^T S \xi$ has Hessian S , of signature $(1, 1)$, and generates the squeeze e^{tR_x} : hyperbolic, the j -world (Fig. 5). But the quantum generator $-i(x\partial_x + \frac{1}{2})$ of dilations is symmetric on $L^2(\mathbb{R}_+)$ ([NUMERICAL] residual $4 \cdot 10^{-6}$), with real continuous spectrum by Mellin transform. *Hyperbolic classical flows are unitarily implemented.* The j -world fails to detect ghosts because the S -forms are indefinite, not because the flows are non-unitary (C102.16). “Dilations of $C_{\mathbb{Q}}$ are the j -world” is [ANALOGY]. **Over $\mathbb{Q}$ [OPEN]:** the analogue of (iii) would need a state, i.e. a positive two-point functional, on a symplectic space carrying the zeros, with a J commuting with the zero operator. Weil’s positivity is the candidate, and it is RH (L101 §5).

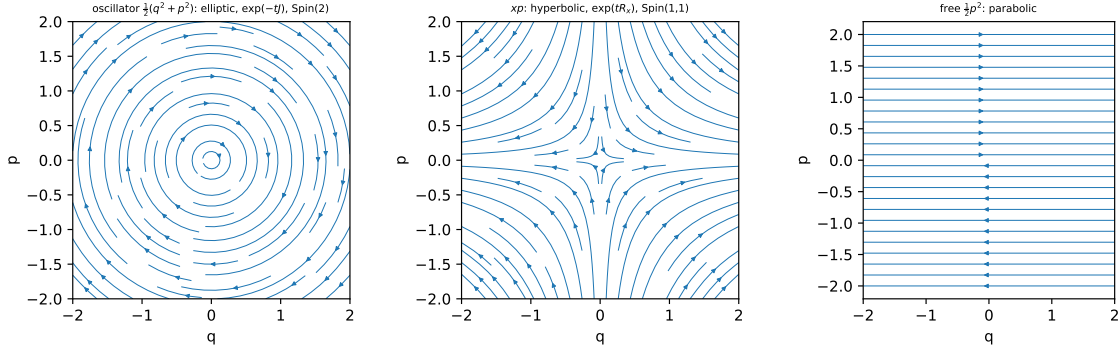


FIGURE 5. Quadratic Hamiltonians: elliptic (Spin(2), HR), hyperbolic (xp , Spin(1, 1)), parabolic.

6. CORRECTIONS TO THE CHAT CLAIMS

C102.1 (J preserves, S reverses ω):

True, but generic: $A^T W A = \det A W$. The real content is that J is *compatible* ($g > 0$); $-J$ also preserves ω and gives $(0, 2)$.

C102.2 (Sylvester):

Correct: $(2g, 0)$ vs. (g, g) .

C102.3 (Hodge index vs. Rosati via the flip):

Correct for true models $((1, 3) \rightarrow (4, 0))$. For ghosts the flip gives $(3, 1)$, not positive definite.

C102.4 ($\omega = \text{cup product}$):

Correct.

C102.5 (Dirac 1925):

Verified: eq. (11), $xy - yx = i\hbar/2\pi [x, y]$.

C102.6 (two-point split):

Correct with $\hbar = 1$: $\langle \xi \xi^T \rangle = \frac{1}{2}(g + i\omega)$. Conventions (factor $\frac{1}{2}$, sign of ω) vary; Wald normalizes by an inner product 2μ subject to eq. (15) of [8].

C102.7 (uncertainty $V + i\Omega \geq 0$):

Missing factor: $V + \frac{i}{2}\Omega \geq 0$ [7], equivalently $g + i\omega \geq 0$ with $g = 2V$.

C102.8 (positivity of $g \iff$ positive-norm Fock):

Correct. For $-J$ the vacuum $e^{x^2/2}$ is not normalizable.

C102.9 (Bogoliubov = change of J ; Shale):

Correct; Shale verified $((T^*T)^{1/2} - I \text{ Hilbert-Schmidt})$. Finite-mode squeezes are always implementable.

C102.10 (Dirac 1928 $\gamma\gamma + \gamma\gamma = 2\eta$).:

Attribution detail: Dirac 1928 writes $\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta_{\mu\nu}$ (eq. (6)). The η -form is the later covariant rewriting.

C102.11 (S, J gamma matrices; $SJ = R_x$; $R_yS = J$; $SJS = -J$).:

Correct; $\text{Cl}(1, 1) \cong M_2(\mathbb{R})$.

C102.12 (t440 path).:

Correct: J swaps the null lines, while S fixes $t = x$ pointwise and reverses $t = -x$; $\det$ obstructs a path. Addition: J and $-J$ lie on different sheets of complex structures.

C102.13 (t443 self-correction).:

CONFIRMED. J.W. (t443): “my statement that the rotation-then-reflection and the S diagonal reflection leave the null rays the same was wrong — the null rays end up opposite.” [THEOREM] ([NUMERICAL]): J swaps the two null lines ($J(1, 1) = (-1, 1)$, $J(1, -1) = (1, 1)$); S fixes $t = x$ pointwise and reverses $t = -x$ ($S(1, -1) = (-1, 1)$). J.W. also noted that by hand he can do S (diagonal flip), then the y -reflection, then a rotation, but not the rotation/reflection sequence, and suspects this is not a coincidence; recorded as [HEURISTIC], left as metaphysical, not pursued. *Added explanation (Grok Bot, not J.W.’s claim):* the hand-flip of the page is R_π about the null diagonal in $\mathbb{R}^3$, which restricts to S and reverses the normal, so $SJS = -J = c$; and $R_yS = J$, i.e. diagonal flip followed by y -reflection is the rotation J (Thm. 4.1(ii), Fig. 4).

C102.14 (“commutation relations”).:

Refined: the commutator gives ω (state-independent). HR needs a commutator *plus* a state, whose symmetric part is $g = \omega(\cdot, J\cdot)$. Lorentz is the Clifford anticommutator η on the *vector* span of (S, J) . $\omega(\cdot, S\cdot)$ on spinors has the same matrix but lives on a different space. Both CCR and CAR need a *positive* form to have Hilbert representations (Thm. 4.1(v)).

C102.15 (Deligne torus on spinors).:

No source verified [OPEN]. Computed version: $e^{\theta J}$ is the spin lift of a 2θ rotation (Thm. 4.1(iii)).

C102.16 (xp squeeze; j -world).:

Classical: correct. Quantum: dilations are unitary. “Ghost-blind” refers to indefinite forms, not non-unitary flows.

C102.17 (ghost = negative norm).:

Correct in the finite models (min eig = -2). Gupta–Bleuler is analogy only.

7. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	Weil positivity as a two-point condition: recast the explicit-formula functional as $g + i\omega \geq 0$ for a Gaussian state (GNS), and test ghost sensitivity on restricted support with the true prime side.	0.5	0.6	0.54
2	Spinor route: seek a self-adjoint anticommuting pair (S, R_x) on a space carrying the zeros whose bivector $J = SR_x$ commutes with the zero operator, so that $\text{Cl}(2, 0)$ rather than $\text{Cl}(1, 1)$.	0.85	0.05	0.53
3	Certified finite-model ghost detector min eig($g + i\omega$) on all L98 models, with interval arithmetic.	0.3	0.85	0.52
4	A Shale-type criterion for the Wick flip on $H^\perp$ in an infinite (adelic) model: when is the flip implementable?	0.6	0.25	0.46
5	xp regularizations as boundary conditions selecting a J (Berry–Keating/Connes absorption).	0.5	0.3	0.42

8. QUESTIONS FOR J.W.

- Q1. Do you accept C102.14: HR is commutator *plus* state (J), and Lorentz is the Clifford anticommutator on vectors?

- Q2. Spinors: in direction 2, what would S and R_x be for ζ , with $J = SR_x$ the bivector?
- Q3. Over $\mathbb{Q}$, is Weil's functional to be read as the two-point function of a state, and if so on which symplectic space?
- Q4. Is the Shale/Hilbert–Schmidt condition the right “infinite-genus” replacement for the finite-model compatibility of J ?
- Q5. L103: direction 1 or direction 2?

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TAA LEMMA 103: WEIL POSITIVITY AS A TWO-POINT CONDITION

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma103_*.py`, data `num/l103/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma103/verified_refs.md`.

ABSTRACT. L102 asked (Q3) whether Weil’s functional is the two-point function of a state, and on which symplectic space. It also asked (Q1) whether the Hodge–Riemann form is exactly “commutator + state”.

Verdicts. (1) *Q3: Restatement.* On real test functions f (log scale), $W(f * \tilde{h})$ is symmetric. No antisymmetric form arises from W , from the odd part, or from the σ -pairing; all of these vanish identically (Thm. 3.1). A nonzero ω appears only after adding a complex structure: multiplication by i on complex test functions, or equivalently the Hilbert transform (positive-frequency split) on real ones. Then $g = \omega(\cdot, J\cdot)$ holds by construction, and $g + i\omega \geq 0 \iff g \geq 0 \iff$ Weil positivity (Thm. 3.2). *The ω part carries no information beyond Weil’s $g \geq 0$* ([RESTATEMENT], checked numerically: complexified spectrum = $\{0\} \cup 2\text{spec}$, kernel of exactly half dimension). (2) *Invariant state* ([THEOREM]): g is scaling-invariant (Toeplitz in log scale). A scaling-invariant state exists iff Weil positivity holds, i.e. iff RH. A ghost makes the invariant form indefinite, so no invariant state exists. This confirms the L102/Q1 sharpening (“ghost = absence of an invariant state”) over $\mathbb{Q}$, but it is RH restated (circular). (3) *Numerics, prime side only* (pole + Γ + primes; no zeros used; [NUMERICAL]). The form on supports $[-L, L]$, $L \leq 16$, is positive for ζ . It agrees with the zero side (1517 zeros) to $|\Delta\mu_{\min}| \leq 1.4 \cdot 10^{-7}$. Synthetic ghosts first become visible at $L^* = 0.90$ ($\gamma = 14.13$, $\delta = 0.2$), 1.15 ($\delta = 0.05$), 1.70 ($\gamma = 30.42$) and 3.45 (the Davenport–Heilbronn location 85.70, $\delta = 0.3085$). In each case $L^* \approx \log(\gamma/2\pi) + O(1)$: a window of log-length L only sees zeros up to height $\approx 2\pi e^L$. All $L^* > \log 2$, and every ghost quartet produces exactly 2 negative directions. (4) *Q1* (§6, [THEOREM]): pure Gaussian states $\leftrightarrow$ compatible J ; mixed states $\leftrightarrow g \geq g_J$ with J non-unique; over $\mathbb{F}_q$, $h(i)$ is the unique F -invariant pure state, and ghosts admit no invariant state.

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GLOSSARY (TERMS RESTATED)

Log scale; test functions.:

$u = \log x$ on $\mathbb{R}_+^\times$. Take $f \in C_c(\mathbb{R})$ with $\text{supp } f \subset [-L/2, L/2]$, $\tilde{f}(u) = \overline{f(-u)}$ and $\hat{f}(r) = \int f(u) e^{iru} du$.

1

Date: 25 October 2026 (approved t445u; questions from L102 Q1, Q3; J.W. t446).

Weil functional.:

$W(\phi) = \sum_{\gamma} \hat{\phi}(\gamma)$, summed over nontrivial zeros $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{C}$, counted with multiplicity. Its prime-side form is the explicit formula (1).

Two-point form.:

For a state on a CCR algebra over (V, ω) , $\langle \xi \xi^T \rangle = \frac{1}{2}(g + i\omega)$, and $g + i\omega \geq 0$ (L102 [TAA-L]).

Ghost.:

A synthetic quartet $\frac{1}{2} \pm \delta \pm i\gamma$ replacing a pair $\frac{1}{2} \pm i\gamma$ (L98–L101 [TAA-L]).

Compatible J ; pure state.:

$J^2 = -1$, $J^T W J = W$, $\omega(\cdot, J\cdot) > 0$. Pure Gaussian states correspond to compatible J (§6).

1. WHAT IS ASKED

L102 Q3 (J.W.): is Weil's $W(f * \tilde{f})$ the symmetric part g of a two-point function, and on which symplectic space? Candidates for ω : the odd part, or the functional-equation pairing σ ($u \mapsto -u$, i.e. $x \mapsto 1/x$). Then test $g + i\omega \geq 0$ on restricted support using the prime side only. Track ghosts at heights 14 and 30 and at the DH location. Find where ghost-sensitivity starts beyond $\log 2$. Decide whether ω is new information or a restatement. Find a natural J on the prime side. Test for an invariant state. Check circularity. L102 Q1 is answered in §6.

2. THE EXPLICIT FORMULA AND WEIL'S CRITERION

Theorem 2.1 (Weil's explicit formula; criterion). [CITED] ([4] *§on Weil's explicit formula, eq. (1); same form in [6], quoting Hejhal 6.7*). Let h be even and analytic in $|\operatorname{Im} r| \leq \frac{1}{2} + \delta$ with $|h(r)| \leq A(1 + |r|)^{-2-\delta}$, and let $g(u) = \frac{1}{2\pi} \int_{\mathbb{R}} h(r) e^{-ir u} dr$. Then

$$(1) \quad \sum_{\gamma} h(\gamma) = h\left(\frac{i}{2}\right) + h\left(-\frac{i}{2}\right) - g(0) \log \pi + \frac{1}{2\pi} \int_{\mathbb{R}} h(r) \operatorname{Re} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ir}{2}\right) dr - 2 \sum_{n \geq 1} \frac{\Lambda(n)}{\sqrt{n}} g(\log n).$$

Weil: RH holds iff the left side is ≥ 0 for all $g = g_0 * g_0$ of the stated type ([4], “Weil showed that the Riemann hypothesis is equivalent to showing that (1) is positive for all such g ”).

For real f and $h = |\hat{f}|^2 = \hat{f}(r)\hat{f}(-r)$ we have $g = f * \tilde{f}$. Define $Q(f) := \sum_{\gamma} h(\gamma)$, computed by the right side of (1). The symmetric bilinear form $Q(f_1, f_2)$ follows by polarization.

3. THE TWO-POINT RECAST

Theorem 3.1 (no antisymmetric part from W or σ). [THEOREM] For real test functions: (i) $B(f_1, f_2) := W(f_1 * \tilde{f}_2)$ is symmetric, because the zero multiset is invariant under $\gamma \mapsto -\gamma$ (complex conjugation and the functional equation), so $W(\phi) = W(\check{\phi})$ with $\check{\phi}(u) = \phi(-u)$, and $(f_1 * \tilde{f}_2)^{\sim} = f_2 * \tilde{f}_1$. (ii) $W(f_1 * f_2)$, the σ -twisted pairing, is symmetric because convolution is commutative. (iii) Hence every bilinear form on real test functions built from W by convolution and σ has zero antisymmetric part: the “odd part” and the “ σ -pairing” candidates for ω are identically 0.

Theorem 3.2 (two-point form = Weil form). [THEOREM] Let V be the complex test functions, viewed as a real space, and $H(f_1, f_2) := W(f_1 * \tilde{f}_2)$ with $\tilde{f}(u) = \overline{f(-u)}$. H is Hermitian. Set $g = \operatorname{Re} H$, $\omega = \operatorname{Im} H$ and $J = \text{multiplication by } i$. Then: (i) ω is antisymmetric, $g(Jx, Jy) = g(x, y)$, and $g = \omega(\cdot, J\cdot)$ identically. (ii) On $V \otimes_{\mathbb{R}} \mathbb{C}$, writing $z = x + \mathbf{i}y$ with $\mathbf{i}$ the formal complex unit, $(g + \mathbf{i}\omega)(z, z) = g(x - Jy, x - Jy)$. Hence $g + \mathbf{i}\omega \geq 0 \iff g \geq 0 \iff H \geq 0 \iff$ Weil positivity. Its kernel is $\{x = Jy\}$, of half dimension: the “state” is pure by construction. (iii) On real test functions the same structure is carried by the Hilbert transform $\mathcal{H}$ (multiplier $-i \operatorname{sgn} r$): $\omega_{\mathcal{H}}(f_1, f_2) := B(f_1, \mathcal{H}f_2)$ is antisymmetric, $\mathcal{H}^2 = -1$ on $\{\hat{f}(0) = 0\}$, and $\mathcal{H}$ commutes with scaling. This is the positive-frequency complex structure of Fock quantization ([5] [CITED], §2).

Proof. (i) $H(x, iy) = iH(x, y)$ gives $\operatorname{Re}H(x, iy) = -\operatorname{Im}H(x, y)$. With the sign convention $\omega(x, y) = -g(x, Jy)$, i.e. $g = \omega(\cdot, J\cdot)$ up to orientation, the claim follows. $g(Jx, Jy) = \operatorname{Re}H(ix, iy) = g(x, y)$. (ii) Expand: $g(x, x) + g(y, y) + 2\omega(x, y) = g(x, x) + g(Jy, Jy) - 2g(x, Jy)$. (iii) $B(f_1, \mathcal{H}f_2) = \sum_{\gamma} \hat{f}_1(-\gamma)\hat{f}_2(\gamma)(-i \operatorname{sgn}\gamma)$ is odd under $1 \leftrightarrow 2$ (on the line; for complex γ , sgn refers to $\operatorname{Re}\gamma$). $\square$

Verdict on Q3 [RESTATEMENT]. Weil’s g is the symmetric part of a two-point function on (V, ω) , with V the test functions on the log-scale line $\mathbb{R} \cong \mathbb{R}_+^\times$, $\omega = \operatorname{Im}H$ (equivalently $B(\cdot, \mathcal{H}\cdot)$) and $J = i$ (equivalently $\mathcal{H}$). But ω is *manufactured from g and J* , so the condition $g + i\omega \geq 0$ is *literally* Weil positivity. No new information arises. The natural J on the prime side is the positive-frequency split; it is natural (scaling-equivariant) but does not by itself bear on RH.

4. INVARIANT STATES AND THE GHOST

Theorem 4.1 (scaling-invariant states). *[THEOREM] Translation $f \mapsto f(\cdot - \tau)$ (scaling $x \mapsto e^\tau x$) leaves $f_1 * f_2$ unchanged, so g and ω are scaling-invariant. Hence: (i) The Weil form defines a scaling-invariant state on the CCR algebra of (V, ω) iff $g \geq 0$ on all test functions, iff RH (Thm. 2.1). (ii) If a ghost quartet exists, g is indefinite. Since the form is invariant, no scaling-invariant state exists. This is the $\mathbb{Q}$ -analogue of §6, Thm. 6.2(iii): a ghost is the absence of an invariant state. (iii) Restricted supports $[-L/2, L/2]$ are not scaling-invariant subspaces. Positivity there is a compression of (i): for $L < \log 2$ no prime enters and positivity is archimedean (Yoshida, reproved by Bombieri [2] [CITED-sec]; [3] [CITED]).*

Circularity check [RESTATEMENT].

- (i) is RH; the two-point language relabels it and supplies no independent positivity.
- The natural J (i or $\mathcal{H}$) makes $g = \omega(\cdot, J\cdot)$ tautological, so “purity” is automatic once positivity holds.
- A non-circular route would need a symplectic space and a J defined *without* W , e.g. from an arithmetic object, as in L101’s degree form, *together with* a reason that W is its two-point function. Neither is available [OPEN].

5. NUMERICS: PRIME SIDE ONLY

Method ([NUMERICAL], num/lemma103_weil.py).

- Basis: hat functions of spacing $h = 0.025$ on $[-L/2, L/2]$, so $g_{jk} = h B_3((v - (u_j - u_k))/h)$ with B_3 the cubic B-spline and $\hat{\phi}_j = h \operatorname{sinc}^2(rh/2)e^{iru_j}$.
- Terms: the pole term $2c^2 \cosh(d/2)$; $-\log \pi$ times the L^2 Gram matrix; the Γ term by quadrature of $\operatorname{Re} \psi(\frac{1}{4} + \frac{ir}{2})$; and all prime powers $n \leq e^{16} \approx 8.9 \cdot 10^6$.
- Generalized eigenvalues μ of (Q, G_{L^2}) . The matrices are Toeplitz, which is the scaling invariance of Thm. 4.1.
- *Validation*: the zero side with 1517 zeros ($\gamma < 2000$, L104 data) matches the rows to 10^{-7} – 10^{-9} , and $|\mu_{\min}^{\text{prime}} - \mu_{\min}^{\text{zero}}| \leq 1.4 \cdot 10^{-7}$ (at $L = 0.5$; $\leq 10^{-9}$ for $L \geq 1$).
- *Ghosts*: the quartet contribution $\sum_{t=\pm\gamma \pm i\delta} h(t)$ minus the removed pair, added analytically (exact). The DH location $0.808517 + 85.699348i$ was verified to be a zero of the Davenport–Heilbronn function (Hurwitz-zeta form, mpmath). Since DH has no Euler product it has no prime side, so it enters only as a ghost *location* in ζ data.

L	$\mu_{\min}$ true	$g_{14}, \delta=0.2$	$g_{14}, \delta=0.05$	$g_{30}, \delta=0.2$	DH loc.
0.5	$3.8 \cdot 10^{-2}$	$+4.9 \cdot 10^{-2}$	$+4.9 \cdot 10^{-2}$	$+3.8 \cdot 10^{-2}$	$+3.8 \cdot 10^{-2}$
$\log 2$	$1.3 \cdot 10^{-3}$	$+1.3 \cdot 10^{-3}$	$+1.4 \cdot 10^{-3}$	$+1.4 \cdot 10^{-3}$	$+1.3 \cdot 10^{-3}$
1	$2.3 \cdot 10^{-6}$	$-1.4 \cdot 10^{-4}$	$+7.8 \cdot 10^{-7}$	$+2.4 \cdot 10^{-6}$	$+2.3 \cdot 10^{-6}$
2	$8.2 \cdot 10^{-8}$	-0.049	-0.0025	-0.013	$+8.2 \cdot 10^{-8}$
4	$5.1 \cdot 10^{-9}$	-0.44	-0.026	-0.42	-0.40
8	$4.9 \cdot 10^{-11}$	-3.9	-0.21	-3.8	-9.7
16	$6.3 \cdot 10^{-12}$	-45	-1.8	-45	-192

Findings ([NUMERICAL]).

- The true form stays positive for all $L \leq 16$, but $\mu_{\min} \rightarrow 0$ rapidly: there are near-null directions. At $L = 16$, $\mu_{\min}$ is about $7 \times$ the zero-side discrepancy.
- Each ghost quartet gives exactly 2 negative eigenvalues (“half the number of off-line zeros”, consistent with Bombieri [2] [CITED-sec]).
- The onset L^* (first L with $\mu_{\min} < -10 \times$ floor, step 0.05) is 0.90, 1.15, 1.70 and 3.45, against $\log(\gamma/2\pi) = 0.81, 0.81, 1.58$ and 2.61 (Fig. 1). [HEURISTIC]: the window $[-L, L]$ resolves zeros up to height $\approx 2\pi e^L$, so ghost sensitivity at height γ needs $L \gtrsim \log(\gamma/2\pi)$, plus an amount that grows as δ shrinks.
- Every $L^* > \log 2$: the primes are needed. Before L^* , ghosts are invisible even though $\|\Delta\|/\|Q\|$ is already 0.14 at $L = \log 2$: the perturbation lies in directions where Q is large.
- This refines L101’s support sweep (perturbation $\sim 10^{-3}$ at $\log 2$; that used a different norm and a zero-side kernel).

Two-point check ([NUMERICAL]). At $L = 1$ and 4, $\min \text{eig}(g + i\omega) = -3 \cdot 10^{-15}$ and $-6 \cdot 10^{-15}$, and the number of zero eigenvalues is exactly $M(39, 160)$: spectrum $\{0\} \cup 2 \text{spec}(Q)$, as Thm. 3.2(ii) predicts.

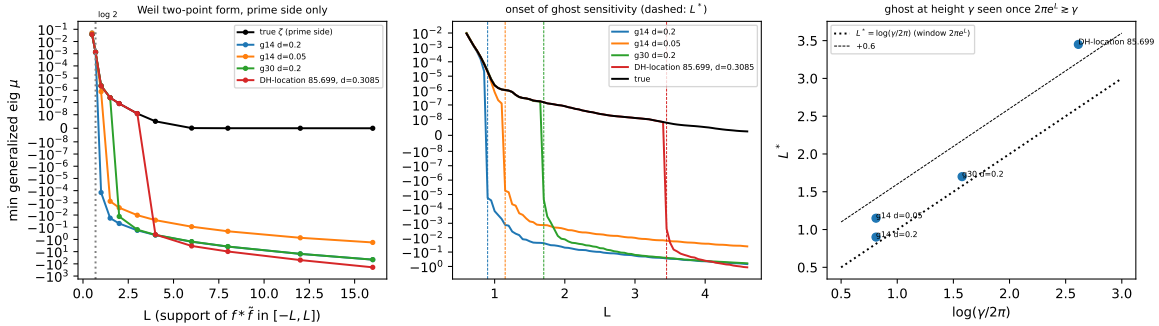


FIGURE 1. Left: $\mu_{\min}$ of the prime-side Weil form vs. L (symlog), true and ghosts. Middle: onset scan. Right: L^* vs. $\log(\gamma/2\pi)$.

6. Q1 ANSWERED: THE HODGE–RIEMANN FORM IS EXACTLY “COMMUTATOR + PURE STATE”

Question (L102 Q1, J.W. t446): is the HR form exactly ω (the commutator) plus a state J ? Are pure Gaussian states the same as compatible complex structures, and mixed states the same as $g \geq \omega(\cdot, J \cdot)$? Over $\mathbb{F}_q$, is $h(i)$ the unique pure state commuting with F ? **Verdict: yes on all counts, with two sharpenings.** (a) Positive definiteness of g alone is not enough. (b) In the mixed case J is not unique. [THEOREM] (Thm. 6.1, Thm. 6.2).

Setting. (V, ω) is real symplectic of dimension $2n$ with $\omega(x, y) = x^T W y$. A centred Gaussian state is determined by its symmetric matrix $g = \langle \xi, \xi^T \rangle$ ($= 2V$, V the noise matrix). Its two-point matrix is $\frac{1}{2}(g + i\omega)$ (L102 Thm. 3.1 [TAA-L]).

Theorem 6.1 (states $\leftrightarrow$ complex structures). *[THEOREM] (from Williamson’s theorem as stated in [1] [CITED], §IX, §XII). (i) g is the g -matrix of a quantum state iff $g + i\omega \geq 0$, iff all symplectic eigenvalues ν_k of g (Williamson: $g = S^T \text{diag}(\nu, \nu) S$, $S \in \text{Sp}(2n, \mathbb{R})$) satisfy $\nu_k \geq 1$. ([1] §IX: realisable iff every Williamson entry of V is $\geq \frac{1}{2}$.) Positive definiteness of g is necessary but not sufficient: $g = \frac{1}{2}I$ is PD, yet $\min \text{eig}(g + i\omega) = -\frac{1}{2}$. (ii) Pure $\iff$ all $\nu_k = 1 \iff \det g = 1$ (given (i)) $\iff A := W^{-1}g$ satisfies $A^2 = -1 \iff g = \omega(\cdot, J\cdot)$ with $J = A$ compatible ($J^2 = -1$, $J^T W J = W$, $WJ > 0$). The map $J \mapsto (\text{vacuum of } J)$ is a bijection between compatible complex structures and centred pure Gaussian states. Both are $\text{Sp}(2n, \mathbb{R})/U(n)$, since Sp acts transitively on Gaussian pure states with stabilizer $U(n)$ ([1] §XII). (iii) Mixed $\iff g \geq \omega(\cdot, J\cdot)$ for some compatible J , with $g \neq \omega(\cdot, J\cdot)$. One choice is $J_g = A(-A^2)^{-1/2}$, the polar part of A , which gives $g - g_{J_g} = S^T \text{diag}(\nu - 1, \nu - 1) S \geq 0$. J is not unique: for the thermal $g = 3I$, every squeezed J_r with $|r| \leq \frac{1}{2} \ln 3$ satisfies $g \geq g_{J_r}$. (iv) Purity: $\text{Tr} \rho^2 = \prod_k \nu_k^{-1} = (\det g)^{-1/2}$.*

Proof sketch. (i)–(ii) Williamson reduces to n independent modes, and for one mode $g + i\omega \geq 0 \iff \det g \geq 1$ (L102). If $\nu = 1$ then $g = S^T S$. Since $S^T W S = W$ gives $W^{-1} S^T = S^{-1} W^{-1}$, we get $A = W^{-1} S^T S = S^{-1} W^{-1} S$ and $A^2 = S^{-1} W^{-2} S = -1$. Conversely $A^2 = -1$ forces $\nu_k = 1$, since the eigenvalues of A are $\pm i\nu_k$. (iii) $g - S^T S = S^T (D - I) S \geq 0$, and $J_g = S^{-1} W^{-1} S$ is the polar part of $A = S^{-1} W^{-1} D S$. (iv) This is the standard thermal-mode product, checked below. $\square$

[NUMERICAL] (num/lemma103_Q1.py, num/1103/Q1.json).

- For 900 random states ($n = 1, 2, 3$, half pure), all satisfy $g + i\omega \geq 0$. All 450 pure ones satisfy $A^2 = -1$, $\det g = 1$, $\nu = 1$ and $g = g_J$. For all 450 mixed ones, J_g is compatible and $g - g_{J_g} \geq 0$.
- For the thermal $g = 3I$, the admissible squeeze range found is $|r| \leq 0.545$ on a 0.005 grid, against the predicted $\frac{1}{2} \ln 3 = 0.549$.
- An independent check in a truncated Fock space (dimension 160) of squeezed thermal states $(\bar{n}, r)$ gives purity $= 1/\sqrt{\det g}$: (0,0): 1; (0,0.6): 1; (0.3,0): 0.625; (0.3,0.5): 0.625; (1,0.3): 0.3333. Also $\langle [x, p] \rangle = i$ throughout.

Theorem 6.2 (uniqueness over $\mathbb{F}_q$). *[THEOREM] Let $F \in GL_{2n}(\mathbb{R})$ be semisimple with no real eigenvalues and $F^T W F = qW$. Let $J_F := h(i) \in \mathbb{R}[F]$ be the complex structure of L101 Prop. 3.1 [TAA-L] (blockwise $(2F_k - t_k)/\sqrt{4q - t_k^2}$, signs chosen so that $\omega(\cdot, J_F \cdot) > 0$). This exists iff all $|\alpha| = \sqrt{q}$ (L100 Thm. 4.1). (i) Every compatible J commuting with F equals J_F , multiplicities allowed (e.g. $E \times E$). So the vacuum of $h(i)$ is the unique F -invariant pure Gaussian state. (ii) F -invariant mixed Gaussian states form a cone with more than one point, e.g. $c g_{J_F}$ for $c \geq 1$. (iii) Ghosts: if some $|\alpha| \neq \sqrt{q}$, there is no F -invariant state at all, pure or mixed.*

Proof. (i) J_F is a polynomial in F , so any J commuting with F commutes with J_F . Then $K = J J_F$ satisfies $K^2 = J^2 J_F^2 = 1$, and K is symplectic. Its eigenspaces $V_\pm$ are ω -orthogonal: $\omega(x, y) = \omega(Kx, Ky) = -\omega(x, y)$ for $x \in V_+$, $y \in V_-$. So $\omega|_{V_+}$ is nondegenerate. On V_+ , $J = K J_F^{-1} = -J_F$, hence $\omega(x, Jx) = -\omega(x, J_F x) < 0$ for $0 \neq x \in V_+$. Compatibility of J therefore forces $V_+ = 0$, i.e. $J = J_F$. (ii) $F' = F/\sqrt{q}$ is symplectic and commutes with J_F , so it preserves g_{J_F} . (iii) An invariant $g > 0$ makes F' an isometry of a Euclidean metric, so all $|\text{eig}(F')| = 1$, i.e. $|\alpha| = \sqrt{q}$. $\square$

[NUMERICAL].

- Random complex structures in the commutant of F were sampled for genus 1 ($a = -3$), genus 2, CM $E \times E$ ($a = -2$, repeated; commutant dimension 8) and a genus-3 model with a repeated trace. All compatible ones found equal $h(i)$: 301, 224, 233 and 179 hits, with 0 exceptions. Symplectic non-compatible ones have inertia $(2n - 2k, 2k)$.
- This is sampling, not exhaustion; the proof above is what decides.
- For the ghost quartet ($\delta = 0.2$), $|\text{eig}(F')| = 0.8187, 1.2214$, and $\|\sum_{|k| \leq N} (F'^k)^T F'^k\|/(2N + 1) = 4.0, 441, 3 \cdot 10^{12}$ for $N = 5, 20, 80$: averaging cannot produce an invariant metric.

What this answers [RESTATEMENT]. “HR = commutator + state” is exact in the following form. The commutator fixes ω . A pure state is the same thing as a compatible J , and the HR metric is

its symmetric part $g_J = \omega(\cdot, J\cdot)$. Over $\mathbb{F}_q$, Frobenius selects *one* pure state, $h(i)$, when RH holds and *no* state when it fails. A ghost is therefore not a negative-norm *state*. It is the *absence* of any invariant state; the (2, 2) forms of L100–L102 are what the would-be J 's give. This refines C102.17.

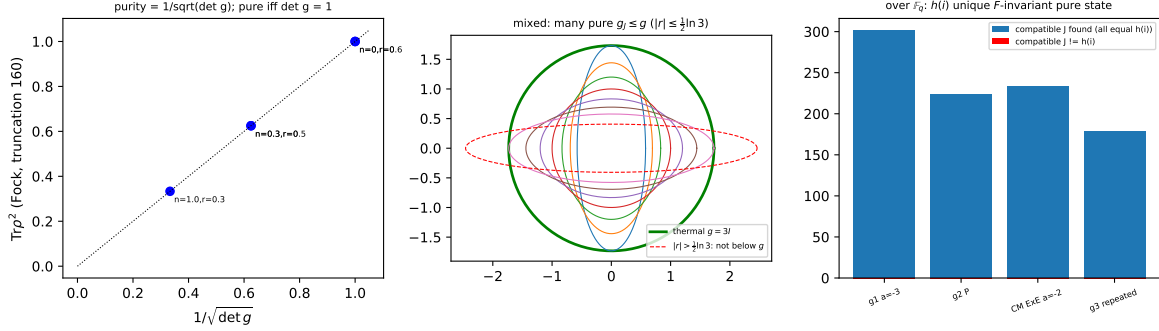


FIGURE 2. Left: $\text{purity} = 1/\sqrt{\det g}$ in Fock-space checks. Middle: a mixed (thermal) state dominates a whole family of pure g_J . Right: over $\mathbb{F}_q$, every compatible J commuting with F that was found equals $h(i)$.

7. CORRECTIONS TO THE CHAT CLAIMS AND TO THE BRIEF

C103.1 (“ ω from the odd part or from the σ -pairing”):

Off: both vanish identically on real test functions (Thm. 3.1). A nonzero ω needs an added complex structure.

C103.2 (“ ω might be new information beyond $g \geq 0$ ”):

No: $g + i\omega \geq 0 \iff g \geq 0$ (Thm. 3.2). [RESTATEMENT].

C103.3 (“natural pure state on the prime side”):

Yes in form: $J = \mathcal{H}$ (positive frequency), scaling-equivariant. But purity is tautological and positivity is RH.

C103.4 (“ghost-sensitivity starts beyond $\log 2$ ”):

Confirmed. The onset is at $L^* \approx \log(\gamma/2\pi) + O(1)$, which depends on the ghost height, not on $\log 2$ alone.

C103.5 (L101’s 10^{-3} at $\log 2$):

Not comparable: different norm. In the generalized-eigenvalue sense ghosts are invisible at $\log 2$ for all four test ghosts.

C103.6 (“DH as a test”):

DH has no Euler product, hence no prime side. It is used only as a ghost location; the location was verified.

C103.7 (L102 C102.17):

Sharpened: a ghost is the absence of an invariant state (§6, Thm. 4.1).

8. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	Spinor route (L104): build (S, R_x) , hence J , from arithmetic data, independent of W . This is the only non-circular use of the two-point language.	0.85	0.05	0.53
2	Make the onset law $L^*(\gamma, \delta)$ precise: an analytic lower bound on the negative eigenvalue of the ghost-perturbed form as a function of L, γ, δ .	0.4	0.6	0.48
3	Near-null directions of the true form ($\mu_{\min} \rightarrow 0$): identify them (prolate-type functions; cf. [3]) and their growth rate.	0.4	0.55	0.46
4	Certified version of the prime-side positivity for $L \leq L_0$ via interval arithmetic (equivalent to zero verification up to $2\pi e^{L_0}$).	0.2	0.8	0.44
5	A symplectic space defined without W (e.g. from the adelic scaling action) carrying W as a two-point function.	0.7	0.05	0.44

9. QUESTIONS FOR J.W.

- Q1. Do you accept C103.1–2: the two-point form is Weil’s form restated, with ω manufactured from J ?
- Q2. Is the onset law $L^* \approx \log(\gamma/2\pi)$ (“window sees zeros up to $2\pi e^L$ ”) the right picture of where primes start to matter?
- Q3. Which arithmetic object should supply (V, ω, J) *before* W (direction 1/5)?
- Q4. Should the near-null directions of the true form be studied as “almost-ghosts”?
- Q5. Proceed to L104 (spinors) as planned?

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TAA LEMMA 104: A SPINOR ROUTE TO J FOR ζ

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma104_*.py`, data `num/1104/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma104/verified_refs.md`.

ABSTRACT. L102 found that the Hodge circle is $\text{Spin}(2)$, i.e. the even part of $\text{Cl}(2, 0)$, generated by two anticommuting reflections (S, R_x) with $J = SR_x$. Q2: can such a *Euclidean Clifford pair* be built on a space carrying the zeros of ζ , with J commuting with the zero operator, without assuming RH? Or is there an obstruction?

Verdicts. (1) *Finite models* ([THEOREM], [NUMERICAL]). A Euclidean Clifford pair (antisymplectic involutions S, R , $SR = -RS$, with $J = SR$ compatible and $[J, F] = 0$) exists *iff* the model satisfies its RH. When it exists, $J = h(i)$ is unique (L103 §Q1), and S is complex conjugation in a g_J -orthonormal eigenbasis, reversing F . Ghosts fail in two distinct ways:

- a real pair $e^{\pm\delta}$ gives commutant generator $X^2 = +1$, so the pair spans $\text{Cl}(1, 1)$ (squeeze, as in L102);
- a quartet $e^{\pm\delta \pm i\theta}$ admits commuting J with $J^2 = -1$, but every symplectic one has g_J of signature $(2, 2)$: a *pseudo-Euclidean* pair.

Checked to 10^{-13} on genus 1, genus 2 and CM $E \times E$. For CM curves the arithmetic $J_{CM} = \iota(i)$ equals $h(i)$. (2) *Candidates for ζ* .

- σ, c commute (Klein group), so they give no Clifford pair. Negative, [THEOREM].
- $W_{\mathbb{R}}$'s j acts trivially on the Hodge type $(0, 0)$ of ζ 's motive: the Weil operator is $C = 1$. Negative, [THEOREM]/[CITED].
- Chirality σc commutes with the scaling generator D ; the grading for D is σ . Negative as a source of J .
- On $L^2(\mathbb{R}_+^\times)$ the pair $(\sigma, \sigma\mathcal{H})$, with $\mathcal{H}$ the Hilbert transform, *is* a Euclidean Clifford pair, and $J = \mathcal{H}$ commutes with scaling: unconditional ([THEOREM], [NUMERICAL]). But in Connes' spectral realization critical zeros are absorption spectrum and noncritical zeros are *resonances* [2]. A J built on the Hilbert space is *ghost-blind*.

(3) *Obstruction (dichotomy)* ([THEOREM] for finite models, [HEURISTIC] for ζ). Clifford pairs always exist algebraically. The content is “commuting + positive for the right form”. With the Weil form, existence $\iff$ RH (L103); with a W -independent form, it is unconditional but ghost-blind (also Q4: Shale HS). The spinor route is the Hilbert–Pólya problem in Clifford dress. *No non-circular J for ζ was found*. The only non-circular mechanism seen in the finite models is arithmetic (CM J , and, beyond genus 1, positivity of Rosati type, [ANALOGY], not opened). (4) Q4 (§3): Shale's HS condition is not the infinite-genus replacement for compatibility.

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Date: 25 October 2026 (approved t445u; questions from L102 Q2, Q4; J.W. t446).

GLOSSARY (TERMS RESTATED)

Symplectic model.:

(V, ω) is a real symplectic space, $\omega(x, y) = x^T W y$. For a curve of genus g over $\mathbb{F}_q$, $V = H^1(\dim 2g)$, and $F = \text{Frob}/\sqrt{q} \in \text{Sp}(V)$. RH for the model means all eigenvalues of F have $|\lambda| = 1$.

Compatible J .:

$J^2 = -1$, $J^T W J = W$, $g_J := WJ > 0$. In L102 conventions $WJ_0 = I$.

Clifford pair.:

Involutions $S^2 = R^2 = 1$ with $SR = -RS$. They generate $\text{Cl}(2, 0) \cong M_2(\mathbb{R})$, and $J = SR$ (the bivector) has $J^2 = -1$. If instead $R^2 = -1$, they generate $\text{Cl}(1, 1)$. *Euclidean* means S, R are g_J -orthogonal reflections, i.e. $g_J > 0$.

Ghost.:

A real pair $e^{\pm\delta}$ or a quartet $e^{\pm\delta \pm i\theta}$ of normalized Frobenius eigenvalues. For ζ : zeros $\frac{1}{2} \pm \delta \pm i\gamma$.

 $\sigma, c, D, \mathcal{H}$.:

$\sigma f(u) = f(-u)$ (i.e. $x \mapsto 1/x$), $cf = \bar{f}$, $D = -i d/du$ (scaling generator), and $\mathcal{H} = -i \text{sgn}(D)$ (Hilbert transform), all in log scale $u = \log x$.

1. THE SPINOR CRITERION IN FINITE MODELS

Theorem 1.1 (spinor criterion). *[THEOREM] Let $F \in \text{Sp}(V, \omega)$. The following are equivalent: (a) there is a Euclidean Clifford pair (S, R) , antisymplectic, with $J = SR$ compatible, $[J, F] = 0$ and $SFS = F^{-1}$; (b) there is a compatible J commuting with F ; (c) F is semisimple with all eigenvalues on $|\lambda| = 1$, i.e. RH for the model. If F has no eigenvalue ± 1 , then $J = h(i)$ is unique (L103, Thm. 6.2).*

Proof. (a) $\Rightarrow$ (b) is trivial. (b) $\Rightarrow$ (c): F is symplectic and commutes with J , so it preserves $g_J = \omega(\cdot, J\cdot) > 0$. Hence F is orthogonal for a positive metric, so semisimple and unitary. (c) $\Rightarrow$ (a): choose an F -adapted symplectic basis in which $F = \bigoplus_k R_{\theta_k}$. Take $J = \bigoplus J_0$ and $S = \bigoplus \text{diag}(1, -1)$. Then $S^T W S = -W$, $SJ_0S = -J_0$ and $SR_{\theta}S = R_{-\theta}$. Put $R = SJ$. Then $R^2 = S(JS)J = -S^2J^2 = 1$, $SR = J$, $RS = SJS = -J$, so $SR = -RS$. Explicitly, $R = \bigoplus \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, a g_J -orthogonal reflection. $\square$

Spinor reading [RESTATEMENT]. The Frobenius is $F = e^{\Theta J}$ with $\Theta = \bigoplus \theta_k$: an element of the $\text{Spin}(2)$'s generated by the bivector $J = SR$. Each block is “a rotation as a product of two reflections”, $F_k = S \cdot (SF_k)$, and SF_k is again an F -reversing reflection. The ghost alternatives:

Theorem 1.2 (how ghosts fail). *[THEOREM] (i) Real pair (2×2 block, $|\text{tr} F| > 2$): the commutant is $\mathbb{R}[X]$ with $X = (F - \frac{t}{2})/\sqrt{t^2/4 - 1}$ and $X^2 = +1$. Any F -reversing involution S anticommutes with X , and $R = SX$ has $R^2 = -1$, so $\langle S, X \rangle = \text{Cl}(1, 1)$ and $F = \cosh r + X \sinh r = e^{rX}$ is a squeeze. No J exists. In the elliptic case the same construction gives $X^2 = -1$ and $\text{Cl}(2, 0)$. (ii) Quartet (4×4 block, eigenvalues $e^{\pm\delta \pm i\theta}$, $\delta > 0$): the real J with $J^2 = -1$ that are functions of F are the four sign choices $\pm i$ on the two conjugate pairs. Two of them are antisymplectic, so they are not admissible. The two symplectic ones satisfy $\text{sig } g_J = (2, 2)$, because they preserve the two Lagrangian eigenplanes ($|\lambda| = e^{\pm\delta}$), and ω pairs these planes off-diagonally.*

Proof of (ii). $\omega(E_+, E_+) = 0 = \omega(E_-, E_-)$ for the generalized eigenplanes $E_{\pm}$ (since $\omega(Fx, Fy) = \omega(x, y)$ forces $\lambda\mu = 1$ for pairing). A J that is a function of F preserves $E_{\pm}$, so g_J has block form $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$, which is indefinite of signature $(2, 2)$. $\square$

Numerics ([NUMERICAL], num/lemma104_spinor.py, Fig. 1).

- *Genus 1*, $q = 101$, $a \in \{-20, -16, \dots, 20\}$ (Hasse range): $X^2 = -1$, $S^2 = R^2 = 1$, $\|SX + XS\| = 0$, $J = J_F$ exactly, and S reverses F . $\text{cond}(g_J)$ grows to 402 at $|t| = 1.99$: the pair degenerates at the Hasse edge. Ghost traces $t = \pm 2 \cosh \delta$ ($\delta = 0.05, 0.1, 0.3$) all give $X^2 = +1$, $R^2 = -1$, i.e. $\text{Cl}(1, 1)$.

- *Genus 2*, 5 random true models (random symplectic conjugation): of the 4 commuting J , exactly one has signature $(4, 0)$, and it equals J_F ($\leq 2 \cdot 10^{-13}$). The others are $(2, 2)$, $(2, 2)$ and $(0, 4)$. The spinor S satisfies all relations to $\leq 10^{-13}$.
- *Quartets*, $\delta = 0.05, 0.2, 0.5$: zero compatible J ; the symplectic ones are $(2, 2)$ and the other two are antisymplectic.
- *Real block $\oplus$ true block*: no J at all.

Proposition 1.3 (CM supplies J arithmetically). *[THEOREM] Let $E/\mathbb{F}_p$ have CM by $\mathbb{Z}[i]$, $p = a^2 + b^2$, with Frobenius $\pi = a + bi$ acting on $H^1 \cong \mathbb{Z}[i]$ by multiplication. Then $\iota(i)$ commutes with F , satisfies $\iota(i)^2 = -1$ and preserves ω , and $F = \cos \theta + \sin \theta \iota(i)$, so $\iota(i) = J_F = h(i)$. On $E \times E$, $J_{CM} \oplus J_{CM} = J_F$. Every other commuting $J = i(P_+ - P_-)$ (orthogonal or oblique, 4000 samples per p) has $\min \text{eig } g_J \leq -1$ ([NUMERICAL]; $p = 13, 101, 10009$).*

Here J comes from an endomorphism, not from the zeros. Positivity is automatic in dimension 2. In higher genus the analogous role is played by a positivity of Rosati type ([ANALOGY]; the standard input of Weil's proof, not opened here, so not cited).

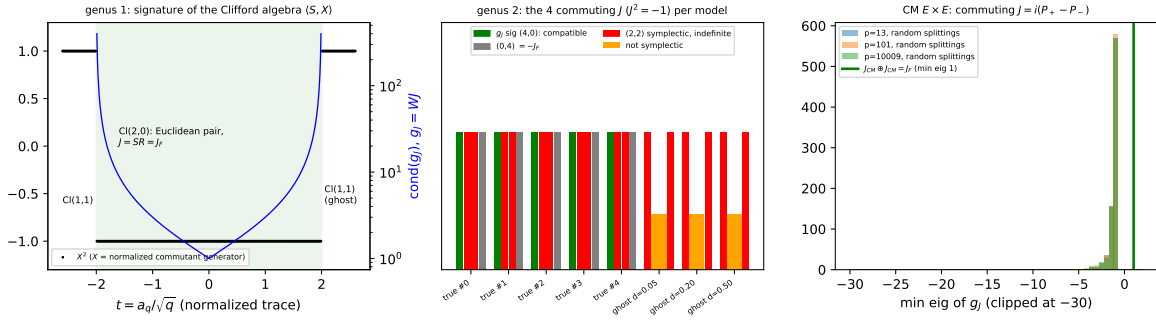


FIGURE 1. Left: genus 1, the sign of X^2 (so $Cl(2, 0)$ vs. $Cl(1, 1)$) and $\text{cond}(g_J)$ vs. normalized trace. Middle: signatures of the 4 commuting J in genus-2 true and ghost-quartet models. Right: CM $E \times E$, min eig g_J for random commuting J ; only $J_{CM} = J_F$ is positive.

2. CANDIDATES FOR ζ

Proposition 2.1 (σ and c). *[THEOREM] $\sigma c = c\sigma$, so $\{1, \sigma, c, \sigma c\}$ is a Klein group. Commuting involutions give $(\sigma c)^2 = +1$, not a J . Both reverse D ($\sigma D \sigma = -D$, $c D c = -D$), so the chirality σc commutes with D . The $\mathbb{Z}_2$ -grading anticommuting with D is σ itself. No Clifford pair arises.*

Proposition 2.2 ($W_{\mathbb{R}}$ and Hodge). *[CITED]/[THEOREM] $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ inside the quaternions [6], so $j^2 = -1$. A Hodge structure is an action of the Deligne torus, and the Weil operator is $C = i^{p-q}$ on $H^{p,q}$ [5]. For ζ (the 1-dimensional structure of type $(0, 0)$, cf. the Tate structures [5]), $C = 1$. The Hodge/Weil-group J for ζ is trivial, and its 1-dimensional Betti space carries no zeros. The finite-model $J = h(i)$ lives on H^1 , which has no finite-dimensional counterpart for ζ .*

Theorem 2.3 (scaling pair on $L^2(\mathbb{R}_+^\times)$). *[THEOREM] On $L^2(\mathbb{R}, du)$, i.e. log scale, set $S = \sigma$ and $R = \sigma \mathcal{H}$ on $\{\hat{f}(0) = 0\}$. Then $S^2 = R^2 = 1$, $SR = -RS$, and $J = SR = \mathcal{H}$ is real, unitary, satisfies $J^2 = -1$, and commutes with all scalings U_τ , while $\sigma U_\tau \sigma = U_{-\tau}$. So $(\sigma, \sigma \mathcal{H})$ is a Euclidean Clifford pair (orthogonal reflections for the L^2 metric) with J commuting with the zero-operator candidate D . This holds unconditionally. Numerically, on a periodic grid ($N = 2048$), all relations hold to $4 \cdot 10^{-16}$ ([NUMERICAL]).*

Why this does not help ([RESTATEMENT]/[HEURISTIC]).

- The positive form here is L^2 , not Weil's.
- In Connes' trace-formula picture [2] (abstract): “critical zeros ... as an absorption spectrum, while eventual noncritical zeros appear as resonances”. RH is reduced there to the validity of the trace formula. A ghost would not be a vector of the Hilbert space on which $J = \mathcal{H}$ acts, so this J neither detects nor excludes ghosts.
- $\mathcal{H}$ is a spectral function of D , so it commutes with every scaling-equivariant map. Formally it therefore descends to Connes-type cokernels ([HEURISTIC]; domain issues not checked).
- Whether σ preserves such a cokernel is not established ([OPEN]; it would need the Fourier/functional-equation operator).

Dirac doubling ([THEOREM]). On $\mathcal{K} \otimes \mathbb{C}^2$, $D_{\text{Dirac}} = D \otimes \sigma_x$ is graded by $1 \otimes \sigma_z$, and $J = 1 \otimes i\sigma_y$ commutes with $D \otimes 1$. This always works and carries no information. Connes–Consani spectral triples [1] are not used beyond what L101 recorded (direction 5).

Theorem 2.4 (obstruction / dichotomy). [THEOREM] (finite) / [HEURISTIC] (ζ). (i) Clifford pairs exist on every even-dimensional real space ($\text{Cl}(2, 0) \cong M_2(\mathbb{R})$). The obstruction is never algebraic. (ii) If the positive form is the Weil form g (L103), a compatible J commuting with scaling exists iff $g \geq 0$, iff RH. Ghosts give $(2, 2)$ blocks (Thm. 1.2(ii)); this is the $\mathbb{Q}$ -version of “a ghost is the absence of an invariant state”. (iii) If the form is W -independent (L^2 , HS-model A of §3), the pair exists unconditionally and is ghost-blind. Hence a non-circular spinor route needs a positivity source for the Weil form that is not $g \geq 0$ itself: the analogue of Rosati positivity (Prop. 1.3). Circularity flag: every construction of J for ζ above either assumes RH or does not see the zeros' real parts.

3. Q4 ANSWERED: SHALE'S HILBERT–SCHMIDT CONDITION IS NOT THE INFINITE-GENUS REPLACEMENT FOR “ J COMPATIBLE”

Question (L102 Q4, J.W. t446): with infinitely many zeros, is Shale's HS condition the right replacement for finite-genus compatibility of J ? **Verdict: No.** In the truncated zero model below: (a) HS-finiteness of $J_\zeta - J_0$ holds *unconditionally* for the true zeros, given the right normalization (Thm. 3.1). (b) It *fails* for the true zeros under other natural normalizations ([NUMERICAL]). (c) It is *blind* to ghosts: one, several, or infinitely many (Thm. 3.2). Ghosts are detected only by *blockwise compatibility* (positivity), which in this model is a [RESTATEMENT] of RH. HS is an additional, independent, global regularity condition: unitary equivalence to the archimedean Fock space. It is not a substitute.

Model ([HEURISTIC] as a model choice). Block n is $(\mathbb{R}^2, \omega)$ with zero operator $D_n = \frac{1}{2} + X_n$, $X_n \in \mathfrak{sp}(2)$. On the line, $X_n = \gamma_n J_n$ with J_n compatible, and the L101 formula $(2D_n - 1)/\sqrt{4\lambda_n - 1}$ returns $\pm J_n$. The reference (archimedean, Γ -factor) vacuum J_0 sits at the points γ_n^0 defined by $\theta(\gamma_n^0) = (n - \frac{3}{2})\pi$, with θ the Riemann–Siegel theta function, the phase of the Γ -factor. So $N_0(\gamma_n^0) = n - \frac{1}{2}$ for the smooth count $N_0 = \theta/\pi + 1$. The two vacua are related by a squeeze $B(r_n) = \text{diag}(e^{r_n}, e^{-r_n})$, $J_n = B(r_n)J_0B(r_n)^{-1}$. Three normalizations: (A) $r_n = \frac{1}{2} \log(\gamma_n/\gamma_n^0)$, frequency ratio (scale-invariant); (B) $r_n = \gamma_n - \gamma_n^0$, additive; (B2) $r_n = (\gamma_n - \gamma_n^0) \log(\gamma_n/2\pi)/2\pi$, unfolded to unit mean spacing. Per block, $\|J_n - J_0\|_F^2 = (e^{2r_n} - 1)^2 + (e^{-2r_n} - 1)^2 \approx 8r_n^2$. Shale's quantity is $\|(B^T B)^{1/2} - 1\|^2 = (e^{r_n} - 1)^2 + (e^{-r_n} - 1)^2 \approx 2r_n^2$, so both are finite iff $\sum r_n^2 < \infty$ [4] [CITED].

Theorem 3.1 (model A is HS-finite unconditionally). [THEOREM] (using the Riemann–von Mangoldt formula with $S(T) = O(\log T)$, [3] [CITED-sec] eqs. (1.3)–(1.4)). $|\gamma_n - \gamma_n^0| = O(1)$, hence $|r_n| \ll 1/\gamma_n$ and $\sum_n r_n^2 \ll \sum_n \gamma_n^{-2} < \infty$. Therefore $J_\zeta = \oplus J_n$ is Shale-implementable on the Fock space of J_0 , with no hypothesis on the real parts (they do not enter model A).

Proof sketch. $N(t) = \theta(t)/\pi + 1 + S(t) + O(1/t)$. Evaluating at γ_n , where $N = n - \frac{1}{2}$ in the mean-of-jumps convention, and subtracting $\theta(\gamma_n^0)/\pi = n - \frac{3}{2}$ gives $\theta(\gamma_n) - \theta(\gamma_n^0) = -\pi S(\gamma_n) + O(1/\gamma_n) = O(\log \gamma_n)$. Since $\theta'(t) = \frac{1}{2} \log(t/2\pi) + O(1/t)$, the mean value theorem gives $|\gamma_n - \gamma_n^0| = O(1)$. Then $r_n = \frac{1}{2} \log(1 + (\gamma_n - \gamma_n^0)/\gamma_n^0) = O(1/\gamma_n)$, and $\gamma_n \gg n/\log n$. $\square$

[NUMERICAL] (num/lemma104_zeros.py, num/lemma104_Q4.py; data num/1104/).

- 1517 zeros with $0 < \gamma < 2000$ were found by sign changes of $Z(t)$. The count matches $N_0(2000) = 1516.99$, and the first and last zeros agree with `mpmath.zetazero`.
- $\max |\gamma_n - \gamma_n^0| = 0.994$ and the rms is 0.343. In model A, $\max |r_n| \gamma_n = 0.499$.
- $\|J_N - J_0\|_{HS}^2$ at $N = 10, 100, 1000, 1517$: model A 0.00615, 0.00900, 0.00965, 0.00968 (converging; tail over $(N/2, N]$ is $6.6 \cdot 10^{-5}$); model B 56.9, 422, 2071, 2753 (diverging); model B2 2.06, 47.8, 785, 1272 (diverging).
- The vacuum overlaps $\prod (\cosh r_n)^{-1/2}$ are 0.99970 (A), $3 \cdot 10^{-19}$ (B) and $6 \cdot 10^{-12}$ (B2) at $N = 1517$.
- So HS-finiteness for the *same true zeros* depends on the normalization chosen for the reference vacuum ([NUMERICAL]; divergence in B and B2 is numerical, with no proof given).

Theorem 3.2 (HS is ghost-blind; compatibility is not). *[THEOREM] Replace the zero pair at block k by a quartet $\frac{1}{2} \pm \delta \pm i \gamma_k$ ($\delta \neq 0$). The block becomes $(\mathbb{R}^4, \omega)$ with $X = \text{diag}(\delta + \gamma_k J_0, -\delta + \gamma_k J_0) \in \mathfrak{sp}(4)$. (i) No compatible J commutes with X . The two symplectic complex structures in its commutant give $\min \text{eig}(g + i\omega) = -2$ ([NUMERICAL]), and there is no X -invariant state at all (L103, uniqueness theorem (iii) [TAA-L]). (ii) Any fixed choice of J on the ghost block changes $\|J - J_0\|_{HS}^2$ by a finite amount. So HS-finiteness, and hence Shale implementability, is unchanged by finitely many ghosts. It is also unchanged by infinitely many ghosts, provided the chosen J 's on ghost blocks are square-summably close to J_0 .*

[NUMERICAL]: ghost at $k = 1$, ghost at $k = 100$, and ghosts at every 50th zero (30 blocks up to $N = 1517$, $J = J_0$ on ghost blocks). The model-A HS^2 changes from 0.009685 by $-1.4 \cdot 10^{-3}$, $-1.0 \cdot 10^{-5}$ and $-3.6 \cdot 10^{-5}$ respectively; all remain finite. Meanwhile the number of blocks failing compatibility is 1, 1 and 30 (growing linearly). For all true blocks $|\min \text{eig}(g + i\omega)| \leq 1.7 \cdot 10^{-16}$.

What this implies.

- (1) [RESTATEMENT]: the infinite-genus analogue of “ J compatible” is *blockwise* compatibility (positivity of the two-point form in every block). In this model it is RH, because the model is built from the zeros.
- (2) [THEOREM]: Shale/HS is a separate condition. It asks whether J_ζ lives in the same Fock space as the archimedean vacuum J_0 . It concerns the tail (asymptotic positions) and is insensitive to finitely many, or sparse, off-line zeros.
- (3) [THEOREM]: in the scale-invariant normalization the HS condition holds *unconditionally* (Riemann–von Mangoldt), so it carries no RH information. In additive or unfolded normalizations it fails even for the true zeros.
- (4) [OPEN]: a canonical, zero-independent construction of the symplectic space and of J_0 over $\mathbb{Q}$ (L103's core question) is still missing. Without it the model is circular, since the zeros were inserted by hand.

4. CORRECTIONS TO THE CHAT CLAIMS AND TO THE BRIEF

C104.1 (“ σ and c as the Clifford pair”):

They commute (Klein group). Their product is an involution, not J (Prop. 2.1).

C104.2 (“ $W_{\mathbb{R}}$'s j gives J for ζ ”):

$j^2 = -1$ in $W_{\mathbb{R}}$, but the Weil operator on ζ 's type- $(0,0)$ structure is 1 (Prop. 2.2).

C104.3 (“chirality $\sigma \circ c$ ”):

It commutes with D . The grading anticommuting with D is σ .

C104.4 (“spinor recovers $h(i)$ for true models, fails for ghosts”):

Confirmed (Thms. 1.1, 1.2), but the construction takes F (the zeros) as input. As an RH route it is circular, except via arithmetic J (CM).

C104.5 (L102: “ghost = Spin(1,1)”):

Refined. A real-pair ghost is $\text{Cl}(1,1)$; a *quartet* ghost has $J^2 = -1$ but signature $(2,2)$, i.e. pseudo-Euclidean, not Lorentzian. The quartet is the ζ -relevant type.

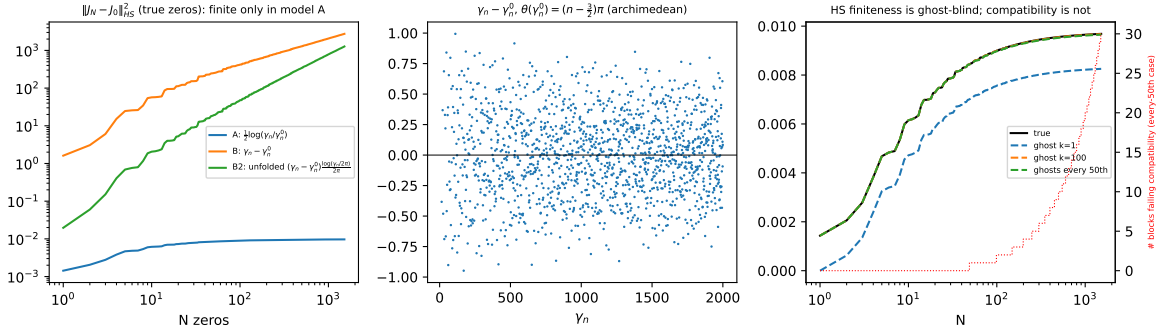


FIGURE 2. Left: $\|J_N - J_0\|_{HS}^2$ for 1517 true zeros in three normalizations. Middle: $\gamma_n - \gamma_n^0$ ($|\cdot| < 1$). Right: ghosts barely move the HS sum but break compatibility in every ghost block.

C104.6 (Connes' spectral approach):

Only the abstract of [2] was opened. The statement “noncritical zeros are resonances” is quoted from it; nothing further is claimed.

5. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	Positivity source of Rosati type for $\mathbb{Q}$ (what CM shows in the finite case): identify a candidate operator algebra or correspondence.	0.95	0.03	0.58
2	$\mathcal{H}$ -type J on spaces where noncritical zeros are spectral (Connes' earlier δ -parametrized approach, mentioned in the abstract of [2]; not opened [OPEN]): test in truncated zero models whether g_J turns (2, 2) exactly at ghosts. This gives a ghost-seeing version of Thm. 2.3.	0.6	0.45	0.54
3	Quartet signature (2, 2) as a certified ghost detector in truncated zero models (L103 numerics in spinor form).	0.3	0.8	0.50
4	Genus ≥ 3 , CM abelian varieties: uniqueness of the arithmetic J and multiplicities.	0.2	0.9	0.48
5	Open Connes–Consani [1] fully: look for a grading/real structure giving an anticommuting pair with W -related form.	0.5	0.4	0.46

6. QUESTIONS FOR J.W.

- Q1. Do you accept the dichotomy (Thm. 2.4): with the Weil form the spinor J is RH-equivalent, and with a W -independent form it is ghost-blind?
- Q2. Is the quartet picture (pseudo-Euclidean (2, 2), not $\text{Spin}(1, 1)$) the right spinor image of an off-line zero?
- Q3. The CM J is arithmetic and equals $h(i)$. Is there a $\mathbb{Q}$ -analogue of “an endomorphism squaring to -1 commuting with Frobenius”?
- Q4. Should direction 2 ($\mathcal{H}$ -type J on a space where noncritical zeros are spectral) be L105?
- Q5. Is your hand-sequence intuition (L102 C102.13) related to $F = S \cdot (SF)$, a rotation as two reflections?

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TAA LEMMA 105: THE SPACE OF HODGE–RIEMANN STATES AND LAX–PHILLIPS: RESONANCES INTO EIGENVALUES

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RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma105_*.py`, data `num/1105/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma105/verified_refs.md`.

ABSTRACT. J.W.’s chat claims t447–t450 are checked under (a)–(d).

Results.

- **(a) Hodge–Riemann states = Siegel space** ([THEOREM]/[CITED]). The compatible J for fixed ω form $\mathbb{H}_g = \mathrm{Sp}(2g, \mathbb{R})/U(g)$. A symplectic F commutes with J_Z iff $F \cdot Z = Z$. For $g = 1$, $\mathbb{H}$ is the upper sheet of $\{aJ_0 + bS + cR_x : a^2 - b^2 - c^2 = 1\}$, i.e. the space of future unit timelike spinor bivectors. Elliptic Frobenius has a timelike axis and hence a fixed point; a ghost has a spacelike axis and two null fixed rays on $\partial\mathbb{H}$. J.W.’s numbers are confirmed: $(-3 + i\sqrt{395})/2$, and 19.93, 5.07 for the fake trace. In genus 2 a true model has a unique fixed $Z \in \mathbb{H}_2$, while a ghost quartet has none (0/200 vs. control 200/200).
- **(b) Scattering phase** ([THEOREM], [NUMERICAL]). $d(r) = -\frac{1}{2\pi}\varphi'/\varphi(\frac{1}{2} + ir)$ is *exactly* an absolutely convergent sum of Lorentzians:

$$d(r) = -\frac{1}{2\pi(r^2+1/4)} + \frac{1}{2\pi} \sum_{\rho} [L_{\beta/2} + L_{(1-\beta)/2}](r - \frac{\gamma}{2}).$$

On the line every zero is a bump of area 1, HWHM exactly $\frac{1}{4}$ and peak $4/\pi$. So J.W.’s “equal widths” is exact. A ghost gives HWHM $\frac{1}{4} \pm \frac{\delta}{2}$ with the same area and $|\varphi| = 1$ unchanged: *counting and unitarity are ghost-blind; only the shape sees ghosts*. Model vs. exact agree to $7 \cdot 10^{-7}$, and J.W.’s peaks lie within 0.01 of $\gamma/2$.

- **(c) Elliptic points.** $S = J_0$ has order 4 in SL_2 and order 2 in PSL_2 . $\xi(2\pi) = 0.06363$, so $s = \pi$ is *not* a pole (corrected). The genuine roles of the elliptic points are: the $\frac{1}{2}$ in $E(z, s)$; the orbifold area $\pi/3$; and the height $a = 1$ of i , which is the edge of the Lax–Phillips cut range.
- **(d) Lax–Phillips** ([CITED] [6, 4], [THEOREM], [NUMERICAL]). The convention is confirmed: $\sigma(B'') = \{(\rho - 1)/2\}$, and $\mathrm{RH} \iff$ uniform damping $\frac{1}{4}$. *Closure fails, for three precise reasons.*
 - No self-adjoint operator can have the resonances as eigenvalues, since $\mathrm{Im} \lambda_{\rho} = \gamma(1 - \beta)/2 \neq 0$ even under RH.
 - B'' is never skew-adjoint (because $\mathrm{Re} \mu < 0$); only $2B'' + \frac{1}{2}$ can be, and that is Hilbert–Pólya (LP80).
 - A phase closure $e^{i\theta}$ at $y = a$ gives real levels with $N \approx N_{\zeta}(2R) + \frac{R}{\pi} \log a$, so no one-to-one match for $a > 1$. At $a = 1$ (height of i) and $\theta = 3\pi/4$ the levels sit within a median 2% of a spacing from $\gamma/2$, but never on them. The misfit is $2\Delta_k + O(1/\gamma)$, where Δ_k is the change of $\arg \zeta$ from $1 + i\gamma_k$ to $\frac{1}{2} + i\gamma_k$ ([NUMERICAL], to $4 \cdot 10^{-3}$).

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GLOSSARY (TERMS RESTATED)

Hodge–Riemann state / compatible J :

$J^2 = -1$, $J^T W J = W$, $g_J = WJ > 0$, with $WJ_0 = I$ (L102–L104). Pure Gaussian states correspond to compatible J (L103 §6).

Spinor basis of $\mathfrak{sl}_2(\mathbb{R})$:

$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, with $(aJ_0 + bS + cR_x)^2 = (-a^2 + b^2 + c^2)I$ (L102).

Modular surface:

$X = SL_2(\mathbb{Z}) \backslash \mathbb{H}$, $\lambda = s(1-s) = \frac{1}{4} + r^2$. Eisenstein constant term $y^s + \varphi(s)y^{1-s}$; $\Lambda(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$; $\rho = \beta + i\gamma$.

Lax–Phillips:

$U(t)$ is the automorphic wave group, $\mathcal{D}_\pm$ the outgoing/incoming subspaces above the cut $y > a$, $Z(t) = P_+ U(t) P_-$ with generator B , and B'' the version with cusp forms removed ([6] §3, §6).

 L_w :

$L_w(x) = w/(w^2 + x^2)$, with $\int L_w = \pi$ and HWHM w .

1. CLAIMS CHECKED

(a) Pure states = Siegel space; Frobenius fixed points; genus 2 and ghosts. (b) Weyl law $\lambda/12$; scattering phase $\varphi = \Lambda(2s-1)/\Lambda(2s)$; $|\varphi| = 1$; poles at $\rho/2$; peaks at $\gamma/2$; “equal widths”; ghost widths $\frac{1}{4} \mp \frac{\delta}{2}$. (c) The stabilizer of i is generated by $S = J$; i is the state of $y^2 = x^3 - x$; the “ $s = \pi$ ” idea. (d) Lax–Phillips core, conventions, ghosts, and the closure boundary condition.

2. (A) THE SPACE OF HODGE–RIEMANN STATES

Theorem 2.1 (Siegel space). *[THEOREM]/[CITED]. (i) $\mathrm{Sp}(2g, \mathbb{R})$ acts on $\mathbb{H}_g = \{Z = X + iY : Z = Z^T, Y > 0\}$ by $Z \mapsto (AZ + B)(CZ + D)^{-1}$, transitively, with the stabilizer of $i1_g$ equal to $U(g)$, so $\mathbb{H}_g = \mathrm{Sp}(2g, \mathbb{R})/U(g)$ [12]. Moreover $\Gamma_g \backslash \mathbb{H}_g$ ($\Gamma_g = \mathrm{Sp}(2g, \mathbb{Z})$) parametrizes principally polarized abelian varieties [12]. (ii) $Z \mapsto J_Z := GJ_0G^{-1}$, with $G = \begin{pmatrix} Y^{1/2} & XY^{-1/2} \\ 0 & Y^{-1/2} \end{pmatrix}$, is a bijection from $\mathbb{H}_g$ onto the compatible J . Indeed $WJ_Z = G^{-T}G^{-1} > 0$, and the inverse reads X, Y off $(WJ)^{-1} = GG^T$. (iii) $F \in \mathrm{Sp}(2g, \mathbb{R})$ commutes with J_Z iff $F \cdot Z = Z$: the stabilizer of Z is $GU(g)G^{-1}$, which is the centralizer of J_Z in Sp .*

Proposition 2.2 ($g = 1$ in spinor form). *[THEOREM] Write $X = aJ_0 + bS + cR_x$. Then $X^2 = -1$ iff $a^2 - b^2 - c^2 = 1$, and $WX = aI + bR_x - cS$ is positive iff $a > 0$. So compatible J form the upper sheet of the unit hyperboloid in $(\mathfrak{sl}_2, \text{Killing}) \cong \mathbb{R}^{1,2}$, with*

$$a = \frac{1+x^2+y^2}{2y}, \quad b = \frac{1-x^2-y^2}{2y}, \quad c = \frac{x}{y} \quad (z = x + iy).$$

F' acts by conjugation, i.e. by $SO^+(1, 2)$. Elliptic F' has a timelike axis, which is its fixed point in $\mathbb{H}$ and is $h(i)$. Hyperbolic F' (ghost) has a spacelike axis, whose orthogonal plane meets the light cone in two null rays. These are its two fixed points on $\partial\mathbb{H}$: $J.W.$ ’s “null rays at infinity”.

Numerics ([NUMERICAL], num/lemma105_siegel.py).

