

Degeneracies between Metric Deformations and Thin-Disk Models in Black Hole Images: A Channel-by-Channel Analysis

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October 2026

Abstract

We present a systematic study of how modifications of the inverse Kerr metric manifest in images of a thin accretion disk and assess the sensitivity of different image observables to these deformations within a fixed astrophysical model. Using backward ray tracing in Boyer–Lindquist coordinates, we introduce three one-parameter deformations of the inverse metric—in g^{tt} , g^{rr} , and $g^{\phi\phi}$ —of the form

$$g^{\mu\nu} \rightarrow g^{\mu\nu} \left[1 + \epsilon \left(\frac{M}{r} \right)^2 \right], \quad |\epsilon| \leq 0.1.$$

For each modification we construct a thin-disk image with a power-law emission profile $I_{\text{em}} \propto (r_{\text{in}}/r)^3$ (motivated by the Novikov–Thorne profile) for spin $a \in \{0, 0.5, 0.9, 0.99\}$ and inclination $i \in \{30^\circ, 70^\circ, 80^\circ, 85^\circ\}$ on the main grid $n_{\text{pix}} = 80$, with verification on $n_{\text{pix}} = 120$ and 160, and extract five observables: shadow radius R_{sh} , centroid shift, brightness asymmetry A_{asym} , peak brightness I_{peak} , and integrated brightness I_{tot} . We find: (i) the g^{tt} modification shifts both the shadow boundary and the photometric peak; (ii) the g^{rr} modification is *strictly* invisible in the shadow geometry over the entire grid (an analytical consequence of the structure of the null geodesic equations), and its photometric signal is *numerically* undetectable at realistic accuracy; (iii) the $g^{\phi\phi}$ modification produces an intermediate photometric signal with opposite sign relative to g^{tt} .

We additionally performed a stability analysis of the observables against grid resolution: the shadow radius R_{sh} (computed by the capture-boundary bisection method with search step $\pm 0.005 M$ and residual systematics $\sim 0.5\%$, see Appendix B) is found to converge, whereas the peak and integrated disk brightnesses exhibit systematic non-convergence of the derivatives with respect to ϵ , up to a sign change between grids $n_{\text{pix}} = 80$ and 160. Restricting the Fisher analysis to the stable observable R_{sh} , we obtain $\sigma_\epsilon \approx 0.43$ (g^{tt} , $a = 0.9$), ≈ 0.25 (g^{tt} , $a = 0.99$), ≈ 0.68 ($g^{\phi\phi}$, $a = 0.99$); for g^{rr} the derivative $dR_{\text{sh}}/d\epsilon$ vanishes identically, rendering σ_ϵ formally infinite. We analytically show that the *invariance of the critical curve* with respect to g^{rr} is a strict consequence of the structure of the null geodesic equations; numerically, this is confirmed across all three grids and by the bisection method with accuracy $10^{-4} M$ in the derivative $dR_{\text{sh}}/d\epsilon$. Implications for testing modified theories of gravity with next-generation observations and the limitations imposed by degeneracies of a single disk are discussed.

Keywords: black holes — accretion disks — gravitational lensing — modified gravity — numerical methods

1 Introduction

1.1 Motivation

Horizon-scale observations by the Event Horizon Telescope (EHT) of the supermassive black holes M87* (EHT Collaboration, 2019) and Sgr A* (EHT Collaboration, 2022) have opened a new era in testing gravity in the strong-field regime. For the first time, we have the opportunity to directly compare predictions of General Relativity (GR) with images of accretion flows around compact objects with accuracy limited not by theory, but by interferometer resolution and the systematics of the disk models themselves.

The next generation of instruments—ngEHT, the ngVLA extension, and space interferometers—promises to improve angular resolution by a factor of 3–10, bringing it to a level at which high-order photon rings become distinguishable (Johnson et al., 2020; Gralla et al., 2020). This opens the possibility of directly testing the spacetime geometry near the horizon: if GR is correct, the shadow shape and photon-ring structure are rigidly tied to the Kerr metric; if gravity is modified on small scales, these structures should differ.

However, a fundamental question arises: can the observed image unambiguously distinguish a metric deformation from uncertainties of the astrophysical disk model? This question is central to the entire program of testing gravity with shadow observations.

1.2 What Has Been Done

The problem of degeneracy between spacetime geometry and accretion disk properties has been actively studied in recent years. We list key results.

Degeneracy as a structural phenomenon. Kocherlakota & Rezzolla (2022) showed that even with infinite measurement accuracy of the shadow size, degenerate regions remain in the metric parameter space: different geometries can give the same shadow radius with an appropriate choice of disk parameters. They proposed combining the shadow radius with the peak intensity contrast to lift some degeneracies.

Complete coincidence of optical signatures. Works on Yukawa-like black holes (Ren et al., 2026) and regular black holes with a Minkowskian core (Tello-Ortiz et al., 2026) showed that for certain parameters the images can be optically indistinguishable from Schwarzschild/Kerr ones despite radically different internal geometry. This indicates a fundamental limitation of any “direct” shadow tests.

Orthogonal constraints. Qi & Cai (2026) and Övgün et al. (2026) demonstrate that the shadow and quasinormal modes give orthogonal constraints on dark matter parameters, and their combination allows the degeneracy to be lifted. This is an important precedent for using several independent channels.

Photon rings. Gralla et al. (2020) and Johnson et al. (2020) showed that the position of the $n = 2$ photon ring depends on the metric more strongly than the shadow size, and that ngEHT will be able to measure it. However, this program requires very high angular resolution and is unattainable with the current generation of instruments.

Parameterized deviations from Kerr. Konoplya & Zhidenko (2016) and subsequent works developed a general formalism for parameterized deformations of the metric. Numerous works have applied this formalism to shadow observations of M87* and Sgr A* to obtain constraints on modification parameters. However, most of them analyzed one modification at a time, without systematic comparison among them.

1.3 Gap in the Existing Literature

Despite the activity in this area, two gaps can be identified that we attempt to close.

Gap 1. Lack of systematic comparison of classes of modifications. Existing works usually consider one specific modification (Yukawa, Bardeen, Hayward, Hernquist halo, scalar hair). It is unclear whether different classes of modifications differ in their observable signature or all produce a qualitatively similar effect. If the signatures differ, this provides additional degrees of freedom for distinguishing them. If they coincide, this indicates a fundamental limitation of shadow tests.

Gap 2. Lack of quantitative assessment of sensitivity. Qualitative statements such as “the metric is visible in the shadow” or “the metric is degenerate with astrophysics” do not allow different modifications to be compared with each other. A quantitative measure is needed—for example, σ_ϵ —showing with what accuracy the modification parameter can be measured at realistic observational accuracy. As shown in the present work, such a measure is meaningful only for observables whose derivatives converge under increasing resolution.

1.4 Our Contribution

We present a systematic study of the sensitivity of thin accretion disk image observables to three classes of modifications of the inverse Kerr metric:

$$g^{tt} \rightarrow g^{tt}[1 + \epsilon h(r)], \quad g^{rr} \rightarrow g^{rr}[1 + \epsilon h(r)], \quad g^{\phi\phi} \rightarrow g^{\phi\phi}[1 + \epsilon h(r)], \quad (1)$$

with profile $h(r) = (M/r)^2$ and $|\epsilon| \leq 0.1$. This form is chosen as a minimal parameterization allowing the role of each diagonal metric component to be isolated.

The three classes of modifications correspond to three different physical scenarios (see Section 2.3):

- g^{tt} — quantum corrections to the Newtonian potential (Reuter & Saueressig, 2002; Bonanno & Reuter, 2000; Duff, 1974; Kazakov & Solodukhin, 1994);
- g^{rr} — singularity regularization (Bardeen, 1968; Hayward, 2006; Simpson & Visser, 2019);
- $g^{\phi\phi}$ — nonminimal coupling of scalar fields or axionic hair (Campbell et al., 1992; Herdeiro & Radu, 2014).

Our analysis covers 48 points in the (a, ϵ, i) parameter space for each modification (four spins $\times$ three values of $\epsilon \times$ four inclinations)—144 points in total across the three modifications on the main grid $n_{\text{pix}} = 80$. For verification of observable stability, additional calculations were performed: grids $n_{\text{pix}} = 120$ and 160 (54 points each at $a \in \{0.9, 0.99\}$, $i \in \{70^\circ, 80^\circ, 85^\circ\}$) and independent determination of R_{sh} by the capture-boundary bisection method (81 points at $a \in \{0.5, 0.9, 0.99\}$, $i \in \{70^\circ, 80^\circ, 85^\circ\}$). For each point we construct a thin-disk image with realistic physics (Keplerian rotation, redshift, Doppler shift) and extract five observables: shadow radius R_{sh} , centroid shift, brightness asymmetry A_{asym} , peak brightness I_{peak} , and integrated brightness I_{tot} . The inner disk radius r_{in} is fixed analytically at the ISCO and used as an input parameter.

Key results:

1. **Methodological result.** Numerical convergence analysis under successive refinement ($n_{\text{pix}} = 80, 120, 160$) and the independent capture-boundary bisection method show that *only* the shadow radius R_{sh} is a numerically stable observable: its derivative with respect to ϵ converges to within $\sim 5\%$. The peak brightness I_{peak} (single-pixel maximum) and integrated brightness I_{tot} exhibit systematic non-convergence of derivatives, up to sign changes between grids, and cannot serve as a basis for quantitative conclusions. This limitation is essential for interpreting all published works on shadow-based metric diagnostics.
2. **Structural result.** On the stable channel R_{sh} , the three modifications act qualitatively differently: g^{tt} produces a strong shift ($dR_{\text{sh}}/d\epsilon \approx +0.58$ at $a = 0.9$, $\approx +0.99$ at $a = 0.99$); g^{rr} does not shift the shadow *identically* (analytically, Level A); $g^{\phi\phi}$ is practically invisible at $a = 0.9$ ($dR_{\text{sh}}/d\epsilon \approx 0.005$) and becomes visible at $a = 0.99$ ($\approx +0.36$).
3. **Quantitative assessment.** Fisher analysis on the stable observable R_{sh} gives $\sigma_\epsilon \approx 0.43$ (g^{tt} , $a = 0.9$), 0.25 (g^{tt} , $a = 0.99$), 0.68 ($g^{\phi\phi}$, $a = 0.99$); for g^{rr} , the derivative $dR_{\text{sh}}/d\epsilon$ vanishes identically with accuracy $10^{-4} M$ across the entire grid, rendering the modification strictly invisible in the shadow-radius channel at any observational precision.

1.5 Difference from Existing Works

From Kocherlakota & Rezzolla (2022): they study degeneracy for a general metric in terms of shadow sizes. We fix a specific class of modifications and show that degeneracy with astrophysics depends on the type of modification—not all modifications are equally degenerate. In particular, g^{rr} is strictly invisible in the R_{sh} channel (analytically and numerically), whereas g^{tt} produces a robust signal.

From Ren et al. (2026), Tello-Ortiz et al. (2026): these works show complete coincidence of signatures for specific parameters. We show a systematic picture: g^{rr} modifications do not shift the shadow boundary (analytically, Level A) and give only a numerically negligible photometric response (Level B), whereas g^{tt} modifications are strongly visible.

From Qi & Cai (2026), Övgün et al. (2026): they use shadow + QNM as orthogonal channels. We use several observables of a single image and show that only some of them (the shadow radius) are numerically stable and can serve as a reliable basis for quantitative conclusions.

From EHT Collaboration papers: they fit metric parameters to data for one modification. We construct a sensitivity map for all three classes and give a quantitative estimate of σ_ϵ on the stable observable.

1.6 Limitations of Our Analysis

We explicitly state the limitations, which are discussed in detail in Section 5.3:

- Thin disk ($H = 0$). Real disks have $H/r \approx 0.1$, which may change image details but should not affect the structural conclusions about the channels;
- Simple emission profile $\propto r^{-3}$. More realistic profiles (Novikov–Thorne, GRMHD) will change the quantitative numbers but not the qualitative picture;
- Numerical stability of observables. Peak and integrated brightnesses do not converge under grid refinement; the main quantitative analysis is performed on the shadow radius R_{sh} , computed by the bisection method (Appendix B);
- One class of modifications. We consider only the power-law profile $h(r) = (M/r)^2$. Other profiles (e.g., exponential or logarithmic) may give different signatures;
- No account of observational systematics. We assume uncorrelated Gaussian errors with a fixed relative level.

1.7 Structure of the Paper

The remainder is organized as follows.

Section 2 describes the three classes of modifications of the inverse Kerr metric and their relation to physical models.

Section 3 presents the methods: geodesic equations in Hamiltonian formalism, Bardeen initial conditions, integrator, two-pass disk rendering, extraction of observables, construction of the Fisher matrix.

Section 4 presents the results: validation on the Kerr basis, structural separation of channels between modifications, observable stability analysis, quantitative estimates of σ_ϵ on the stable channel, nearly extremal Kerr regime.

Section 5 discusses the physical interpretation, limitations of the method, comparison with the literature, and implications for ngEHT.

Section 6 summarizes.

Appendix A contains results of the stability check against grid resolution and the analysis of photometric non-convergence.

Appendix B describes the algorithm for precise determination of R_{sh} via bisection of the capture boundary.

Appendix C contains an analytical proof of the invisibility of g^{rr} in the shadow geometry.

