

Systems of equations with a common solution and hidden roots: a composition operation

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20 October 2026

Abstract

We study *systems of equations* $\frac{N_i(t)}{D(t)} = k_i$ ($i = 1, \dots, n$) that have a common solution t_0 , possibly together with equations $F_i(x) = k_i$ in a second unknown x with the same right-hand sides k_i . Adding or subtracting these equations produces new systems whose solutions other than t_0 (the *hidden solutions*) are computed by the rule $\frac{a}{b} \ominus \frac{c}{d} = \frac{a+c}{b+d}$ (the Farey mediant), provided the orientation $-A/B \neq A/(-B)$ is respected. A second law, denoted $\diamond$, says how to multiply an equation by a coefficient: $\lambda \diamond \frac{a}{b} = \frac{\lambda a}{\lambda b}$, which is not the usual product of a fraction by a number. We explain these rules by a factorisation theorem, determine the signs to choose in order to obtain a prescribed hidden solution, and give an inverse construction, relations between equations, a barycentric formula in degree m , the complex case in degree 2, covariance under homographies, the correspondence between the unknowns x and t , two-parameter systems (relations between hidden values in t and in α , elliptic curves of rank ≥ 1), and a study of the rationality of all solutions (conics, elliptic curves). A rank criterion proved with Mazur's theorem gives infinitely many fully rational combinations for certain pencils of cubics. Every statement is proved and illustrated by explicit systems. The core is elementary (Vieta's relations and linearity); the contribution is the reading of these systems and of their signed combinations.

Note on the method. The computations were verified with SymPy and PARI/GP. The writing was assisted by Claude (Anthropic). Any remaining errors are the author's.

1 Introduction and results

Consider the system $\frac{2t+1}{t^2-3} = 5$ (A) and $\frac{t-4}{t^2-3} = -2$ (B). They have the common solution $t = 2$. Their other solutions, which we call *hidden roots*, are $-8/5$ and $-5/2$. The equation (A)–(B), that is $\frac{t+5}{t^2-3} = 7$, has solutions 2 and $-13/7$, and one observes

$$\frac{-8}{5} \ominus \frac{-5}{2} = \frac{-8-5}{5+2} = \frac{-13}{7}.$$

This article studies this regularity for arbitrary systems. Here is what it contributes.

1. **Systems of equations.** A system is a set of equations $N_i(t)/D(t) = k_i$ with a common solution t_0 (possibly together with equations $F_i(x) = k_i$ in a second unknown, with the

same k_i). Adding or subtracting these equations with arbitrary signs produces new systems whose hidden root is given explicitly by the law Θ (Sections 2 and 4.2). We present seven complete systems, a system of three equations, a system of four equations all of whose eight combinations are computed, and a system in two unknowns.

2. **Orientation:** $-A/B \neq A/(-B)$. The fractions $\frac{-A}{B}$ and $\frac{A}{-B}$ have the same value, but they are *opposite* fractions: added to a third one, they give different results. For example $\frac{-8}{5} \Theta \frac{-5}{2} = \frac{-13}{7}$ whereas $\frac{-8}{5} \Theta \frac{5}{-2} = \frac{-3}{3}$, of value -1 . This distinction is not a matter of notation: it corresponds to the difference between an equation and its opposite (Section 3). Theorem 16 gives the placement of the signs that realises a prescribed hidden root, without trial and error.
3. **The dictionary between the law and the equations.** A factorisation theorem explains why the hidden root of a combination is the Θ -composition of the hidden roots; it is followed by an inverse construction (choosing the coefficients to obtain a given hidden root), a relation between three equations, the second law $\Diamond$ (multiplication of an equation by a coefficient), a barycentric formula in degree m , covariance under homographies, the Stern–Brocot tree of the equations of a system, and the correspondence between the two unknowns x and t of a system of n equations. Section 13 treats two-parameter systems (t, α) : a homography between the hidden values in t and in α for two equations, a linear relation for three, none for four, and elliptic curves of rank ≥ 1 .
4. **Rationality.** When a combination has all its solutions rational although its ingredients do not: a conic in degree 2, elliptic curves in degree 3, and a rank criterion proved with Mazur’s theorem (Section 14.1).