- With $F = \begin{pmatrix} a & -q \\ 1 & 0 \end{pmatrix} / \sqrt{q}$ (companion basis) the fixed points solve $z^2 - az + q = 0$. For $a = -3$, $q = 101$: $z = -1.5 + 9.93730i$, with $[J_z, F] = 2 \cdot 10^{-16}$ and $\mathrm{spec} g_J = \{0.098, 10.17\}$. For $a = 25$: $z = 19.9330$ and 5.0670 , real.
- Genus 2 (4 random true models): exactly one compatible commuting J . The resulting $Z \in \mathbb{H}_2$ satisfies $\|F \cdot Z - Z\| \leq 2.4 \cdot 10^{-15}$ and the round trip $J \rightarrow Z \rightarrow J$ holds to $2 \cdot 10^{-15}$.
- Ghost quartets ($\delta = 0.1, 0.3$): the commuting J are $(2, 2)$ or antisymplectic, and no fixed point was found in $\mathbb{H}_2$ from 200 starts. The same search on a true model finds it 200/200 times.

Basis caveat ([THEOREM], [NUMERICAL]). The fixed point depends on the $\mathbb{Z}$ -lattice. The companion basis sees the order $\mathbb{Z}[\pi]$: for $a = -3$ the fixed point is a CM point of discriminant -395 , whose class number is $h = 8$ (8 reduced forms). Over $\mathbb{F}_{101}$ the $j = 1728$ curves $y^2 = x^3 + Dx$ have traces $\{\pm 2, \pm 20\}$ (brute force), and $y^2 = x^3 - x$ has trace -2 . Its companion fixed point is $1 \pm 10i \sim 10i$ (order $\mathbb{Z}[10i]$), *not* i . Traces ± 20 give $10 \pm i \sim i$. With the $\mathbb{Z}[i]$ -module lattice of L104 Prop. 1.3, the state is i for all of them.

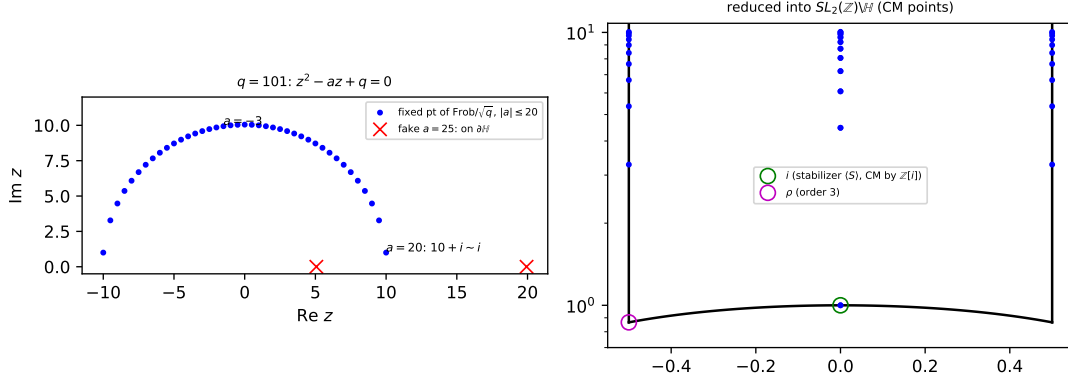


FIGURE 1. Left: fixed points of normalized Frobenius ($q = 101$, companion basis) for $|a| \leq 20$; the fake trace $a = 25$ lies on $\partial\mathbb{H}$. Right: the same points reduced into $SL_2(\mathbb{Z})\backslash\mathbb{H}$; i and ρ marked.

3. (B) THE LAPLACIAN AND THE SCATTERING PHASE

Theorem 3.1 (known facts). [CITED].

- $\varphi(s) = \Lambda(2s-1)/\Lambda(2s)$ [8, (24)]; equivalently LP's $s(z) = \Gamma(\frac{1}{2})\Gamma(iz)\zeta(2iz)/(\Gamma(\frac{1}{2}+iz)\zeta(1+2iz))$ with $s = \frac{1}{2} + iz$ [6, (4.7)'].
- Poles in $\text{Re } s < \frac{1}{2}$ are resonances [8], located at $\rho/2$. The pole at $s = 1$ has residue $3/\pi$ [10], and area $= \pi/3$ [6, §8], so $3/\pi = 1/\text{area}$ (the L97 shadow $3/(\pi\gamma)$).
- Weyl law: $N(T) = \#\{t_j \leq T\} \sim T^2/12$ [7, (1.1)]; in general $N(\lambda) \sim \frac{\text{area}}{4\pi}\lambda$ [8, (25)]. J.W.'s $\lambda/12$ is confirmed.
- The winding identity is $\sum_{|t_j| \leq T} 1 - \frac{1}{2\pi} \int_{-T}^T \frac{\varphi'}{\varphi}(\frac{1}{2} + it) dt \sim \frac{\text{area}}{2\pi} T^2$ [8, App. 3]. This fixes the density on $r \geq 0$ as $d(r) = -\frac{1}{2\pi} \varphi'/\varphi$; J.W.'s $-\frac{1}{4\pi}$ is the full-line normalization.

$|\varphi(\frac{1}{2} + ir)| = 1$ follows from $\varphi(s)\varphi(1-s) = 1$ and $\overline{\varphi(\bar{s})} = \varphi(s)$ ([THEOREM]).

Theorem 3.2 (exact Lorentzian decomposition). [THEOREM] For real r , with the sum over all nontrivial zeros $\rho = \beta + i\gamma$ (absolutely convergent),

$$d(r) = -\frac{1}{2\pi} \frac{1}{r^2 + \frac{1}{4}} + \frac{1}{2\pi} \sum_{\rho} \left[L_{\beta/2}(r - \frac{\gamma}{2}) + L_{\frac{1-\beta}{2}}(r - \frac{\gamma}{2}) \right].$$

Proof. Let $\Xi(w) = \frac{1}{2}w(w-1)\Lambda(w)$ (entire, order 1). Then $\varphi(s) = \frac{\Xi(2s-1)}{\Xi(2s)} \cdot \frac{s}{s-1}$, and by Hadamard (classical; not re-opened here) $\text{Re } \Xi'/\Xi(w) = \sum_{\rho} \text{Re } \frac{1}{w-\rho}$, absolutely convergent. At $s = \frac{1}{2} + ir$: $\text{Re}[\frac{1}{s-(\rho+1)/2} - \frac{1}{s-\rho/2}] = -L_{\beta/2}(x) - L_{(1-\beta)/2}(x)$ with $x = r - \gamma/2$, and $\text{Re}[\frac{1}{s} - \frac{1}{s-1}] = \frac{1}{r^2+1/4}$. Multiply by $-\frac{1}{2\pi}$. $\square$

Corollary 3.3 (widths). [THEOREM] (i) On-line zero ($\beta = \frac{1}{2}$): the bump is $\frac{1}{\pi} L_{1/4}(r - \gamma/2)$, with area 1, HWHM exactly $\frac{1}{4}$ (the distance of the pole $\rho/2$ from $\text{Re } s = \frac{1}{2}$) and peak $4/\pi$. “Equal widths”

is exact. (ii) A ghost pair $\frac{1}{2} \pm \delta + i\gamma$ gives $\frac{1}{\pi}[L_{1/4+\delta/2} + L_{1/4-\delta/2}]$: J.W.'s $\frac{1}{4} \mp \frac{\delta}{2}$ is exact, and each of the two zeros carries both widths with weight $\frac{1}{2}$. (iii) The area, and hence the winding count, is the same as for an on-line double zero, and the synthetic φ_{ghost} still has $|\varphi| = 1$. Counting and unitarity are ghost-blind; only the shape is ghost-sensitive.

Numerics ([NUMERICAL], num/lemma105_scatter.py, Fig. 2).

- Exact d (mpmath) vs. the Lorentzian sum (1517 zeros < 2000 plus a density tail) on $r \in [0.3, 25]$: $\max|\Delta| = 6.9 \cdot 10^{-7}$.
- J.W.'s peaks agree with $\gamma/2$ within 0.0096 (grid 0.02).
- Isolated bumps $k = 1..10$: HWHM $0.25041 \rightarrow 0.25003$, area $1.0019 \rightarrow 1.0002$, centre to $2 \cdot 10^{-5}$ (small finite-window bias).
- Ghost at γ_1 , $\delta = 0.1, 0.2, 0.4$: peak 2.65, 3.03, 7.07 vs. 2.55 for an on-line double zero, with equal area and $\max||\varphi_g| - 1| = 9 \cdot 10^{-16}$.
- $\int_0^{25} d = 9.23$ vs. $N_\zeta(50) = 10$; the difference is the $-\frac{1}{2}$ pole term and the cut bump at 24.89.

Circularity [RESTATEMENT]. φ is built from ζ , so reading widths off φ is locating zeros. Self-adjointness of Δ gives $|\varphi| = 1$, which is ghost-blind (Cor. 3.3(iii)). RH is the statement that all resonance widths equal $\frac{1}{4}$.

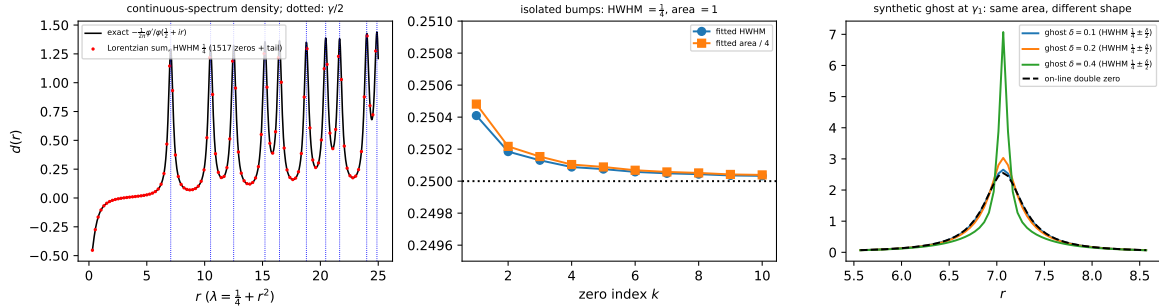


FIGURE 2. Left: exact density vs. Lorentzian sum. Middle: fitted HWHM and area/4 of isolated bumps. Right: ghost vs. on-line double zero at γ_1 .

4. (C) ELLIPTIC POINTS

Proposition 4.1. [CITED]/[THEOREM]. (i) $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = J_0$ satisfies $S^2 = -I$, so it has order 4 in $SL_2(\mathbb{Z})$ and order 2 in $PSL_2(\mathbb{Z}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ [11]. It fixes i ; i corresponds to $\mathbb{C}/\mathbb{Z}[i]$, $j = 1728$, the curve $y^2 = x^3 - x$ ($j = 1728 \cdot 4a^3/(4a^3 + 27b^2)$ with $a = -1$, $b = 0$). (ii) $\Lambda(2\pi) = 0.0636341$, so $\varphi(\pi) = 1.16070$ is finite. $s = \pi$ is not a pole (J.W.'s value 0.0636 is right; the conclusion is corrected). (iii) Genuine versions:

- $E(z, s) = \frac{1}{2} \sum_{(m,n)=1} y^s / |mz + n|^{2s}$ [10]: the $\frac{1}{2}$ is the centre $\pm I$, i.e. the double cover.
- The orbifold area is $2\pi(2g - 2 + \#cusps + \sum(1 - \frac{1}{m_i})) = 2\pi(-2 + 1 + \frac{1}{2} + \frac{2}{3}) = \frac{\pi}{3}$ (Gauss–Bonnet, [THEOREM]; matches [6]).
- LP's trace formula has elliptic terms indexed by the order m , with $2\cos(\pi/m) = \text{tr}$ [6, §8].
- LP require the cut at $1 < a_0 < a$ [6, (4.8)]. The excluded height $a = 1$ is the height of i , and it is exactly where the closure of §5 becomes Gram-like ([HEURISTIC]).

5. (D) LAX–PHILLIPS: RESONANCES INTO EIGENVALUES

Theorem 5.1 (Lax–Phillips, Faddeev–Pavlov). [CITED] [6] (expository account of [5], which was not opened).

- $Z(t) = P_+U(t)P_-$, $t \geq 0$, is a strongly continuous semigroup on $\mathcal{K}$ (Thm. 3.1). Its generator B has compact resolvent (Thm. 3.2) and pure point spectrum (Cor. 3.3), and the poles of $s(z)$ lie in $-i\sigma(B)$ (Cor. 4.3).
- For the modular group the poles of S' are the zeros of $\zeta(1+2iz)$, so under RH they lie on $\text{Im } z = \frac{1}{4}$.
- B'' has spectrum $\{(\kappa - 1)/2\}$ over the nontrivial zeros κ , and “ $2B'' + 1/2$ is the kind of operator ... envisaged by Hilbert; all that is missing is an appropriate inner product ... which makes [it] skew-hermitian” (§6).
- Pavlov–Faddeev: $RH \iff Z''(t)$ decays exponentially. LP: RH holds if $\limsup t^{-1} \log \|Z''(t)f\|_E \leq -\frac{1}{4}$ on a dense set (6.19). (The FP paper [3] was not opened; this is quoted from LP80.)
- The energy form is indefinite (§2, §5).
- [4]: under RH “the spectrum of B'' [lies] on the line $\text{Re } \lambda = -1/4$ ”. The authors add that they “do not believe that this approach will be helpful in verifying the Riemann hypothesis”.
- [9] (abstract): the spectrum of the generator coincides with the poles of the scattering matrix, with multiplicities.

Checks of J.W.’s statements ([THEOREM]).

- $\mu = \rho/2 - \frac{1}{2} = (\rho - 1)/2$ is exactly LP’s convention.
- Independence of the cut a : $\sigma(B'')$ equals the set of poles, which does not involve a ([CITED], [9, 6]). We did *not* compute B on the surface; the poles were computed instead (honest limitation).
- “Contraction gives only $\text{Re } \mu \leq 0$ ”: yes. Unconditionally $0 < \beta < 1$ gives $-\frac{1}{2} < \text{Re } \mu < 0$.
- $RH \iff$ uniform damping $\frac{1}{4}$; a ghost gives damping $\frac{1}{4} \mp \frac{\delta}{2}$ (Fig. 3).

Theorem 5.2 (why closure cannot work). [THEOREM] (i) The resonance eigenvalue $\lambda_\rho = s(1-s)$ at $s = \rho/2$ has $\text{Im } \lambda_\rho = \frac{\gamma(1-\beta)}{2} \neq 0$. No self-adjoint realization of Δ , under any boundary condition, has resonances as eigenvalues, even under RH. (ii) A skew-adjoint generator has $\text{Re } \mu = 0$, but $\text{Re } \mu = (\beta-1)/2 < 0$. So B'' is skew-adjoint for no inner product. Only the shifted $2B'' + \frac{1}{2}$ can be, and finding that inner product is the Hilbert–Pólya problem: if it exists, all $\text{Re } \kappa = \frac{1}{2}$ [RESTATEMENT]. (iii) Phase closure. Gluing the zeroth mode at $y = a$, $Aa^{1/2+ir}e^{i\theta} = \varphi(\frac{1}{2} + ir)Aa^{1/2-ir}$, means $\Phi_a(r) := \arg \varphi(\frac{1}{2} + ir) - 2r \log a \equiv \theta$. Since $|\varphi| = 1$ the levels are real. Equivalently $\text{Im}(e^{i\alpha} a^{ir} \Lambda(1 + 2ir)) = 0$: these are zeros of a function on $\text{Re } s = 1$, not on $\text{Re } s = \frac{1}{2}$. That this defines a self-adjoint operator (a pseudo-Laplacian type) is [HEURISTIC]; it was not checked in the literature here.

Numerics ([NUMERICAL], num/lemma105_lp.py, _delta.py).

- The winding $-\arg \varphi(\frac{1}{2} + iR)/2\pi$ is 1.26, 6.25, 13.07, 20.92, 29.11, 38.14 vs. $N_\zeta(2R) = 1, 6, 13, 21, 29, 38$ for $R = 10, \dots, 60$.
- Closure levels up to $R = 60$: 37, 39, 45, 50, 68 for $a = 1, 1.1, 1.5, 2, 5$, vs. 38 zeros. The predicted excess is $\frac{R}{\pi} \log a = 0, 1.8, 7.7, 13.2, 30.7$. For $a > 1$ (LP’s range) the closure spectrum is too dense to match the zeros.
- The phase each zero would need, $\theta_k = \Phi_a(\gamma_k/2)$, has circular concentration 0.93, 0.16, 0.06, 0.34, 0.40. Only $a = 1$ is nearly constant.
- At $a = 1$, $\theta^* = 2.364 \approx 3\pi/4$: levels lie within median 0.020 and max 0.095 of a local spacing from $\gamma_k/2$, and none within 10^{-3} .
- Identity ([THEOREM] asymptotic via the Riemann–Siegel θ , [NUMERICAL] to $4.4 \cdot 10^{-3}$ for $k \leq 20$): $\frac{1}{2}(\theta_k - \frac{3\pi}{4}) = \Delta_k + O(1/\gamma_k)$, where $\Delta_k = \arg \zeta(\frac{1}{2} + i\gamma_k) - \arg \zeta(1 + i\gamma_k)$ along the horizontal.

Hitting every zero needs θ to depend on the zero through Δ_k , i.e. through the critical strip: circular.

Spinor reading ([ANALOGY]). On the 2-dimensional channel $\{y^{1/2+ir}, y^{1/2-ir}\}$ a closure is a phase, i.e. Spin(2), a rotation. Resonance decay is a boost $e^{-t/4}$, i.e. Spin(1, 1). RH says all boosts are equal, so a single global rescaling $e^{t/4}$ removes them (Hilbert’s $2B'' + \frac{1}{2}$). A ghost leaves unequal boosts that no single rescaling removes. A phase can never produce a boost; this is the content of Thm. 5.2(i)–(ii).

Comparisons ([CITED]/[ANALOGY]).

- Connes' cut-off is $R_\Lambda = \hat{P}_\Lambda P_\Lambda$ ([2] §V, (15)–(16)): a compression by two projections, structurally like $Z(t) = P_+ U(t) P_-$. In it critical zeros are absorption lines and noncritical zeros are resonances.
- Berry–Keating [1, §6] regularize XP by truncating X and P ; this reproduces the smooth zero count including $7/8$. They call the boundary condition that would make XP Hermitian with discrete spectrum “a central unsolved problem”, and note that a Bloch–Floquet phase $e^{i\alpha}$ gives different self-adjoint extensions.
- J.W.'s $e^{i\theta}$ is BK's α . Our $a = 1$ result shows such a phase reproduces the smooth count, not the zeros.

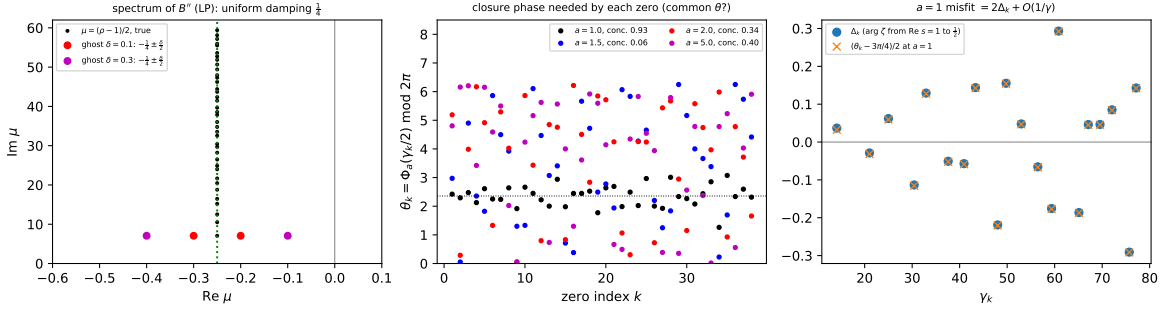


FIGURE 3. Left: $\sigma(B'')$ (true) and ghost eigenvalues. Middle: closure phase θ_k required by each zero for $a = 1, 1.5, 2, 5$. Right: at $a = 1$ the misfit equals $2\Delta_k$.

6. CORRECTIONS TO THE CHAT CLAIMS (T447–T450)

C105.1 (a).:

Confirmed (Thm. 2.1, Prop. 2.2), with a caveat: the fixed point depends on the lattice. The companion basis sees $\mathbb{Z}[\pi]$; the $a = -3$ point is a CM point of discriminant -395 ($h = 8$).

C105.2 (b) normalization.:

The density on $r \geq 0$ is $-\frac{1}{2\pi}\varphi'/\varphi$; J.W.'s $-\frac{1}{4\pi}$ is the full-line version. Weyl $\lambda/12$, $|\varphi| = 1$ and poles at $\rho/2$ are confirmed; the peaks are confirmed to 0.01.

C105.3 (b) “equal widths”.:

Confirmed *exactly*: HWHM $\frac{1}{4}$ in r , area 1 per zero. Ghost widths $\frac{1}{4} \mp \frac{\delta}{2}$ are confirmed exactly, with weight $\frac{1}{2}$ each per zero. Caveat: area and $|\varphi|$ are ghost-blind.

C105.4 (c) $s = \pi$.:

Corrected: $\xi(2\pi) = 0.0636 \neq 0$, so there is no pole. Genuine versions are in Prop. 4.1(iii).

C105.5 (c) $i \leftrightarrow y^2 = x^3 - x$.:

True over $\mathbb{C}$. Over $\mathbb{F}_{101}$ in the companion basis, $y^2 = x^3 - x$ (trace -2) sits at $10i$; only traces ± 20 sit at i .

C105.6 (d).:

The convention $\mu = (\rho - 1)/2$, “contraction is trivial”, “RH = uniform damping $\frac{1}{4}$ ” and “ghost = unequal damping” are all confirmed (LP80, LP74).

C105.7 (d) closure.:

Fails, as J.W. expected, for the three precise reasons of Thm. 5.2, plus the counting obstruction for $a > 1$.

C105.8 (title).:

“Resonances into eigenvalues”: they become eigenvalues of the non-self-adjoint B'' (LP), never of a self-adjoint operator.

7. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	Inner product for $2B'' + \frac{1}{2}$ (LP80's “missing” piece): test in a Sz.-Nagy–Foiias model with synthetic Blaschke data whether uniform damping is equivalent to such an inner product, and how ghosts obstruct it.	0.9	0.05	0.56
2	The $a = 1$ closure: statistics of Δ_k and the relation to Gram's law; is the cut through i distinguished?	0.4	0.7	0.52
3	Certified width spectroscopy: extract resonance widths from φ on finite windows (equivalent to zero verification).	0.3	0.8	0.50
4	Ghost-sensitivity of LP's criterion (6.19) on explicit dense data in a model.	0.45	0.5	0.47
5	Frobenius fixed points as CM/Heegner points across q , and their equidistribution.	0.3	0.6	0.42

8. QUESTIONS FOR J.W.

- Q1. Does the spinor picture (the space of J as a hyperboloid; ghosts as spacelike axes with null fixed rays) match your “null rays at infinity”?
- Q2. Is the exact width statement (HWHM $\frac{1}{4}$, area 1, ghost $\frac{1}{4} \pm \frac{\delta}{2}$ with equal area) what you meant by “equal widths”?
- Q3. Your closure phase is Berry–Keating's α . Do you accept the three-part obstruction (Thm. 5.2) and the counting excess $\frac{R}{\pi} \log a$?
- Q4. Is the near-success at $a = 1$ (the height of i), with misfit $2\Delta_k$, the “orbit-closure” effect you had in mind?
- Q5. Should L107 pursue direction 1, the missing inner product, in a finite model?

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TAA LEMMA 106: ROSATI-TYPE POSITIVITY FOR $\mathbb{Q}$

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma106_*.py`, data `num/1106/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma106/verified_refs.md`.

ABSTRACT. Over $\mathbb{F}_q$, the Rosati involution $\dagger$ makes $\text{Tr}(\varphi\varphi^\dagger)$ positive definite on $\text{End}^0(A)$, and this gives RH for curves. We ask what a $\mathbb{Q}$ -analogue needs: an algebra $\mathcal{A}$, an involution $\dagger$, and a trace Tr with $\text{Tr}(\varphi\varphi^\dagger) \geq 0$ *proved by counting*.

Results.

- **Where the positivity comes from** ([CITED] [5], [THEOREM], [NUMERICAL]). It comes from geometry, not from arithmetic. In Milne’s proof, $\text{Tr}(\alpha\alpha^\dagger)$ is a positive multiple of the intersection number $(D^{g-1} \cdot \alpha^{-1}D)$, which is positive because the intersection is non-empty: it is a count of points. In genus 1 this is literally $\text{Tr}_{H^1}(\varphi\varphi^\dagger) = 2 \deg \varphi = 2\#\ker \varphi$. We check this by brute-force counts, e.g. $\#E(\mathbb{F}_{101^2}) = 10235 = N(1 - \alpha^2)$ and $\#\ker(1 + \pi) = 89 = p + 1 + a$. Frobenius enters only through $\pi\pi^\dagger = q$.
- **Dictionary theorem** ([THEOREM]). For $F \in \text{GSp}(2g)$ and $\mathcal{A} = \mathbb{R}[F]$ with $F^\dagger = qF^{-1}$, three conditions are equivalent: Rosati positivity on $\mathcal{A}$; a compatible J with $[J, F] = 0$ (the L104 spinor criterion); and RH for the model. Exactly on the commutant of J , $\dagger$ is the Cartan involution of J (numerically to 10^{-11} , against 148 off the commutant). The signature defect is $n_- = \#\{\text{real ghost pairs}\} + 2\#\{\text{quartets}\}$, which matches Bombieri’s count of two negative directions per quartet. Calibration ([NUMERICAL]): true models give $(2, 0)$, $(4, 0)$, $(8, 0)$; ghosts give $(1, 1)$, $(2, 2)$, $(3, 1)$, $(4, 4)$. On all of $\text{End}(H^1)$ the same form is indefinite, $(2, 2)$ and $(8, 8)$: *positivity needs a small algebra*.
- **Candidates for $\mathbb{Q}$** ([NUMERICAL], [CITED]). Seven candidates were tested: the Weil distribution, L^2/degree traces, Bost–Connes, Hecke, Connes’ trace formula, the Connes–Consani Sonin trace, and the scaling-site square. *None* gives a trace that is positive, a count, and ghost-sensitive.
 - The Weil trace is ghost-sensitive but is not a count. The prime term is indefinite (signature $(98, 221)$ at $L = 8$), and positivity is a cancellation: $\mu_{\min} = 5 \cdot 10^{-11}$ out of terms of size 17.
 - Counting traces (degree n of μ_n , $\sigma(n)$ for T_n , $\zeta(\beta) = \sum n^{-\beta}$) are positive but ghost-blind.
 - The one zero-free positivity of the right shape is Connes–Consani’s $W_\infty(g * g^*) \geq \text{Tr}(\vartheta(g)S\vartheta(g)^*) \geq 0$ [3]. It holds only for support $L \leq \log 2$. There, all 20 tested ghosts ($\gamma \leq 99$, $\delta \leq 0.49$) are invisible; the strongest becomes visible at $L^* = 0.75$, just 8% beyond $\log 2$.
- **Circularity** ([RESTATEMENT]). “Weil positivity = Rosati positivity for $\mathbb{Q}$ ” is a restatement. What is missing is the analogue of Weil’s 1940 lemma $2m_2 = \text{Tr}(X \circ X')$ [5]: an identity equating the ghost-sensitive Lefschetz/Weil trace with a degree count.

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Date: 25 October 2026 (approved t452u; top L104 direction, score 0.58).

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GLOSSARY (TERMS RESTATED)

Polarization, Rosati.:

$\lambda : A \rightarrow A^\vee$ is an isogeny of the form φ_D with D ample. Then $\alpha^\dagger = \lambda^{-1}\alpha^\vee\lambda$ on $\text{End}^0(A) = \text{End}(A) \otimes \mathbb{Q}$ [5, p.21]. On H^1 it is the symplectic adjoint $\varphi^\dagger = \omega^{-1}\varphi^T\omega$, since $e_\lambda(\alpha x, y) = e_\lambda(x, \alpha^\dagger y)$.

Compatible J , Cartan involution.:

$J^2 = -1$, $\omega(J\cdot, J\cdot) = \omega$, and $g_J = \omega(\cdot, J\cdot) > 0$ (L102–L105). The Cartan involution is $\varphi^\theta = g_J^{-1}\varphi^T g_J$, and $\text{Tr}(\varphi\varphi^\theta) = \|\varphi\|_{HS}^2$.

Ghost.:

A fake Frobenius root set that violates $|\alpha| = \sqrt{q}$ but keeps the functional equation $\alpha \mapsto q/\alpha$: either a real pair $\sqrt{q}e^{\pm\delta}$ or a quartet $\sqrt{q}e^{\pm\delta \pm i\theta}$. For ζ , a ghost is an off-line quartet $\frac{1}{2} \pm \delta + i(\pm\gamma)$.

Weil form.:

$Q_L(f) = W(f * \tilde{f})$ for f supported in $[-\frac{L}{2}, \frac{L}{2}]$ (log scale), computed from the prime side: pole + archimedean $-\log \pi \cdot L^2 + \text{primes}$. RH $\iff Q_L \geq 0$ for all L ([6], L103).

Count.:

A non-negative integer that is the cardinality of a finite set: a kernel, an intersection, or a fibre.

1. CLAIMS CHECKED (BRIEF FOR L106)

(R1) Over $\mathbb{F}_q$, positivity of $\text{Tr}(\varphi\varphi^\dagger)$ on $\text{End}(\text{Jac}) \otimes \mathbb{Q}$ “comes from arithmetic”. (R2) $\dagger$ comes from the polarization. (R3) In the CM case $\iota(i) = h(i)$. (R4) A $\mathbb{Q}$ -analogue needs $(\mathcal{A}, \dagger, \text{Tr})$ with $\text{Tr}(\varphi\varphi^\dagger) \geq 0$ by counting, and candidates are Hecke, Bost–Connes, convolution on $\mathbb{Q}_+^\times$ or the adèle class space, Connes’ cross product, and Weil’s positivity. (R5) Without a count, “Weil = Rosati for $\mathbb{Q}$ ” is a restatement.

2. WHERE ROSATI POSITIVITY COMES FROM OVER $\mathbb{F}_q$

Theorem 2.1 (Weil, Rosati, Castelnuovo; as in [5]). *[CITED] Let $A/\mathbb{F}_q$ be an abelian variety with polarization λ from an ample D . (i) [5, Thm. 1.27]: $\text{Tr}(\alpha\alpha^\dagger) = \frac{2g}{(D^g)}(D^{g-1} \cdot \alpha^{-1}D) > 0$ for $\alpha \neq 0$. The proof cuts with hyperplanes $H_1, \dots, H_{g-1}$ to get a non-empty 0-dimensional intersection, so the intersection number is > 0 . (ii) [5, 1.29–1.30]: $\pi\pi^\dagger = q$ by the Weil pairing. On the semisimple $\mathbb{Q}[\pi]$, a positive involution must be complex conjugation, so $|\rho(\pi)| = q^{1/2}$. (iii) [5, pp. 5–6]: Weil’s 1940 “important lemma” states $2m_2 = \text{Tr}(X \circ X')$ for an (m_1, m_2) -correspondence with $m_1 = g$; hence $\text{Tr}(X \circ X') \in \mathbb{Z}_{\geq 0}$. The number of coincidences is $m_1 + m_2 - \text{Tr}(X)$. Weil’s 1941 trace is $\sigma(X) = d_1(X) + d_2(X) - (X \cdot \Delta)$. (iv) [5, p. 13]: Castelnuovo: $\sigma(D, D) = 2d_1d_2 - (D^2) \geq 0$.*

Proposition 2.2 (genus 1: trace = twice a count). *[THEOREM] On H^1 of an elliptic curve, the symplectic adjoint of a 2×2 matrix is its adjugate, so $\varphi\varphi^\dagger = \det \varphi \cdot I$ and $\text{Tr}_{H^1}(\varphi\varphi^\dagger) = 2 \deg \varphi = 2\#\ker \varphi$ for separable φ . On $\mathbb{Q}[\pi]$ with basis $(1, \pi)$ the Gram matrix is $\begin{pmatrix} 2 & a \\ a & 2q \end{pmatrix}$, with determinant $4q - a^2$. Positivity of the count $\deg(m - n\pi) = m^2 - amn + qn^2$ is therefore the Hasse bound (L101; [8]).*

Numerics ([NUMERICAL], num/lemma106_rosati.py).

- *True curves.* Brute-force counts over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ agree exactly with the degrees.
 - $p = 13$, $y^2 = x^3 + 2x + 3$ ($a = -4$): $\#E(\mathbb{F}_{13}) = 18 = N(1 - \alpha)$, $\#E(\mathbb{F}_{169}) = 180 = N(1 - \alpha^2)$, $\#\ker(1 + \pi) = 10 = p + 1 + a$.

- $p = 101, y^2 = x^3 - x$ ($a = -2$): 104, 10400, 100.
- $p = 101, y^2 = x^3 + x + 7$ ($a = -13$): 115, 10235, 89.

The kernel of $1 + \pi$ is counted directly: $1 + \#\{f(x) = 0\} + 2\#\{f(x) \text{ non-square}\}$.

- *Ghosts.* A fake trace $a = 25, q = 101$ gives $\deg(10 - \pi) = -49$, which is a *negative count* and hence impossible. The minimum over $|m| \leq 60, n < 20$ is -1951 .

3. ROSATI = CARTAN ON THE COMMUTANT: THE DICTIONARY

Theorem 3.1 (dictionary). *[THEOREM] Let (V, ω) be symplectic of dimension $2g$. Let F satisfy $\omega(Fx, Fy) = q\omega(x, y)$, put $\mathcal{A} = \mathbb{R}[F]$, and use the symplectic adjoint $\dagger$, so $F^\dagger = qF^{-1}$ and $\mathcal{A}^\dagger = \mathcal{A}$. The following are equivalent: (i) $\text{Tr}(\varphi\varphi^\dagger) > 0$ for all $0 \neq \varphi \in \mathcal{A}$; (ii) F is semisimple and all of its eigenvalues have $|\alpha| = \sqrt{q}$ (RH for the model); (iii) there is a compatible J with $[J, F] = 0$ (L104 spinor criterion; equivalently a Euclidean Clifford pair). Moreover, if J is compatible, then $\varphi^\dagger = \varphi^\theta$ (Cartan) for every φ commuting with J . In general, n_- of the form on $\mathcal{A}$ equals $\#\{\text{real pairs } \alpha \neq q/\alpha\} + 2\#\{\text{quartets}\}$, counted with Jordan-free multiplicity one.*

Proof. Write $g_J = \omega J$ up to the sign that makes it positive. Then $\varphi^\theta = g_J^{-1}\varphi^T g_J = J^{-1}\varphi^\dagger J$, which equals $\varphi^\dagger$ iff $\varphi^\dagger$ commutes with J . Since $J^\dagger = -J$, this holds for every φ in the commutant of J .

(iii) $\Rightarrow$ (i): $\text{Tr}(\varphi\varphi^\theta) = \|\varphi\|_{HS}^2 > 0$.

(i) $\Rightarrow$ (ii): this is the argument of [5, 1.29]. A positive involution forces $\mathcal{A}$ to be semisimple and acts as complex conjugation on each factor $\mathbb{C}$, so $\alpha\bar{\alpha} = q$.

(ii) $\Rightarrow$ (iii): L104 Thm. 3.1.

Signature: $\mathcal{A} \otimes \mathbb{C}$ splits over the orbits of the roots under $\{\bar{\cdot}, \alpha \mapsto q/\alpha\}$. A unit pair gives $\mathbb{C}$ with $x\bar{y}$, signature $(2, 0)$. A real pair gives $\mathbb{R} \oplus \mathbb{R}$ with the swap, $x_1y_2 + x_2y_1$, signature $(1, 1)$. A quartet gives $\mathbb{C} \oplus \mathbb{C}$ with the swap, signature $(2, 2)$. $\square$

Proposition 3.2 (CM $E \times E$). *[THEOREM] $\text{End}^0(E \times E) = M_2(K)$ with K imaginary quadratic. For the product polarization, $\dagger$ is the conjugate transpose and the form is $\text{Tr}_{K/\mathbb{Q}} \text{tr}(\varphi\bar{\varphi}^T)$, signature $(8, 0)$. Here $\iota(i) \in K \otimes \mathbb{R} = \mathbb{C}$ acts as a scalar, so it is central in $\text{End}^0 \otimes \mathbb{R} = M_2(\mathbb{C})$, and $\dagger$ is the Cartan involution of $J = \iota(i)$ on the whole algebra. This is the L104 statement $\iota(i) = J_F = h(i)$ in Rosati form. The real-quadratic ghost analogue ($\dagger = \text{Galois} \circ \text{transpose}$) gives $(4, 4)$.*

Numerics ([NUMERICAL]).

- *Genus 2, $p = 11$, from brute-force counts over $\mathbb{F}_{11}$ and $\mathbb{F}_{121}$.*
 - $y^2 = x^5 + x + 3$: (13, 141), $P = T^4 + T^3 + 10T^2 + 11T + 121$.
 - $y^2 = x^5 + 2x^3 + x + 5$: (10, 142), $P = T^4 - 2T^3 + 12T^2 - 22T + 121$.
 - $y^2 = x^5 + x^4 + 3x^2 + 2$: (7, 147), $P = T^4 - 5T^3 + 25T^2 - 55T + 121$.
- In all three, $|\alpha|/\sqrt{p} = 1$ to 10^{-12} and the Rosati form on $\mathbb{Q}[\pi]$ is $(4, 0)$, with $\lambda_{\min} = 3.14, 2.74, 0.77$.
- *Ghost models.* The quartet $(\delta = 0.01, 0.05, 0.15, 0.3)$ is $(2, 2)$ with $\lambda_{\min} = -0.014, -0.36, -3.3, -14.8$. A unit pair plus a real pair is $(3, 1)$.
- *Cartan check.* For a random $\text{Sp}(4)$ conjugate of $\sqrt{11}(R_{1,0} \oplus R_{2,2})$ with J : on $\mathbb{R}[F]$, $\max |\dagger - \theta| = 1.2 \cdot 10^{-11}$; for a random X the difference is 148.
- *Full algebra.* $\text{Tr}(XX^\dagger)$ on all of $M_4(\mathbb{R})$ is $(8, 8)$, and on $M_2(\mathbb{R})$ it is $(2, 2)$ ($= 2 \det$).

Remark 3.3 (two positivities). *[THEOREM]/[TAA-L] Rosati positivity on $\mathcal{A}$ (Thm. 3.1) and the Hodge index on $\text{NS}(C \times C)$ (L101: $(1, 3)$ true and $(2, 2)$ ghost, i.e. Castelnuovo, Thm. 2.1(iv)) are Weil’s two proofs (1940 and 1941; [5, p. 6]). The first lives on End^0 (torsion), the second on the surface $C \times C$. For $\mathbb{Q}$ the second needs a “square of $\text{Spec } \mathbb{Z}$ ”, which is exactly what [3] says is “not yet in shape” (intro, refs. [12, 16]).*

Corollary 3.4 (circularity). *[RESTATEMENT] Let $\mathcal{A}$ be any algebra containing the scaling flow, with $\dagger$ the σ -involution $f \mapsto f^\dagger$ and Tr the Weil distribution. Then “Rosati positivity” $\iff$ Weil*

positivity $\iff$ RH ([6]; [3, (2)]). By Thm. 3.1 it is also $\iff$ the existence of the L104 J. Connes makes the same point about Goldfeld’s Weil-distribution inner product: its positivity “is of course equivalent to” RH “but this does not give any clue how to prove” it [4, p. 50]. A proof needs the trace to be a count, as in Prop. 2.2.

4. CANDIDATES FOR $\mathbb{Q}$: POSITIVE? A COUNT? GHOST-SENSITIVE?

Candidate $(\mathcal{A}, \dagger)$	Tr	pos.	count	ghost-sens.	verdict
$C_c(\mathbb{R}_+^\times)$ or $\mathbb{Q}_+^\times$ -conv., $\bar{f}^\sharp$	Weil distribution W (explicit formula)	$\iff$ RH	no	yes	circular
same	L^2 trace $\delta_1: \sum f ^2$	yes	trivially	no	blind
$\mathbb{Q}[\mathbb{N}^\times] / \text{Bost-Connes}, \mu_n^\dagger$	degree: $\tau(\mu_n \mu_m^\dagger) = n \delta_{nm}$ ($\# \ker n _{\mathbb{Q}/\mathbb{Z}} = n$)	yes	yes	no	blind
Bost-Connes C^* -system	partition function $\zeta(\beta) = \sum n^{-\beta}, \beta > 1$ [4, III]	yes	yes	no	blind
Hecke T_n on $L^2(X)$, ad-joint	Hilbert trace; $\deg T_n = \sigma(n)$ cosets [7, (30)]	yes	yes	no*	blind
Adèle class space, cross product [4]	trace of scaling = W	$\iff$ RH	no	yes	circular
Sonin compression [3]	$\text{Tr}(\vartheta(g)S\vartheta(g)^*)$	yes	op. pos.	no (window)	partial

* ζ enters only through the Eisenstein part, where $|\varphi| = 1$ and the count are ghost-blind (L105).

4.1. The Weil trace is not a count. [NUMERICAL] (num/lemma106_qcand.py, the L103 prime-side machinery, $h = 0.025$). On the minimizer of Q_L we split the form into its terms:

L	pole	archimedean	$-\log \pi \cdot L^2$	primes	$\mu_{\min}$
$\log 2$	1.107	0.039	-1.145	$-3 \cdot 10^{-6}$	$1.3 \cdot 10^{-3}$
2	2.791	-1.038	-1.145	-0.608	$8 \cdot 10^{-8}$
8	16.66	-2.865	-1.145	-12.65	$5 \cdot 10^{-11}$

The prime-power term matrix is indefinite: (12, 12) at $L = 1$, (40, 39) at $L = 2$, (98, 221) at $L = 8$. The pole term has signature (1, 1). Positivity is a cancellation to about 10^{-12} relative, not a sum of counts.

This mirrors the finite-field situation. There $\sigma(X)$ also contains the point count with a minus sign, and positivity comes from the separate degree identity of Thm. 2.1(iii).

4.2. Counting traces are ghost-blind. [THEOREM] The degree trace on $\mathbb{Q}[\mathbb{N}^\times]$, the L^2 trace and the Hecke Hilbert trace are positive for structural reasons (diagonal, or $\text{Tr}(TT^*)$ for normal T , [7, (31)]). They do not involve the zeros of ζ , so a ghost cannot change them. The Ramanujan bound $\|T_p\| \leq 2$ [7, (32)] is the Rosati-type statement *at p* for GL_2 , not for ζ [ANALOGY].

4.3. The Connes–Consani window. [CITED] [3, Thm.1, (4)]: if $g \in C_c^\infty$ has support in $[2^{-1/2}, 2^{1/2}]$ and $\hat{g}(\frac{i}{2}) = \hat{g}(0) = 0$, then $W_\infty(g * g^*) \geq \text{Tr}(\vartheta(g)S\vartheta(g)^*) \geq 0$. Here S is the projection onto Sonin’s space. This is a genuine, zero-free positivity coming from a positive operator, which is the closest analogue of Thm. 2.1(i). No primes enter, and the window is our $L \leq \log 2$.

[NUMERICAL] *Ghost test inside the window.* We insert 20 quartets: $\gamma \in \{14.13, 21.02, 25.01, 30.42, 98.83\}$, $\delta \in \{0.05, 0.2, 0.4, 0.49\}$, each replacing a double on-line zero (the L103 single-zero convention gives the same signs). At $L = \log 2$ all 20 stay positive. The strongest ($\gamma_1, \delta = 0.49$) lowers $\mu_{\min}$ from $1.34 \cdot 10^{-3}$ to $5.6 \cdot 10^{-4}$. The onsets are:

ghost	$\gamma_1, \delta = 0.49$	$\gamma_1, \delta = 0.2$	$\gamma_2, \delta = 0.49$	$\gamma_2, \delta = 0.2$
L^*	0.75	0.875	0.975	1.25

So [HEURISTIC]: the proved window is ghost-blind, but only by a margin of 8% in L . Ghost sensitivity begins essentially where the first prime ($\log 2$) enters.

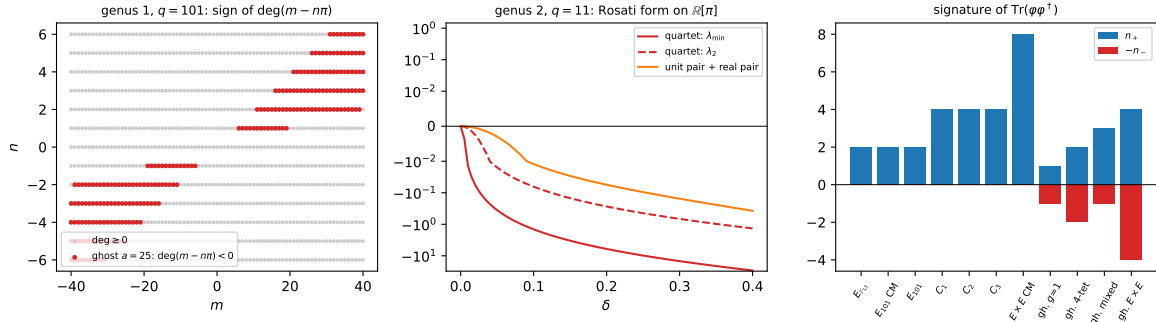


FIGURE 1. [NUMERICAL] Finite models. (a) Sign of the “degree” $m^2 - amn + qn^2$: the true curve ($a = -13$) has none negative, while the ghost $a = 25$ has many (a negative count). (b) Genus 2: smallest eigenvalues of the Rosati form on $\mathbb{R}[\pi]$ versus δ ; two negative directions per quartet and one per real pair. (c) Signatures: true models are definite, ghosts are indefinite.

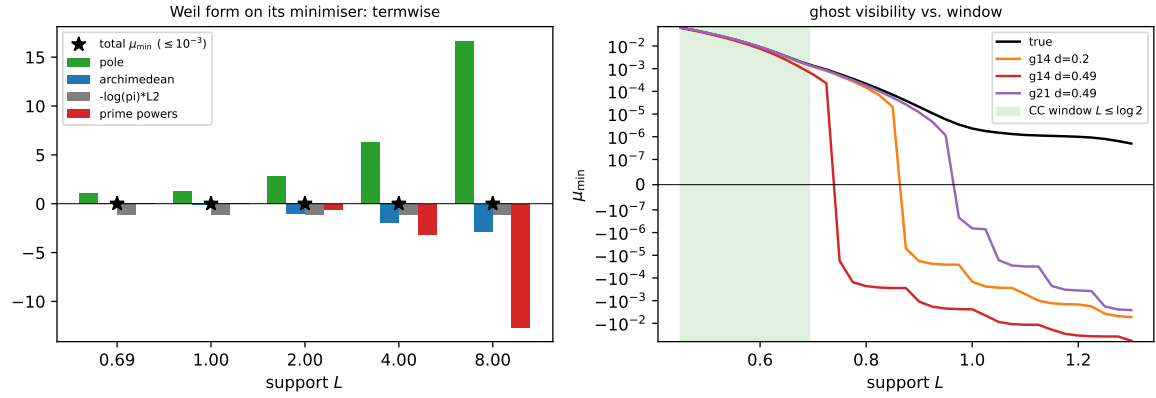


FIGURE 2. [NUMERICAL] $\mathbb{Q}$. (a) Termwise Weil form on its minimizer: a large pole term cancels the prime and archimedean terms, leaving a total $\leq 10^{-3}$. (b) $\mu_{\min}(L)$ for the true zeros and three ghosts. The proved Connes–Consani window (shaded) ends at $\log 2$; the strongest ghost becomes visible at 0.75.

5. WHAT A $\mathbb{Q}$ -ANALOGUE WOULD NEED

Reading Thm. 2.1 as a recipe, a $\mathbb{Q}$ -analogue would need four ingredients:

- (N1) A *small* algebra $\mathcal{A} \ni$ scaling. On all of $\text{End}(H^1)$ the Rosati form is indefinite (Thm. 3.1; (8, 8)), so $\mathcal{A}$ must play the role of “algebraic” correspondences.
- (N2) $\dagger$ with $\vartheta_\lambda^\dagger = \vartheta_{\lambda^{-1}}$ up to the modulus, the analogue of $\pi\pi^\dagger = q$. *Available* (the σ -involution $\bar{f}^\sharp$, [3]).
- (N3) A degree $\deg : \mathcal{A} \rightarrow \mathbb{R}$ with $\deg(\phi\phi^\dagger)$ a count. *Available but ghost-blind*: $n = \# \ker(n|_{\mathbb{Q}/\mathbb{Z}})$, or $\sigma(n)$ for T_n .
- (N4) A Lefschetz–degree identity, analogous to $2m_2 = \text{Tr}(X \circ X')$, equating the ghost-sensitive Weil trace of $\phi\phi^\dagger$ with the degree count. *Missing* [OPEN].

Conjecture 5.1 (shape only). [OPEN]/[CONJECTURE] RH would follow from (N1)–(N4) with (N4) proved independently of the zeros. Any version of (N4) must fail for the ghost models of §3, where

it would produce a negative count (Prop. 2.2, $\deg(10 - \pi) = -49$). It must also be invisible for supports below $\log 2$, where it cannot differ from [3].

6. CORRECTIONS TO THE CHAT CLAIMS (L106 BRIEF)

C106.1 (R1) “from arithmetic”:

Corrected: from *geometry*. Positivity is the positivity of a non-empty intersection (Thm. 2.1(i)). Arithmetic enters only via $\pi\pi^\dagger = q$ (1.30). The $\mathbb{F}_q$ -point count enters Weil’s trace with a minus sign.

C106.2 (R2):

Confirmed: $\alpha^\dagger = \lambda^{-1}\alpha^\vee\lambda$. On H^1 it is the symplectic adjoint.

C106.3 trace:

Confirmed with a precision. $\mathrm{Tr}_{E/\mathbb{Q}}$ (Milne), Tr_{V_ℓ} and $\mathrm{Tr}_{K/\mathbb{Q}}$ tr differ by positive constants on the simple factors, so positivity is the same.

C106.4 (R3) $\iota(i) = h(i)$:

Confirmed as a model statement (L104). In $E \times E$ it is central, and $\dagger$ is its Cartan involution (Prop. 3.2). Caveat: over $\mathbb{F}_q$ there is no Hodge structure; $h(i)$ means the J of L101/L104 (or of a CM lift).

C106.5 (R4) “by counting”:

Confirmed literally in genus 1 ($\#\ker$). In general it is a positive intersection number. For correspondences it is Castelnuovo/Hodge index, which is not a bare count.

C106.6 (R4) candidates:

None of the listed candidates satisfies all three criteria (table, §4). The Connes–Consani Sonin trace is the partial exception, valid only where ghosts are invisible.

C106.7 (R5):

Confirmed (Cor. 3.4). The precise missing piece is (N4).

7. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	Function-field calibration of (N4): compute Weil’s σ -trace, degrees and $(X \cdot \Delta)$ for correspondences on $C \times C$ (genus 1/2, ghosts), linking the L101 NS lattice and Thm. 3.1.	0.4	0.9	0.60
2	Search for a Lefschetz–degree identity on $C_c(\mathbb{R}_+^*) \rtimes \mathbb{Q}_+^\times$ or Bost–Connes (N4), with the ghost test of Conj. 5.1.	0.9	0.1	0.58
3	Semi-local window $\{\infty, 2\}$: push the [3] positivity past $\log 2$ numerically and track the ghost onset against the window.	0.7	0.35	0.56
4	Hecke/Ramanujan as “Rosati at p ”: ghost models for T_p eigenvalues; what plays J ?	0.4	0.6	0.48
5	An off-diagonal “Lefschetz” functional on the Bost–Connes algebra reproducing W (KMS-state side).	0.6	0.25	0.46

8. QUESTIONS FOR J.W.

- Q1. Do you accept that over $\mathbb{F}_q$ the positivity is geometric (a non-empty intersection, $\#\ker$) rather than arithmetic?
- Q2. Is the dictionary of Thm. 3.1 what you wanted: Rosati positive on $\mathbb{R}[F] \iff$ commuting compatible $J \iff$ RH-model, with $n_- = \#\text{real pairs} + 2\#\text{quartets}$, matching Bombieri?
- Q3. Do you accept the verdict that counting traces are ghost-blind and that the ghost-sensitive trace is the Weil trace and hence circular, with the Connes–Consani window ($L \leq \log 2$, ghost onset ≥ 0.75) as the only partial exception?

- Q4. Which object should carry the “degree” for $\mathbb{Q}$: $\# \ker(n|_{\mathbb{Q}/\mathbb{Z}}) = n$ (Bost–Connes) or $\deg T_n = \sigma(n)$ (Hecke)?
- Q5. Should L107 pursue direction 1 (function-field (N4) calibration) or direction 2?

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TAA LEMMA 107: THE LOGICAL FORM OF RH AND TAA'S CERTIFICATION STRUCTURE

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves, or claims to prove, the Riemann Hypothesis, and nothing here suggests that RH is independent of the usual axioms. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (script `num/lemma107_logic.py`, data `num/l107/logic.json`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma107/verified_refs.md`.

ABSTRACT. J.W. asked (t451) what Gödel's incompleteness theorems imply for TAA, and (t453) whether a computable RH-inequality relates to “two reflections make a boost, two boosts make a rotation”. **Verdicts.** (1) RH is a Π_1^0 sentence: “for every n , a checkable property of n holds” ([THEOREM]/[CITED], via Lagarias and Davis–Matiyasevich–Robinson). Hence: if RH is false, already the weak arithmetic Q proves it false; so if ZFC does not refute RH, RH is *true*; and a consistent ZFC can never prove “RH is independent of ZFC” ([THEOREM], Thm. 1.2). (2) This does *not* exclude independence: $\text{Con}(\text{ZFC})$ is also Π_1^0 and is unprovable in ZFC (Gödel II). The famous “natural” independent statements (Paris–Harrington, Goodstein) are Π_2^0 and independent of PA only, not of ZFC ([CITED]/[THEOREM]). (3) The Gödel-II reading of L103–L104 (“positivity cannot certify itself from inside; a certificate must come from a strictly larger arena”) is an *analogy* ([ANALOGY]), not a theorem. It is useful as a design rule ([HEURISTIC]): it points to L106 (Rosati-type positivity, where positivity is a *count*). (4) Each zero contributes $x^\rho/\rho = e^{\beta u} e^{i\gamma u}/\rho$, $u = \log x$: a growth (“boost”, [ANALOGY]) factor times a rotation. A ghost's excess $e^{(\beta-1/2)u}/|\rho|$ beats the on-line sum at $u^* \approx (\beta - \frac{1}{2})^{-1} \log(|\rho|A/2)$, a height-dependent onset of the same logarithmic type as L103's window law ([RESTATEMENT]/[NUMERICAL]/[HEURISTIC]). (5) Boost facts ([THEOREM]/[NUMERICAL]): in 1+1, rapidities add; two non-collinear boosts in 2+1 give a boost *and* a Wigner rotation that is not small (up to 90°); two boosts are *never* a pure rotation (except trivially); three boosts closing a triangle are.

GLOSSARY (TERMS RESTATED)

Formal system / theory F ::

Axioms plus mechanical rules; “ $F \vdash A$ ” means A has a proof in F . PA = Peano arithmetic; Q = Robinson arithmetic (PA without induction); ZFC = set theory, enough for ordinary mathematics [5].

Σ_1^0 , Π_1^0 , Π_2^0 ::

Counting unbounded quantifiers over integers, with a mechanically decidable core P : $\Sigma_1^0 = “\exists n P(n)”$ (a search that stops if it succeeds); $\Pi_1^0 = “\forall n P(n)”$ (“no counterexample exists”); $\Pi_2^0 = “\forall m \exists n P(m, n)”$ (“every search terminates”).

Consistent / sound / independent::

Consistent: F proves no contradiction. Sound (for a class): what F proves in that class is true. A is independent of F if $F \not\vdash A$ and $F \not\vdash \neg A$.

$\text{Con}(F)$::

The arithmetic sentence “no number codes a proof of $0 = 1$ in F ”, a Π_1^0 sentence [5].

Boost / rotation::

A boost $B(\zeta) = \begin{pmatrix} \cosh \zeta & \sinh \zeta \\ \sinh \zeta & \cosh \zeta \end{pmatrix}$ (rapidity ζ) is hyperbolic; a rotation is trigonometric (compact). Here “boost part of a zero” is the real growth factor $e^{\beta u}$: an *analogy*, since $e^{\beta u}$ is a scaling, not a Lorentz map.

1. RH IS A Π_1^0 STATEMENT

Proposition 1.1 (three arithmetic forms). *[CITED]* (i) Lagarias [2, Thm. 1.1]: $RH \iff$ for all $n \geq 1$, $\sigma(n) \leq H_n + e^{H_n} \log H_n$, with equality only for $n = 1$. Here $\sigma(n) = \sum_{d|n} d$ and $H_n = \sum_{j \leq n} 1/j$. (ii) Robin (as stated in [2, (1.2), Prop. 3.1–3.2]; [4] not opened): $RH \iff \sigma(n) < e^\gamma n \log \log n$ for all $n \geq 5041$. If RH fails, there are $0 < \beta' < \frac{1}{2}$, $C > 0$ with $\sigma(n) \geq e^\gamma n \log \log n + Cn \log \log n / (\log n)^{\beta'}$ for infinitely many n , and β' can be any value in $(1 - b, \frac{1}{2})$, where $b = \Re \rho > \frac{1}{2}$ for an off-line zero ρ . (iii) Davis–Matiyasevich–Robinson [1], via [3, Thm. 2]: with $\eta(p^k) = p$, $\eta(j) = 1$ otherwise, and $\delta(x) = \prod_{n < x} \prod_{j \leq n} \eta(j)$, $RH \iff$ for all $n \geq 1$, $(\sum_{k \leq \delta(n)} \frac{1}{k} - \frac{n^2}{2})^2 < 36n^3$. Every quantity here is rational, so each instance is decidable by exact arithmetic. By MRDP, RH is equivalent to the unsolvability of one explicit Diophantine equation [3, §2], [5, §4.4].

Why (i) and (ii) are Π_1^0 ([THEOREM], our check). (iii) is Π_1^0 on its face. In (i) and (ii), the right-hand sides are computable reals. A strict inequality between computable reals can be confirmed in finite time (interval arithmetic), but an equality cannot in general. In (i), a violation is a strict “ $>$ ” for some $n \geq 2$, so “ $\neg RH$ ” is Σ_1^0 and RH is Π_1^0 . In (ii) as stated, a violation would be “ $\geq$ ”. But by the second sentence of (ii), if RH fails then some violation is *strict*. So $RH \iff \forall n \geq 5041 \neg(\sigma(n) > e^\gamma n \log \log n)$, which is Π_1^0 .

Theorem 1.2 (what Π_1^0 buys). *[THEOREM]* (from Σ_1^0 -completeness of Q , [5, §2.5]: a true Σ_1^0 sentence is provable in Q ; equivalently a false Π_1^0 sentence is refutable in Q .) Let $F \supseteq Q$ be any axiomatizable theory, e.g. PA or ZFC . (a) If RH is false, then $Q \vdash \neg RH$, and so $F \vdash \neg RH$. The refutation is the computation on the counterexample n . (b) Hence “ $F \not\vdash \neg RH$ ” implies RH is true. In particular, “ RH is independent of ZFC ” implies RH . No soundness assumption is needed: an inconsistent F refutes everything. (c) If ZFC is consistent, ZFC cannot prove “ RH is independent of ZFC ”. Indeed (a) is provable inside ZFC , so $ZFC \vdash \neg(ZFC \not\vdash \neg RH)$ would give $ZFC \vdash RH$, contradicting the claimed independence. An independence proof could only live in a stronger theory T' , and T' would then prove RH .

Numerics ([NUMERICAL]; not evidence for RH beyond the finite range).

- Direct exact $\sigma(n)$ for $n \leq 2 \cdot 10^6$: (i) holds with relative margin ≥ 0.0268 (minimum at $n = 12$); (ii) holds for $5041 \leq n \leq 2 \cdot 10^6$. The exceptions to (ii) in $3 \leq n \leq 5040$ are 26 numbers, the largest being 5040.
- 200 449 greedy colossally abundant numbers (by the critical exponents of [2, §2], stopping before any prime beyond the sieve could be needed), up to $\log n = 2.75 \cdot 10^6$: all have $\sigma(n)/(n \log \log n) < e^\gamma$ after 5040. The scaled gap $(e^\gamma - \text{ratio})\sqrt{\log n} \log \log n$ stays in $[1.30, 1.64]$ for $\log n > 100$ (Fig. 1a).
- (iii) for $n \leq 2 \cdot 10^5$, using $\log \delta(n) = \psi_1(n) = \int_1^n \psi$ [3, Lemma 3] and $H_N = \log N + \gamma + O(1/N)$: $\max |\cdot|/n^{3/2} = 0.622$, against the allowed 6.

2. GÖDEL I AND II, AND NATURAL INDEPENDENCE

Theorem 2.1 (Gödel–Rosser; Gödel II). *[CITED]* [5, §§1.1, 2.5, 3.1]. Let F be a consistent axiomatizable theory containing Q . (I) There is a sentence R_F with $F \not\vdash R_F$ and $F \not\vdash \neg R_F$. Gödel’s own sentence G_F is provably equivalent to the Π_1^0 sentence $\forall x \neg \text{Prf}_F(x, \ulcorner G_F \urcorner)$, and it is true. (II) If F contains a little more arithmetic (e.g. PRA), then $F \not\vdash \text{Con}(F)$. Löb [5, §4.3]: $F \vdash (\text{Prov}_F(\ulcorner A \urcorner) \rightarrow A)$ iff $F \vdash A$. F can endorse “if I prove A then A ” only for A it already proves.

Proposition 2.2 (logical complexity of the natural examples). *[CITED]/[THEOREM]*. The Paris–Harrington principle (a finite Ramsey statement, 1977) and Goodstein’s theorem (independence: Kirby–Paris 1982) are unprovable in PA and provable in stronger systems (ZFC ; second-order arithmetic for Goodstein) [5, §4.5], [6]. Both have the form “for every input, a search terminates”, so

they are Π_2^0 ([THEOREM], by inspection of the statements). Goodstein’s theorem gives a total computable function that PA cannot prove total [6]. By contrast $\text{Con}(\text{PA})$, $\text{Con}(\text{ZFC})$ and G_F are Π_1^0 and independent of their own theories.

Reading ([RESTATEMENT]/[HEURISTIC]).

- The Π_1^0 form rules out “RH false but unprovably so”. It does not rule out “RH true but unprovable in ZFC”: $\text{Con}(\text{ZFC})$ is exactly such a Π_1^0 truth.
- The natural PA-independent examples we checked are Π_2^0 and are settled by ZFC. We know of no result suggesting that RH is independent of ZFC ([HEURISTIC]: an absence claim, limited to what we opened).
- Incompleteness concerns *formal derivability*. Nothing in Gödel’s theorems obstructs a ZFC proof of RH.

3. ANALOGY: SELF-CERTIFICATION AND L103–L104

Logic ([CITED])	TAA ([TAA-L]; the pairing is [ANALOGY])
$F \not\vdash \text{Con}(F)$: a theory cannot certify its own consistency.	L103/L104 dichotomy: a J built from the Weil form certifies positivity only by assuming it (circular); a W -independent J is ghost-blind.
$\text{Con}(\text{PA})$ is provable from a <i>stronger</i> principle (ε_0 -induction, Gentzen) [5, §4.5].	Over $\mathbb{F}_q$, positivity on $H^1(C)$ comes from the <i>larger</i> arena $C \times C$ (Hodge index, <i>proved</i> ; not the open Hodge conjecture). The extra direction turns a reflection into a rotation (L100–L102).
Löb: self-reflection is endorsed only for what is already proved.	“ J commuting with F and positive” is available only if RH for the model holds (L104 Thm. 1.1).
Finite verification: a Π_1^0 statement is checked one n at a time and can be refuted by one n .	L103 onset law ([HEURISTIC]): a window L sees heights $\lesssim 2\pi e^L$; here a ghost of height γ shows at $u^* \sim \log \rho /(\beta - \frac{1}{2})$ (§4).

Where the analogy breaks ([RESTATEMENT]).

- Gödel II is a theorem about provability predicates. The L103–L104 circularity is a mathematical dependence (a construction uses its conclusion as input), not a metamathematical one.
- Gentzen’s proof is genuinely stronger than PA. Weil’s $C \times C$ argument is not “stronger axioms”; it is a larger *geometric object*, inside ordinary mathematics.
- There is no theorem saying a non-circular positivity source for ζ must exist, or must not.

Design rule ([HEURISTIC]). Certificates that worked (Hodge index on $C \times C$; Rosati positivity, *not opened*, [ANALOGY]) express positivity as a *count*: intersection numbers and traces of $\phi\phi^\dagger$, which are ≥ 0 because they count something. The logic analogy suggests looking for such a count on a strictly larger structure than the one carrying the zeros. That is L106’s brief (Rosati-type positivity for $\mathbb{Q}$).

4. BOOST PART $\times$ ROTATION PART (T453)

Proposition 4.1 (zero contributions). [RESTATEMENT] (*standard explicit formula; normalization ours*). For $x > 1$ not a prime power, $\psi(x) - x = -\sum_\rho x^\rho/\rho - \log 2\pi - \frac{1}{2} \log(1 - x^{-2})$. With $u = \log x$ and $\rho = \beta + i\gamma$,

$$\frac{x^\rho}{\rho\sqrt{x}} = \underbrace{e^{(\beta-\frac{1}{2})u}}_{\text{growth (“boost”, [ANALOGY])}} \cdot \underbrace{e^{i\gamma u}}_{\text{rotation}} \cdot \frac{1}{\rho}.$$

Under RH every growth factor is 1: $(\psi(x) - x)/\sqrt{x}$ is a sum of rotations with weights $1/|\rho|$. This is the “equal damping” statement in the explicit-formula picture (L105, in preparation, tests the Lax–Phillips version). A ghost quartet $\beta + i\gamma$, $1 - \beta + i\gamma$ ($\beta > \frac{1}{2}$, with conjugates) adds a term of envelope $2e^{(\beta-\frac{1}{2})u}/|\rho|$. It exceeds an on-line sum of size A at $u^* = (\beta - \frac{1}{2})^{-1} \log(|\rho|A/2)$.

The *rigorous* link from a ghost to a violation of (i)/(ii) is Robin’s Prop. 3.2 [2] (which goes through Nicolas’ method), not this envelope picture ([HEURISTIC]).

Numerics ([NUMERICAL], Fig. 1b).

- The first 1000 zeros (mild $\cos^2$ taper) reproduce the sieved $(\psi(x) - x)/\sqrt{x}$ on $u \in [8, 16]$ with rms error 0.048, against signal rms 0.215. Their sum stays within $|A| \leq 0.70$ on $u \leq 45$.
- A synthetic ghost $\beta = 0.62$, $\gamma = 40$ crosses that band at $u^* = 22.0$ ($x \approx e^{22}$), matching the formula. The $1/|\rho|$ weight makes higher ghosts appear later: logarithmically in height, as in L103’s window law ([HEURISTIC]).

Proposition 4.2 (composition of boosts). [THEOREM]/[CITED]. (a) In 1+1, $B(\zeta_1)B(\zeta_2) = B(\zeta_1 + \zeta_2)$ (numerically $3 \cdot 10^{-14}$). (b) In 2+1 or more, two non-collinear boosts compose to a boost times a Thomas–Wigner rotation; $\text{SO}(2, 1)^+$ already shows it [7]. For perpendicular boosts of equal rapidity ζ we find (polar decomposition) the angle $\theta = 2 \arctan(\tanh^2(\zeta/2))$, agreeing with [7] to $4 \cdot 10^{-15}$. So $\theta = 85.8^\circ$ at $\zeta = 4$, and $\theta \rightarrow 90^\circ$: not small (Fig. 1c). (c) Two boosts are never a pure rotation unless $B_1 = B_2^{-1}$: if $B_1 B_2 = R$ then $B_1 = R B_2^{-1}$, and uniqueness of the polar decomposition (positive-definite symmetric $\times$ orthogonal) forces $R = 1$. (d) Three boosts closing a velocity triangle give a pure rotation [7]. Numerically, $B_{ac}^{-1} B_b B_a$ is orthogonal with $(0, 0)$ entry 1 to $5 \cdot 10^{-15}$, with angle 23.8° for rapidities 0.9, 1.1.

Reading ([ANALOGY]). The compact circle generated by boosts needs (i) at least two independent boost directions and (ii) a closed loop: three boosts, or a commutator. Over $\mathbb{Q}$, the “growth parameters” β of zeros form a single real direction (the u -line). An arena with two independent hyperbolic directions (e.g. the L100 wish for something like $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$) is what (b)–(d) would need. No such arena is constructed here ([OPEN]).

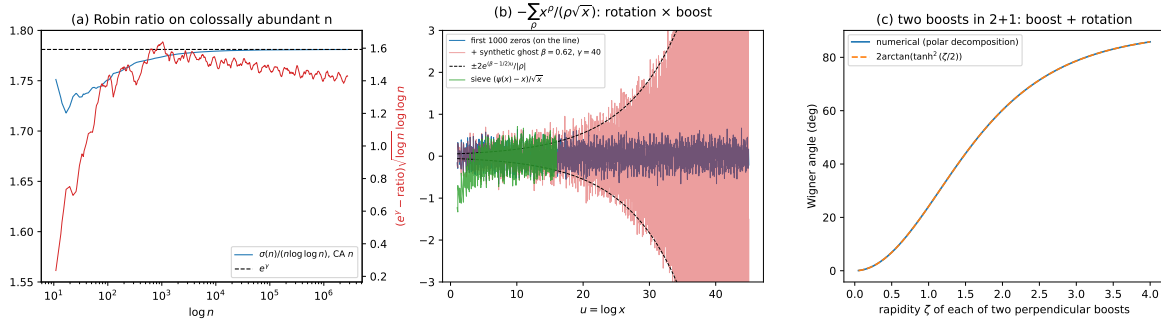


FIGURE 1. (a) $\sigma(n)/(n \log \log n)$ on greedy colossally abundant n (blue) vs. e^γ (dashed); red: scaled gap. (b) Normalized explicit-formula sum from the first 1000 zeros (blue), with a synthetic ghost quartet $\beta = 0.62$, $\gamma = 40$ added (red) and its envelope (dashed); green: sieved $(\psi(x) - x)/\sqrt{x}$ for $u \leq 16$. (c) Wigner angle for two perpendicular boosts: numerics vs. closed form.

5. CORRECTIONS TO THE CHAT CLAIMS AND TO THE BRIEF

C107.1 (t453: “two boosts in different directions make a boost plus a *small* rotation”).:

Not small in general: up to 90° for perpendicular boosts (Prop. 4.2b). “At least two space directions” is right; 2+1 suffices.

C107.2 (“two boosts make a rotation”).:

Never a pure rotation (Prop. 4.2c); three boosts closing a loop can be (d). L99’s title “Two Boosts Make a Rotation” should be read as “boost-generated rotation” (Cartan involution), not literal composition.

C107.3 (t453: “each zero adds a term x^ρ ”):

The term is $-x^\rho/\rho$. The $1/\rho$ weight matters: it delays high ghosts (Prop. 4.1).

C107.4 (t453: “a ghost’s extra growth eventually beats any fixed bound, so $\sigma(n)$ breaks the inequality”):

Correct in conclusion, but the proof is Robin’s Prop. 3.2 (Nicolas/Landau-type argument), not the single-term envelope. Cancellation among many zeros has to be excluded, and that is what Robin’s proof does.

C107.5 (brief: “Robin’s inequality is Π_1^0 ”):

True, but needs the strict-excess form (Prop. 1.1(ii) and the paragraph after it). The literal “ $\geq$ ” violation is not semi-decidable.

C107.6 (brief: Lagarias “strict except $n = 1$ ”):

Confirmed: $\leq$, with equality only at $n = 1$ [2]. Journal: Amer. Math. Monthly 109 (2002) 534–543.

C107.7 (t451: “if RH is independent of a sound theory, it is true”):

Stronger: “not refutable in ZFC” already implies RH; no soundness needed. And a consistent ZFC cannot prove the independence (Thm. 1.2c).

C107.8 (t451: Gödel II analogy, “positivity can’t be self-certified from within”):

Analogy only (§3). Gödel II forbids neither a ZFC proof of RH nor a non-circular positivity source.

C107.9 (brief: DMR as the Diophantine form):

The DMR paper was not opened. The statement is taken from Nayebi’s exposition (opened) and is labelled [CITED-sec]/[BIB].

6. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	L106: positivity as a <i>count</i> on a larger structure (Rosati type for $\mathbb{Q}$).	0.95	0.03	0.58
2	Unify the ghost-onset laws: explicit-formula $u^* = (\beta - \frac{1}{2})^{-1} \log(\rho A/2)$ vs. L103’s $L^* \approx \log(\gamma/2\pi)$; certified finite-window statements.	0.40	0.80	0.56
3	Robin-ratio and Lagarias-margin diagnostics with injected ghosts (a synthetic σ -analogue from a modified ψ) to see the violation onset in $\log n$.	0.30	0.75	0.48
4	Two-direction hyperbolic arena for $\mathbb{Q}$ where Wigner-type closure produces a compact circle (§4, reading).	0.60	0.10	0.40
5	A Löb-style formulation: which positivity statements an arena can certify about itself (purely as an organizing language).	0.25	0.40	0.31

7. QUESTIONS FOR J.W.

- Q1. Is Thm. 1.2 (independence $\Rightarrow$ truth; independence cannot be proved inside ZFC) the “nuance” you sensed in t451, or did you have something about the *form* of a proof in mind?
- Q2. Do you accept that the Gödel-II/L104 link stays an analogy (§3), used only as the design rule “certify from a larger arena”?
- Q3. Should direction 2 (one onset law for windows and explicit-formula sums) be folded into L105’s damping test?
- Q4. Is the Wigner reading (rotation needs two boost directions and a closed loop) the picture you want for the Spec $\mathbb{Z} \times \text{Spec } \mathbb{Z}$ idea?
- Q5. C107.2: rename the L99 slogan to “boost-generated rotation” in the volume text, or keep it with a footnote?