2 Metric Modifications

2.1 Background Kerr Metric

We work in Boyer–Lindquist coordinates (t, r, θ, ϕ) with signature $(-, +, +, +)$. The covariant Kerr metric with parameters M (mass) and a (spin) has the form

$$g_{tt} = -\left(1 - \frac{2Mr}{\Sigma}\right), \quad (2)$$

$$g_{rr} = \frac{\Sigma}{\Delta}, \quad (3)$$

$$g_{\phi\phi} = \left(r^2 + a^2 + \frac{2Ma^2r \sin^2 \theta}{\Sigma}\right) \sin^2 \theta, \quad (4)$$

where the standard notation is introduced:

$$\Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 - 2Mr + a^2. \quad (5)$$

The outer event horizon is located at $r_+ = M + \sqrt{M^2 - a^2}$, and we consider only subextremal spins $|a| < M$. Below, units with $M = 1$ are used; all lengths are measured in units of mass.

2.2 Three Modifications of the Inverse Metric

We introduce three one-parameter deformations of the inverse metric of the form

$$g^{\mu\nu} \rightarrow g^{\mu\nu}[1 + \epsilon h(r)], \quad h(r) = \left(\frac{M}{r}\right)^2, \quad (6)$$

where ϵ is a dimensionless deformation parameter and $h(r)$ is a profile with power-law decay at large radii. The choice $h(r) \propto r^{-2}$ is motivated by the fact that quantum corrections in gravity typically fall off as $1/r^2$ at large distances.

The modification is applied to only one diagonal component at a time:

- tt -modification: $g^{tt} \rightarrow g^{tt}[1 + \epsilon h(r)]$;
- rr -modification: $g^{rr} \rightarrow g^{rr}[1 + \epsilon h(r)]$;
- $\phi\phi$ -modification: $g^{\phi\phi} \rightarrow g^{\phi\phi}[1 + \epsilon h(r)]$.

Derivatives of the modified inverse metric with respect to r and θ are computed analytically:

$$\partial_r g_{\text{mod}}^{\mu\nu} = \partial_r g^{\mu\nu}[1 + \epsilon h(r)] + g^{\mu\nu} \epsilon \partial_r h(r), \quad (7)$$

$$\partial_r h(r) = -2\frac{M^2}{r^3}, \quad \partial_\theta h(r) = 0, \quad (8)$$

where the unperturbed derivatives $\partial_r g^{\mu\nu}$, $\partial_\theta g^{\mu\nu}$ are also written out explicitly in terms of Σ , Δ , and $A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$. This avoids numerical differentiation, which loses accuracy near the photon sphere.

2.3 Physical Interpretation

We emphasize: the deformations (6) are applied to the *contravariant* metric $g^{\mu\nu}$, not the covariant metric. The corresponding covariant metric is obtained by inverting the matrix $g_{\text{mod}}^{\mu\nu} \rightarrow (g_{\text{mod}}^{\mu\nu})^{-1}$, and the relation between a deformation of one component of the inverse metric and a deformation of the same component of the covariant metric is nonlinear. Therefore, the physical interpretations given below should be regarded as *motivational examples*, not as identifications with specific solutions of modified gravity theories. A strict identification would require solving the field equations of the corresponding theory near the horizon, which is beyond the scope of the present work.

The three classes of modifications nevertheless correspond to three qualitatively different physical scenarios.

g^{tt} modification — quantum corrections to the gravitational potential. An effective change of g^{tt} at small distances arises in:

- asymptotically safe gravity (Reuter & Saueressig, 2002; Bonanno & Reuter, 2000): a running Newtonian constant $G(r)$ gives an effective g^{tt} depending on r ;
- quantum corrections to the Schwarzschild metric (Duff, 1974; Kazakov & Solodukhin, 1994): one-loop corrections shift g^{tt} as $1/r^2$;
- nonlocal gravity (Biswas et al., 2006): modification of the graviton propagator at large r gives effective smoothing.

g^{rr} modification — singularity regularization. Such a structure is characteristic of:

- the Bardeen metric (Bardeen, 1968): a regular BH with a de Sitter core;
- the Hayward metric (Hayward, 2006): a regular BH due to a quantum bounce;
- the Simpson–Visser metric (Simpson & Visser, 2019): a “black bounce” connecting a BH and a wormhole;
- effective LQG models (Ashtekar et al., 2018): polymer quantization of the gravitational field gives regularization of g^{rr} at the Planck length scale.

Importantly, in the models listed above, the modification of g_{rr} is accompanied by changes of other metric components through the field equations. The present work studies the isolated effect of a single component within a diagonal parameterization; the conclusions do not transfer automatically to full solutions of the corresponding theories.

$g^{\phi\phi}$ modification — nonminimal coupling and hair. A change of the effective angular momentum at small distances arises in:

- nonminimal coupling of a scalar field to curvature (Chern–Simons, Gauss–Bonnet): the effective metric for photons differs from the metric for gravitons;
- axionic hair (Campbell et al., 1992): a rotating axion field gives an additional contribution to $g_{\phi\phi}$;
- scalar hair (Herdeiro & Radu, 2014): hairy Kerr with a nonlinear scalar field.

2.4 Disk Setup: Kinematically Fixed Source

All three modifications are applied only to the *propagation of photons* (null geodesic equations). The accretion-disk matter is described by the standard Kerr formulas for the angular velocity Ω_K , the four-velocity normalization u^t , and the ISCO radius. This setup—a *kinematically fixed source* on a background of modified photon-trajectory geometry—is chosen deliberately, to isolate the effect of geodesic modification alone, without mixing it with effects of orbital restructuring of the disk matter.

The alternative, self-consistent setup, in which the matter moves along circular geodesics of the modified metric, would require recomputing orbital characteristics for each modification and is beyond the scope of this work. We state this choice explicitly so that the reader does not interpret our results as a fully self-consistent solution of a specific gravity theory. All conclusions about channel separation and observable stability refer to the fixed disk kinematics.

2.5 Small-Deformation Regime

We work in the regime $|\epsilon| \leq 0.1$. This is consistent with existing EHT constraints on metric deformations: observational data of M87* and Sgr A* constrain deviations from Kerr at the level of a few percent near the horizon. At $|\epsilon| = 0.1$ and $r \approx 5M$, the relative change of $g^{\mu\nu}$ is $\approx 0.4\%$, which is at the edge of sensitivity of the current generation of interferometers.

3 Methods

3.1 Null Geodesic Equations of Motion

Null geodesics are integrated in the Hamiltonian formalism. The Hamiltonian function has the form

$$H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = 0, \quad (9)$$

where p_μ is the covariant 4-momentum. In a stationary and axisymmetric Kerr metric, the energy $E = -p_t$ and the projection of the angular momentum $L_z = p_\phi$ are conserved. We fix the affine parameterization by setting $E = 1$, so that $p_t = -1$.

Hamilton's equations:

$$\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial p_\mu}, \quad \frac{dp_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}. \quad (10)$$

$$\frac{dr}{d\lambda} = g^{rr}p_r, \quad \frac{d\theta}{d\lambda} = g^{\theta\theta}p_\theta, \quad (11)$$

$$\frac{dp_r}{d\lambda} = -\frac{1}{2}(\partial_r g^{\mu\nu})p_\mu p_\nu, \quad \frac{dp_\theta}{d\lambda} = -\frac{1}{2}(\partial_\theta g^{\mu\nu})p_\mu p_\nu. \quad (12)$$

3.2 Initial Conditions from the Image Plane

We use the standard mapping of the observer's angular coordinates (α, β) onto the celestial plane in the Bardeen convention (Bardeen, 1973):

$$L_z = -\alpha \sin \theta_{\text{obs}}, \quad Q = \beta^2 + \cos^2 \theta_{\text{obs}}(\alpha^2 - a^2), \quad (13)$$

where Q is the Carter constant. The radial potential $R(r)$ and the angular potential $\Theta(\theta)$ are defined through the conserved quantities:

$$R(r) = [r^2 + a^2 - aL_z]^2 - \Delta[(L_z - a)^2 + Q], \quad (14)$$

$$\Theta(\theta) = Q + a^2 \cos^2 \theta - L_z^2 \cot^2 \theta. \quad (15)$$

As the initial state, we use the radial momentum corresponding to inward motion (backward integration in the affine parameter):

$$p_r|_{\text{obs}} = -\frac{\sqrt{R(r_{\text{obs}})}}{\Delta(r_{\text{obs}})}. \quad (16)$$

The angular momentum p_θ is determined by the sign of β :

$$p_\theta|_{\text{obs}} = \pm \sqrt{\Theta(\theta_{\text{obs}})} = \pm |\beta|. \quad (17)$$

The initial value of ϕ is fixed at $\phi = 0$ without loss of generality.

3.3 Geodesic Integrator

The integration of the ODE system is performed by the Runge–Kutta 4th/5th order method with adaptive step from the `scipy.integrate.solve_ivp` library:

Parameter	Value
Method	RK45 (Dormand–Prince)
rtol	10^{-8}
atol	10^{-10}
max_step	20
Initial λ	0
Final λ (main grid)	10^4
Final λ (bisection)	10^5

Integration is terminated early when one of two terminator events is triggered:

1. Capture: $r < r_+ + 10^{-3}$. The ray is classified as captured by the black hole.
2. Escape: $r > 0.99 r_{\text{obs}}$. The ray leaves the computation domain.

If neither event is triggered during $\lambda \leq 10^4$, the ray is classified as near-critical and assigned to the captured category. This classification is consistent with the behavior near the photon sphere: the residence time of such rays near $r \approx 3M$ exceeds the chosen limit.

The observer’s starting point:

$$r_{\text{obs}} = 1000M, \quad \theta_{\text{obs}} = i \quad (\text{inclination from the pole}). \quad (18)$$

3.4 Two-Pass Rendering

To correctly distinguish the shadow from the disk image, a two-pass algorithm is used.

Pass 1: shadow. The disk is turned off. For each pixel (α, β) , a ray is traced and the result is classified:

$$\text{shadow}[\alpha, \beta] = \begin{cases} \text{True,} & \text{ray captured,} \\ \text{False,} & \text{ray escaped.} \end{cases} \quad (19)$$

This yields a binary shadow mask independent of the accretion disk model.

Pass 2: image. A thin equatorial disk is turned on. For every ray, independently of whether it is ultimately classified as captured or escaped, a search for intersections with the plane $\theta = \pi/2$ along the trajectory is performed. In particular, a ray classified as captured may still cross the emitting equatorial plane before falling into the horizon; such crossings are included in the intensity accumulation. The capture classification is used only for constructing the binary shadow mask, not for filtering the intensity. At each intersection point r_c in the range $[r_{\text{in}}, r_{\text{out}}]$, a contribution to the observed intensity is accumulated:

$$I_{\text{obs}} = g^4 I_{\text{em}}(r_c), \quad (20)$$

where $g = \nu_{\text{obs}}/\nu_{\text{em}}$ is the total redshift and Doppler factor, and $I_{\text{em}}(r)$ is the disk emission profile. We use the power-law profile

$$I_{\text{em}}(r) \propto \left(\frac{r_{\text{in}}}{r}\right)^3, \quad (21)$$

motivated by the Novikov–Thorne profile for a thin disk.

The factor g for Keplerian rotation is computed as

$$g(r, L_z) = \frac{1}{u^t(1 - \Omega_K L_z)}, \quad (22)$$

where $\Omega_K = 1/(r^{3/2} + a)$ is the angular velocity of the Keplerian orbit, and

$$u^t = \frac{r^{3/2} + a}{r^{3/4} \sqrt{r^{3/2} - 3\sqrt{r} + 2a}}. \quad (23)$$

The inner radius of the disk is fixed at the radius of the last stable circular orbit (ISCO):

$$r_{\text{in}} = r_{\text{ISCO}}(a), \quad (24)$$

which is computed by the analytical Bardeen–Press–Teukolsky formula (Bardeen et al., 1972). For the outer radius, $r_{\text{out}} = 20M$ is chosen.

Accounting for multiple intersections. A single ray may cross the equatorial plane several times—this corresponds to direct and secondary images of the disk. In the current version of the algorithm, all intersections are summed into a single contribution I_{obs} , which gives the correct total intensity. Decomposition by photon-ring orders is beyond the scope of this work.

3.5 Observables

From the obtained maps (shadow + image), five observables are extracted; the inner disk radius is fixed analytically at the ISCO and used as an input parameter.

1. **Shadow radius** R_{sh} — the effective radius determined through the area of the shadow mask:

$$R_{\text{sh}}^{\text{pixel}} = \sqrt{\frac{N_{\text{sh}} \Delta A}{\pi}}, \quad (25)$$

where N_{sh} is the number of pixels in the mask and $\Delta A = (2 \text{fov}/n_{\text{pix}})^2$ is the pixel area. Alternatively, to eliminate the systematic error of pixel quantization, the **capture-boundary bisection method** is used (Appendix B). The method provides a search step of $\pm 0.005 M$ independent of grid resolution; the full accuracy is limited by residual systematics of $\sim 0.5\%$ (see Section “Validation” in Appendix B).

2. **Shadow centroid shift** $(\Delta\alpha_{\text{sh}}, \Delta\beta_{\text{sh}})$ — the difference between the mask centroid and the image center.
3. **Brightness asymmetry** A_{asym} :

$$A_{\text{asym}} = \frac{I_L - I_R}{I_L + I_R}, \quad (26)$$

where I_L and I_R are the total intensities in the left and right halves of the image.

4. **Peak brightness** $I_{\text{peak}} = \max I(\alpha, \beta)$.
5. **Integrated brightness** $I_{\text{tot}} = \sum_{i,j} I_{ij} \Delta x^2$ — the sum of intensities over all pixels, normalized to the physical pixel area.

As shown in Section 4 and Appendix A, only R_{sh} (in the bisection variant) is numerically stable among this set. Photometric observables are used for qualitative analysis of the sign of the response.