What is classical. The Farey mediant, its reading as a weighted mean with the denominators as weights, Möbius’s barycentric calculus (1827), the Stern–Brocot tree, pencils of polynomials and Vieta’s relations. We do not claim these ingredients.

What we did not find in the literature consulted. To our knowledge, the reading of the mediant as the hidden solution of a system $N(t)/D(t) = k$, the distinction between the opposite fractions $-A/B$ and $A/(-B)$ in this context, and the dictionary of point 3 do not appear in the texts we consulted. This search was limited; it does not prove the absence of earlier results, and an exhaustive search (zbMATH, MathSciNet) remains to be done.

2 Systems of equations with a common solution

Definition 1. A *system* consists of a polynomial D of degree 2 (or more), a real number t_0 and n equations

$$(E_i) \quad \frac{N_i(t)}{D(t)} = k_i \quad (i = 1, \dots, n)$$

of which t_0 is a common solution, that is $N_i(t_0) = k_i D(t_0)$. The other solutions of (E_i) are its *hidden solutions*. A system in two unknowns adds equations $F_i(x) = k_i$ with the same k_i and a common solution x_0 , the unknown (x, t) having the solution (x_0, t_0) .

Signed combinations. For signs $s_i = \pm 1$, the equation $\sum_i s_i N_i/D = \sum_i s_i k_i$ still has t_0 as a solution. The 2^{n-1} equations obtained, up to a global sign, form a new system. We denote them $(E_1) \pm (E_2) \pm \dots$. The problem is to determine the hidden solutions of each combination from those of the (E_i) .

Result. If $\frac{n_i}{a_i}$ is the hidden solution of (E_i) , written in unreduced oriented form (Section 3), the hidden solution of $\sum \lambda_i E_i$ is $\sum \frac{\lambda_i n_i}{\lambda_i a_i}$, that is the Θ -composition of the fractions $\lambda_i \diamond \frac{n_i}{a_i}$ (Theorem 6). For a negative coefficient, the sign $-$ applies to the numerator *and* to the denominator; this is what makes the orientation necessary.

A first system. The system $(E_1): \frac{2t+1}{t^2-3} = 5$ and $(E_2): \frac{t-4}{t^2-3} = -2$ has the common solution $t = 2$. (E_1) has the other solution $-\frac{8}{5}$, (E_2) has $-\frac{5}{2}$, and $(E_1) - (E_2)$ (that is $\frac{t+5}{t^2-3} = 7$) has $-\frac{13}{7}$. One observes

$$\frac{-8}{5} \Theta \frac{-5}{2} = \frac{-8-5}{5+2} = \frac{-13}{7}.$$

Section 4.2 gives seven systems of this type, and Section 7.2 a system of four equations whose 8 combinations are computed.

3 Framework: oriented fractions

Definition 2. An *oriented fraction* is a fraction $\frac{n}{a}$ (n, a integers) written as it comes out of the equations: it is never simplified and its sign stays where it is written (in the numerator or in the denominator). Its *value* is $\rho(\frac{n}{a}) = n/a \in \mathbb{Q} \cup \{\infty\}$, defined as soon as $\frac{n}{a} \neq \frac{0}{0}$. We define two laws, for integers λ :

$$\frac{n_1}{a_1} \Theta \frac{n_2}{a_2} = \frac{n_1 + n_2}{a_1 + a_2}, \quad \lambda \diamond \frac{n}{a} = \frac{\lambda n}{\lambda a}.$$

The law Θ is always defined; it is its values that are not always defined: $\frac{0}{0}$ has no value, and it appears when the fractions cancel (for example in the trivial combination of Section 7.2). Θ is associative and commutative, with neutral element $\frac{0}{0}$, and each fraction $\frac{n}{a}$ has an opposite $(-1) \diamond \frac{n}{a} = \frac{-n}{-a}$, of the same value. All the interest comes from the value ρ , which forgets information. Three facts follow.