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TAA LEMMA 108: GHOSTS AS S -OBJECTS: NULL RAYS AT INFINITY, CUSP REFLECTION, AND THE MISSING S -SYMMETRY

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma108_*.py`, data `num/1108/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma108/verified_refs.md`.

ABSTRACT. J.W.'s t455 answer (items 1–5) is checked.

Results.

- **(1) Dictionary confirmed and sharpened** ([THEOREM], [NUMERICAL]).
 - *Genus 1.* A ghost has $F' = \pm(\cosh r + S \sinh r)$, with $S = (2F' - t)/\sqrt{t^2 - 4}$ and $r = \operatorname{arccosh}(|t|/2)$. This S satisfies $S^2 = 1$ and $[S, F] = 0$. It is *anti*-symplectic ($S^T \omega S = -\omega$, $\det S = -1$; L102). It is the *unique* traceless involution commuting with F , up to sign. Its null eigenlines are F 's boundary fixed points: 5.0670 and 19.9330 for $a = 25$, $q = 101$.
 - *Roles swap.* True F commutes with J and is reversed by an anti-symplectic S ($SFS = F^{-1}$; L104). A ghost commutes with S and is reversed by a compatible J ($JFJ^{-1} = F^{-1}$, error $2 \cdot 10^{-10}$).
 - *Genus 2.* A quartet carries a commuting split-complex $S = \operatorname{sgn} \log |\lambda|$, unique up to sign, together with a commuting J -phase. k real pairs give 2^k choices. A *mixed* model (unit pair plus ghost) has *no* commuting anti-symplectic involution; only the partial $E = \operatorname{sgn} \log |\lambda|$ exists ($E^3 = E$, E^2 the projector onto the ghost part).
- **(2) No invariant state** ([THEOREM], [NUMERICAL]). Every orbit of a ghost converges to the attracting null ray $x_+ = 19.9330$, and $d_{\mathbb{H}}(z_n, z_0)$ grows at slope 1.36964, against the theoretical $2r = 1.36964$. True orbits stay on hyperbolic circles about the CM point (to $4 \cdot 10^{-16}$).
- **(3) Cusp conventions confirmed** ([CITED], [THEOREM]). The constant term is $y^s + \varphi y^{1-s}$, with $\varphi = \Lambda(2s - 1)/\Lambda(2s)$ and $\varphi(\frac{1}{2}) = -1$. With LP's time convention, y^s is incoming and φy^{1-s} outgoing. Poles at $\rho/2$ come from *all* zeros, with damping $\frac{1}{2} - \operatorname{Re} s = \frac{1}{4} \mp \frac{\delta}{2}$.
- **(4) Bridge between the two infinities** ([THEOREM], new). A ghost's null rays are conjugate real-quadratic points: the axis of S is the axis of $\gamma = uF + kI \in SL_2(\mathbb{Z})$ from Pell's equation. For $a = 25$, $\gamma = F - 5$, so it projects to a *closed geodesic* of length $2 \log \frac{15 + \sqrt{221}}{2} = 5.407$. A null ray is a *cusp* only for $\{\alpha, q/\alpha\} = \{1, q\}$ ($a = q + 1$), the $H^0 \oplus H^2$ “pole” eigenvalues. So ghosts sit on the geodesic side of the trace formula, not at the cusp.
- **(5) Main question: a restatement, with three precise negatives** ([RESTATEMENT], [THEOREM], [NUMERICAL]).
 - $s \mapsto \frac{1}{2} - \bar{s}$ always preserves the resonance set. Its 2-cycles are exactly the ghosts, so a mode-selective split-complex E exists formally, and $\text{RH} \iff E = 0$. But E is defined from pole positions.
 - The cusp's geometric S (the reflection $z \mapsto -\bar{z}$, $\det = -1$) commutes with *everything* and is ghost-blind.
 - φ on the line is a $\operatorname{Re} s = 1$ object, $d(r) = \frac{2}{\pi} \operatorname{Re} \Lambda' / \Lambda(1 + 2ir)$ (to 10^{-31}), and $\frac{1}{2} - \bar{s}$ is not a symmetry of φ : $\varphi(\frac{1}{2} - \bar{s}) = \Lambda(1 + 2s)/\Lambda(2s)$. Reading E off the scattering data is the explicit formula.

With synthetic ghosts, $|\varphi_g| = 1$ holds to 10^{-15} ; only the bump shape (peak $\times \frac{1}{1-4\delta^2}$) and the broken $\operatorname{Re} s = 1$ identity see them.

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GLOSSARY (TERMS RESTATED)

F' , t ::

The normalized Frobenius is $F' = F/\sqrt{q} \in SL_2(\mathbb{R})$ (companion basis $F = \begin{pmatrix} a & -q \\ 1 & 0 \end{pmatrix}$), with $t = a/\sqrt{q}$. It is elliptic if $|t| < 2$ (true) and hyperbolic if $|t| > 2$ (ghost).

J , S (spinor basis, L102)::

$J^2 = -1$ (timelike in $(\mathfrak{sl}_2, \text{Killing}) \cong \mathbb{R}^{1,2}$, symplectic), $S^2 = +1$ (spacelike, $\det S = -1$, anti-symplectic). $\mathbb{R}[J] = \mathbb{C}$ (complex i); $\mathbb{R}[S] = \mathbb{R} \oplus \mathbb{R}$ (split-complex j).

States.:

Compatible J correspond to points of $\mathbb{H}$ (L105). F acts by $z \mapsto (az + b)/(cz + d)$, and null rays correspond to points of $\partial\mathbb{H} = \mathbb{P}^1(\mathbb{R})$.

Cusp, resonances.:

$X = SL_2(\mathbb{Z}) \backslash \mathbb{H}$, with cusp $i\infty$; $E(z, s)$ has constant term $y^s + \varphi(s)y^{1-s}$. Resonances are poles of φ in $\text{Re } s < \frac{1}{2}$. The LP damping of a mode is $\frac{1}{2} - \text{Re } s = -\text{Re } \mu$, with $\mu = (\rho - 1)/2$ (L105).

Ghost for ζ ::

An off-line pair $\frac{1}{2} \pm \delta + i\gamma$ (with conjugates), compared with a double on-line zero.

1. CLAIMS CHECKED (T455)

(1) Ghost = $\pm$ boost, commuting with S , with fixed points equal to S 's null eigenlines; true = $e^{\theta J}$. Is S unique, and is it anti-symplectic? What happens in genus 2? (2) Ghost iteration flows to the attracting null ray, so there is no invariant state. (3) Constant term = incoming + reflected; resonances at $\rho/2$ for all zeros; damping $\frac{1}{4} \pm \frac{\delta}{2}$. (4) Two infinities: is there a bridge? (5) Is there a mode-selective S in cusp scattering, and is "RH $\iff$ no S -symmetry" non-circular?

2. (1) THE J/S DICTIONARY

Theorem 2.1 (genus 1). *[THEOREM] Let $F' \in SL_2(\mathbb{R})$ with $t = \text{tr } F'$, $|t| \neq 2$, and put $X = (2F' - t)/\sqrt{|t^2 - 4|}$. (i) $X^2 = \text{sgn}(t^2 - 4)$ and $[X, F'] = 0$. (ii) Ghost case ($|t| > 2$): $S := X$ has $\det S = -1$, so $\omega(Sx, Sy) = -\omega(x, y)$, and $F' = \text{sgn}(t)(\cosh r + S \sinh r)$ with $r = \text{arccosh}(|t|/2)$. (iii) True case ($|t| < 2$): $J := X$ is symplectic and $F' = e^{\theta J}$ (L101/L104). (iv) Uniqueness. The commutant of F' is $\mathbb{R}[F'] = \{x + yX\}$. Its traceless elements with $(x + yX)^2 = \pm 1$ are exactly $\pm X$. (v) The eigenlines of S are those of F' , i.e. its two fixed points $x_{\pm} \in \partial\mathbb{H}$, which are null rays of the Killing form. (vi) Reversal. True F' : $SF'S = F'^{-1}$ for every unit spacelike $S \perp J$. Ghost F' : $JF'J^{-1} = F'^{-1}$ for every compatible $J \perp S$, i.e. $JS = -SJ$.*

Proof. (i) follows from Cayley–Hamilton: $(F' - \frac{t}{2})^2 = \frac{t^2-4}{4}$. (ii) A traceless 2×2 matrix S with $S^2 = 1$ has $\det S = -1$, and $\omega(Sx, Sy) = \det S \omega(x, y)$. Then $\cosh r = |t|/2$ and $\sinh r S = \operatorname{sgn}(t)(F' - \frac{t}{2})$. (iv) $(x + yX)^2 = x^2 \pm y^2 + 2xyX$, and the trace condition forces $x = 0$. (vi) In Killing terms, $F' = \exp(\text{axis})$, and conjugation by any unit vector orthogonal to the axis reverses the exponent. $\square$

Numerics ([NUMERICAL], num/lemma108_finite.py).

- *Ghosts.* $a = 25$: $r = 0.68482$, $|F' - (\cosh r + S \sinh r)| = 0$, $S^T \omega S = -\omega$, null lines $\{5.06697, 19.93303\}$ equal the roots of $z^2 - 25z + 101$; the reversing J has $|JF'J^{-1} - F'^{-1}| = 1.9 \cdot 10^{-10}$. $a = 21$: $\{7.4586, 13.5414\}$, $r = 0.2982$.
- *True.* $a = -3, -13$: $SF'S = F'^{-1}$ to 10^{-13} , with S anti-symplectic.
- *Uniqueness.* A grid search confirms that $\pm X$ are the only solutions.

Theorem 2.2 (genus 2). *[THEOREM] Let $F' \in \operatorname{Sp}(4, \mathbb{R})$ be semisimple with distinct eigenvalues, so its commutant is $\mathbb{R}[F']$. Split V into ω -orthogonal blocks over the orbits of $\{\cdot, \lambda \mapsto \lambda^{-1}\}$. (a) Quartet $\{\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}\}$: the eigenspaces for $|\lambda| > 1$ and for $|\lambda| < 1$ are Lagrangian. So $S = \operatorname{sgn} \log |\lambda|$ is an anti-symplectic involution in $\mathbb{R}[F']$, unique up to sign. It commutes with the phase structure $J = \operatorname{sgn} \arg \lambda$ (on the complexification), and JS satisfies $(JS)^2 = -1$. Locally $F' = e^{\delta S + \theta J}$: bicomplex, boost $\times$ rotation. (b) k real pairs: anti-symplectic involutions in the commutant are $\bigoplus_i (\pm S_i)$, giving 2^k of them. (c) Mixed (unit pair + real pair): the commutant contains no anti-symplectic involution, because on the unit block the only involutions are ± 1 , which are symplectic. The canonical $E = \operatorname{sgn} \log |\lambda| \in \mathbb{R}[F']$ satisfies $E^3 = E$; E^2 is the projector onto the ghost block, and E is anti-symplectic there and 0 elsewhere. (d) $E = 0 \iff$ all $|\lambda| = 1 \iff$ a compatible J commutes with F' (L104). [RESTATEMENT]: E is a spectral function of F' .*

Numerics ([NUMERICAL]). We took random $\operatorname{Sp}(4)$ conjugates of four models and enumerated the involutions $V \operatorname{diag}(\pm 1) V^{-1}$ over the conjugation orbits:

- quartet ($\delta = 0.15$, $\theta = 1$): exactly 2 anti-symplectic involutions, signs $(1, -1)$ and $(-1, 1)$;
- two real pairs: 4;
- mixed: 0;
- true: 0.

In all four models $E^3 = E$, $[E, F] = 0$ and $E \in \mathbb{R}[F]$ to 10^{-14} .

3. (2) ITERATION: NO INVARIANT STATE

Proposition 3.1. *[THEOREM] If F' is hyperbolic with attracting eigenline x_+ , then for every $z \in \mathbb{H}$ we have $F'^m z \rightarrow x_+$, and $d_{\mathbb{H}}(F'^m z, z) = 2nr + O(1)$. There is therefore no F' -invariant state, and no invariant probability measure on $\mathbb{H}$ (L103 Q1). If F' is elliptic, its orbits lie on hyperbolic circles about z^* .*

Proof. Conjugate F' to $\operatorname{diag}(e^r, e^{-r})$, which acts by $z \mapsto e^{2r} z$. The translation length along the axis is $\log N(\gamma) = 2r$ [2, § on (1)]. An invariant measure would have to be supported on fixed points, and there are none in $\mathbb{H}$. $\square$

[NUMERICAL] For $a = 25$, $q = 101$ and four starting points, all orbits end at $19.933034373660 \pm 10^{-12}$; the fitted slope is 1.369644 against $2r = 1.369636$. For $a = -3$, the orbit distance to $z^* = -1.5 + 9.9373i$ is constant to $4 \cdot 10^{-16}$ (Fig. 1).

4. (3) THE CUSP: CONVENTIONS

Proposition 4.1. *[CITED]/[THEOREM] (i) $E(z, s) = y^s + \varphi(s)y^{1-s} + \dots$ with $\varphi = \Lambda(2s-1)/\Lambda(2s)$ [3, 2]. Its poles in $0 < \operatorname{Re} s < \frac{1}{2}$ lie at $\rho/2$ [3], for every nontrivial zero. (ii) $\varphi(s)\varphi(1-s) = 1$, $|\varphi| = 1$ on the line, and $\varphi(\frac{1}{2}) = -1$, from $\Lambda(w) \sim -1/w$ and $\Lambda(w) \sim 1/(w-1)$. Numerically: $\varphi(\frac{1}{2} + 10^{-6}) = -1.0000039$, and $\varphi(s)\varphi(1-s) = 1$ to 10^{-31} . (iii) In LP80 (2.8)–(2.11), x -independent waves solve $u_{tt} = y^2 u_{yy} + u/4$; $v = u/\sqrt{y}$ in $\log y$ is the classical wave. So with e^{irt} time dependence,*

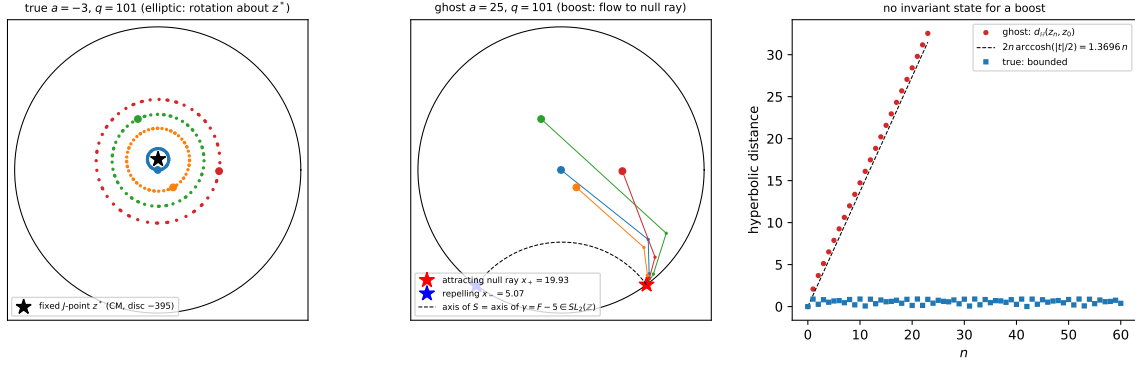


FIGURE 1. [NUMERICAL] Disk model $w = (z - 10i)/(z + 10i)$. Left: true $a = -3$, where orbits rotate about the fixed J -point (a CM point of disc -395). Middle: ghost $a = 25$, where every orbit flows to the attracting null ray x_+ ; the dashed line is the axis of S , which is also the axis of $\gamma = F - 5 \in SL_2(\mathbb{Z})$ (§5). Right: hyperbolic distance grows at the rate $2r$.

$y^{1/2+ir}$ is a function of $\log y + t$, i.e. left-moving and incoming, and $\phi y^{1/2-ir}$ is outgoing. The labels swap with $t \mapsto -t$. The common factor $\sqrt{y}$ is the $\frac{1}{4}$. (iv) A pole $s = \rho/2$ has $r = (s - \frac{1}{2})/i$ with $\text{Im } r = (1 - \beta)/2$. The damping is $\frac{1}{2} - \text{Re } s = \frac{1}{4} - \frac{\delta}{2}$ for $\beta = \frac{1}{2} + \delta$, and its partner $1 - \bar{\rho}$ gives $\frac{1}{4} + \frac{\delta}{2}$ (L105).

5. (4) TWO INFINITIES: THE BRIDGE

Theorem 5.1 (ghost null rays are geodesic endpoints, not cusps). [THEOREM] Let $F = \begin{pmatrix} a & -q \\ 1 & 0 \end{pmatrix}$ with $D = a^2 - 4q > 0$. (i) $x_{\pm} = (a \pm \sqrt{D})/2$ are the roots of the form $(1, -a, q)$. (ii) If D is not a square, let (t, u) be the least solution of $t^2 - Du^2 = 4$, and put $\gamma = uF + \frac{t-au}{2}I$. Then $\gamma \in SL_2(\mathbb{Z})$, $\gamma \in \mathbb{Z}[F]$ commutes with F and with S , and γ fixes $x_{\pm}$. So $\gamma = \pm(\cosh \frac{\ell}{2} + S \sinh \frac{\ell}{2})$ with the same S . The axis of S projects to a closed geodesic of $X(1)$ of length $\ell = 2 \log \frac{t+u\sqrt{D}}{2}$ [2, lengths $2 \log \epsilon_d$]. (iii) $x_{\pm} \in \mathbb{P}^1(\mathbb{Q})$, i.e. they are cusps, iff D is a square iff $\alpha \in \mathbb{Z}$. For $q = p$ prime this forces $\{\alpha, q/\alpha\} = \{1, q\}$, i.e. $a = \pm(q + 1)$: the $H^0 \oplus H^2$ eigenvalues, the pole terms of $Z(T)$. (iv) True F ($D < 0$) fixes a CM point whose stabilizer in Γ is finite (L105).

Proof. (ii) γ has integer entries, since $t \equiv au \pmod{2}$ follows from $t^2 \equiv a^2 u^2 \pmod{4}$. Its determinant is $\frac{t^2 - Du^2}{4} = 1$. It commutes with F because it lies in $\mathbb{Z}[F]$, and its eigenlines are those of F . For the length, use (Sar03): $\ell = \log N(\gamma)$ with $N(\gamma)^{1/2} = \epsilon$. (iii) $D = m^2$ gives integer eigenvalues α with $\alpha \cdot q/\alpha = p$. $\square$

[NUMERICAL] Pell data:

(a, q)	D	(t, u)	γ	ℓ
(25, 101)	221	(15, 1)	$\begin{pmatrix} 20 & -101 \\ 1 & -5 \end{pmatrix} = F - 5$	5.407
(21, 101)	37	(146, 24)	$24F - 179$	9.967
(8, 13)	12	(4, 1)	$F - 2$	2.634
(102, 101)	100^2	—	fixed points 1 and 101 (cusps)	—

Remark 5.2. [ANALOGY] In the Selberg trace formula for $X(1)$ [2], the cusp (through φ) is on the *spectral* side and closed geodesics are on the *geometric* side. Thm. 5.1 therefore puts finite-model ghosts on the geometric side. The cusp corresponds instead to the universal boost $\{1, q\}$ that is removed by normalizing $F' = F/\sqrt{q}$; for ζ this is the factor $\sqrt{y}$, the $\frac{1}{4}$. J.W.'s two infinities are thus

related, but they are *not* the same: the cusp is the null ray of the trivial-motive boost, while ghost null rays are irrational.

6. (5) IS THERE A MODE-SELECTIVE S IN CUSP SCATTERING?

Theorem 6.1 (resonance involution). *[THEOREM]/[RESTATEMENT] Let $\mathcal{R}$ be the set of poles of φ in $0 < \operatorname{Re} s < \frac{1}{2}$, so $\mathcal{R} = \rho/2$. (i) $\iota(s) = \frac{1}{2} - \bar{s}$ maps $\mathcal{R}$ to itself, since $\rho \mapsto 1 - \bar{\rho}$ preserves the zeros. Its fixed points are $\mathcal{R} \cap \{\operatorname{Re} s = \frac{1}{4}\}$, and its 2-cycles are exactly the ghosts (damping $\frac{1}{4} \pm \frac{\delta}{2}$). On the LP spectrum, ι is $\mu \mapsto -\frac{1}{2} - \bar{\mu}$. (ii) On the span of the resonance modes, $E := \operatorname{sgn}(\text{damping} - \frac{1}{4})$ commutes with B'' . It vanishes on on-line modes and is split-complex ($E^2 = 1$) on each ghost pair. This is the exact analogue of Thm. 2.2(c). (iii) $\operatorname{RH} \iff \iota|_{\mathcal{R}} = \operatorname{id} \iff E = 0 \iff 2B'' + \frac{1}{2}$ is skew-adjoint on the resonance span in some inner product (L105). [RESTATEMENT]: E is defined from pole positions.*

Proposition 6.2 (the geometric S of the cusp is ghost-blind). *[THEOREM] The reflection $\sigma(z) = -\bar{z}$, i.e. $\operatorname{diag}(1, -1)$ with $\det = -1$ (L102's anti-symplectic R_x), normalizes Γ and fixes the cusp. It commutes with the scaling boost $\operatorname{diag}(e^{t/2}, e^{-t/2})$, whose null rays 0 and $i\infty$ include the cusp. $E(\sigma z, s) = E(z, s)$ for all s , since the Fourier expansion has only $\cos(2\pi n x)$ [3]. So σ acts as $+1$ on every Eisenstein and resonance mode, true or ghost. The cusp's S -symmetry is the universal one.*

Proposition 6.3 (the scattering phase is a $\operatorname{Re} s = 1$ object). *[THEOREM] $d(r) = -\frac{1}{2\pi} \varphi' / \varphi(\frac{1}{2} + ir) = \frac{2}{\pi} \operatorname{Re} \frac{\Lambda'}{\Lambda}(1 + 2ir)$, because $\varphi' / \varphi = 2\Lambda' / \Lambda(2ir) - 2\Lambda' / \Lambda(1 + 2ir)$ and $\Lambda' / \Lambda(w) = -\Lambda' / \Lambda(1 - w)$. Moreover, $\varphi(\frac{1}{2} - \bar{s}) = \Lambda(1 + 2s) / \Lambda(2s) \neq \varphi(s)$, so the ratio $\varphi(s) / \varphi(\frac{1}{2} - \bar{s}) = \Lambda(2s - 1) / \Lambda(2s + 1)$ has no poles on $\mathcal{R}$. Thus ι is a symmetry of the pole set but not of the scattering matrix: it is inherited from ζ 's functional equation through the shift $2s \pm 1$, i.e. through $\operatorname{Re} s = 1 \leftrightarrow 0$.*

[NUMERICAL] (`num/lemma108_cusp.py`, `mpmath` 30 digits). The identity holds at $r \in \{0.7, 3, 7.07, 10.51, 14.2, 25\}$ to $\leq 2 \cdot 10^{-31}$, and the ι -identity to $1.4 \cdot 10^{-31}$.

Synthetic ghosts. We split a double zero at γ_1 into $\frac{1}{2} \pm \delta$ using a Blaschke factor. The candidate detectors behave as follows:

- $||\varphi_g| - 1| \leq 1.1 \cdot 10^{-15}$, and the change in area is $\leq 1.6 \cdot 10^{-12}$: blind.
- σ : blind by Prop. 6.2.
- ι -invariance of $\mathcal{R}_g$: holds, with a 2-cycle $\{0.25 \pm \frac{\delta}{2}\}$; it sees the ghost only by locating it.
- Bump shape: the peak ratio is $1/(1 - 4\delta^2) = 1.042, 1.190, 2.778$ for $\delta = 0.1, 0.2, 0.4$.
- Violation of the $\operatorname{Re} s = 1$ identity: $\max |\Delta d| = 0.106, 0.485, 4.52$.

Only the last two see ghosts, and both amount to locating zeros or comparing with primes (Fig. 2).

Verdict on Q5 ([RESTATEMENT], honest negative). “ $\operatorname{RH} \iff$ cusp reflection has no S -symmetry” is a *restatement*, for three reasons:

- (a) the cusp's natural S exists but commutes with all modes (Prop. 6.2);
- (b) the mode-selective E always exists formally and vanishes iff RH , but it is a spectral function of the resonances (Thm. 6.1);
- (c) any construction of E from φ on the line passes through $\operatorname{Re} s = 1$, which is the explicit formula (Prop. 6.3; L103, L106).

A non-circular version would need E built geometrically, independently of φ 's poles. Thm. 5.1 suggests looking on the geodesic side ([HEURISTIC]).

7. CORRECTIONS TO THE CHAT CLAIMS (T455)

C108.1 (1):

Confirmed, with sharpenings. The ghost is $\operatorname{sgn}(t) \times$ a boost; S is unique up to sign and anti-symplectic. The dictionary also has a *reversal* half: J reverses ghosts and S reverses true F .

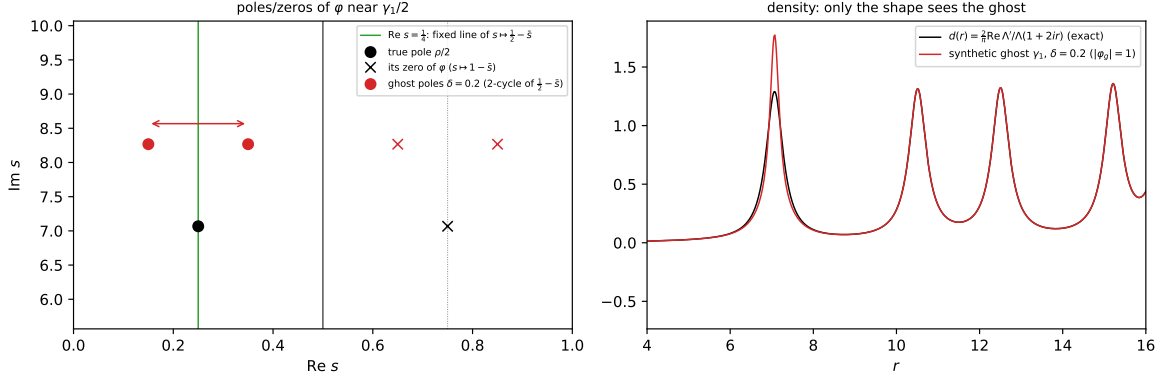


FIGURE 2. [NUMERICAL] Left: poles (●) and zeros (×) of φ near $\gamma_1/2$; the ghost pair is drawn offset upward for clarity. $s \mapsto 1 - \bar{s}$ (unitarity) pairs each pole with a zero; $s \mapsto \frac{1}{2} - \bar{s}$ pairs ghost poles with each other and fixes $\text{Re } s = \frac{1}{4}$. Right: exact $d(r)$ and the ghost-modified density, which keeps the same area and $|\varphi| = 1$.

C108.2 (1) genus 2.:

A quartet gives a bicomplex structure (S and J commute), not a purely split one. Real pairs give 2^k choices. Mixed models have *no* commuting S , only the partial E .

C108.3 (2).:

Confirmed, with rate $2r$ per step.

C108.4 (3).:

Confirmed. The incoming/outgoing labels depend on the sign of time (LP80 (2.11)). $\varphi(\frac{1}{2}) = -1$.

Damping is $\frac{1}{2} - \text{Re } s$.

C108.5 (4).:

The analogy is upgraded to a partial theorem: ghost null rays are closed-geodesic endpoints, while the cusp corresponds to $\{1, q\}$. The two infinities are *different*.

C108.6 (5).:

Restatement, for the three reasons above. No non-circular S -statement was found.

8. RANKED DIRECTIONS (SCORE = 0.6·SIG+0.4·PROB)

	Direction	sig	prob	score
1	Geodesic-side S : for $X(1)$, relate the split structures of hyperbolic classes ($\epsilon_d, h(d)$) to the ghost E through the trace formula. Is there a geometric E ?	0.8	0.25	0.58
2	Bicomplex genus- g classification: the moduli of commuting (J, S, E) for $\text{GSp}(2g)$, and the Rosati signature (L106) as $\text{rk } E^2$.	0.35	0.9	0.57
3	Finite-model trace formula: realize the $\{1, q\}$ cusp and the ghost geodesics in one model (Hecke at q on $X(1)$, Eichler–Selberg), and test where ghost terms appear.	0.5	0.6	0.54
4	Late-time LP decay as an E -detector: certify $e^{t/4} \ Z''(t)\ $ bounds on finite windows (equivalent to verification).	0.3	0.7	0.46
5	$\varphi(s)/\varphi(\frac{1}{2} - \bar{s}) = \Lambda(2s - 1)/\Lambda(2s + 1)$: is there an operator realization of this “shifted” reflection?	0.5	0.3	0.42

9. QUESTIONS FOR J.W.

- Q1. Do you accept the reversal half of the dictionary (true: commute J /reverse S ; ghost: commute S /reverse J)?
- Q2. Genus 2: is the bicomplex quartet structure $F' = e^{\delta S + \theta J}$ the “split-complex structure” you meant?
- Q3. The cusp corresponds to $\{1, q\}$ and ghosts to closed geodesics: does this match your “two infinities”?
- Q4. Do you accept the verdict that “RH $\iff$ no S -symmetry” is a restatement, for the three reasons (a)–(c)?
- Q5. Should L109 take direction 1 (a geometric E on the geodesic side) or direction 2 (bicomplex classification)?

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- [3] Wikipedia, “Real analytic Eisenstein series”. [opened: Fourier expansion, poles at $\rho/2$]

TAA LEMMA 109: A GEODESIC-SIDE E ? BUILDING THE SPLIT-COMPLEX STRUCTURE FROM HYPERBOLIC CLASSES

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma109_*.py`, data `num/l109/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma109/verified_refs.md`.

ABSTRACT. **Question** (L108, direction 1). Can the split-complex structure E of L108, or its rank, be built from closed geodesics and hyperbolic classes, without first knowing where the zeros are?

Answer. Yes, in the sense that a computable object has rank E^2 as its signature. No, in the sense that matters: every detector that works is Weil/Hasse positivity restated, and every *count-positivity* or *trace-formula* detector is blind to ghosts. In detail:

- **(A) Point-count Toeplitz form = Rosati form** ([THEOREM], [NUMERICAL]). For a curve over $\mathbb{F}_q$ put $\sigma_m = q^{-m/2}(q^m + 1 - N_m)$ and $T_N = [\sigma_{j-k}]_{j,k < N}$. Then $n_-(T_N) = \# \text{real pairs} + 2 \# \text{quartets} = \frac{1}{2} \text{rk } E^2$ for $N \geq 2g$ (Thm. 2.1). This is computable from $N_1, \dots, N_g$ alone. But $T_N \geq 0$ is Weil's theorem (L106 Rosati positivity), so it is a [RESTATEMENT].
- **(B) Ghost classes carry trace-formula weight 0** ([CITED]+[THEOREM], [NUMERICAL]). In the Popa–Zagier/Eichler–Selberg weighting of 2×2 integer matrices of determinant q , a ghost trace ($a^2 - 4q > 0$ non-square) is a non-split hyperbolic class. Its centralizer contains L108's Pell element $\gamma = uF + kI$, so it is infinite and the weight is $1/|\Gamma_M| = 0$. Only elliptic (true) classes and the split $\{1, q\}$ class contribute. Deuring's count $\#\{\text{curves with trace } r\} = H(4p - r^2)$ was checked exactly for $p = 13, 101$ (0 mismatches).
- **(C) Positivity of closed-point counts is ghost-blind** ([THEOREM], [NUMERICAL]). Genus 1, $q = 101$: all 204 integer traces $a \in [-101, 102]$ give $a_d \geq 0$ for $d \leq 40$ (Hasse allows only $[-20, 20]$). Genus 2, $q = 11$: 2935 ghost (s_1, s_2) have all closed-point counts ≥ 0 .
- **(D) Frobenius boost $\neq$ geodesic** ([THEOREM]). Both act along the same axis S , but their lengths are commensurable iff $q \mid a$. For $a = 25$: $\ell = 5.407$ and $2r = 1.370$. Among the 81 ghost traces $21 \leq |a| \leq 101$ only $a = \pm q$ qualify; there $\gamma = F^2/q$.
- **(E) Over $\mathbb{Q}$: a restatement, precisely** ([RESTATEMENT], [NUMERICAL]). The prime-side Weil form has $n_-(Q_L) \rightarrow 2 \# \text{quartets}$ (L103 machinery: 0/2/4/6 for 0–3 synthetic quartets). That this vanishes for all L is Weil's criterion. On $X(1)$ the geodesics are real-quadratic classes (lengths $2 \log \epsilon_d$, multiplicity $h(d)$) and not rational primes. ζ enters the Selberg trace formula only through φ'/φ , an $\text{Re } s = 1$ object (L108).
- **Erratum to L106** ([NUMERICAL]). L106's "double-zero replacement" convention leaves the on-line multiplicity at γ equal to -1 , which is unphysical. It inflates n_- at large L (4 instead of 2 per quartet) but moves the onsets by at most 0.025. L106's conclusions stand.

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GLOSSARY (TERMS RESTATED)

Ghost:

A Frobenius-like F with $\det F = q$ but some eigenvalue off the circle $|\alpha| = \sqrt{q}$. Genus 1: an integer trace $|a| > 2\sqrt{q}$. Over $\mathbb{Q}$: an off-line zero, which comes in a quartet $\frac{1}{2} \pm \delta \pm i\gamma$ (L103–L108).

 E :

The partial split-complex structure $E = \operatorname{sgn} \log |\lambda|$ on the eigenspaces of F (L108). $E^3 = E$, and E^2 is the projector onto the ghost part.

Hyperbolic / elliptic class:

A matrix $M \in M_2(\mathbb{Z})$ with $\det M = n$ and $t^2 - 4n > 0$ (hyperbolic) or < 0 (elliptic). It is *split* if $t^2 - 4n$ is a square.

Pell element:

$\gamma = uF + kI \in SL_2(\mathbb{Z})$, with $t^2 - Du^2 = 4$, $D = a^2 - 4q$, $k = (t - au)/2$ (L108). It commutes with F and projects to a closed geodesic of length $\ell = 2 \log \frac{t+u\sqrt{D}}{2}$.

 $H(D)$:

The Hurwitz class number, extended (Popa–Zagier) by $H(0) = -\frac{1}{12}$, $H(-u^2) = -\frac{u}{2}$, and $H = 0$ at other negative arguments.

 a_d :

The number of closed points of degree d , $a_d = \frac{1}{d} \sum_{e|d} \mu(d/e) N_e$. A genuine curve has $a_d \geq 0$.

1. CLAIMS CHECKED

The brief asked for three things:

- (i) a construction of E (or its trace or rank) from hyperbolic classes or geodesics;
- (ii) a count over hyperbolic classes, in finite models, that detects ghosts non-circularly from $\#C(\mathbb{F}_{q^n})$, together with a verdict on whether it is a positivity statement or Hasse restated;
- (iii) the $\mathbb{Q}$ case, with a precise reason why it is a restatement.

Circularity flag (applies throughout). In a finite model, “without zero locations” is *vacuous*. Newton’s identities recover $P(T)$, and hence all zeros, from $N_1, \dots, N_g$. So any detector computed from point counts is a function of the zeros. The real question is which *structural* statements (positivity, trace-formula weights) see ghosts. Over $\mathbb{Q}$ it is not vacuous, because the prime data are infinite.

2. FINITE MODELS: WHAT POINT COUNTS SEE

2.1. The Toeplitz detector.

Theorem 2.1 ([THEOREM]). *Let $F \in \mathrm{GSp}_{2g}$ have characteristic polynomial P with distinct roots α_i and $\det F = q^g$. Put $\lambda_i = \alpha_i/\sqrt{q}$, so the multiset $\{\lambda_i\}$ is closed under $\lambda \mapsto 1/\lambda$ and under complex conjugation. Let $\sigma_m = \sum_i \lambda_i^m$ (so $\sigma_{-m} = \sigma_m \in \mathbb{R}$, and $\sigma_m = q^{-m/2}(q^m + 1 - N_m)$ for a curve) and $T_N = [\sigma_{j-k}]_{0 \leq j,k < N}$. Then for $N \geq 2g$:*

$$(n_+, n_-)(T_N) = (\# \text{unit} + \# \text{real pairs} + 2\# \text{quartets}, \# \text{real pairs} + 2\# \text{quartets}),$$

where unit eigenvalues are counted individually. Hence $n_-(T_N) = \frac{1}{2} \mathrm{rk} E^2$, and $T_N \geq 0$ for all N iff F is ghost-free.

Proof. Let $v(\lambda) = (\lambda^j)_{j < N}$. Then $T_N = \sum_i v(\lambda_i)v(\lambda_i^{-1})^T$. Group the terms by orbit:

- A unit λ has $\lambda^{-1} = \bar{\lambda}$; together with $\bar{\lambda}$ it contributes $vv^* + \bar{v}v^T$, which is positive of rank 2.
- A real pair $\{\lambda, \lambda^{-1}\}$ contributes $vw^T + wv^T$, of signature $(1, 1)$.
- A quartet $\{\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}\}$ contributes the real part of $2(vw^* + wv^*)$ with v, w complex, of signature $(2, 2)$.

For $N \geq 2g$ the $2g$ vectors $v(\lambda_i)$ are independent (Vandermonde), so the signatures add. $\mathrm{rk} E^2$ is the number of off-circle eigenvalues, $2\#\text{real pairs} + 4\#\text{quartets}$. $\square$

Remark 2.2 ([RESTATEMENT], [TAA-L]). T_N is the normalized Rosati form $B_{jk} = \sum \alpha^j (q/\alpha)^k$ of L106, written in point counts. Its positivity for genuine curves is the Rosati/Castelnuovo positivity that *proves* Weil's RH over $\mathbb{F}_q$. So (A) is a restatement of Weil, not a new detector. Numerically (geo.json, block C):

- true curves over $\mathbb{F}_{101}$ give $(2, 0)$ and a true genus-2 model $(4, 0)$;
- a ghost with $a = 25$ gives $(1, 1)$ from $N = 2$;
- a quartet gives $(2, 2)$ from $N = 4$;
- a mixed model gives $(3, 1)$;
- two quartets in $g = 4$ give $(4, 4)$ from $N = 8$ (Fig. 1, middle).

2.2. Hyperbolic classes carry weight zero.

Proposition 2.3 ([CITED]+[THEOREM]). *Popa and Zagier [4] assign weights to $SL_2(\mathbb{Z})$ -classes of $M \in M_2(\mathbb{Z})$ with $\det M = n$, $\mathrm{tr} M = t$:*

- scalar: $\frac{1}{6}$;
- elliptic: $-1/|\Gamma_M|$;
- split hyperbolic: 1;
- non-split hyperbolic and parabolic: 0.

With the extended H this gives the Kronecker–Hurwitz relation $\sum_{t \in \mathbb{Z}} H(4n - t^2) = \sigma(n)$. For $n = q$ prime, a ghost trace a ($a^2 - 4q > 0$, non-square) is non-split hyperbolic. Its centralizer contains the infinite cyclic group generated by L108's Pell element γ ([THEOREM], L108), so its weight is 0. Ghosts therefore contribute nothing to the class-number side, and only true traces and the split $\{1, q\}$ class ($a = \pm(q+1)$) appear.

Check ([NUMERICAL], [CITED] [1]). We enumerated all (A, B) with $y^2 = x^3 + Ax + B$ over $\mathbb{F}_p$, for $p = 13$ and $p = 101$. In every case $\#\{(A, B) : \mathrm{tr} = r\} = \frac{p-1}{2}H(4p - r^2)$, using the (A, B) multiplicities $(p-1)/2$, $(p-1)/4$ ($j = 1728$) and $(p-1)/6$ ($j = 0$) from [1, Thm. 7, Cor. 1]. There were 0 mismatches over 15 and 41 traces, and no ghost trace ever occurs. Further checks:

- Deuring's sum: $\sum_{r^2 < 4p} H(4p - r^2) = 2p$ gives 26 and 202.
- Kronecker–Hurwitz: $\sum_t H(4q - t^2) = \sigma(q) = q + 1$ gives 14 and 102. Here $2q$ is the elliptic part and $2H(-(q-1)^2) = -(q-1)$ the split part, and the ghost part is 0 (Fig. 2, left).

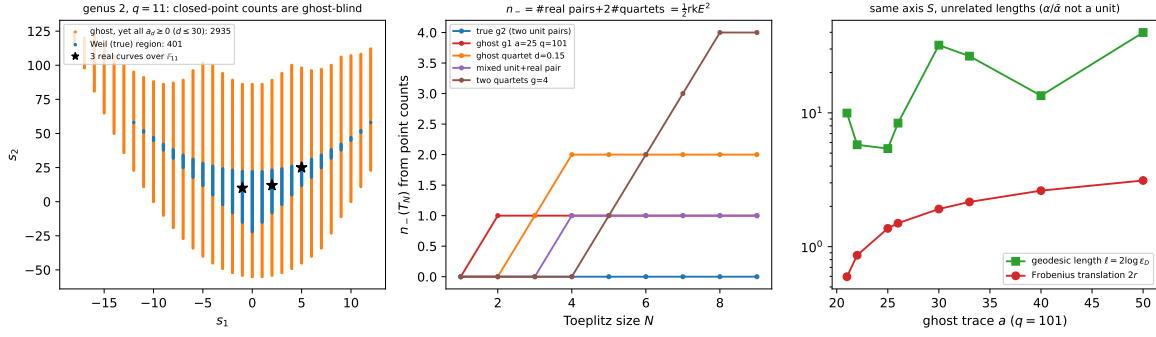


FIGURE 1. Finite models. Left: genus 2, $q = 11$; the Weil region (blue) sits inside the much larger set of points whose closed-point counts are all nonnegative (orange), and stars mark three real curves (L106). Middle: n_- of the point-count Toeplitz form T_N stabilizes at $\# \text{real pairs} + 2\# \text{quartets} = \frac{1}{2} \text{rk } E^2$ (Thm. 2.1). Right: Pell geodesic length vs. Frobenius translation for ghosts with $q = 101$; they share an axis but have unrelated lengths (Prop. 2.5).

Reading ([RESTATEMENT]). The “geodesic side” of the GL_2 trace formula is exactly where ghosts would live, and it is weighted to zero. That is Deuring/Hasse built into the weights. It is not a count that *detects* ghosts; it is a count that *cannot contain* them.

2.3. Closed-point positivity is blind.

Proposition 2.4 ([THEOREM], [NUMERICAL]). *In genus 1, $a_d = \frac{1}{d} \sum_{e|d} \mu(d/e)(q^e + 1 - \alpha^e - \beta^e)$. As $d \rightarrow \infty$, $a_d \sim (q^d - \alpha^d - \beta^d)/d$, which is eventually positive iff $\max(|\alpha|, |\beta|) < q$.*

- For $q = 101$, all $a \in [-101, 102]$ pass for $d \leq 40$: 204 traces, against Hasse’s 41.
- In genus 2 over $\mathbb{F}_{11}$ ($s_1 \in [-40, 40]$, $s_2 \in [-80, 249]$) there are 401 Weil points and 3336 count-positive points. 2935 of the count-positive points are ghosts, and no Weil point fails (Fig. 1, left).

The prime analogue over $\mathbb{Q}$ is $\Lambda(n) \geq 0$, which is equally blind: positivity of *counts* is not positivity of the *form*.

2.4. Frobenius boost versus Pell geodesic.

Proposition 2.5 ([THEOREM]). *For a genus-1 ghost ($D = a^2 - 4q > 0$ non-square), $F/\sqrt{q}$ and γ commute and translate along the same axis (L108’s S). Their translation lengths $2r = 2 \operatorname{arccosh} \frac{|a|}{2\sqrt{q}}$ and $\ell = 2 \log \epsilon$ are commensurable iff $q \mid a$.*

Proof. $(F/\sqrt{q})^{2m} = \pm \gamma^k$ is equivalent to $(\alpha/\bar{\alpha})^m = \epsilon^{2k}$ being a unit. Now $\alpha/\bar{\alpha} = \alpha^2/q$ has norm 1 and trace $(a^2 - 2q)/q$. If some power of it is an algebraic integer, so is $\alpha/\bar{\alpha}$ itself, which needs $q \mid a^2$, i.e. $q \mid a$. Conversely, if $a = \pm q$ then $\alpha^2/q = \pm(\alpha \mp 1)$ is a unit, so $\gamma = F^2/q$ up to sign. $\square$

Data ([NUMERICAL], `comm.json`). For $q = 101$, all ghost traces $21 \leq a \leq 101$ were checked. The only commensurable case is $a = 101$, with $\ell/2r = 2$ exactly. Elsewhere, for example:

a	ℓ	$2r$	$\ell/2r$
21	9.967	0.596	16.71
25	5.407	1.370	3.948
50	39.85	3.123	12.76

So the geodesic shares E ’s *axis* but not its *boost*: the axis is visible on the geodesic side and the rapidity is not.

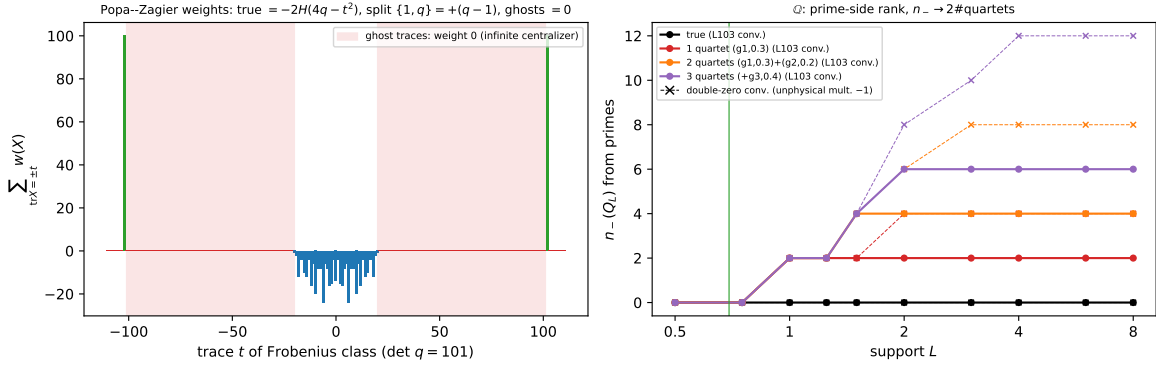


FIGURE 2. Left: Popa–Zagier weights summed over classes of trace $\pm t$, $\det = 101$. True traces give $-2H(4q - t^2)$, the split class $t = \pm 102$ gives $q - 1 = 100$, and ghost traces (shaded) give *exactly* 0. Right: $n_-(Q_L)$ of the prime-side Weil form for 0–3 synthetic quartets; solid lines use the L103 convention and dashed lines the L106 double-zero convention (multiplicity -1 , unphysical). The green line is $L = \log 2$.

3. OVER $\mathbb{Q}$: WHY THE GEODESIC SIDE RESTATES

Proposition 3.1 ([RESTATEMENT], [NUMERICAL]). *Let Q_L be the Weil quadratic form on test functions supported in $[-L/2, L/2]$, computed from primes and the archimedean term (L103 machinery, hat basis $h = 0.025$). Then:*

- (a) *With k synthetic quartets inserted (L103 convention: each quartet replaces one on-line zero), $n_-(Q_L)$ rises to $2k$ and stays there. For $k = 0, 1, 2, 3$ we get $0, 2, 4, 6$ at $L \geq 2$ (Fig. 2, right). This is consistent with Bombieri’s “half the off-line zeros” [2] and equals $\frac{1}{2} \text{rk } E^2$ as in Thm. 2.1.*
- (b) *$n_-(Q_L) = 0$ for all $L \iff \text{RH}$. This is Weil’s criterion [3], and it is a [RESTATEMENT].*

Why this is a restatement and not a geodesic construction.

- *The geodesics of $X(1)$ are not the primes.* In Selberg’s trace formula [5, App. 3, (1)] the hyperbolic classes $\{P\}$ are primitive closed geodesics with lengths $2 \log \epsilon_d$ and multiplicity $h(d)$, indexed by real-quadratic discriminants. Rational primes enter only through the scattering term φ'/φ on the spectral side.
- *That term is an $\text{Re } s = 1$ object.* By L108, $d(r) = \frac{2}{\pi} \text{Re } \Lambda'/\Lambda(1 + 2ir)$. Rewriting it in terms of zeros is the explicit formula.
- *The prime-side form exists only through Weil.* The only prime-side object whose signature counts ghosts is Weil’s form, so “building E from primes” means evaluating Weil’s functional.

The “primes $\leftrightarrow$ geodesics” dictionary is an [ANALOGY] between Selberg’s and Riemann’s zeta functions, not an identity on $X(1)$. Unlike the finite case, finite- L data see only low ghosts. The onset is $L \approx 0.78$ for $(\gamma_1, \delta = 0.49)$ and 1.25 for $(\gamma_2, 0.2)$, so no finite computation certifies RH.

Erratum to L106 ([NUMERICAL]). L106 used the *double-zero replacement* (a quartet replaces two on-line zeros). Since ζ ’s zeros are simple, this leaves multiplicity -1 at γ , an unphysical negative term. In the control, *removing* one zero (multiplicity 0) never produces negativity up to $L = 3$, because the form stays a sum of $|\hat{g}|^2$. Effect of the convention:

- at large L it doubles n_- (Fig. 2, dashed);
- it moves L106’s onsets $0.75/0.875/0.975/1.25$ to $0.775/0.875/1.0/1.25$.

L106’s window conclusions are unchanged. Its PDF is not modified.

4. CORRECTIONS TO THE CHAT CLAIMS

- (1) “*Primes $\leftrightarrow$ geodesics*” on $X(1)$. Corrected: the geodesics are real-quadratic classes $(2 \log \epsilon_d, h(d))$ [5]. Primes appear only via φ'/φ , so the dictionary is an [ANALOGY].
- (2) *The Pell axis $\gamma = uF + \frac{t-au}{2}I$ as a carrier of E* . Confirmed for the axis: γ is integral because $t \equiv au \pmod{2}$ always. Not for the boost (Prop. 2.5): $\ell = 5.407 \neq 2r = 1.370$ at $a = 25$, and the two are commensurable only when $q \mid a$.
- (3) *A count over hyperbolic classes detecting ghosts*. Negative. Ghost classes have weight 0 (Prop. 2.3), and closed-point positivity is blind (Prop. 2.4). The one working detector is the Toeplitz/Rosati form, which is Weil restated.
- (4) “*Sarnak class-number/geodesic results.*” Only the description in [5, App. 3] was opened and is used. Sarnak’s class-number asymptotics (1982) were *not* opened and are *not* cited.
- (5) *Ruelle/Selberg zeta*. Not used: no source was opened for the zeros of the Selberg zeta function at scattering poles. Listed as direction 3.
- (6) “*Expected answer for $\mathbb{Q}$ is a restatement.*” Confirmed, with the precise reason in Prop. 3.1.

5. RANKED DIRECTIONS (SCORE = 0.6 sig + 0.4 prob)

#	Direction	sig	prob	score
1	L110 (approved): classify the commuting (J, S, E) and prove $\text{rk } E^2 = 2n_-$ by Williamson normal forms, in genus g .	0.6	0.9	0.72
2	A Toeplitz/Carathéodory reading of Weil’s form over $\mathbb{Q}$: is Q_L a Toeplitz form in log-scale? Use Carathéodory–Toeplitz to bound $n_-(Q_L)$ by the ghost count below a height.	0.6	0.5	0.56
3	Selberg zeta $Z_{X(1)}$: open Hejhal and check whether its zeros at $\rho/2$ give a geodesic-side object with $\text{rk} = 2n_-$. Expected to be the scattering determinant again ([RESTATEMENT]).	0.5	0.4	0.46
4	A $q \mid a$ analysis: for $a = \pm q$, $\gamma = F^2/q$. Do the “commensurable ghosts” have any role (supersingular-like)?	0.3	0.6	0.42
5	Higher-rank weights: extend Prop. 2.3 to GSp_4 orbital integrals. Expected: ghost classes again have non-compact centralizer.	0.4	0.3	0.36

6. QUESTIONS FOR J.W.

- Q1:** Is (A) (point-count Toeplitz = Rosati, $n_- = \frac{1}{2} \text{rk } E^2$) the “geodesic-side E ” you had in mind, given that it is Weil restated?
- Q2:** Accept the reading of Prop. 2.3: hyperbolic-class weights *exclude* ghosts rather than detect them?
- Q3:** Accept the L106 erratum (adopt the L103 convention as standard from now on)?
- Q4:** Should the commensurable ghosts $a = \pm q$ be pursued (direction 4), or dropped as a curiosity?
- Q5:** For $\mathbb{Q}$, is direction 2 (Toeplitz structure of Q_L) worth a lemma, knowing that its best case is an effective restatement?

REPRODUCIBILITY

- `num/lemma109_geo.py`: blocks A–E, writes `geo.json` and `g2grid.npy`; about 3 s.
- `num/lemma109_qrank.py`: writes `qrank.json`; about 7 s.
- `num/lemma109_comm.py`: writes `comm.json`.
- `num/lemma109_figs.py`: figures.

Never touched: `lemma1.pdf`, `volume*`, `submit` directories, `num/lemma11_ghost.py`, `briefs/prequantization-brief.md`, `num/lemma34_string.py`, and the L88–L108 PDFs and builds.

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- [2] E. Bombieri, Remarks on Weil's quadratic functional in the theory of prime numbers I, Rend. Lincei (2000). [abstract only; CITED-sec, carried from L103]
- [3] M. Ram Murty, Prime numbers, explicit formulas and the Riemann hypothesis (lecture notes). [opened in L103: Weil's criterion]
- [4] A. A. Popa, D. Zagier, A simple proof of the Eichler-Selberg trace formula, arXiv:1711.00327. [opened: p. 2, class weights and the extended H]
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TAA LEMMA 110: CLASSIFYING COMMUTING (J, S, E) IN GENUS g AND THE ROSATI SIGNATURE

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma110_*.py`, data `num/l110/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma110/verified_refs.md`.

ABSTRACT. For a normalized $F \in \mathrm{Sp}(2g, \mathbb{R})$ with distinct eigenvalues, write its type as (k_u, k_r, k_c) : k_u unit pairs, k_r real pairs and k_c quartets, with $g = k_u + k_r + 2k_c$. We classify every commuting structure:

- compatible complex structures J ;
- symplectic complex structures (any J with $J^2 = -1$);
- anti-symplectic involutions S ;
- symplectic involutions;
- partial split-complex $E \in \mathfrak{sp}$ with $E^3 = E$.

Theorem 1.1 ([THEOREM]; brute force [NUMERICAL]: 63 normal-form models in $g = 1\text{--}4$, 778 Weil polynomials over $\mathbb{F}_{11}$, 240 random $e^{\Omega H}$; 0 mismatches). The counts are:

Structure	Count	Condition
compatible J	1	iff $k_r = k_c = 0$ (true case)
symplectic J	$2^{k_u+k_c}$	iff $k_r = 0$
anti-symplectic S	$2^{k_r+k_c}$	iff $k_u = 0$
symplectic involutions	$2^{k_u+k_r+k_c}$	always
E (any rank)	$3^{k_r+k_c}$	always
E of maximal rank	$2^{k_r+k_c}$	ghost part non-zero

E always vanishes on the unit part. The canonical choice $E = \mathrm{sgn} \log |F|$ is the unique maximal-rank E with $E \log |F| \geq 0$.

Consequences.

- L108's counts $2/4/0/0$ (quartet / two real pairs / mixed / true) and “ 2^k ” are the case $k_u = 0$ ([THEOREM]).
- $\mathrm{rk} E^2 = 2k_r + 4k_c = 2n_-$, where n_- is the Rosati/Toeplitz index of L106/L109. This is Bombieri's “half of the off-line zeros” in finite form ([THEOREM], [CITED-sec]).
- A quartet has two commuting symplectic J , but neither is compatible.
- For degenerate spectra the counts become manifolds ([THEOREM] sketch, [NUMERICAL]): compatible J form $\prod U(p, q)/U(p) \times U(q) \times \prod$ Siegel spaces (dimensions 2, 6, 12 checked), and anti-symplectic S for an m -fold real pair form components of dimension $2k(m - k)$ (checked: 2, 4).

Over $\mathbb{Q}$ ([ANALOGY], [RESTATEMENT]). ζ has no real zeros in $(0, 1)$ ([THEOREM], via η), so $k_r = 0$, and it has infinitely many unit-type zeros (Hardy, [CITED-sec]). So ζ is always of *mixed* type: no global S , only a partial E , and “a commuting compatible J exists $\iff$ RH” is a restatement.

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GLOSSARY (TERMS RESTATED)

Normalized F :

$F = \text{Frob}/\sqrt{q} \in \text{Sp}(2g, \mathbb{R})$, with $F^T \Omega F = \Omega$. Its eigenvalues come in orbits under $\lambda \mapsto \bar{\lambda}, 1/\lambda$.

Type (k_u, k_r, k_c) :

The orbit counts:

- unit pairs $\{e^{\pm i\theta}\}$ (true);
- real pairs $\{\lambda, 1/\lambda\}$ with $\lambda \in \mathbb{R} \setminus \{\pm 1\}$ (ghost);
- quartets $\{\lambda, \bar{\lambda}, 1/\lambda, 1/\bar{\lambda}\}$ (ghost).

Compatible J :

$J^2 = -1$, $J^T \Omega J = \Omega$, and $g_J(u, v) = \Omega(u, Jv)$ positive definite.

Anti-symplectic S :

$S^2 = 1$ and $S^T \Omega S = -\Omega$ (L102, L108).

Partial E :

$E \in \mathfrak{sp}(2g)$ ($E^T \Omega + \Omega E = 0$) with $E^3 = E$ and $[E, F] = 0$. E^2 is a projector.

 n_- :

The negative index of the Rosati form $B_{jk} = \sum \alpha^j (q/\alpha)^k$ (L106), equivalently of the point-count Toeplitz form T_N (L109, Thm. 2.1).

Krein sign:

For a unit eigenvalue $\lambda \neq \pm 1$, the sign of the Hermitian form $\hat{Q}^\lambda$ [3, Def. 16, Prop. 17].

1. THE CLASSIFICATION

1.1. Normal form and commutant. By Gutt's normal form [3, Thm. 1, Thm. 9, Thm. 15] ([CITED]), F is a symplectic direct sum of its restrictions to the real symplectic subspaces $V_{[\lambda]}$, where $V_{[\lambda]}^{\mathbb{C}} = E_\lambda \oplus E_{1/\lambda} \oplus E_{\bar{\lambda}} \oplus E_{1/\bar{\lambda}}$. Moreover $\Omega(E_\lambda, E_\mu) = 0$ unless $\lambda\mu = 1$ [3, Cor. 4].

For off-circle λ , Gutt's normal form is $\text{diag}(A^{-1}, A^T)$:

- $A = \lambda$ for a real pair;
- $A = r R(\phi)$, with $R(\phi)$ the rotation by ϕ , for a quartet.

Our numerical models are built exactly this way and then conjugated by a random $e^{\Omega H}$.

With distinct eigenvalues the commutant of F is $\mathbb{R}[F]$. An element $X \in \mathbb{R}[F]$ acts on each eigenvector v_μ by a scalar x_μ , subject to two conditions:

- reality: $x_{\bar{\mu}} = \bar{x}_\mu$;
- the symplectic adjoint ($\Omega(Xu, v) = \Omega(u, X^*v)$, $F^* = F^{-1}$) acts by $x_\mu^* = x_{1/\mu}$.

So, by Cor. 4, the three conditions below are each *blockwise*:

$$X^T \Omega X = \pm \Omega \iff x_\mu x_{1/\mu} = \pm 1, \quad X \in \mathfrak{sp} \iff x_\mu + x_{1/\mu} = 0.$$

Theorem 1.1 ([THEOREM]). *Let $F \in \mathrm{Sp}(2g, \mathbb{R})$ have distinct eigenvalues and type (k_u, k_r, k_c) . The commuting structures are, block by block:*

block	sympl. J ($x = \pm i$)	compatible J	anti-sympl. S	sympl. invol.
unit pair $\{e^{\pm i\theta}\}$	2	1 (Krein sign)	0	2
real pair $\{\lambda, \lambda^{-1}\}$	0	0	2	2
quartet	2	0	2	2

Global counts are products over blocks: compatible J : $[k_r = k_c = 0]$; symplectic J : $2^{k_u+k_c}[k_r = 0]$; anti-symplectic S : $2^{k_r+k_c}[k_u = 0]$; symplectic involutions: $2^{k_u+k_r+k_c}$.

For E : on a unit pair $E = 0$ is forced; on a real pair or quartet $E \in \{0, \pm\}$. So there are $3^{k_r+k_c}$ partial E 's, of which $2^{k_r+k_c}$ have maximal rank $\mathrm{rk} E^2 = 2k_r + 4k_c$. The canonical $E = \mathrm{sgn} \log |F|$ (sign + on $|\mu| > 1$) is the unique maximal-rank E with $E \log |F| = |\log |F|| \geq 0$. On the ghost subspace $V_{\mathrm{gh}} = \bigoplus_{\mathrm{off-circle}} V_{[\lambda]}$, $E|_{V_{\mathrm{gh}}}$ is an anti-symplectic involution.

Proof. Each item reduces to a condition on the scalars x_μ .

Symplectic J . Here $x_\mu = \pm i$ and $x_\mu x_{1/\mu} = 1$, so $x_{1/\mu} = -x_\mu$, while reality gives $x_{\bar{\mu}} = -x_\mu$.

- Unit pair: $\bar{\mu} = 1/\mu$, and the two conditions agree, leaving 2 choices.
- Real pair: $\bar{\mu} = \mu$ would force $x_\mu = -x_\mu$, so there is none.
- Quartet: the choice of x_λ fixes the other three, leaving 2 choices.

Compatibility. On a unit block, $g_J(v, v)$ is $\pm$ (the Krein sign) times a positive quantity [3, Prop. 17], so exactly one sign is compatible. On a quartet, $P = (E_\lambda \oplus E_{\bar{\lambda}})_{\mathbb{R}}$ is Lagrangian (Cor. 4: $\lambda \bar{\lambda} \neq 1$, $\lambda^2 \neq 1$) and J -invariant, so $g_J|_P = 0$ and J is not compatible.

Anti-symplectic S . Here $x = \pm 1$ and $x_\mu x_{1/\mu} = -1$, so $x_{1/\mu} = -x_\mu$, while reality gives $x_{\bar{\mu}} = x_\mu$.

- Unit pair: $\bar{\mu} = 1/\mu$ gives $x = -x$, so there is none.
- Real pair and quartet: 2 choices each.

Symplectic involutions. Here $x_{1/\mu} = x_\mu = x_{\bar{\mu}}$, leaving 2 choices on every block.

Partial E . Here $x \in \{0, \pm 1\}$ with $x_{1/\mu} = -x_\mu$ and $x_{\bar{\mu}} = x_\mu$. On a unit pair this forces $x = 0$; on a real pair or quartet the value x_λ is free. The rest is bookkeeping. $\square$

Corollary 1.2 ([THEOREM]; confirms L108). *In genus 2 the anti-symplectic counts are 2 (quartet), 4 (two real pairs), 0 (mixed) and 0 (true). In genus 1 a ghost has $\pm S$. “ 2^k for k real pairs” holds iff there is no unit pair; in general the count is $2^{k_r+k_c}[k_u = 0]$.*

Theorem 1.3 ([THEOREM], [TAA-L] L106/L109). $n_-(\text{Rosati}) = k_r + 2k_c = \frac{1}{2} \mathrm{rk} E_{\max}^2$, and $n_+ = k_u \cdot 2 + k_r + 2k_c$ (unit eigenvalues counted individually).

Proof. L109, Thm. 2.1: unit pairs contribute (2, 0), real pairs (1, 1) and quartets (2, 2). Then $\mathrm{rk} E^2 = 2k_r + 4k_c$ by Thm. 1.1. $\square$

Bombieri ([CITED-sec] [2]). The number of negative directions of Weil's form is half the number of off-line zeros. Thm. 1.3 is the finite statement “ $n_- = \frac{1}{2} \# \{\text{off-circle eigenvalues}\} = \frac{1}{2} \mathrm{rk} E^2$ ”. Numerically, all 63 models satisfy $n_-(T_{2g}) = \frac{1}{2} \mathrm{rk} E^2$ exactly (60-digit Toeplitz signatures; Fig. 1, right).

1.2. Degenerate spectra: counts become moduli.

Proposition 1.4 ([THEOREM] sketch, [NUMERICAL]). *Suppose F is semisimple but has repeated eigenvalues.*

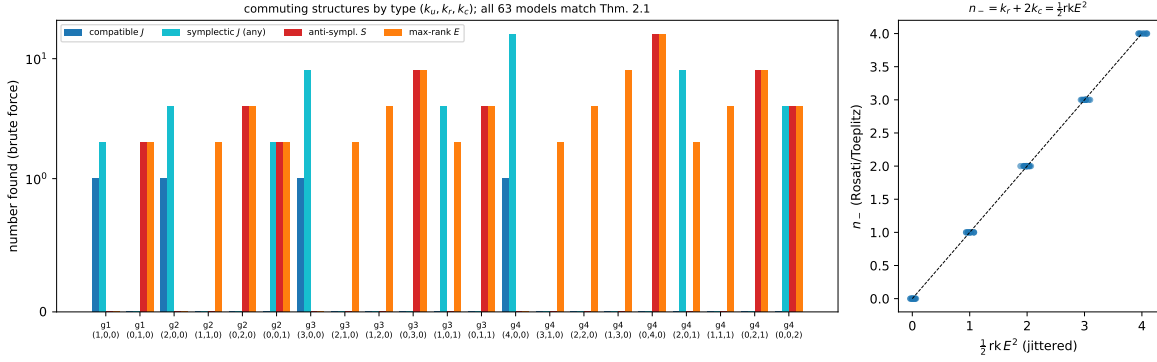


FIGURE 1. Left: brute-force counts of commuting structures for each type (k_u, k_r, k_c) in $g = 1-4$ (symlog scale); every count equals Thm. 1.1. Right: n_- of the 60-digit Toeplitz form against $\frac{1}{2} \text{rk } E^2$ for all 63 models (Thm. 1.3).

- Compatible J . The commuting compatible J form $\prod_{\lambda \in S^1 \setminus \pm 1} U(p_\lambda, q_\lambda) / (U(p_\lambda) \times U(q_\lambda)) \times \prod_{\lambda = \pm 1} \mathfrak{H}_{m_\lambda}$. Here (p_λ, q_λ) is the Krein signature and $\mathfrak{H}_m$ is Siegel space of dimension $m(m+1)$, so the dimension is $\sum 2p_\lambda q_\lambda + \sum m_\lambda(m_\lambda + 1)$. Uniqueness therefore holds iff every repeated unit eigenvalue $\neq \pm 1$ has definite Krein signature and ± 1 is not an eigenvalue.
- Anti-symplectic S . For an m -fold real pair the commuting S form $m+1$ components, indexed by the signature of $S|_{E_\lambda}$, of dimension $2k(m-k)$.

Tangent-space dimensions computed numerically (*degen.json*):

Structure	Case	dim
compatible J	same Krein sign	0
compatible J	opposite Krein sign	2
compatible J	$F = -I$, $g = 2$	6
compatible J	$F = -I$, $g = 3$	12
anti-symplectic S	2-fold real pair, $k = 1$	2
anti-symplectic S	3-fold real pair, $k = 1$	4
anti-symplectic S	$k = 0$ base point	0

So the canonical character of J (true) and of E (ghost) depends on genericity. Supersingular-type $F = \pm I$ has a whole Siegel space of commuting J , which is L105's Siegel space.

2. ENUMERATION OVER STRUCTURED AND RANDOM MODELS

Structured ([NUMERICAL], *ensembles.json*). Take all integer $(s_1, s_2) \in [-40, 40] \times [-80, 249]$ for $P(T) = T^4 - s_1 T^3 + s_2 T^2 - 11s_1 T + 121$. By type:

Type	Count
true (2, 0, 0)	400
mixed (1, 1, 0)	15100
two real pairs (0, 2, 0)	2084
quartet (0, 0, 1)	9127
degenerate	19

A 3% subsample (778 polynomials), realized in Gutt normal form, was brute-forced with 0 mismatches. The type map (Fig. 2, left) shows that the Weil region is the small blue lens. Mixed type, where neither J nor S exists, dominates.

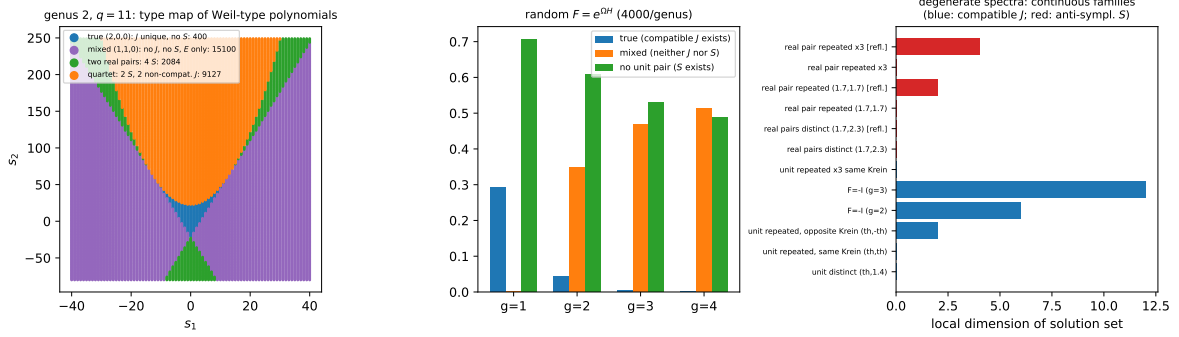


FIGURE 2. Left: genus-2 type map over $\mathbb{F}_{11}$ (Weil-type integer polynomials). Middle: type frequencies for random $e^{\Omega H}$. Right: local dimension of the sets of commuting compatible J (blue) and anti-symplectic S (red) for degenerate spectra (Prop. 1.4).

Random ([NUMERICAL]). For $F = e^{\Omega H}$ with H Gaussian symmetric (4000 per genus), the fraction of true type is:

g	1	2	3	4
true fraction	29%	4.3%	0.3%	0%

Brute-force checks: 240, with 0 mismatches. The existence of a compatible J is a measure-shrinking condition ([HEURISTIC]): it is an open condition, but the open set shrinks quickly with g in this ensemble).

3. THE $\mathbb{Q}$ DICTIONARY

Proposition 3.1 ([THEOREM]). ζ has no zeros on the real segment $(0, 1)$. Hence “real pairs” never occur for ζ : $k_r = 0$, and n_- (Bombieri) is even.

Proof. By [4], $\zeta(s) = \eta(s)/(1 - 2^{1-s})$ for $\text{Re } s > 0$. For real $s \in (0, 1)$, $\eta(s) = \sum (-1)^{n-1} n^{-s}$ is an alternating series with decreasing terms, so $\eta(s) \geq 1 - 2^{-s} > 0$, while $1 - 2^{1-s} < 0$. $\square$

Reading ([ANALOGY], [RESTATEMENT]). Infinitely many zeros of ζ lie on the line (Hardy 1914, [CITED-sec] via [4]), so formally $k_u = \infty$. By Thm. 1.1:

- ζ admits *no* global commuting anti-symplectic S , whatever the truth of RH;
- a commuting compatible J exists iff $k_c = 0$, which is RH;
- E is the only ghost-sensitive structure, and $\text{rk } E^2 = 4k_c = 2n_-$.

This is a restatement: there is no operator F for ζ , and the dictionary is an analogy. Real pairs would be the analogue of Siegel zeros of Dirichlet L -functions (zeros in $(\frac{1}{2}, 1)$, [4]), which is where S could exist.

4. CORRECTIONS TO THE CHAT CLAIMS

- (1) “Use Williamson normal forms.” Williamson’s theorem diagonalizes positive-definite *quadratic forms* symplectically ([CITED], [1, §IX], opened in L103); it does not give the conjugacy classes of symplectic matrices. The right tool is the Krein-sign normal form [3], used here.
- (2) *L108 counts 2/4/0/0 and 2^k .* Confirmed (Cor. 1.2). The general formula is $2^{k_r+k_c}[k_u = 0]$: quartets also double the count, and one unit pair kills it.
- (3) “The commuting compatible J ” (*singular*). Unique only for generic or definite-Krein spectra. For $F = \pm I$ it is a Siegel space of dimension $g(g+1)$ (Prop. 1.4).

- (4) *L108's bicomplex* $e^{\delta S + \theta J}$ for a quartet. Its J is a commuting *symplectic* complex structure, but it is *never compatible* (Lagrangian invariant plane).
- (5) “*The partial E.*” Algebraically E is not unique: there are $3^{k_r+k_c}$ solutions, $2^{k_r+k_c}$ of maximal rank. It is canonical only with the dynamical normalization $E = \text{sgn} \log |F|$.
- (6) $n_- = \text{real pairs} + 2 \cdot \text{quartets} \leftrightarrow \text{Bombieri}$. Confirmed as $n_- = \frac{1}{2} \text{rk } E^2$. For ζ the real-pair term is absent (Prop. 3.1).