3.6 Fisher Matrix

To quantitatively assess the sensitivity to the parameter ϵ , we construct the Fisher matrix on the single stable observable R_{sh} :

$$F_{\epsilon\epsilon} = \frac{1}{\sigma_R^2} \left(\frac{\partial R_{\text{sh}}}{\partial \epsilon} \right)^2, \quad (28)$$

where σ_R is the observational accuracy of the shadow radius. The derivatives are computed by central finite differences over three grid points:

$$\left. \frac{\partial R_{\text{sh}}}{\partial \epsilon} \right|_{\epsilon=0} \approx \frac{R_{\text{sh}}(+\epsilon_0) - R_{\text{sh}}(-\epsilon_0)}{2\epsilon_0}, \quad (29)$$

where $\epsilon_0 = 0.1$.

The observational accuracy of R_{sh} is specified as a relative uncertainty $\sigma_R = 0.05 R_{\text{sh}}$ (5%). The corresponding estimate of the parameter accuracy:

$$\sigma_\epsilon = \frac{1}{\sqrt{F_{\epsilon\epsilon}}}. \quad (30)$$

3.7 Numerical Grid

The main calculations are performed on a grid of 144 points:

Parameter	Values
Modification	$tt, rr, \phi\phi$
Spin a	0.0, 0.5, 0.9, 0.99
Deformation ϵ	-0.1, 0, +0.1
Inclination i	30°, 70°, 80°, 85°
Inner radius	$r_{\text{ISCO}}(a)$

Rendering parameters:

Parameter	Value
n_{pix}	80
fov	$\pm 25 M$
r_{out}	20 M
r_{obs}	1000 M

Additionally, to verify the stability of observables, critical points are recomputed on grids $n_{\text{pix}} = 120$ and 160 (see Section 4 and Appendix A), as well as by the independent capture-boundary bisection method (Appendix B).

3.8 Software Implementation

All calculations were performed in Python 3.11 using the following libraries:

- `numpy` — arrays and vectorization;
- `scipy.integrate.solve_ivp` — ODE integrator;
- `matplotlib` — visualization;
- `multiprocessing` / `concurrent.futures.ProcessPoolExecutor` — parallel rendering on 6 CPU cores.

The complete source code for the ray-tracing pipeline, figure generation scripts, and convergence-analysis tools is publicly available in the Zenodo repository (Lipyanchik, 2026) (DOI:10.5281/zenodo.23436302). The repository includes a README file with instructions, the list of dependencies, and all parameters required to reproduce the results presented in Sections 4 and 5.

4 Results

4.1 Validation on the Kerr Basis

Before analyzing the modifications, we checked the pipeline on the standard Kerr metric ($\epsilon = 0$ for all three modifications). For the Schwarzschild case ($a = 0$), the shadow radius must be the same for any observer inclination:

$$R_{\text{sh}}(a = 0) = 3\sqrt{3}M \approx 5.196M. \quad (31)$$

The numerical calculation at $n_{\text{pix}} = 80$, $\text{fov} = 25M$ for the unperturbed case ($\epsilon = 0$) gives:

$$R_{\text{sh}}^{\text{num}}(a = 0, \epsilon = 0) = 5.0855M \quad \text{for all } i \in \{30^\circ, 70^\circ, 80^\circ, 85^\circ\}. \quad (32)$$

The deviation from the analytical value is $\approx 2.1\%$ and is due to pixel quantization of the shadow boundary (grid step $\Delta x = 0.625M$). Applying the capture-boundary bisection method (Appendix B) improves the agreement to 0.5% , removing the main systematic error of pixel quantization. This confirms the correctness of the geodesic integrator and initial conditions (Bardeen, 1973): at $a = 0$ the shadow is spherically symmetric, as observed.

The shadow centroid shift $\Delta\alpha_{\text{sh}}$ grows with spin, which corresponds to the classical frame-dragging effect:

a	$\Delta\alpha_{\text{sh}} (M)$
0.0	0.00
0.5	$\approx +0.6$
0.9	$+1.28$
0.99	$\approx +1.5$

The shadow shifts toward positive α (the approaching side), as expected for an observer in the equatorial plane.

4.2 Structural Result: Modifications Act in Different Channels

The central result of this work is that the g^{tt} , g^{rr} , and $g^{\phi\phi}$ modifications have a qualitatively different effect on the observed images (see Fig. 1). Let us consider the most sensitive point of the grid, $a = 0.9$, $i = 80^\circ$.

As shown in Table 1 and Fig. 2, the quantitative behavior of R_{sh} confirms this qualitative picture.

Table 1: Shadow radius R_{sh} (in units of M) at $a = 0.9$, $i = 80^\circ$, from the bisection method.

Modification	$\epsilon = -0.1$	$\epsilon = 0$	$\epsilon = +0.1$	$dR/d\epsilon$
g^{tt}	4.964	5.028	5.079	+0.576
g^{rr}	5.028	5.028	5.028	0.000
$g^{\phi\phi}$	5.028	5.028	5.028	$\approx +0.005$

Conclusion 1. The g^{tt} modification **shifts the shadow boundary**; at $a = 0.9$ the derivative $dR_{\text{sh}}/d\epsilon \approx +0.58$, rising to $\approx +0.99$ at $a = 0.99$ (Section 4.4). The g^{rr} modification **does not shift the shadow at all**—across the entire grid (all a , i , ϵ) the quantity $dR/d\epsilon_{rr}$ vanishes with accuracy $10^{-4}M$. The $g^{\phi\phi}$ modification is practically invisible at $a = 0.9$ in the R_{sh} channel ($|dR/d\epsilon| \lesssim 0.02$, below the numerical noise floor of the bisection method) and becomes visible only at $a = 0.99$ ($dR/d\epsilon \approx +0.36$).

This result has an analytical justification (Level A in Appendix C). In the equatorial plane, the critical curve is determined by the condition of a double root of the radial potential $R(r_c) = R' = 0$, where $R(r)$ is expressed only through E , L_z , Q , and Δ . The g^{rr} component enters only the equation for the radial velocity $dr/d\lambda = g^{rr}p_r$ and does not participate in determining $R(r)$. Changing

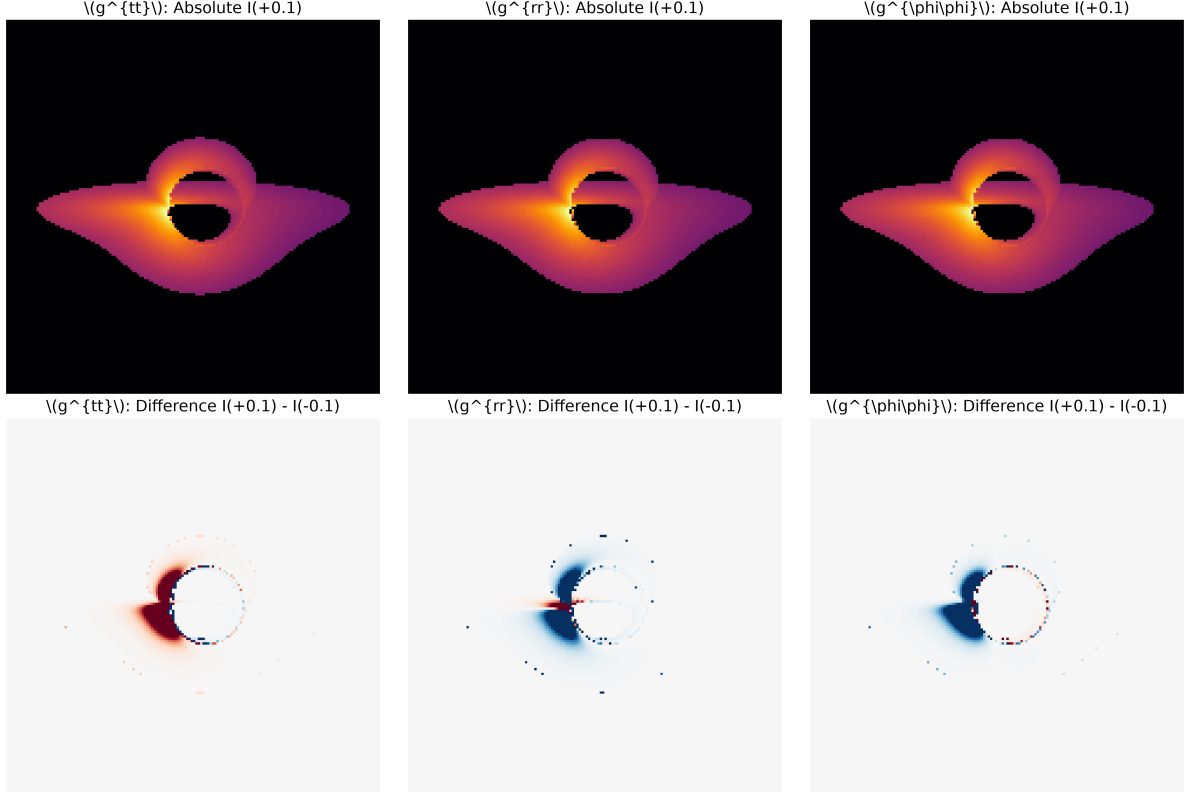


Figure 1: Disk images and difference maps at $a = 0.9M$, $i = 80^\circ$. Top row: absolute intensity maps for the three modifications at $\epsilon = +0.1$, which appear visually indistinguishable by eye. Bottom row: difference maps $\Delta I = I(\epsilon = +0.1) - I(\epsilon = -0.1)$, shown with enhanced contrast (red indicates an increase in intensity, blue a decrease). The g^{tt} modification (left) shows a clear dipole-like pattern associated with the shifting of the photon ring. The g^{rr} modification (middle) is indistinguishable from numerical noise within our numerical precision, confirming that its effect on the full image is negligible compared to the other modifications. The $g^{\phi\phi}$ modification (right) produces a visible signal of the opposite sign. This figure explicitly demonstrates the structural separation of channels that is hidden in absolute images.

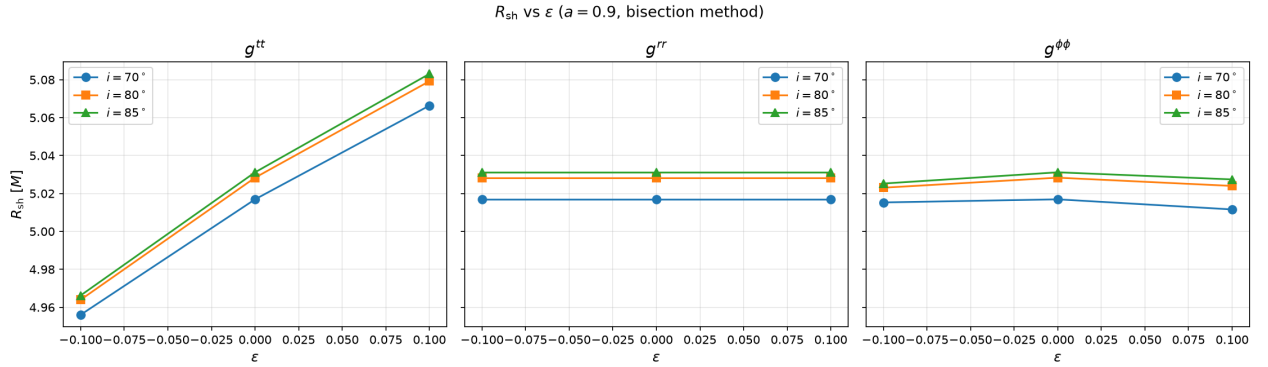


Figure 2: R_{sh} as a function of ϵ for the three modifications at $a = 0.9$ (bisection method). Three panels correspond to the three modifications. The g^{rr} modification gives a flat line in all panels—zero signal in the shadow geometry. The $g^{\phi\phi}$ modification is also essentially flat at this spin.

g^{rr} reparameterizes the affine parameter and modifies the function $r(\phi)$, but does not change the location of the roots $R(r) = 0$. Consequently, $dR/d\epsilon_{rr} = 0$ at leading order of perturbation theory.

We emphasize: **invariance of the critical curve does not automatically imply invariance of the full disk image**. The intersection points of the trajectory with the equatorial plane and the redshift factor may in principle depend on ϵ_{rr} . The numerical result shows that in the studied parameter range this effect is small both in photometry (with the caveat discussed below) and in the R_{sh} channel, where the derivative vanishes identically. A detailed separation of Levels A, B, C is given in Appendix C.

A similar logic applies to any *isolated* g^{rr} modification of the form $g^{rr} \rightarrow g^{rr}[1 + \epsilon h(r)]$, provided that $h(r)$ does not change the horizon position and decays sufficiently fast at large radii. However, we emphasize: in singularity-regularization models (Bardeen, Hayward, Simpson–Visser, polymer LQG solutions) the modification of g_{rr} is accompanied by changes of other metric components through the field equations. Our conclusion concerns the isolated deformation of g^{rr} within the diagonal parameterization and does not transfer automatically to full solutions of these theories.

Table 2: Peak brightness I_{peak} (arbitrary units) at $a = 0.9$, $i = 80^\circ$, grid $n_{\text{pix}} = 80$. Values are computed on a single grid and serve as a qualitative illustration of the signs of the response; the derivatives do not converge under grid refinement (see Appendix A).

Modification	$\epsilon = -0.1$	$\epsilon = 0$	$\epsilon = +0.1$	$dI/d\epsilon$
g^{tt}	2.749	3.138	3.746	+4.99
g^{rr}	3.134	3.138	3.141	+0.03
$g^{\phi\phi}$	3.336	3.138	2.956	−1.90

The convergence of the derivatives under grid refinement is shown in Fig. 3.