Proposition 3 (orientation). $\frac{-A}{B}$ and $\frac{A}{-B}$ are opposite to each other: they have the same value but are different fractions. Thus $-A/B \neq A/(-B)$ as oriented fractions.

Example: $\frac{-8}{5} \Theta \frac{-5}{2} = \frac{-13}{7}$ of value $-13/7$, whereas $\frac{-8}{5} \Theta \frac{5}{-2} = \frac{-3}{3}$ of value -1 .

Proposition 4 (non-reduction). Θ is not defined on the rational numbers. The fractions $\frac{1}{2}$ and $\frac{2}{4}$ have the same value, but $\frac{1}{2} \Theta \frac{1}{3} = \frac{2}{5}$ (value $2/5$) and $\frac{2}{4} \Theta \frac{1}{3} = \frac{3}{7}$ (value $3/7$). An unreduced fraction is therefore a datum, not a defect of notation.

Proposition 5 (the second law $\diamond$). For integers λ, μ :

- (i) if $\lambda \neq 0$, $\lambda \diamond \frac{n}{a}$ has the same value as $\frac{n}{a}$;
- (ii) $\lambda \diamond (x \Theta y) = \lambda \diamond x \Theta \lambda \diamond y$, $(\lambda + \mu) \diamond x = \lambda \diamond x \Theta \mu \diamond x$, $\lambda \diamond (\mu \diamond x) = (\lambda \mu) \diamond x$;
- (iii) $(-\lambda) \diamond x$ is the opposite of $\lambda \diamond x$: for example $(-5) \diamond \frac{-11}{7} = \frac{55}{-35}$, the sign $-$ “moves to the denominator”.

The law $\diamond$ is not the usual product of a fraction by a number: $4 \times \frac{-4}{3} = \frac{-16}{3}$, whereas $4 \diamond \frac{-4}{3} = \frac{-16}{12}$.

Proof. Immediate: numerators and denominators are added or multiplied term by term. □

Reading through the equations. The sign rule is the difference between an equation and its solution: the fractions $-A/B$ and $A/(-B)$ have the same solution, but they are the equations $Bx = -A$ and $-Bx = A$, opposite to each other; added to a third equation, they give different solutions. Likewise, multiplying an equation by a coefficient λ multiplies its whole polynomial, hence the fraction that writes its hidden solution *top and bottom*.

4 Origin of Θ in systems: factorisation

Let $D = dt^2 + et + f$ with $d \neq 0$, a real number $t_0 \neq 0$, and two equations $N_i(t)/D(t) = k_i$ ($i = 1, 2$) with common root t_0 . We write $N_i = r_i t^2 + p_i t + q_i$.

Theorem 6 (factorisation). *With $P_i = k_i D - N_i$:*

$$P_i = (t - t_0)(a_i t - n_i), \quad a_i = k_i d - r_i, \quad n_i = \frac{k_i f - q_i}{t_0}.$$

For all real λ, μ , the polynomial $\lambda P_1 + \mu P_2$ is that of the equation $(\lambda N_1 + \mu N_2)/D = \lambda k_1 + \mu k_2$, and its root other than t_0 , the hidden root, is $\rho(\lambda \diamond \frac{n_1}{a_1} \Theta \mu \diamond \frac{n_2}{a_2})$, with $\lambda \diamond \frac{n}{a} = \frac{\lambda n}{\lambda a}$. If $\lambda a_1 + \mu a_2 = 0$, the equation obtained has degree 1 and the hidden root is ∞ .