5. RANKED DIRECTIONS (SCORE = 0.6 sig + 0.4 prob)

#	Direction	sig	prob	score
1	Krein collisions: by Krein–Gelfand–Lidskii-type theory, true→ghost transitions happen only where unit eigenvalues of <i>opposite</i> Krein sign collide (Prop. 1.4, dim 2). Test the ζ analogue: zeros leave the line only through double zeros (de Bruijn–Newman flow). Source to open first.	0.6	0.6	0.60
2	A Pontryagin-space E on Weil’s form: a canonical E on the negative space of Q_L (L109 dir. 2).	0.6	0.5	0.56
3	Hodge reading: compatible $J \leftrightarrow$ Siegel/Hodge point (L105); ghosts are exactly the F with no Hodge point. Is that Rosati positivity restated? (expected [RESTATEMENT])	0.5	0.5	0.50
4	Non-semisimple F (Jordan blocks, Gutt Thm. 10): count structures at collisions.	0.3	0.7	0.46
5	Siegel zeros as real pairs: a formal S for Dirichlet L -functions.	0.4	0.4	0.40

6. QUESTIONS FOR J.W.

- Q1:** Adopt Thm. 1.1’s table as the TAA reference classification (replacing L108’s case list)?
- Q2:** Do you accept the correction that the quartet’s J is symplectic but never compatible?
- Q3:** Should the canonical E be fixed as $\text{sgn} \log |F|$ (dynamical), given that algebra alone leaves $3^{k_r+k_c}$ choices?
- Q4:** Pursue direction 1 (Krein collisions $\leftrightarrow$ double zeros) next?
- Q5:** Is the $\mathbb{Q}$ reading (“ ζ is always mixed: no global S ”) the intended endpoint of the S -programme?

REPRODUCIBILITY

- `num/lemma110_classify.py`: 63 models; about 3 s.
- `num/lemma110_degen.py`.
- `num/lemma110_ensembles.py`: about 20 s.
- `num/lemma110_figs.py`.

Never touched: `lemma1.pdf`, `volume*`, `submit` directories, `num/lemma11_ghost.py`, `briefs/prequantization_brief.md`, `num/lemma34_string.py`, and the L88–L109 PDFs and builds.

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- [1] Arvind, B. Dutta, N. Mukunda, R. Simon, The real symplectic groups in quantum mechanics and optics, *Pramana* 45 (1995), arXiv:quant-ph/9509002. [opened in L103, §IX (Williamson); carried]
- [2] E. Bombieri, Remarks on Weil’s quadratic functional, *Rend. Lincei* (2000). [abstract only; CITED-sec, carried]
- [3] J. Gutt, Normal forms for symplectic matrices, *Portugaliae Math.* 71 (2014) 109–139, arXiv:1307.2403. [opened: Thm. 1, Lemma 3, Cor. 4, Thm. 9, Thm. 15, Def. 16, Prop. 17]
- [4] Wikipedia, “Riemann hypothesis” (raw source). [opened: $\zeta = \eta/(1 - 2^{1-s})$; Hardy 1914 infinitely many zeros on the line; Siegel zeros under GRH]

TAA LEMMA 111: KREIN COLLISIONS AND DOUBLE ZEROS: HOW A TRUE PAIR WOULD HAVE TO LEAVE THE LINE

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma111_*.py`, data `num/1111/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma111/verified_refs.md`.

ABSTRACT. **(a) Finite models.**

- *Strong stability* ([THEOREM], Thm. 1.2). For semisimple $F \in \mathrm{Sp}(2g, \mathbb{R})$ with all eigenvalues on the unit circle, F is strongly stable (no small symplectic perturbation creates a ghost) iff two conditions hold:
 - the Krein form is definite on every eigenspace E_λ , i.e. no eigenvalue carries both signs;
 - ± 1 is not an eigenvalue.

This is Krein–Gelfand–Lidskii in matrix form.

- *Where it is Hasse restated.* For distinct eigenvalues it is automatic, and it *is* the openness of the Weil/Hasse condition; in genus 1, strongly stable $\iff |t| < 2$ ([RESTATEMENT]).
- *Where it adds content.* Only at collisions. A polynomial on the boundary of the Weil region is realized by several Sp-classes (Krein labelings), and only the opposite-sign one is an exit point. For example, $E \times E$ is a boundary polynomial but is strongly stable.

(b) Families ([NUMERICAL]). Over 300 random symplectic paths in genus 2 and 3:

- all 710 ghost births and all 204 ghost deaths occur at collisions of *opposite* Krein sign (562 at ± 1 , giving real pairs; 148 elsewhere, giving quartets);
- none of the 34 close same-sign encounters produced a ghost.

Perturbations of size ε (400 per size, 7 sizes) give:

- distinct or same-sign double eigenvalue: 0 ghosts in 2800 trials each;
- opposite-sign double eigenvalue or $\pm I$: about 70% ghosts, with splitting $\propto \varepsilon^{1.00}$.

(c) Over $\mathbb{Q}$. For the de Bruijn–Newman flow H_t we located the first collisions exactly ([NUMERICAL]):

- zeros 4 and 5 meet at $t^* = -3.95479$, $x^* = 66.1975$; zeros 7 and 8 meet at $t^* = -4.03382$;
- afterwards the pair leaves the line with $|\mathrm{Im}| \propto (t^* - t)^{0.51}$.

Three “Krein signs” are possible for zeros of ζ ([THEOREM], [RESTATEMENT]):

- the *graphical* one, $\mathrm{sgn} H'_t(x_j)$ (Kollár–Miller), alternates, so every adjacent pair is “opposite” and nothing is protected;
- the Weil-form one gives every on-line zero sign $+$, but it is not conserved by any known flow and has no gap, and “definite” there $\iff$ RH (circular);
- the Hermite–Biehler phase is left open.

Rodgers–Tao ($\Lambda \geq 0$) says that ζ is *not* strongly stable under the heat flow: if RH holds, it sits exactly at a collision threshold. Lehmer pairs are near-collisions. For the pair at $\gamma \approx 7005.06$ (normalized gap 0.042) the two-body estimate is $t^* \approx -7.1 \cdot 10^{-4}$ ([HEURISTIC]).

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GLOSSARY (TERMS RESTATED)

Krein form:

$h(u, w) = i \Omega(\bar{u}, w)$ on $\mathbb{C}^{2g}$. It is Hermitian and F -invariant ($h(Fu, Fw) = h(u, w)$), and it is Gutt's $\hat{Q}_1^\lambda$ [3, Def. 16].

Krein sign:

The Krein sign of a unit eigenvalue λ is the sign of h on E_λ ; it is mixed if $h|_{E_\lambda}$ is indefinite. The eigenvalues λ and $\bar{\lambda}$ always have opposite signs.

Strongly stable:

There is $\delta > 0$ such that every $F' \in \mathrm{Sp}(2g, \mathbb{R})$ with $\|F' - F\| < \delta$ has all eigenvalues on the circle and is semisimple. In TAA terms: no nearby ghost.

Krein collision:

Two unit eigenvalues of opposite Krein sign meet; generically they then leave the circle (Hamiltonian–Hopf).

 H_t :

$\int_0^\infty e^{tu^2} \Phi(u) \cos(zu) du$, with $H_0(z) = \frac{1}{8} \xi(\frac{1}{2} + \frac{iz}{2})$ [6, (1)–(3)]. A zero x corresponds to $\gamma = x/2$.

Graphical Krein sign:

The sign of the slope of the eigenvalue branch at a real characteristic value [4, (9)].

1. FINITE MODELS: KREIN–GELFAND–LIDSKII IN $\mathrm{Sp}(2g, \mathbb{R})$

Lemma 1.1 ([THEOREM]). *If $Fv = \mu v$ with $|\mu| \neq 1$, then $h(v, v) = 0$.*

Proof. $h(v, v) = h(Fv, Fv) = |\mu|^2 h(v, v)$. □

Theorem 1.2 ([THEOREM]; [CITED] [4] for the Hamiltonian version). *Let $F \in \mathrm{Sp}(2g, \mathbb{R})$ be semisimple with all eigenvalues on S^1 .*

- (i) *If $\pm 1 \notin \mathrm{spec} F$ and $h|_{E_\lambda}$ is definite for every λ , then F is strongly stable.*
- (ii) *If some $h|_{E_\lambda}$ is indefinite, or $\pm 1 \in \mathrm{spec} F$, then arbitrarily small perturbations $F e^{\varepsilon \Omega H}$ create ghosts.*

Proof. (i) Let Γ_λ be a small circle around λ and P_ε the Riesz projector of F' onto the eigenvalues inside Γ_λ . It is continuous, so h stays definite on $\mathrm{Im} P_\varepsilon$ for $\|F' - F\|$ small, since definiteness is an open condition. By Lemma 1.1 that image contains no eigenvector of modulus $\neq 1$. Definiteness also excludes Jordan chains: if $(F' - \mu)w = v$ with $|\mu| = 1$, then $h(v, v) = 0$.

(ii) If $\lambda = \pm 1$, then E_λ is real, so $h(v, v) = 0$ for real v . For an indefinite E_λ , take u, w with $h(u, u) > 0 > h(w, w)$. On the real symplectic span of the pair, the rotation is perturbed by a

boost (L108), and the eigenvalues split into a quartet linearly in ε . Kollár–Miller state the Hamiltonian version: opposite signs are necessary for Hamiltonian–Hopf bifurcation and sufficient for some admissible deformation, while “points with the same Krein signatures cannot leave the imaginary axis even if they collide” [4, pp. 5–6]; see also [2]. Our numerics give about 70% ghosts at every $\varepsilon \in [10^{-4}, 10^{-1}]$, with splitting exponent 1.006 (Fig. 1, top right). $\square$

Corollary 1.3 ([RESTATEMENT]). (a) *In genus 1, $F \in SL_2(\mathbb{R})$ is strongly stable iff $|\mathrm{tr} F| < 2$. That is Hasse’s strict inequality $|a| < 2\sqrt{q}$.*
 (b) *A true F with distinct eigenvalues is always strongly stable. This is the openness of the Weil region, so Hasse/Weil robustness is restated, not strengthened.*
 (c) *Krein theory adds information only at collisions: a double unit root, or a root at $\pm\sqrt{q}$.*

Proposition 1.4 ([THEOREM]; L110). *Suppose F commutes with a compatible J (L110). Then h is positive on $V^{1,0}$ (the $-i$ -eigenspace of J , with our sign convention) and negative on $V^{0,1}$, so the Krein+ eigenvalues are exactly $\mathrm{spec}(F|_{V^{1,0}}) = \{e^{i\phi_1}, \dots, e^{i\phi_g}\}$. An opposite-sign collision means $e^{i\phi_j} = e^{-i\phi_k}$. If $F = e^{\Omega H}$ with $H > 0$ and Williamson frequencies ω_j [1, §IX], this happens iff $\omega_j + \omega_k \in 2\pi\mathbb{Z}$. The condition $\lambda_j + \lambda_k \neq 0$ quoted for Krein–Gelfand–Lidskii is the same statement in Hamiltonian form.*

Weil boundary by Krein class ([THEOREM], [NUMERICAL]; Fig. 1, bottom). In genus 2, a true point has angles $0 < \theta_1 < \theta_2 < \pi$ and two Krein labelings:

- labeling $(+, +)$: the only opposite collisions are at $\theta \in \{0, \pi\}$, i.e. roots at $\pm\sqrt{q}$ (real-pair births). The exit distance is $\min(\theta_1, \pi - \theta_2)$.
- labeling $(+, -)$: $\theta_1 = \theta_2$ (the double-root curve $s_2 = s_1^2/4 + 2q$, quartet births) is also an exit. The exit distance is $\min(\theta_2 - \theta_1, \theta_1, \pi - \theta_2)$.

So $(T^2 - aT + q)^2 (E \times E)$ is on the Weil boundary but has a strongly stable realization (same sign). Its image in polynomial space is a *one-sided* neighbourhood.

Circularity ([RESTATEMENT]). Krein definiteness presupposes that all eigenvalues are already on the circle. It is a *stability* statement, not a *positivity* statement, and Frobenius is not deformed. Nothing here can prove Weil; the proof of Weil remains Rosati positivity (L106/L109).

2. FAMILIES: GHOSTS ARE BORN ONLY AT KREIN COLLISIONS

Paths ([NUMERICAL], `krein.json`). Each path is $F(s) = \exp(\Omega((1-s)H_0 + sH_1))$, with $H_0 > 0$ (a true start) and H_1 indefinite, sampled at 1501 steps. We ran 150 paths in each of $g = 2, 3$ and tracked Krein signs of the unit eigenvalues.

	births (opposite-sign)	deaths (opposite-sign)	same-sign encounters < 0.02 (ghosts)
$g = 2$	276 (276; 231 at ± 1)	65 (65)	0 (0)
$g = 3$	434 (434; 331 at ± 1)	139 (139)	34 (0)

Fig. 1 (top left) shows one path. Krein+ (red) and Krein− (blue) branches meet at $\arg \lambda = 0$ and become a real pair (black). At $\arg = \pm\pi$ a red and a blue branch pass through each other: the eigenvalues reach -1 but stay on the circle.

3. OVER $\mathbb{Q}$: HEAT FLOW, DOUBLE ZEROS, LEHMER PAIRS

3.1. Verified facts.

- ([CITED] [6], [5]) H_t has only real zeros iff $t \geq \Lambda$, and $\mathrm{RH} \iff \Lambda \leq 0$.
- Rodgers–Tao prove $\Lambda \geq 0$ [6, Thm. 1]; Polymath15 prove $\Lambda \leq 0.22$ [5, Thm. 1.1].
- For $\Lambda < t \leq 0$ the zeros move by $\partial_t x_k = 2 \sum_{j \neq k}' (x_k - x_j)^{-1}$ [6, Thm. 11]. Zeros repel forward in t and attract backward. Gaps obey [6, Lemma 12(i)].
- Lower bounds on Λ came from Lehmer pairs, via the method of Csordas–Smith–Varga [6, Table 1 and p. 3].

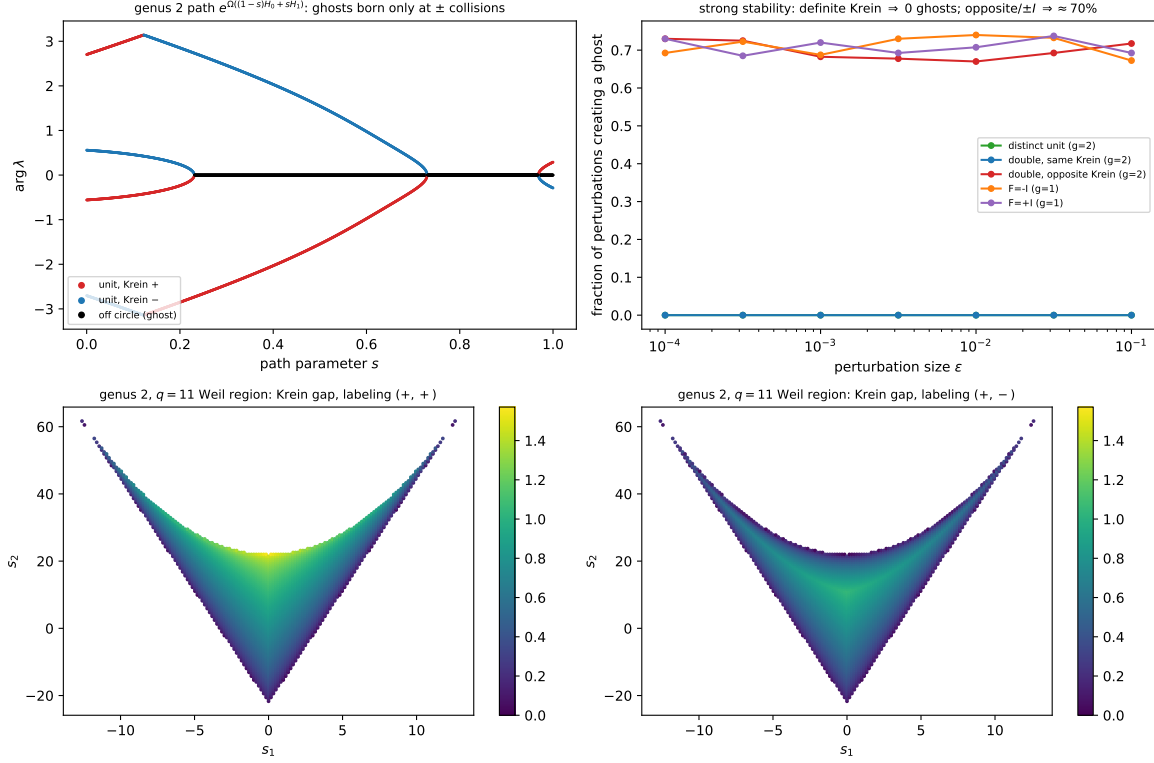


FIGURE 1. Top left: eigenvalue arguments along a genus-2 symplectic path, coloured by Krein sign; ghosts (black) appear only where opposite colours meet. Top right: fraction of ε -perturbations creating a ghost. Distinct and same-sign double eigenvalues never do (the two curves coincide at 0); opposite-sign and $\pm I$ do about 70% of the time. Bottom: genus-2 Weil region ($q = 11$), coloured by the distance to the nearest opposite-sign collision for the two Krein labelings.

3.2. Numerics. These are [NUMERICAL], from `heat.json` and `collide.json`. We used 40-digit Gauss–Legendre quadrature with 1200 nodes.

- *Check at $t = 0$.* The zeros of H_0 match $2\gamma_n$ to $7 \cdot 10^{-31}$ for $n \leq 8$.
- *Collisions.* Running backward to $t = -6$, the first collisions are the pairs (4, 5) and (7, 8):

pair	t^* (exact)	x^*	t^* (estimate, Lemma 12(i))
(4, 5)	-3.954793	66.1975	-3.948
(7, 8)	-4.033821	86.8057	-4.018

The exact values solve $H_t = \partial_x H_t = 0$. The estimate uses a frozen neighbour sum; the two-body values are -3.15 and -2.90 .

- *After collision.* The pair becomes $x \pm iy$ (a quartet with $-x \pm iy$), with $y \approx 1.45(t^* - t)^{0.509}$ and $1.46(t^* - t)^{0.511}$. This is the square-root law of a generic (non-semisimple) Krein collision (Fig. 2).
- *Lehmer pair.* For $\gamma_{6709} = 7005.06287$ and $\gamma_{6710} = 7005.10056$ the normalized gap is 0.042, and the heuristic collision time is $t^* \approx -7.1 \cdot 10^{-4}$ ([HEURISTIC]; not a bound).

Proposition 3.1 ([THEOREM]). *Simple real zeros of a real function have alternating derivative signs. So the graphical Krein signature $\text{sgn } H'_t(x_j)$ [4, (9)] alternates along the zeros: $-, +, -, \dots$ at $t = 0$, as computed. Every adjacent pair is therefore of opposite sign, and Krein theory gives no*

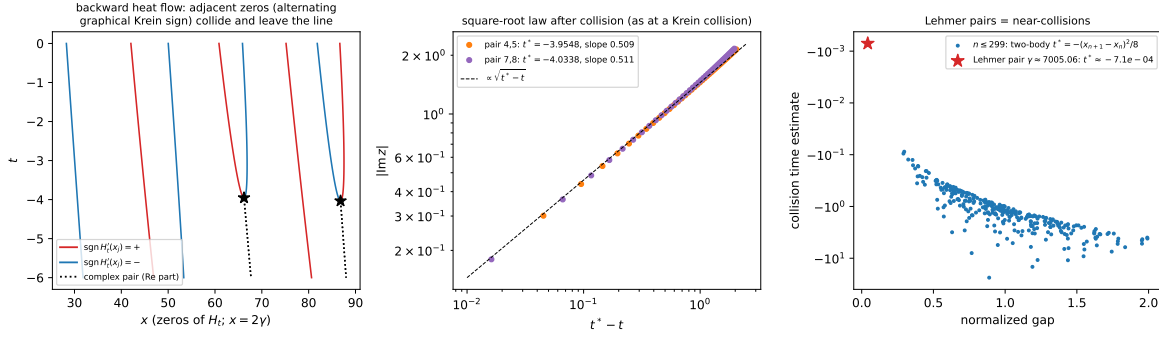


FIGURE 2. Left: zeros $x_1, \dots, x_8$ of H_t for $-6 \leq t \leq 0$, coloured by graphical Krein sign; stars mark the exact collisions, and dotted lines are the real parts of the complex pairs. Middle: $|\operatorname{Im} z|$ against $t^* - t$, with slope $\approx \frac{1}{2}$. Right: two-body collision-time estimates for $n \leq 299$ and the Lehmer pair at $\gamma \approx 7005$.

obstruction to their colliding and leaving the line. By KM’s intermediate-value argument, the only protected configuration would be two adjacent zeros of the same slope sign, which is impossible for a scalar function.

Is there a Krein sign for ζ that would forbid collisions? ([RESTATEMENT], [OPEN]) We considered three candidates:

- (i) *Graphical* (Prop. 3.1): it alternates, so nothing is protected.
- (ii) *Weil/Pontryagin*. Each on-line zero contributes $|\hat{g}(\gamma)|^2$ to Weil’s form, so all have the same sign. But this form is not conserved by the heat flow, nor by any known flow, and its positive directions have no spectral gap. Indeed the L103 onset $L(\delta) \rightarrow \infty$ as $\delta \rightarrow 0$, so a newborn quartet is invisible at any fixed L . “All +” is RH itself, which is circular.
- (iii) *Hermite–Biehler phase*. If ξ is the real part of a Hermite–Biehler function, the zeros are where a monotone phase crosses $\frac{\pi}{2} + \pi\mathbb{Z}$: a definite “Krein” structure. No source was opened, so this is left [OPEN] (direction 1).

Reading. $\Lambda \geq 0$ says that ζ ’s zeros are *not* strongly stable under backward heat flow: if RH holds, $t = 0$ is exactly a collision threshold. This is Newman’s “barely true” in Krein language, and the opposite of the finite-model picture, where true Frobenius with distinct eigenvalues is strongly stable.

4. CORRECTIONS TO THE CHAT CLAIMS

- (1) “Unit pairs carry Krein signs.” Signs belong to *eigenvalues*: in each unit pair, λ and $\bar{\lambda}$ have opposite signs. They are invariants of the Sp-realization, not of the polynomial [3, Thm. 2].
- (2) “All true pairs of the same sign $\Rightarrow$ strongly stable.” The correct condition is that no eigenvalue carries both signs and ± 1 is not an eigenvalue (Thm. 1.2). With distinct eigenvalues it holds automatically, and then it is Hasse/Weil openness (Cor. 1.3).
- (3) “Leaving the line needs a double zero with opposite signs.” Confirmed in finite models; real pairs are born at ± 1 , where opposite signs are automatic. For H_t , a double zero is confirmed (t^* exact), but the opposite sign is automatic, so it is no constraint. Departure happens backward in t .
- (4) “Polymath $\Lambda \leq 0.2$.” The verified bound is $\Lambda \leq 0.22$ [5, Thm. 1.1]. Rodgers–Tao $\Lambda \geq 0$ is confirmed.
- (5) “Lehmer pairs as near-collisions.” Confirmed heuristically ([HEURISTIC]). The rigorous use is the Csordas–Smith–Varga method cited in [6].

- (6) *Sources*. Ekeland, MacKay and Yakubovich–Starzhinskii were *not* opened; Gelfand–Lidskii and Krein are cited only through [4, 2].

5. RANKED DIRECTIONS (SCORE = 0.6 sig + 0.4 prob)

#	Direction	sig	prob	score
1	L112 (approved): Toeplitz–Carathéodory reading of the Weil form.	0.6	0.8	0.68
2	A Hermite–Biehler/de Branges phase as a definite Krein sign for ξ (open Lagarias/de Branges first). Expected: definiteness $\iff$ RH ([RESTATEMENT]).	0.6	0.4	0.52
3	CM type = Krein labeling of Frobenius (Shimura–Taniyama; source to open). Which labeling is arithmetic?	0.5	0.5	0.50
4	Krein-gap statistics in families (Katz–Sarnak) vs. Lehmer-pair statistics.	0.5	0.5	0.50
5	A flow on the Weil distribution that conserves a Krein form (would make (ii) dynamical).	0.5	0.3	0.42

6. QUESTIONS FOR J.W.

- Q1:** Accept Cor. 1.3: generic strong stability *is* Hasse/Weil openness, and Krein adds content only on the boundary?
- Q2:** Accept that ζ is *not* strongly stable under the heat flow (Rodgers–Tao), i.e. that the finite-model analogy breaks here?
- Q3:** Should the graphical sign (alternating) be recorded as the TAA “Krein sign of a zero”?
- Q4:** Pursue the Hermite–Biehler phase (direction 2)?
- Q5:** Is the CM-type question (direction 3) in scope?

REPRODUCIBILITY

- `num/lemma111_krein.py`: about 20 s.
- `num/lemma111_heat.py`: about 5 min.
- `num/lemma111_collide.py`: about 2 min.
- `num/lemma111_figs.py`.

Never touched: `lemma1.pdf`, `volume*/`, `submit` directories, `num/lemma11_ghost.py`, `briefs/prequantization_brief.md`, `num/lemma34_string.py`, and the L88–L110 PDFs and builds (including L106).

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- [5] D. H. J. Polymath, Effective approximation of heat flow evolution of the Riemann ξ function, and a new upper bound for the de Bruijn–Newman constant, arXiv:1904.12438. [opened: abstract, Thm. 1.1]
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TAA LEMMA 112: THE TOEPLITZ–CARATHÉODORY READING OF THE WEIL FORM: VERBLUNSKY COEFFICIENTS, LI’S CRITERION, AND WHAT THE MACHINERY BUYS

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Every equivalence with RH below is a *reformulation*. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma112_*.py`, data `num/l112/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma112/verified_refs.md`.

ABSTRACT. **Machinery** ([CITED]). Carathéodory–Toeplitz/Herglotz: a sequence (c_k) is positive definite iff it is the moment sequence of a positive measure on $\mathbb{T}$. Verblunsky: such measures correspond bijectively to sequences $\alpha_n \in \mathbb{D}$, and $D_n = \prod_j (1 - |\alpha_j|^2)^{n-j}$ [5]. Bochner and Krein (extension from an interval) are the continuous analogues [6, 3].

Finite models.

- Genus 1: $\alpha_0 = a/(2\sqrt{q})$, so $|\alpha_0| < 1$ is the Hasse bound ([THEOREM]).
- Genus g : the angle measure has $2g$ atoms, so its Verblunsky sequence stops with $|\alpha_{2g-1}| = 1$. Then $T_{2g} \succ 0 \iff \text{RH}$, and $n_-(T_{2g}) = k_r + 2k_c = \#\{j < 2g : E_j < 0\}$ ([THEOREM]; [NUMERICAL]: genus 2, $q = 11$, 26,711 polynomials, 0 mismatches).
- The Frobenius polynomials form a g -dimensional *equal-weight locus* inside the Verblunsky cube $(-1, 1)^{2g-1}$.

Over $\mathbb{Q}$. The L103 hat-basis Weil matrix is Toeplitz, and on the zero side its entries are the moments of $\mu_h = \sum_{\rho} h^2 \text{sinc}^4(\gamma h/2) \delta_{e^{i\gamma h}}$.

- *Discrete Weil criterion* ([THEOREM] modulo the explicit formula; [RESTATEMENT]). For every $h > 0$ outside a countable set, RH holds iff $|\alpha_n(h)| < 1$ for all n .
- Levinson–Durbin computes the α_n from primes in $O(M^2)$ ([NUMERICAL]).
 - True world: $\max |\alpha_n| = 0.384$ up to $L = 4$.
 - Each L103 ghost world: some $|\alpha_n|$ jumps above 1 (up to 14) at the L103 onsets.
 - The sign pattern reproduces the inertia exactly.

Li ([THEOREM] sketch, [NUMERICAL]). By Bombieri–Lagarias [4], Li’s Gram matrix $K_N = [\lambda_i + \lambda_j - \lambda_{|i-j|}]$ is the Toeplitz form of the Cayley-mapped zero measure on zero-sum vectors. So $\text{RH} \iff n \mapsto \lambda_n$ is *negative definite* on $\mathbb{Z}$.

- $K_N \succeq 0$ implies $\lambda_1, \dots, \lambda_N \geq 0$, so it is a strictly stronger finite test.
- In ghost models it fires at $N = 4$ –105, where the diagonal Li test needs $n = 248$ to > 6000 .

Assessment ([RESTATEMENT]). The machinery buys:

- effective step-by-step tests;
- a stronger finite form of Li;
- a dictionary (ghost $\leftrightarrow |\alpha| > 1$, Szegő $\leftrightarrow$ entropy).

It does not buy a positivity mechanism. Szegő theory is inapplicable because μ_h is pure point. Killip–Nenciu/CUE concerns random-weight spectral measures, not equal-weight or zero measures.

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GLOSSARY (TERMS RESTATED)

Toeplitz matrix: $T_N = [c_{j-k}]_{j,k=0}^{N-1}$ with $c_{-k} = \bar{c}_k$.

Positive definite (PD) sequence: Every $T_N \succeq 0$.

Carathéodory function: $F(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)$. It is analytic on $\mathbb{D}$ with $\Re F > 0$ [5, (2.2)–(2.3)].

Verblunsky (Schur, reflection) coefficients: $\alpha_n = -\overline{\Phi_{n+1}(0)}$ for the monic orthogonal polynomials, via the Szegő recursion [5, (2.17)–(2.18)]. Levinson–Durbin computes them from $c_0, \dots, c_n$ in $O(n^2)$ [7].

E_n : The prediction error $D_{n+1}/D_n = \|\Phi_n\|^2$. It satisfies $E_{n+1} = E_n(1 - |\alpha_n|^2)$ [5, (2.19)].

Ghost: An off-line zero. L103 convention: insert a quartet $\frac{1}{2} \pm \delta \pm i\gamma$ and delete one on-line zero.

k_u, k_r, k_c : The numbers of unit pairs, real pairs and quartets (L110).

1. THE CLASSICAL MACHINERY (VERIFIED)

Theorem 1.1 (Carathéodory–Toeplitz–Herglotz; [CITED]). *A sequence $(c_k)_{k \in \mathbb{Z}}$ with $c_{-k} = \bar{c}_k$ is positive definite iff $c_k = \int e^{-ik\theta} d\mu$ for a positive finite measure μ on $\mathbb{T}$. Equivalently, $F(z) = c_0 + 2 \sum_{k \geq 1} c_k z^k$ is a Carathéodory function.*

Sources: [5, (2.2)–(2.3), (2.8)] (Herglotz representation; moments c_n), and [6] (“Bochner’s theorem in the special case of the discrete group $\mathbb{Z}$ is often referred to as Herglotz’s theorem”).

Theorem 1.2 (Verblunsky; Szegő recursion; [CITED]). • *The map from nontrivial probability measures on $\mathbb{T}$ to sequences $(\alpha_n) \in \mathbb{D}^{\mathbb{N}}$ is a bijection [5, Thm. 2.3].*

- *A measure with exactly N atoms has $\alpha_0, \dots, \alpha_{N-2} \in \mathbb{D}$ and $|\alpha_{N-1}| = 1$ [5, §2].*
- *$\|\Phi_{n+1}\|^2 = (1 - |\alpha_n|^2) \|\Phi_n\|^2$ and $\det T_{n+1} = \prod_{j < n} (1 - |\alpha_j|^2)^{n-j}$ for $c_0 = 1$ [5, (2.19)–(2.20), (8.1)]. Hence $T_{n+1} \succ 0 \iff |\alpha_0|, \dots, |\alpha_{n-1}| < 1$.*

Proposition 1.3 (Inertia from the reflection pattern; [THEOREM]). *Suppose all leading minors of a real symmetric Toeplitz T_N are nonzero. Run Levinson–Durbin formally; then $\alpha_n \in \mathbb{R}$ may leave $[-1, 1]$. In that case*

$$n_-(T_N) = \#\{n < N : E_n < 0\}, \quad \text{sgn } E_n = \text{sgn } c_0 \cdot (-1)^{\#\{j < n : |\alpha_j| > 1\}}.$$

Proof. Jacobi/Sylvester: the number of sign changes in $1, D_1, \dots, D_N$ equals n_- . Then use $E_n = D_{n+1}/D_n$ and $E_{n+1} = E_n(1 - \alpha_n^2)$. $\square$

So the step test both certifies PD and *counts* negative directions: each $|\alpha_j| > 1$ flips the sign of all later E_n .

Continuous analogues ([CITED]).

- *Bochner*: a continuous PD function on $\mathbb{R}$ is the Fourier–Stieltjes transform of a positive finite measure [6].
- *Krein (1940)*: a continuous PD function on $(-2a, 2a)$ has a continuous PD extension to $\mathbb{R}$, in general not unique; Raikov gave a direct proof, and Artemenko extended it to discontinuous functions [3, §1].

The Weil distribution has atoms at $\pm \log p^k$, so the continuous Krein theorem does not apply verbatim. The distributional version is used below only as [HEURISTIC]. Krein’s papers were not opened.

2. FINITE MODELS: FROBENIUS ANGLE MEASURES

For a curve of genus g over $\mathbb{F}_q$:

- let $z_i = \alpha_i / \sqrt{q}$ for $i = 1, \dots, 2g$;
- let $c_m = \frac{1}{2g} \sum_i z_i^m = \sigma_m / 2g$ (L109, Thm. 2.1, [TAA-L]);
- the multiset $\{z_i\}$ is closed under $z \mapsto \bar{z}$ and $z \mapsto 1/z$.

Theorem 2.1 ([THEOREM]). *Assume the z_i are distinct.*

- (i) *The inertia of T_{2g} is $n_+ = 2k_u + k_r + 2k_c$ and $n_- = k_r + 2k_c$. Hence $T_{2g} \succ 0 \iff \text{all } |z_i| = 1$ (RH for the curve) $\iff |\alpha_0|, \dots, |\alpha_{2g-2}| < 1$.*
- (ii) *Under RH, the representing measure is the angle measure $\mu = \frac{1}{2g} \sum \delta_{z_i}$. It is trivial with $2g$ atoms, so $|\alpha_{2g-1}| = 1$, and T_N is singular for $N > 2g$.*
- (iii) *Genus 1: $\alpha_0 = c_1/c_0 = a/(2\sqrt{q})$, so $|\alpha_0| < 1 \iff a^2 < 4q$ (Hasse).*

Proof. (i) We have $x^* T_{2g} x = \frac{1}{2g} \sum_i \overline{P(z_i)} P(1/\bar{z}_i)$ with $P(z) = \sum_k x_k \bar{z}^k$; for a pair $\{z, 1/\bar{z}\}$ the two terms combine into $2\Re$.

- A unit atom contributes $|P(z_i)|^2$.
- An off-circle pair $\{z, 1/\bar{z}\}$ contributes $2\Re(\overline{P(z)} P(1/\bar{z}))$, a form of signature $(1, 1)$.

The $2g$ functionals $P \mapsto P(z_i)$ are independent (Vandermonde). A real pair is one off-circle pair; a quartet is two. The last equivalence is Thm. 1.2. (ii) This is Thm. 1.2. (iii) Direct. $\square$

Part (i) is L110 Thm. 2.3 read through Toeplitz ([RESTATEMENT]). What is new is the *step-by-step* form: α_0 tests the trace bound, and $\alpha_1, \dots, \alpha_{2g-2}$ test the remaining coefficients.

Numerics ([NUMERICAL], `num/lemma112_finite.py`).

- *Genus 1* ($q = 101$, all $|a| \leq 25$): $\alpha_0 = a/(2\sqrt{q})$ to machine precision, and $|\alpha_0| < 1 \iff \text{Hasse}$.
- *Genus 2* ($q = 11$; integer grid $|s_1| \leq 40$, $-80 \leq s_2 < 250$; 26,711 polynomials with distinct roots):
 - “true $\iff |\alpha_0|, |\alpha_1|, |\alpha_2| < 1$ ” holds with 0 mismatches;
 - on true points $|\alpha_3| = 1$ to 10^{-11} .
- *Ghosts*, by the first failing index:
 - α_0 (trace bound): 17,802;
 - α_1 : 8,131;
 - α_2 : 388;
 - singular leading minor ($|\alpha_1| = 1$ exactly): 35.
- The inertia rule of Prop. 1.3 equals the L106 count $n_- = k_r + 2k_c$ except at 6 points. All 6 are among the 35 singular-minor points, where Sylvester’s rule does not apply.

See Fig. 1, left.

Remark 2.2 (Equal-weight locus; [THEOREM]). Measures invariant under conjugation have real α_n .

- The symmetric measures with $2g$ atoms are parametrized by $(\alpha_0, \dots, \alpha_{2g-2}) \in (-1, 1)^{2g-1}$ and $\alpha_{2g-1} = \pm 1$, which is $2g - 1$ parameters (g angles and g weights, minus normalization).
- Frobenius measures have *equal* weights $1/2g$, so they fill only a g -dimensional locus.

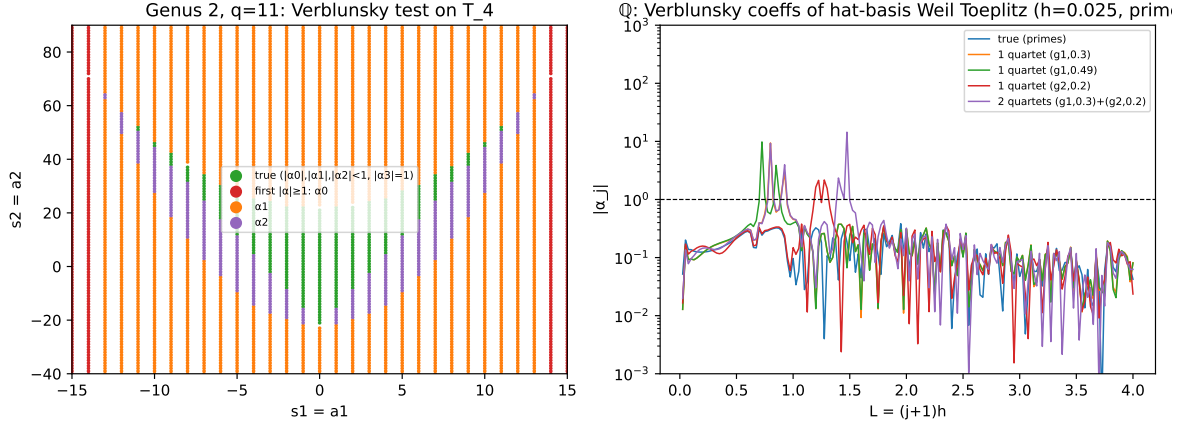


FIGURE 1. Left: genus 2, $q = 11$. Green points are true polynomials (exactly $|\alpha_0|, |\alpha_1|, |\alpha_2| < 1$). Other colours mark the first $|\alpha_j| \geq 1$ (red: $\alpha_0 = \text{trace bound}$). Right: $|\alpha_j|$ of the hat-basis Weil Toeplitz over $\mathbb{Q}$ computed from primes ($h = 0.025$, $L = (j+1)h$). The true world stays below 0.4; every L103 ghost world crosses 1. [NUMERICAL]

This matters for random-matrix links: Killip–Nenciu [5, Thm. 5.7] gives independent α_j with density $\propto (1 - |\alpha|^2)^{(n-j-2)}$ for the *spectral measure of a Haar $U(n)$ matrix and a fixed vector*. That measure has random (Dirichlet) weights, not equal weights, and $U(n)$ is not $USp(2g)$. So Katz–Sarnak equidistribution does not translate into independent Verblunsky coefficients for Frobenius measures without further work ([OPEN]).

3. OVER $\mathbb{Q}$: THE WEIL FORM AS A POSITIVE-DEFINITE SEQUENCE

3.1. The hat basis is a Toeplitz basis. Take the L103 basis of hats $g_j(x) = \Lambda((x - jh)/h)$, which has $\hat{g}_j(t) = e^{-ijht} h \text{sinc}^2(th/2)$. Then:

- $Q_{jk} = W(g_j * \tilde{g}_k)$ depends only on $j - k$, since $g_j * \tilde{g}_k$ is a translated cubic B-spline;
- $Q_M = [c_{j-k}]$ is Toeplitz;
- c_m is computed from primes $p^k \leq e^{(|m|+2)h}$, the archimedean term and the pole (L103, [TAA-L]).

The explicit formula (Weil 1952, carried; [2]) gives the zero side

$$(1) \quad c_m = \sum_{\rho} h^2 \text{sinc}^4(\gamma_{\rho} h/2) e^{i\gamma_{\rho} m h}, \quad \rho = \frac{1}{2} + i\gamma_{\rho}.$$

This series converges absolutely ($\text{sinc}^4 \sim \gamma^{-4}$). Numerically, (1) with 1517 zeros ($\gamma \leq 2000$) matches the prime side to 1.8×10^{-5} relative ([NUMERICAL]).

Theorem 3.1 (Discrete Weil criterion; [THEOREM] given (1), [RESTATEMENT]). *(i) If RH holds, then for every $h > 0$ the sequence (c_m) is PD. Its representing measure is $\mu_h = \sum_{\rho} h^2 \text{sinc}^4(\gamma h/2) \delta_{e^{i\gamma h}}$, which is pure point.*

(ii) If RH fails, then (c_m) is not PD for every h outside a countable set. In Verblunsky terms, some $|\alpha_n(h)| \geq 1$.

Proof. (i) Every γ is real, so (1) is the moment sequence of $\mu_h \geq 0$; apply Thm. 1.1.

(ii) Suppose $\gamma = \gamma_0 + i\delta$ with $\delta > 0$.

- $\sin(\gamma h/2) \neq 0$ for nonreal γ , so the ghost weight is nonzero.
- The ghost contributes terms $\propto e^{\delta|m|h} e^{\pm i\gamma_0 m h}$.

- If δ is maximal among ghosts, these terms cannot be cancelled by other zeros unless their frequencies alias, i.e. $\gamma_0 h \equiv \pm \gamma'_0 h \pmod{2\pi}$ with matching weights. Each such coincidence is a nontrivial analytic condition in h , so it holds only for countably many h .
- Hence $\limsup |c_m| = \infty$. But PD forces $|c_m| \leq c_0$.

□

Thm. 3.1 is a *reformulation* of RH. It is close to Weil’s criterion restricted to a lattice of test functions, plus Thm. 1.1. We have not seen this exact form in the sources we opened, but it is likely folklore (Q1).

Krein’s view at fixed L ([HEURISTIC])..

- Positivity of Q_M only says that $c_0, \dots, c_M$ extend to *some* positive measure on $\mathbb{T}$.
- The extensions are parametrized by the free tail $\alpha_M, \alpha_{M+1}, \dots \in \mathbb{D}$ (Thm. 1.2): primes beyond $e^{(M+2)h}$ are the “future” degrees of freedom.
- RH says that the tail *actually dictated by the primes* never leaves $\mathbb{D}$.
- This is the discrete form of Krein’s non-unique extension. Nothing at finite L forces RH.

3.2. Numerics: ghosts push $|\alpha_n|$ out of the disc. Script `num/lemma112_weil.py`: $h = 0.025$, $M \leq 160$ ($L = 4$), 3.4 s. Prime-side c_m ; ghosts inserted on the zero side.

World (L103 convention)	first $ \alpha_j \geq 1$	$L = (j+1)h$	inertia at $L = 4$
true (primes)	none ($\max \alpha = 0.384$)	—	(160, 0)
quartet ($\gamma_1, \delta = 0.3$)	$j = 30$	0.775	(158, 2)
quartet ($\gamma_1, \delta = 0.49$)	$j = 28$	0.725	(158, 2)
quartet ($\gamma_2, \delta = 0.2$)	$j = 47$	1.200	(158, 2)
two quartets	$j = 31$	0.800	(156, 4)

- At $M = 20, 40, 60, 80, 120, 160$, the inertia from the E_n signs (Prop. 1.3) equals the eigenvalue inertia in every world.
- The onsets agree with L103/L109 within one grid step h . L109 used an eigenvalue tolerance 10^{-8} ; Verblunsky detects the exact sign change.
- Each quartet produces *one* pair of $|\alpha| > 1$ events (peaks up to $|\alpha| \approx 14$), giving $n_- = 2$ per quartet, as predicted.
- In the true world $|\alpha_j| \approx 0.1$ does not decay up to $L = 4$. This is consistent with μ_h being pure point (Szegő condition $\sum |\alpha_j|^2 < \infty$ fails, §5).

See Fig. 1, right.

4. LI’S CRITERION AS TOEPLITZ POSITIVITY

Verified sources. Li’s criterion is $\text{RH} \iff \lambda_n \geq 0$ for all n (Li 1997, record and zbMATH review only, carried from L30/L98). Bombieri–Lagarias [1] give $\lambda_n = \sum_{\rho} [1 - (1 - 1/\rho)^n]$ for arbitrary multisets, an arithmetic formula, and the link to Weil (abstract). Lagarias [4, lines on BL’s three observations; Thm. 3.1] gives, for $G_n(s) = 1 - (1 - 1/s)^n$,

$$\langle G_n, G_m \rangle_W = \lambda_n + \lambda_{-m} - \lambda_{n-m}, \quad \|G_n\|_W^2 = 2\Re \lambda_n.$$

Li positivity is verified numerically to $n = 10^5$ per [8] ([CITED-sec]).

Theorem 4.1 ([THEOREM] sketch). *Let $z_{\rho} = 1 - 1/\rho$ (Cayley map; $|z_{\rho}| = 1 \iff \Re \rho = \frac{1}{2}$). For a zero multiset symmetric under $\rho \mapsto 1 - \rho$ and $\rho \mapsto \bar{\rho}$, $\lambda_{-n} = \lambda_n \in \mathbb{R}$. Set $K_N = [\lambda_i + \lambda_j - \lambda_{|i-j|}]_{i,j=1}^N$ with $\lambda_0 = 0$.*

- (i) *Under RH, $K_N = \sum_{\rho} v_{\rho} v_{\rho}^*$ with $v_{\rho} = (1 - z_{\rho}^i)_i$, so $K_N \succeq 0$ for all N . Equivalently, $n \mapsto \lambda_n = \sum_{\rho} (1 - \Re z_{\rho}^n)$ is a negative-definite function on $\mathbb{Z}$ (Schoenberg/Lévy–Khinchine) with Lévy measure $\sum_{\rho} \delta_{\arg z_{\rho}}$, which has infinite mass but $\sum \theta_{\rho}^2 < \infty$.*

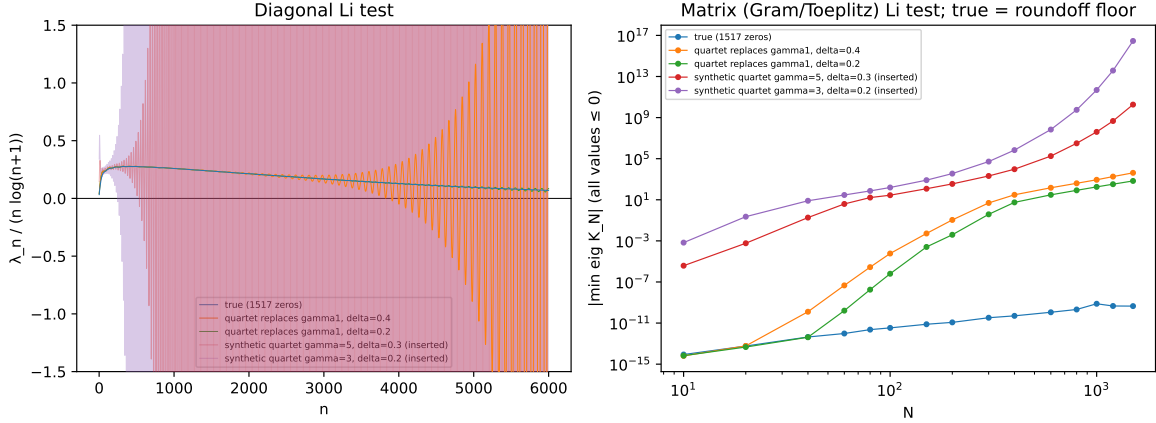


FIGURE 2. Left: diagonal Li test, $\lambda_n/(n \log(n+1))$ for the truncated true model and four ghost worlds. Right: $|\min \text{eig } K_N|$ (all minima are ≤ 0). The true model stays at the roundoff floor, while ghost worlds become visibly indefinite one to three orders of magnitude earlier in N than the diagonal test. [NUMERICAL]

- (ii) K_N is the Toeplitz form $[c_{i-j}]$ with $c_k = -\lambda_k$, restricted to vectors $(x_0, \dots, x_N)$ with $\sum x_i = 0$. It is conditional positive definiteness, because the Cayley measure has infinite mass.
- (iii) $K_N \succeq 0$ implies $\lambda_n = \frac{1}{2}(K_N)_{nn} \geq 0$ for $n \leq N$. So the matrix test is at least as strong as the diagonal Li test at every finite N .
- (iv) If RH fails, then $K_N \not\succeq 0$ for large N : ghost terms grow like $|z|^n$, which is incompatible with the $O(n^2)$ growth of negative-definite functions on $\mathbb{Z}$ (generic non-aliasing as in Thm. 3.1).

Parts (i)–(iii) follow from Lagarias Thm. 3.1. Part (iv) is a sketch ([HEURISTIC] for the non-cancellation step).

Numerics ([NUMERICAL], `num/lemma112_li.py`, 30s). The model zero multiset is the first 1517 zeros ($\gamma \leq 2000$). BL's criterion applies to any multiset, and the truncation gives $\lambda_1 = 0.02256$ (vs. 0.0230957 for ζ).

World	first n : $\lambda_n < 0$	first N : $K_N \not\succeq 0$	$\min \text{eig } K_{150}$
true (1517 zeros)	none (≤ 6000)	none (≤ 1500)	-7.7×10^{-12} (roundoff)
quartet replaces γ_1 , $\delta = 0.4$	3906	73	-5.3×10^{-3}
quartet replaces γ_1 , $\delta = 0.2$	none (≤ 6000)	105	-2.6×10^{-4}
quartet inserted at $\gamma = 5$, $\delta = 0.3$	535	6	-1.2×10^2
quartet inserted at $\gamma = 3$, $\delta = 0.2$	248	4	-8.1×10^2

- The onset threshold is $\min \text{eig} < -10^{-8}|\lambda_N|$.
- The roundoff floor of the true model stays below 10^{-9} up to $N = 1500$, while ghost minima reach -10^1 to -10^{16} (Fig. 2).

The Gram test sees a ghost at γ_1 with $\delta = 0.2$ at $N \approx 105$, where the diagonal test has not fired by $n = 6000$.

Mechanism ([HEURISTIC]). The diagonal entries carry the bulk $\sim \frac{n}{2} \log n$, while K_N can project the bulk out along vectors adapted to $\arg z_{\text{ghost}}$.

Caveat. For the actual ζ , low-lying ghosts are already excluded by verified zero computations (not re-cited here). A ghost at height γ has $|z| - 1 \approx \delta/\gamma^2$ (checked: 1.0020 for γ_1 , $\delta = 0.4$), so both tests need $N \gtrsim \gamma^2/\delta$ ([HEURISTIC]).

5. WHAT THE MACHINERY BUYS, AND WHAT A PROOF WOULD NEED

- (a) *Effective tests* ([THEOREM]/[NUMERICAL]). Levinson–Durbin gives an $O(M^2)$ step test from primes, which certifies PD and counts n_- (Prop. 1.3). It is equivalent to eigenvalues but stable and incremental: one new prime window gives one new α .
- (b) *New necessary conditions* ([THEOREM]). These are the joint conditions $K_N \succeq 0$ (Thm. 4.1(iii)), strictly stronger than $\lambda_1, \dots, \lambda_N \geq 0$ at finite N , and $|\alpha_n(h)| < 1$ for all h . They are new as *finite* tests, but equivalent to RH in the limit ([RESTATEMENT]). Bombieri’s “negative eigenvalues = half the off-line zeros” (Bom00, abstract, carried) is the Toeplitz inertia statement.
- (c) *Szegő/entropy is inapplicable*. Under RH, μ_h is pure point, so $\prod(1 - |\alpha_j|^2) = 0$ and the Szegő condition [5, (8.5), (8.9)] fails. Proving PD via a Szegő/entropy inequality would require an absolutely continuous part that the zero measure does not have. The observed non-decay of $|\alpha_j|$ agrees.
- (d) *CUE via Verblunsky: no direct link*. Killip–Nenciu describes random-weight spectral measures of Haar unitaries. Neither Frobenius (equal weights) nor μ_h (deterministic sinc⁴ weights) is of that form. A Montgomery–Odlyzko statement about γ does not imply independence of the $\alpha_n(h)$ ([OPEN]).
- (e) *What a proof needs* ([RESTATEMENT]). PD of (c_m) is equivalent (spectral theorem/GNS) to the existence of a Hilbert space H , a unitary U_h and a vector v with $c_m = \langle U_h^m v, v \rangle$. This is the discrete Hilbert–Pólya statement. In function fields, $H = H^1$ with Frobenius/ $\sqrt{q}$, and positivity comes from geometry (Castelnuovo–Hodge index). The GNS construction from c itself is circular. A proof needs a *non-tautological* H in which the primes appear as U -orbit data. This is exactly what TAA has lacked since L18.

6. CORRECTIONS TO THE CHAT CLAIMS

- (1) “ $|\alpha_n| < 1 \iff PD$.” The condition is *joint*: $T_{n+1} \succ 0 \iff |\alpha_j| < 1$ for all $j < n$. A single $|\alpha_j| > 1$ flips the sign of all later E_n , and n_- counts the odd prefixes (Prop. 1.3). It requires nonsingular leading minors (35 genus-2 exceptions).
- (2) “ $T_N PSD \iff \text{angles real} \iff RH$; the measure is the angle measure.” Confirmed, with two refinements. The test stabilizes at $N = 2g$ (Thm. 2.1). The angle measure is trivial, so the Verblunsky sequence *terminates* with $|\alpha_{2g-1}| = 1$ and T_N is singular for $N > 2g$; PD in the strict sense fails there even for true curves.
- (3) “*Szegő/entropy argument*.” Inapplicable to zero measures, which are pure point (§5(c)).
- (4) “*Link to CUE via Verblunsky*.” Killip–Nenciu concerns random-weight spectral measures of $U(n)$, not equal-weight eigenvalue measures, and not $USp(2g)$. The symplectic analogue was not opened.
- (5) “*Krein’s continuous analogue for the Weil form*.” Krein’s theorem is for continuous PD functions. The Weil distribution has atoms, and the extension is non-unique, so it yields no constraint at finite L ([HEURISTIC]).
- (6) *Sources*.
 - Li 1997: record/zbMATH only.
 - BL 1999: abstract, plus Lagarias math/0404394 opened.
 - Yoshida 1992 and Bombieri 2000: abstracts only (carried).
 - Krein 1940, Raikov, Killip–Nenciu originals: not opened; cited via [3, 5].

7. RANKED DIRECTIONS (SCORE = 0.6 sig + 0.4 prob)

#	Direction	sig	prob	score
1	Li Gram test on the actual ζ : compute λ_n from primes (BL arithmetic formula) to $n \sim 10^4$ and certify $K_N \geq 0$. This is a strictly stronger finite check than published diagonal checks.	0.5	0.8	0.62
2	Equal-weight locus: describe the Weil region in Verblunsky coordinates and push Sato–Tate measure to the law of $(\alpha_0, \dots, \alpha_{2g-2})$.	0.5	0.7	0.58
3	Non-tautological unitary dilation: build (H, U_h, v) with $c_m = \langle U^m v, v \rangle$ from adelic/prolate data (Connes–Consani).	0.9	0.1	0.58
4	Statistics of $\alpha_n(h)$ for μ_h vs. random-weight CUE (Killip–Nenciu) predictions; does pair correlation leave a trace?	0.6	0.5	0.56
5	Krein strings/de Branges for the continuous Weil form on $[0, L]$ (open Krein/de Branges sources first).	0.7	0.3	0.54

8. QUESTIONS FOR J.W.

- Q1:** Record Thm. 3.1 (discrete Weil criterion) as [THEOREM] given the explicit formula, with the countable aliasing exception? Do you know prior literature for this exact form?
- Q2:** Pursue direction 1 (Li Gram test on true λ_n)? It needs high-precision λ_n , which is days of computation.
- Q3:** Accept that Szegő/entropy and Killip–Nenciu give *no* route here (pure point; weight mismatch)?
- Q4:** Is the equal-weight locus (direction 2) worth a lemma, or is it Sato–Tate bookkeeping?
- Q5:** The L106 erratum is still pending. Is the L103 convention used here acceptable as the standing convention?

REPRODUCIBILITY

- `num/lemma112_weil.py` (3s) $\rightarrow$ `num/l112/weil_verblunsky.json`.
- `num/lemma112_finite.py` (2s) $\rightarrow$ `finite.json`, `g2alpha.npy`.
- `num/lemma112_li.py` (30s) $\rightarrow$ `li.json`, `li_lambda.npy`.
- `num/lemma112_figs.py`.

Never touched: `lemma1.pdf`, `volume*/`, `submit` directories, `num/lemma11_ghost.py`, `briefs/prequantization_brief.md`, `num/lemma34_string.py`, and the L88–L111 PDFs and builds (including L106).

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- [5] B. Simon, OPUC on one foot, *Bull. AMS* 42 (2005), *arXiv:math/0502485*. [opened: (2.2)–(2.4), (2.8), (2.17)–(2.20), Thm. 2.3, Thm. 5.7, (8.1), (8.5), (8.9)]
- [6] Wikipedia, “Bochner’s theorem” (raw text, accessed 2026-10-25). [opened; secondary]
- [7] Wikipedia, “Levinson recursion” and “Toeplitz matrix” (accessed 2026-10-25). [opened; secondary]
- [8] Wikipedia, “Li’s criterion” (accessed 2026-10-25). [opened; secondary: zero-sum form, BL generalization, numerical verification to $n = 10^5$]

TAA LEMMA 113: FROBENIUS PLACE BY PLACE: CARTAN FACTORIZATION, NEWTON OVER HODGE, AND THE ARCHIMEDEAN SHADOW

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Over $\mathbb{Q}$ every equivalence with RH below is a *reformulation*. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma113_*.py`, data `num/1113/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma113/verified_refs.md`.

ABSTRACT. The same inequality appears at every place. Write a matrix as $k_1 a k_2$ (compact $\times$ diagonal $\times$ compact). Then:

- the diagonal part a (singular values over $\mathbb{R}$; elementary divisors over $\mathbb{Q}_p$) bounds the eigenvalues;
- over $\mathbb{R}$ this is Weyl’s inequality [10];
- over $\mathbb{Q}_p$, for Frobenius, it is Mazur’s “Newton above Hodge” [7], which Kottwitz–Rapoport call *Mazur’s inequality*, also in group form [6, 9].

Weil’s proof in this language ([THEOREM], Thm. 2.1). Rosati positivity gives a metric in which $FF^\dagger = q$. So all singular values equal $\sqrt{q}$: the Hodge polygon is flat. By Weyl the Newton polygon is then flat too, and that is RH.

Ghosts ([THEOREM], Thm. 2.2; [NUMERICAL]). A ghost with eigenvalue $\sqrt{q} e^r$ forces rapidity $\geq r$ in *every* metric. The minimum over metrics is exactly r , and it is attained exactly where F is normal; there the whole Cartan vector equals the Newton vector. Numerically:

- true models reach rapidity 0 (error $< 10^{-6}$);
- ghost and mixed models reach exactly r .

Adelic frame ([THEOREM], Thm. 5.1). Ghosts are invisible at every finite place and to the product formula. They break only the individual archimedean absolute values: the archimedean shadow.

Also included.

- The three “Hodges”, kept apart.
- Dieudonné–Manin, $B(G)$ and $B(G, \mu)$, and Fargues–Fontaine (Newton $\leftrightarrow$ Harder–Narasimhan, with a sign and *not* an equivalence of categories).
- A tropical/Graeffe archimedean polygon.
- p -adic slopes from brute-force point counts.
- Corrections C113.1–8.

Over $\mathbb{Q}$, “find the metric” is a [RESTATEMENT] of RH (Deninger’s Hodge-* programme; the L106 (N4) / L112 dilation problem). A restatement can still open machinery, and §7 says which.

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GLOSSARY (FOR A PHYSICIST)

Place: A way to measure size of rational numbers: the usual $|\cdot|_\infty$, or the p -adic $|x|_p = p^{-\text{ord}_p x}$, one for each prime p .

KAK / Cartan factorization: $g = k_1 a k_2$ with k_i in a maximal compact subgroup K and a diagonal. Over $\mathbb{R}$, $K = O(n)$ and a holds the singular values (SVD), so this is the polar decomposition [11]. Over $\mathbb{Q}_p$, $K = \text{GL}_n(\mathbb{Z}_p)$ and $a = \text{diag}(p^{a_i})$ (invariant-factor/Smith normal form) [7, p. 662].

Rapidity: log of a singular value. In $\text{SL}_2(\mathbb{R})$, K consists of rotations (the TAA J -side) and A of boosts $\text{diag}(e^r, e^{-r})$ (the S -side).

Newton polygon: The convex polygon of the sorted eigenvalue sizes ($\log |\lambda|$ or $\text{ord}_p \lambda$).

Hodge polygon: The same polygon built from the a -part.

Dominance: “ $\mu \geq \nu$ ” means: equal total, and every partial sum of the sorted entries of μ is at least that of ν .

F-crystal: A $W(k)$ -lattice M ($W =$ Witt vectors, the p -adic integers for $k = \mathbb{F}_p$) with an injective σ -semilinear F ($\sigma =$ Frobenius on scalars). An *isocrystal* is the same over $W[1/p]$ [7, §4].

Ghost: A Frobenius eigenvalue off the circle $|\alpha| = \sqrt{q}$. L103 convention: insert a quartet, delete one on-line zero.

1. CARTAN FACTORIZATION AT EVERY PLACE

Theorem 1.1 (Weyl's inequality; [CITED]). *Let $A \in \text{GL}_n(\mathbb{C})$ have eigenvalues $|\lambda_1| \geq \dots \geq |\lambda_n|$ and singular values $\sigma_1 \geq \dots \geq \sigma_n$ (the a -part of $A = k_1 a k_2$). Then*

$$|\lambda_1 \cdots \lambda_k| \leq \sigma_1 \cdots \sigma_k \quad (k < n), \quad \text{with equality for } k = n.$$

So $\text{Newt}_\infty(A) \leq \text{Hdg}_\infty(A)$ in dominance order.

Source: [10], inequality (3) $\lambda_1 \leq \kappa_1$ applied to the exterior powers $A^{[k]}$, giving (4) “with the equality sign prevailing in the last”. Also [12].

- In $\text{SL}_2(\mathbb{R})$: $\sigma = e^{\pm r}$, and $|\lambda| \leq e^r$.
- Equality throughout holds iff A is normal.

Theorem 1.2 (p -adic Cartan and Mazur's inequality; [CITED]). *(i) (Invariant factors; [7, p. 662], [9, Ex. 4.3]) Every injective W -linear $F : H \rightarrow M$ between free W -modules of rank n has the form $\text{GL}_n(W) \text{diag}(p^{a_1}, \dots, p^{a_n}) \text{GL}_n(W)$. The multiplicities $h^j = \#\{i : a_i = j\}$ are the Hodge numbers of the span.*

- (ii) (Mazur’s Lemma; [7, p. 662]) The Newton polygon of an F -crystal lies on or above its Hodge polygon, with the same endpoints. The proof uses exterior powers $\Lambda^t M$.
- (iii) (Mazur’s Theorem 2; [7, p. 662]) Suppose X/W is projective and smooth and $H^q(X, \Omega^p)$ are torsion-free. Then the Hodge numbers of the span of F on $H_{DR}^m(X/W)$ equal the geometric $h^{j,m-j}$. Together with (ii) this gives Katz’s conjecture (Newton above geometric Hodge), proved in general by Ogus [15, §1.1].
- (iv) (Kottwitz–Rapoport) For an isocrystal (N, F) with Newton vector ν and a lattice M with Hodge vector $\mu(M) = \text{inv}(M, FM)$, “Mazur’s inequality states that $\mu(M) \geq \nu(N, F)$ ”. Conversely, every $\mu \geq \nu$ is realized (Thm. A, Thm. 4.11) [6, Intro, §4]. The group version (“the group theoretic version of Mazur’s theorem”) defines $B(G, \mu)$ [6, §3] and is proved in [9, Thm. 4.2].

Remark 1.3 (One inequality, two places; [RESTATEMENT]). Weyl’s proof and Mazur’s are the same proof: the a -part bounds the largest eigenvalue (Weyl (3); Mazur (*)), applied to every exterior power. Over $\mathbb{R}$ the size is σ , over $\mathbb{Q}_p$ it is p^{-a} . Mazur’s version allows σ -semilinear F ; over $\mathbb{F}_q$, F^a is linear [7, p. 655]. So “Newton above Hodge is the p -adic Weyl inequality” is correct as a statement about KAK ([RESTATEMENT]). The difference is what the compact group knows:

- over $\mathbb{R}$ the compact part can be changed by changing the metric (§2);
- over $\mathbb{Q}_p$ the lattice is fixed by geometry (crystalline cohomology).

2. WEIL’S PROOF IN POLYGON LANGUAGE

Theorem 2.1 ([THEOREM]; ingredients [CITED] from [8]). Let $A/\mathbb{F}_q$ be an abelian variety with Frobenius π and Rosati involution $\dagger$, and let $\mathcal{A} = \mathbb{Q}[\pi] \otimes \mathbb{R}$.

- (i) $\langle x, y \rangle = \text{Tr}(xy^\dagger)$ is an inner product on $\mathcal{A}$, by Rosati positivity [8, Thm. I.14.3, Lemma II.1.3].
- (ii) $\pi^\dagger \pi = q$ [8, Lemma II.1.2]. So left multiplication L_π satisfies $L_\pi^* = L_{\pi^\dagger}$ and $L_\pi L_\pi^* = q$: every singular value of L_π is $\sqrt{q}$, and Hdg_∞ is flat of slope $\frac{1}{2} \log q$.
- (iii) By Thm. 1.1, $\text{Newt}_\infty \leq \text{Hdg}_\infty$ with equal endpoints, and a flat polygon dominates only itself. So every eigenvalue of L_π (every conjugate of π) has $|\alpha| = \sqrt{q}$. This is RH for A .

Theorem 2.2 (Minimal boost; [THEOREM]). Let $F \in \text{Sp}(2g, \mathbb{R})$ be semisimple with normalized eigenvalues $e^{\pm r_i \pm i\theta_i}$, and let $r_{\max} = \max_i |r_i|$. Over all metrics (conjugations $Q \in \text{Sp}(2g, \mathbb{R})$):

- (i) $\min_Q \log \sigma_1(QFQ^{-1}) = r_{\max}$;
- (ii) the minimum is attained iff QFQ^{-1} is normal, and then the whole Cartan vector equals the Newton vector;
- (iii) in particular a metric with $FF^* = 1$ exists iff $r_{\max} = 0$ (RH for the model). A ghost forces rapidity $\geq r$ in every metric.

Proof. The bound $\geq$ is Thm. 1.1 with $k = 1$, i.e. spectral radius $\leq$ norm. For attainment, F is Sp -conjugate to its real normal form, which is normal: rotations, $\text{diag}(e^r, e^{-r})$, and $e^r R_\theta \oplus e^{-r} R_\theta$. Equality in Weyl forces normality. $\square$

Numerics ([NUMERICAL], `num/lemma113_cartan.py`). Each model is conjugated by a random symplectic matrix (“random basis”); then we minimize $\|QFQ^{-1}\|_F$ over $Q = \exp(\Omega S)$.

Model	max rap., random basis	min over metrics	Rosati average G_N
true $g = 2$ (angles 0.7, 2.1)	2.635	0.000000	converges; rapidity 2×10^{-4}
true $g = 1$	1.410	0.000000	converges; 1.6×10^{-4}
ghost $g = 1$, $r = 0.3$	0.348	0.300000	overflow at $N = 768$
mixed $g = 2$ (unit + real pair 0.3)	1.803	0.300000	overflow at $N = 765$
quartet $g = 2$, $r = 0.2$	1.262	0.200000	overflow at $N = 1147$

- The “Rosati metric” is $G_N = \frac{1}{N} \sum_{k < N} (F^k)^T F^k$. It exists, i.e. $F^T G F = G$, iff F generates a bounded group. For ghosts $\log_{10} \|G_N\|$ grows at $2r/\ln 10$ per step.

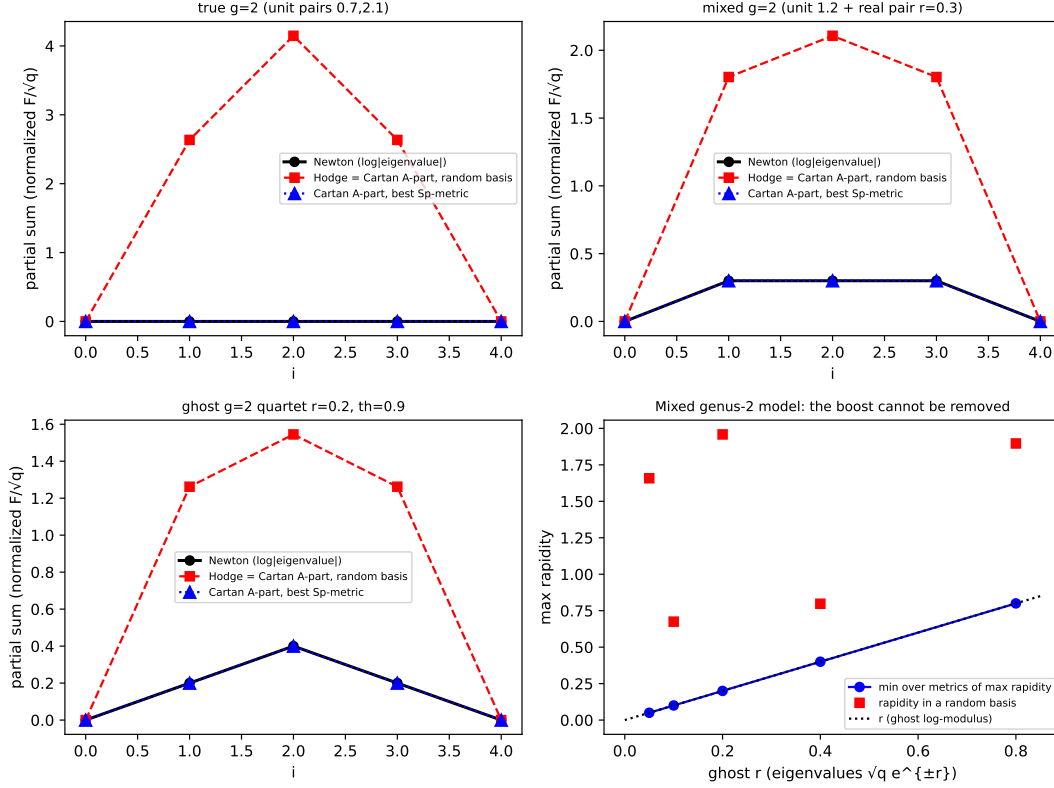


FIGURE 1. Newton (black) vs. Hodge = Cartan A -part (red: random basis; blue: best metric), as partial sums of sorted log values of $F/\sqrt{q}$. Weyl: red $\geq$ black always. True model: the best metric flattens both. Mixed/quartet: the best metric only reaches the kinked Newton polygon. Lower right: the minimal boost equals the ghost’s r . [NUMERICAL]

- Scan, mixed model with $r \in \{0.05, 0.1, 0.2, 0.4, 0.8\}$: the minimum equals r to 10^{-14} .
- At the optimum, the Cartan vector equals the Newton vector entrywise: $(0.3, 0, 0, -0.3)$ and $(0.2, 0.2, -0.2, -0.2)$.

See Fig. 1.

Over $\mathbb{Q}$ ([RESTATEMENT]).

- “There is a metric in which (Frobenius)/ $\sqrt{q}$ is unitary” is equivalent to RH for the model. For ζ the analogous statement (a Hilbert space on which the generator is $\frac{1}{2} + \text{skew}$) is Deninger’s “Hodge $*$ -operator” hypothesis [3, §3]. It is also the L112 unitary dilation.
- Producing that metric from structure is the L106 (N4) Lefschetz–degree problem.
- A restatement is still useful when it imports machinery. Here the machinery is KAK at every place: the archimedean “Hodge polygon” becomes a definite target (flatness), not a slogan.

3. THE THREE HODGES, KEPT DISTINCT

(H1) *Hodge index theorem* ([CITED], [13]; Hartshorne V.1.9, carried). On a surface the intersection form on $NS \otimes \mathbb{R}$ has signature $(1, \rho - 1)$ (Sylvester): Lorentzian, with an ample class timelike. This is the light-cone/ S -side. Weil’s inequality follows from it on $C \times C$ (Bombieri’s Clay

description, “an easy consequence of the algebraic index theorem for surfaces” [1]; [8, Ch. III §11]).