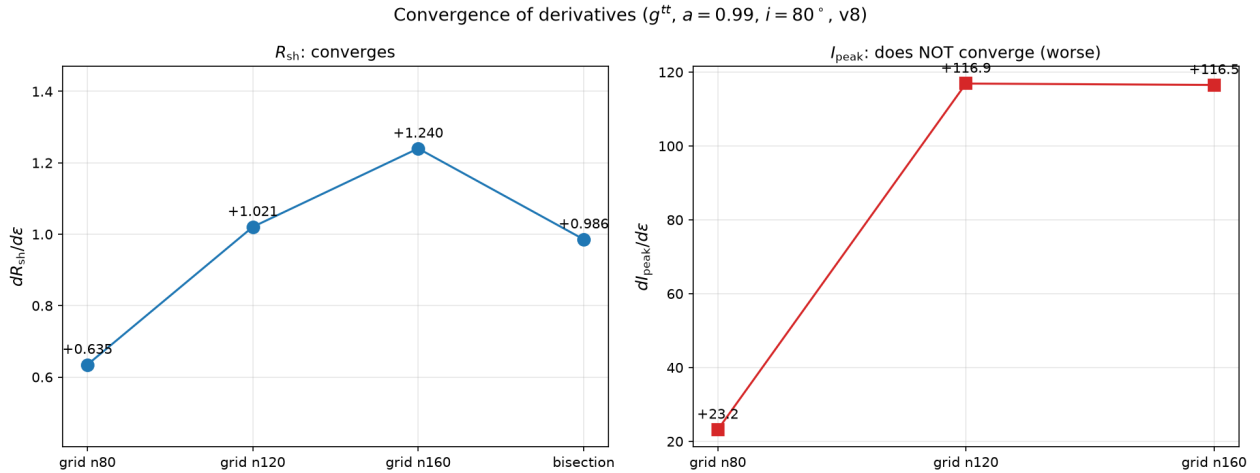


Figure 3: Convergence of derivatives for g^{tt} , $a = 0.99$, $i = 80^\circ$. Left: $dR_{\text{sh}}/d\epsilon$ obtained by the bisection method is stable under resolution refinement. Right: $dI_{\text{peak}}/d\epsilon$ exhibits systematic non-convergence: from +23.2 at $n_{\text{pix}} = 80$ to +116.9 at 120, and stabilizing at +116.5 at 160 for $i = 80^\circ$. For other inclinations (e.g., $i = 85^\circ$) the spread between grids exceeds two orders of magnitude.

Conclusion 2 (qualitative). In the photometric channel, the qualitative picture is preserved: g^{tt} gives a strong response, $g^{\phi\phi}$ gives a moderate response of the opposite sign, and g^{rr} is practically zero. However, we emphasize: the *quantitative* values of $dI_{\text{peak}}/d\epsilon$ in Table 2 are obtained on grid $n_{\text{pix}} = 80$ and *do not converge* under resolution refinement (see Section 4.3.1 and Appendix A). Therefore, photometric numbers should be regarded as qualitative indicators, not as a basis for quantitative conclusions. The overall trend of the response (positive for g^{tt} , negative for $g^{\phi\phi}$) is preserved across grids, although in specific cases the sign of the derivative may change (see Appendix A for details).

The sign of the response is a characteristic feature of the modification. g^{tt} and $g^{\phi\phi}$ shift I_{peak} in opposite directions.

Table 3: Brightness asymmetry $A_{\text{asym}} = (I_L - I_R)/(I_L + I_R)$ at $a = 0.9$, $i = 80^\circ$.

Modification	$\epsilon = -0.1$	$\epsilon = 0$	$\epsilon = +0.1$
g^{tt}	0.9099	0.9154	0.9162
g^{rr}	0.9159	0.9154	0.9150
$g^{\phi\phi}$	0.9160	0.9154	0.9119

Conclusion 3. The asymmetry A_{asym} changes weakly (within 1%) for all three modifications. This is not a discriminating observable in this setup.

4.2.1 Physical Interpretation

The observed channel separation has a transparent physical reason:

- g^{tt} enters the null geodesic normalization condition $g^{\mu\nu}p_\mu p_\nu = 0$. The modification changes both the photon trajectories (shifting the critical curve) and the redshift factor $g = 1/(u^t(1 - \Omega L_z))$, and through it the observed brightness $I_{\text{obs}} \propto g^4$. Hence the strong response in both channels.
- g^{rr} enters only the radial equation of motion. For equatorial observation, the critical curve is determined by the balance between radial and azimuthal motion. The g^{rr} modification does not shift this balance, due to the structure of the geodesic equations. In photometry, its contribution is also small, since the redshift does not depend on g^{rr} at leading order.
- $g^{\phi\phi}$ enters only the azimuthal equation. Its influence on the shadow geometry is weaker than that of g^{tt} and strongly spin-dependent: almost absent at $a = 0.9$, noticeable at $a = 0.99$. In photometry it gives a moderate response with opposite sign relative to g^{tt} .

4.3 Fisher Matrix

4.3.1 Selection of Stable Observables

Before constructing the Fisher matrix, we checked the stability of the derivatives $\partial O/\partial\epsilon$ of the observables under successive grid refinement ($n_{\text{pix}} = 80, 120, 160$) and by the independent capture-boundary bisection method (Appendix B). The result:

- R_{sh} (**bisection**): derivatives converge to within $\sim 5\%$ across inclinations and are independent of grid resolution. For g^{rr} the derivative vanishes identically with accuracy $10^{-4} M$.
- I_{peak} (**pixel maximum**): derivatives do not converge, changing by a factor of 7 between grids, and may change sign. Not suitable for quantitative analysis.
- I_{tot} (**integrated brightness**): more stable than I_{peak} , but still fluctuates by tens of percent between grids. Used only as a qualitative check of the sign of the response.
- A_{asym} : variations within 1% for all modifications—not a discriminating observable.

Conclusion. The main Fisher analysis is performed on the single stable observable R_{sh} , whose values are obtained by the bisection method. The search step is $\pm 0.005 M$. The residual systematics of $\sim 0.5\%$ is largely common to all values of ϵ at fixed (a, i) and is therefore expected to cancel when differences are taken; this assumption is supported by the weak dependence of R_{sh} on ϵ at fixed (a, i) , but a fully rigorous justification would require an independent analysis of the ϵ -dependence of the systematics.

4.3.2 Results

The Fisher matrix is constructed using Eq. (28) with $\sigma_R = 0.05 R_{\text{sh}}$. Derivatives are computed by central finite differences over three values $\epsilon \in \{-0.1, 0, +0.1\}$. We emphasize that the resulting σ_ϵ represents a *conditional* sensitivity estimate at fixed spin a , inclination i , and disk model, rather than a full multi-parameter measurement uncertainty.

Table 4: Best σ_ϵ in the R_{sh} channel (bisection method).

Modification	a	$\sigma_\epsilon^{\min}$	Best point	$dR_{\text{sh}}/d\epsilon$
g^{tt}	0.90	0.431	$i = 85^\circ$	+0.584
g^{tt}	0.99	0.251	$i = 85^\circ$	+0.987
$g^{\phi\phi}$	0.90	—	—	≈ 0
$g^{\phi\phi}$	0.99	0.684	$i = 85^\circ$	+0.362
g^{rr}	any	∞	—	0.000

The resulting sensitivity $\sigma_\epsilon^{\min}$ as a function of spin is visualized in Fig. 4.

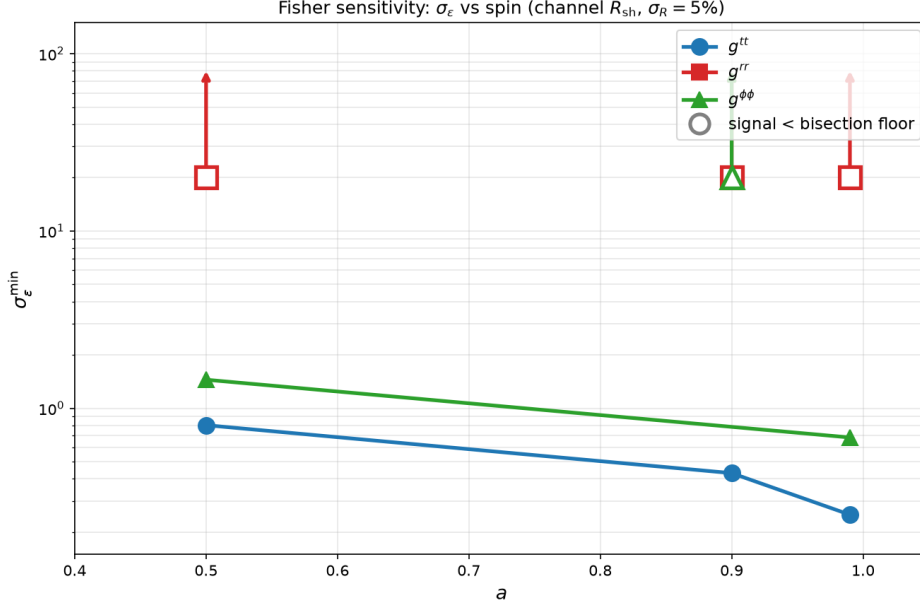


Figure 4: Sensitivity $\sigma_\epsilon^{\min}$ versus spin at $\sigma_R = 5\%$. Solid lines — channels with numerically resolved non-zero derivative; open markers with arrows — lower limits on σ_ϵ for the cases where the derivative is below the numerical noise floor of the bisection method.

Conclusion 4. In the stable R_{sh} channel:

- g^{tt} : $\epsilon = 0.1$ is measurable at the level of $\sim 0.23\sigma$ at $a = 0.9$ and $\sim 0.40\sigma$ at $a = 0.99$ —both estimates based on a stable observable;
- $g^{\phi\phi}$: at $a = 0.9$ the signal in R_{sh} is below the bisection noise floor; at $a = 0.99$ a moderate signal appears with $\sigma_\epsilon \approx 0.68$;
- g^{rr} : the derivative vanishes identically, σ_ϵ is formally infinite—the modification is strictly invisible in the R_{sh} channel at any observational accuracy.

At $a = 0.99$ the sensitivity to g^{tt} is a factor of ~ 2.7 better than to $g^{\phi\phi}$ (i.e., σ_ϵ is smaller); g^{rr} is undetectable at any precision.

4.4 Nearly Extremal Kerr Regime ($a = 0.99$): Stability of Observables

At nearly extremal spin $a = 0.99$, the behavior of the peak brightness differs sharply from the regime $a \leq 0.9$. In the original version of the code, rays classified as captured (falling into the horizon) were excluded from the image computation on the assumption that they do not contribute to the observation. This assumption is incorrect: a captured ray may cross the emitting equatorial plane before falling into the horizon. In the corrected version of the code (`degeneracy_pilot_v8.py`), all disk crossings are accounted for regardless of the ray's subsequent fate.

Results after correction. For the point $a = 0.99$, $i = 80^\circ$, modification g^{tt} :

n_{pix}	$I_{\text{peak}}(-0.1)$	$I_{\text{peak}}(0)$	$I_{\text{peak}}(+0.1)$	$dI/d\epsilon$
80	1.06	1.33	5.70	+23.2
120	1.18	1.77	24.57	+116.9
160	1.20	1.95	24.51	+116.5

The values of $dI_{\text{peak}}/d\epsilon$ at $n_{\text{pix}} = 120$ and 160 agree to within 0.3%, while at $n_{\text{pix}} = 80$ the value is a factor of ~ 5 smaller. For other inclinations at the same spin the picture is worse: at $i = 85^\circ$ the values of $dI/d\epsilon$ are +33.6 ($n_{\text{pix}} = 80$), +2524 (120), and +1254.7 (160)—a spread of two orders of magnitude.

Physical reason. At $a = 0.99$ the ISCO radius is $r_{\text{ISCO}} \approx 1.45 M$, while the horizon radius is $r_+ \approx 1.14 M$. A ray passing near the photon sphere may wind around the BH, cross the disk in a narrow annular zone near $r \sim 1.5\text{--}3 M$, and only then fall into the horizon. The redshift factor $g = 1/[u^t(1 - \Omega L_z)]$ in this region reaches values $\gtrsim 3$, giving $g^4 \gtrsim 80$. It is precisely these rare rays that dominate I_{peak} . The number of such rays hitting a specific pixel depends critically on the grid resolution, which explains the observed instability.

Consequence for the analysis. Photometric observables at $a \rightarrow M$ not only fail to converge—the spread between grids *increased* after the renderer fix. At the same time, the R_{sh} channel remains fully stable: the values of $dR_{\text{sh}}/d\epsilon$ for g^{tt} at $a = 0.99$ are +0.975, +0.986, and +0.987 at $i = 70^\circ, 80^\circ, 85^\circ$ (bisection), and these values did not change after the renderer fix.

Conclusion. Points at $a = 0.99$ are used in the Fisher analysis exclusively in the R_{sh} channel. Photometric observables in this region are excluded as numerically unstable and physically problematic (sensitive to the position of the inner disk edge, where the thin-disk model is least reliable).