- (H2) *Hodge–Riemann / Rosati* ([CITED], [8]). The form is *definite* on the primitive complement, and $\text{Tr}(\alpha\alpha^\dagger) > 0$. This is the *J*-side.
- (H3) *Mazur’s Hodge polygon* ([CITED], [7]). Filtration dimensions $h^{j,m-j}$, equal to the elementary divisors of the span (Thm. 1.2(iii)). There is *no positivity* at all: it is a $\mathbb{Z}_p$ -lattice invariant.

Proposition 3.1 (J/S duality; [RESTATEMENT]). *(H1) and (H2) are one statement: Lorentzian overall plus definite on the complement of the timelike (ample, fibre) classes.*

- *The S-side (boost/light cone) is the one positive direction.*
- *The J-side (rotation/compact) is the definiteness of the rest.*
- *Thm. 2.1 uses (H2).*
- *The p-adic side (H3) never sees a sign, which is why it cannot detect ghosts (Thm. 5.1).*

4. THE p -ADIC FROBENIUS

- *Dieudonné–Manin* ([CITED], [7, p. 657]; [14]; [4, §2.5.3]). Over $\bar{\mathbb{F}}_p$, an isocrystal decomposes uniquely as $\bigoplus A/A(T^s - p^r)$. The slopes r/s and their multiplicities determine it. Over $\mathbb{F}_q$ the slopes are the ord_q of the eigenvalues of F^a .
- *Kottwitz’s $B(G)$* ([CITED], [5, §1.7]). $B(G)$ is the set of σ -conjugacy classes $b \sim gb\sigma(g)^{-1}$ in $G(L)$. It is classified by the Newton point (slope decomposition, [5, §4]) together with a $\pi_1(G)$ -invariant. $B(G, \mu) \subset B(G)$ is the finite subset cut out by the group-theoretic Mazur inequality ([6, §3] citing Kottwitz II §6, not opened; [9, Thm. 4.2]).
- *Fargues–Fontaine* ([CITED], [4, Prop. 8.2.6, Thm. 8.2.10]).
 - The functor $\mathcal{E}(D, \varphi) = \bigoplus_i (\widehat{D \otimes B})^{\varphi=\pi^i}$ sends an isocrystal to $\bigoplus_\lambda \mathcal{O}_X(-\lambda)^{m_\lambda}$, where m_λ is the multiplicity of slope λ .
 - It is essentially surjective, and gives an equivalence between isoclinic isocrystals of slope $-\lambda$ and semistable bundles of slope λ .
 - So Newton slopes $\leftrightarrow$ Harder–Narasimhan slopes *up to sign*, and on isomorphism classes.
 - It is not an equivalence of all categories: $\text{Hom}(\mathcal{O}(\lambda), \mathcal{O}(\mu)) \neq 0$ for $\lambda < \mu$, while isocrystals of different slopes have no maps.
- *An archimedean analogue for $\mathbb{Q}$* is [OPEN]. Deninger’s foliated-space programme [3] and Connes–Consani’s (arXiv:2106.01715, carried) are *programmes*, not results used here.

Numerics ([NUMERICAL], `num/lemma113_places.py`). Point counts are by brute-force enumeration of $y^2 = f(x)$ over $\mathbb{F}_p$, plus the point at infinity, for $p \leq 43$:

- $y^2 = x^3 + x$ is supersingular ($a_p = 0$, slopes $\frac{1}{2}, \frac{1}{2}$) exactly for $p \equiv 3 \pmod{4}$ in our range ($p = 7, 11, 19, 23, 31, 43$).
- $y^2 = x^3 + 1$ is supersingular exactly for $p \equiv 2 \pmod{3}$ ($p = 5, 11, 17, 23, 29, 41$).
- $y^2 = x^3 - x + 1$ is ordinary for all tested p .
- For example $\#E(\mathbb{F}_{13}) = 20$, $a_{13} = -6$ for $y^2 = x^3 + x$ (slopes 0, 1).
- For every curve, the elementary divisors of F on any F -stable lattice are $(1, p)$, i.e. Hodge $(0, 1)$, because $\det = p$ and the entries are integral. Newton lies on or above it, as Mazur predicts. As Mazur notes [7, p. 657], for curves this follows from the functional equation alone.

5. THE ADELIC FRAME: WHAT A GHOST DOES AT EACH PLACE

Theorem 5.1 ([THEOREM]; cf. [7, p. 654]). *Let $P(T) = \prod_{i=1}^{2g} (1 - \alpha_i T) \in \mathbb{Z}[T]$ satisfy the functional equation $\{\alpha_i\} = \{q/\alpha_i\}$, so that $\prod \alpha_i = q^g$. Let α be any root.*

- (i) $\ell \neq p$: α and q/α are algebraic integers, so $|\alpha|_v = 1$ for all $v \mid \ell$. This holds for ghosts too.

- (ii) p : the slopes $\text{ord}_q(\alpha) \in [0, 1]$ are read from the p -adic Newton polygon of P , symmetric under $s \mapsto 1 - s$. Ghost polynomials have the same constraints: e.g. $T^2 - 33T + 11$ (real ghost, $|\alpha| = 11^{-0.454}, 11^{1.454}$) has slopes $\frac{1}{2}, \frac{1}{2}$ like a supersingular curve.
- (iii) Product formula: $\prod_v |\alpha|_v = 1$ holds for every nonzero algebraic number, so it cannot characterize Weil numbers. Moreover $\sum_i \log_q |\alpha_i|_\infty = g = \sum_i \text{ord}_q \alpha_i$ for every such P , ghost or not.
- (iv) ∞ : RH is the statement that each embedding has $|\iota\alpha| = \sqrt{q}$. A ghost keeps the sum of archimedean weights and spreads them, $\frac{1}{2} \pm r/\log q$.

Proof. (i) is Mazur’s argument [7, p. 654]. (ii) α and q/α are integral. (iii) Product formula; $\prod \alpha_i = q^g$. (iv) Definition. $\square$

Numerics at $q = 11$ ([NUMERICAL]).. Archimedean weights $\log_{11} |\alpha|$, and p -adic slopes:

model	factors over $\mathbb{Q}$	arch. weights	slopes
$g=2$ true (2, 10)	irreducible	.5, .5, .5, .5	0, 0, 1, 1
mixed (8, 30)	irreducible	.399, .5, .5, .601	0, 0, 1, 1
mixed (14, 70)	$(x^2 - 8x + 11)(x^2 - 6x + 11)$	.237, .763 .5, .5	0, 1 0, 1
quartet (0, 23)	irreducible	.437, .437, .563, .563	0, 0, 1, 1
two real pairs (0, -30)	irreducible	.327, .327, .673, .673	0, 0, 1, 1
$g=1, a = 12$	$(x - 1)(x - 11)$	0 1	0 1

- The p -adic and ℓ -adic data are identical for true and ghost models.
- A single Galois orbit can mix unit and ghost roots, as in (8, 30). So “ghost” is a property of embeddings, not of the orbit.

What is missing for $\mathbb{Q}$ ([OPEN]/[RESTATEMENT])..

- There is no finite-dimensional H^1 and no single prime playing the role of p .
- The analogue of the product formula is the explicit formula, which is ghost-blind in the same sense (L104–L106).
- The only “place” where RH lives is ∞ , so the finite places offer no lever.

6. THE ARCHIMEDEAN NEWTON POLYGON: TROPICAL, GRAEFFE, FIEDLER

Proposition 6.1 ([THEOREM]). *Let $\text{Newt}_\infty(P)$ be the polygon of the sorted $\log |\alpha_i|$.*

- (i) RH for the model holds iff Newt_∞ is a single straight segment of slope $\frac{1}{2} \log q$.
- (ii) A ghost creates kinks at $\frac{1}{2} \log q \pm r$.
- (iii) The tropical polygon (upper hull of $(i, \log |e_i|)$, e_i the coefficients) of the k -th Graeffe iterate (roots α^{2^k}), divided by 2^k , converges to Newt_∞ with error $\leq \max_i \log \binom{n}{i} / 2^k$.

Part (iii) is Ostrowski-type and elementary: $|e_i|$ lies within a factor $\binom{n}{i}$ of the product of the top i moduli. Plain tropicalization ($k = 0$) does not force $|\alpha| = \sqrt{q}$ (L23, L42 Thm. 8.5, [TAA-L]).

Numerics ([NUMERICAL], Fig. 2): for $g = 2$ models at $q = 11$, the error for ghost and mixed models is 0.17–1.22 at $k = 0$ (true model: 0) and ≤ 0.0035 at $k = 8$, always within the bound $\log 6/2^k$. True models give the straight line; ghosts give visible kinks.

Fiedler exponents ([TAA-L] L43; [ANALOGY]). L43 used Fiedler’s linearization $f'(e)$ of $\dot{w} = f(w)$ at a zero e : sink, center or source as $\Re f'(e) < 0, = 0, > 0$ (Fiedler (1.24)). For $f = i\Xi$, a simple on-line zero is a center, and an off-line pair has $(\lambda, -\bar{\lambda})$. This is a *local* index, not a polygon, and Broughan’s criterion adds a separatrix condition. Here it is context only.

Context. Deligne purity [2, Thm. 1.6]: Frobenius eigenvalues on H^i have $|\alpha| = q^{i/2}$, i.e. Newt_∞ of each H^i is straight ([CITED]). Griffiths transversality and Simpson’s correspondence are [ANALOGY] only and are not cited. The Ramis–Malgrange irregular-ODE Newton polygon was *not verified* and is omitted.

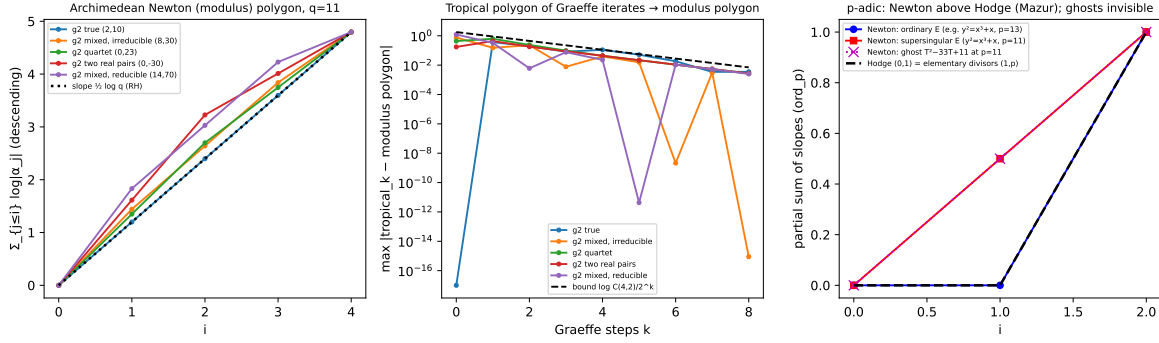


FIGURE 2. Left: archimedean Newton polygons ($q = 11$, genus 2). True = the dotted line of slope $\frac{1}{2} \log q$; ghost and mixed models kink. Centre: Graeffe/tropical convergence with the bound $\log 6/2^k$. Right: p -adic Newton vs. Hodge $(0, 1)$. Ordinary coincides with Hodge; supersingular and the ghost $T^2 - 33T + 11$ give the *same* polygon. [NUMERICAL]

7. WHAT THE PLACE-BY-PLACE VIEW BUYS

- One theorem at all places: KAK plus exterior powers gives $\text{Newton} \leq \text{Hodge}$ ([RESTATEMENT], but it unifies Weyl and Mazur).
- A sharp archimedean target: $\text{RH} \iff \text{minimal boost } 0$, and ghosts cost exactly r (Thm. 2.2).
- A negative result: the finite places and the product formula are ghost-blind (Thm. 5.1). Any proof must use ∞ -positivity.
- Machinery imported:
 - $B(G, \mu)$ and its converse [6]: for each Newton vector, every Hodge vector above it is realized by a lattice;
 - Fargues–Fontaine: Newton as Harder–Narasimhan, i.e. slopes as a stability problem.

An archimedean version would ask: which “Hodge (metric) data” are realizable over a given Newton point? Thm. 2.2 answers this for $\text{Sp}(2g, \mathbb{R})$: exactly those dominating ν , with ν itself attained ([THEOREM], the archimedean analogue of KR Thm. A).

8. CORRECTIONS TO THE CHAT CLAIMS (T464/T465)

- C113.1 “*Mazur’s Hodge polygon = elementary-divisor polygon of Frobenius.*” Correct with hypotheses. Mazur *defines* the abstract Hodge polygon of an F-crystal by the invariant factors of its span (which forgets σ). His Thm. 2 identifies it with the geometric Hodge numbers for X/W projective smooth with torsion-free $H^q(\Omega^p)$ [7, p. 662]. The inequality for F-crystals is his “very easy Lemma”. The geometric version in general is due to Ogus [15].
- C113.2 “*Kottwitz–Rapoport call it Mazur’s inequality.*” Verified: “By Mazur’s inequality we have $\mu(M) \geq \nu$ ”, §4 is titled “A converse to Mazur’s inequality”, and the term is used for non-quasisplit G too. The group form is called “the group theoretic version of Mazur’s theorem” and attributed to Kottwitz II and Rapoport–Richartz [6].
- C113.3 “*Product formula characterizes Weil numbers.*” False. The product formula holds for every nonzero algebraic number, and ghosts satisfy it (Thm. 5.1(iii)). The correct adelic description is Mazur’s: an ℓ -adic unit, plus question (I) (complex sizes, i.e. RH) and question (II) (p -adic sizes).

- C113.4 “*Fargues–Fontaine: Newton $\leftrightarrow$ HN polygon.*” Correct up to the sign $\lambda \mapsto -\lambda$ and on isomorphism classes. The functor is essentially surjective but not an equivalence of categories [4, 8.2.6, 8.2.10].
- C113.5 “*Newton above Hodge is the p -adic Weyl inequality.*” Correct as [RESTATEMENT]: same proof by exterior powers.
- C113.6 “*The p -adic side gives positivity.*” No: (H3) has no sign. The positivity is (H1)/(H2), and ghosts are invisible at p .
- C113.7 “*Tropical/Graeffe polygon from L109/L112.*” The tropical polygon work is in L23/L42 (Thm. 8.5), not L109/L112. Here it is recomputed with Graeffe.
- C113.8 *Sources.* Kottwitz II (B(G, μ) §6), Katz 1979, Manin 1963 and Mazur 1973 were not opened; they are cited via [6, 9, 7, 15]. The Ramis–Malgrange polygon was dropped.

9. RANKED DIRECTIONS (SCORE = 0.6 sig + 0.4 prob)

#	Direction	sig	prob	score
1	Archimedean $B(G, \mu)$: for the Weil-form Toeplitz (L112) define a “Hodge vector” (Cartan part in the prime-side metric) and test Weyl-type dominance numerically against ghost worlds.	0.6	0.7	0.64
2	Harder–Narasimhan reading of Thm. 2.2: Kempf–Ness/moment-map (minimal-norm metric = stable point) for Frobenius in $\mathrm{Sp}(2g, \mathbb{R})$; ghosts as unstable directions.	0.6	0.6	0.60
3	Deninger’s Hodge-* as a minimal-boost problem on finite truncations of the explicit formula.	0.8	0.2	0.56
4	Supersingular-type ghosts: classify Weil-type polynomials by $(\mathrm{Newton}_p, \mathrm{Newton}_\infty)$ pairs and count them (Honda–Tate side; sources to open).	0.4	0.8	0.56
5	Fargues–Fontaine curve for $\mathbb{R}$ (twistor $\mathbb{P}^1$; Simpson) as an archimedean analogue ([ANALOGY]; sources to open).	0.7	0.2	0.50

10. QUESTIONS FOR J.W.

- Q1:** Record Thm. 2.2 as the archimedean analogue of KR’s converse (Hodge vectors realizable over a Newton point)?
- Q2:** Accept Thm. 5.1: ghosts are invisible at all finite places and to the product formula?
- Q3:** Should the three Hodges get fixed TAA names (Index, Riemann, Mazur), to stop the conflation?
- Q4:** Pursue direction 2 (Kempf–Ness), which would make “find the metric” a GIT problem?
- Q5:** Is the Fargues–Fontaine twistor analogy (direction 5) worth sourcing, or out of scope?

REPRODUCIBILITY

- `num/lemma113_cartan.py` (1 s) $\rightarrow$ `num/l113/cartan.json`.
- `num/lemma113_places.py` (1 s) $\rightarrow$ `places.json`.
- `num/lemma113_figs.py`.

Never touched: `lemma1.pdf`, `volume*/`, `submit` directories, `num/lemma11_ghost.py`, `briefs/prequantization_brief.md`, `num/lemma34_string.py`, and all earlier builds and PDFs (including L106, L88/89).

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TAA LEMMA 114: PYTHAGORAS, THE PLATONIC SOLIDS, AND THE CUSP: LORENTZIAN POLYNOMIALS, THE MODULAR TOWER, AND Δ

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Over $\mathbb{Q}$ every equivalence with RH below is a *reformulation*. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma114_*.py`, data `num/1114/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma114/verified_refs.md`.

ABSTRACT. J.W. asked (pt472u) whether the weight-12 discriminant Δ is a “diagonalization” of the Pythagorean/Platonic side. The verified answer (§8) has three parts.

- **Platonic** ([THEOREM]). Klein’s icosahedral forms satisfy $T^2 = H^3 - 1728F^5$ (checked symbolically). The modular forms satisfy $E_6^2 = E_4^3 - 1728\Delta$ (checked to q^{1000}). The dictionary is $H \leftrightarrow E_4$ (faces, $j = 0$), $T \leftrightarrow E_6$ (edges, $j = 1728$), $F^5 \leftrightarrow \Delta$ (vertices = cusps). So Δ is the *vertex form* of the tower $(2, 3, N)$ in the limit $N \rightarrow \infty$. It is not a diagonalization.
- **Pythagorean** ([THEOREM]). At the cusp an elliptic curve degenerates to a nodal cubic. Its smooth part is the hyperbola $\mathbb{G}_m$ (split node) or the circle, the norm-one torus (non-split node). Over $\mathbb{R}$ this is crunode versus acnode; over $\mathbb{F}_p$ the counts are $p - 1$ versus $p + 1$.
- **“Diagonal”** ([RESTATEMENT]). $\Delta = g_2^3 - 27g_3^2$ is a discriminant, i.e. it vanishes where roots collide. Weil’s diagonal $\Delta_C \subset C \times C$ is a different object; only an analogy links them.

New sections (t473–t479).

- **Anomaly** ([THEOREM]). E_2^* is $6i/\pi^2$ times the z -derivative of Kronecker’s constant term. Its $3/(\pi y)$ comes from the residue $3/\pi = 1/\text{vol}$ of the scattering term $\varphi(s)y^{1-s}$ at $s = 1$. Verdict: a genuine $\Delta \rightarrow \text{Eisenstein bridge}$, but at the edge $s = 1$. Extending it to all s just gives back $\varphi(s)$ ([RESTATEMENT]).
- **Velocity** ([THEOREM]). $E(z, \frac{1}{2}) \equiv 0$ and $\partial_s E(z, s)|_{1/2} = 2E^*(z, \frac{1}{2})$. Its constant term $2\sqrt{y}(\log y + \gamma - \log 4\pi)$ changes sign at $y^* = 4\pi e^{-\gamma}$. This is exactly the Lagarias–Suzuki threshold where the constant term acquires off-line real zeros. The velocity is sign-indefinite.
- **Null rays** ([THEOREM] over $\mathbb{F}_q$). Fibre classes are null. Γ_F has light-cone coordinates $(q, 1)$ and Δ has $(1, 1)$. Weil’s bound is Euclidean Cauchy–Schwarz in the transverse space. A ghost is a second time direction. Numerically, all 401 true genus-2 models stay in the cone and all 2935 count-positive ghosts exit it.

Section A ([THEOREM]/[NUMERICAL]). For the homogenized Jensen polynomials of ξ , being Lorentzian is equivalent to the Turán inequalities. These hold unconditionally, so the Lorentzian family is strictly weaker than RH. Hyperbolicity tests up to $d = 30$ are blind to the L103 ghost.

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GLOSSARY (FOR A PHYSICIST)

The three Δ ’s: (kept apart throughout). (1) $\Delta_C \subset C \times C$, the *diagonal curve* (Weil’s proof). (2) The *diagonal functor / discriminant* Disc, the locus where roots of a polynomial collide; for a cubic, $\text{Disc} = g_2^3 - 27g_3^2$. (3) The *modular discriminant* $\Delta(z) = \eta(z)^{24} = q \prod (1 - q^n)^{24}$, a weight-12 cusp form. Item (3) is the discriminant (2) of the lattice cubic. Item (1) shares only the letter.

Lorentzian polynomial: A homogeneous polynomial with nonnegative coefficients whose quadratic derivatives have Lorentzian signature (at most one positive eigenvalue), with M-convex support [5].

Hyperbolic polynomial: Here: a real polynomial with only real roots.

Jensen polynomial: $J_\gamma^{d,n}(X) = \sum_j \binom{d}{j} \gamma(n+j) X^j$ [11].

Cusp: The point $i\infty$ of $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$, where the lattice degenerates.

Null / timelike: $v^2 = 0$ / $v^2 > 0$ for a form of signature $(1, \rho - 1)$.

1. A. COMBINATORIAL HODGE THEORY AND LORENTZIAN POLYNOMIALS

1.1. What is proved.

- **McMullen 1993** [CITED] [16]. For a *simple* polytope P , the polytope algebra has weight spaces of dimension $h_r(P)$. It satisfies hard Lefschetz and Hodge–Riemann–Minkowski inequalities. This proves the necessity of the g -theorem conditions for all simple polytopes, rational or not. (Sufficiency is Billera–Lee [BIB]; the rational case is Stanley 1980 [BIB].)
- **Icosahedron/dodecahedron** [THEOREM]. The icosahedron is simplicial with $f = (12, 30, 20)$. So $h = (1, 12 - 3, 30 - 24 + 3, 20 - 30 + 12 - 1) = (1, 9, 9, 1)$ and $g = (1, 8)$. The dodecahedron is *simple*; its h -vector (defined through the dual) is the same $(1, 9, 9, 1)$. Unimodality and $h_i = h_{3-i}$ are the hard-Lefschetz/Poincaré shadows [29].

- **AHK 2018** [CITED] [1], Thm. 1.4. The Chow ring of a matroid satisfies hard Lefschetz and Hodge–Riemann. This gives log-concavity of the characteristic-polynomial coefficients (Heron–Rota–Welsh).
- **Brändén–Huh** [CITED] [5], abstract and Prop. 2.2. The Hessian of a nonzero Lorentzian polynomial has *exactly one positive eigenvalue at every point of the positive orthant*. Stable implies Lorentzian. By Ex. 2.26/Thm. 2.25, a bivariate $\sum a_k w_1^k w_2^{d-k}$ is Lorentzian iff (a_k) is nonnegative, ultra log-concave ($a_k^2 / \binom{d}{k}^2 \geq a_{k-1} a_{k+1} / \binom{d}{k-1} \binom{d}{k+1}$) and has no internal zeros. Example: the cubic family with parameter θ is Lorentzian iff $0 \leq \theta \leq 9$ and stable iff $0 \leq \theta \leq 8$ (Ex. 2.3).
- **Pólya / GORZ** [CITED] [11]. GORZ write $(-1 + 4z^2)\Lambda(\frac{1}{2} + z) = \sum_n \gamma(n) z^{2n} / n!$; this equals $8\xi(\frac{1}{2} + z)$ ([NUMERICAL]: $\gamma(0) = 3.97696622\dots = 8\xi(\frac{1}{2})$).
 - Pólya: $\text{RH} \iff$ every $J_\gamma^{d,n}$ is hyperbolic.
 - GORZ Thm. 1: for each d , $J_\gamma^{d,n}$ is hyperbolic for all large n .
 - GORZ Thm. 2: hyperbolic for all n when $d \leq 8$.
 - Earlier work covered $d \leq 3$ (Csordas–Norfolk–Varga for $d = 2$, the Turán inequalities; Dimitrov–Lucas). Chasse: $J_\gamma^{d,0}$ is hyperbolic for $d \leq 2 \cdot 10^{17}$.

1.2. Is the Weil/Li form the Hessian of a Lorentzian family?

Theorem 1.1 ([THEOREM]). *Let $f_{d,n}(w_1, w_2) = \sum_k \binom{d}{k} \gamma(n+k) w_1^k w_2^{d-k}$ be the homogenized Jensen polynomial. Then*

$$f_{d,n} \text{ Lorentzian } \forall d, n \iff \gamma(m)^2 \geq \gamma(m-1)\gamma(m+1) \quad \forall m \geq 1 \iff J_\gamma^{2,n} \text{ hyperbolic } \forall n.$$

The Turán inequalities hold unconditionally (Csordas–Norfolk–Varga, as recorded in [11]). So every $f_{d,n}$ is Lorentzian, whether or not RH holds. The Hessian test reduces to the 2×2 matrix $\begin{pmatrix} \gamma(m) & \gamma(m+1) \\ \gamma(m+1) & \gamma(m+2) \end{pmatrix}$ having $\det \leq 0$.

Proof. All $\gamma(n) > 0$, because $\gamma(n) = 64 \frac{n!}{(2n)!} \int_0^\infty \Phi(u) (2u)^{2n} du$ with $\Phi > 0$ (the Rodgers–Tao kernel [19]). Ultra log-concavity of $a_k = \binom{d}{k} \gamma(n+k)$ is exactly $\gamma(n+k)^2 \geq \gamma(n+k-1)\gamma(n+k+1)$. Then apply [5, Thm. 2.25]. The discriminant of $J_\gamma^{2,m}$ is $4(\gamma(m+1)^2 - \gamma(m)\gamma(m+2))$. $\square$

Contrast: Hilbert versus Lorentz ([THEOREM], [NUMERICAL]). The moments $\mu_n = \int \Phi(2u)^{2n}$ form a Stieltjes moment sequence. Cauchy–Schwarz gives $\mu_{n+1}^2 \leq \mu_n \mu_{n+2}$, so the Hankel matrix is PSD: L112’s “Hilbert-space” positivity. Turán is the *opposite* inequality for γ . Since $\gamma(n) = 64 n! \mu_n / (2n)!$, it reads $\mu_{n+1}^2 / (\mu_n \mu_{n+2}) \geq (2n+1)/(2n+3)$. Cauchy–Schwarz is thus almost tight. Numerically the ratio is 0.3583 at $n = 0$ vs. $1/3$, and 0.99904 at $n = 400$ vs. 0.99751. The L112 Li/Toeplitz Gram form is of PSD (Hilbert) type. It is *not* the Hessian of a Lorentzian family. The Lorentzian family is controlled by Turán, which is unconditional. **Verdict** ([RESTATEMENT]): no Lorentzian $\Leftrightarrow$ RH statement arises at the level of single-variable Jensen data. Any such statement would need a genuinely multivariate family, and that is Open (D4).

1.3. Numerics with ghosts. [NUMERICAL] (num/lemma114_gamma.py, lemma114_jensen.py, lemma114_jensen_bigd.py).

- *Data.* $\gamma(0.460)$ by Gauss–Legendre quadrature of the Φ -moments at 70 digits; $\gamma(1)$ agrees with $8\xi''(\frac{1}{2})/2$ to 70 digits.
- *Ghosts* (L103 convention: insert a quartet, delete *one* on-line zero). We divide the series $F(u) = 8\xi(\frac{1}{2} + \sqrt{u})$ by $(1 + u/\gamma_1^2)$ and multiply by $Q(u)/Q(0)$, where $Q(u) = (u - \delta^2 + \gamma^2)^2 + 4\delta^2\gamma^2$. The division is done stably: a forward recursion below the crossover $n^* = 4$ and a convergent backward series above it, with identity residual 10^{-34} .
- *Pitfall.* A naive forward recursion manufactures false Turán failures for $n \gtrsim 40$ by catastrophic cancellation. We record this so that nobody repeats it.

model	Turán fails	non-hyperbolic $J^{d,n}$ ($d \leq 30, n \lesssim 400$)	first d : $J^{d,0}$ non-real
true ξ	none	none	none up to $d = 80$
L103 quartet $\gamma_1, \delta=0.3$	none	none	> 80 (unresolved)
quartet $\gamma=5, \delta=0.3$	none	none	61–80 (one pair)
quartet $\gamma=2, \delta=0.45$	none	only $n = 0, d \geq 6$	≤ 6

Finding ([NUMERICAL], [RESTATEMENT]).

- At fixed degree the Jensen tests are *ghost-blind* except for very low ghosts, and only at $n = 0$. Large shifts n never see a ghost, consistent with GORZ Thm. 1 (its hypotheses concern the asymptotics of $\gamma(n)$, which a finite ghost does not change).
- A ghost at height γ becomes visible only once d is large enough that $J^{d,0}(z^2/d)$ resolves height γ : $d \leq 40$ for $\gamma = 2$, $d \in (60, 80]$ for $\gamma = 5$, and $d > 80$ for $\gamma_1 = 14.13$. *Precision limit*: with $\gamma(n)$ known to 60 digits, root-finding reports false non-real roots for the *true* ξ from $d = 100$ (26 roots), so only $d \leq 80$ is trusted.
- Hence Pólya’s criterion is a [RESTATEMENT] of RH. Its content sits at $d \rightarrow \infty$, not in any fixed- d theorem.

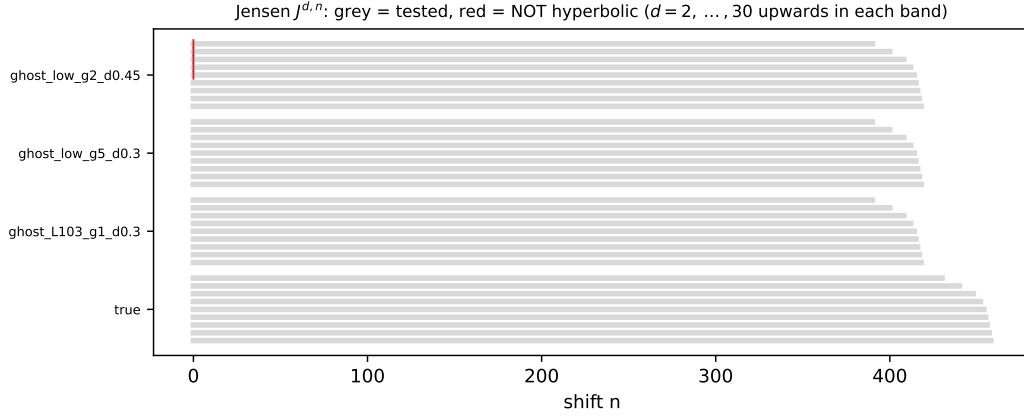


Fig. 2. Jensen hyperbolicity map ([NUMERICAL]). Grey: tested (d, n) ; red: non-hyperbolic.

1.4. Batyrev mirror symmetry (brief). [CITED] [2]. If P is a reflexive polytope then so is its polar P^* , and the Calabi–Yau families $\mathcal{F}(P)$, $\mathcal{F}(P^*)$ are mirror (Batyrev writes Δ ; renamed here to avoid a fourth Δ). The cube $[-1, 1]^3$ and the octahedron $\text{conv}(\pm e_i)$ form a reflexive polar pair ([THEOREM], direct check). The *regular* icosahedron and dodecahedron are not lattice polytopes ([THEOREM]). For a lattice embedding the centroid would be rational, so R^2/edge^2 would be rational; but for the icosahedron $R^2/a^2 = (10+2\sqrt{5})/16$. So the $(2, 3, 5)$ solids sit outside Batyrev’s framework.

2. B. PYTHAGORAS

Theorem 2.1 ([CITED] [20, 22]). $C(\mathbb{Q}) = \{x^2 + y^2 = 1\}(\mathbb{Q})$ is a group under angle addition, and

$$C(\mathbb{Q}) \cong \mathbb{Z}/4 \oplus \bigoplus_{p \equiv 1 \pmod{4}} \mathbb{Z}.$$

The torsion is generated by $(0, 1)$ (that is, i). For $p = a^2 + b^2$ the free generator is $((a^2 - b^2)/p, 2ab/p) = (a + bi)^2/p$, one for each prime $p \equiv 1 \pmod{4}$, with the Gaussian prime chosen up to conjugation and units.

[NUMERICAL] (num/lemma114_cm.py): for all 722 primitive triples with $m < 60$, every prime factor of the hypotenuse is $\equiv 1 \pmod{4}$. The proof maps $\mathbb{Z}[i] \setminus 0 \rightarrow C(\mathbb{Q})$ by $m + ni \mapsto (m + ni)^2/N(m + ni)$ [20].

Hyperbola versus circle ([THEOREM]). $x^2 - y^2 = 1$ is the *split* torus: $(x, y) \mapsto x + y$ identifies it with $\mathbb{G}_m$, with points $\mathbb{Q}^\times$. The circle is the norm-one torus of $\mathbb{Q}(i)/\mathbb{Q}$. Over $\mathbb{F}_p$ ([NUMERICAL], $p < 60$ exhaustive):

- the affine hyperbola has $p - 1$ points;
- the affine circle has $p - \left(\frac{-1}{p}\right)$ points: $p - 1$ when $p \equiv 1 \pmod{4}$, where $i \in \mathbb{F}_p$ and the circle splits, and $p + 1$ when $p \equiv 3 \pmod{4}$.

CM curve $E : y^2 = x^3 - x$ ([NUMERICAL] for $p < 200$; [THEOREM] classically).

- $a_p = 0$ for $p \equiv 3 \pmod{4}$ (supersingular).
- For $p \equiv 1 \pmod{4}$, write $p = A^2 + B^2$ with A odd and B even, and choose signs so that $\pi = A + Bi \equiv 1 \pmod{2 + 2i}$, i.e. $A + B \equiv 1 \pmod{4}$. Then

$$a_p = \pi + \bar{\pi} = 2A \quad \text{for every } p < 200 \text{ (e.g. } a_5 = -2, a_{13} = 6, a_{17} = 2, a_{29} = -10).$$

The opposite sign rule $a_p = -2A$ fails at every such p .

- Frobenius is the Gaussian integer π with $\pi\bar{\pi} = p$. The Hasse bound $|a_p| \leq 2\sqrt{p}$ is then the triviality $A^2 \leq A^2 + B^2$: *Pythagoras is RH for this curve*.

Link ([ANALOGY]). The free generators of $C(\mathbb{Q})$ (Theorem 2.1) and the Frobenius elements of E are the same Gaussian primes. Their squares give the rational circle points; they themselves are Frobenius eigenvalues of norm p .

3. C. PLATONIC SOLIDS AND THE MODULAR TOWER

- [CITED] [27]. The finite subgroups of $SO(3)$ are the cyclic groups C_n , dihedral D_n , $T \cong A_4$, $O \cong S_4$, $I \cong A_5$.
- [CITED] [26]. For (l, m, n) the triangle with angles $\pi/l, \pi/m, \pi/n$ is spherical, Euclidean or hyperbolic as $\kappa = \frac{1}{l} + \frac{1}{m} + \frac{1}{n} - 1$ is > 0 , $= 0$ or < 0 (Gauss–Bonnet).
 - Spherical: $(2, 2, n)$, $(2, 3, 3)$ (tetrahedron), $(2, 3, 4)$ (cube/octahedron), $(2, 3, 5)$ (dodecahedron/icosahedron).
 - Euclidean: $(2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$.
 - The modular group $\mathrm{PSL}_2(\mathbb{Z})$ is the limit $(2, 3, \infty)$, with $\kappa = -1/6$.
- Orbifold area $2\pi|\kappa| = \pi/3 = \mathrm{vol}(\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H})$ [THEOREM] (also [15, §8]).

Theorem 3.1 ([THEOREM]; [NUMERICAL] group check). For $N = 3, 4, 5$, $\mathrm{PSL}_2(\mathbb{Z}/N) \cong A_4, S_4, A_5$ (orders 12, 24, 60). The element-order profiles computed are $\{1:1, 2:3, 3:8\}$, $\{1:1, 2:9, 3:8, 4:6\}$ and $\{1:1, 2:15, 3:20, 5:24\}$, matching A_4, S_4, A_5 . Moreover:

- $X(N)$ has index $\mu = 12, 24, 60$, cusps $\mu/N = 4, 6, 12$ and genus $1 + \mu/12 - \mu/(2N) = 0$.
- The cover $j : X(N) \cong \mathbb{P}^1 \rightarrow X(1) \cong \mathbb{P}^1$ is Galois with group $G = \mathrm{PSL}_2(\mathbb{Z}/N) \subset \mathrm{PGL}_2(\mathbb{C}) \cong SO(3)$ (rotation group of the solid). It is branched only over $j = \infty, 0, 1728$ with ramification $N, 3, 2$.
- So the fibres are G -orbits with stabilisers C_N, C_3, C_2 , of sizes $|G|/N, |G|/3, |G|/2$:
 - $X(3)$: $(4, 4, 6) = \text{tetrahedron vertices, dual-tetrahedron vertices, edge midpoints}$;
 - $X(4)$: $(6, 8, 12) = \text{octahedron vertices, cube vertices, edge midpoints}$;
 - $X(5)$: $(12, 20, 30) = \text{icosahedron vertices, dodecahedron vertices (= icosahedron face centres), edge midpoints}$.
- Riemann–Hurwitz gives $2g - 2 = -2$ in all three cases. The cusps (fibre over ∞) are the vertices of the tetrahedron, octahedron and icosahedron, and j is a Belyi map.

Wikipedia records the $X(5)$ /icosahedron case and $X(7) =$ Klein quartic with 24 cusps [25]; we recomputed the latter ($\mu = 168, g = 3$).

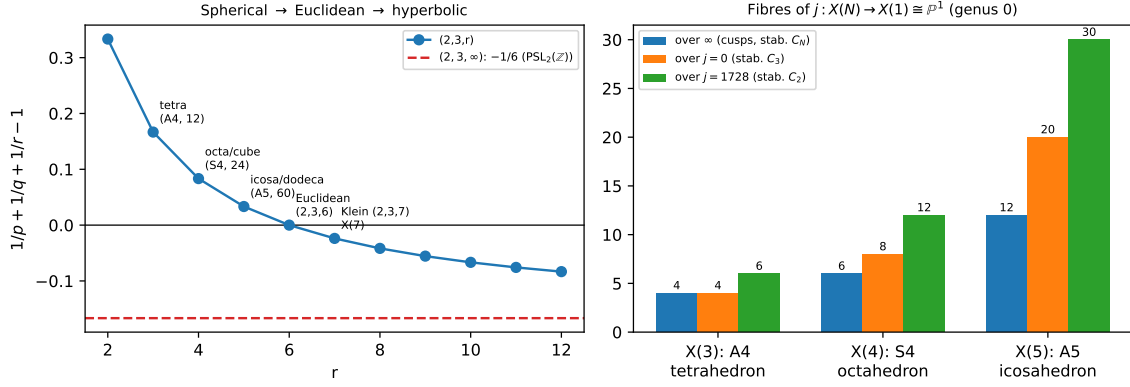


Fig. 1. Left: curvature $\frac{1}{2} + \frac{1}{3} + \frac{1}{r} - 1$ from spherical (Platonic) through Euclidean $(2, 3, 6)$ to hyperbolic, with the modular limit $-1/6$. Right: fibres of j on $X(N)$.

4. D. THE CUSP AND Δ

4.1. Why weight 12.

Theorem 4.1 ([CITED] [21, Prop. 2, (6), (22), (23)]). *For nonzero $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$,*

$$\sum_{P \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} \frac{\mathrm{ord}_P f}{n_P} + \mathrm{ord}_\infty f = \frac{k}{12}, \quad n_i = 2, \quad n_\rho = 3, \quad n_P = 1 \text{ otherwise.}$$

Hence:

- E_4 ($k/12 = 1/3$) has one simple zero, at ρ ; E_6 ($1/2$) has one simple zero, at i .
- A cusp form needs $\mathrm{ord}_\infty \geq 1$, so $k \geq 12$. $\Delta = q \prod (1 - q^n)^{24} = (E_4^3 - E_6^2)/1728$ is nowhere zero on $\mathbb{H}$, with a simple zero at the cusp.

The orders 2 and 3 of the elliptic points force the denominator 12 in the valence formula. Equivalently ([THEOREM]), weight- k forms are sections of a line bundle of orbifold degree $k/12$. The orbifold Euler characteristic $\kappa = -1/6$ gives $\deg K(\log \infty) = 1/6$ (weight 2). Weight 12 is the first with degree $1 = \deg \mathcal{O}(\infty)$: $\omega^{\otimes 12} \cong \mathcal{O}(\text{cusp})$ on the coarse space. On the open moduli stack, $\mathrm{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12$, generated by the Hodge bundle (Mumford; [BIB], not opened). Coarse check: $\dim M_{12} = 2 = h^0(\mathcal{O}(1))$.

Notation coincidence ([THEOREM]). For the lattice cubic $4x^3 - g_2x - g_3$, the discriminant (up to the factor 16) is $g_2^3 - 27g_3^2$. With $g_2 = \frac{4\pi^4}{3}E_4$ and $g_3 = \frac{8\pi^6}{27}E_6$ this gives $(2\pi)^{12}\Delta$. So the modular discriminant is the root-collision discriminant Disc of E_z . Its nonvanishing on $\mathbb{H}$ says that E_z never degenerates there. Degeneration happens only at the cusp.

4.2. The cusp is Pythagorean: Tate curve and nodal cubics. [CITED] [28]: $E_q = \mathbb{C}^\times / q^\mathbb{Z}$, with $j = q^{-1} + \dots$. Over a p -adic field the Tate curve has split multiplicative reduction; every semistable curve is a Tate curve up to quadratic twist. As $q \rightarrow 0$, $E_q \rightarrow \mathbb{C}^\times = \mathbb{G}_m$, the *split node*.

Theorem 4.2 ([THEOREM]; [NUMERICAL]). *Let $Y_a : y^2 = x^2(x + a)$ with $a \neq 0$ real, and let Y_a^{ns} be its smooth locus.*

- $a > 0$ (crunode, real tangents $y = \pm\sqrt{a}x$): $\phi(P) = (y - \sqrt{a}x)/(y + \sqrt{a}x)$ is an isomorphism $Y_a^{ns}(\mathbb{R}) \cong \mathbb{R}^\times$. Two components; this is the hyperbola, the split torus.
- $a < 0$ (acnode, tangents $y = \pm i\sqrt{|a|x}$): $\phi(P) = (y - i\sqrt{|a|x})/(y + i\sqrt{|a|x})$ lands in $U(1)$ and is an isomorphism $Y_a^{ns}(\mathbb{R}) \cong S^1$. One component, since the node is isolated; this is the circle, the norm-one torus.
- Numerically the homomorphism error under the chord-tangent law is $\leq 1.4 \cdot 10^{-12}$ (200 random pairs each), and $|\phi| = 1$ holds to $3 \cdot 10^{-13}$.

- Over $\mathbb{F}_p$ ($p < 60$ exhaustive): $\#Y_a^{ns}(\mathbb{F}_p) = p - 1$ if a is a square (split) and $p + 1$ if not (non-split).

So the boundary of the modular curve, where Δ vanishes, is exactly where elliptic curves become the Pythagorean objects of §2: hyperbola or circle. **L111 link** ([ANALOGY]): ghosts are born at discriminant zeros, where two real zeros collide (Krein collision). At the cusp, two branches of the curve collide at the node. It is the same “Disc = 0” geometry in different spaces.

4.3. $\tau(n)$, **Deligne, Sato–Tate**. [NUMERICAL] (num/lemma114_modular.py):

- $E_4^3 - E_6^2 = 1728\Delta$ holds coefficientwise (exact integers) up to q^{1000} ;
- $\tau(1..10) = 1, -24, 252, -1472, 4830, -6048, -16744, 84480, -113643, -115920$;
- multiplicativity, the Hecke relation $\tau(p^2) = \tau(p)^2 - p^{11}$, and $\tau \equiv \sigma_{11} \pmod{691}$ hold in range.

Ramanujan–Petersson is a **Theorem**: Deligne [8, Théorème 8.2], $|a_p| \leq 2p^{(k-1)/2}$; for Δ , $|\tau(p)| \leq 2p^{11/2}$. Over the 168 primes $p < 1000$, $\max |\tau(p)|/2p^{11/2} = 0.9594$. Sato–Tate moments are $\langle \cos^2 \theta \rangle = 0.234$ and $\langle \cos^4 \theta \rangle = 0.114$ (limits 1/4 and 1/8; [NUMERICAL], small sample).

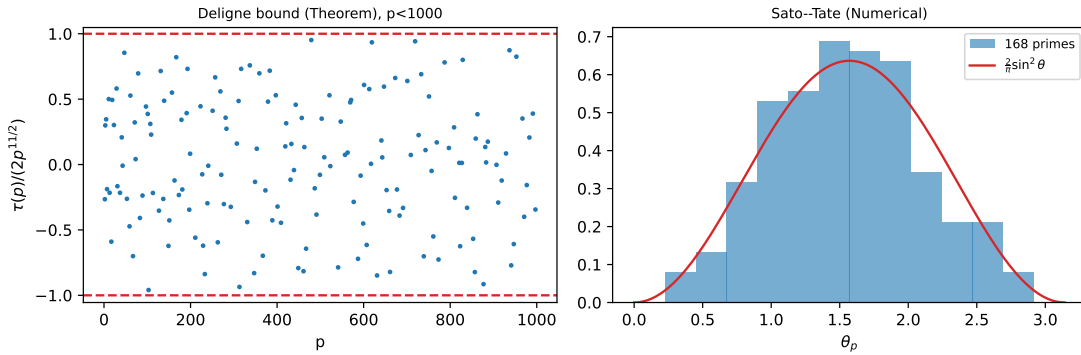


Fig. 3. Normalized $\tau(p)$ and the Sato–Tate histogram.

4.4. **Rankin–Selberg, exactly**. Take $E(z, s) = \frac{1}{2} \sum_{(c,d)=1} y^s |cz + d|^{-2s}$ [23, 14]. The constant term is $y^s + \varphi(s)y^{1-s}$ with $\varphi(s) = \Lambda(2s - 1)/\Lambda(2s)$ and $\Lambda(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$; the residue at $s = 1$ is $3/\pi$.

Theorem 4.3 ([THEOREM] by unfolding; [NUMERICAL] check).

$$\int_{\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}} y^{12} |\Delta(z)|^2 E(z, s) \frac{dx dy}{y^2} = \frac{\Gamma(s + 11)}{(4\pi)^{s+11}} \sum_{n \geq 1} \frac{\tau(n)^2}{n^{s+11}} \quad (\Re s > 1).$$

At $s = 3$: LHS by quadrature over $\mathcal{F}$ (lattice-sum E) = $2.70938940 \cdot 10^{-6}$; RHS = $2.70938890 \cdot 10^{-6}$; ratio 1.0000002 (truncation level).

Euler factors ([THEOREM], checked symbolically for $p = 2, 3, 5, 7$ to order 8):

$$\sum_k \tau(p^k)^2 X^k = \frac{1 + p^{11} X}{(1 - \alpha^2 X)(1 - p^{11} X)(1 - \beta^2 X)},$$

which gives $\sum \tau(n)^2 n^{-s} = \zeta(s - 11) L(\mathrm{Sym}^2 \Delta, s) / \zeta(2s - 22)$.

Diagonal pullback ([ANALOGY]). $y^{12} |\Delta|^2$ is the restriction of $\Delta \boxtimes \overline{\Delta}$ (a function on $\mathbb{H} \times \mathbb{H}$) to the diagonal $z_1 = z_2$. Rankin–Selberg integrates this diagonal restriction against E , which is formally like Weil’s $\Delta_C \cdot \Delta_C$. Positivity of $|\Delta|^2$ gives Landau-type nonvanishing and zero-free regions near $\Re = 1$. It does *not* give RH for $L(\mathrm{Sym}^2, s)$ or for ζ .

Where are the zeros of ζ ? ([THEOREM]). They sit in $\varphi(s) = \Lambda(2s - 1)/\Lambda(2s)$: zeros of $\zeta(2s)$ are poles of φ , at $s = \rho/2$ (on $\Re s = \frac{1}{4}$ under RH; L105 and [15, §6]). Cusp forms are orthogonal to Eisenstein series, so $\langle \Delta, E \rangle$ -type pairings kill exactly the term $\varphi(s)y^{1-s}$. The missing ingredient is a positivity that *sees* φ , i.e. the continuous spectrum. This is what §§5–7 probe.

5. THE QUASIMODULAR ANOMALY: FROM Δ INTO THE EISENSTEIN CONSTANT TERM (T473)

Theorem 5.1 ([CITED] [21, (17)–(21), Prop. 6, Prop. 7 proof]; [NUMERICAL]). *Let $E_2 = 1 - 24 \sum \sigma_1(n)q^n$.*

- (1) $E_2 = \frac{1}{2\pi i} \frac{d}{dz} \log \Delta$. Numerically $|E_2 - \frac{1}{2\pi i} (\log \Delta)'| \leq 1.1 \cdot 10^{-31}$ at two points.
- (2) $E_2(\frac{az+b}{cz+d}) = (cz+d)^2 E_2(z) + \frac{6c(cz+d)}{\pi i}$; in particular $E_2(-1/z) = z^2 E_2(z) + \frac{6z}{\pi i}$. Derivation: Zagier's $G_2(\gamma z) = (cz+d)^2 G_2 - \pi i c(cz+d)$ with $E_2 = 6G_2/\pi^2$. Numerically the error is $\leq 2 \cdot 10^{-31}$ at three points; the wrong constant $12z/(\pi i)$ is off by ≈ 2 .
- (3) $E_2^* = E_2 - \frac{3}{\pi y}$ transforms with weight 2 and is non-holomorphic. Errors $\leq 3 \cdot 10^{-30}$ for S and for $(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix})$.
- (4) Hecke's trick. $G_{2,\varepsilon} = \frac{1}{2} \sum' (mz+n)^{-2} |mz+n|^{-2\varepsilon}$ converges for $\varepsilon > 0$ and $\lim_{\varepsilon \rightarrow 0} G_{2,\varepsilon} = G_2 - \pi/(2y)$. The extra term is $I(\varepsilon)\zeta(1+2\varepsilon)/y^{1+2\varepsilon}$ with $I(0) = 0$ and $I'(0) = -\pi$. It is a zero of the archimedean integral times the pole of ζ at 1.

Theorem 5.2 ([CITED] [24]; [NUMERICAL]). With $E_W(z, s) = \sum'_{(m,n)} y^s |mz+n|^{-2s} = 2\zeta(2s)E(z, s)$,

$$E_W(z, s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log(\sqrt{y}|\eta(z)|^2)) + O(s-1), \quad \log(\sqrt{y}|\eta|^2) = \frac{1}{12} \log(y^6 |\Delta(z)|).$$

Numerically we use the Fourier expansion and a symmetric difference at $s = 1 \pm 10^{-8}$. At $z = 0.1 + 1.1i, -0.3 + 0.95i, 0.25 + 1.7i$ the constant term agrees with the right side to 10^{-16} , and the residue is π to 10 digits. The Fourier formula itself matches the direct lattice sum at $s = 3$ to 10^{-10} .

Proposition 5.3 ([THEOREM]; [NUMERICAL]). Write $\text{CT}_1(z)$ for the constant term in Theorem 5.2. Then

$$\partial_z \log(y^{12} |\Delta|^2) = 2\pi i E_2^*, \quad \text{so} \quad E_2^*(z) = \frac{6i}{\pi^2} \partial_z \text{CT}_1(z).$$

At $z = 0.13 + 1.21i$ both sides equal $-0.0143935 - 0.3332684i$. The anomaly term has a precise origin:

- $\partial_z \log y^{12} = -6i/y$ produces the $-3/(\pi y)$;
- the $\log y$ inside $\log \sqrt{y}$ comes from expanding the scattering term $2\zeta(2s)\varphi(s)y^{1-s} = \frac{\pi}{s-1}y^{1-s} = \frac{\pi}{s-1} - \pi \log y + \dots$ at its pole.

So $3/(\pi y)$ is the residue of φ at $s = 1$ ($3/\pi$ in the normalization of E) times the s -derivative of y^{1-s} . It is not “ φ evaluated at a point”: φ has a pole there.

- $3/\pi = 1/\text{vol}$ ([THEOREM]). $\text{vol}(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}) = \pi/3$, by Gauss–Bonnet with $K = -1$: $2\pi(1 - \frac{1}{2} - \frac{1}{3} - 0)$, the $(2, 3, \infty)$ curvature of Fig. 1. Also $\text{Res}_{s=1} E = 3/\pi$ [23].
- **Global, not local** ([THEOREM]). $\partial_z E_2^* = \frac{3i}{2\pi} y^{-2}$ is a constant multiple of the hyperbolic area density, uniform over $\mathcal{F}$. Near the cusp, $E_2^* = 1 - \frac{3}{\pi y} + O(e^{-2\pi y})$. The anomaly decays only like $1/y$ (a power law, part of the zeroth Fourier coefficient). It is the cusp's bookkeeping of a global curvature.
- **Arakelov context** ([THEOREM], short computation). For weight- k forms with Petersson metric $\|f\|^2 = y^k |f|^2$, $c_1 = -\frac{i}{2\pi} \partial \bar{\partial} \log y^k = \frac{k}{4\pi} \frac{dx dy}{y^2} > 0$. Its total mass is $\frac{k}{4\pi} \cdot \frac{\pi}{3} = \frac{k}{12}$, the valence formula again. Weight 12 has degree 1, matching $\omega^{12} \cong \mathcal{O}(\text{cusp})$ (up to the logarithmic singularity of the metric at the cusp, Mumford's “good” metrics, [BIB]). Faltings and Hriljac proved the arithmetic Hodge index theorem (the Arakelov pairing is minus the Néron–Tate pairing on degree-0 classes) [12, 9]. It is the proved arithmetic-surface analogue of Weil positivity, but for curves over number fields, not for “ $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ ”.

Verdict ([RESTATEMENT]/[HEURISTIC]).

- The anomaly is a genuine $\Delta \rightarrow$ Eisenstein bridge (Proposition 5.3). But it lives at $s = 1$, the pole edge where Kronecker evaluates. That is “near $\Re s = 1$ ” positivity, the same regime as zero-free regions.
- To reach the critical line one would need an anomaly-type correction for every s . The natural candidate is the s -family of non-holomorphic constant-term pieces, and that family *is* $\varphi(s)y^{1-s}$. So the extension is a [RESTATEMENT] of the scattering problem $\varphi(s)$ (L105), not new input.

6. A VELOCITY IN s : THE CRITICAL POINT $s = \frac{1}{2}$ (T474)

Theorem 6.1 ([THEOREM]; [NUMERICAL]). For $\mathrm{SL}_2(\mathbb{Z})$ with $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$:

- (1) $\varphi(\frac{1}{2}) = -1$ and $\varphi'(\frac{1}{2}) = 2(\gamma - \log 4\pi) = -3.9076171641\dots$ (numerically identical to 12 digits).
Reason: $\Lambda(u) = -1/u + c_0 + O(u)$ and $\Lambda(1+u) = 1/u + c_0 + O(u)$ with $c_0 = \frac{1}{2}(\gamma - \log 4\pi)$.
- (2) $E(z, \frac{1}{2}) \equiv 0$. The constant term is $y^{1/2} - y^{1/2}$, and the non-constant terms carry $1/\Lambda(2s) \rightarrow 0$.
Numerically $|E(z, \frac{1}{2} + 10^{-25})| \approx 4 \cdot 10^{-25}$ at four points.
- (3) Since $E = E^*/\Lambda(2s)$ with $E^* = \Lambda(2s)E$ regular at $\frac{1}{2}$ [14], the velocity is

$$V(z) := \partial_s E(z, s)|_{s=1/2} = 2E^*(z, \frac{1}{2}) = 2\sqrt{y}(\log y + \gamma - \log 4\pi) + 8\sqrt{y} \sum_{n \geq 1} d(n)K_0(2\pi ny) \cos 2\pi nx.$$

The constant term equals $2y^{1/2} \log y + \varphi'(\frac{1}{2})y^{1/2}$. The closed form matches the finite difference to 14 digits, and $\Delta_{\mathbb{H}}V = \frac{1}{4}V$ (ratio 0.25 to 12 digits).

Sign ([NUMERICAL], [THEOREM]). On a 41×184 grid of $\mathcal{F}$ ($y \leq 10$), V ranges over $[-3.92, 2.21]$. It is negative below and positive above the curve $y = 7.0552$, flat in x because the K_0 terms are $\sim e^{-44}$ there. The zero is $y^* = 4\pi e^{-\gamma} = 7.0555$. This is exactly the Lagarias–Suzuki threshold [14, Thm. 3]. For $1 \leq y \leq y^*$ all zeros of $a_0(y, s) = \Lambda(2s)y^s + \Lambda(2-2s)y^{1-s}$ lie on $\Re s = \frac{1}{2}$; for $y > y^*$ there are exactly two real off-line zeros. Our V ’s constant term is $2a_0(y, \frac{1}{2})$. So the velocity is not sign-definite, and its sign change marks the birth of a pair of off-line zeros of the constant term (a double zero at $\frac{1}{2}$ when $y = y^*$): a Krein-type collision in the sense of L111 ([ANALOGY]). V is not L^2 (it grows like $\sqrt{y} \log y$), so “ V as a quadratic form” is meaningful only after truncation. The Maass–Selberg Gram form of truncated Eisenstein series is PSD, which is again Hilbert type ([RESTATEMENT]).

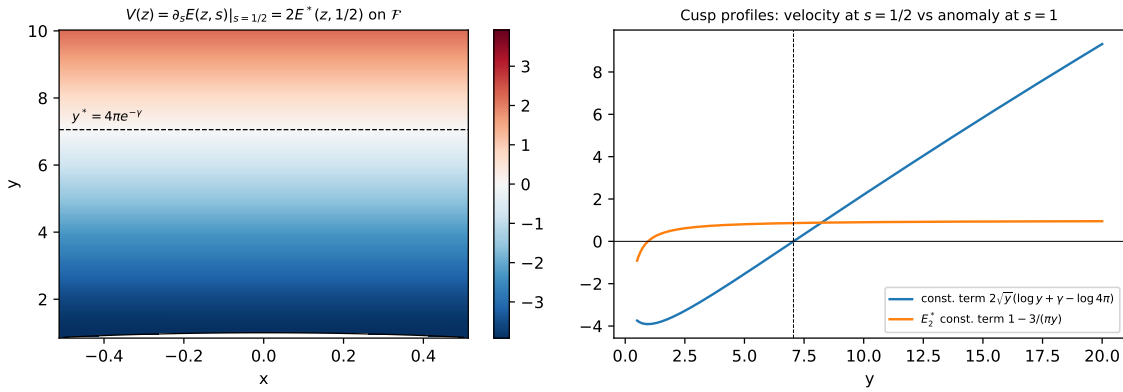


Fig. 4. $V = \partial_s E|_{1/2}$ on $\mathcal{F}$ with the sign change at $y^* = 4\pi e^{-\gamma}$ (left). Constant-term profiles at $s = \frac{1}{2}$ (velocity) and at $s = 1$ (anomaly $1 - 3/\pi y$) (right).

Klein–Gordon dictionary ([RESTATEMENT] for the formulas, [ANALOGY] for the physics words).

- $\Delta_{\mathbb{H}}y^s = s(1-s)y^s$, and $s = \frac{1}{2} + it$ gives $\lambda = \frac{1}{4} + t^2$: a dispersion relation with “mass²” $\frac{1}{4}$. The bottom of the continuous spectrum of $\mathbb{H}^n$ is $(n-1)^2/4$, i.e. $\frac{1}{4}$ for $K = -1$, $n = 2$ (Selberg’s $\frac{1}{4}$ [30]).

- *Correction to the picture:* ζ -zeros are *not* eigenvalues at $s = \frac{1}{2} + i\gamma$. They are *resonances* (poles of φ, E) at $s = \rho/2$, which lie on $\Re s = \frac{1}{4}$ under RH [15, §6]. A ghost $\rho = \frac{1}{2} \pm \delta + i\gamma$ sits at $\Re s = \frac{1}{4} \pm \delta/2$: the L105 uniform damping $\frac{1}{4}$ and its ghost shift. Off-line means complex t , i.e. modes that grow or decay in the wave group (“tachyon-like” only as a metaphor).
- The two special points are fixed by curvature, in different ways:
 - $s = 1$ ($\lambda = 0$): the residue $3/\pi = 1/\text{vol}$ needs the global Gauss–Bonnet data $(2, 3, \infty)$;
 - $s = \frac{1}{2}$ ($\lambda = \frac{1}{4}$): fixed by the *local* curvature $K = -1$ alone ([THEOREM]).
- The other genuine “zero velocity” dynamics is de Bruijn–Newman heat flow (L111). $\Lambda \geq 0$ is a Theorem [19] and $\Lambda \leq 0.22$ [18]; RH $\iff \Lambda \leq 0$ ($\Rightarrow \Lambda = 0$).

Open ([OPEN]). Whether any velocity-type object at $s = \frac{1}{2}$ carries the missing positivity is open. The computation above shows the naive one does not: it is sign-indefinite, and its sign change is a constant-term phenomenon unrelated to ζ ’s zeros.

7. NULL RAYS AND WEIL’S PROOF (T478, T479)

Let $C/\mathbb{F}_q$ have genus g (geometric points throughout), $V = C \times C$, $f_1 = [C \times \text{pt}]$, $f_2 = [\text{pt} \times C]$. For a divisor D put $d_1 = D \cdot f_1$, $d_2 = D \cdot f_2$. This is Milne’s notation, with $C_1 = C \times \text{pt}$ [17, §1, Thm. 1.5].

- Theorem 7.1** ([THEOREM] [17]). (1) Null fibres. $f_1^2 = f_2^2 = 0$ and $f_1 \cdot f_2 = 1$. So $H = f_1 + f_2$ has $H^2 = 2 > 0$ (timelike); it is ample.
- (2) Reverse Cauchy–Schwarz. By Hodge index, $\text{NS}(V)_{\mathbb{R}}$ has signature $(1, \rho - 1)$. For $H^2 > 0$ and any D , $(D \cdot H)^2 \geq D^2 H^2$. *Proof:* write $D = tH + D'$ with $D' \cdot H = 0$, so $D'^2 \leq 0$ and $(D \cdot H)^2 = t^2(H^2)^2 \geq (t^2 H^2 + D'^2)H^2$. This is the reversed triangle inequality of special relativity (the twin paradox, [ANALOGY]). Conversely, if reverse Cauchy–Schwarz holds for all D against one timelike H , then $H^\perp$ is negative semidefinite, so the two statements are equivalent.
- (3) Light-cone decomposition. $D = d_2 f_1 + d_1 f_2 + D_{\text{prim}}$ with $D_{\text{prim}} \cdot f_i = 0$, $D^2 = 2d_1 d_2 + D_{\text{prim}}^2$, and $D_{\text{prim}}^2 \leq 0$. Hence Castelnuovo–Severi: $D^2 \leq 2d_1 d_2$ [17, Thm. 1.5]. The mass shell $m^2 = 2p^+ p^- - p_\perp^2$ is the [ANALOGY]. The defect $\text{def}(D) = 2d_1 d_2 - D^2 = -D_{\text{prim}}^2 \geq 0$ is the transverse length².
- (4) Intersection numbers (Milne Ex. 1.7 and adjunction):
- graph Γ_F of Frobenius: $(d_1, d_2) = (q, 1)$ and $\Gamma_F^2 = q(2 - 2g)$;
 - diagonal Δ_C : $(d_1, d_2) = (1, 1)$ and $\Delta_C^2 = 2 - 2g$;
 - $\Gamma_F \cdot \Delta_C = \#C(\mathbb{F}_q) = q + 1 - a$.
- Therefore

$$\Gamma_{\text{prim}}^2 = -2gq, \quad \Delta_{\text{prim}}^2 = -2g, \quad \Gamma_{\text{prim}} \cdot \Delta_{\text{prim}} = (q + 1 - a) - (q \cdot 1 + 1 \cdot 1) = -a.$$

- (5) Weil’s bound as Euclidean Cauchy–Schwarz. On the negative-definite transverse space, $(\Gamma_{\text{prim}} \cdot \Delta_{\text{prim}})^2 \leq \Gamma_{\text{prim}}^2 \Delta_{\text{prim}}^2$, i.e. $a^2 \leq 4g^2 q$. Equivalently $\text{def}(m\Gamma_F + n\Delta_C) = 2gq m^2 + 2a mn + 2g n^2 \geq 0$ for all m, n [17, Cor. 1.6]. Applied to F^k for all k , this gives RH for C . All of J.W.’s t479 numbers are correct as stated, provided “light-cone coordinates $(1, q)$ ” is read as (d_2, d_1) ; in Milne’s order $(d_1, d_2) = (q, 1)$.

Rapidity ([THEOREM], with one correction). On the hyperbolic plane $\langle f_1, f_2 \rangle$ the form is $2uv$ in the coordinates $uf_1 + vf_2$. A boost of rapidity η is $(u, v) \mapsto (e^\eta u, e^{-\eta} v)$. The two classes have hyperbolic parts

$$\Gamma_{\text{hyp}} = f_1 + qf_2 \quad \text{and} \quad \Delta_{\text{hyp}} = f_1 + f_2.$$

Then $\Gamma_{\text{hyp}} = \sqrt{q} \cdot B_{-\frac{1}{2} \log q}(\Delta_{\text{hyp}})$: a dilation by $\sqrt{q}$ composed with a boost of rapidity $\frac{1}{2} \log q$. Dividing by $\sqrt{q}$ fixes the norm ($\Gamma_{\text{hyp}}^2/q = 2 = \Delta_{\text{hyp}}^2$) but does *not* undo the boost; the rapidity $\frac{1}{2} \log q$ remains.

Conceptually, $\Gamma_F = \{(x, F(x))\} = (F \times 1)^* \Delta_C$. The map $(F \times 1)^*$ multiplies the intersection form by $\deg(F \times 1) = q$ and acts on $\langle f_1, f_2 \rangle$ by $f_1 \mapsto f_1$, $f_2 \mapsto qf_2$ (the pullback of $\text{pt} \times C$ is $F^{-1}(\text{pt}) \times C$).

So $\Delta_{\text{hyp}} \mapsto f_1 + qf_2 = \Gamma_{\text{hyp}}$, and $q^{-1/2}(F \times 1)^*$ is an *isometry* of $\text{NS}(V)_{\mathbb{R}}$. On the hyperbolic plane it is exactly the boost by $\frac{1}{2} \log q$. On the negative-definite transverse space an isometry is a rotation, with eigenvalues of modulus 1. That is the $\sqrt{q}$ -normalized Frobenius being unitary, i.e. RH ([RESTATEMENT] of Weil’s argument via Hodge index; we do not claim more). The number $\frac{1}{2} \log q$ is L113’s archimedean slope $\log |\alpha| = \frac{1}{2} \log q$ ([THEOREM], same number). That it is the same boost is [ANALOGY].

Ghosts as extra time directions ([RESTATEMENT] over $\mathbb{F}_q$, [NUMERICAL]). The transverse Gram matrix of $(\Gamma_{\text{prim}}^{(k)}, \Delta_{\text{prim}})$ is $-\begin{pmatrix} 2gq^k & a_k \\ a_k & 2g \end{pmatrix}$. This matrix is defined for any “model” $(q, \{\alpha_i\})$ with $\alpha \mapsto q/\alpha$ symmetry, since $\sum(\alpha + q/\alpha) = 2a$. It has a positive eigenvalue (a timelike transverse direction) iff $|a_k| > 2g q^{k/2}$. If some $|\alpha| \neq \sqrt{q}$, then $\limsup |a_k|/q^{k/2} = \infty$, so this happens for some k . Hence, for models, $RH \iff$ the span of all $\Gamma^{(k)}, \Delta$ keeps signature $(1, \cdot)$: exactly one time direction. For geometric curves Hodge index guarantees it ([THEOREM]). Numerics (`num/lemma114_null.py`):

- genus 1, $q = 101$ (all a that pass L109’s count-positivity, $-101 \leq a \leq 102$): in-cone for all $k \leq 60$ exactly when $|a| \leq 20$ (Hasse); every ghost exits at $k = 1$;
- genus 2, $q = 11$ (L109 grid): all 401 Weil models stay in the cone for $k \leq 80$, and all 2935 count-positive ghosts exit. Onset histogram k^* : 1:184, 2:2001, 3:628, 4:73, 5:35, 6:4, 7:6, 8:2, 9:2. The latest is $(s_1, s_2) = (-8, 31)$ at $k^* = 9$, back inside at $k = 10$: exits can be intermittent;
- the true L106 curve $(s_1 = -1, s_2 = 10)$ has normalized transverse minimal eigenvalue ratio 0.090, 0.0067, 0.00075 for $k = 1, 2, 3$ (inside, approaching the cone as $q^{-k/2}$).

This is the light-cone version of the L109 point-count Toeplitz detector n_- . Each exit adds one timelike transverse direction. Over $\mathbb{Q}$, Bombieri’s theorem that the number of negative eigenvalues of the truncated Weil form equals half the number of off-line zeros for large truncation [4] is the analogue ([ANALOGY]): one quartet gives two extra directions of the wrong sign.

For $\mathbb{Q}$ ([CITED] [3, §V]; [RESTATEMENT]+[ANALOGY]). Weil’s criterion, as stated by Bombieri: $RH \iff$ the right side of the explicit formula is ≤ 0 for all $f(x) = \int_0^\infty g(xy)g(y)dy$ with

$$\int_0^\infty g(x) \frac{dx}{x} = \int_0^\infty g(x) dx = 0,$$

i.e. $\hat{g}(0) = \hat{g}(1) = 0$ in Mellin terms. These are the conditions killing the pole terms $\hat{f}(0), \hat{f}(1)$ (the poles of ζ at $s = 0, 1$). Bombieri places this directly beside the algebraic index theorem: the intersection form is negative semidefinite on divisors of degree 0. So the two pole conditions play the role of $D \cdot f_1 = D \cdot f_2 = 0$, i.e. of projecting out the two null fibre classes. Connes’s dictionary pairs RH with Castelnuovo positivity in the same way [7]. For the Eisenstein constant term $y^s + \varphi(s)y^{1-s}$, the two “null directions” y^s and y^{1-s} are exchanged by $s \mapsto 1 - s$ ([ANALOGY]). L108’s “the cusp is the $\{1, q\}$ part” is the analogue of the two fibre classes [TAA-L].

The missing statement ([OPEN]). An *arithmetic Castelnuovo–Severi inequality* on “ $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ ”. One needs a Lorentzian intersection form on a space of “correspondences” containing the Frobenius-scaling graphs. In it the two classes attached to the poles $s = 0, 1$ should be null, with pairing 1. The transverse part should be negative definite: $D^2 \leq 2(D \cdot f_1)(D \cdot f_2)$, with f_1, f_2 realized by the $\hat{g}(0), \hat{g}(1)$ functionals. This is sharper than “positivity”: it names the inequality and its null coordinates. It is still a [RESTATEMENT] of RH via Weil’s criterion. Connes–Consani’s scaling site states as a long-term goal “an adequate version of the Riemann–Roch theorem in characteristic 1, suitable to transplant the pRH proof of Weil to the Riemann zeta function” [6, §1]. They prove Riemann–Roch on the periodic orbits C_p , not a Hodge index theorem ([CITED]).

Pointer only ([ANALOGY]; not developed). Gupta–Bleuler/BRST quantization of a massless vector, and the Goddard–Thorn no-ghost theorem in string theory [10], have the same shape. Removing the two null polarizations leaves a positive transverse space. This is specific to massless (light-like) kinematics. If J.W. approves, it becomes its own lemma.

8. ANSWER TO PT472U: IS Δ A “DIAGONALIZATION” OF THE PYTHAGOREAN/PLATONIC SIDE?

Theorem 8.1 ([THEOREM]; [NUMERICAL] symbolic). *Let $F = xy(x^{10} - 11x^5y^5 - y^{10})$ (12 icosahedron vertices), H (degree 20, face centres) and T (degree 30, edge midpoints), as in [13]. Then:*

- $\text{Hess}(F) = -121H$ and $J(F, H) = -20T$, and $T^2 = H^3 - 1728F^5$ (all verified symbolically);
- the degree-60 invariant map $z \mapsto H^3/1728F^5$ is the quotient $\mathbb{P}^1 \rightarrow \mathbb{P}^1/A_5$, branched over three points with orbits 12, 20, 30 and ramification 5, 3, 2;
- by Theorem 3.1 and uniqueness of the $A_5 \subset \text{PGL}_2(\mathbb{C})$ action up to conjugacy, this is $j : X(5) \rightarrow X(1)$ up to Möbius coordinates.

The dictionary is

$$\begin{aligned} H &\leftrightarrow E_4 \ (j = 0), & T &\leftrightarrow E_6 \ (j = 1728), & F^5 &\leftrightarrow \Delta \ (\text{cusps}), \\ j &= \frac{H^3}{F^5} \leftrightarrow \frac{E_4^3}{\Delta}, & T^2 &= H^3 - 1728F^5 \leftrightarrow E_6^2 = E_4^3 - 1728\Delta. \end{aligned}$$

Answer. In the precise sense available, Δ is the *vertex (cusp) form* of the Platonic tower $(2, 3, N)$, and the identity $1728\Delta = E_4^3 - E_6^2$ is Klein’s syzygy in the limit $N \rightarrow \infty$ ([THEOREM] for $N = 5$; the $N \rightarrow \infty$ reading is [ANALOGY]). The cusp itself is Pythagorean: degenerate curves there have smooth part $\mathbb{G}_m$ (hyperbola) or the norm-one torus (circle) (Theorem 4.2). And Δ is “diagonal” only as a discriminant, the locus where roots collide. “Diagonalization” in the linear-algebra sense is not supported ([HEURISTIC] at most). The Weil-diagonal analogy (Rankin–Selberg as a diagonal restriction) is [ANALOGY].