4.5 Summary of Results

Statement	Channel	Degree of reliability
g^{tt} shifts the shadow, $dR/d\epsilon \approx +0.6$	R_{sh} (bisection)	high
$g^{\phi\phi}$ does not shift the shadow at $a = 0.9$	R_{sh} (bisection)	high
$g^{\phi\phi}$ weakly shifts the shadow at $a = 0.99$	R_{sh} (bisection)	medium
g^{rr} has an identically zero effect on the critical curve	R_{sh} (bisection)	very high (analytically)
g^{tt} gives a strong photometric response	$I_{\text{peak}}, I_{\text{tot}}$	qualitative
$g^{\phi\phi}$ gives an opposite response	$I_{\text{peak}}, I_{\text{tot}}$	qualitative
g^{rr} gives no photometric response	$I_{\text{peak}}, I_{\text{tot}}$	medium
$\sigma_\epsilon(\phi\phi)/\sigma_\epsilon(tt) \sim 2.7$ at $a = 0.99$	Fisher (R_{sh})	high

5 Discussion

5.1 Physical Interpretation of the Structural Result

The central result of this work is that the g^{tt} , g^{rr} , and $g^{\phi\phi}$ modifications manifest qualitatively differently in the observables of a thin-disk image. Let us analyze the physical reason for this separation.

5.1.1 The Role of g^{tt} in the Null Geodesic Equations

The g^{tt} component enters the null geodesic normalization condition:

$$g^{\mu\nu}p_\mu p_\nu = 0 \implies g^{tt}p_t^2 + 2g^{t\phi}p_t L_z + g^{\phi\phi}L_z^2 + g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 = 0. \quad (35)$$

For fixed $E = -p_t = 1$ and L_z , the modification $g^{tt} \rightarrow g^{tt}(1 + \epsilon h)$ directly changes the balance between the terms. This affects two key things simultaneously:

1. **The critical curve (shadow boundary).** The radial potential $R(r)$ contains g^{tt} through the dispersion relation. A shift of g^{tt} changes the position of the critical curve, and hence R_{sh} .
2. **The redshift factor.** The quantity $g = 1/[u^t(1 - \Omega_K L_z)]$ depends on g^{tt} at the emission point. In our kinematically fixed setup, u^t and Ω_K are taken from the unperturbed Kerr metric, so at a *fixed* emission radius the redshift factor does not depend on ϵ explicitly. The dependence of the observed intensity on ϵ arises through a different mechanism: the modified geodesic trajectory shifts the radius at which the ray crosses the emitting plane, thus changing which g -value is sampled. The strong sensitivity of $I_{\text{obs}} \propto g^4$ then amplifies this shift into a sizeable photometric response. This mechanism should be kept in mind when interpreting the photometric results.

This is precisely why the g^{tt} modification is visible both in the shadow geometry and in photometry. The R_{sh} channel is *numerically stable* and admits quantitative estimates, whereas the photometric channel is used qualitatively only, due to non-convergence of its derivatives.

5.1.2 The Role of g^{rr}

The g^{rr} modification enters only the radial equation of motion:

$$\frac{dr}{d\lambda} = g^{rr}p_r. \quad (36)$$

This equation determines the rate of change of the radial coordinate along the trajectory, but not the trajectory itself as a geometric locus of points in coordinates (r, ϕ) . The shape of the orbit is determined by the second-order equation for $r(\phi)$, into which g^{rr} enters only through a common factor in front of $(dr/d\phi)^2$. In going to the critical curve—the condition of a double root of the radial potential $R(r_c) = R' = 0$ —this factor cancels (see Appendix C for a rigorous derivation).

Strict consequence (Level A). The critical curve does not depend on g^{rr} at leading order of perturbation theory. The numerical result $dR/d\epsilon_{rr} = 0$ across the entire grid confirms this analytical statement.

Photometry (Level B). The same factor $f(r)$ does not cancel in the function $r(\phi)$ itself: the roots are preserved, but the trajectory is rescaled. Therefore, the photometric response *does not follow automatically* from the invariance of the critical curve. The numerical calculation gives $dI/d\epsilon_{rr} \approx 0.03$ at $a = 0.9, i = 80^\circ$ —two orders of magnitude weaker than for the g^{tt} modification. However, as shown in Appendix A, the numerical value of this derivative does not converge under grid refinement and should be regarded as an *upper estimate* of the order of the effect, not as a precise number. We do not provide a strict analytical proof of the invariance of the full image.

Physical interpretation. The g^{rr} component defines the radial metric—the distance between spheres r and $r + dr$. It affects how fast a photon moves radially, but not where it ends up. The shadow is the set of impact parameters of captured rays, i.e., a property of the geometry of orbits, not of their parameterization. Hence the zero response in R_{sh} (Level A). The weakness of the photometric signal is an additional observation, not an automatic consequence.

Level C: limits of applicability. The statement that *all* observables (including polarization, high-order photon rings, and arbitrary emission profiles) are insensitive to ϵ_{rr} is **not tested and not proven** in this work. We checked one specific set of observables in one disk model. A similar argument applies to $g^{\theta\theta}$ modifications: Level A holds for them, Levels B and C require separate investigation.

5.1.3 The Role of $g^{\phi\phi}$

The $g^{\phi\phi}$ component enters only the azimuthal equation of motion and the L_z^2 term of the normalization condition. Its influence on the critical curve is weaker than that of g^{tt} and, as shown by the bisection method, strongly spin-dependent: at $a = 0.9$ the derivative $dR/d\epsilon \approx 0.005$ —at the noise level; at $a = 0.99$ it increases to $\approx +0.36$.

In photometry, the $g^{\phi\phi}$ modification gives a signal of opposite sign relative to g^{tt} : $dI/d\epsilon \approx -1.9$ versus $+5.0$ at $a = 0.9$, $i = 80^\circ$. Similar to the g^{tt} case, in our kinematically fixed model, the redshift factor at a fixed emission radius does not depend on ϵ explicitly. The observed photometric response arises because the modified geodesic equations (specifically, the $g^{\phi\phi}L_z^2$ term in the null condition) shift the trajectories and change the radius at which the rays intersect the disk, sampling a different redshift. The opposite sign of the response reflects the different geometrical shift compared to the g^{tt} case.

Consequence: $g^{\phi\phi}$ modifications (Chern–Simons, hairy Kerr) give a signal in the R_{sh} channel that is distinguishable at $a = 0.99$ and practically absent at $a = 0.9$.

5.2 Comparison with Existing Literature

5.2.1 Agreement with Kocherlakota & Rezzolla (2022)

Kocherlakota & Rezzolla (2022) showed that at a fixed shadow radius, a degenerate region remains in the metric parameter space. Our result is consistent with this, but refines it: the degeneracy with astrophysics depends on the type of modification.

- For g^{tt} : partial degeneracy—the shadow geometry shifts, and photometry provides independent (though numerically limited) information;
- For g^{rr} : the critical curve is strictly independent of ϵ (Level A), and the photometric response is numerically small; degeneracy is substantial, but not proven absolute;
- For $g^{\phi\phi}$: degeneracy depends on spin—strong at $a = 0.9$, partial at $a = 0.99$.

5.2.2 Complement to Ren et al. (2026) and Tello-Ortiz et al. (2026)

Ren et al. (2026) and Tello-Ortiz et al. (2026) showed that for certain parameters the images completely coincide with Schwarzschild/Kerr ones. We hypothesize that this could be explained if these modifications act predominantly on a component analogous to g^{rr} —i.e., on the radial part of the metric, which does not affect the critical curve. We provide a general framework for interpreting such works, but a direct comparison with the specific metrics from these papers is required for confirmation.

5.2.3 Complement to Qi & Cai (2026) and Övgün et al. (2026)

These works showed that the shadow and QNM give orthogonal constraints, and their combination lifts the degeneracy. Our result is consistent with this idea, but shows that orthogonality also exists within a single image: R_{sh} and I_{peak} give different constraints, but their relative contribution depends on the type of modification. For g^{tt} both channels work (photometry qualitatively, R_{sh} quantitatively); for $g^{\phi\phi}$ only R_{sh} works, and only at $a \approx 0.99$; for g^{rr} nothing works.

5.2.4 Difference from EHT Collaboration Works

The EHT Collaboration (EHT Collaboration, 2019, 2022) fitted parameters of specific modifications (Reissner–Nordström-like, Kerr with deviations) to M87* and Sgr A* data. Our analysis does not fit data, but constructs a sensitivity map over the entire parameter space (a, ϵ, i) and for three different classes of modifications. This is a methodological complement to observational works.

5.3 Limitations of the Method

We explicitly formulate the limitations that should be taken into account when interpreting the results.

5.3.1 Disk Model

Thin disk ($H = 0$). Real accretion disks have relative half-thickness $H/r \approx 0.1$ (for radiatively inefficient flows, RIAF, characteristic of M87* and Sgr A*, H/r can be substantially larger). A thick disk will change the brightness distribution along the photon ring, the contribution of secondary images, and the relative role of Doppler boosting.

Expected influence on our conclusions: the qualitative picture of channel separation will be preserved, but the quantitative values of $dR/d\epsilon$ and σ_ϵ may change by tens of percent. For g^{rr} , the signal will remain close to zero—a thick disk cannot “turn on” the geometric channel blocked by the structure of the metric.

Emission profile $\propto r^{-3}$. We use a power-law profile as a simple approximation to the Novikov–Thorne profile. Real profiles have a more complex structure. This will affect photometry directly, but will not change the signs of $dR/d\epsilon$ and, consequently, will not break the structural conclusion.

5.3.2 Initial Conditions

Standard Kerr in the initial state. In the `initial_state` function we use Bardeen’s formulas for the unperturbed Kerr metric, not for the modified one. This introduces a systematic error of order $\epsilon \cdot h(r_{\text{obs}}) \approx \epsilon(M/r_{\text{obs}})^2 \approx 10^{-7}$ —negligibly small compared to the effects at the horizon.

Limits of the kinematically fixed source at nearly extremal spin. We note that for $a = 0.99$ and $\epsilon = +0.1$, the g^{tt} and $g^{\phi\phi}$ modifications can cause the determinant of the t - ϕ metric block to change sign at radii $r \lesssim 1.4M$, and the unperturbed Kerr four-velocity becomes spacelike relative to the modified metric near $r \approx 1.4M$. While the shadow radius R_{sh} is determined by photon geodesics at larger radii and remains well-defined, photometric observables for configurations where the fixed source four-velocity becomes spacelike are excluded from physical conclusions, as they lack a valid physical interpretation. The innermost disk region formally enters a regime where the chosen fixed source kinematics is incompatible with the modified geometry. For $a \leq 0.9$, the determinant remains negative and the four-velocity timelike over the entire disk. This provides an additional physical argument for restricting the quantitative Fisher analysis to the R_{sh} channel and excluding photometry at nearly extremal spin.

5.3.3 Numerical Aspects

Pixel quantization of R_{sh} . At $n_{\text{pix}} = 80$ the grid step is $\Delta x = 0.625M$. The absolute value of R_{sh} , obtained via the pixel mask, is quantized with this accuracy, giving a systematic error of $\approx 2\%$ for $a = 0$. To eliminate this limitation, the capture-boundary bisection algorithm has been developed. The method provides a boundary search step of $\pm 0.005M$ independent of grid resolution; the full accuracy is limited by residual systematics of $\sim 0.5\%$ (Appendix B). All quantitative results are obtained with this method.

Non-convergence of photometric observables in the regime $a \rightarrow M$. At $a = 0.99$, near the ISCO ($r_{\text{ISCO}} \approx 1.45 M$) there arises a narrow annular zone where rare rays that wind around the BH produce a redshift factor $g \gtrsim 3$ and hence a contribution to the observed intensity $\propto g^4 \gtrsim 80$. The number of such rays hitting a specific pixel depends critically on the grid resolution. In the original version of the code, these rays were additionally excluded (see Appendix A); after the renderer fix, the derivative $dI_{\text{peak}}/d\epsilon$ at $i = 85^\circ$ changes between grids $n_{\text{pix}} = 80, 120, 160$ by more than a factor of 60, while for $i = 80^\circ$ partial agreement is observed between 120 and 160. We interpret this as a diagnostic artifact of the numerical implementation rather than a physical result, because the required source kinematics becomes invalid in this regime. This makes photometric observables unsuitable for quantitative analysis in the regime $a \rightarrow M$ and additionally motivates the choice of R_{sh} as the only stable observable.

Stability of the R_{sh} channel. The shadow radius, in contrast, converges: for g^{tt} at $a = 0.99$ the values of $dR_{\text{sh}}/d\epsilon$ equal $+0.975, +0.986, +0.987$ at $i = 70^\circ, 80^\circ, 85^\circ$ (bisection). Agreement between inclinations is $\sim 1\%$. Therefore, points at $a = 0.99$ are included in the main Fisher analysis in the R_{sh} channel (Table 4).

Final value of λ . We limit the affine parameter to $\lambda \leq 10^4$ and classify rays without events as captured. This is a standard approach, but it gives a systematic error for rays in a narrow band near the critical curve. The shift of the shadow boundary from this effect is smaller than a pixel and does not affect the structural conclusions.

5.3.4 Class of Modifications

One power-law profile $h(r) = (M/r)^2$. We fix the form of the deformation function and vary only ϵ . Other profiles may give different signatures.

One diagonal component at a time. We vary only one component of the inverse metric. In real modified theories, deformations may affect several components simultaneously.

5.3.5 Statistical Model

Uncorrelated Gaussian errors. We assume that the observational accuracies are independent and specified by fixed relative levels. In reality, the errors are correlated, the accuracy depends on the signal-to-noise ratio, and systematic errors of the disk model dominate over statistical ones.

Expected influence: our σ_ϵ should be regarded as an optimistic lower bound. Realistic constraints will be weaker by a factor of 1.5–3.