9. CORRECTIONS TO THE CHAT CLAIMS (T467–T479)

The chat claims reached this lemma as summaries in the briefs. Each item gives the verified form and flags where the relayed wording overstates.

C114.1 (circle points): $C(\mathbb{Q}) \cong \mathbb{Z}/4 \oplus \bigoplus_{p \equiv 1(4)} \mathbb{Z}$. The torsion is $\mathbb{Z}/4$, generated by $i = (0, 1)$.

There is one free generator *per prime* $p \equiv 1 \pmod{4}$, namely $(a + bi)^2/p$; triples are products of these. Primes $p \equiv 3 \pmod{4}$ and 2 contribute nothing free.

C114.2 (CM normalization): For $y^2 = x^3 - x$: $a_p = 2A$ where $\pi = A + Bi \equiv 1 \pmod{2 + 2i}$, so the sign is fixed by primarity (verified $p < 200$), and $a_p = 0$ for $p \equiv 3 \pmod{4}$. “ $a_p = \pm 2a$ with $a^2 + b^2 = p$ ” is incomplete without this sign rule.

C114.3 ($X(N)$ cusps): The cusps of $X(3), X(4), X(5)$ are *vertices* of the tetrahedron, octahedron and icosahedron (4, 6, 12). Face centres lie over $j = 0$ and edge midpoints over $j = 1728$. All three curves have genus 0, and $X(7)$ has genus 3 with 24 cusps.

C114.4 (McMullen): McMullen 1993 proves hard Lefschetz and Hodge–Riemann–Minkowski for *simple* polytopes via the polytope algebra, hence necessity in the g -theorem for all simple polytopes. Sufficiency is Billera–Lee. The icosahedron is simplicial; the dodecahedron is the simple one.

C114.5 (GORZ): GORZ prove hyperbolicity for $d \leq 8$ (all n) and for each d at large n . This is not RH, since Pólya needs all (d, n) . Our ghost tests show fixed- d results are ghost-blind.

C114.6 (BH): The exact statement is that the Hessian of a nonzero Lorentzian polynomial has exactly one positive eigenvalue *at every point of the positive orthant*. The definition requires Lorentzian signature of the *quadratic derivatives* together with M-convex support. It is not “one positive eigenvalue everywhere”.

C114.7 (h -vector): Icosahedron (1, 9, 9, 1), $g = (1, 8)$. The simple dodecahedron has the same h -vector via polar duality.

C114.8 (real nodal dichotomy): Crunode means split, $\mathbb{R}^\times$, two components; acnode means non-split, S^1 , one component. Over $\mathbb{F}_p$ the counts are $p - 1$ and $p + 1$. Only the acnode is the circle; the crunode is the hyperbola.

C114.9 (Rankin–Selberg): The exact shape is $\int y^{12} |\Delta|^2 E d\mu = \Gamma(s + 11)(4\pi)^{-s-11} \sum \tau(n)^2 n^{-s-11}$, with E normalized by $\frac{1}{2} \sum_{(c,d)=1}$. The Sym^2 relation carries $\zeta(s-11)/\zeta(2s-22)$.

C114.10 (labels): The following are [ANALOGY], not Theorems: “ Δ diagonalizes the Platonic side”; “ghosts are born at discriminant zeros”; “Rankin–Selberg is Weil’s diagonal”.

C114.11 (t473): Several corrections:

- The constant $6z/(\pi i)$ is correct.
- The $3/(\pi y)$ is the *residue* $3/\pi$ of φ at $s = 1$ times $\partial_s y^{1-s}$, not “ φ at a special point”. In Hecke’s ε -language it is $\zeta(1 + 2\varepsilon)$ ’s pole times $I(\varepsilon)$ ’s zero.
- Kronecker’s constant is $2\pi(\gamma - \log 2 - \log \sqrt{y}|\eta|^2)$ for the *full* lattice sum. For E (coprime, $\frac{1}{2}$) divide by $2\zeta(2s)$.
- The anomaly decays only like $1/y$. It sits in the cusp’s zeroth Fourier coefficient, while its $\bar{\partial}$ is the uniform area form.

C114.12 (t474): Several corrections:

- $E(z, \frac{1}{2}) \equiv 0$ and $\varphi(\frac{1}{2}) = -1$ are correct.
- The velocity is $2E^*(z, \frac{1}{2})$, a known object ([RESTATEMENT]), not a new one. It is *not* sign-definite: it changes sign at $y^* = 4\pi e^{-\gamma}$ (Lagarias–Suzuki).
- ζ -zeros are resonances at $s = \rho/2$ ($\Re = \frac{1}{4}$), not Laplace eigenvalues at $\frac{1}{2} + i\gamma$.
- $\frac{1}{4}$ is local curvature; $3/\pi$ is global.

C114.13 (t478): Castelnuovo–Severi is $D^2 \leq 2d_1 d_2$ (Milne Thm. 1.5). “Reverse Minkowski” is loose. Precisely, the orthogonal complement of the hyperbolic plane $\langle f_1, f_2 \rangle$ is negative definite (Hodge index), which is equivalent to reverse Cauchy–Schwarz against $f_1 + f_2$. The bound $|a| \leq 2g\sqrt{q}$ uses the defect form (Cor. 1.6), with $\text{def}(\Delta) = 2g$ and $\text{def}(\Gamma_F) = 2gq$.

C114.14 (t479): The intersection numbers are all correct, and so are the coordinates in Milne’s order $(d_1, d_2) = (q, 1)$. One correction: *dividing by $\sqrt{q}$ does not undo the boost*. $\Gamma_{\text{hyp}} = \sqrt{q} B_{-\frac{1}{2} \log q} \Delta_{\text{hyp}}$; the division normalizes the length and the rapidity $\frac{1}{2} \log q$ remains. It is undone by $q^{-1/2}(F \times 1)^*$ inverse, not by scaling. Finally, “RH $\iff$ one time direction” is a Theorem for curves (Hodge index $\Rightarrow$ RH) and a [RESTATEMENT] for models and for $\mathbb{Q}$.

10. RANKED DIRECTIONS (SCORE = 0.6 sig + 0.4 prob)

	Direction	sig	prob	score
D1	<i>Arithmetic Castelnuovo–Severi in null coordinates.</i> Realize f_1, f_2 as the $\hat{g}(0), \hat{g}(1)$ functionals in Weil’s form. Compute the “transverse Gram” of scaling graphs numerically (L112 machinery) and check that a ghost adds exactly one timelike pair (Bombieri count).	0.90	0.30	0.66
D2	<i>Lagarias–Suzuki as a ghost-birth laboratory.</i> At $y^* = 4\pi e^{-\gamma}$ the constant term $a_0(y, s)$ has a double zero at $\frac{1}{2}$ and the velocity changes sign. Compare quantitatively with L111 Krein collisions (de Bruijn–Newman) and with Weng’s rank-2 zeta.	0.60	0.70	0.64
D3	<i>The anomaly family is φ.</i> Make the [RESTATEMENT] of §5 into a theorem: Maass–Selberg Gram of truncated $E(\cdot, s)$ as the “all- s anomaly” and its relation to L105’s B'' .	0.50	0.65	0.56
D4	<i>Multivariate Lorentzian families from ξ</i> (several shifts or derivatives as variables), whose Lorentzian property is strictly stronger than Turán. Test with the stable ghost recursion.	0.60	0.40	0.52
D5	<i>Klein towers beyond $N = 5$.</i> $X(7)$ (Klein quartic, genus 3): the syzygies of $\text{PSL}_2(\mathbb{F}_7)$ invariants versus E_4, E_6, Δ . What survives of the “vertex form = Δ ” dictionary when $g > 0$?	0.45	0.60	0.51

11. QUESTIONS FOR J.W.

- Q1:** Should the next lemma be D1? This means fixing the explicit test-function space and writing “ $D^2 \leq 2(D \cdot f_1)(D \cdot f_2)$ ” as a numerical inequality on Weil’s form, knowing in advance that it is equivalent to RH.
- Q2:** Is the Klein syzygy dictionary ($F^5 \leftrightarrow \Delta$) the “diagonalization” you had in mind? Or did you mean the discriminant/diagonal sense (§8)?
- Q3:** The velocity $V = 2E^*(\cdot, \frac{1}{2})$ is sign-indefinite. Do you want a truncated or weighted version tested (e.g. $\int V d\mu$ over horocycles $y \leq Y$)?
- Q4:** Gupta–Bleuler / Goddard–Thorn as its own lemma (pointer in §7)?
- Q5:** For the Rankin–Selberg “diagonal”: should a future lemma take the Faltings–Hriljac arithmetic Hodge index theorem as the model, and so work over curves over number fields rather than $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$?

REPRODUCIBILITY

- `num/lemma114_gamma.py` (1 min) $\rightarrow$ `num/l114/gamma.json`.
- `num/lemma114_jensen.py` (stable ghost solve) $\rightarrow$ `jensen.json`.
- `num/lemma114_jensen_bigd.py` $\rightarrow$ `jensen_bigd.json`.
- `num/lemma114_cm.py` $\rightarrow$ `cm.json`.
- `num/lemma114_modular.py` $\rightarrow$ `modular.json`, `tau.json`.
- `num/lemma114_eis.py` (Kronecker, E_2^* , Rankin–Selberg, nodal) $\rightarrow$ `eis.json`.
- `num/lemma114_velocity.py` $\rightarrow$ `velocity.json`.
- `num/lemma114_null.py` $\rightarrow$ `null.json`.
- `num/lemma114_figs.py`.

Never touched: `lemma1.pdf`, `volume*/`, `submit` directories, `num/lemma11_ghost.py`, `briefs/prequantization_brief.md`, `num/lemma34_string.py`, and all earlier builds and PDFs (including L106, L88/89). L109’s `g2grid.npy` was read only.

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TAA LEMMA 115: AN ARITHMETIC NO-GHOST THEOREM? LIGHT-CONE GAUGE, GUPTA-BLEULER, AND THE WEIL FORM

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Over $\mathbb{Q}$ every equivalence with RH below is a *reformulation*. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma115_*.py`, data `num/1115/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma115/verified_refs.md`. Ghosts follow the L103 convention.

ABSTRACT. J.W. asked (t480u, frame t482) whether the physicists’ *no-ghost* mechanism (Gupta–Bleuler, BRST, Goddard–Thorn) gives a route to Weil positivity for $\mathbb{Q}$. Short answer: it gives an exact *dictionary* but no new positivity. Main findings:

- **Over $\mathbb{F}_q$** ([THEOREM], Thm. 2.1). Take a fibre class n (null, $n^2 = 0$). Then $\mathbb{Q} \xrightarrow{1 \mapsto n} \text{NS}_{\mathbb{R}} \xrightarrow{D \mapsto D \cdot n} \mathbb{Q}$ is a complex: $d_1 d_0 = n^2 = 0$, so $Q^2 = 0$ is *exactly nullness*. Its cohomology $H^1 = n^\perp / \langle n \rangle$ is the transverse space. On the Frobenius graphs, its form is *minus the L109 point-count Toeplitz matrix*. Hence $n_+(H^1) = k_r + 2k_e$, the L106/L110 ghost count. [NUMERICAL]: 0 mismatches over 26,698 genus-2 and 405 genus-1 models.
- **Over $\mathbb{Q}$** ([THEOREM] given the explicit formula, Thm. 2.4). The pole terms of Weil’s functional are a hyperbolic plane, $2 \text{Re } \hat{g}(0) \hat{g}(1)$. One subsidiary condition, $\hat{g}(1) = 0$, already kills both pole terms. So $\text{RH} \iff W \leq 0$ on $\{\hat{g}(1) = 0\}$, which is Gupta–Bleuler’s single weak condition. A null state n_ξ with $\hat{n}_\xi(s) = (1-s)\xi(s)$ realises the BRST complex exactly (Prop. 2.5). Scaling acts as a boost on the pole plane and as a rotation on the zeros (Prop. 2.6). [NUMERICAL]: in the hat basis, true primes give $n_+(W) = 1$ and $n_+(W|_{\hat{g}(1)=0}) = 0$; one/two L103 quartets give 3/5 and 2/4.
- **“Critical dimension”** ([THEOREM] for models). $\sum |\alpha_i|^2 - 2gq \geq 0$, with equality $\iff$ RH. This is the Hermitian trace minus the Rosati trace (always $2gq$). It is the only finite-model quantity we found that is *sign-definite and vanishes exactly on the physical side*.
- **Goddard–Thorn is a transfer theorem** ([CITED] Borchers’ sketch). It carries positivity from a light-cone space V that is positive by hypothesis to P^1/N , and $c = 26$ enters only to keep spurious states spurious. An arithmetic no-ghost theorem would need an independently positive transverse model of H^1 . That is Hilbert–Pólya, so the route is a [RESTATEMENT] (§5).
- **Lagarias–Suzuki laboratory** ([CITED]+[NUMERICAL]). For $a_0(y, s)$, Maass–Selberg positivity puts every non-real zero on the line for $y \geq 1$. Off-line zeros are exactly one real pair, which exists iff $y > y^* = 4\pi e^{-\gamma}$. They are born in a collision at $s = \frac{1}{2}$ (exponents 0.49/0.51) of opposite graphical Krein signs. The “critical-dimension” quantity is $\kappa(y) = \log(y^*/y)$, the velocity sign.
- **Frobenius/Verschiebung** (t483–t484; [THEOREM]/[RESTATEMENT]). On the constant term, $\text{Fr}\phi(y) = \phi(py)$ and $\text{Ve}\phi(y) = p\phi(y/p)$ satisfy $\text{Fr Ve} = \text{Ve Fr} = p$ and $T_p = p^{-1/2}(\text{Fr} + \text{Ve})$, so $T_p E = (p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s})E$ (checked to 10^{-30}). $\text{Ve} = \text{Fr}^\dagger$ on y^s iff $\text{Re } s = \frac{1}{2}$. Corrections: the null rays are eigenlines of *both* Fr and Ve, with swapped eigenvalues. For $a_y = y^s + \varphi y^{1-s}$ the centre becomes a *triple* zero at y^* (it is double for Lagarias–Suzuki’s a_0).

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GLOSSARY (FOR A PHYSICIST)

Weil form W : For g on $\mathbb{R}_+$, set $\hat{g}(s) = \int_0^\infty g(x)x^{s-1}dx$ and $f = g * \tilde{g}$, $f(x) = \int g(xy)\overline{g(y)}dy$. Then $\hat{f}(s) = \hat{g}(s)\hat{g}(1-\bar{s})$ and the explicit formula [1] gives $W(g) := \hat{f}(0) + \hat{f}(1) - \sum_\rho \hat{f}(\rho) = (\text{prime} + \text{archimedean side})$. Weil: RH $\iff W(g) \leq 0$ whenever $\hat{g}(0) = \hat{g}(1) = 0$ [1, §V].

Null pair / hyperbolic plane: A 2-plane with form $2uv$ (or $2\text{Re } u\bar{v}$): signature $(1, 1)$, null axes $u = 0, v = 0$.

Ghost: *Physics:* a negative-norm state. *Here:* a wrong-sign direction in the transverse space, coming from an off-line zero (L103 convention: a quartet $\frac{1}{2} \pm \delta \pm i\gamma$ replacing the zero at γ).

k_u, k_r, k_c : Numbers of unit-circle pairs, off-circle real pairs and off-circle quartets of a finite model (L110).

1. THE PHYSICS TEMPLATE

1.1. **Gupta–Bleuler.** [CITED] [13, 8]. One-photon states $|k, \epsilon_\mu\rangle$ have null momentum $k^2 = 0$ and four polarizations $\mu = 0, \dots, 3$. The form $\langle k; \epsilon_\mu | k'; \epsilon_\nu \rangle = -\eta_{\mu\nu} 2|\vec{k}| \delta(\vec{k} - \vec{k}')$ (signature $+- --$) is positive on spatial polarizations and negative on the time-like one. The physical photon has only two (transverse) polarizations.

- The condition $k \cdot \epsilon = 0$ leaves three polarizations: two transverse and one longitudinal, $\epsilon \propto k$. On that subspace the form is only *semidefinite*. Its zero-norm part is exactly the gauge direction.
- Physical space := (3-polarization subspace)/(zero-norm subspace). It is positive definite.
- Multi-photon: the condition is imposed *weakly*, $\langle \chi | \partial^\mu A_\mu | \psi \rangle = 0$, which is not the operator identity $\partial^\mu A_\mu = 0$. String theory uses the same device for L_n , $n > 0$: $L_n|\text{phys}\rangle = 0$ [8, (2.6)].

In light-cone coordinates the four polarizations are a *null pair* (time-like $\pm$ longitudinal) plus two transverse. The condition $k \cdot \epsilon = 0$ removes one member of the null pair. The quotient removes the other, which has become null and orthogonal to everything left.

1.2. **BRST.** [CITED] [9]. The BRST charge is nilpotent, $Q^2 = 0$, and Hermitian, and the pre-physical space is a Krein space. Physical states are $\mathcal{H}_{\text{phys}} = \ker Q / \text{im } Q$. In QED (Lorenz gauge) the Kugo–Ojima quartet is time-like photon, longitudinal photon, ghost and antighost. It is trivial in cohomology: the two unphysical photon polarizations are not in $\ker Q$, and ghost/antighost are Q -exact. In string theory the bc ghosts carry $c = -26$, and Weyl anomaly cancellation forces $D = 26$ free bosons [8, §5.2–5.3]. Tong calls “critical dimension” something of a misnomer: the requirement is *total central charge*.

1.3. **Goddard–Thorn 1972.** [CITED] [4, 2, 14]. *Abstract* [4] (opened via INSPIRE): the counting of states coupling in the Veneziano model at $d = 26$ ($d = 10$ for Neveu–Schwarz) shows that “a simple consequence of this counting is the absence of ghosts for $d \leq 26$ ($d \leq 10$)”. *Borcherds’ form* [2, Thm. 5.1]. Let V carry a Virasoro action with $L_i^\dagger = L_{-i}$ and central charge 24, with finite-dimensional L_0 -eigenspaces V^i , $i \geq 0$. Let $P^1 \subset V \otimes V_{II_{1,1}}$ be the weight-1 primaries ($L_0 v = v$, $L_i v = 0$ for $i > 0$), and P_r^1 those of momentum $r \in II_{1,1}$. Then $P_r^1 / \text{null} \cong V^{1-(r,r)/2}$ for $r \neq 0$. *Proof structure* (Borcherds’ sketch “taken from Goddard and Thorn”):

- (1) Fix a **norm-zero** $w \in II_{1,1}$ with $(r, w) \neq 0$. Build K_i (DDF-type operators) from $e_{-2}^{-w} e^w$. They satisfy $[L_i, K_j] = -j K_{i+j}$ and $[K_i, K_j] = 0$ *because w has norm 0*.
- (2) Spaces: P (primaries), S (spurious $= P^\perp$), $N = S \cap P$ (radical), T (transverse: P killed by K_i , $i > 0$), K generated from T by the K_i .
- (3) Lemmas 5.1–5.3: T is nondegenerate and $T \cong V e^r$ (light-cone states). Lemma 5.4: the algebra of $L_{i < 0}$ is generated by maps sending S^1 into S , namely L_{-1} and $L_{-2} + \frac{3}{2} L_{-1}^2$. **This is the only place $c = 26$ ($= 24 + 2$) is used.** Lemma 5.5: $P^1 = T^1 \oplus N^1$.
- (4) **Mass shell:** $L_0 = 1$ (weight 1, intercept $a = 1$ [8, §2.1.2]). It enters via $V^{1-(r,r)/2}$, i.e. level matching.

Key point ([RESTATEMENT] of the source). The theorem **transfers** positivity. If V is positive (in GT, V comes from a positive-definite lattice), then P_r^1 / N has no ghosts. It never manufactures positivity.

1.4. **Why massless is special.** [CITED] [8, pp. 41–42]. Massive particles form representations of the little group $SO(D-1)$; massless ones only of $SO(D-2)$. So a massless vector has $D-2$ states (the photon has 2), while a massive spin-1 particle has 3. In light-cone gauge only $D-2$ transverse oscillators remain, and the Hilbert space is positive “by construction” [8, §2.2]. The $(D-2)$ first excited states fit into a Lorentz representation only if they are massless, which forces $D = 26$. *Arithmetic reading* ([ANALOGY]): null momentum is what makes $k \cdot \epsilon = 0$ contain $\epsilon = k$. Without nullness the gauge direction is not inside its own orthogonal complement and the quotient trick fails (Thm. 2.1(i)).

2. THE ARITHMETIC DICTIONARY

2.1. **Over $\mathbb{F}_q$: a solved no-ghost theorem.** Let $C/\mathbb{F}_q$ have genus g . Write $f_1 = \{\text{pt}\} \times C$, $f_2 = C \times \{\text{pt}\}$ and Γ_j = graph of F^j ($\Gamma_0 = \Delta_C$). Then $f_i^2 = 0$, $f_1 \cdot f_2 = 1$, $\Gamma_j \cdot f_1 = 1$, $\Gamma_j \cdot f_2 = q^j$, $\Gamma_j^2 = q^j(2-2g)$, and $\Gamma_j \cdot \Gamma_k = q^{\min(|j-k|, 1-a_{|j-k|})}$ for $j \neq k$, where $a_m = q^m + 1 - N_m$ [6] (L114).

Theorem 2.1 ([THEOREM]). *Let $n \in \text{NS}(C \times C)_{\mathbb{R}}$ and define $d_0 : \mathbb{R} \rightarrow \text{NS}_{\mathbb{R}}$, $1 \mapsto n$, and $d_1 : \text{NS}_{\mathbb{R}} \rightarrow \mathbb{R}$, $D \mapsto D \cdot n$.*

- (i) $d_1 d_0 = n^2$. *So $(\mathbb{R} \rightarrow \text{NS}_{\mathbb{R}} \rightarrow \mathbb{R})$ is a complex iff n is null. For $h = f_1 + f_2$ ($h^2 = 2$, ample type), i.e. the hard-Lefschetz direction, it is not a complex.*
- (ii) *For $n = f_2$ (or f_1), $H^1 = n^\perp / \langle n \rangle$ is isometric to the transverse space $\langle f_1, f_2 \rangle^\perp$. On the classes Γ_j its Gram matrix is*

$$T_{jk} = -q^{\min(j,k)} a_{|j-k|} \quad (a_0 = 2g), \quad T = -\text{diag}(q^{j/2}) [\sigma_{|j-k|}] \text{diag}(q^{k/2}),$$

with $\sigma_m = q^{-m/2} a_m$, the L109 point-count Toeplitz sequence.

(iii) Hence (L110 Thm. 2.3, L112 Thm. 2.1) $n_+(H^1|_\Gamma) = k_r + 2k_c$ for distinct roots. H^1 is negative semidefinite (“no ghost”) iff RH for the model. For curves this holds by the Hodge index theorem [16, 1].

Proof. (i) is immediate. (ii) $\Gamma_j - (\Gamma_j \cdot f_1)f_2 - (\Gamma_j \cdot f_2)f_1$ is orthogonal to both f_i . Its products give $\Gamma_j \cdot \Gamma_k - q^j - q^k = -q^{\min} a_{|j-k|}$ (for $j = k$: $q^j(2 - 2g) - 2q^j$). Any $D \in n^\perp$ is $D = un + D_\perp$ with $D^2 = D_\perp^2$, independent of u . (iii) follows from (ii) and the cited inertia theorems. $\square$

Exact checks ([NUMERICAL], sympy, `finite.json`): for $(g, q) = (1, 101)$ with $a = 10$ and 25, and for $(2, 11)$ with $(s_1, s_2) = (1, 10), (8, 30), (0, -30)$: $d_1 d_0 = 0$, transverse Gram = T exactly, and full inertia = H^1 inertia + $(1, 1)$.

Remark 2.2 (Which ingredient plays BRST/Virasoro over $\mathbb{F}_q$?). [RESTATEMENT]. The *BRST structure* (null n , $Q^2 = 0$, quotient) is linear algebra and needs only a null class: the fibres. Hard Lefschetz ([CITED] Deligne 1980 for ℓ -adic [17]) gives the $\mathfrak{sl}_2$ decomposition $h^\perp \oplus \mathbb{R}h$ with h ample. That splits off one *time-like* direction, not a null pair, and gives no sign. The *positivity* input, playing the role of “ V positive” in Goddard–Thorn, is the algebraic index theorem. Bombieri [1]: it is essentially due to Severi (1906); the proof uses Riemann–Roch on the surface and finiteness of families of curves; no other algebraic proof is known. Hodge’s analytic proof covers characteristic 0 only. Equivalently, it is Rosati positivity $\text{Tr}(xx^\dagger) > 0$ (L113 Thm. 2.1). The answer to the brief’s “ample class / Lefschetz $\mathfrak{sl}_2$?” is therefore *no*: Lefschetz is the gauge-fixing, and Riemann–Roch/Severi is the positivity.

Proposition 2.3 (“Critical-dimension” identity; [THEOREM]). Let $\{\alpha_i\}_{i=1}^{2g}$ be the roots of a real Weil-type polynomial (multiset stable under $\alpha \mapsto q/\alpha$ and $\alpha \mapsto \bar{\alpha}$). Then

$$\sum_i |\alpha_i|^2 - 2gq \geq 0, \quad \text{with equality} \iff |\alpha_i| = \sqrt{q} \ \forall i.$$

Here $\sum \alpha_i \cdot (q/\alpha_i) = 2gq$ always; it is the trace of $FF^\dagger$ for the Rosati involution, since $FF^\dagger = q$ [7, Lemma 1.2, p. 76]. So the defect is (Hermitian trace) – (Rosati trace).

Proof. The multiset is stable under $\iota : \alpha \mapsto q/\bar{\alpha}$. So $\sum |\alpha|^2 = \frac{1}{2} \sum (|\alpha|^2 + q^2/|\alpha|^2) \geq \frac{1}{2} \cdot 2g \cdot 2q$ by AM–GM, with equality iff $|\alpha|^2 = q$. $\square$

[NUMERICAL]: zero mismatches of “defect = 0 $\iff$ RH” on 26,698 distinct genus-2 models ($q = 11$). The minimum ghost defect is 0.166 and true models give $\leq 2.7 \cdot 10^{-13}$. For genus 1, $q = 101$, the minimum ghost defect is 37 ($a = 21$). [ANALOGY]: the defect plays the role of $D - 26$. It is ≥ 0 automatically, and “no ghosts” is the borderline = 0, unlike GT’s open range $d \leq 26$.

2.2. Over $\mathbb{Q}$: the pole plane, one subsidiary condition, a null state.

Theorem 2.4 ([THEOREM] given Weil’s explicit formula and criterion [1]). For g in Bombieri’s class,

$$W(g) = 2 \text{Re}(\hat{g}(0)\overline{\hat{g}(1)}) - \sum_\rho \hat{g}(\rho)\overline{\hat{g}(1-\bar{\rho})}.$$

The pole part is a hyperbolic plane in the null coordinates $u = \hat{g}(0)$, $v = \hat{g}(1)$. Moreover

$$RH \iff W(g) \leq 0 \text{ for all } g \text{ with } \hat{g}(1) = 0 \iff W(g) \leq 0 \text{ for all } g \text{ with } \hat{g}(0) = 0.$$

Proof. $\hat{f}(0) = \hat{g}(0)\overline{\hat{g}(1)}$ and $\hat{f}(1) = \hat{g}(1)\overline{\hat{g}(0)}$, so a single condition $\hat{g}(1) = 0$ kills both pole terms. ($\Rightarrow$) Under RH, $1 - \bar{\rho} = \rho$, so $W(g) = -\sum |\hat{g}(\rho)|^2 \leq 0$. ($\Leftarrow$) If RH fails, Weil’s criterion gives g with $\hat{g}(0) = \hat{g}(1) = 0$ and $W(g) > 0$; this g satisfies $\hat{g}(1) = 0$. $\square$

This is Gupta–Bleuler exactly. One weak condition on one null coordinate suffices, and the other null coordinate $\hat{g}(0)$ drops out of the form. Bombieri’s second condition is redundant for the equivalence. We claim no priority: it is an immediate consequence of the cited formulae.

Proposition 2.5 (A BRST null state for $\mathbb{Q}$; [THEOREM]/[RESTATEMENT]). *Let n_ξ be the inverse Mellin transform of $(1-s)\xi(s)$, with $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Then n_ξ is smooth and $O(x^N)$ at 0 and ∞ for every N , because ξ decays exponentially on vertical lines. It satisfies $\hat{n}_\xi(1) = 0$, $\hat{n}_\xi(0) = \xi(0) = \frac{1}{2}$ and $\hat{n}_\xi(\rho) = 0$ for all ρ . Hence $W(n_\xi) = 0$ and $W(g, n_\xi) = \frac{1}{2}\hat{g}(1)$. So $\mathbb{C} \xrightarrow{1 \mapsto n_\xi} \mathcal{W} \xrightarrow{g \mapsto W(g, n_\xi)} \mathbb{C}$ is a complex ($Q^2 = 0$) with $H^1 = \{\hat{g}(1) = 0\}/\mathbb{C}n_\xi$, and $RH \iff -W$ is positive semidefinite on H^1 (Thm. 2.4). The construction uses only ξ and is non-circular. The positivity is RH itself ([RESTATEMENT]).*

Proposition 2.6 (Scaling = boost $\oplus$ rotation; [THEOREM]). *Let $(U_\lambda g)(x) = \lambda^{1/2}g(\lambda x)$. Then $\widehat{U_\lambda g}(s) = \lambda^{\frac{1}{2}-s}\hat{g}(s)$, $U_\lambda g * \widehat{U_\lambda g} = g * \hat{g}$, and $W(U_\lambda g) = W(g)$. The Weil form is scaling-invariant; this is why it is Toeplitz in the hat basis (L112). On the pole plane, $(u, v) \mapsto (\lambda^{1/2}u, \lambda^{-1/2}v)$ is a boost of rapidity $\frac{1}{2}\log \lambda$. On an on-line zero $\rho = \frac{1}{2} + i\gamma$ it acts by the phase $\lambda^{-i\gamma}$ (a rotation). On a ghost $\rho = \frac{1}{2} + \delta + i\gamma$ it acts as a boost $\lambda^{\mp\delta}$ on the pair $(\rho, 1 - \bar{\rho})$. This is the $\mathbb{Q}$ -version of L114’s “ $q^{-1/2}(F \times 1)^* = \text{boost on the hyperbolic plane, rotation on the transverse space}$ ”, with $\lambda = q$ giving rapidity $\frac{1}{2}\log q$ (C114.14).*

Remark 2.7 ($\mathfrak{sl}_2$ / Virasoro / Hecke? [OPEN]). The symmetry we can verify is the abelian scaling group (Prop. 2.6) together with $x \mapsto 1/x$, which comes from the functional equation. The oscillator/metaplectic $\mathfrak{sl}_2$ (TAA §3 in STATE; Mp_2 cocycle and Mehler sign, [TAA-L]) is spanned by x^2 , ∂_x^2 and $x\partial_x + \frac{1}{2}$ on $L^2(\mathbb{R})$. Only the third generator preserves W ; we found no reason for the other two to be symmetries. Hecke operators act on the modular side (§4), not on W . *Dimension count.* GT ’s count matches $\dim P_r^1/N$ with $\dim V^{1-(r,r)/2}$ level by level. The finite-field analogue $\dim H^1 = \rho - 2$ is trivially true and signature-blind. For $\mathbb{Q}$ there is no grading with finite-dimensional pieces on which to count. Riemann–von Mangoldt counts zeros but cannot see their real parts. An arithmetic central charge would have to be an *anomaly whose cancellation forces the decoupled null sector to leave a positive quotient*. Over $\mathbb{Q}$ the null sector already decouples for free (Thm. 2.4). So whatever plays “ $c = 26$ ” is not where the difficulty lies; the difficulty is the missing positive V (§5).

3. NUMERICS

3.1. Finite models. [NUMERICAL] (lemma115_finite.py, 60-digit inertia). **Genus 1**, $q = 101$, all $|a| \leq 2q$ (405 models): $n_+(H^1) = k_r + 2k_c$ in all cases. The 41 true models ($|a| \leq 20$) have $n_+ = 0$ and every ghost has $n_+ \geq 1$. Example $a = 25$: H^1 inertia $(1, 1, 2)$, defect 221. **Genus 2**, $q = 11$, the L109 grid of 26,730 polynomials (26,698 with distinct roots): $n_+(H^1) = k_r + 2k_c$ with **0 mismatches**. True models have inertia $(0, 4, 2)$, i.e. rank $4 = 2g$, with 2 null from linear dependence. Ghosts split as $n_+ = 1$: 15,100 (mixed) and $n_+ = 2$: 11,211 (two real pairs or a quartet), matching L110’s census. All 2935 count-positive ghosts have $n_+(H^1) > 0$. A first version flagged 13 mismatches. All were Weil-true polynomials with *repeated* roots on the circle, mislabelled by floating-point root-finding; they are now excluded by an exact discriminant test. *Which physical-state condition does a ghost violate?* It is not $D \cdot n = 0$, which every class can satisfy after subtracting multiples of f_1, f_2 . A ghost violates *positivity of the quotient*. Equivalently it violates the “mass-shell” $|\alpha| = \sqrt{q}$, i.e. the critical identity of Prop. 2.3, or “ $F^* = F^\dagger$ ” (Hermitian adjoint = Rosati adjoint).

3.2. Truncated Weil form for $\mathbb{Q}$ with the pole plane. [NUMERICAL] (lemma115_weil.py; L112 hat basis, $h = 0.025$, primes $\leq e^4$). Write Q_M for L112’s Toeplitz zero-side matrix computed from the prime side (zero vs prime agreement $1.8 \cdot 10^{-5}$). With $c = h(\sinh(h/4)/(h/4))^2$, the hats have $\hat{g}_j(0) = ce^{-jh/2}$ and $\hat{g}_j(1) = ce^{jh/2}$. So the pole matrix is $uv^T + vu^T$ exactly, and $W_M = uv^T + vu^T - Q_M$ is Bombieri’s prime/archimedean side. On $v^\perp$ the pole form vanishes to $7 \cdot 10^{-18}$.

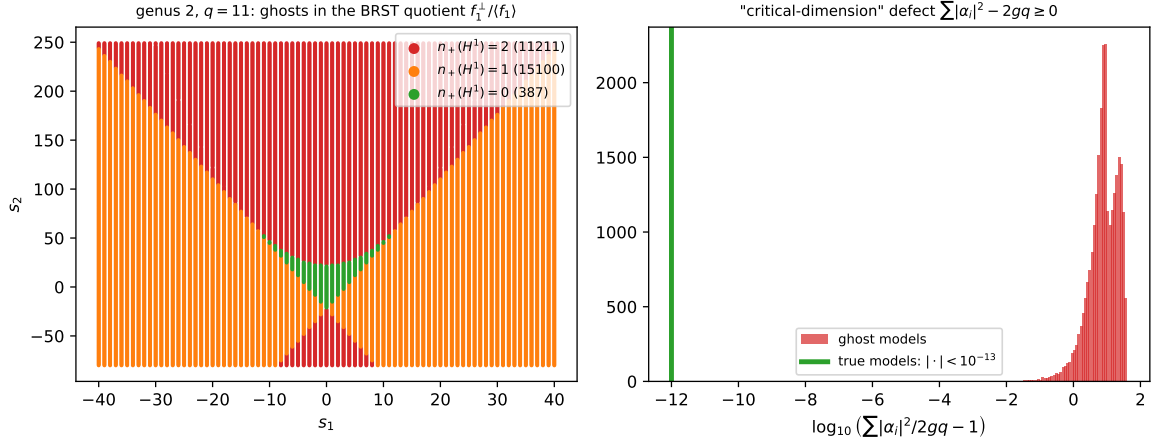


FIGURE 1. Left: genus 2, $q = 11$. Ghost number n_+ of the BRST quotient $f^\perp/\langle f \rangle$ (green = true). Right: the critical-dimension defect $\sum |\alpha_i|^2 - 2gq$ is 0 on true models and ≥ 0.166 on ghosts.

world ($L = 4, M = 160$)	$n_-(Q_M)$	$n_+(W_M)$	$n_+(W _{\hat{g}(1)=0})$	$n_+(W _{\hat{g}(0)=\hat{g}(1)=0})$
true primes	0	1	0	0
1 quartet $(\gamma_1, 0.3), (\gamma_1, 0.49), (\gamma_2, 0.2)$	2	3	2	2
2 quartets	4	5	4	4

True: $n_+(W_M) = 1$ from $L = 0.4$ on (the time-like direction of the pole plane), and 0 on $\{\hat{g}(1) = 0\}$ at every L . *Masking at onset:* at $L = 1.0$, $(\gamma_1, 0.3)$ already has $n_-(Q) = 2$ but $n_+(W) = 1$ and $n_+(W|_{v^\perp}) = 1$. The time-like pole direction temporarily absorbs a ghost direction, as Cauchy interlacing permits. At $L \geq 1.5$ the count is the asymptotic $1 + 2k$.

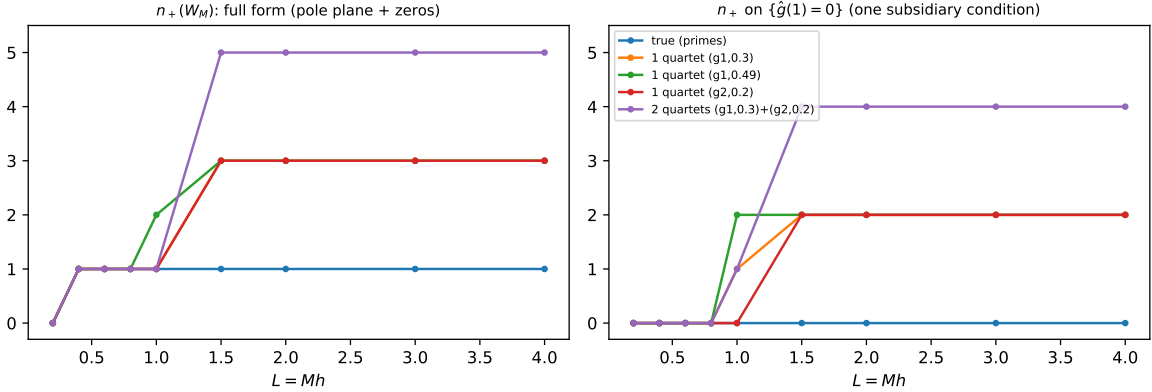


FIGURE 2. Positive index of the truncated Weil form: full (left) and under the single subsidiary condition $\hat{g}(1) = 0$ (right). True primes: 1 and 0.

3.3. Lagarias–Suzuki laboratory. Source [CITED] [5, Thm. 3]. For each $y \geq 1$, $a_0(y, s) = \zeta^*(2s)y^s + \zeta^*(2-2s)y^{1-s}$ satisfies the modified RH, with critical value $y^* = 4\pi e^{-\gamma} = 7.055507\dots$: (1) all zeros lie on the critical line for $1 \leq y \leq y^*$; (2) for $y > y^*$ there are *exactly two* zeros off the line. They are real and simple, $\rho_y, 1 - \rho_y$ with $\frac{1}{2} < \rho_y < 1$, and ρ_y is nondecreasing with $\rho_y \rightarrow 1$.

Hejhal: for $0 < y < 1$ there are complex off-line zeros. *Mechanism* (LS Lemma 1, Hejhal): the Maass–Selberg relation $(s - \bar{s})(1 - s - \bar{s}) \int |E_T^*|^2 = \dots$. Its left side is a genuine norm, nonzero off the line *unless s is real* ($s - \bar{s} = 0$). So positivity confines non-real zeros to the line and is blind to real zeros. LS Lemma 3: $F'(y, \frac{1}{2}) = 0$ exactly at y^* , and $G(y, \frac{1}{2}) \propto (\log 4\pi - \gamma - \log y)\sqrt{y}$. (Normalization note: LS’s (44) and (45) differ by a factor 2. We use (45), $H = (s - 1)\xi(2s)y^s + s\xi(2s - 1)y^{1-s}$; numerically $H/(s - \frac{1}{2})|_{1/2} = \frac{1}{2}(\log 4\pi - \gamma - \log y)\sqrt{y}$. Zero sets are unaffected.)

Numerics [NUMERICAL] (lemma115_ls.py). Real zeros in $(\frac{1}{2}, 1)$: none for $y \in \{1, 2, 4, 6, 7, 7.05\}$, and one for $y = 7.06$ (0.51544), 7.5 (0.64732), 8 (0.70550), 10 (0.81337), 20 (0.93305), 100 (0.98959), 1000 (0.99903). Argument principle on $0.52 \leq \operatorname{Re} s \leq 1.6$, $|\operatorname{Im} s| \leq 30$: 0 zeros for $y = 0.9, 1, 3, 7$; 1 for $y = 7.2, 8, 20$; 12 for $y = 0.5$ (Hejhal’s regime). For $y = 0.9$ there are none in this box. **Birth at the centre.** For $y = y^*(1 - \epsilon)$ the lowest on-line zero has $t_1 \sim \epsilon^{0.51}$. For $y = y^*(1 + \epsilon)$, $\rho_y - \frac{1}{2} \sim \epsilon^{0.49}$. This is a square-root collision at $s = \frac{1}{2}$. **What plays the critical dimension?** [ANALOGY]/[THEOREM]: $\kappa(y) = \log(y^*/y) = (\text{velocity constant term})/(-2\sqrt{y})$ (L114). The toy has *two* conditions:

- $y \geq 1$ (“unitarity”: Maass–Selberg applies); it closes the complex channel;
- $\kappa \geq 0$ (“critical”); it closes the real channel.

For ζ the real channel is closed (no zeros in $(0, 1)$, L110 Prop. 4.1). What is missing is a Maass–Selberg-type norm closing the complex channel ([HEURISTIC]).

4. FROBENIUS, VERSCHIEBUNG AND THE CUSP (T483–T484)

1. Algebraic ([CITED]). Dieudonné modules: $D = W(k)\{F, V\}/(FV - p)$, $FV = VF = p$ [11]. For an abelian variety over $\mathbb{F}_q$ and a polarization, the Rosati involution satisfies $\pi \circ \pi^\dagger = q$ [7, Lemma 1.2]. Hence $\pi^\dagger = q\pi^{-1} = V$: *the Rosati adjoint of Frobenius is Verschiebung*. Modular forms [15]: T_m acts on coefficients by $b(n) = \sum_{d|(m,n)} d^{k-1}a(mn/d^2)$. For $m = p$ this is $T_p = U_p + p^{k-1}V_p$ with $U_p : \sum a(n)q^n \mapsto \sum a(pn)q^n$ and $V_p : f(\tau) \mapsto f(p\tau)$. Then $U_p V_p = \text{id}$, but $V_p U_p \neq \text{id}$ (it projects onto $p \mid n$). [NUMERICAL]: $T_p \Delta = \tau(p)\Delta$ via this formula for $p = 2, 3, 5, 7$ (to $n \leq 200/133/80/57$), and $U_p V_p = \text{id}$. At the cusp V_N is $y \mapsto Ny$: a scaling, i.e. a boost of rapidity $\frac{1}{2} \log N$ in the normalization of Prop. 2.6.

2. Eisenstein constant term ([THEOREM]). Take $T_p f(z) = p^{-1/2}[f(pz) + \sum_{b=0}^{p-1} f(\frac{z+b}{p})]$. On functions of y alone set $\text{Fr } \phi(y) = \phi(py)$ and $\text{Ve } \phi(y) = p\phi(y/p)$ (the constant term of the b -average). Then

$$\text{Fr Ve} = \text{Ve Fr} = p, \quad T_p|_{\text{const}} = p^{-1/2}(\text{Fr} + \text{Ve}),$$

the exact analogue of $F + V = a_p$, $FV = p$ for an elliptic curve. On y^s : $(\text{Fr}, \text{Ve}) = (p^s, p^{1-s})$. On y^{1-s} : (p^{1-s}, p^s) . Both give $T_p = p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s} = \alpha + \beta$ with $\alpha\beta = 1$ (Satake). [NUMERICAL]: $T_p E(z, s)/E(z, s) = p^{s-1/2} + p^{1/2-s}$ to relative error $\leq 2 \cdot 10^{-30}$ for $p = 2, 3, 5$ and $s = 0.8, \frac{1}{2} + 7.3i, 0.31 + 2.2i$, using the Wikipedia Fourier expansion [12] (S-invariance $\leq 6 \cdot 10^{-31}$). *Correction to the chat claim* (C115.9). The two null rays are *not* “the F-line and the V-line”. Each null ray is an eigenline of both Fr and Ve , and S ($s \leftrightarrow 1 - s$) swaps the *eigenvalue assignments*. The Fr/Ve split is between the two pieces of T_p . [ANALOGY]: null rays $\leftrightarrow$ the two orderings (α, β) of the Satake pair.

3. Rosati and the critical line ([RESTATEMENT], exact). $FV = q$ always holds and is ghost-blind (L113 Thm. 5.1). For a finite model, $\text{RH} \iff$ there is a *positive* involution with $F^\dagger = V$: ($\Leftarrow$) is L113 Thm. 2.1; ($\Rightarrow$) take a metric making $F/\sqrt{q}$ unitary. For curves the Rosati involution supplies it. On the Eisenstein side, Ve on y^s equals $\overline{\text{Fr}}$, i.e. $p^{1-s} = \overline{p^s}$, iff $1 - s - \bar{s} \in \frac{2\pi i}{\log p} \mathbb{Z}$, i.e. iff $\operatorname{Re} s = \frac{1}{2}$ (the left side is real). [NUMERICAL]: $s = \frac{1}{2} + 7.3i$, $p = 3$: $\text{Fr} = -0.28601 + 1.70827i$, $\text{Ve} = \overline{\text{Fr}}$, $\text{Fr Ve} = 3$. On the line $1 - s = \bar{s}$, so the functional-equation reflection coincides with complex conjugation; Hardy’s $Z(t)$ is real there [18]. *Statement*: the critical line is exactly where “Verschiebung = adjoint of Frobenius”. This restates Ramanujan for $E(\cdot, s)$ (tempered $\iff \operatorname{Re} s = \frac{1}{2}$), not RH for ζ . *What*

it opens: the L110 split-complex E and the Krein reading (L111) can be phrased with (Fr, Ve), i.e. $E = \text{sgn} \log |\text{Fr}/\text{Ve}|^{1/2}$ on non-tempered lines.

4. J.W.'s graph: the centre zero ([THEOREM]+[NUMERICAL]). Write $a_y(s) = y^s + \varphi(s)y^{1-s}$ with $\varphi = \Lambda(2s-1)/\Lambda(2s) = -s\xi(2s-1)/((1-s)\xi(2s))$, regular at $\frac{1}{2}$. Then $\varphi(\frac{1}{2}) = -1$ (numerically -1.0), so $a_y(\frac{1}{2}) = 0$ for all y . Also $a'_y(\frac{1}{2}) = 2\sqrt{y}(\log y + \gamma - \log 4\pi)$, the velocity constant term (checked: -2.60378 at $y^*/2$, 5.20757 at $2y^*$). At $y = y^*$: $a'(\frac{1}{2}) \approx 10^{-31}$, $a''(\frac{1}{2}) \approx 5 \cdot 10^{-31}$ and $a'''(\frac{1}{2}) = -85.0085$. So the centre is a **triple** zero of a_y : the fixed zero plus the colliding pair $\frac{1}{2} \pm it_1$. In LS's $a_0 = \Lambda(2s)a_y$, the pole of $\Lambda(2s)$ at $\frac{1}{2}$ cancels the fixed zero, which is why LS speak of a *multiple (double)* zero at y^* . For $y > y^*$ a real pair splits off: a_y has real zeros 0.68131 at $1.1y^*$ and 0.89201 at $2y^*$, agreeing with ρ_y . *Side and number* ([CITED] LS Thm. 3): off-line zeros occur for $y > y^*$, exactly two, real. (For $0 < y < 1$: Hejhal, complex.) *Krein reading* ([THEOREM]+[NUMERICAL]). $G(y, \frac{1}{2} + it)$ is real and even in t , so the graphical Krein signs $\text{sgn } \partial_t G$ at $\pm t_1$ are opposite. This is forced, and verified at $y/y^* = 0.5, 0.9, 0.99$ (-1 at $+t_1$, $+1$ at $-t_1$). It is L111's opposite-sign collision, here of real-pair type (k_r), the type ζ itself cannot have. *Verschiebung drives the birth* ([ANALOGY]). $V_N : y \mapsto Ny$ translates $\log y$ by $\log N$. The no-ghost window $[1, y^*]$ has log-width $\log y^* = 1.9538$. So V_2 pushes 35% of it past y^* and V_8 pushes all of it (V_7 : 99.6%). But V_N is a *parameter flow* on the family a_y , not a symmetry: it is not Γ -invariant, which is why T_N averages it with Ve.

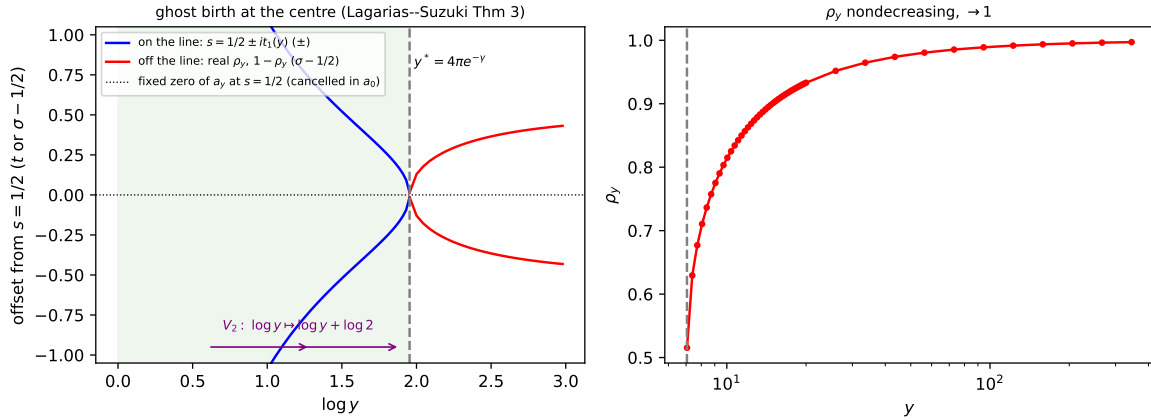


FIGURE 3. Zeros of the Lagarias–Suzuki constant term near $s = \frac{1}{2}$. Blue: on-line pair $\frac{1}{2} \pm it_1(y)$. Red: the real pair $\rho_y, 1 - \rho_y$ born at y^* . Dotted: fixed zero of a_y . Green band: the no-ghost window $1 \leq y \leq y^*$. Purple: V_2 . Right: $\rho_y \uparrow 1$.

5. p -adic side ([THEOREM]; L113). $T^2 - 33T + 11$ over $\mathbb{F}_{11}$ has 11-adic slopes $\frac{1}{2}, \frac{1}{2}$ (Newton polygon from $(0,1), (2,0)$), i.e. F and V are symmetric: supersingular-type. Yet its roots 32.663 and 0.33677 give $|\alpha|^2/q = 96.99$ and 0.0103 , a real ghost pair ($k_r = 1$). F/V symmetry at p is slope symmetry, not positivity. Only the archimedean modulus sees the ghost.

5. VERDICT

- (1) [THEOREM] The Gupta–Bleuler/BRST *structure* exists exactly: over $\mathbb{F}_q$ (Thm. 2.1; $Q^2 = 0 \iff$ fibre null) and over $\mathbb{Q}$ (Thm. 2.4, Prop. 2.5).
- (2) [RESTATEMENT] “No ghosts in H^1 ” $\iff$ RH, in both settings. Over $\mathbb{F}_q$ the positivity comes from Severi/Riemann–Roch (or Rosati), *not* from the BRST structure and not from Lefschetz $\mathfrak{sl}_2$.

- (3) [RESTATEMENT]/[ANALOGY] Goddard–Thorn’s mechanism is a *transfer*: positivity of a light-cone model V plus an isomorphism $V \cong P^1/N$ (dimension count, $c = 26$). For $\mathbb{Q}$ the null sector already decouples (Thm. 2.4). The missing input is a positive V with $V \cong H^1$, which is the Hilbert–Pólya/Connes problem. **The no-ghost route is a restatement, not a non-circular route**, unless such a V is built independently.
- (4) [OPEN] An arithmetic “critical dimension” would have to be a sign-definite quantity that vanishes exactly when no ghosts exist (models: $\sum |\alpha|^2 - 2gq$, Prop. 2.3). Equivalently it is an anomaly whose cancellation forces Hermitian = Rosati. The $\mathbb{Q}$ -analogue would be $\sum_\rho |\hat{g}(\rho)|^2 - \sum_\rho \hat{g}(\rho)\hat{g}(1-\bar{\rho}) \equiv 0$, which is again RH. In the LS toy it is $\kappa(y) = \log(y^*/y)$.
- (5) [HEURISTIC] The family argument J.W. wants (transport positivity from $s = 1$ to $s = \frac{1}{2}$) works in the LS toy *only because* Maass–Selberg is a genuine norm for every $y \geq 1$. That is the template: a one-parameter family of honest norms, blind only to a channel closed by another argument.

6. CORRECTIONS (T479–T484 CHAT CLAIMS)

- C115.1:** “Dividing by $\sqrt{q}$ undoes the boost”: **no**. See L114 C114.14: the division fixes the norm and the rapidity $\frac{1}{2} \log q$ remains. Prop. 2.6 gives the $\mathbb{Q}$ -version: unitary scaling by λ is a boost of rapidity $\frac{1}{2} \log \lambda$.
- C115.2:** Gupta–Bleuler: the null pair is not removed by one condition. $k \cdot \epsilon = 0$ removes the time-like combination. The longitudinal one survives as a zero-norm state and is quotiented out. The condition is weak ($\langle \chi | \partial A | \psi \rangle = 0$), not $\partial A = 0$ [13].
- C115.3:** Goddard–Thorn: the published claim is the absence of ghosts for $d \leq 26$ [4]. In Borchers’ proof sketch, $c = 26$ is used in Lemma 5.4 (spurious stays spurious) and the null vector w makes $[K_i, K_j] = 0$. The mass shell is $L_0 = 1$. The light-cone “Lorentz invariance $\Rightarrow D = 26$ ” argument [8] is a different derivation. That ghosts appear for $d > 26$ is standard but was not verified from an opened source here.
- C115.4:** “BC high-temperature phase” ([ANALOGY]). Bost–Connes: $Z(\beta) = \zeta(\beta)$, a unique KMS state for $0 < \beta \leq 1$, and symmetry breaking for $\beta > 1$ [3, 10]. β is a real inverse temperature. The transition at $\beta = 1$ is the pole of ζ , which matches the cusp’s $s = 1$ residue $3/\pi$ (the null/anomaly part), not the line $\operatorname{Re} s = \frac{1}{2}$. Any link to transverse positivity is analogy only.
- C115.5:** t482 frame. “ y^s and φy^{1-s} are the null pair, swapped by S ”: S acts on s (functional equation $E(z, s) = \varphi(s)E(z, 1-s)$), not as $z \mapsto -1/z$ on the two terms. “ $3/\pi$ null part, Kronecker first transverse term”: [RESTATEMENT] as in L114. “RH is transverse positivity on the critical line”: yes, precisely Thm. 2.4.
- C115.6:** “Lefschetz $\mathfrak{sl}_2$ is the BRST operator over $\mathbb{F}_q$ ”: not so. Lefschetz splits off a time-like h ($h^2 > 0$), not a null pair, and gives no sign (Rem. 2.2).
- C115.7:** “A dimension count could give positivity”: counts are signature-blind. GT’s count needs V positive as input.
- C115.8:** (t484.2) “Hecke eigenvalue $p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s}$ ”: correct, in the $p^{-1/2}$ -normalized T_p .
- C115.9:** (t484.2) “The two null rays are the F and V eigenlines”: corrected in §4.2. Both rays are eigenlines of both operators, and S swaps the eigenvalues.
- C115.10:** (t484.3) “RH $\iff V = F^\dagger$ in a positive metric”: correct as a [RESTATEMENT] for models. Note that $V = F^\dagger$ holds automatically for the Rosati involution, so the content is its positivity. On the Eisenstein side “ $\operatorname{Ve} = \operatorname{adjoint}$ ” $\iff \operatorname{Re} s = \frac{1}{2}$ is Ramanujan for E , not RH for ζ .
- C115.11:** (t484.4) “ $s = \frac{1}{2}$ becomes a double zero at y^* ”: for $a_y = y^s + \varphi y^{1-s}$ it is **triple** ($a' = a'' = 0$, $a''' = -85.0085$); it is double for LS’s a_0 . The off-line zeros are on the side $y > y^*$, exactly two, real. Also complex ones for $0 < y < 1$ (Hejhal).
- C115.12:** (t484.4) “Verschiebung drives the birth”: as a parameter flow, yes ([ANALOGY]). It is not a symmetry of the zero set.

7. RANKED DIRECTIONS

- (1) **Maass–Selberg as the transverse norm** (0.62). Find a truncated-Eisenstein/Lax–Phillips family of honest norms whose blind channel for ζ is the real one (already closed). L105 resonances at $\rho/2$ are the natural place.
- (2) **Arithmetic Castelnuovo–Severi with genuine null classes** (0.56). Find a surface-like object where f_1, f_2 are classes, not functionals (L114 dir. 1; Connes–Consani).
- (3) **Hermitian-minus-Rosati trace for $\mathbb{Q}$** (0.52). Find a regularized version of Prop. 2.3 built from primes, to test whether any piece is sign-definite unconditionally.
- (4) **Fr/Ve Krein structure on non-tempered Eisenstein lines** (0.48). Do this via $E = \text{sgn log |Fr/Ve|}$; it links L110/L111 to the LS toy.
- (5) **Borcherds-type input** (0.34). Is there any $c = 24$ -like positive module whose character involves ζ ? Speculative.

8. QUESTIONS FOR J.W.

- Q1:** Is Thm. 2.4 (one subsidiary condition) worth a short note, after a literature check for priority (Bombieri 2000, Yoshida)?
- Q2:** Should L116 build the Maass–Selberg family (direction 1)?
- Q3:** Is the Lagarias–Suzuki “two conditions” reading (unitarity $y \geq 1$, criticality $y \leq y^*$) the analogy you want to push?
- Q4:** Do you want the $\mathbb{Q}$ null state n_ξ computed and plotted (it is explicit from θ)?
- Q5:** Your BC remark: did you mean $\beta \leq 1$ ($\leftrightarrow s \leq 1$, edge of the strip) or something on the critical line?

REPRODUCIBILITY

- `num/lemma115_finite.py` (5 min, mpmath 60 digits, sympy exact) $\rightarrow$ `num/l115/finite.json`, `finite_g2.npy`.
- `num/lemma115_weil.py` $\rightarrow$ `weil.json`.
- `num/lemma115_ls.py` $\rightarrow$ `ls.json`, `ls_branch.npy`, `ls_t1.npy`. The $y = 1000$ root was rebracketed on $(0.99, 1 - 10^{-8})$, as noted in the json.
- `num/lemma115_vfrob.py` $\rightarrow$ `vfrob.json`.
- `num/lemma115_figs.py` $\rightarrow$ `build115/figs/f1_finite`, `f2_weil`, `f3_ls` {pdf,png}.

Never touched: earlier builds and PDFs (including L106, L88/89), submit directories. L109’s `g2grid.npy`, L104’s `zeros.json` and L112’s code were read only (the hat-basis code was copied).

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TAA LEMMA 116: THERMODYNAMIC GEOMETRY MEETS THE LIGHT CONE. BOST–CONNES, FISHER–RUPPEINER, CONTACT GEOMETRY, BISOGNANO–WICHMANN, AND THE LAGUERRE INEQUALITY

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Every equivalence with RH below is a *reformulation*. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma116_*.py`, data `num/1116/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma116/verified_refs.md`. Ghosts follow the L103 convention: the zero pair $\frac{1}{2} \pm i\gamma$ is replaced by the quartet $\frac{1}{2} \pm \delta \pm i\gamma$.

ABSTRACT. J.W. asked (t481u) whether thermodynamic geometry (Fisher/Ruppeiner/Weinhold metrics, contact geometry) and relativistic thermodynamics (four-temperature, Bisognano–Wichmann, Unruh, Tomita–Takesaki) can say anything about RH. In particular, is there a thermodynamic *stability* statement equivalent to Weil positivity, or to L115’s one condition $\hat{g}(1) = 0$? Short answer: there is *exactly one* natural stability statement that sees zeros. It is necessary but not sufficient, and its completion to an equivalence is a known [RESTATEMENT].

- **The Bost–Connes Fisher metric is ghost-blind** ([THEOREM]+[NUMERICAL]). $g(\beta) = (\log \zeta)''(\beta) = \text{Var}_\beta(\log n) > 0$ for $\beta > 1$ (checked to 15 digits at $\beta = 2, 3$). Any 2D Gibbs family whose field couples *multiplicatively* (Euler product) is separable, hence Ruppeiner-flat ([THEOREM], $K = 0$ numerically). A non-multiplicative χ_4 field gives curvature of *both* signs, $K \in [-1.46, 14.47]$. That curvature measures non-multiplicativity, not zeros. Ghost- ζ has positive $(\log \zeta_{\text{gh}})''$ on $(1, 10]$ as well.
- **The complexified Fisher form is Lorentzian and flat** ([THEOREM]). $\text{Re}[(\log \zeta)''(s) ds^2]$ is the Hessian of the harmonic function $\log |\zeta|$. It is traceless, with $\det = -|(\log \zeta)'|^2$, signature $(1, 1)$ and Gaussian curvature 0 for every holomorphic ζ , ghost or not ([NUMERICAL]: $|K| < 10^{-26}$). Its *light cone* is not ghost-blind, though. On the critical line its $\sigma\sigma$ component is $\Lambda(t) = (\log \xi)''(\frac{1}{2} + it) = L_1(\Xi)(t)/\Xi(t)^2$, the first Laguerre expression.
- **Key test** (§5). The stability statement “ ∂_σ is time-like along $\sigma = \frac{1}{2}$ ”, i.e. $\Lambda > 0$, is implied by RH ([THEOREM]). A ghost of offset δ at γ contributes $-2/\delta^2$ at $t = \gamma$. In the 1517-zero model it is detected at every $\gamma_j \lesssim 564$, for every $\delta < \frac{1}{2}$. It is *not* sufficient: [NUMERICAL] ghosts at close zero pairs are Laguerre-invisible, e.g. $\gamma \approx 1329.04, 1329.21$ for $\delta \geq 0.3$, and $\gamma \approx 947.08$ for $\delta = 0.49$. The full generalized Laguerre tower is equivalent to RH (Csordas–Varga), which is a [RESTATEMENT]. Sondow–Dumitrescu monotonicity is equivalent to RH but circular. $\hat{g}(1) = 0$ has no thermodynamic counterpart: it is the pole, i.e. the BC transition $\beta = 1$.
- **Where Hölder fails** ([THEOREM], Prop. 5.2). $\log \xi(\frac{1}{2} + \tau)$ is convex in real τ unconditionally, because $\Xi(i\tau) = 8 \int \Phi(u) \cosh(2\tau u) du$ with $\Phi > 0$. We derive $L_1(F)(x) = \frac{1}{4} \iint \Phi(u) \Phi(v) [(u+v)^2 \cos x(u-v) + (u-v)^2 \cos x(u+v)]$ ([NUMERICAL] to 10 digits). At $x = 0$ the integrand is positive, which is Hölder/Cauchy–Schwarz. At height $x \neq 0$ the weights are *signed*. That is exactly where log-convexity stops transferring.
- **Contact and relativistic layers** ([THEOREM]/[CITED]/[ANALOGY]). Mrugała’s metric $G = dp_i dx^i + \theta \otimes \theta$ has signature $(n+1, n)$ (computed). It restricts to the positive Fisher metric on Legendre submanifolds. This is the same “indefinite ambient, definite on a distinguished subspace” pattern as Hodge index/Hodge–Riemann and the L115 BRST quotient. BW says the vacuum is KMS at $\beta = 2\pi$ for boosts. The BC flow $\sigma_t(\mu_n) = n^{it} \mu_n$ is a scaling flow whose imaginary-time value at $\beta = \frac{1}{2}$ is the L113 rapidity $\frac{1}{2} \log p$. Yet nothing in BC distinguishes $\beta = \frac{1}{2}$.

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GLOSSARY (FOR A PHYSICIST)

ξ, Ξ, Φ : $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$; $\Xi(t) = \xi(\frac{1}{2}+it)$ is real and even. $\Phi(u) = \sum_{m \geq 1} (2\pi^2 m^4 e^{9u} - 3\pi m^2 e^{5u}) e^{-\pi m^2 e^{4u}}$; $\Xi(t) = 8 \int_0^\infty \Phi(u) \cos(2tu) du$. (Classical, after Pólya; source not opened, [NUMERICAL] checked at $t = 0, 3, 7, \gamma_1$ to 12 digits.)

Laguerre expressions: $L_1(f) = f'^2 - ff''$. More generally $L_n(f) = \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!} \binom{2n}{j} f^{(j)} f^{(2n-j)}$ [6].

Fisher / Ruppeiner / Weinhold: For a Gibbs family $\rho \propto e^{p_i F_i}$ with $\psi = \log Z(p)$, the Fisher metric is $g_{ij} = \partial_i \partial_j \psi = \text{Cov}(F_i, F_j)$ [2, (7.4)]. Ruppeiner is $g^R = -\text{Hess } S$ in extensive variables, and Weinhold is $g^W = \text{Hess } U$ with $ds_R^2 = ds_W^2/T$ [7].

KMS at β : An equilibrium state for a flow α_t : $\phi(A\alpha_\beta(B)) = \phi(BA)$ [10, 11].

Ghost: An off-line zero (L103 quartet). “Ghost-blind” means the quantity cannot tell a ghost model from the true one.

1. BOST–CONNES: $\zeta(\beta)$ AS A PARTITION FUNCTION, AND ITS FLOW AS A BOOST

Proposition 1.1 ([CITED] [1], (3.19) and following). *On $\mathcal{H} = \ell^2(\mathbb{N}^\times)$ with basis ϵ_k , the BC algebra acts by $\mu_n \epsilon_k = \epsilon_{nk}$. The time evolution is implemented by $H \epsilon_k = \log k \epsilon_k$, and $Z(\beta) = \text{Tr } e^{-\beta H} = \zeta(\beta)$ ($\beta > 1$). Extremal KMS $_\beta$ states: for $0 < \beta \leq 1$ there is a unique KMS state. For $1 < \beta \leq \infty$ they are indexed by $\widehat{\mathbb{Z}}^*$, with $\phi_{\beta, \rho}(e(r)) = \zeta(\beta)^{-1} \sum n^{-\beta} \rho(\zeta_r^n)$. $\text{GL}_1(\widehat{\mathbb{Z}})$ acts freely and transitively below the critical temperature.*

Proposition 1.2 ([THEOREM], from Prop. 1.1). $\sigma_t(x) = e^{itH} x e^{-itH}$ gives $\sigma_t(\mu_n) = n^{it} \mu_n$. In imaginary time, $\sigma_{-i\beta}(\mu_n) = n^\beta \mu_n$.

Proof. $e^{itH} \mu_n e^{-itH} \epsilon_k = e^{-it \log k} e^{it \log nk} \epsilon_{nk} = n^{it} \mu_n \epsilon_k$. Continue analytically. $\square$

Remark 1.3 ([ANALOGY], “the flow as a boost”). In (log-energy, time) the operator μ_n shifts $\log k$ by $\log n$. The flow multiplies it by the phase n^{it} , and in imaginary time by the real factor $e^{\beta \log n}$. A real exponential of a generator with real spectrum is the boost form, and $\log n$ plays the role of rapidity per unit β . At $\beta = \frac{1}{2}$ the factor is $p^{1/2} = e^{\frac{1}{2} \log p}$, the L113/L114 rapidity $\frac{1}{2} \log q$ ([TAA-L];

$q^{-1/2}\text{Fr}$ is a boost by $\frac{1}{2}\log q$ on the null plane). **But** ([THEOREM] from Prop. 1.1) nothing in BC singles out $\beta = \frac{1}{2}$: the KMS state is unique on all of $(0, 1]$. The only distinguished temperature is $\beta = 1$, the pole of ζ , i.e. the point $s = 1$ in L115's hyperbolic pole plane (C115). This confirms L84 (BC is ghost-blind) and L99 (BC is solvable with no Cartan involution).

Remark 1.4 ([ANALOGY], Fisher/Lee–Yang zeros). The zeros of ζ are zeros of the BC partition function at *complex* β . In the language of the primon gas these are Lee–Yang/Fisher zeros [14]. The BC KMS theory is a theory of *real* β only, so the critical line is invisible to it. That is a structural explanation of L84's ghost-blindness.

2. FISHER, RUPPEINER, WEINHOLD

2.1. 1D family. With $\theta = -\beta$ as the natural parameter and $F = \log n$:

Proposition 2.1 ([THEOREM]; [CITED] [2] (7.4)). $g(\beta) = \partial_\beta^2 \log \zeta(\beta) = \text{Var}_\beta(\log n) = \zeta''/\zeta - (\zeta'/\zeta)^2 > 0$ for $\beta > 1$. A one-dimensional Riemannian metric is always flat. [NUMERICAL]: at $\beta = 2$ both sides equal 0.884481833963524; at $\beta = 3$, 0.172280711506.

Continued to $\beta < 1$, $(\log |\zeta|)''$ is *no longer* a variance. It changes sign at $\beta_0 = -0.385528083743$ and is negative on our whole grid in $(-7, \beta_0)$ (trivial zeros excluded), e.g. -2.39 at $\beta = -6.9$ ([NUMERICAL], Fig. 1a). So positivity of the Fisher metric is a property of the convergent (Gibbs) region, not of ζ .

2.2. 2D families and Ruppeiner curvature. For a Hessian metric $g = \text{Hess } \psi$ in two variables, the Gaussian curvature is $K = -\frac{1}{4D^2} \det \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{22} \\ \psi_{111} & \psi_{112} & \psi_{122} \\ \psi_{112} & \psi_{122} & \psi_{222} \end{pmatrix}$ with $D = \det g$. We checked this formula against Brioschi at one point, getting 0.022230507 both ways ([NUMERICAL]).

Proposition 2.2 ([THEOREM]). If $\psi(\theta, h) = A(\ell_1(\theta, h)) + B(\ell_2(\theta, h))$ for two independent linear forms ℓ_1, ℓ_2 , then $K = 0$. The Euler-factor field family $Z(\beta, h) = \zeta(\beta)(1 - 2^{-\beta})/(1 - e^{h2^{-\beta}})$, which weights n by $e^{h v_2(n)}$, has $\psi = [\log \zeta(\beta) + \log(1 - 2^{-\beta})] + [-\log(1 - e^{h-\beta \log 2})]$ and is therefore flat.

Proof. In coordinates $u = \theta$, $v = \theta \log 2 + h$ (linear, so Hessian metrics transform tensorially), $\psi = A(u) + B(v)$. Then $g = A'' du^2 + B'' dv^2$ is a product of two 1D metrics, hence flat. $\square$

This is the arithmetic version of the folklore “Ruppeiner is flat for non-interacting systems” [7]. The primon gas is free [14], and multiplicativity is non-interaction.

Proposition 2.3 ([NUMERICAL]). The χ_4 field family is $Z(\beta, h) = \sum_n n^{-\beta} e^{h\chi_4(n)} = 4^{-\beta} [e^h \zeta(\beta, \frac{1}{4}) + e^{-h} \zeta(\beta, \frac{3}{4}) + \zeta(\beta, \frac{1}{2}) + \zeta(\beta, 1)]$. Here $e^{h\chi_4(n)}$ is not multiplicative, and the family is curved. On a 25×21 grid with $\beta \in [1.2, 6]$, $h \in [-2, 2]$: $K \in [-1.461, 14.47]$. $K > 0$ for $h \lesssim 0.4$, and $K < 0$ in a wedge of large positive h (Fig. 1b). Sample values: $K(2, 0) = 0.0719$, $K(3, 0.5) = 0.0222$, $K(4, 1) = 0.1013$. The Euler-field family gives $K = 0$ at all four test points.

Remark 2.4 (Ghost test, honest version). (i) *Circularity*. A Gibbs family needs a positive Dirichlet series. A ghost ζ , i.e. ζ times the Hadamard ratio of the quartet to the pair, is *not* a Dirichlet series with non-negative coefficients. So a “ghost Gibbs family” does not exist, and a comparison could only be formal. (ii) *Blindness*. Formally, $(\log \zeta_{\text{gh}})'' > 0$ on $(1, 10]$ for the L103 ghosts $(\gamma_1, 0.3)$, $(\gamma_1, 0.49)$, $(\gamma_2, 0.2)$. The ratio to the true value lies in $[1.000004, 6.36]$: the size changes but the sign does not ([NUMERICAL]). The ratio is largest at $\beta \approx 10$, where g itself is tiny. (iii) *What curvature measures*. Prop. 2.2 and Prop. 2.3 show that Ruppeiner curvature detects *non-multiplicativity of the field*. Zeros of ζ never enter.

2.3. Complexified Fisher form: a flat Lorentzian metric.

Proposition 2.5 ([THEOREM]). *Let f be holomorphic and non-vanishing on a domain U , and set $\psi = \log |f| = \operatorname{Re} \log f$ and $\phi = (\log f)''$. Then the real Hessian of ψ in (σ, t) is*

$$\operatorname{Hess} \psi = \begin{pmatrix} \operatorname{Re} \phi & -\operatorname{Im} \phi \\ -\operatorname{Im} \phi & -\operatorname{Re} \phi \end{pmatrix} = \operatorname{Re}(\phi ds^2).$$

It is traceless (ψ is harmonic), with $\det = -|\phi|^2$, so its signature is $(1, 1)$ wherever $\phi \neq 0$. Its Gaussian curvature is 0.

Proof. The Cauchy–Riemann equations give the matrix. For flatness, near a point with $\phi \neq 0$ take $w = \int \sqrt{\phi} ds$. Then $\phi ds^2 = dw^2$ and $\operatorname{Re}(dw^2) = du^2 - dv^2$, which is the flat Minkowski plane. $\square$

[NUMERICAL]: at $s \in \{0.3 + 5i, 0.7 + 10i, 1.5 + 3i, 0.5 + 17i\}$, for ζ and for the ghost $(\gamma_1, 0.3)$, Brioschi gives $|K| \leq 6 \times 10^{-27}$ and $\det g = -|\phi|^2$ to all printed digits. Signature and curvature are therefore *ghost-blind*. What is not ghost-blind is the *orientation* of the light cone relative to the coordinate axes. The null directions are $\arg ds = -\frac{1}{2} \arg \phi \pm \frac{\pi}{4}$, and ∂_σ is time-like iff $\operatorname{Re} \phi > 0$.

3. CONTACT GEOMETRY: MRUGAŁA’S METRIC

Proposition 3.1 ([CITED] [2] (6.15)–(6.18); signature [THEOREM]). *On the thermodynamic phase space $M^{2n+1} \ni (w, p_i, x^i)$ with $\theta = dw - x^i dp_i$, Mrugała notes that $dp_i dx^i$ is degenerate on M . He removes the degeneracy by setting $G := dp_i dx^i + \theta \otimes \theta$ (6.16). On every Legendre submanifold S ($\theta|_S = 0$, i.e. $w = \psi(p)$, $x^i = \partial_i \psi$) one has $G|_S = dp_i dx^i|_S = \partial_i \partial_j \psi dp_i dp_j$ (6.17). For $\psi = \log Z$ this is the Fisher metric (7.4). **Computed here:** G is non-degenerate of signature $(n+1, n)$. Also, his alternative (6.18), $\mathcal{H} = dw dw - 2x^i dw dp_i + x^i x^j dp_i dp_j + dx^i dp_i$, expands identically to $\theta \otimes \theta + dx^i dp_i = G$ ([THEOREM], algebra).*

Proof of signature. In the coframe (θ, dp_i, dx^i) , $G = \theta^2 + \sum_i dp_i \odot dx^i$. Each $dp_i \odot dx^i$ is a hyperbolic plane of signature $(1, 1)$, and θ^2 adds one positive direction. $\square$

Remark 3.2 ([ANALOGY]: three “indefinite ambient, definite on a distinguished subspace” structures). (a) Mrugała: G of signature $(n+1, n)$, positive on Legendre submanifolds of convex ψ . These are maximal $d\theta$ -isotropic and n -dimensional, so they are *not* maximal positive subspaces. (b) Hodge index on a surface: signature $(1, \rho-1)$, negative on the primitive part (L113 H1/H2, [TAA-L]). (c) L115 BRST: indefinite form on NS, with $H^1 = n^\perp/n$ definite iff RH ([TAA-L]). The analogy is precise up to one point. In (a), positivity on S is convexity of $\psi = \log Z$, which holds because Z is a positive sum (Hölder). In (b) and (c) the analogous positivity is the theorem to be proved. A Legendre submanifold for ξ along the critical line would need $\log \Xi$ to be concave, which is the content of §5.

4. RELATIVISTIC THERMODYNAMICS: FOUR-TEMPERATURE, BW, UNRUH, TOMITA–TAKESAKI

- [CITED] [4]: if entropy is a Lorentz scalar, $\partial S / \partial P^\mu = u_\mu / T \equiv \beta_\mu$ (the four-temperature), and $dS = \beta \cdot dP$. A thermometer at rest, in contact with a system moving with Lorentz factor γ , reads T/γ in the Einstein–Planck option that Becattini adopts. Equivalently $1/T_{\text{thermometer}} = \beta \cdot v$, with v the thermometer’s four-velocity.
- [CITED] [3] (§5, and around (5.8)–(5.10)): for a wedge U the vacuum modular operator is $\Delta_\Omega = \exp(-2\pi K)$, with K the boost generator. So Ω restricted to the wedge is KMS at inverse temperature 2π with respect to boosts (Bisognano–Wichmann).
- [CITED] [9]: a uniformly accelerated observer sees $T = \hbar a / (2\pi c k_B)$, which is the 2π above in proper-time units.
- [CITED] [10]: Takesaki–Winnink: a faithful normal state is KMS for its own modular group (at $\beta = 1$ after rescaling).

Remark 4.1 ([ANALOGY]/[THEOREM] comparison with BC and L113–L114). In BW the temperature is *fixed* (2π) by geometry, because the modular flow is a boost. In BC the flow is fixed, the scaling n^{it} , and β is free. If one insists on a BW-type normalization “modular flow = scaling, $\beta = 1$ ”, then the BC state at β has modular group $\sigma_{-\beta t}$ (sign conventions vary) ([THEOREM], from Takesaki–Winnink applied to ϕ_β for $\beta > 1$). The rapidity per unit modular time is $\beta \log n$. The L113/L114 value $\frac{1}{2} \log q$ corresponds to $\beta = \frac{1}{2}$, which lies in the *unique-state* phase where $\zeta(\beta)$ diverges, the KMS state is not a Gibbs state, and there is no trace formula (Prop. 1.1). The four-temperature has no arithmetic analogue that we can identify. BC has no momentum, only energy $\log n$, so β^μ reduces to the scalar β .

5. THE KEY TEST: THERMODYNAMIC STABILITY VERSUS WEIL POSITIVITY

5.1. Three candidate stability statements.

- (S1) *Real-axis convexity* (Hölder). $\log \xi(\frac{1}{2} + \tau)$ is convex in real τ .
- (S2) *Light-cone stability on the critical line* (first Laguerre). $\Lambda(t) := \operatorname{Re}(\log \xi)''(\frac{1}{2} + it) = -(\log \Xi)''(t) = L_1(\Xi)(t)/\Xi(t)^2 > 0$ for all real t that are not zeros.
- (S3) *Monotonicity* (Sondow–Dumitrescu). $|\xi(\sigma + it)|$ is increasing in $\sigma > \frac{1}{2}$ [5].

Theorem 5.1 (status of (S1)–(S3)). (a) (S1) holds unconditionally and is ghost-blind ([THEOREM]; [NUMERICAL]: $(\log \xi)''$ on the real axis is $0.0462 \rightarrow 0.0361$ for $\tau \in [0, 8]$ for ξ , and $0.0562 \rightarrow 0.0400$ for the ghost).

(b) $RH \Rightarrow (S2)$ ([THEOREM]). For a single L103 ghost at γ_j , $\Lambda_{\text{gh}}(\gamma_j) = S_j - 2/\delta^2 + O(\gamma_j^{-2})$, where $S_j = \sum_{k \neq j} [(\gamma_j - \gamma_k)^{-2} + (\gamma_j + \gamma_k)^{-2}]$. So the ghost violates (S2) iff $\delta < \delta_j^* \approx \sqrt{2/S_j}$ (at $t = \gamma_j$).

(c) (S2) does not imply RH ([NUMERICAL], and [CITED] [6] for the general fact).

(d) The generalized Laguerre inequalities $L_n(\Xi) \geq 0$ for all n are equivalent to $\Xi \in \mathcal{LP}$, hence to RH ([CITED] [6], Thm. 1.2 = Csordas–Varga 1990, Thm. 2.9; Ξ has order 1, genus 0 in t^2). This is a [RESTATEMENT].

(e) (S3) $\iff$ RH ([CITED] [5], Thm. 1 and Cor. 1), but it is circular as a test: it fails at the first point where $|\xi|$ vanishes off the line ([NUMERICAL]: at $t = \gamma_1$ the ghost $(\gamma_1, 0.3)$ gives $\partial_\sigma \log |\xi| < 0$ on $(\frac{1}{2}, 0.8)$, e.g. -7.99 at $\sigma = 0.7$, and a pole of $\log$ at 0.8).

Proof of (b). By Hadamard, $(\log \xi)''(s) = -\sum_\rho (s - \rho)^{-2}$. Under RH, at $s = \frac{1}{2} + it$ each zero $\frac{1}{2} \pm i\gamma$ contributes $(t \mp \gamma)^{-2} > 0$. A quartet $\frac{1}{2} \pm \delta \pm i\gamma$ contributes, for the $+\gamma$ pair, $2((t - \gamma)^2 - \delta^2)/((t - \gamma)^2 + \delta^2)^2$, which equals $-2/\delta^2$ at $t = \gamma$. $\square$

Proposition 5.2 ([THEOREM] given $\Phi > 0$: where Hölder stops). Let $F(x) = \int_0^\infty \Phi(u) \cos(xu) du$, so that $\Xi(t) = 8F(2t)$. Then

$$L_1(F)(x) = \frac{1}{4} \iint_{[0, \infty)^2} \Phi(u)\Phi(v) \left[(u+v)^2 \cos x(u-v) + (u-v)^2 \cos x(u+v) \right] du dv.$$

At $x = 0$, and for $\cos \rightarrow \cosh$ (the real axis, (S1)), the integrand is ≥ 0 , which is Hölder/Cauchy–Schwarz. For $x \neq 0$ the weights $\cos x(u \pm v)$ are signed, and nothing forces positivity.

Proof. $F' = -\int u\Phi \sin$ and $F'' = -\int u^2\Phi \cos$. Write $C_k = \int u^k \Phi \cos(xu)$ and $S_k = \int u^k \Phi \sin(xu)$. Then $L_1 = S_1^2 + C_0 C_2$. Expanding $\cos x(u \mp v) = CC \pm SS$ inside the double integral gives $\frac{1}{4}[(2C_0 C_2 + 2C_1^2 + 2S_0 S_2 + 2S_1^2) + (2C_0 C_2 - 2C_1^2 - 2S_0 S_2 + 2S_1^2)] = C_0 C_2 + S_1^2$. $\square$

[NUMERICAL]: direct and kernel evaluations agree to 10 digits at $x = 0, 5, 20, 28.27$ (e.g. $4.46087142 \times 10^{-5}$ at $x = 0$). $\Phi > 0$ on a 301-point grid in $[0, 3]$.

Remark 5.3 (Hadamard three-lines). The complex form of Hölder is the three-lines/Phragmén–Lindelöf convexity [12]. For ζ it gives $\mu(\frac{1}{2}) \leq \frac{1}{4}$ (Lindelöf convexity) [13], a statement about growth,

not zeros. Log-convexity of the maximum modulus on vertical lines is ghost-blind in exactly the same way as (S1).

Remark 5.4 (Relation to Weil positivity and to $\hat{g}(1) = 0$). (S2) is a *pointwise* positivity of $-(\log \Xi)''$. It is the test function $\hat{f}$ concentrated at one height t in the Weil form's zero side. Weil positivity asks for the whole quadratic form, which is why it is equivalent to RH and (S2) is not. The L115 condition $\hat{g}(1) = 0$ removes the hyperbolic pole plane. Thermodynamically, the pole $s = 1$ is the BC transition $\beta = 1$ (C115). So " $\hat{g}(1) = 0$ " translates to "stay away from the Hagedorn-like point", which is a gauge condition rather than a stability statement ([ANALOGY]). **Verdict of the key test** ([RESTATEMENT]): the only thermodynamic stability statements that are equivalent to RH are the full Laguerre tower and Sondow–Dumitrescu monotonicity. Both are restatements. The thermodynamic ones in the Gibbs sense (Fisher positivity, Ruppeiner signature, Hölder) hold unconditionally and are ghost-blind, because they live at real β / real τ .

6. NUMERICS

All data are in `num/1116/`. Scripts: `lemma116_thermo.py` (N1–N4), `lemma116_tail.py` (S1, S3, kernel identity), `lemma116_thresh_all.py`, `lemma116_thresh_scan.py`, `lemma116_controot.py`, `lemma116_phicheck.py`, `lemma116_figs.py`. The zero list is L104's 1517 zeros up to $T = 2000$; the tail beyond T uses the density $\frac{1}{2\pi} \log \frac{\gamma}{2\pi}$.

- **Λ validation:** zero sum plus tail vs direct mpmath $(\log \xi)''$. Relative error 6.9×10^{-8} ($t = 3$), 3.1×10^{-8} (10), 1.4×10^{-8} (17.5), 5.7×10^{-10} (30), 1.7×10^{-10} (50).
- **Ghosts:** $\Lambda_{\text{gh}}(\gamma)$ is -22.15 for $(\gamma_1, 0.3)$, -8.26 for $(\gamma_1, 0.49)$, -49.86 for $(\gamma_2, 0.2)$ and -799.9 for $(\gamma_1, 0.05)$. The minimum is at $t = \gamma$ in each case. The negative t -intervals have lengths $0.595, 0.965, 0.395, 0.095$.
- **Threshold:** $\delta^*(\gamma_1) \approx 5.44$, $\delta^*(\gamma_2) = 3.79$, so every strip ghost is visible there. Over all zeros with $\gamma < 1800$: 40 have $\delta^* < \frac{1}{2}$. The first is at $\gamma = 564.16$; the minimum is $\delta^* = 0.2233$ at $\gamma = 1329.205$; the median over the last 200 is 0.79 . All of these sit at close neighbour pairs (S_j up to 40.1).
- **Local invisibility:** scanning $|t - \gamma| \leq 1.5$ (no tail, which only adds positive terms), ghosts with $\delta \in \{0.3, 0.4, 0.45, 0.49\}$, $\delta > \delta^*$, have $\min \Lambda_{\text{gh}} \geq 0$ at 17 of the 40 zeros. Examples: $\gamma = 947.08$ ($\delta = 0.49$); 1054.78, 1055.00, 1415.59, 1415.78 ($\delta \geq 0.4$); 1329.04, 1329.21 ($\delta \geq 0.3$). These ghosts satisfy (S2), so (S2) $\not\Rightarrow$ RH even inside the 1517-zero model.
- **Light-cone map:** Fig. 2 shows the sign of $\text{Re}(\log \xi)''$ (Hadamard, 1517 zeros, no tail) on $[0, 1] \times [5, 40]$. In both models the space-like (orange) bands sit at the zero heights. In the true model every band is *cut* at $\sigma = \frac{1}{2}$, so ∂_σ is time-like along the whole line. With the ghost, the band at γ_1 *crosses* the line. For ζ itself (not ξ) the fraction of a $[-0.5, 1.5] \times [0.5, 30]$ grid with $\text{Re}(\log \zeta)'' > 0$ is 0.734 (0.764 with the ghost), so the ζ light cone tilts throughout the strip and its sign there is uninformative.

7. VERDICT

- (1) [THEOREM]: every *Gibbs-type* object (BC Fisher metric, Ruppeiner/Weinhold, Mrugała on Legendre submanifolds, Hölder, three-lines) is positive unconditionally and ghost-blind, because it lives at real β . Real β never meets a non-trivial zero.
- (2) [THEOREM]: the complexified Fisher form is a *flat Lorentzian* metric for every ζ , ghost or not. Its light cone relative to ∂_σ on $\sigma = \frac{1}{2}$ is the first Laguerre inequality. That is necessary for RH, detects most single ghosts, and misses ghosts at close pairs ([NUMERICAL]).
- (3) [RESTATEMENT]: the stability statements equivalent to RH are the full Laguerre tower (Csordas–Varga) and Sondow–Dumitrescu monotonicity. Neither is new, and neither is thermodynamic in the Gibbs sense.

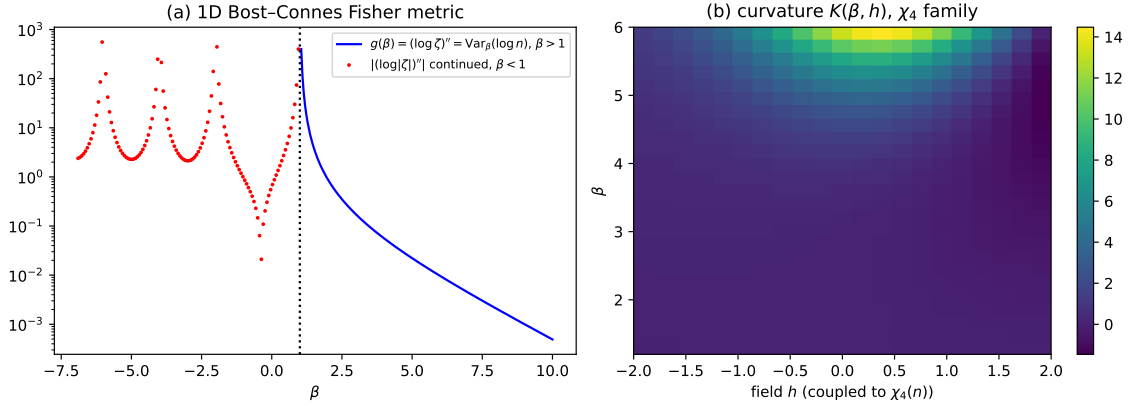


FIGURE 1. (a) BC Fisher metric $g(\beta) = \text{Var}_\beta(\log n)$, and the modulus of its continuation $(\log |\zeta|)''$ to $\beta < 1$ (negative on $(-7, -0.3855)$). (b) Gaussian curvature of the χ_4 field family (both signs). The Euler-factor family is flat.

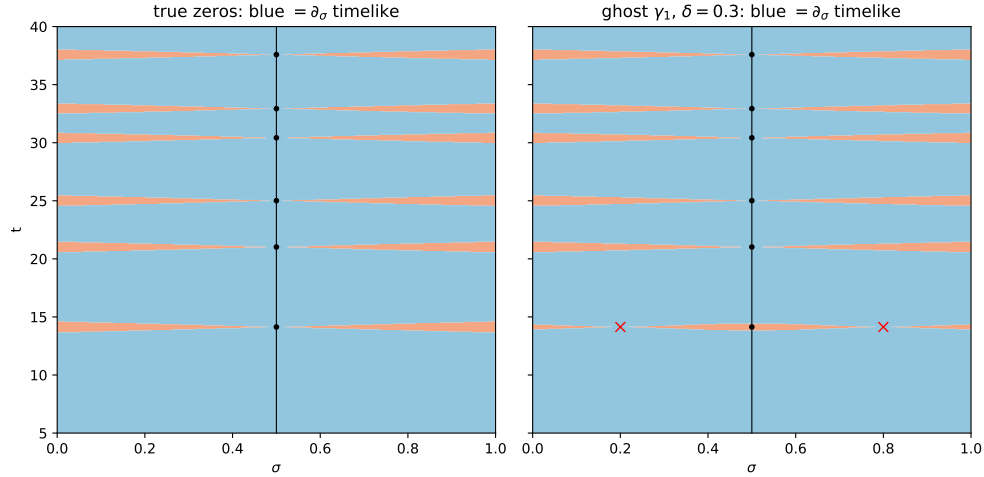


FIGURE 2. Light cone of the complexified Fisher form $\text{Re}[(\log \xi)'' ds^2]$: blue where ∂_σ is time-like ($\text{Re}(\log \xi)'' > 0$). Left: true zeros (dots). Right: ghost quartet at $\gamma_1, \delta = 0.3$ (crosses).

- (4) [ANALOGY]: the contact picture (indefinite G , definite on Legendre submanifolds) is the same shape as Hodge index and BRST. But on the thermodynamic side, definiteness is *free* (convexity of $\log$ of a positive sum). The arithmetic side would need $\log \Xi$ concave on the line, which is (S2) again.
- (5) [OPEN]: a thermodynamic system whose KMS states live at *complex* β on the critical line, or whose modular flow is the boost at $\beta = \frac{1}{2}$ with a genuine Gibbs structure, would be needed. We know of none (cf. L84/L99).

8. CORRECTIONS

Chat claims (t480).

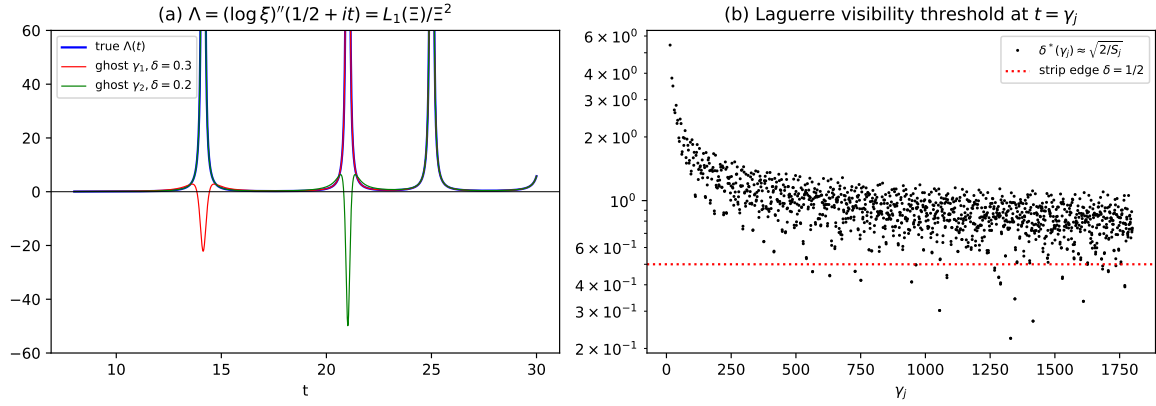


FIGURE 3. (a) $\Lambda(t) = (\log \xi)''(\frac{1}{2} + it) = L_1(\Xi)/\Xi^2$, true vs ghosts. (b) Laguerre visibility threshold $\delta^*(\gamma_j)$ for all 1517 zeros. Below the red line, strip ghosts with $\delta \in (\delta^*, \frac{1}{2})$ escape detection at $t = \gamma_j$.

- C116.1 *Mrugała metric.* The metric is $G = dp_i dx^i + \theta \otimes \theta$ on M^{2n+1} [2, (6.16)], not $dp_i dx^i$ alone, which is degenerate. It is indefinite of signature $(n+1, n)$ (our computation; the source says only that it removes the degeneracy). Only $G|_S$ is the positive Fisher metric. Mrugała’s (6.18) coincides algebraically with G for $\theta = dw - x^i dp_i$.
- C116.2 “*BC high-temperature phase*” = *critical strip*. No. For $0 < \beta \leq 1$ there is a unique KMS state, but $\text{Tr } e^{-\beta H}$ diverges: $\zeta(\beta)$ is not a partition function there. The phase boundary is the pole $\beta = 1 \leftrightarrow s = 1$ (C115), and the real segment $(0, 1)$ contains no zeros. The zeros are at complex β , where BC has no states.
- C116.3 “*Ruppeiner = Fisher.*” Only up to conventions and limits: Fisher = $\text{Hess log } Z$ in natural parameters [2, (7.4)]. Ruppeiner = $-\text{Hess } S$ in extensive variables, which is the Legendre-dual Hessian. Ruppeiner is the thermodynamic (large-system) limit of Fisher, and for small systems may not exist [7]. Weinhold = $\text{Hess } U$, conformal to Ruppeiner by $1/T$ [7]. For BC (a single system, not a thermodynamic limit) we use Fisher.
- C116.4 “*Temperature 4-vector* T^μ .” The covariant object is the inverse four-temperature $\beta^\mu = u^\mu/T$ [4], not a vector T^μ . A moving thermometer reads T/γ (Einstein–Planck option, which Becattini adopts; the alternative option gives the converse factor). BW’s 2π is a KMS parameter for boosts [3], not a component of β^μ .

Internal.

- C116.5 The planned claim “ χ_4 family has positive curvature” is false: K has both signs (Prop. 2.3).
- C116.6 The planned claim “a ghost of any $\delta < \frac{1}{2}$ makes $\Lambda(\gamma) < 0$ ” holds only where $\delta_j^* > \frac{1}{2}$, i.e. at all zeros up to $\gamma \approx 564$ in our model. At 40 close pairs below 1800 the threshold drops below $\frac{1}{2}$ (min 0.223).
- C116.7 “The continuation of the Fisher metric stays positive for $\beta < 1$ ” is false. It is negative on $(-7, -0.3855)$ (at grid points). It is a variance only for $\beta > 1$.
- C116.8 Script pitfalls (fixed): mpmath raises at $\zeta(1)$ when $\tau = \frac{1}{2}$ hits $s = 1$ (special value $\xi(1) = \frac{1}{2}$ now used), and $\sigma = 0.8$ sits exactly on the ghost zero for (S3) (grid moved).
- C116.9 *Normalization.* The relation $\Xi(t) = 8 \int_0^\infty \Phi(u) \cos(2tu) du$ is used with the L114 normalization of Φ . It was checked numerically, not taken from an opened source.

9. RANKED DIRECTIONS

- (1) **Laguerre tower on Laguerre-invisible ghosts** (0.58). For the ghosts at $\gamma \approx 1329$ (and the other close pairs), find the smallest n with $L_n(\Xi_{\text{gh}}) < 0$, and measure how n grows with pair closeness. This is an honest “order of stability” functional, and it connects to L114’s Jensen-degree onsets.
- (2) **Light-cone tilt functional** (0.52). $\Theta(t) = \frac{1}{2} \arg(\log \xi)''$ near the line, as a Krein-type index (L115 Krein signs). Count crossings of ∂_σ through the null cone as t grows.
- (3) **Legendre submanifold for ξ** (0.45). Find a contact embedding in which (S2) is Mrugała-positivity on a Legendre surface, and ask which additional contact condition gives the full tower.
- (4) **Lee–Yang structure** (0.40). Can ξ be the partition function of a ferromagnetic system in a variable related to $s - \frac{1}{2}$? (Primon gas pointer [14]; no source opened beyond that.)
- (5) **Ruppeiner curvature as a multiplicativity detector** (0.28). Classify 2D Dirichlet families by $K \equiv 0 \iff$ separable. This is clean but has no RH content.

10. QUESTIONS FOR J.W.

- Q1. Is (S2), “ ∂_σ time-like along the critical line”, the stability statement you had in mind, or did you want a statement at real β (which we show is always ghost-blind)?
- Q2. Should L117 compute the Laguerre tower L_n on the 17 locally invisible ghost sites (direction 1)?
- Q3. Do you want the BW normalization ($\beta = 2\pi$ fixed, flow rescaled) pursued for BC at $\beta = \frac{1}{2}$, despite the KMS state there not being Gibbs?
- Q4. Is a Lee–Yang angle on ξ worth a literature pass (sources to be verified first)?
- Q5. For the contact analogy, should Hodge–Riemann be modelled as “Legendre + convexity” (and if so, convexity of what on the arithmetic side)?

REPRODUCIBILITY

From /workspace/taa: python3 num/lemma116_thermo.py (about 20 min),
then lemma116_tail.py, lemma116_thresh_all.py, lemma116_thresh_scan.py,
lemma116_controot.py, lemma116_phicheck.py, lemma116_figs.py. Outputs:
num/l116/{thermo,tail,thresh_all,thresh_scan,cont_root,phicheck}.json,
g_cont.npy, Ref2_grid.npy, Kchi.npy, laguerre_threshold{,_all}.npy. Figures:
build116/figs/f{1_fisher,2_cone,3_laguerre}. {pdf,png}. mpmath at 20–40 digits.

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TAA LEMMA 117: THE ARITHMETIC CASTELNUOVO–SEVERI INEQUALITY AS A SUBLUMINAL RECURSION. SZEGŐ BOOSTS, VERSCHIEBUNG PAIRING, HERMITE–BIEHLER DOMINANCE, $E(i, s)$, AND KRAMERS–KRONIG

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Every equivalence with RH below is a *reformulation*. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma117_*.py`, data `num/1117/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma117/verified_refs.md`. Ghosts follow the L103 convention: the pair $\frac{1}{2} \pm i\gamma$ is replaced by the quartet $\frac{1}{2} \pm \delta \pm i\gamma$.

ABSTRACT. J.W. proposed (t485, approved t487) to read Weil positivity as an *arithmetic Castelnuovo–Severi inequality*: every step of a recursion must be subluminal. We make each part of that picture precise. Our conclusion is that the picture is a coherent *dictionary*, and every RH-equivalent statement in it is a [RESTATEMENT].

- **Szegő steps are boosts** ([THEOREM], Thm. 1.1). Simon’s transfer matrix factors as $A(z, \alpha) = B(\alpha) \text{diag}(z, 1)$. Here $B(\alpha) \in SU(1, 1)$ is a pure boost of velocity $|\alpha|$ and rapidity $\text{artanh } |\alpha|$, and $\text{diag}(z, 1)$ is a rotation. On $|z| = 1$ the vector (φ_n, φ_n^*) is *null*, and rapidities add as Doppler factors: $\varphi_n(1) = e^{-\sum \text{artanh } \alpha_k}$ for real α ([NUMERICAL] 2×10^{-16}). A step with $|\alpha| > 1$ is a Krein swap composed with a subluminal boost of velocity $1/|\alpha|$. [NUMERICAL] (L112 data): the true max is $|\alpha| = 0.3836$ (rapidity 0.4043). Ghosts *jump over* c (e.g. $-0.92 \rightarrow -9.79$). Superluminal steps come in pairs (4 per quartet), and the number of negative E_n is 2 per quartet = n_- .
- **Verschiebung pairing** ([THEOREM]). With the pairing $\rho \leftrightarrow 1 - \bar{\rho}$, $\text{Ve} = \text{Fr}^\dagger$ always. An on-line zero is self-paired and gives a *definite* square, not a null direction. An off-line pair gives a hyperbolic plane on which $p^{-1/2} \text{Fr}$ is a boost of rapidity $\delta \log p$. This yields L115’s count $n_+(W) = 1 + 2\#\{\text{quartets}\}$, and the finite “critical dimension” becomes $\sum |\alpha_i|^2 - 2gq = 4q \sum_i \sinh^2 r_i$.
- **Lagarias–Suzuki dominance is a Schur statement** ([CITED] LS Thm. 4; [THEOREM]). $\Theta_T = \xi(2s-1)T^{1-2s}/\xi(2s)$ is a Schur function on $\text{Re } s > \frac{1}{2}$. *Doubling is the same as shifting by the full unconditional strip half-width* $c = \frac{1}{2}$, which is why it is unconditional and ghost-blind ([NUMERICAL]: $\sup |\Theta_1| = 0.9981$, ghost 0.9977). The undoubled target is: $\text{RH} \iff |\xi(s+c)| > |\xi(s-c)|$ on $\text{Re } s > \frac{1}{2}$ for *every* $c > 0$ ([THEOREM]). [NUMERICAL]: the ghost $\delta = 0.3$ violates this exactly for $c < 0.3$ (sup 7.6 at $c = 0.1$; $|\gamma_1| = 1.92$ in the Schur algorithm).
- $E(i, s) = 2\zeta(s)L(s, \chi_{-4})/\zeta(2s)$ ([THEOREM]+[NUMERICAL], relative error $< 10^{-29}$ at five s , including inside the strip). Also $\sum' (m^2 + n^2)^{-3} = 4.658913615604$ and $E(\rho, s) = 3(\frac{\sqrt{3}}{2})^s \zeta(s)L(s, \chi_{-3})/\zeta(2s)$. RH-type behaviour of E at these CM points is RH+GRH ([RESTATEMENT]). In contrast, $z = i\sqrt{5}$ and $z = 0.1 + 1.2i$ show an off-line zero pair below $t = 20$ ([NUMERICAL]). We prove $E^*(z, \frac{1}{2}) = \sqrt{y}(\log y + \gamma - \log 4\pi) + 4\sqrt{y} \sum d(n)K_0(2\pi ny) \cos 2\pi nx$. Where it is positive, a real off-line zero is forced (32% of $\mathcal{F} \cap \{y \leq 10\}$, threshold on the imaginary axis = $y^* = 4\pi e^{-\gamma}$).
- **Kramers–Kronig** ([THEOREM]/[RESTATEMENT]). $-(\log \xi)''$ is a susceptibility of *lossless* Lorentz oscillators. The functional equation swaps retarded and advanced. KK on $\text{Re } s = \frac{1}{2} + d$ holds iff no zero lies to its right; holding for all $d > 0$ is equivalent to RH. With $\tau_d = \text{Re } \xi'/\xi = \frac{d}{dt} \arg \xi$ (group delay up to sign convention), $\tau_d > 0$ for all $d > 0 \iff \text{RH}$, and $\tau_d = d\Lambda + O(d^2)$ links this to L116’s light cone. A ghost is an all-pass, non-minimum-phase factor with excess (engineering) delay $2\delta/(\delta^2 + x^2)$ per zero and total 4π per quartet, independent of δ . Källén–Lehmann positivity holds automatically, so what RH controls is pole *location* (causality), not spectral weight.

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GLOSSARY (FOR A PHYSICIST)

Verblunsky/Schur coefficients α_n : The reflection coefficients of a positive measure on the circle (Levinson–Durbin); $\|\Phi_{n+1}\|^2 = \prod_{j \leq n} (1 - |\alpha_j|^2)$ [3, 1]. Geronimus: they equal the Schur parameters of the Schur function [1, Thm. 3.1].

$U(1, 1)$, $SU(1, 1)$: 2×2 matrices preserving $|u|^2 - |v|^2$ ($J = \text{diag}(1, -1)$). $SU(1, 1) \cong SL(2, \mathbb{R})$ double-covers $SO^+(2, 1)$.

Rapidity: $v = \tanh w$; collinear rapidities add [5].

Schur function: Holomorphic on a disk or half-plane with $|f| \leq 1$ [4]. *Hermite–Biehler*: E with $|E^\#| < |E|$ on a half-plane, i.e. $E^\# / E$ is Schur.

Minimum phase / all-pass: Minimum phase: zeros and poles in the stable half-plane; all-pass: $|A| = 1$ on the boundary [13, 14].

$\mathcal{F}$: The standard fundamental domain of $SL_2(\mathbb{Z})$; $\rho = e^{2\pi i/3}$.

1. SZEGŐ STEPS AS $U(1, 1)$ BOOSTS

Simon writes the Szegő recursion for orthonormal OPUC as $(\varphi_{n+1}, \varphi_{n+1}^*)^T = A(z, \alpha_n)(\varphi_n, \varphi_n^*)^T$ with $A(z, \alpha) = \rho^{-1} \begin{pmatrix} z & -\bar{\alpha} \\ -z\alpha & 1 \end{pmatrix}$, $\rho = \sqrt{1 - |\alpha|^2}$, $\det A = z$ [1, (2.28)–(2.29)].

Theorem 1.1 ([THEOREM]). *Let $\tilde{B}(\alpha) = \begin{pmatrix} 1 & -\bar{\alpha} \\ -\alpha & 1 \end{pmatrix}$ and $J = \text{diag}(1, -1)$.*

- (a) $\tilde{B}(\alpha)^* J \tilde{B}(\alpha) = (1 - |\alpha|^2) J$ for every $\alpha \in \mathbb{C}$.
- (b) For $|\alpha| < 1$: $A(z, \alpha) = B(\alpha) D(z)$ with $B = \rho^{-1} \tilde{B} \in SU(1, 1)$ positive Hermitian, and $D(z) = \text{diag}(z, 1)$. Writing $\alpha = \tanh \eta e^{i\theta}$, $B(\alpha) = \cosh \eta I - \sinh \eta \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}$: a pure boost of rapidity $\eta = \text{artanh } |\alpha|$ (velocity $|\alpha|$) in direction θ , acting on $\mathbb{C}^{1,1}$. For $|z| = 1$, $D(z) = z^{1/2} \text{diag}(z^{1/2}, z^{-1/2})$ is a rotation up to phase. So the n -step transfer matrix is an alternating product of boosts and rotations in $U(1, 1)$.
- (c) Under $SU(1, 1) \rightarrow SO^+(2, 1)$, $B(\alpha)$ maps to a boost of rapidity $2 \text{artanh } |\alpha|$, the Poincaré-disk distance from 0 to $|\alpha|$. Non-collinear boosts compose to a boost plus a rotation (the Wigner rotation [6]).

- (d) Null transport. For $|z| = 1$, $|\varphi_n^*(z)| = |\varphi_n(z)|$, so (φ_n, φ_n^*) is J -null. If all α_k are real, at $z = 1$ each step multiplies this null vector by the Doppler factor $\rho_k^{-1}(1 - \alpha_k) = e^{-\operatorname{artanh} \alpha_k}$, hence $\varphi_n(1) = \exp(-\sum_{k < n} \operatorname{artanh} \alpha_k)$.
- (e) Superluminal steps. For real $|\alpha| > 1$: $\tilde{B}(\alpha) = -\alpha S \tilde{B}(1/\alpha)$ with $S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is a subluminal boost of velocity $1/|\alpha|$ followed by the swap S , which reverses the sign of J (a Krein-sign flip). In Levinson–Durbin, $E_{n+1} = E_n(1 - \alpha_n^2)$ changes sign exactly at such steps.

Proof. (a) is a direct multiplication. For (b), $\tilde{B}$ is Hermitian, so $B^*JB = J$ with $\det B = 1$; the cosh/sinh form follows from $\cosh \eta = \rho^{-1}$ and $\sinh \eta = |\alpha|\rho^{-1}$. (c) is the standard spin- $\frac{1}{2}$ relation (half-rapidities in $SU(1, 1)$). For (d), $\varphi^*(z) = z^n \overline{\varphi(1/\bar{z})}$ gives equal moduli on $|z| = 1$; real coefficients give $\varphi_n^*(1) = \varphi_n(1)$, and then $\varphi_{n+1}(1) = \rho_n^{-1}(1 - \alpha_n)\varphi_n(1)$ with $\rho_n^{-1}(1 - \alpha_n) = \sqrt{(1 - \alpha_n)/(1 + \alpha_n)}$. (e) follows from $S\tilde{B}(1/\alpha) = \begin{pmatrix} -1/\alpha & 1 \\ 1 & -1/\alpha \end{pmatrix}$ and $SJS = -J$. $\square$

[NUMERICAL] (`lemma117_boost.py`): the identities (a), (b) and (e) hold to 3.6×10^{-15} on 250 random α . Doppler identity (d) is checked against an *independent* Gram–Schmidt OPUC of a smooth even weight ($n \leq 12$): 2.2×10^{-16} .

Corollary 1.2 ([RESTATEMENT], with L112 Thm. 3.1 [TAA-L]). *For the L112 hat-basis Toeplitz form at step h , RH implies that every Szegő step is subluminal for every h and n . Conversely, if RH fails, some step is superluminal for every h outside a countable aliasing set. “Weil positivity $\iff$ every step subluminal” is therefore a restatement of Toeplitz positive-definiteness (Verblunsky/Sylvester), not a new inequality.*

world ($h = 0.025$, $n < 160$)	$\max \alpha $ pre-onset	rapidity	onset n (L)	$\alpha_{n-1} \rightarrow \alpha_n$	#superlum./# $E_n < 0$
true primes	0.3836	0.4043	none	–	0/0
quartet ($\gamma_1, 0.3$)	0.5715	0.6497	30 (0.775)	$-0.571 \rightarrow -1.106$	4/2
quartet ($\gamma_1, 0.49$)	0.9239	1.6151	28 (0.725)	$-0.924 \rightarrow -9.792$	4/2
quartet ($\gamma_2, 0.2$)	0.6486	0.7728	47 (1.2)	$-0.649 \rightarrow -1.625$	4/2
two quartets	0.9137	1.5498	31 (0.8)	$-0.914 \rightarrow -9.261$	8/4

TABLE 1. L112 Verblunsky data read as boosts. Each world’s #negative E_n equals its Toeplitz n_- (L112 inertia checks), i.e. 2 per quartet.

2. VERSCHIEBUNG PAIRING $\rho \leftrightarrow 1 - \bar{\rho}$

Recall from L115 ([TAA-L]): $\operatorname{Fr} y^s = p^s y^s$, $\operatorname{Ve} y^s = p^{1-s} y^s$, $\operatorname{Fr} \operatorname{Ve} = p$, and the Weil form’s zero side is $-\sum_{\rho} \hat{g}(\rho) \overline{\hat{g}(1 - \bar{\rho})}$.

Proposition 2.1 ([THEOREM]). *Pair y^s with $y^{s'}$ by $\langle y^s, y^{s'} \rangle \neq 0$ iff $s' = 1 - \bar{s}$ (the Mellin–Plancherel pairing).*

- (a) $\langle \operatorname{Fr} y^s, y^{1-\bar{s}} \rangle = \langle y^s, \operatorname{Ve} y^{1-\bar{s}} \rangle$, so $\operatorname{Ve} = \operatorname{Fr}^\dagger$ for this pairing, always. Ve is the adjoint on the same vector iff $s = 1 - \bar{s}$, i.e. $\operatorname{Re} s = \frac{1}{2}$ (L115).
- (b) An on-line zero is self-paired. Its contribution $-\lvert \hat{g}(\rho) \rvert^2$ is a definite (negative) square, not a null direction.
- (c) An off-line zero $\rho = \frac{1}{2} + \delta + i\gamma$ and its partner $1 - \bar{\rho} = \frac{1}{2} - \delta + i\gamma$ contribute $-2\operatorname{Re} \hat{g}(\rho) \overline{\hat{g}(1 - \bar{\rho})}$: a hyperbolic plane with null axes $y^\rho, y^{1-\bar{\rho}}$. On it $p^{-1/2} \operatorname{Fr} = p^{i\gamma} \operatorname{diag}(p^\delta, p^{-\delta})$, a boost of rapidity $\delta \log p$ times a rotation. The ghost offset is a rapidity per unit $\log p$.
- (d) Hence $n_+(W) = 1 + 2k$ for k L103 quartets (pole plane plus two hyperbolic planes per quartet), as found numerically in L115.

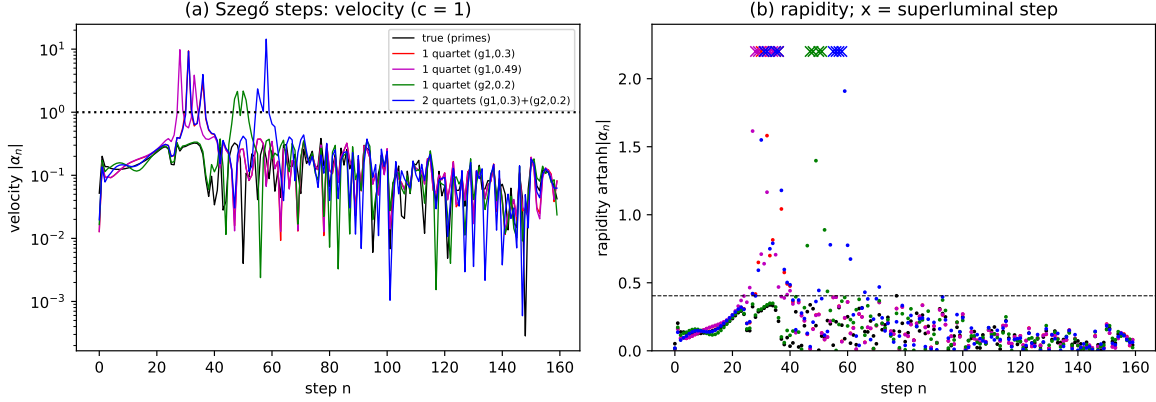


FIGURE 1. (a) Step velocities $|\alpha_n|$ (log scale; dotted line = c). (b) Rapidities of subluminal steps; crosses mark superluminal steps. Dashed: the true maximum 0.4043.

Proposition 2.2 ([THEOREM], finite critical dimension as boost energy). *For a genus- g model whose roots come in pairs $\alpha, q/\bar{\alpha}$ with $|\alpha|^2 = qe^{2r}$: $\sum_i |\alpha_i|^2 - 2gq = 4q \sum_{\text{pairs}} \sinh^2 r_i \geq 0$, with equality iff all rapidities $r_i = 0$, i.e. RH (L115 Prop. 2.2). [NUMERICAL]: random models, error 1.4×10^{-14} .*

Proof. Each pair contributes $q(e^{2r} + e^{-2r}) - 2q = 2q(\cosh 2r - 1) = 4q \sinh^2 r$. $\square$

3. LAGARIAS–SUZUKI DOMINANCE AS A SCHUR-FUNCTION STATEMENT

Theorem 3.1 ([CITED] [2, Thm. 4]). *Let F be entire of genus ≤ 1 , real on $\mathbb{R}$, with $F(s) = \pm F(1-s)$ and all zeros in $|\text{Re } s - \frac{1}{2}| < a$. Then for $c \geq a$, $|F(s+c)/F(s-c)| > 1$ on $\text{Re } s > \frac{1}{2}$, and $F(s+c) + e^{i\theta} F(s-c)$ has all zeros on $\text{Re } s = \frac{1}{2}$.*

LS apply this to $F(s) = \xi(2s - \frac{1}{2})$ with $a = c = \frac{1}{4}$. This gives $|\xi(2s)/\xi(2s-1)| > 1$ on $\text{Re } s > \frac{1}{2}$, and hence RH for $I(T, s)$, $T \geq 1$ [2, Thm. 2, (41)–(43)].

Proposition 3.2 ([THEOREM]). (a) $\Theta_T(s) = \xi(2s-1)T^{1-2s}/\xi(2s)$ is holomorphic with $|\Theta_T| < 1$ on $\text{Re } s > \frac{1}{2}$, and $|\Theta_T| = 1$ on $\text{Re } s = \frac{1}{2}$: a Schur (indeed inner-type) function. $\xi(2s)T^s$ is Hermite–Biehler for the axis $\frac{1}{2}$.

(b) Doubling is the unconditional shift. With $w = 2s - \frac{1}{2}$, $\xi(2s)/\xi(2s-1) = \xi(w + \frac{1}{2})/\xi(w - \frac{1}{2})$, and $\text{Re } s > \frac{1}{2} \iff \text{Re } w > \frac{1}{2}$. So the doubled statement is Theorem 3.1 for $F = \xi$ with $c = a = \frac{1}{2}$, the unconditional half-width of the critical strip. In the s -variable the zeros of $\xi(2s)$ sit at $\rho/2$, inside $0 < \text{Re } s < \frac{1}{2}$, symmetric about $\frac{1}{4}$. The stability boundary $\frac{1}{2}$ is the edge of the doubled strip, so no zero can reach it, ghost or not.

(c) Undoubled target. $\text{RH} \iff |\xi(s+c)| > |\xi(s-c)|$ on $\text{Re } s > \frac{1}{2}$ for every $c > 0$.

Proof. (a) From Theorem 3.1, $|\Theta_1| < 1$, and $|T^{1-2s}| = T^{1-2\sigma} \leq 1$. On the line, $\xi(2it) = \xi(1-2it) = \overline{\xi(1+2it)}$. (b) is a substitution. For (c), RH gives a arbitrarily small, so Theorem 3.1 applies for every $c > 0$. Conversely, if $\rho_0 = \frac{1}{2} + \delta + i\gamma$ with $\delta > 0$, take $c < \delta$ and $s = \rho_0 - c$: then $\text{Re } s > \frac{1}{2}$ and $\xi(s+c) = 0$. $\square$

[NUMERICAL] (lemma117_schur.py). Doubled: $\sup |\Theta_1| = 0.99815$ (ghost $(\gamma_1, 0.3)$: 0.99774) and $\sup |\Theta_2| = 0.97085$ (ghost 0.97046) on a grid in $\text{Re } s \in [0.52, 2]$, $t \leq 30$. This is ghost-blind, because the ghost ξ still satisfies the hypotheses of Thm. 3.1 with $a = \frac{1}{2}$. Undoubled, near γ_1 : true ξ has $\sup |\xi(s-c)/\xi(s+c)| < 1$ for all $c \in \{0.1, \dots, 0.5\}$. The ghost gives 7.61, 7.35, 1.59 for

$c = 0.1, 0.2, 0.29$ and < 1 for $c \geq 0.31$, so it breaks the condition exactly at $c < \delta$. *Schur algorithm* (chart $s = \frac{1}{2} + i\gamma_1 + \frac{1}{2}\frac{1+z}{1-z}$): for the undoubled ghost at $c = 0.1$ and 0.2 , the Schur parameter $|\gamma_1| = 1.92$, resp. 1.48 , is superluminal at the *first* step. All other cases keep $|\gamma_n| < 1$ for $n \leq 10$ (true doubled: $0.686, 0.824, 0.929, \dots$). For $n \geq 12$ the parameters approach 1 (inner-type function), and precision is lost there. By Geronimus, Schur parameters *are* Verblunsky coefficients, so §3 and §1 are the same subluminality condition in two charts.

4. $E(i, s)$ AND $E(\rho, s)$

With $E(z, s) = \frac{1}{2} \sum_{(m,n)=1} y^s |mz + n|^{-2s}$ [7], summing over all $(m, n) \neq 0$ multiplies by $\zeta(2s)$ [7]. For $z = i$, $\sum' (m^2 + n^2)^{-s} = \sum r_2(n) n^{-s} = 4\zeta(s)\beta(s)$ [8, 9]. Likewise for $z = \rho$, $\sum' (m^2 - mn + n^2)^{-s} = 6\zeta_{\mathbb{Q}(\sqrt{-3})}(s) = 6\zeta(s)L(s, \chi_{-3})$ (unit count 6, class number 1; $\zeta_K = \zeta L(\chi)$ [10]).

Proposition 4.1 ([THEOREM]; [NUMERICAL]). $E(i, s) = \frac{2\zeta(s)L(s, \chi_{-4})}{\zeta(2s)}$ and $E(\rho, s) = \frac{3(\sqrt{3}/2)^s \zeta(s)L(s, \chi_{-3})}{\zeta(2s)}$. [NUMERICAL]: the lattice sum ($|m|, |n| \leq 3000$ plus tail) is 4.658913615604 , against $4\zeta(3)\beta(3) = 4.6589136156038$. The Fourier expansion [7] matches both formulas at $s = 3, 2+i, 0.7+5i, 0.5+10i, 0.3+3i$ with relative error $\leq 2.5 \times 10^{-30}$. $E(i, 3) = 2.28974560780$.

Proposition 4.2 ([THEOREM]). $E^*(z, s) := \Lambda(2s)E(z, s)$ satisfies $E^*(z, \frac{1}{2}) = \sqrt{y}(\log y + \gamma - \log 4\pi) + 4\sqrt{y} \sum_{n \geq 1} d(n)K_0(2\pi ny) \cos 2\pi nx$. As $s \rightarrow 1^-$, $E^*(z, s) \rightarrow -\infty$. So $E^*(z, \frac{1}{2}) > 0$ forces a real zero of $E(z, \cdot)$ in $(\frac{1}{2}, 1)$, off the line.

Proof. Put $u = 2s - 1$ in the Fourier expansion. The constant terms are $\sqrt{y}[\Lambda(1+u)y^{u/2} + \Lambda(1-u)y^{-u/2}]$. With $\Lambda(1 \pm u) = \pm u^{-1} + c_0 + O(u)$ and $2c_0 = \gamma - \log 4\pi$ (cf. L114/L115, $\varphi'(\frac{1}{2})$), the limit is $\sqrt{y}(\log y + 2c_0)$. For the non-constant terms, $n^0 \sigma_0(n) = d(n)$ and K_0 . Near $s = 1$, $\Lambda(2s - 1) \sim (2s - 2)^{-1}$. $\square$

[NUMERICAL] (`lemma117_eis.py`). The closed form equals the Fourier limit to 12 digits at $i, \rho, 0.1 + 1.2i$. $E^*(i, \frac{1}{2}) = -1.9501$ and $E^*(\rho, \frac{1}{2}) = -1.9605$, so no real off-line zero is forced at the CM points. On the imaginary axis the sign flips at $y = 7.05550795545 = 4\pi e^{-\gamma}$, which is LS's y^* to all digits shown (the K_0 terms are $O(e^{-44})$ there). 32% of $\mathcal{F} \cap \{y \leq 10\}$ has a forced real off-line zero, starting at $y \approx 7.15$ on our grid (Fig. 2). Zero counts in the box $[-3, 4] \times [1, 20]$ (argument principle) versus sign changes on $\text{Re } s = \frac{1}{2}$, $t \in [1, 20]$: $z = i$: 6/6; $z = \rho$: 5/5; $z = i\sqrt{5}$ (form $m^2 + 5n^2$, class number 2): 12/10; $z = 0.1 + 1.2i$: 7/5; $z = 0.31 + 0.97i$: 6/6. So the CM points with class number 1 behave RH-type in this window, while $i\sqrt{5}$ and $0.1 + 1.2i$ each have an off-line pair of zeros below height 20 ([NUMERICAL]).

Remark 4.3 ([RESTATEMENT]/[ANALOGY]). At $z = i$, RH for $E(i, \cdot)$ (zeros of $\Lambda(2s)E$ on the line) is equivalent to RH for ζ plus GRH for $L(s, \chi_{-4})$; at ρ , likewise with χ_{-3} . The positivity that controls E (Maass–Selberg) acts on the *constant term*, i.e. in y after integrating out x , and it is lost on specializing to a point. $E(i, s)$ is the first natural object with zeros at ρ and poles at $\rho/2$. It carries the undoubled information, but no positivity is known that controls it.

5. THE UNDOUBLED TARGET

Collecting §§1–4 and L115, an object $\mathcal{O}$ with resonances at ρ itself (not $\rho/2$) and a positivity that forces RH would need all of the following.

- (U1) *Shift zero*. A Hermite–Biehler/Schur bound for every shift $c > 0$ (Prop. 3.2(c)). The doubled LS bound is the $c = \frac{1}{2}$ case.
- (U2) *One gauge condition*. Negativity of W on $\{\hat{g}(1) = 0\}$ only (L115 Thm. 2.3): the pole plane is removed by one subsidiary condition, and the transverse form must then be definite.

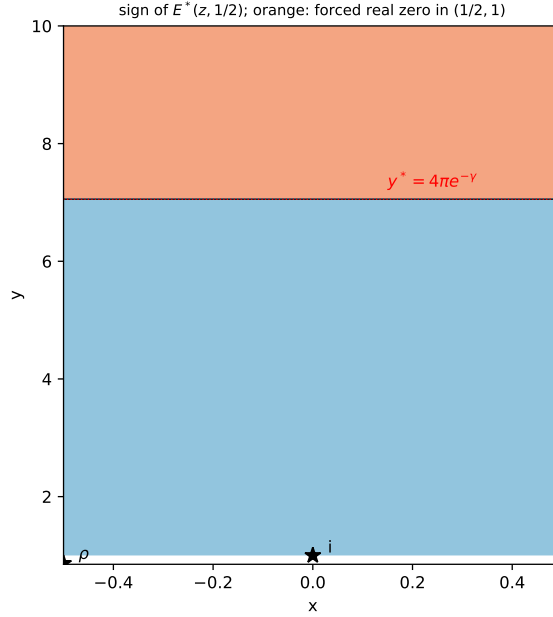


FIGURE 2. Sign of $E^*(z, \frac{1}{2})$ on $\mathcal{F}$, $y \leq 10$. Orange: a real zero in $(\frac{1}{2}, 1)$ is forced. Stars: CM points i, ρ . Red: y^* .

- (U3) *Sign-definite critical dimension.* A quantity like $4q \sum \sinh^2 r_i$ (Prop. 2.2) that is ≥ 0 for structural reasons (Rosati/Hodge over $\mathbb{F}_q$) and vanishes exactly on the physical side.
- (U4) *All steps subluminal.* Equivalently, a passive (positive-real) realization of the Carathéodory function (§6).

Each of (U1)–(U4) is *equivalent* to RH ([RESTATEMENT]). None comes with an independent source of positivity, which is the same missing ingredient as L115’s “positive $V \cong H^1$ ”.

6. KRAMERS–KRONIG, CAUSALITY AND THE FRONT

6.1. Susceptibility. By Hadamard, $\xi'/\xi(s) = \sum_{\rho} (s - \rho)^{-1}$ (paired) and $G := (\log \xi)'' = -\sum_{\rho} (s - \rho)^{-2}$. Set $s = \frac{1}{2} + d + it$ and view t as frequency. A zero $\frac{1}{2} + \delta' + i\gamma$ is a resonance at γ with width $\delta' - d$. On the critical line ($d = 0$), RH zeros are *undamped* Lorentz oscillators [11]. The functional equation $\xi'/\xi(1-s) = -\xi'/\xi(s)$ maps the right half-plane to the left: it *swaps retarded and advanced* ([THEOREM]). Analyticity on one side is therefore equivalent to analyticity on the other, and under RH all poles are on the boundary: a lossless medium.

Proposition 6.1 ([THEOREM] for finite pole models; [RESTATEMENT] for ξ). *For $G_N = -\sum_{|\gamma| \leq T} (s - \rho)^{-2}$, the boundary function $t \mapsto G_N(\frac{1}{2} + d + it)$ has Fourier support on one half-line (KK holds) iff no pole lies in $\text{Re } s > \frac{1}{2} + d$. The acausal part equals exactly the sum of the terms with $\text{Re } \rho > \frac{1}{2} + d$. For ξ itself, KK on every line $\text{Re } s = \frac{1}{2} + d$, $d > 0$, in subtracted form (since ξ'/ξ grows like $\frac{1}{2} \log t$), is equivalent to RH ([RESTATEMENT]). On $\text{Re } s = \frac{1}{2}$ itself classical KK fails for every model, true or ghost, because the poles lie on the line.*

[NUMERICAL] (lemma117_kk.py; 100 zeros, $t \in [-1500, 1500]$, $\Delta t = 0.005$). FFT and exact split agree to 4–10 digits. True model: acausal fraction $\leq 1.5 \times 10^{-5}$ (discretization). The $d = 0.02$ value 6.5×10^{-5} comes from poles 0.02 from the line. Ghost $(\gamma_1, 0.3)$: 0.0019, 0.0090, 0.036, 0.28, 0.998 at $d = 0.02, 0.05, 0.1, 0.2, 0.29$, and $\leq 10^{-5}$ for $d \geq 0.4$. The value 0.018 at $d = 0.31$ is a grid artefact

(pole 0.01 from the line). Ghost $(\gamma_2, 0.2)$ has its pole *on* the line at $d = 0.2$, where FFT gives 0.71 (boundary case). Fig. 3a.

6.2. Group delay, all-pass factors, Bode.

Proposition 6.2 ([THEOREM]). *Let $\tau_d(t) = \operatorname{Re} \xi' / \xi(\frac{1}{2} + d + it) = \partial_\sigma \log |\xi| = \frac{d}{dt} \arg \xi(\frac{1}{2} + d + it)$ (“group delay” in our sign convention; the engineering group delay for frequency $\omega = t$ is $\tau_{\text{eng}} = -\tau_d$).*

(a) $\text{RH} \iff \tau_d(t) > 0$ for all $d > 0$ and all t . (This is a known criterion; we prove it from Hadamard.)

(b) $\tau_d(t) = d \Lambda(t) + O(d^2)$, with $\Lambda = (\log \xi)''(\frac{1}{2} + it)$ the L116 light-cone component.

(c) A quartet factors as $\xi_{\text{gh}} = M \cdot A$, where M has the off-line zeros reflected to $\operatorname{Re} < \frac{1}{2}$ (minimum phase for the axis $\frac{1}{2}$) and $A = \prod \frac{s - \rho_+}{s - \rho_-}$ is all-pass ($|A| = 1$ on $\operatorname{Re} s = \frac{1}{2}$). It changes τ_0 by $-2\delta [(\delta^2 + (t - \gamma)^2)^{-1} + (\delta^2 + (t + \gamma)^2)^{-1}]$, i.e. it adds that much positive engineering delay (Bode: the minimum-phase factor has the least delay [13]). The total excess delay is 4π for every δ , a winding of 2π per zero moved across the axis.

Proof. (a) Under RH each paired term $\operatorname{Re}(s - \rho)^{-1} = d/(d^2 + (t - \gamma)^2) > 0$. If some $\rho_0 = \frac{1}{2} + \delta + i\gamma$ has $\delta > 0$, take $s = \rho_0 - \epsilon$: the ρ_0 term $-1/\epsilon$ dominates. (b) $|\xi(\sigma + it)| = |\xi(1 - \sigma + it)|$ gives $\tau_0 = 0$ off zeros, and $\partial_\sigma \tau = \operatorname{Re}(\log \xi)''$. (c) On $\operatorname{Re} s = \frac{1}{2}$, $s - \rho_+ = -\delta + ix$ and $s - \rho_- = \delta + ix$ ($x = t - \gamma$), with $\frac{d}{dx} \arg(a + ix) = a/(a^2 + x^2)$; and $\int 2\delta/(\delta^2 + x^2) = 2\pi$. $\square$

[NUMERICAL]: $\min_t \tau_d$ on $[10, 30]$ (1517 zeros, no tail). True: 0.00105, 0.0105, 0.0521 at $d = 0.01, 0.1, 0.5$, always > 0 . Ghost $(\gamma_1, 0.3)$: $-0.22, -2.49, -7.99, -98.2$ at $d = 0.01, 0.1, 0.2, 0.29$ and > 0 at $d \geq 0.31$. Ghost $(\gamma_2, 0.2)$: $-0.50, -6.65$ at $d = 0.01, 0.1$ and > 0 at $d \geq 0.2$. At $d = 10^{-3}$, $\tau/d = 0.10515, 0.22903, 5.7774$ against L116’s $\Lambda = 0.10623, 0.23011, 5.7785$ at $t = 10, 17.5, 30$; the gap is the missing tail plus $O(d)$. Total excess delay is 12.5663 for $\delta = 0.05, 0.3, 0.49$ ($4\pi = 12.56637$), with peak $\approx 2/\delta$.

6.3. Källén–Lehmann, complex poles, Brune, front velocity.

- *Spectral positivity is automatic* ([THEOREM]). The Källén–Lehmann density must be positive [15]. The residues of ξ'/ξ are multiplicities (≥ 1) in *every* model, ghosts included. What a ghost changes is the pole *position*: a complex-conjugate pair $\gamma \pm i\delta$ in the t -plane, one damped and one anti-damped. This is the configuration physicists associate with Lee–Wick-type complex poles (named only; no source opened). RH is a causality/location statement, not a spectral-weight statement.
- *Brune and layered media* ([CITED]+[ANALOGY]). L112’s Carathéodory function $F = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_h$ has $\operatorname{Re} F > 0$ [1, (2.2)–(2.3)]. After a Cayley map it is a positive-real impedance, and a rational PR impedance is realizable by passive elements iff PR (Brune [16]). The Schur algorithm is “layer stripping” [4], so the α_n are reflection coefficients of a layered medium. A ghost makes μ_h non-positive, and no passive realization exists: some layer has gain ($|\alpha| > 1$, Thm. 1.1(e)).
- *Front velocity* ([CITED]+[ANALOGY]). In physics the front travels at c in every medium. Superluminal group velocity is a narrow-band artefact and violates nothing [17, 18]. The arithmetic analogue of the front is the abscissa $\Theta = \sup \operatorname{Re} \rho$, the growth exponent of $\psi(x) - x$ (von Koch: RH gives the optimal error [19]). *The analogy breaks here*: in physics anomalous (even negative) group delay is harmless, while for ξ a sign reversal of τ_d on $\operatorname{Re} s > \frac{1}{2}$ is equivalent to $\neg\text{RH}$ (Prop. 6.2(a)). The reason is that the arithmetic medium is lossless with unit-weight poles, so any sign reversal must come from a pole on the wrong side.

7. TIE-IN: THE LIGHT-CONE TILT AS A KREIN INDEX (L116 DIRECTION 2)

By Prop. 6.2(b), L116’s (S2) condition “ ∂_σ time-like on the critical line” ($\Lambda > 0$) is the *first-order* germ at $d = 0$ of positive group delay. Lagarias-type positivity for all $d > 0$ is its *global* version and is

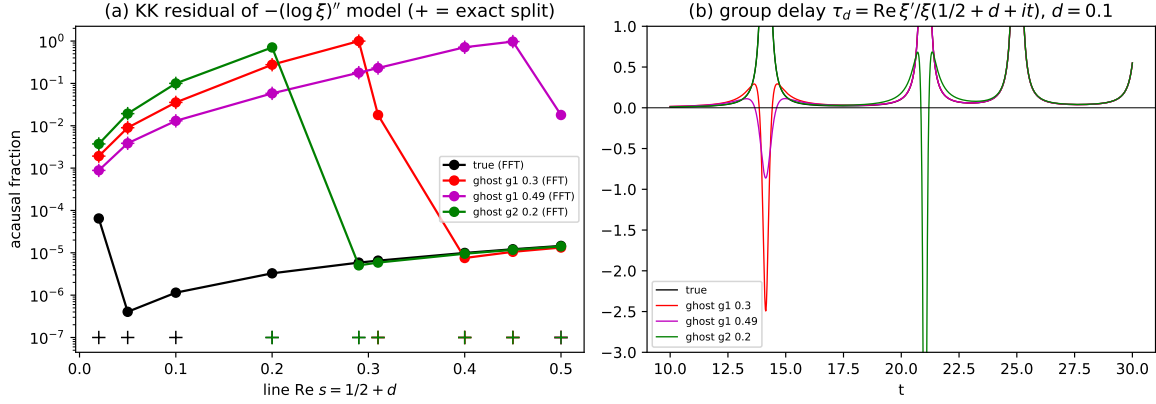


FIGURE 3. (a) Acausal fraction of $-(\log \xi)''$ (100-zero model) on lines $\text{Re } s = \frac{1}{2} + d$. Dots: FFT; +: exact split (zero for true; plotted at 10^{-7}). (b) Group delay at $d = 0.1$: negative dips at ghosts.

equivalent to RH; (S2) is necessary only (L116: 17 Laguerre-invisible ghost sites). In the tilt language, $\Theta(t) = \frac{1}{2} \arg(\log \xi)''(\frac{1}{2} + it)$ crosses the null cone exactly where Λ changes sign. The count of crossings is 0 for true ξ (RH) and 2 per *detected* ghost. This makes it a Krein-type index that is *weaker* than the Toeplitz inertia n_- of §1, which counts every quartet (2 each) ([ANALOGY]/[THEOREM]).

8. VERDICT

- (1) [THEOREM]: the subluminal picture is *exact*. Szegő/Schur steps are $U(1, 1)$ boosts, the critical dimension is boost energy $4q \sum \sinh^2 r$, off-line pairs are hyperbolic planes on which $p^{-1/2} \text{Fr}$ boosts by $\delta \log p$, and group delay / KK give the causal reading.
- (2) [THEOREM]: the Lagarias–Suzuki unconditional positivity is the $c = \frac{1}{2}$ case of a shift-dominance family. Doubling is no more than the unconditional strip width, which is why it is ghost-blind.
- (3) [RESTATEMENT]: every RH-equivalent version (all steps subluminal, all shifts $c > 0$, KK for all $d > 0$, $\tau_d > 0$, minimum phase, PR realization) is a restatement. “Arithmetic Castelnuovo–Severi” remains an *analogy*: no arithmetic surface or intersection pairing is produced (L114/L115).
- (4) [OPEN]: an undoubled object with an *independent* positivity, such as a Maass–Selberg-type identity at shift $c \rightarrow 0$ or a passive realization of the Weil Carathéodory function built from primes, is the missing piece.

9. CORRECTIONS

Chat claims t485–t486.

- C117.1 “Weil positivity $\iff$ every step subluminal.” True for the discretized hat-basis Toeplitz form, in the precise sense of Cor. 1.2 (RH $\Rightarrow$ all h ; converse off a countable aliasing set, L112 Thm. 3.1). It is a restatement, not a new inequality.
- C117.2 “Rapidity ≈ 0.405 .” $\text{artanh } 0.38361 = 0.4043$.
- C117.3 “Ghosts hit c at onset.” They jump over c : $-0.571 \rightarrow -1.106$, $-0.924 \rightarrow -9.792$, $-0.649 \rightarrow -1.625$ (Table 1). Superluminal steps come in pairs, and n_- (2 per quartet) is the number of negative E_n , not the number of superluminal steps (4 per quartet).
- C117.4 “Velocity $|\alpha|$, rapidity $\text{artanh } |\alpha|$.” Correct on $\mathbb{C}^{1,1}$ (Thm. 1.1(b)). In $SO^+(2, 1)$ the rapidity is $2 \text{artanh } |\alpha|$.
- C117.5 “On-line zero = null.” An on-line zero gives a definite (negative) square. Null directions occur only inside off-line hyperbolic planes and the pole plane.

- C117.6 “*Doubling puts zeros at $\rho/2$, hence unconditional.*” The mechanism is LS Thm. 4 with $a = c = \frac{1}{4}$, equivalently $F = \xi$ with $c = \frac{1}{2}$ (Prop. 3.2(b)). It is ghost-blind ([NUMERICAL]). Symmetry axis $\frac{1}{4}$ (zeros of $\xi(2s)$) versus stability boundary $\frac{1}{2}$ is the right picture.
- C117.7 “ $E(i, s) = 2\zeta(s)L(s, \chi_{-4})/\zeta(2s)$.” Correct for $E = \frac{1}{2} \sum_{\text{coprime}}$. Summing over all $(m, n) \neq 0$ gives $4\zeta(s)L(s, \chi_{-4})$, i.e. $\zeta(2s)$ times more.
- C117.8 *RH-type behaviour of E .* At i and ρ it is RH+GRH ([RESTATEMENT]), not a property of $E(z, s)$ in general: real off-line zeros are forced on 32% of $\mathcal{F} \cap \{y \leq 10\}$ (Prop. 4.2), and complex off-line pairs occur at $i\sqrt{5}$ and $0.1 + 1.2i$ below $t = 20$, consistent with LS’s remark that Epstein zetas generically violate RH [2].
- C117.9 “ $KK \iff RH$ via $\text{Re } s > \frac{1}{2}$.” This needs subtraction (ξ'/ξ grows like $\frac{1}{2} \log t$; use $(\log \xi)''$ or subtracted relations). KK must hold on *every* line $\text{Re } s = \frac{1}{2} + d$, $d > 0$. On the critical line itself classical KK fails for all models.
- C117.10 *Källén–Lehmann.* Positivity of spectral weights is automatic (residues = multiplicities) and does not distinguish ghosts. RH controls pole location.
- C117.11 *Superluminal group velocity.* In physics it is not a violation (front at c). For ξ , a sign reversal of τ_d on $\text{Re } s > \frac{1}{2}$ is equivalent to $\neg\text{RH}$, because the medium is lossless with unit-weight poles. The physical intuition does not transfer.
- C117.12 “*Arithmetic Castelnuovo–Severi.*” Over $\mathbb{F}_q$ Castelnuovo–Severi is a theorem (Hodge index, L114). Over $\mathbb{Q}$ the subluminal recursion is a restatement of Toeplitz positivity: an analogy, not an inequality on a surface.

Internal.

- C117.13 The Schur parameters are reliable only for $n \leq 10$. They tend to 1 because Θ is inner-type on the boundary line, and for $n \geq 12$ the computed values exceed 1 even for the true ξ (precision loss, not a ghost signal).
- C117.14 FFT KK residuals are unreliable when a pole is within $\approx 2\Delta t$ of the line (the $d = 0.31$ and $d = 0.5$ entries) or exactly on it ($(\gamma_2, 0.2)$ at $d = 0.2$).

10. RANKED DIRECTIONS

- (1) **Shift-dominance flow** $c \downarrow 0$ (0.60). Track $\sup |\xi(s - c)/\xi(s + c)|$ and its Schur parameters as c decreases, on the true zeros up to height T . The first c at which the Schur algorithm loses contractivity would be a quantitative “distance to RH” in the doubling chart.
- (2) **Prime-side passive realization** (0.55). Build the L112 Carathéodory function’s layered medium from primes alone, and test whether reflection coefficients from the explicit-formula side stay < 1 for larger L and $h \rightarrow 0$.
- (3) **Maass–Selberg at a point** (0.48). Find a positivity for $E(z, s)$ at CM points that does not pass through the constant term (cf. forced zeros for $y > y^*$).
- (4) **Rapidity statistics** (0.44). Compare the distribution of $\text{artanh } |\alpha_n|$ for true primes with random-measure (CUE/Killip–Nenciu) predictions (L112 direction).
- (5) **Krein index from light-cone tilt** (0.40). Formalize the §7 crossing count and compare it with the Toeplitz n_- on the L116 invisible ghost sites.

11. QUESTIONS FOR J.W.

- Q1. Is the shift-dominance family $|\xi(s + c)| > |\xi(s - c)|$ ($c \downarrow 0$) the “undoubled target” you had in mind, or did you want an object with *poles* at ρ (like $E(i, s)$ has zeros at ρ and poles at $\rho/2$)?
- Q2. Should the boost picture be developed on $\mathbb{C}^{1,1}$ (rapidity $\text{artanh } |\alpha|$) or on $SO^+(2, 1)$ (rapidity doubled)? The Doppler identity favours $\mathbb{C}^{1,1}$.
- Q3. For KK: is the critical line or the family of lines $\text{Re } s = \frac{1}{2} + d$ the physical frequency axis in your picture?

- Q4. Do you want the CM-point E tests extended to class number > 1 forms (Epstein zetas with known off-line zeros)?
- Q5. Should L118 pursue direction 1 (shift flow with Schur parameters) or direction 2 (prime-side passive realization)?