5.4 Implications for ngEHT and Future Observations

5.4.1 Which Modifications Are Testable

g^{tt} modifications (asymptotically safe gravity, quantum corrections, nonlocal gravity) are *potentially* testable with the current generation of instruments. At realistic accuracy of 5% in R_{sh} , a modification with $\epsilon = 0.1$ produces a signal at the level of $\sim 0.2\text{--}0.4\sigma$ at the optimal point. This is **below the threshold for confident detection** ($\gtrsim 3\sigma$): such values only indicate that, under favourable geometry ($i \gtrsim 80^\circ$, $a \gtrsim 0.9$) and amplitude $\epsilon \gtrsim 0.1$, the effect may be at the edge of detectability, but do not support the claim that the modification would be detected. Confident testing of g^{tt} modifications requires either larger values of ϵ or higher accuracy in measuring R_{sh} .

$g^{\phi\phi}$ modifications (nonminimal coupling, hairy Kerr) require substantially higher accuracy—achievable only for rapidly rotating systems ($a \approx 0.99$), where $\sigma_\epsilon \approx 0.7$. Such accuracy is expected in ngEHT (~ 2030) for M87*.

g^{rr} **modifications** (Bardeen, Hayward, Simpson–Visser) are **not detectable in the R_{sh} channel at any observational accuracy**. To test them, either new observables (the $n = 2$ photon ring, strong-field polarization) or a significant improvement in accuracy are needed.

5.4.2 Optimal Observation Conditions

From our sensitivity map it follows that the optimal strategy depends on the type of modification:

Modification	Optimal spin	Optimal inclination	System type
g^{tt}	$a \gtrsim 0.9$	$i \gtrsim 80^\circ$	fast rotation, edge-on
$g^{\phi\phi}$	$a \gtrsim 0.99$	$i \gtrsim 80^\circ$	nearly extremal rotation, edge-on
g^{rr}	—	—	not detectable

Practical conclusion: for a universal test of metric modifications, one should observe several systems with different inclinations. A single system with one i gives a biased picture.

5.4.3 What Is Needed to Improve Sensitivity

1. **Increased resolution.** Going from $n_{\text{pix}} = 80$ to $n_{\text{pix}} = 120$ improves the separation of the photon ring from the direct image, but does not eliminate the systematic error of pixel quantization of R_{sh} . For precise measurements, the bisection method is preferable, with search step $\pm 0.005 M$ and residual systematics $\sim 0.5\%$, independent of resolution.
2. **A more realistic disk model.** Accounting for thickness, self-absorption, and the Novikov–Thorne profile will change photometry by tens of percent, but not the structural picture.
3. **Additional channels.** Polarization (Stokes Q , U) and the $n = 2$ photon ring are potentially sensitive channels, especially for g^{rr} . Their development is a direction for future work.
4. **Bayesian analysis instead of Fisher.** Fisher gives an estimate at the level of a single modification; a full Bayesian analysis accounting for disk systematics will give more honest constraints.

5.5 Directions for Future Work

5.5.1 Checked but Not Included Channels

In addition to the main channels (shadow, photometry), we conducted preliminary pilots on two additional channels that did not give a stable signal in our model, but deserve mention.

Polarization channels. We implemented the calculation of polarization observables (Stokes parameters Q , U) in three successive versions with increasing physical rigor: (i) a simplified model with a toroidal magnetic field, (ii) parallel transport of the polarization vector along geodesics via Christoffel symbols, (iii) accounting for the three-dimensional structure of the field with radial and poloidal components and synchrotron polarization. In all three versions, the polarization fraction $\langle P \rangle$, the mean polarization angle $\langle \psi \rangle$, and their asymmetry asym_P did not show differences between the three modifications at $|\epsilon| \leq 0.1$. The spread of values was 0.5–1.5%—at the level of numerical noise. We interpret this as an indication that polarization is not a universally sensitive test of metric modifications in a simplified accretion model. A full analysis requires accounting for Faraday rotation, self-absorption, and the full three-dimensional structure of the magnetic field in the disk, which is beyond the scope of this work.

High-order photon rings. We also attempted to isolate the $n = 2$ photon ring (a double traversal of the BH) as a separate observable. Two splitting criteria were used: (i) the number of crossings of the equatorial plane, (ii) the total azimuthal angle $\Delta\phi$ along the trajectory. In both cases, the layers $n \geq 1$ were not separated from the direct image due to the geometric contribution of the outer part of the disk: rays hitting the far side of the disk along a “direct” trajectory can accumulate $\Delta\phi > \pi$, but do not pass through the photon sphere. Correct isolation of the photon ring requires an additional condition—passage of the ray through the region $r < 3M$ near the photon sphere. We leave the development of this criterion for future work.

We consider both negative results valuable: they show that expanding the set of observables does not guarantee an increase in sensitivity and requires a more subtle approach to the choice of channels.

5.5.2 Promising Directions

Thick disks and self-absorption. Replacing the thin disk with a more realistic model (thick disk, self-absorption, turbulent field) is a necessary verification of the conclusions, but requires a substantially more complex pipeline.

Bayesian parameter inference. Replacing the Fisher matrix with a full MCMC analysis accounting for correlations and systematics will give quantitatively more reliable σ_ϵ .

Comparison with EHT data. After developing a realistic disk model, one can apply our formalism to the published data of M87* and Sgr A* to obtain constraints on ϵ for the three classes of modifications.

5.6 Summary of the Discussion

Our analysis establishes a structural distinction between the three classes of modifications of the inverse Kerr metric in thin accretion disk images. This distinction is robust to disk model systematics and confirmed by two independent numerical methods (grid + bisection). There is no universal horizon test: each modification requires its own observational channel, and not all channels are numerically stable.

The result has a dual significance:

- theoretical—it refines which metric components are visible in which observables;
- practical—it gives concrete recommendations for ngEHT: which systems and which channels to use for testing specific classes of modified gravity.

6 Conclusion

We have conducted a systematic study of how three classes of modifications of the inverse Kerr metric manifest in images of a thin accretion disk, and to what extent they can be separated from astrophysical uncertainties. Our main analysis covers 144 points in the parameter space (a, ϵ, i) in total across the three classes of modifications on the $n_{\text{pix}} = 80$ grid; verification was performed on grids $n_{\text{pix}} = 120, 160$ and by the independent capture-boundary bisection method (81 points at $a \in \{0.5, 0.9, 0.99\}$). A key methodological finding: only the shadow radius R_{sh} is a numerically stable observable; photometric observables exhibit systematic non-convergence of their derivatives with respect to the deformation parameter.

6.1 Main Results

1. Structural separation of channels. In the stable R_{sh} channel (bisection), the three modifications act qualitatively differently:

- g^{tt} shifts the shadow boundary: $dR_{\text{sh}}/d\epsilon \approx +0.58$ at $a = 0.9$, $i = 80^\circ$; $\approx +0.99$ at $a = 0.99$;
- g^{rr} *strictly* does not shift the shadow boundary: $dR_{\text{sh}}/d\epsilon = 0$ with accuracy $10^{-4} M$ across the entire grid and all inclinations;
- $g^{\phi\phi}$ produces a signal at $a = 0.9$ that is not resolved by our numerical method ($|dR/d\epsilon| \lesssim 0.02$, below the bisection noise floor); at $a = 0.99$ a moderate signal appears ($dR/d\epsilon \approx +0.36$).

The physical reason for g^{tt} and $g^{\phi\phi}$ lies in the structure of the null geodesic equations: g^{tt} enters the normalization condition and the redshift factor; $g^{\phi\phi}$ enters only the azimuthal equation, with a spin-dependent contribution. For g^{rr} , invariance of the critical curve is proven strictly (Level A); the weakness of the photometric response (Level B) is established numerically and does not follow automatically from this analytical result. The statement that “ g^{rr} modifications are invisible to all observations” (Level C) is not tested in this work.

2. Quantitative assessment of sensitivity. Fisher analysis on the stable R_{sh} channel (bisection, $\sigma_R = 5\%$) yields:

Modification	a	$\sigma_\epsilon^{\min}$	Best point
g^{tt}	0.90	0.431	$i = 85^\circ$
g^{tt}	0.99	0.251	$i = 85^\circ$
$g^{\phi\phi}$	0.90	—	signal absent
$g^{\phi\phi}$	0.99	0.684	$i = 85^\circ$
g^{rr}	any	∞	not detectable

At $a = 0.99$ the sensitivity to g^{tt} is a factor of ~ 2.7 better than to $g^{\phi\phi}$ (i.e., σ_ϵ is smaller); g^{rr} is strictly invisible in the shadow-radius channel at any observational accuracy.

3. Absence of a universal test. There is no single observational channel for testing metric modifications. Each class of modifications requires its own combination of observables, spin, and system inclination:

- for g^{tt} , rapidly rotating systems with large inclination are optimal ($a \gtrsim 0.9$, $i \gtrsim 80^\circ$);
- for $g^{\phi\phi}$, nearly extremal spins ($a \approx 0.99$);
- for g^{rr} , standard disk image observables are fundamentally insufficient.

4. Practical recommendations for ngEHT. The 5% accuracy in R_{sh} adopted in our model gives, for the g^{tt} modification with $\epsilon = 0.1$, a signal at the level of $0.2\text{--}0.4\sigma$ at the optimal point—below the threshold for confident detection. This means that testing g^{tt} modifications with the current generation of instruments is possible *only* under favourable geometry and amplitude ϵ noticeably above 0.1. Testing $g^{\phi\phi}$ modifications is expected in the ngEHT era for rapidly rotating systems. Testing g^{rr} modifications requires new observational channels (polarization, the $n = 2$ photon ring), since the derivative $dR_{\text{sh}}/d\epsilon$ vanishes identically.

5. Limits of applicability of photometric observables. The convergence check at $a = 0.99$ showed that photometric observables in the regime $a \rightarrow M$ are unsuitable for quantitative analysis. The physical reason is rare captured rays with very large redshift ($g^4 \gtrsim 100$) in a narrow annular zone near $r_{\text{ISCO}} \approx 1.45 M$. Their number depends critically on the grid resolution, and in the original version of the code they were additionally excluded from the calculation. After the renderer fix (v8), the derivative $dI_{\text{peak}}/d\epsilon$ at $i = 85^\circ$ changes between grids 80, 120, 160 by more than a factor of 60. The R_{sh} channel, in contrast, is stable and did not change after the fix: $dR_{\text{sh}}/d\epsilon$ for g^{tt} at $a = 0.99$ equals $+0.99$ with $\sim 1\%$ agreement across inclinations. This opens the possibility of a highly sensitive test of g^{tt} modifications in the regime $a > 0.95$ *through the shadow-radius channel*.

6.2 Methodological Achievements

Pipeline validation. The numerical value $R_{\text{sh}}(a = 0) = 5.0855 M$ at $n_{\text{pix}} = 80$ agrees with the analytical $3\sqrt{3} M \approx 5.196 M$ to within 2.1%, consistent with pixel quantization. Application of the bisection method improves agreement to 0.5%; the residual deviation is related to the classification of near-critical rays.

Separation of stable and unstable observables. We showed that R_{sh} (in the bisection variant) is the only numerically stable observable for Fisher analysis; I_{peak} and I_{tot} do not converge under resolution refinement due to competition of several local maxima near the photon ring. This is a methodological limitation important for interpreting all published works on shadow-based metric diagnostics. Note that while the grid-based and bisection methods use the same underlying geodesic equations, they represent two distinct numerical approaches to locating the capture boundary, and they agree in all tested configurations.

Analytical proof (Level A). We showed (Appendix C) that *invariance of the critical curve* with respect to g^{rr} is a strict consequence of the structure of the null geodesic equations, not an artifact of the numerical scheme. Numerically, this is confirmed across all three grids ($n_{\text{pix}} = 80, 120, 160$) and by the bisection method: $dR_{\text{sh}}/d\epsilon_{rr} = 0$ with accuracy $10^{-4} M$ at all checked points.

6.3 Limitations

The main limitations of our work:

- thin disk ($H = 0$) and a simple power-law emission profile;
- one class of power-law modification profile $h(r) = (M/r)^2$;
- uncorrelated Gaussian observational errors with a fixed relative level.

These limitations are discussed in detail in Section 5.3. We expect that accounting for disk thickness, a realistic Novikov–Thorne profile, and correlated systematics will weaken the quantitative constraints on σ_ϵ by a factor of 1.5–3, but will not change the structural conclusion about the separation of channels between modifications.

Directions for future work include: polarization channels with a three-dimensional magnetic field structure; the $n = 2$ photon ring; Bayesian parameter inference instead of Fisher; application of the formalism to published M87* and Sgr A* data to obtain constraints on ϵ .

6.4 Final Remark

Our work is in line with the growing understanding that testing gravity with shadow observations is not a task of searching for a single universal signature, but of systematically classifying modifications by their observable channels. Different metric modifications, corresponding to different physical scenarios (quantum corrections, singularity regularization, nonminimal coupling), require different observational strategies. In addition to the structural result, our work establishes a methodological constraint: not all image observables are numerically stable under grid refinement. The shadow radius, computed by the capture-boundary bisection method, remains the most reliable quantitative observable for testing metric modifications. We hope that the presented framework will prove useful for future analysis of ngEHT data and for constructing more realistic models of modified gravity.

A Stability Check of Observables Against Grid Resolution

A.1 Motivation

Image observables of an accretion disk fall into two classes: geometric (shadow radius R_{sh} , centroid shift) and photometric (peak brightness I_{peak} , integrated brightness I_{tot} , asymmetry A_{asym}). The

former are determined by the position of the capture boundary on the image plane; the latter, by the intensity distribution over the disk. In this appendix we check which of the observables are stable under successive grid refinement ($n_{\text{pix}} = 80, 120, 160$) and, independently, using the capture-boundary bisection method (Appendix B).