REPRODUCIBILITY

From `/workspace/taa: python3 num/lemma117_boost.py, lemma117_schur.py` (≈ 10 min), `lemma117_eis.py` (≈ 30 min with zero counts), `lemma117_kk.py`, then `lemma117_figs.py`. Outputs in `num/l117/`: `boost.json`, `rap_world0..4.npy`, `abs_alpha.json`, `schur.json`, `eis.json`, `Ehalf_grid.npy`, `kk.json`, `tau_curves.npy`. Figures: `build117/figs/f1_rapidity`, `f2_eis`, `f3_kk` (pdf, png). Inputs: `num/l112/weil_verblunsky.json`, `num/l104/zeros.json`, `num/l116/thermo.json`.

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**TAA LEMMA 118: PASSIVITY AND THE ARROW OF TIME. OSCILLATOR
BATHS, SZ.-NAGY–FOIAS AND LAX–PHILLIPS SEMIGROUPS, THE
MODULAR SURFACE, DARLINGTON FROM PRIMES, AND
PUSZ–WORONOWICZ**

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

RH is not proved. Nothing here proves or claims to prove the Riemann Hypothesis. Every equivalence with RH below is a *reformulation*. Labels: [THEOREM], [RESTATEMENT], [ANALOGY], [HEURISTIC], [CONJECTURE], [OPEN], [NUMERICAL] (scripts `num/lemma118_*.py`, data `num/1118/`); sources [CITED], [CITED-sec], [BIB], [TAA-L]. References: `lit/lemma118/verified_refs.md`. Ghosts follow the L103 convention: the pair $\frac{1}{2} \pm i\gamma$ is replaced by the quartet $\frac{1}{2} \pm \delta \pm i\gamma$.

ABSTRACT. J.W. (t488) proposed: light is passive, the arrow of time is passive, and passivity of the arrow emerges from a global collection of oscillators. We give the rigorous version and test it on ζ .

- **Arrow from a bath** ([CITED] FKM, CL; [THEOREM]; [NUMERICAL]). A finite linear system of oscillators is reversible and almost periodic, so it recurs and cannot damp. Irreversibility of a part needs *infinitely many* modes. In the Lax–Phillips (LP) form, a passive semigroup is the compression $Z(t) = P_+ U(t) P_-$ of a unitary group to the complement of infinite-dimensional incoming/outgoing channels. Finite channels are impossible (Prop. 1.2). [NUMERICAL]: a Caldeira–Leggett bath of N modes damps at rate 0.1032 ($\eta/2 = 0.1$), then revives at $2\pi N/\Omega + 9.2$ for every N . Mean recurrence times of M random modes and of M prime phases $t \log p$ both grow geometrically (Fig. 1).
- **The modular surface is a passive system, unconditionally** ([CITED] LP80, Uetake; [THEOREM]). The LP scattering matrix is $\phi(s) = \Lambda(2s-1)/\Lambda(2s)$ ([NUMERICAL] 2×10^{-31}). Its reduced form $\Phi = \xi(2s-1)/\xi(2s)$ is *inner* on $\text{Re } s > \frac{1}{2}$. The LP semigroup Z'' is a contraction with no input about zeros. Resonances sit at $\rho/2$, symmetric about $\frac{1}{4}$. The arrow of time is the *gap* $\frac{1}{4}$ between the symmetry axis and the stability boundary $\frac{1}{2}$. Under RH every mode decays at exactly rate $\frac{1}{4}$. A ghost splits the rates to $\frac{1}{4} \mp \frac{\delta}{2} > 0$ and is never a gain mode, so doubled passivity is *ghost-blind* ([NUMERICAL]). This is L117’s “doubling = shift $c = \frac{1}{2}$ ”.
- **Reversible + passive = lossless** ([THEOREM], Thm. 4.1). If the resonance set is symmetric about the boundary (the functional equation), the system is passive iff every resonance lies *on* the boundary. An off-line pair straddles it, and one member is a gain mode. Gram inertia $n_- = 2$ per ghost pair ([NUMERICAL]). Undoubled, RH therefore means the arithmetic system is *lossless*, with no arrow at all ([RESTATEMENT]). The arrow exists only after doubling.
- **Darlington from primes** ([CITED] Darlington/Arov/Douglas–Helton; [THEOREM]; [NUMERICAL]). Each Euler factor is a lossless resonator (a delay loop of length $\log p$; $\tanh(\frac{s}{2} \log p)$ is a shorted lossless stub). A true chain cascade of lossless sections never loses passivity. The Euler product, however, *multiplies impedances*, and loses positive-reality at σ_N : 0.394 (2 primes), 0.998 ($p \leq 29$), 1.1895 ($p \leq 10^7$), converging to van de Lune’s $\sigma_0 = 1.19234733718619 \dots$ ([NUMERICAL], 25 digits). Finite cascades never vanish and are ghost-blind. The Hilbert-space (passive) boundary of prime-side objects is $\sigma = \frac{1}{2}$ (ℓ^2 , HLS), but every manifestly positive prime-side kernel evaluates ζ at *doubled* arguments. Circularity is exact: the layer-stripping cascade of the arithmetic target has passive sections iff RH (L112/L117).
- **Pusz–Woronowicz** ([CITED]). Gibbs/KMS states are completely passive, automatically. Primon-gas and Bost–Connes passivity uses ζ only at real $\beta > 1$, and is ghost-blind (L116). Light is *lossless*; its arrow is a boundary condition plus an infinite reservoir.

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GLOSSARY (FOR A PHYSICIST)

Passive / lossless: Passive: output energy $\leq$ input energy (contraction, Schur or positive-real transfer function) [19]. Lossless: equality (unitary; all-pass/inner on the boundary).

Schur / inner / positive-real (PR): Schur: analytic, $|S| \leq 1$ on the right half-plane. Inner: Schur with $|S| = 1$ a.e. on the boundary. PR: $\text{Re } Z \geq 0$; $Z \mapsto (Z - 1)/(Z + 1)$ maps PR to Schur [20].

Dilation, characteristic function: Every contraction T is PU^nP for a unitary U [17, 18]. For a completely non-unitary T , the Sz.-Nagy–Foiias characteristic function Θ_T is a Schur function and a complete unitary invariant.

Lax–Phillips system: A unitary group $U(t)$ with incoming/outgoing subspaces $\mathcal{D}_\mp$; $Z(t) = P_+U(t)P_-$; the scattering matrix S relates the incoming to the outgoing translation representations [3].

KMS / passive state: Equilibrium at inverse temperature β . Passive: no work is extractable by cyclic processes [7].

1. FORD–KAC–MAZUR AND CALDEIRA–LEGGETT: THE ARROW FROM A BATH

Sources ([CITED]). FKM [1], abstract: “a system of coupled harmonic oscillators can be made a model of a heat bath. Thus a particle coupled harmonically to the bath ... will (in an appropriate limit) exhibit Brownian motion.” Their program (i)–(iii) solves the full reversible equations of motion, randomizes the bath initial data, and derives a stochastic process for the particle; the limit is $N \rightarrow \infty$ with a frequency cutoff. Caldeira–Leggett [2] (Physica A 121 (1983) 587, title and abstract verified) apply the Feynman–Vernon influence functional to a particle linearly coupled to an oscillator bath with spectral density $J(\omega)$. Poincaré recurrence for finite-volume measure-preserving flows: [14].

Proposition 1.1 (finite baths recur). *[THEOREM] Let $H = \frac{1}{2}p^T p + \frac{1}{2}q^T K q$ with $K > 0$ of size $N + 1$. Every coordinate is $x(t) = \sum_k a_k \cos \omega_k t + b_k \sin \omega_k t$, an almost periodic function. Hence for every $\epsilon, T > 0$ there is $t > T$ with $|x(t) - x(0)| < \epsilon$, and $x(t) \rightarrow 0$ forces $x \equiv 0$. A finite system cannot damp.*

Proof. Diagonalize K . For a finite sum of almost periodic terms, Kronecker/Dirichlet simultaneous approximation of $(\omega_k t \bmod 2\pi)_k$ gives returns, and the Bohr mean $\lim T^{-1} \int_0^T |x|^2 = \frac{1}{2} \sum (a_k^2 + b_k^2)$ vanishes only for $x \equiv 0$ [15]. $\square$

Proposition 1.2 (the arrow needs infinite channels). *[THEOREM] Let $U(t)$ be a unitary group on $\mathcal{H}$ and $\mathcal{D}_+ \subset \mathcal{H}$ a closed subspace with $U(t)\mathcal{D}_+ \subset \mathcal{D}_+$ ($t \geq 0$) and $\bigcap_t U(t)\mathcal{D}_+ = \{0\}$ (LP (1.11)(i)–(ii) [3]). If $\dim \mathcal{D}_+ < \infty$, then $\mathcal{D}_+ = \{0\}$. Consequently a nontrivial LP semigroup, which is the rigorous “passive arrow”, exists only with infinite-dimensional channels; by the translation representation these are $L^2(\mathbb{R}_+, \mathcal{N})$ [3].*

Proof. $U(t)$ is injective, so $U(t)\mathcal{D}_+ \subset \mathcal{D}_+$ with equal finite dimension forces $U(t)\mathcal{D}_+ = \mathcal{D}_+$, and then $\bigcap_t U(t)\mathcal{D}_+ = \mathcal{D}_+$. $\square$

This is the rigorous form of J.W.’s “passivity emerges from a global collection of oscillators” [THEOREM]: *the whole is unitary; the part, compressed away from infinite outgoing channels, is a contraction semigroup.* FKM/CL are concrete instances: the bath continuum is the translation representation.

Numerics ([NUMERICAL], lemma118_bath.py). We use a CL star bath with ohmic density $J(\omega) = \eta\omega$ ($\omega < \Omega$), $\eta = 0.2$, $\Omega = 4$, $\omega_0 = 1$, N equally spaced modes $\omega_j = (j - \frac{1}{2})\Omega/N$, $c_j^2 = \frac{2}{\pi}\eta\omega_j^2\Delta\omega$, the counterterm, and a relaxed bath. The solution is exact via normal modes.

N	decay rate (fit)	min envelope before revival	revival peak t	$2\pi N/\Omega$
25	0.1031	4.0×10^{-2}	48.48	39.27
50	0.1032	1.5×10^{-3}	87.76	78.54
100	0.1032	2.5×10^{-5}	166.30	157.08
200	0.1032	7.4×10^{-6}	323.38	314.16
400	0.1032	3.8×10^{-6}	637.54	628.32

The damping rate is N -independent and equals the Markov value $\eta/2$ up to a cutoff correction. The revival (peak $|x| = 0.747$) occurs at the Heisenberg time $2\pi/\Delta\omega$ plus a constant 9.2, which matches the Wigner delay $\approx 2/\eta = 10$ of the resonance ([HEURISTIC]). For incommensurate (random) frequencies, Poincaré recurrence into $|\omega_k t \bmod 2\pi| < 0.5$ has mean return time 3.1, 13.9, 94.9, 271, 1847, 8180, 52632 for $M = 1..7$ modes, growing geometrically (Fig. 1b). The phases $t \log p$ of the first M primes behave the same way ($8.4 \rightarrow 38462$), because $\{\log p\}$ is $\mathbb{Q}$ -independent (Kronecker [16]). *A truncated Euler product is a finite bath* (§4.1).

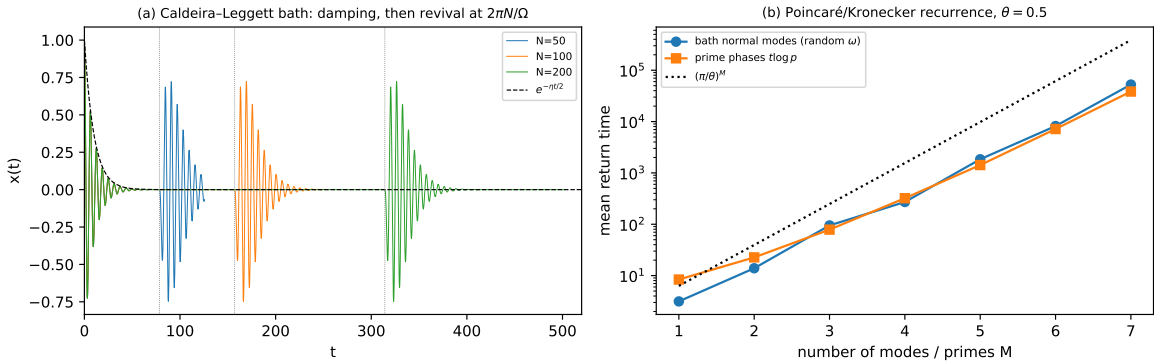


FIGURE 1. (a) CL bath, $x(t)$ for $N = 50, 100, 200$: damping $e^{-\eta t/2}$ then revival near $2\pi N/\Omega$ (dotted). (b) Mean recurrence time vs number of modes: random bath frequencies and prime phases $t \log p$.

2. SZ.-NAGY–FOIAS DILATION AND LAX–PHILLIPS SCATTERING

Facts ([CITED]/[CITED-sec]). (i) Sz.-Nagy (1953): every contraction T on H has a minimal unitary dilation, $T^n = PU^n P$; a completely non-unitary (c.n.u.) contraction dilates to a direct sum

of bilateral shifts [17, 18]. (ii) LP [3, §1]: $U(t)$ unitary, with $\mathcal{D}_\mp$ satisfying $U(t)\mathcal{D}_- \subset \mathcal{D}_-$ ($t < 0$), $U(t)\mathcal{D}_+ \subset \mathcal{D}_+$ ($t > 0$), trivial intersections, and dense unions. $Z(t) = P_+U(t)P_-$ is a strongly continuous semigroup on $\mathcal{K} = \mathcal{H} \ominus (\mathcal{D}_- \oplus \mathcal{D}_+)$ [Thm. 3.1]. If $\mathcal{D}_- \perp \mathcal{D}_+$, S is causal and $S(z)$ is analytic in the lower half-plane. (iii) The LP S -matrix is “essentially the characteristic function” of the cogenerator of Z , a compression of the shift whose characteristic function is inner; singularities of S correspond to the spectrum of the generator (Adamyán–Arov, as presented in [6, §1]).

Proposition 2.1 (passivity is structural). *[THEOREM] In any LP system, $\|Z(t)\| \leq 1$ and $Z(t) \rightarrow 0$ strongly. The scattering matrix is a Schur (indeed inner) function. No spectral information enters.*

Proof. $Z(t)$ is a product of a unitary and orthogonal projections, hence a contraction. For strong decay, $\bigcup_t U(t)\mathcal{D}_+$ is dense and Z annihilates $\mathcal{D}_+$ after transport. The Schur and inner properties follow from (iii). $\square$

Physics reading ([ANALOGY]). The unitary group is the reversible microscopic dynamics; $\mathcal{D}_-/\mathcal{D}_+$ are the remote past and future (incoming and outgoing radiation); $P_\pm$ is the *retarded boundary condition* “nothing comes in, what goes out is gone”. The arrow is $P_+(\cdot)P_-$, not the dynamics.

3. LAX–PHILLIPS AND FADDEEV–PAVLOV ON THE MODULAR SURFACE

Exact statements ([CITED] [3], opened). LP use $L = L_0 + \frac{1}{4}$ and the wave equation $u_{tt} = y^2\Delta u + u/4 = Lu$ (1.9). The zero Fourier coefficient of the Eisenstein data is $\{y^{1/2+iz}, izy^{1/2+iz}\} + s(z)\{y^{1/2-iz}, \dots\}$ (4.7) with

$$s(z) = \frac{\sqrt{\pi}\Gamma(iz)\zeta(2iz)}{\Gamma(\frac{1}{2}+iz)\zeta(1+2iz)} \quad (4.7)'$$

Thm. 6.5: the spectral representer of the scattering operator is $S'(\sigma) = -s(\sigma)$. Verbatim: “The poles of S' are the nontrivial zeroes of $\zeta(1+2iz)$. Thus if the Riemann hypothesis is true these poles will lie on the line $\text{Im } z = 1/4$.” On Faddeev–Pavlov [5] (cited through LP; original not opened): “Pavlov and Faddeev have formulated the Riemann hypothesis in terms of properties of Z'' : The Riemann hypothesis is true if and only if $Z''(t)$ decays exponentially. Specifically, in [6] we have shown: The Riemann hypothesis is true if $\limsup t^{-1} \log \|Z''(t)f\|_E \leq -1/4$ for a set of f dense in $\mathcal{K}_C'$ ” (6.19). The OCR of the inequality sign is ambiguous; see C118.7. “The infinitesimal generator B'' of Z'' has as its spectrum the values $\{(\kappa-1)/2\}$ ”, and $2B'' + \frac{1}{2}$ “is the kind of operator ... envisaged by Hilbert; all that is missing is an appropriate inner product ... which makes (6.20) skew-hermitian.” In §5, E is positive on $\mathcal{K}_E$ and $U(t)$ preserves it. Uetake [4] (opened): the spectrum of the generator equals the poles of the scattering matrix, with multiplicity. With $\mathcal{D}_\pm''$ the matrix is $-\frac{s-1/2}{s+1/2} \xi(2s)/\xi(-2s)$ (his variable), and “RH is equivalent to saying that all the poles ... lie on $\text{Re } s = -1/4$ ”.

Checks ([NUMERICAL], lemma118_lp.py). With $s = \frac{1}{2} + iz$: $s(z) = \phi(s) := \Lambda(2s-1)/\Lambda(2s)$, $\Lambda = \pi^{-s/2}\Gamma(s/2)\zeta$, relative error 2.0×10^{-31} . $\max_t |\phi(\frac{1}{2} + it) - 1| = 3 \times 10^{-31}$. $\text{Res}_{s=1}\phi = 0.95492966 = 3/\pi$ (the constant eigenfunction). $\Phi := \xi(2s-1)/\xi(2s) = \phi \cdot \frac{s-1}{s}$ removes that pole (10^{-31}).

Theorem 3.1 (the doubled system is passive, unconditionally). (a) *[THEOREM] (given LP §5) $\|Z''(t)\|_E \leq 1$ for all $t \geq 0$.* (b) *[THEOREM] Φ is inner on $\text{Re } s > \frac{1}{2}$. It is analytic there because $\xi(2s) \neq 0$ for $\text{Re } 2s > 1$ (Euler product). It is unimodular on $\text{Re } s = \frac{1}{2}$ by $\xi(2s-1) = \xi(2-2s) = \overline{\xi(2\bar{s})}$. A hypothetical zero $1 + i\gamma_0$ would put a numerator and a denominator zero at the same boundary point $\frac{1}{2} + \frac{i}{2}\gamma_0$, which cancel. The bound $|\Phi| \leq 1$ is the LP structure (Prop. 2.1). [NUMERICAL]: $\sup |\Phi| = 0.99815$ on the L117 grid (0.99774 with a ghost).* (c) *[THEOREM] The poles of Φ are at $\rho/2$, with $\text{Re } \rho \in (0, \frac{1}{2})$, symmetric about $\frac{1}{4}$. The decay rate of mode ρ is $(1-\beta)/2$. **RH** $\iff$ **every rate equals** $\frac{1}{4}$ [RESTATEMENT]. Each mode decays $\iff \beta < 1$ (Hadamard–de la Vallée Poussin). A uniform rate would force $\sup \beta < 1$ (open).* (d) *[THEOREM] A ghost quartet with $0 < \delta < \frac{1}{2}$ gives rates $\frac{1}{4} \pm \frac{\delta}{2} > 0$. It is never a gain mode, so doubled passivity is ghost-blind.*

Proof. (a) On $\mathcal{K}'_C$, E is positive definite and U -invariant (LP §5), so U is unitary there; $P''_{\pm}$ are E -orthogonal. (b) As stated; the cancellation uses that $\rho \mapsto 1 - \bar{\rho}$ preserves multiplicity. (c) From B'' -spectrum $\{(\kappa - 1)/2\}$. (d) Immediate. $\square$

The arrow is the gap ([THEOREM]/[ANALOGY]). In L117's chart $w = 2s - \frac{1}{2}$, $\Phi(s) = \Psi_{1/2}(w)$ with $\Psi_c(w) := \xi(w - c)/\xi(w + c)$ (identity checked in L117). In the w chart the symmetry axis is $\operatorname{Re} w = \frac{1}{2} - c$, the boundary is $\frac{1}{2}$, and the *gap* is c . For Ψ_c , every on-line mode decays at rate c and a ghost at rates $c \pm \delta$. The gap is exactly the irreversibility. Doubling buys the gap $c = \frac{1}{2}$, the full strip half-width (in LP's time units it is $\frac{1}{4}$). This is why it is unconditional and ghost-blind. J.W.'s “symmetry axis $\frac{1}{4}$ vs boundary $\frac{1}{2}$ ” is exactly this.

Gram/inertia test ([NUMERICAL]). For resonances λ_k , the model-space Gram matrix is $G_{jk} = -1/(\bar{\lambda}_j + \lambda_k)$, the $L^2(0, \infty)$ Gram of $e^{\lambda_k t}$. Its inertia is $n_-(G) = \#\{\operatorname{Re} \lambda > 0\}$, and when $G > 0$ the compressed translation $Z(t)$ has $\|Z(t)\|_G = 1$ exactly (a contraction, reached in the limit). We use 30 zeros + conjugates.

world	doubled $\lambda = \frac{\rho-1}{2}$ n_- / rates	undoubled $\lambda = \rho - \frac{1}{2} - c$: $n_-(G)$ at $c =$				
		0.05	0.15	0.25	0.29	0.31
true	0 / {0.25}	0	0	0	0	0
ghost ($\gamma_1, 0.3$)	0 / {0.1, 0.25, 0.4}	2	2	2	2	0
ghost ($\gamma_2, 0.2$)	0 / {0.15, 0.25, 0.35}	2	2	0	0	0
ghost ($\gamma_1, 0.49$)	0 / {0.005, 0.25, 0.495}	2	2	2	2	2

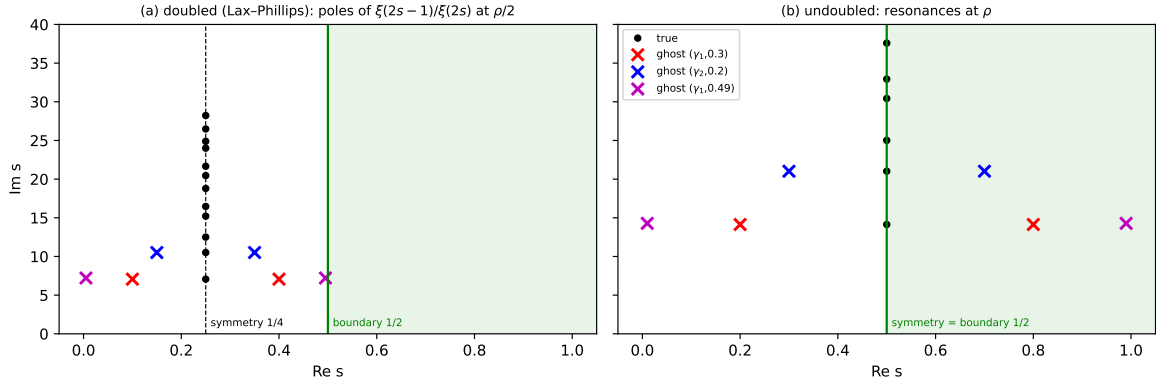


FIGURE 2. Resonances. (a) Doubled (LP): poles of Φ at $\rho/2$; symmetry axis $\frac{1}{4}$ (dashed) $\neq$ boundary $\frac{1}{2}$ (green). Ghost pairs straddle $\frac{1}{4}$ but stay stable. (b) Undoubled: symmetry axis = boundary, and every ghost pair puts one member in the unstable (green) half-plane.

4. THE UNDOUBLED PASSIVE SYSTEM

Theorem 4.1 (reversible + passive $\Rightarrow$ lossless). *[THEOREM] Let $S = \prod_k b_{z_k} / \prod_k b_{p_k}$ be a (finite or convergent) ratio of half-plane Blaschke factors with resonance set $R = \{p_k\}$. Assume time-reversal symmetry $R = \sigma(R)$, $\sigma(s) = 1 - \bar{s}$ (the functional equation), and $|S| = 1$ on $\operatorname{Re} s = \frac{1}{2}$. Then S is passive (Schur on $\operatorname{Re} s > \frac{1}{2}$) iff $R \subset \{\operatorname{Re} s = \frac{1}{2}\}$ (boundary resonances are understood as limits of the shifted family Ψ_c , $c \downarrow 0$). An off-line pair $\{\rho, \sigma(\rho)\}$ straddles the boundary, and its right member is a gain mode with rate $\beta - \frac{1}{2}$. The model-space Gram matrix has $n_- = \#\{\text{gain modes}\}$ (2 per ghost quartet).*

Proof. A pole in $\operatorname{Re} s > \frac{1}{2}$ violates Schur. By symmetry, poles off the line come in pairs, one on each side. The inertia follows from the Lyapunov identity $\Lambda^*G + G\Lambda = -\mathbf{1}\mathbf{1}^*$ (rank one) and the inertia theorem; it is checked numerically above. $\square$

Corollary 4.2. [RESTATEMENT] For $\Psi_c = \xi(\cdot - c)/\xi(\cdot + c)$: Ψ_c is unimodular on $\operatorname{Re} s = \frac{1}{2}$ for every c (FE). It is passive (inner) iff every zero has $\beta < \frac{1}{2} + c$. “Passive for all $c > 0$ ” $\iff$ RH $\iff |\xi(s + c)| > |\xi(s - c)|$ on $\operatorname{Re} s > \frac{1}{2}$ for all $c > 0$ (L117) $\iff \tau_d = \operatorname{Re} \xi'/\xi(\frac{1}{2} + d + it) > 0$ for all $d > 0$ (L117) $\iff$ the L112 Carathéodory function is positive-real. As $c \downarrow 0$ the gap closes: RH says the undoubled arithmetic system is lossless, a reversible medium with no arrow (like light).

de Branges form ([THEOREM]). The Pick kernel of Ψ_c is
$$\frac{\xi(w + c)\overline{\xi(z + c)} - \xi(w - c)\overline{\xi(z - c)}}{(w + \bar{z} - 1)\xi(w + c)\overline{\xi(z + c)}}.$$

Using $\xi(s - c) = E_c^\#(s)$ for $E_c := \xi(\cdot + c)$ (FE), passivity of Ψ_c is exactly the Hermite–Biehler property of E_c , i.e. the existence of the de Branges space $\mathcal{H}(E_c)$. This is the Sz.-Nagy–Foiás model space $H^2 \ominus \Psi_c H^2$ (§2).

Connes’ absorption picture as a passive shadow ([CITED] [8]; [THEOREM]). Connes: critical zeros form an absorption spectrum (Thm. 1, with a Sobolev parameter $\delta > 1$, seeing *only* critical zeros); “eventual noncritical zeros appear as resonances” entering through “their harmonic potential with respect to the critical line”. In the cutoff limit (Lemma 3, char. p), each zero contributes $\int \hat{f} d\mu_\rho$ with μ_ρ the harmonic measure (Poisson kernel) of ρ on $i\mathbb{R}$. This limit Δ_∞ is of positive type for every zero set. The global trace formula, i.e. $\Delta_\infty = \text{Weil’s } \Delta$, is equivalent to RH for all L -functions with Grössencharakter (Thm. 5 in char. p ; asserted analogously for number fields). *New identity* [THEOREM]: for an off-line pair at distance δ , the excess group delay of its all-pass factor (L117: $2\delta/(\delta^2 + x^2)$) is 2π times the Poisson density $\frac{1}{\pi}\delta/(\delta^2 + x^2)$. So Connes’ harmonic measure *is* the normalized excess delay. Δ_∞ is the passive (minimum-phase) shadow of Δ , and RH says the shadow equals the original. Connes also notes (§I(B)) that “the Riemann flow cannot satisfy time reversal symmetry” [8].

4.1. Darlington synthesis and the prime-by-prime cascade. Sources ([CITED]/[BIB]). Darlington (1939): every PR function is the impedance of a lossless two-port terminated in one resistor [21]. Brune: PR is necessary and sufficient for passive realizability; Richards’ theorem and unit elements [22, 21]. Non-rational case: the scalar Darlington problem (embed a as an entry of a 2×2 inner matrix) is solvable iff $\|a\|_\infty \leq 1$ and a is pseudocontinuable of bounded type (Arov 1971; Douglas–Helton 1973 [10]; as stated in [9, §1,§4]).

Proposition 4.3 (what a Darlington realization from primes needs). [THEOREM] (a) Φ and every passive Ψ_c are inner, hence pseudocontinuable of bounded type (continuation $1/\overline{\Psi(1 - \bar{s})}$). Their Darlington realization is lossless (no resistor): the LP system itself, or the model space $\mathcal{H}(E_c)$. (b) The obstruction is never pseudocontinuation; it is $\|a\|_\infty \leq 1$, which is RH for $c \downarrow 0$ (Cor.). (c) A chain cascade (product of J -inner transfer matrices of lossless sections, e.g. unit elements of delay $\frac{1}{2} \log p$) is J -inner. Its input impedance with resistive termination is PR for all $\sigma > 0$, at every truncation: a genuine cascade never loses passivity. (d) The Schur (layer-stripping) algorithm is the Darlington cascade. For the L112 target its section parameters are the Verblunsky coefficients, and $|\alpha_n| < 1 \ \forall n \iff$ RH (L112/L117). **Circularity is exact:** a passive cascade realizing the arithmetic target exists iff RH. A non-circular construction would need sections indexed by primes whose passivity is manifest and whose cascade provably equals the target.

Proposition 4.4 (the Euler product is not a passive cascade). [THEOREM]/[CITED] (a) Each factor $f_p(s) = (1 - p^{-s})^{-1}$ has $\operatorname{Re} f_p > \frac{1}{2}$ on $\operatorname{Re} s > 0$ and poles at $2\pi ik/\log p$ on $\operatorname{Re} s = 0$: a lossless resonator (delay loop of length $\log p$, unit feedback). $Z_p = \tanh(\frac{s}{2} \log p) = (1 - p^{-s})/(1 + p^{-s})$ is a shorted lossless stub. (b) The product $P_N = \prod_{p \leq N} f_p$ multiplies impedances, which is not a network interconnection. P_N is PR on $\operatorname{Re} s = \sigma$ for all t iff $\sum_{p \leq N} \arcsin p^{-\sigma} \leq \pi/2$. Proof: $\arg f_p$ ranges

over $[-\arcsin p^{-\sigma}, \arcsin p^{-\sigma}]$, and Kronecker makes $(t \log p)$ dense on the torus. As $N \rightarrow \infty$ this is van de Lune's theorem: $\operatorname{Re} \zeta > 0$ for $\sigma \geq \sigma_0$, where $\sum_p \arcsin p^{-\sigma_0} = \pi/2$ [11]. (c) P_N never vanishes, and is almost periodic in t (it recurs, Fig. 1b). Every finite prime cascade is zero-free and ghost-blind.

Numerics ([NUMERICAL], lemma118_euler.py). PR threshold σ_N :

N	3	5	7	13	29	100	10^3	10^5	10^7	∞
σ_N	0.3939	0.6449	0.7796	0.9014	0.9979	1.0851	1.1508	1.1827	1.1895	1.19234733718619

σ_0 was computed with prime-zeta acceleration ($\sum_k c_k P((2k+1)\sigma)$). It agrees with the published 1.19234733718619320289... [11] to 25 digits. Check with $\{2, 3, 5\}$ ($\sigma_5 = 0.64489$): the torus infimum of $\operatorname{Re} P_5$ is $-0.0619, -0.0147$ at $\sigma_5 - 0.02, \sigma_5 - 0.005$, and $+0.0142, +0.0540$ at $\sigma_5 + 0.005, +0.02$. A t -scan on $[0, 2 \times 10^5]$ finds $\operatorname{Re} P_5 = -0.0619$ at $t = 86530.61$ for $\sigma_5 - 0.02$ and $\min = +0.054$ for $\sigma_5 + 0.02$. *Bohr point.* The lift $s \mapsto (p^{-s})_p$ leaves ℓ^1 at $\sigma = 1$ (partial sums at $\sigma = 1$: 2.20, 2.71, 3.04 for $N = 10^{3,5,7}$, $\sim \log \log N$) and ℓ^2 at $\sigma = \frac{1}{2}$ ($\sum p^{-2\sigma}$: 2.89, 4.15, 5.48 at $\sigma = 0.45$; 1.73, 1.94, 2.02 at 0.55). Accordingly, the Hilbert space $\mathcal{H}^2$ of Dirichlet series with square-summable coefficients consists of functions analytic on $\operatorname{Re} s > \frac{1}{2}$, with kernel $\zeta(z + \bar{w})$ [12] ([CITED]). *Observation* ([THEOREM]): this manifestly positive prime-side kernel evaluates ζ at $z + \bar{w}$, i.e. at *doubled* real part > 1 . It lives in the Euler-product region, sits at shift $c = \frac{1}{2}$, and is ghost-blind, the same as Φ and the fermionic primon gas $\zeta(\beta)/\zeta(2\beta)$ [23].

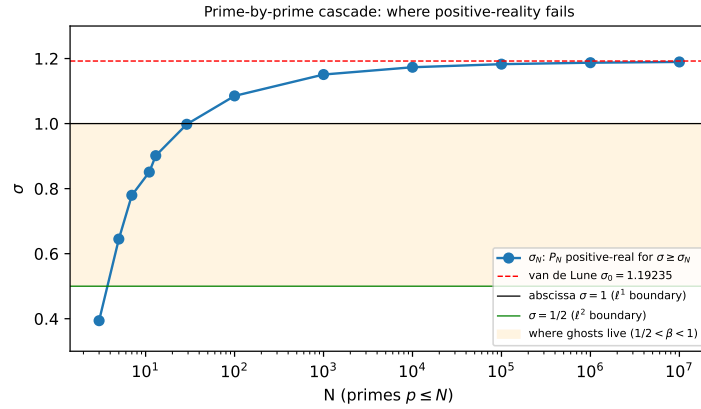


FIGURE 3. Where the truncated Euler product stops being positive-real: σ_N rises from 0.39 (two primes) through 1 at $p \leq 29$ to van de Lune's 1.19235. Ghosts live in $\frac{1}{2} < \beta < 1$ (shaded), invisible to every finite truncation (zero-free).

Recurrence = RH ([CITED] [13, Thm. B], after Bagchi). RH holds iff for every compact $K \subset \{\frac{1}{2} < \operatorname{Re} s < 1\}$ with connected complement and every $\epsilon > 0$, $\liminf_T \nu_T\{\tau \in [0, T] : \sup_K |\zeta(s + i\tau) - \zeta(s)| < \epsilon\} > 0$ (strong recurrence). *Reading* ([ANALOGY]/[RESTATEMENT]): the finite prime bath recurs (Kronecker), and FKM-type irreversibility would mean losing recurrence in the limit. RH is equivalent to the infinite prime bath *still recurring* in the strip, which again says lossless, not dissipative (Thm. 4.1).

5. PUSZ–WORONOWICZ, PRIMON GAS, BOST–CONNES

Source ([CITED] [7], abstract and introduction). “We characterize the equilibrium states ... by a condition saying that the systems are unable to perform mechanical work in cyclic processes. This condition called ‘passivity’ is suggested by the second principle.” “Ground states and β -KMS states

for all inverse temperatures $\beta \geq 0$ are completely passive. We prove that these states are the only completely passive ones.” Primon gas [23]: energies $\log n$, $Z(\beta) = \zeta(\beta)$, Hagedorn divergence at $\beta = 1$; fermionic version without square factors. Bost–Connes: partition function $\zeta(\beta)$ [24]; KMS [25].

Proposition 5.1. [THEOREM] *The Gibbs state of the primon gas at real $\beta > 1$ is completely passive (PW). Its definition and passivity use ζ only at real $\beta > 1$ (and derivatives). Hence thermodynamic passivity of primon/BC states is unconditional and ghost-blind (cf. L116: BC ghost-blind).*

Reading ([ANALOGY]). Zeros of ζ are Fisher zeros of $Z(\beta)$ at complex β with $\operatorname{Re} \beta = \frac{1}{2}$, below the Hagedorn point, where no Gibbs state exists. A Lee–Yang-type statement (“all Fisher zeros on a line”) is RH [26]. PW passivity is a property of states at real temperature, so it cannot see them.

6. LIGHT: LOSSLESS, AND WHERE ITS ARROW COMES FROM

[THEOREM]/[CITED]. Vacuum Maxwell (or the wave equation outside an energy-conserving obstacle) generates a *unitary* group on the energy space: “since the initial-boundary value problem is reversible in time ... the operators $U(t)$ are actually unitary and form a group” [3, §1]. Light is therefore *lossless*, which is stronger than passive. The arrow enters only through (i) an infinite reservoir of modes (the continuum, i.e. LP channels; Prop. 1.2) and (ii) the choice of retarded/outgoing data ($P_{\pm}$; radiation condition; retarded potentials [27]). Wheeler–Feynman (verified [28]): a time-symmetric theory in which a complete absorber turns half-retarded plus half-advanced fields into the observed pure retarded field; the “arrow of time” becomes a property of the absorber boundary condition. Dispersive *media* are passive but not lossless (KK, L117). **Dictionary** ([ANALOGY]): $\text{RH} \leftrightarrow$ the undoubled arithmetic medium is lossless like vacuum; LP doubling $\leftrightarrow$ adding a reservoir and outgoing condition, which produces decay at rate $\frac{1}{4}$.

7. VERDICT

- (1) [THEOREM]: J.W.’s idea has an exact rigorous form. A passive arrow is the compression of a unitary (lossless, reversible) group away from infinite-dimensional channels (Sz.-Nagy–Foiás/LP). Finite systems recur (Props. 1.1, 1.2).
- (2) [THEOREM]: The modular surface is such a system with no hypothesis (Thm. 3.1). Its arrow is the gap $\frac{1}{4}$ between symmetry axis and boundary; RH is “every mode decays at the same rate $\frac{1}{4}$ ” ([RESTATEMENT]). The construction is ghost-blind.
- (3) [THEOREM]: Reversible + passive $\Rightarrow$ lossless (Thm. 4.1). Undoubled, RH is *losslessness* (no arrow), not dissipation ([RESTATEMENT]). Every RH-equivalent passivity is a restatement.
- (4) [THEOREM]/[NUMERICAL]: The Euler product is a product of lossless resonators, but multiplying impedances is not a passive interconnection: PR fails at $\sigma_N \rightarrow 1.19235$, finite truncations are zero-free, and prime-side positive kernels live at doubled arguments. [OPEN]: a Darlington realization from primes needs sections of manifest passivity whose cascade provably equals the target; every known candidate is either ghost-blind or circular.

8. CORRECTIONS

Chat claims t488–t489.

C118.1 “The doubled system is passive unconditionally, requiring only zeros off $\operatorname{Re} s = 1$ (roughly PNT strength).” Passivity (contractivity of Z'' , Schur/inner Φ) requires *no* zero information. It uses only LP’s unitarity, $\xi \neq 0$ on $\operatorname{Re} s > 1$, and the FE. A zero on $\operatorname{Re} s = 1$ would cancel in Φ (Thm. 3.1(b)). PNT strength ($\beta < 1$) is what makes *each mode decay*. A uniform decay rate would need $\sup \beta < 1$ (open). Rate exactly $\frac{1}{4}$ is RH.

- C118.2 “Symmetry axis $\frac{1}{4}$ vs boundary $\frac{1}{2}$.” Correct (in s ; LP’s $\text{Im } z = \frac{1}{4}$ vs the real axis; $\text{Re } w = 0$ vs $\frac{1}{2}$ in L117’s chart). The gap *is* the decay rate, and ghosts move rates to $\frac{1}{4} \pm \frac{\delta}{2}$ without ever crossing.
- C118.3 “Undoubled passivity is exactly RH.” Correct as a restatement, once “undoubled” is defined as the family Ψ_c , $c \downarrow 0$ (at $c = 0$, $\Psi_0 \equiv 1$ is degenerate; the limiting object is τ_d , L117). Added content: by Thm. 4.1 it means *lossless*, so the undoubled system has no arrow.
- C118.4 “Every geometric cusp surface gives $\zeta(2s)$.” Unsupported, and false as stated. The $\zeta(2s)$ comes from the coprime-pair sum in the $SL_2(\mathbb{Z})$ Eisenstein series (L117: $E = \frac{1}{2} \sum_{\text{coprime}}$, LP (4.7’)). Other groups give other scattering determinants. For congruence subgroups these are built from Dirichlet L -functions, and for non-arithmetic cofinite groups no Euler product is known ([HEURISTIC]; these sources were not opened).
- C118.5 “Connes’ absorption picture reduces to Weil positivity.” Partly. Connes reduces RH *for all L -functions with Grössencharakter* (not ζ alone) to validity of a global trace formula. He shows the formula implies positivity of the Weil distribution (Thm. 5, char. p ; analogous for number fields). The absorption spectrum itself (Thm. 1) is unconditional, sees only critical zeros, and is ghost-blind by construction. Non-critical zeros enter by harmonic measure, which is $\frac{1}{2\pi} \times$ the L117 excess delay (§4).
- C118.6 “Light is lossless.” Correct, and stronger than “passive”: the group is unitary. The arrow is the boundary condition plus an infinite reservoir (§6). Material media are only passive.

Internal.

- C118.7 In LP (6.19) the OCR of the inequality sign is ambiguous. We read “ $\leq -\frac{1}{4}$ ”, since under RH the generic rate is exactly $-\frac{1}{4}$. LP state “if” and attribute “iff exponential decay” to Faddeev–Pavlov.
- C118.8 At $c = \delta$ exactly (e.g. $(\gamma_2, 0.2)$, $c = 0.2$) a resonance lies on the boundary and G is singular. A first run gave a spurious $n_- = 4$; such c are excluded from the table.
- C118.9 The CL revival offset (+9.2) is attributed to the Wigner delay only heuristically ([HEURISTIC]).
- C118.10 Bagchi’s theorem is cited through Nakamura’s Theorem B; Bagchi’s originals were not opened. Faddeev–Pavlov, Adamyan–Arov, Arov and Douglas–Helton are cited through LP80, Strauss and Garcia respectively.

9. RANKED DIRECTIONS

- (1) **de Branges space from primes** (0.55; L117 direction 2 sharpened). Compare the manifestly positive HLS kernel $\zeta(z + \bar{w})$ (doubled) with the Pick kernel of Ψ_c , i.e. $\mathcal{H}(\xi(\cdot + c))$. Look for a prime-side identity for the latter at $c < \frac{1}{2}$. The first c at which such an identity needs zeros measures the circularity.
- (2) **Inertia flow $n_-(c)$** (0.50). Track the model-space Gram inertia of Ψ_c on true zeros up to height T as $c \downarrow 0$, against ghosts. This is an operator-theoretic “distance to loss of passivity”.
- (3) **Passive shadow functional** (0.45). Make $\Delta_\infty - \Delta$ (Connes, harmonic measure = excess delay) quantitative for ghosts: size, sign structure, and detectability with band-limited f .
- (4) **Recurrence numerics** (0.40). Measure Bagchi’s strong-recurrence density for ζ on small discs in the strip, and compare with the truncated-product (finite-bath) recurrence of Fig. 1b.
- (5) **Fisher zeros / Lee–Yang** (0.30). Primon gas and its fermionic version $\zeta(\beta)/\zeta(2\beta)$: is there a Lee–Yang-type mechanism, and does the doubling reappear?

10. QUESTIONS FOR J.W.

- Q1. Should “the arrow of time” be the LP gap c (doubled system, unconditional, ghost-blind), or do you want an arrow in the undoubled system (which Thm. 4.1 forbids if RH holds)?

- Q2. Do you accept “reversible + passive $\Rightarrow$ lossless”? It turns RH into a *losslessness* statement rather than a dissipation statement.
- Q3. For Darlington from primes, is the target the L112 Carathéodory function, the Ψ_c family, or the de Branges function $E_c = \xi(\cdot + c)$?
- Q4. Should Bagchi’s strong recurrence (RH $\iff$ the infinite prime bath still recurs) be developed as the counterpart of FKM?
- Q5. For L119: direction 1 (de Branges from primes) or direction 3 (passive shadow functional)?

REPRODUCIBILITY

From `/workspace/taa:` `python3 num/lemma118_bath.py` (≈ 2 min), `lemma118_lp.py`, `lemma118_euler.py` (≈ 1 min), then `lemma118_figs.py`. Outputs in `num/l118/`: `bath.json`, `bath_curves.json`, `lp.json`, `euler.json` (logs `*.log`). Figures: `build118/figs/f1_bath`, `f2_lp`, `f3_euler` (pdf, png). Inputs: `num/l104/zeros.json`; L117 grid values from `num/l117/schur.json`.

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APPENDIX A

Consolidated corrections

Every note ends with a corrections section that records which claims made in the working conversation were wrong, and how they were fixed. Those claims came from pasted texts, from J.W.’s hunches stated as claims, or from the assistant’s own chat replies. Chapter texts were not edited for this volume. The table points to each section and gives its gist; the full text is in the chapter. “Note p.” is the page inside the note; “vol. p.” is the volume page. Corrections that a later chapter *proposes* for an earlier one, and that J.W. has not yet accepted, are not applied anywhere; they are listed with their exact effect in Appendix B, §B.1.

Ch.	Section	note p.	vol. p.	Gist
86	9. Corrections	11	11	J1–J4: the central sign is the leading Taylor coefficient, counted over $(\frac{1}{2}, \infty)$; Gershgorin gives $M \leq 6$; “ $M \approx 20$ ” is 18.75; Kohnen–Zagier/Kohnen restrictions. T1–T5 on the t370 random-matrix claims.
87	9. Corrections	13	26	J1–J9: off-line cones are rotated by $-\frac{1}{2} \arg c$; cones are spacelike, not timelike; the lemniscate rotates, not the cone; the t376 κ claim is a re-statement, not a quadratic form; a self-corrected wall height.
88	8. Corrections	10	39	K1–K7: “saddle cones sum to (n, n) ” true but empty; quartet convention for κ ; Lemma 56 simple-critical-point fix; ρ formula sourced; Bombieri’s date; Tate $\Leftrightarrow$ BSD now cited. Further corrections to this chapter are proposed in Ch. 89 and are pending (Appendix B, §B.1).
89	8. Corrections	8	48	Prior art: Ch. 88 Theorem A is classical (Silverman V.1.1–V.1.2, Shioda 2008, Milne 1968). C89.1–C89.4 are proposed errata to Ch. 88, <i>not applied</i> (pending, §B.1); C89.5 upgrades $\mathrm{NS}(C \times C)$ to [CITED]; C89.6–C89.8 sources, scope of L84, provenance.
90	10. Corrections to chat claims	11	62	L90.C1–C8: Yoneda needs naturality; $\mathbb{N}^\times$ is effective, not ample; $C_K \cong C_K^1 \times \mathbb{R}_{>0}$ splits for every number field and function field; the S^0 cover is trivial.
91	8. Corrections to chat claims and to earlier lemmas	9	72	L91.C1–C11: t391u items confirmed with nuances; base change $\neq Z(T^2)$ for curves; C9: Ch. 89’s “ $n_- = 1$ per ghost” is the even class (proposed erratum, pending, §B.1); C11: an aliased scan withdrawn.

Ch.	Section	note p.	vol. p.	Gist
92	5. Corrections	12	85	L92.C1–C9: Ch. 91’s offset 0.4–0.58 is a basis artefact; slope $\approx \frac{1}{2}$; “blind for $a \leq \frac{1}{2} \log 2$ ” imprecise; sign $Q = -2s$; naive Arb Cholesky fails; pole-degenerate fakes.
93	8. Corrections to chat claims	11	98	L93.C1–C9: “ ζ^2 gives only H^0, H^2 ” is wrong (it contains $H^1 \oplus H^1$); the violating class is $(\alpha, q/\alpha)$ glued by conjugation; Deninger predicts the duality pairing only.
94	8. Corrections	11	110	C94.1–C94.11: weak P4/P9 exist but are ghost-blind; $\text{Spec } \mathbb{Z}^{\times 4}$ collapses; ample = positive cone for all abelian surfaces; the difference map is birational iff C is non-hyperelliptic; Artin contractibility fails.
95	6. Corrections	10	122	C95.1–C95.11: the Hecke-corrected span is that of single T_r (rank 30); $FV = p$ only in weight 2 (p^{11} for Δ); a fake Satake parameter violates Hodge index only in the Frobenius lattice; Ch. 94 Q1 answered.
96	6. Corrections to the chat claims (t406–t408)	9	132	C96.1–C96.7: the E_2 constant; Harer–Zagier is for $M_{g,1}$; scope of Atiyah’s theorem (APS $\eta \neq$ Dedekind η); E_2 is the connection form; the Eisenstein side is mixed.
97	Corrections to the t419 chat claims	8	141	C97.1–C97.6 and C97.m1–m4: the covariant raising operator is $\vartheta_k + \frac{k}{12} E_2^*$; Lion–Vergne link unverified; Deninger’s local spaces are infinite-dimensional; Rankin–Selberg on GL_1 returns ζ .
	6. Corrections to the chat claims (t414–t417)	8	141	
98	7. Corrections to the chat claims (t420–t423)	11	154	C98.1–C98.19: closed geodesics have length $2 \log \varepsilon_d$, not $\log p$; positivity is free only on the compact part; for Yoneda, representability is the whole battle.
99	6. Corrections to the chat claims (t424–t425)	7	163	C99.1–C99.13: $R(W) = ik$; Casimir sign; outer versus inner Cartan involution; the precise Deligne criterion (via Milne); Lefschetz signs.
100	7. Corrections to the chat claims	9	173	C100.1–C100.14: σc is a reflection, not a boost; $\mathfrak{so}(2, 2)$ is ambient only; J must be RH-independent; the $W_{\mathbb{R}}$ circle is trivial on ζ .
101	8. Corrections to the chat claims	7	182	C101.1–C101.15: $\ df\ ^2$ gives $\Delta \geq 0$, not $\geq \frac{1}{4}$; a genus qualifier; S is an anti-isometry.
102	6. Corrections to the chat claims	6	190	C102.1–C102.17: determinant genericity; a factor $\frac{1}{2}$; Dirac’s 1928 form; Hodge–Riemann = commutator plus state; hyperbolic $\neq$ non-unitary; J.W.’s t443 self-correction confirmed.
103	7. Corrections to the chat claims and to the brief	6	198	C103.1–C103.7: the two-point form restates Weil’s; ω carries no new information; onset beyond $\log 2$ confirmed.
104	4. Corrections to the chat claims and to the brief	5	204	C104.1–C104.6: a quartet ghost is pseudo-Euclidean $(2, 2)$, not $\text{Spin}(1, 1)$.
105	6. Corrections to the chat claims (t447–t450)	6	212	C105.1–C105.8: $\xi(2\pi) \neq 0$, so $s = \pi$ is not a pole; “equal widths” made exact; the closure obstruction.

Ch.	Section	note p.	vol. p.	Gist
106	6. Corrections to the chat claims (L106 brief)	6	219	C106.1–C106.7: over $\mathbb{F}_q$ the positivity comes from geometry, not arithmetic. An erratum to this chapter’s numerical convention is proposed in Ch. 109 and is pending (§B.1).
107	5. Corrections to the chat claims and to the brief	4	224	C107.1–C107.9: the Wigner rotation is not small; $x^\rho \rightarrow x^\rho/\rho$; ghost $\Rightarrow$ Robin violation needs Lagarias’s Prop. 3.2; reading of the Ch. 99 slogan.
108	7. Corrections to the chat claims (t455)	5	231	C108.1–C108.6: the ghost is $\text{sgn}(t) \times$ a boost, S unique up to sign; the dictionary has a reversal half; the cusp is the universal boost.
109	Erratum to L106	5	238	Erratum to Ch. 106 (proposed, pending, §B.1). C109.1–C109.6: primes $\neq$ geodesics on $X(1)$; Pell axis yes, boost no; class-number asymptotics not cited.
	4. Corrections to the chat claims	6	239	
110	4. Corrections to the chat claims	5	245	C110.1–C110.6: Williamson is for quadratic forms (use Gutt/Krein); 2^k needs $k_u = 0$; the quartet’s J is symplectic, never compatible; E is not algebraically unique.
111	4. Corrections to the chat claims	5	251	(1)–(6): Krein signs belong to eigenvalues; strong stability condition corrected; Polymath bound $\Lambda \leq 0.22$, not 0.2; Lehmer pairs heuristic.
112	6. Corrections to the chat claims	7	259	(1)–(6): $ \alpha_n < 1$ is a joint condition; the angle measure is trivial; Szegő/entropy inapplicable; Killip–Nenciu is not the equal-weight case.
113	8. Corrections to the chat claims (t464/t465)	7	267	C113.1–C113.8: the product formula does not characterize Weil numbers; the p -adic polygon carries no sign; Newton above Hodge is the p -adic Weyl inequality.
114	9. Corrections to the chat claims (t467–t479)	12	281	C114.1–C114.14: the E_2 anomaly coefficient is $6z/(\pi i)$; dividing by $\sqrt{q}$ does not undo the boost; Klein’s syzygy is a discriminant, not a diagonalization.
115	6. Corrections (t479–t484 chat claims)	9	293	C115.1–C115.12: Gupta–Bleuler is a weak condition; Goddard–Thorn needs $d \leq 26$, $c = 26$ only in Borchers’s Lemma 5.4; Lefschetz is not BRST; a triple, not double, zero.
116	8. Corrections	7	302	C116.1–C116.9: Mrugała’s metric is indefinite; the Bost–Connes high-temperature phase is not the critical strip; inverse four-temperature; the χ_4 curvature takes both signs.
117	9. Corrections	8	313	C117.1–C117.14: “subluminal $\Leftrightarrow$ RH” is a restatement; ghosts jump over c ; on-line pairs are definite, not null; Kramers–Kronig needs a subtraction.
118	8. Corrections	8	323	C118.1–C118.10: passivity needs no zero information; “every cusp surface gives $\zeta(2s)$ ” unsupported; Connes’s reduction covers all Hecke L ; OCR sign in LP (6.19).

Corrections across chapters. Some later chapters correct or sharpen earlier ones; the later chapter states this in its corrections section, and the earlier chapter is left as it was. Accepted in the later chapter’s text (no decision needed): Ch. 92 replaces Ch. 91’s support-threshold offset (a basis artefact) and slope; Ch. 94 (C94.4) explains Ch. 91’s “primes never detect” as a range effect; Ch. 95 answers Ch. 94’s Q1; Ch. 100’s addendum and Ch. 107 (C107.2) refine the reading of Ch. 99’s slogan; Ch. 102 (C102.13) records J.W.’s own t443 self-correction; Ch. 110 corrects Ch. 108’s case list (the quartet’s J is symplectic, never compatible); Ch. 114 (C114.14) and Ch. 115 (C115.1) fix the $\sqrt{q}$ -boost remark. *Pending* (proposed, not accepted): the errata to Ch. 88 from Ch. 89, to Ch. 89 from Ch. 91, and to Ch. 106 from Ch. 109; see §B.1.

APPENDIX B

Pending decisions and open questions

B.1. Pending author decisions: proposed errata

Three later notes propose corrections to earlier notes. J.W. McGreevy has *not yet decided* whether to accept them, so in this volume:

- the earlier notes (Chapters 88, 89 and 106) are reproduced exactly as compiled, without the corrections;
- each item below is a **proposed correction, pending author review**. It is stated here exactly as proposed, with what it would change;
- no result of the volume depends on the decision, except where an item says so.

The notes themselves ask for these decisions: Ch. 89, Questions (“update Ch. 88 now or wait”), Ch. 91 Q3, Ch. 92 Q1, Ch. 93 Q5 and Ch. 95 Q5 (Ch. 89 erratum and C89.4); Ch. 109 Q3 and Ch. 112 Q5 (Ch. 106 erratum).

Proposed errata to Chapter 88 (proposed in Chapter 89, corrections C89.1–C89.4).

Source: Ch. 89, §8, C89.1–C89.4 (note p. 10, vol. p. 50).

- E1. **C89.1 (proof of Ch. 88 Theorem A (b) and (d)).** *Proposed correction, pending author review.* The proofs of (b) and (d) of Theorem A (Ch. 88, Thm. 3.1 (note p. 3, vol. p. 32)) cite Schütt–Shioda (Thm. 11.5/§11.6 and Cor. 6.13) for $E \times E \rightarrow E$, but that source excludes products (“ S is not isomorphic to a product $E \times C$ ”). *What it changes:* (b) would be proved instead by the intersection computation $D_\varphi = \Gamma_\varphi - (\deg \varphi)H - V$, $D_\varphi^2 = -2 \deg \varphi$; (d) by $\mathrm{NS}(E \times E) \cong \mathbb{Z}^2 \oplus \mathrm{End}(E)$ (Milne, *Abelian Varieties*, III 6.3); and “Shioda height” in Ch. 88 would be read as $-D_\varphi^2$, which agrees with Shioda’s height wherever his definition applies. The statements of Theorem A and the numerical check are unchanged.
- E2. **C89.2 (attribution of Ch. 88 Theorem A).** *Proposed correction, pending author review.* *What it changes:* Ch. 88 §3 would state that Theorem A is classical and add the attribution “(classical; Silverman V.1.1–V.1.2, Shioda 2008 §2, Milne 1968)”. The Ch. 84 genus-1 statement “trace form PD $\Leftrightarrow$ RH” would likewise be credited as Silverman V.1.2 in Gram form. No mathematical statement changes; the novelty claim for Theorem A is withdrawn. (“What this volume does and does not advance”, item C1, already treats Theorem A as classical.)
- E3. **C89.3 (Ch. 88 Prop. 3.3 and direction D4).** *Proposed correction, pending author review.* *What it changes:* the genus- g trace pairing on $\mathrm{Hom}(A, B)$ is in Milne 1968a ([CITED-sec]); Prop. 3.3 (Ch. 88, Prop. 3.3 (note p. 4, vol. p. 33)) stays [Heuristic] until Milne 1968a/b is opened, but direction D4 becomes “locate in Milne 1968a” instead of “discover”.
- E4. **C89.4 (status of the Hodge-type standard conjecture).** *Proposed correction, pending author review.* Ch. 88 §2 (Ch. 88, §2, Hodge-type item (note p. 2, vol. p. 31)) says, following Milne 2015, that the Hodge-type standard conjecture in characteristic p is “known essentially only for surfaces”. That was accurate for that source. Since then an unrefereed preprint (OpenAI, dated 6 Oct. 2026) claims the rational Hodge conjecture for CM abelian varieties and, via two theorems of Milne, the Hodge standard conjecture for abelian varieties in every characteristic; an expository preprint by van Hoften and Sempliner and notes on Milne’s website report the claim. Only abstracts and introductions were read; the proofs were not

checked. *What it changes:* append “abelian varieties: claimed Oct. 2026, unrefereed preprint” to the Hodge-type item of Ch. 88 §2. It does not touch ζ , RH, Ch. 88 §6, or any result of Ch. 89.

Corrections C89.5–C89.7 of Ch. 89 are not errata: C89.5 and C89.6 upgrade the status of sources used in Ch. 88 (Milne III Cor. 6.3, Milne’s AV notes for Tate’s theorem), and C89.7 clarifies the scope of Ch. 84’s “ $\#_{\text{neg}} = \kappa$ ” (squarefree P ; otherwise count distinct roots). They are recorded in Appendix A and need no decision.

Proposed erratum to Chapter 89 (proposed in Chapter 91, correction L91.C9).

- E5. **L91.C9 (Ch. 89 N3/Prop. D: “ $n_- = 1$ per ghost”).** *Proposed correction, pending author review.* Source: Ch. 91, §8, L91.C9 (note p. 10, vol. p. 73); the statement concerned is in Ch. 89, §5, N3 (note p. 6, vol. p. 46). *What it changes:* Ch. 89’s “ $n_- = 1$ per ghost” and “ $n_- \leq \kappa_\zeta$ ” would be qualified as holding for R-even test functions (the class Ch. 89 uses). On the full real class (Connes–Consani’s), the count is 2 per quartet ($= \frac{1}{2}Z_{\text{off}}$), attained, in agreement with Ch. 88 Thm. B (Ch. 91, Prop. 3.1 (note p. 7, vol. p. 70)). Ch. 89 is not wrong for its class, and its “0–2” broad-basis fragility is the support threshold explained in Ch. 92.

Proposed erratum to Chapter 106 (proposed in Chapter 109).

- E6. **Double-zero convention in Ch. 106’s numerics.** *Proposed correction, pending author review.* Source: Ch. 109, “Erratum to L106” (note p. 5, vol. p. 238). Ch. 106’s ghost experiments (onsets table, Ch. 106, §4 (note p. 4, vol. p. 217)) let a synthetic quartet replace *two* on-line zeros. Since ζ ’s zeros are simple, this leaves multiplicity -1 at γ , an unphysical negative term. The proposal is to adopt the Ch. 103 convention (each quartet replaces *one* on-line zero) as the standing convention. *What it changes* [Numerical]:
- at large L the double-zero convention doubles n_- (Ch. 109, Fig. 2, dashed); with the Ch. 103 convention k quartets give $n_- = 2k$;
 - Ch. 106’s four onsets 0.75/0.875/0.975/1.25 become 0.775/0.875/1.0/1.25.
- Ch. 106’s window conclusions are unchanged; its PDF is not modified.
- E7. **The standing convention.** Accepting E6 would make the Ch. 103 single-zero replacement the standard for later ghost experiments (Ch. 109 Q3, Ch. 112 Q5). Ch. 109 and later chapters already use it, and show the Ch. 106 convention only as a dashed comparison.

B.2. Open questions in the chapters

Each note ends with “Questions for J.W.” (usually Q1–Q5). They are reproduced in full in the chapters; this table lists their topics and locations. **Status.** Unless marked, no answer from J.W. is recorded in the notes, so the questions are open. Questions of the form “which direction next” were in effect settled by the following lemma requests. Questions marked *pending* concern the errata of §B.1.

Ch.	§	note p.	vol. p.	Topics
86	11	12	12	Accept Thm. 3.1 and the blindness verdict; corrections J/T; “KZ/GZ positivity is a parity theorem”; the Bloch–Kato verdict and the half Tate twist; next M1/M3/M2.
87	11	15	28	Thm. 3.2 as the light-cone form of (H); corrections J1–J9; the landing clock and κ -landing restatement; the no-genus verdict; next N1/N2/N3.
88	7	9	38	The algebraicity/positivity split as frame; D1/D2/D3 for Ch. 89; queue Thm. B as a Vol. 3 addendum; “Hodge $\Rightarrow$ RH” has no mechanism; the (n, n) claim.

Ch.	§	note p.	vol. p.	Topics
89	7	8	48	Verbatim t380/t381 missing; “least common divisors”; the product-structure verdict (Thm. 6.1); start Ch. 90; C89.4: update Ch. 88 now or wait (<i>pending</i> , §B.1).
90	9	10	61	Base category for F ; weaken W1; replace W2 by a Riemann–Roch axiom; the S^0 cover; verbatim t385/t386/t390.
91	7	9	72	Z1 or Z2 next; form of the threshold answer; write the Ch. 89 erratum into the note? (<i>pending</i> , §B.1); meaning of “Frobenius-squared”; verbatim t391u/t393u.
92	7	13	86	The Ch. 89 erratum (<i>pending</i>); an all-test-function certificate; reference ghost model; units for thresholds; priorities A1–A5.
93	7	11	98	Meaning of “a zero pairs with a zero”; W2 or W1; a “zeta of the square”; W_∞ as continuous spectrum; the Ch. 89 erratum (<i>pending</i>).
94	10	13	112	Does RH imply (RR)?; the genus-1 band; an additive structure making deg quadratic (P9); detection scaling; the Perron class.
95	8	11	123	Retarget P9 to a Frobenius splitting; a non-pull-back pairing on $\mathbb{N}^3$; certify D2; weight-12 ring vs. cycles; an h^0 over $\mathbb{Q}$; errata decision (<i>pending</i>).
96	8	10	133	Which anomaly; move Rosati to the odd part; the three-part test; which cohesive topos; direction for Ch. 97.
97	8	9	142	Weight grading presupposes $\text{Re } \rho$; $h^1 \oplus h^1$ vs. $h^1 \otimes h^1$; signed tropical setting; Lion–Vergne source; direction for Ch. 98.
98	9	12	155	Rapidity normalization; non-spectral SOS; involution or ample class first; “positive part scalar” in the Rosati metric; direction for Ch. 99.
99	8	8	164	Rosati = restricted Cartan; rotation vs. boost spectrum; the Deligne-torus circle for ζ ; ample-class reflection; direction for Ch. 100.
100	9	10	174	Thm. 4.1 as the Hodge-circle statement; the shifted mirror; circles on families; Deninger’s circles at the primes; direction for Ch. 101.
101	10	8	183	Formal J carries no content; what is counted over $\mathbb{Q}$; mechanism for $\frac{1}{4}$; the spacelike complement; direction for Ch. 102.
102	8	7	191	HR = commutator plus state; S , R_x for ζ ; Weil’s functional as a two-point function; Shale as replacement; direction for Ch. 103.
103	9	7	199	The two-point form restates Weil; the onset law $L^* \approx \log(\gamma/2\pi)$; which object supplies (V, ω, J) ; near-null directions; proceed to Ch. 104.
104	6	6	205	The dichotomy (Thm. 2.4); the quartet picture; a $\mathbb{Q}$ -analogue of CM; direction 2 as Ch. 105; the hand-sequence intuition.
105	8	7	213	Hyperboloid picture of J ; exact widths; Berry–Keating closure; near-success at $a = 1$; the missing inner product.
106	8	6	219	Positivity over $\mathbb{F}_q$ is geometric; the dictionary (Thm. 3.1); counting traces are ghost-blind; degree for $\mathbb{Q}$; direction.
107	7	5	225	Independence $\Rightarrow$ truth (Thm. 1.2); Gödel-II link as analogy; one onset law; Wigner reading; rename the Ch. 99 slogan.
108	9	7	233	The reversal half; the bicomplex quartet; cusp vs. geodesics as “two infinities”; “no S -symmetry” is a restatement; direction.
109	6	6	239	Point-count Toeplitz = Rosati; weights exclude ghosts; accept the Ch. 106 erratum? (<i>pending</i> , §B.1); commensurable ghosts; Toeplitz structure of Q_L .

Ch.	§	note p.	vol. p.	Topics
110	6	6	246	Thm. 1.1 as reference classification; the quartet's J ; canonical E ; Krein collisions next; “ ζ is always mixed”.
111	6	6	252	Strong stability = Hasse openness; ζ not strongly stable under heat flow; graphical Krein sign; Hermite–Biehler phase; CM type.
112	8	8	260	Discrete Weil criterion (Thm. 3.1) and prior literature; Li Gram test; Szegő/Killip–Nenciu; equal-weight locus; the Ch. 106 erratum (<i>pending</i>).
113	10	8	268	Thm. 2.2 as archimedean converse; Thm. 5.1; names for the three Hodges; Kempf–Ness; Fargues–Fontaine twistors.
114	11	14	283	D1 as next lemma; Klein syzygy; the velocity V ; Gupta–Bleuler lemma; Faltings–Hriljac as model.
115	8	10	294	Thm. 2.4 (one subsidiary condition) and priority; Maass–Selberg family; Lagarias–Suzuki reading; the null state n_ξ ; the Bost–Connes remark.
116	10	9	304	(S2) as the stability statement; Laguerre tower on invisible ghosts; Bisognano–Wichmann normalization; Lee–Yang; contact analogy.
117	11	9	314	Shift dominance as undoubled target; $\mathbb{C}^{1,1}$ vs. $\mathrm{SO}^+(2,1)$; the frequency axis for KK; CM tests beyond class number 1; direction.
118	10	9	324	Which arrow of time; “reversible + passive $\Rightarrow$ lossless”; the Darlington target; Bagchi’s strong recurrence; direction for Lemma 119.

APPENDIX C

Consolidated notation

Each chapter defines its notation at the start of the note. This appendix lists the global conventions and the symbols that different chapters use differently.

Global conventions

- $s = \sigma + it$; nontrivial zeros $\rho = \beta + i\gamma$, also written $\rho = \frac{1}{2} + \delta + i\gamma$; $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$; $\Xi(t) = \xi(\frac{1}{2} + it)$.
- *Ghost*. An off-line zero. A complex ghost comes as a quartet $\{\rho, \bar{\rho}, 1-\rho, 1-\bar{\rho}\}$, i.e. $\frac{1}{2} \pm \delta \pm i\gamma$. A *synthetic* ghost is a quartet inserted by hand into a list of computed zeros of ζ (or a fake factor of a finite model). Two insertion conventions occur: *added* (the quartet is added to all true zeros; e.g. Ch. 92 Thm. 4.1(c)) and *replacement* (the quartet replaces on-line zeros). Ch. 103 replaces one on-line zero; Ch. 106 replaces two (the “double-zero” convention, whose correction is pending, §B.1).
- *Davenport–Heilbronn*. f_{DH} is the Davenport–Heilbronn combination of the two L -functions mod 5 (as in Volume IV); its first off-line zero is $0.80851718 + 85.69934849i$.
- *Over $\mathbb{F}_q$* . F is Frobenius on H^1 ; $F^\dagger = qF^{-1}$ is the Rosati (duality) partner. A *model* is a q -palindromic polynomial P ; it is *true* if all roots have $|\alpha| = \sqrt{q}$, otherwise a *fake* or ghost model. After normalization $F/\sqrt{q} \in \text{Sp}(2g, \mathbb{R})$ has type (k_u, k_r, k_c) : k_u unit pairs, k_r real pairs, k_c quartets, $g = k_u + k_r + 2k_c$ (Chs. 110–113).
- *Structures*. J : a complex structure ($J^2 = -1$), *compatible* if $\omega(\cdot, J\cdot)$ is positive definite (a Hodge circle $h(e^{i\theta}) = \cos \theta + J \sin \theta$). S : an anti-symplectic involution ($S^2 = 1$, $S^T \omega S = -\omega$). E : a (partial) split-complex structure ($E^3 = E$ in $\mathfrak{sp}$). σ : the duality involution $\rho \mapsto 1-\rho$ ($\alpha \mapsto q/\alpha$); c : complex conjugation.
- *Classes on $C \times C$* . $H = \{\text{pt}\} \times C$, $V = C \times \{\text{pt}\}$ (written V_0 in Ch. 89), Δ the diagonal, Γ_F or Γ_φ graphs; $\text{NS}(C \times C) \cong \mathbb{Z}H \oplus \mathbb{Z}V \oplus \text{End}(J)$.

Symbols used differently in different chapters

Symbol	Meanings
a	Support parameter of test functions, supported in $[e^{-a}, e^a]$ (Chs. 91–92, 98); trace of Frobenius, $P = T^2 - aT + q$ (Chs. 94–95, 101, 105, 108–109).
a^*	Ghost-detection threshold in the support (Chs. 91–92); in Ch. 92 over $\mathbb{F}_q$, $a^* = g \log q$.
κ	Number of off-line reciprocal pairs (negative index of the Weil-type form), as in Volume IV (Chs. 88–89); κ_ζ in Ch. 89. In Ch. 88 also compared with the saddle counter τ and Z_{off} .
$n_\pm$	Positive/negative index of a form. For the Weil form with k synthetic quartets (single-zero replacement), $n_- = 2k$ (Chs. 103, 109); Ch. 89 counts 1 per ghost on R-even test functions (see E5 in §B.1).
Q, W	The Weil form. Q_L : truncation at log-support L (Chs. 103, 106, 109); $W(g)$: Weil’s functional with the pole terms $\hat{g}(0), \hat{g}(1)$ (Ch. 115).
$s(f, f), D \bullet D$	Connes–Consani self-intersection on the scaling site (Ch. 91).

Symbol	Meanings
δ	Offset $\beta - \frac{1}{2}$ of a zero; in Ch. 105 the Lorentzian widths are $\frac{1}{4} \pm \frac{\delta}{2}$.
h	Spline or hat-basis spacing (Chs. 92, 112); the Hodge circle $h(e^{i\theta})$, $J = h(i)$ (Chs. 99–104).
c	Complex conjugation (Chs. 100–104); saddle Hessian coefficient $-\Xi''/\Xi$ (Ch. 87); shift in $\xi(s \pm c)$ (Chs. 117–118).
σ_0, σ_N	Van de Lune’s constant 1.19234... and the positive-real thresholds of truncated Euler products (Ch. 118); elsewhere $\sigma = \operatorname{Re} s$, and σ the duality involution.
Λ, λ	Λ : completed L -function (Chs. 86–87). λ : scaling parameter, $\lambda = e^a$ (Ch. 92); Laplacian eigenvalue $\lambda = s(1-s)$ (Chs. 101, 105).
L	Log-support cutoff (Chs. 103, 106, 109); L -function (Chs. 86, 95, 97).
Δ	Diagonal of $C \times C$; the discriminant cusp form $\Delta = \eta^{24}$ (Chs. 95–97, 114); the Laplacian (Ch. 105); in Ch. 96 all three, kept apart.
Φ, Θ_T, Ψ_c	Scattering functions of the doubled Lax–Phillips system (Ch. 118), the Lagarias–Suzuki function (Ch. 117), and $\xi(\cdot - c)/\xi(\cdot + c)$ (Ch. 118).

Claim tags

The tags are mapped to volume classes in the table after the Statement of Intellectual Honesty. All chapters use the long forms ([THEOREM], [CITED], [CITED-sec], [BIB], [TAA-L], [RE-STATEMENT], [ANALOGY], [CONJECTURE], [OPEN], [NUMERICAL], [HEURISTIC]). Local variants: Ch. 92 labels its certified results “computer-assisted, rigorous”; Ch. 87 has [CONSTRUCTION].