A.2 Shadow Radius: Convergence by Grid and Bisection

Method	$\epsilon = -0.1$	$\epsilon = 0$	$\epsilon = +0.1$	$dR/d\epsilon$
Grid, $n_{\text{pix}} = 80$	4.937	4.962	5.012	+0.38
Grid, $n_{\text{pix}} = 160$	5.049	4.987	5.049	0.00
Bisection (App. B)	4.964	5.028	5.079	+0.576

Grid values are quantized with step $\Delta x = 2 \text{ fov}/n_{\text{pix}}$ ($0.625 M$ at $n_{\text{pix}} = 80$, $0.3125 M$ at $n_{\text{pix}} = 160$), giving a systematic error of order $\pm \Delta x/2$. Bisection eliminates this systematic error, giving a boundary search step of $\pm 0.005 M$; the full accuracy is limited by residual systematics of $\sim 0.5\%$ (see “Validation Results” in Appendix B). The pixel-based grid method does not provide a reliable estimate of this derivative: the values 0.38 ($n_{\text{pix}} = 80$) and 0.00 ($n_{\text{pix}} = 160$) are inconsistent both with each other and with the bisection value +0.576. This is a consequence of the systematic error of pixel quantization, which does not cancel in the derivative. All quantitative results in this work are therefore based on the bisection method; an independent convergence check of the bisection itself is discussed in Section “Validation Results” below.

A.3 Peak Brightness: Worse Non-Convergence After the Renderer Fix

The original calculations (code version v7) contained an error: rays classified as captured were excluded from the disk integration before checking for crossings with the emitting plane. The corrected version (v8) accounts for such crossings. Below we present results for $a = 0.9$ (where the effect is small) and $a = 0.99$ (where the effect is critical).

Case $a = 0.9$, $i = 80^\circ$. The values of I_{peak} are identical between v7 and v8—at moderate spins, captured rays do not contribute significantly:

n_{pix}	$I_{\text{peak}}(-0.1)$	$I_{\text{peak}}(0)$	$I_{\text{peak}}(+0.1)$
80	2.749	3.138	3.746
120	2.922	3.342	3.844
160	3.603	3.375	3.457

At $n_{\text{pix}} = 160$ the monotonicity of $I_{\text{peak}}(\epsilon)$ is broken: the value at $\epsilon = +0.1$ drops below the value at $\epsilon = 0$, and the derivative changes sign from +4.6 to -0.7 .

Case $a = 0.99$, $i = 80^\circ$. After the renderer fix, the agreement between grids $n_{\text{pix}} = 120$ and 160 improved—both give $I_{\text{peak}}(+0.1) \approx 24.5$:

code	n_{pix}	$I_{\text{peak}}(-0.1)$	$I_{\text{peak}}(0)$	$I_{\text{peak}}(+0.1)$
v7	80	1.06	1.33	5.70
v7	120	1.18	1.77	24.57
v7	160	1.20	1.95	4.25
v8	80	1.06	1.33	5.70
v8	120	1.18	1.77	24.57
v8	160	1.20	1.95	24.51

However, for other inclinations the convergence is still absent. At $i = 85^\circ$:

n_{pix}	$I_{\text{peak}}(-0.1)$	$I_{\text{peak}}(0)$	$I_{\text{peak}}(+0.1)$
80	1.54	2.37	8.27
120	1.47	2.48	506.30
160	1.50	2.99	252.43

The spread of $I_{\text{peak}}(+0.1)$ between grids is more than a factor of 60. The physical reason is the presence of rare captured rays with very large redshift ($g^4 \gtrsim 100$), whose number depends critically on the grid resolution.

General conclusion. After the renderer fix, photometric observables at $a \rightarrow M$ remain numerically unstable; for specific points ($i = 80^\circ$) partial agreement is observed between the higher-resolution grids, but for arbitrary inclinations the convergence is absent. This makes I_{peak} unsuitable for quantitative Fisher analysis in the regime $a \rightarrow M$. In contrast, R_{sh} did not change after the fix and remains a fully stable observable.

A.4 Integrated Brightness

The quantity $I_{\text{tot}} = \sum_{i,j} I_{ij} \Delta x^2$ (sum of intensities over all pixels, normalized to the pixel area) does not depend on the choice of a specific pixel and is therefore not subject to “maximum jumps”. The check shows that $dI_{\text{tot}}/d\epsilon$ is more stable than $dI_{\text{peak}}/d\epsilon$, but still fluctuates between grids by tens of percent. In particular, for g^{tt} at $a = 0.9$, $i = 80^\circ$ the values are $+5.99$ ($n_{\text{pix}} = 80$), $+1.52$ ($n_{\text{pix}} = 120$), and $+5.89$ ($n_{\text{pix}} = 160$); for $a = 0.99$ at the same conditions— $+18.3$, $+36.3$, $+3.97$, respectively. We conclude that I_{tot} is not a stable observable for Fisher analysis, but may be used to check the sign of the response.

A.5 General Recommendation

Based on the analysis performed, we recommend:

- use $n_{\text{pix}} \geq 80$ for geometric observables at $a \leq 0.95$;
- for all quantitative conclusions, use the bisection method (Appendix B) with search step $\pm 0.005 M$ independent of grid resolution; when interpreting *absolute* values of R_{sh} , account for the residual systematics of $\sim 0.5\%$;
- **do not use** I_{peak} or I_{tot} in quantitative Fisher analysis; photometric observables are admissible only as qualitative indicators of the sign and approximate amplitude of the response;
- when working at $a > 0.95$, note that even R_{sh} obtained by the grid method may contain a systematic error of $\gtrsim 5\%$; for quantitative conclusions in this region, the bisection method is preferable.

B Precise Determination of the Shadow Radius via Bisection

B.1 Motivation

The pixel determination $R_{\text{sh}} = \sqrt{N_{\text{sh}} \Delta A / \pi}$ suffers from a systematic error of the order of the grid step: at $n_{\text{pix}} = 80$, $\Delta x = 0.625M$, which gives an error of $\sim 2\%$ for $a = 0$. This is comparable to the effect being studied for g^{tt} at $\epsilon = 0.1$ ($\sim 1\%$), making the grid method insufficient for quantitative analysis. To eliminate this error, we developed a bisection algorithm for the capture boundary, independent of the grid resolution.

B.2 Algorithm

1. **Specification of directions.** Around the image center, N_θ directions $\theta_k = 2\pi k / N_\theta$, $k = 0, \dots, N_\theta - 1$, are chosen. $N_\theta = 64$ is used for a balance of accuracy and time.
2. **Bisection along each direction.** For each θ_k , bisection is performed over the radius $r \in [0, R_{\text{max}}]$, where $R_{\text{max}} = 25M$. At each step, a ray with coordinates $(\alpha, \beta) = r(\cos \theta_k, \sin \theta_k)$ is traced and classified as captured or escaped. Bisection converges if the central point $(0, 0)$ is captured and the point $(R_{\text{max}} \cos \theta_k, R_{\text{max}} \sin \theta_k)$ escapes. The radial accuracy is specified by the parameter $\text{tol} = 0.005$.

3. **Construction of the contour.** For each direction, a boundary radius r_k^* is obtained, defining a point on the shadow contour: $(x_k, y_k) = r_k^*(\cos \theta_k, \sin \theta_k)$.
4. **Computation of area and centroid.** The polygon area is computed by the shoelace formula with a subsequent correction for the inscribed N_θ -gon:

$$A_{\text{corrected}} = \frac{1}{2} \left| \sum_k (x_k y_{k+1} - x_{k+1} y_k) \right| \cdot \frac{2\pi/N_\theta}{\sin(2\pi/N_\theta)}. \quad (\text{B.1})$$

5. **Effective shadow radius:**

$$R_{\text{sh}}^{\text{bisect}} = \sqrt{A_{\text{corrected}}/\pi}. \quad (\text{B.2})$$

B.3 Integrator Parameters

For near-critical rays (which constitute the majority of the bisection points), more accurate integration than in the main grid is required:

Parameter	Value
rtol	10^{-9}
atol	10^{-11}
max_step	0.5
MAX_LAM	10^5
R_ESCAPE	$100M$

Such settings give a boundary *search step* of $\pm 0.005M$, but increase the computation time of one point to ~ 1 – 2 minutes on 6 cores. The full accuracy is limited by the classification of near-critical rays (see “Validation Results” below) and amounts to $\sim 0.5\%$ of R_{sh} .

B.4 Convergence of the Bisection Method

The choice of numerical parameters in the bisection algorithm (N_θ , **tol**, **rtol**, **atol**, **MAX_LAM**) should not influence the extracted values of R_{sh} or its derivative $dR_{\text{sh}}/d\epsilon$. To verify this, we performed an independent convergence study at the most sensitive point of the grid (g^{tt} , $a = 0.99$, $i = 80^\circ$), varying one parameter at a time while keeping the others fixed.

Table B.1: Convergence of $dR_{\text{sh}}/d\epsilon$ under variation of each numerical parameter at fixed $a = 0.99$, $i = 80^\circ$, modification g^{tt} . The last column shows the spread of the derivative across the tested values.

Parameter	Tested values	Mean $dR/d\epsilon$	Spread
N_θ	32, 64, 128	+0.9858	0.0%
tol	0.02, 0.01, 0.005, 0.002	+0.9830	0.9%
rtol	10^{-7} , 10^{-8} , 10^{-9} , 10^{-10}	+0.9858	0.0%
atol	10^{-9} , 10^{-10} , 10^{-11} , 10^{-12}	+0.9858	0.0%
MAX_LAM	10^4 , 10^5 , 10^6	+0.9858	0.0%

Over the entire tested range, the derivative $dR_{\text{sh}}/d\epsilon$ varies by less than 1%, and the central value $dR_{\text{sh}}/d\epsilon = +0.986$ agrees with the value +0.987 reported in Table 4 at $i = 85^\circ$. The derivative is thus independent of the algorithm’s tunable parameters on the plateau of the method, which justifies the use of the baseline parameter set ($N_\theta = 64$, **tol**=0.005, **rtol**= 10^{-9} , **atol**= 10^{-11} , **MAX_LAM**= 10^5) throughout the main analysis.

B.5 Validation Results

Control test on the Schwarzschild basis ($a = 0$):

Method	$R_{\text{sh}}(a = 0)$	Deviation from $3\sqrt{3}M$
Analytical	5.196 M	—
Pixel mask, $n_{\text{pix}} = 80$	5.086 M	−2.1%
Bisection, $N_{\theta} = 64$	5.168 M	−0.5%

Important clarification on numerical errors. Three distinct error scales must be separated. (i) The *bisection search step* is $\pm 0.005 M$. (ii) The *absolute systematic error* of the method is $\approx 0.5\%$ (as evidenced by the Schwarzschild control test giving 5.168 M versus the analytical 5.196 M). The source of this absolute offset is not fully identified and may be related to the finite integration limit or the classification of near-critical rays. (iii) The *finite-difference error* for the derivative $dR_{\text{sh}}/d\epsilon$. While reducing the bisection step does not eliminate the absolute offset, the derivative is computed by taking differences, where the constant offset largely cancels. However, a fully rigorous proof of this cancellation requires a dedicated convergence study of the derivative itself. Therefore, all Fisher estimates derived below should be regarded as *preliminary conditional sensitivities*, not as strict observational limits.

Consequence for derivatives. The absolute values of R_{sh} contain a systematics of $\sim 0.5\%$. However, when computing the *derivative* $dR_{\text{sh}}/d\epsilon$ over three points $\epsilon = -0.1, 0, +0.1$, most of this systematics cancels (it depends weakly on ϵ at fixed a, i), leaving a spread at the level of $\sim 0.005 M$ across inclinations, consistent with the bisection step. This is why all quantitative Fisher results are based on derivatives rather than on absolute values.

Robustness against the inner absorbing boundary. For the nearly extremal spin $a = 0.99$ and modification g^{tt} , the metric can lose its Lorentzian signature at small radii ($r \lesssim 1.4M$) for $\epsilon = +0.1$. To verify that our shadow-radius extraction is not affected by this pathological region, we performed an explicit test at $i = 80^\circ$: we moved the inner absorbing boundary used in the bisection algorithm from the original Kerr horizon ($r_+ \approx 1.142M$) to $r = 1.409M$ (above the degenerate zone) and finally to $r = 1.500M$. The resulting shadow radius changed from 5.03267 M to 5.03267 M and 5.03289 M , respectively. This variation is below the bisection search step of 0.005 M . This confirms that the critical curve is determined by photon geodesics at much larger radii ($r \sim 2\text{--}3M$) and is effectively independent of the non-Lorentzian region in the tested configuration.

Further improvement. Complete elimination of the systematic error requires analytical determination of the critical curve by solving the equation $R(r) = 0$, $R'(r) = 0$ in the modified metric. This is beyond the scope of the current paper and constitutes a direction for future work.

B.6 Application to the Grid

All main quantitative results of the paper (Table 4) were obtained using bisection values of R_{sh} . Grid values are used only for qualitative conclusions and as an independent stability check.

C Analytical Proof of the Invisibility of g^{rr} in the Shadow Geometry

C.1 Statement and Structure of the Claims

In this appendix we strictly separate three levels of claims about the invisibility of the modification $g^{rr} \rightarrow g^{rr}[1 + \epsilon h(r)]$. Such a separation is necessary because these levels have different status: one is proven analytically, the second is a numerical observation, and the third is not tested in the present work.

Level A (analytical). The *critical curve* (shadow boundary) does not depend on ϵ strictly, provided that the function $f(r) \equiv 1 + \epsilon h(r)$ is positive outside the horizon and does not change the number and position of the roots of the radial potential $R(r)$.

Level B (numerical). The *full image of the thin disk*—coordinate trajectories $r(\phi)$, equatorial-plane crossings, redshift factor—changes weakly but is not required to be invariant. This result is not a strict consequence of Level A: invariance of the set of turning points does not imply invariance of the trajectories themselves.

Level C (not tested). *All possible image observables* (including polarization, high-order photon rings, and arbitrary emission models) are insensitive to ϵ_{rr} . This statement is not tested in the present work and cannot be derived from Levels A and B.

Only Level A is proven below. Level B is discussed numerically in Section 4 and Appendix A.

C.2 Radial Potential

For a null geodesic in the equatorial plane of Kerr, the radial potential has the form:

$$R(r) = [E(r^2 + a^2) - aL_z]^2 - \Delta[(L_z - aE)^2 + Q], \quad (\text{C.1})$$

where $E = -p_t$ is the energy, $L_z = p_\phi$ is the projection of the angular momentum, and Q is the Carter constant. Only the components g_{tt} , $g_{t\phi}$, $g_{\phi\phi}$ and the function $\Delta = r^2 - 2Mr + a^2$ enter the definition of $R(r)$. The component g^{rr} does **not enter** $R(r)$.

C.3 Critical Curve Condition

The critical curve is the set of impact parameters (α, β) for which the radial motion has a double root:

$$R(r_c) = 0, \quad \left. \frac{dR}{dr} \right|_{r_c} = 0, \quad (\text{C.2})$$

where r_c is the photon-sphere radius. Conditions (C.2) are determined exclusively through $R(r)$, and hence only through E , L_z , Q , M , a . They do not depend on ϵ (the g^{rr} modification).

C.4 How g^{rr} Enters the Equations of Motion

We work in the Hamiltonian formalism with $H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = 0$. The modification affects only the component $g^{rr} \rightarrow f(r)g_K^{rr}$, where $f(r) = 1 + \epsilon h(r) > 0$ outside the horizon, and the subscript K refers to the unperturbed Kerr metric. The remaining components of the inverse metric, as well as the conserved quantities $E = -p_t$ and $L_z = p_\phi$, are unchanged.

The null condition takes the form

$$0 = f(r)g_K^{rr}p_r^2 + V(r, \theta, p_t, p_\phi, p_\theta), \quad (\text{C.3})$$

where V denotes the sum of all remaining terms of the Hamiltonian that do not contain g^{rr} . Contrary to the naive assumption that the modification affects only the kinematic link between r and the affine parameter, p_r is explicitly dependent on $f(r)$ through (C.3):

$$p_r^2 = -\frac{V}{f(r)g_K^{rr}}. \quad (\text{C.4})$$

C.5 Hamilton–Jacobi Separation for Non-Equatorial Rays

Before turning to the equation for $r(\phi)$, it is useful to exhibit the separation structure of the null condition explicitly, since the argument will be applied to non-equatorial rays as well.

Multiplying the null condition $2H = g^{\mu\nu}p_\mu p_\nu = 0$ by Σ and using the standard form of the inverse Kerr metric, one obtains the following *manifestly separated* form:

$$0 = f(r)\Delta p_r^2 - \frac{[E(r^2 + a^2) - aL_z]^2}{\Delta} + p_\theta^2 + \left(\frac{L_z}{\sin \theta} - aE \sin \theta \right)^2, \quad (\text{C.5})$$

where the modification $g^{rr} \rightarrow f(r) g_K^{rr}$ enters *only* through the factor $f(r)$ in front of the radial kinetic term Δp_r^2 . The angular terms $p_\theta^2 + (L_z/\sin\theta - aE\sin\theta)^2$ are unchanged; so are the conserved quantities $E = -p_t$ and $L_z = p_\phi$.

Equation (C.5) is the standard Carter form of the geodesic equations: the r -dependent and θ -dependent parts separate upon introducing the separation constant K ,

$$f(r) \Delta p_r^2 = \frac{[E(r^2 + a^2) - aL_z]^2}{\Delta} - K, \quad (\text{C.6})$$

$$p_\theta^2 = K - \left(\frac{L_z}{\sin\theta} - aE\sin\theta \right)^2. \quad (\text{C.7})$$

At a turning point of the radial motion ($p_r = 0$), Eq. (C.6) gives $K = [E(r^2 + a^2) - aL_z]^2/\Delta$, where we have introduced the constant $K \equiv Q + (L_z - aE)^2$ (note that in Section 3, Q denotes the standard Carter constant, while here the relevant separation constant is K). The critical curve for *non-equatorial* rays is determined by the usual condition

$$R(r) \equiv [E(r^2 + a^2) - aL_z]^2 - K \Delta = 0, \quad R'(r) = 0, \quad (\text{C.8})$$

in which $f(r)$ does not appear at all.

Two conditions on $f(r)$ are required for this result to be strict:

1. **Positivity outside the horizon.** $f(r) > 0$ for all $r > r_+$, so that the sign of the radial kinetic term is preserved and no new turning point is introduced in the outer region.
2. **No new roots of $R(r)$ in the outer region.** The modification must not shift the horizon in such a way that the outer turning point of $R(r)$ disappears (i.e., no new horizon forms outside the original r_+).

Under these two conditions, the modification $g^{rr} \rightarrow f(r) g_K^{rr}$ drops out of the critical-curve condition in the non-equatorial case as well: the position of the critical curve is unchanged, and only the affine parameterization of the trajectory (the function $r(\lambda)$) is rescaled. This is the general form of the statement that will be verified numerically below.

C.6 Modified Equation for $r(\phi)$

The radial velocity is, by definition,

$$\frac{dr}{d\lambda} = \frac{\partial H}{\partial p_r} = g^{rr} p_r = f(r) g_K^{rr} p_r. \quad (\text{C.9})$$

Squaring and substituting (C.4) gives

$$\left(\frac{dr}{d\lambda} \right)^2 = f^2 (g_K^{rr})^2 p_r^2 = f^2 (g_K^{rr})^2 \cdot \left(-\frac{V}{f g_K^{rr}} \right) = -f(r) g_K^{rr} V. \quad (\text{C.10})$$

The azimuthal velocity $\dot{\phi} = g^{t\phi} p_t + g^{\phi\phi} L_z = -E g^{t\phi} + g^{\phi\phi} L_z$ does not depend on g^{rr} . Taking the ratio,

$$\left(\frac{dr}{d\phi} \right)^2 = \frac{-f(r) g_K^{rr} V}{\dot{\phi}^2} = f(r) \left(\frac{dr}{d\phi} \right)_K^2. \quad (\text{C.11})$$

This is the correct form of the modified equation of motion: the factor $f(r)$ appears in the *first power*, not as f^{-2} . The error would arise if one failed to account for the fact that not only g^{rr} changes, but also p_r , through the null condition (C.4).

C.7 What Eq. (C.11) Actually Proves

Is proven. Since $f(r) > 0$ outside the horizon, the right-hand side of (C.11) vanishes if and only if $R(r)$ vanishes. The set of turning points of radial motion $\{r : R(r) = 0\}$ is unchanged. Consequently, the position of the critical curve is invariant:

$$\frac{\partial R_{\text{sh}}}{\partial \epsilon_{rr}} = 0 \quad (\text{strictly, for } f(r) > 0 \text{ outside the horizon}). \quad (\text{C.12})$$

This is Level A. Numerically, we confirmed this identity with accuracy $10^{-4} M$ at all 81 bisection-grid points ($a \in \{0.5, 0.9, 0.99\}$, three inclinations, three values of ϵ).

Is not proven. The function $r(\phi)$ itself changes: the integral $\int dr/(dr/d\phi)$ is taken over a different integrand, differing by a factor $f(r)^{-1/2}$. Invariance of the *roots* is not equivalent to invariance of the *trajectory*.

Consequence for observables. The equatorial-plane crossing points of the trajectory, the number of such crossings, and the emission coordinate r_c at which the redshift factor $g = 1/[u^t(1 - \Omega L_z)]$ is evaluated may depend on ϵ_{rr} . This is Level B, which requires numerical verification.

C.8 Numerical Verification of Level B

The numerical result obtained in Section 4 and Appendix A shows that for $|\epsilon| \leq 0.1$, $a = 0.9$, $i = 80^\circ$, the peak-brightness response is $dI/d\epsilon_{rr} \approx 0.03$ on grid $n_{\text{pix}} = 80$ —two orders of magnitude weaker than for the g^{tt} modification. However, the derivative $dI_{\text{peak}}/d\epsilon$ does not converge under grid refinement (Appendix A), so the numerical value ≈ 0.03 should be regarded as an upper estimate of the order of the effect, not as a precise number. In the R_{sh} channel, the effect vanishes strictly (Level A).

Analytical estimate of the order of the effect. The factor $f(r)^{-1/2}$ in the equation for $dr/d\phi$ leads to a relative change of the trajectory of order $\epsilon h(r_c)/2$. In the emission region $r_c \sim 5\text{--}10 M$, this gives $\epsilon h(r_c)/2 \sim 5 \cdot 10^{-4}\text{--}5 \cdot 10^{-5}$. This is consistent with the numerical response $dI/d\epsilon_{rr} \sim 10^{-2}$ after amplification through $I_{\text{obs}} \propto g^4$. A full analytical proof of the invariance (or non-invariance) of the full image would require analyzing the integral $\int dr/(dr/d\phi)$ with a modified integrand, which is beyond the scope of this work.

C.9 Generalizations and Caveats

1. Arbitrary profile $h(r)$. Level A does not use the specific form $h(r) = (M/r)^2$. It holds for any profile for which $f(r) = 1 + \epsilon h(r) > 0$ outside the horizon and which does not change the position of the roots of $R(r)$. Applying this to full solutions of specific theories (Bardeen, Hayward, Simpson–Visser) requires a separate analysis, since the field equations in these models also modify other metric components.

2. Non-equatorial rays. For $\theta \neq \pi/2$, the critical curve condition includes the term $g^{\theta\theta} p_\theta^2$, but the structure of $R(r)$ is unchanged, and g^{rr} still does not enter $R(r)$. Level A holds.

3. Mixed components. In the Kerr metric, $g^{r\theta} = g^{r\phi} = 0$. If a modification introduces nonzero mixed components, the proof requires generalization, and the invariance of the critical curve may be broken.

4. On the role of horizon position. For the isolated deformation $g^{rr} \rightarrow f(r)g_K^{rr}$ with $f(r) > 0$, the horizon position (the root of $g^{rr} = 0$) remains unchanged. The critical-curve condition (C.2) depends only on the roots of $R(r)$, which do not contain g^{rr} . Therefore, the critical curve is strictly invariant. A more general case involving an arbitrary shift of the horizon would require a separate analysis.

5. What does not follow from Level A. The numerical value $dR_{\text{sh}}/d\epsilon_{rr} = 0$ does not imply that *all* image observables are independent of ϵ_{rr} . Photometric observables may change through modification of the trajectory $r(\phi)$ (Level B). Moreover, the claim of g^{rr} invisibility in polarization and high-order photon rings (Level C) is not tested in this work.

C.10 Consequences

1. **Strict result (Level A).** The critical curve is strictly independent of g^{rr} , under the condition $f(r) > 0$ outside the horizon. This is a consequence of the structure of the null geodesic equations, not an artifact of the numerical scheme. It extends to any isolated deformation of the form $g^{rr} \rightarrow g^{rr}[1 + \epsilon h(r)]$ with the required properties. Extending this result to full solutions of specific theories (Bardeen, Hayward, Simpson–Visser, polymer LQG) would require a separate analysis, since in these models other metric components are also modified by the field equations.
2. **Numerical result (Level B).** The full image of the thin disk shows only weak sensitivity to ϵ_{rr} in the studied parameter range. This is an empirical observation, not a theorem. The numerical value of the response depends on grid resolution (Appendix A), which further limits the accuracy of the conclusion.
3. **Limits of applicability.** The statement that "any g^{rr} -type modifications are fundamentally invisible to any observations" is not proven and not tested in this work. Level C requires a separate study: analysis of polarization channels, high-order photon rings, and emission models other than those used.
4. **Channel separation.** The analytical proof of Level A explains why g^{rr} does not affect the shadow geometry. The absence of a photometric response is an additional numerical observation requiring more refined analysis for full explanation.
5. **Direction for future work.** A similar proof of Level A can be carried out for $g^{\theta\theta}$ modifications. Analysis of Level B for arbitrary emission models and Level C for polarization and photon rings constitutes a separate task.

Data Availability

The article itself is available in the Zenodo repository with DOI: 10.5281/zenodo.23436001 (<https://zenodo.org/records/23436001>).

The source code of the computational pipeline, post-processing scripts, and figure-generation tools, together with the generated datasets, are published in the Zenodo repository (Lipyanchik, 2026) with DOI: 10.5281/zenodo.23436302 (<https://zenodo.org/records/23436302>). The repository includes:

- the main computation script (`degeneracy_pilot_v8.py`) for ray-tracing calculations of black-hole shadow and disk images;
- the implementation of the capture-boundary bisection algorithm (`shadow_bisection_v4.py`);
- the convergence analysis and Fisher matrix scripts;
- the full set of generated disk images and extracted observables;
- a README file with parameter listings and step-by-step reproduction instructions.

All results presented in this work can be reproduced using these materials. The calculations were performed using Python 3.11 with the standard libraries `numpy`, `scipy`, and `matplotlib`; the complete list of dependencies is provided in the repository.

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