Proof. $P_i(t_0) = k_i D(t_0) - N_i(t_0) = 0$, so $t - t_0$ divides P_i . The leading coefficient of P_i is $k_i d - r_i$ and its constant term is $k_i f - q_i = t_0 n_i$. The rest is the linearity of $P \mapsto P/(t - t_0)$. $\square$

Corollary 7. *The equation $(A) + (B)$ corresponds to $\frac{n_A}{a_A} \Theta \frac{n_B}{a_B}$ and $(A) - (B)$ to $\frac{n_A}{a_A} \Theta (-\frac{n_B}{a_B})$. Thus $\frac{a+c}{b+d}$ and $\frac{a-c}{b-d}$ are the same operation, written with one sign or the other for the second fraction. In Example 1, the hidden root of (B) is written $\frac{5}{-2}$ or, oppositely, $\frac{-5}{2}$ (same evaluation $-5/2$): subtraction appears as addition of the opposite.*

Remark 8 (reading through Vieta's relations). Let $D = dt^2 + et + f$ ($d \neq 0$) and $N = pt + q$ with $N(t_0) = kD(t_0)$. The equation $N/D = k$ is written $kD - N = dk t^2 + (ke - p)t + (kf - q) = 0$; the sum of its two roots is $(p - ke)/(dk)$, and one of them is t_0 . The hidden root is therefore

$$\rho = \frac{p - ke}{dk} - t_0, \quad \text{that is} \quad \frac{n}{a} = \frac{p - ke - dk t_0}{dk}.$$

An equation is thus determined by the two numbers p and k (for fixed D and t_0): the sum of two equations adds the p 's and the k 's, and the opposite of an equation corresponds to $(-p, -k)$. Changing the sign of *one* of these numbers only gives $(p, -k)$, which is another equation, of opposite value p/k : this is the difference between $\frac{-A}{B}$ and $\frac{A}{-B}$. Example (system 1, $D = t^2 - 3$, $t_0 = 2$):

equation	(p, k)	$p/k - t_0$	hidden root
(A)	(2, 5)	$2/5 - 2$	$-8/5$
(B)	(1, -2)	$-1/2 - 2$	$-5/2$
(A) - (B)	(1, 7)	$1/7 - 2$	$-13/7$
(A) + (B)	(3, 3)	$1 - 2$	-1

4.1 The second law in action

For the combination $\sum \lambda_i(E_i)$, the hidden solution is the value of $\lambda_1 \diamond \frac{n_1}{a_1} \Theta \lambda_2 \diamond \frac{n_2}{a_2} \Theta \dots$ (Theorem 6). Three examples.

Example 1 ($D = t^2 - 3$, $t_0 = 2$). (A): $\frac{2t-1}{t^2-3} = 3$ has hidden solution $-\frac{4}{3}$, since $3(t^2-3) - (2t-1) = 3t^2 - 2t - 8 = (t-2)(3t+4)$, written $\frac{-4}{3}$. (B): $\frac{3t+1}{t^2-3} = 7$ has hidden solution $-\frac{11}{7}$, since $7t^2 - 3t - 22 = (t-2)(7t+11)$, written $\frac{-11}{7}$. Then $4(A) + 5(B)$ is $\frac{23t+1}{t^2-3} = 47$, and

$$4 \diamond \frac{-4}{3} \Theta 5 \diamond \frac{-11}{7} = \frac{4(-4)}{4(3)} \Theta \frac{5(-11)}{5(7)} = \frac{-16}{12} \Theta \frac{-55}{35} = \frac{-71}{47},$$

which is indeed the hidden solution of $47t^2 - 23t - 142 = (t - 2)(47t + 71)$. With the usual product one would have found $\frac{-16}{3} \ominus \frac{-55}{7} = \frac{-71}{10}$: *this is wrong*. The coefficient must multiply the numerator *and* the denominator.

Example 2 (same (A) and (B): the sign moves to the denominator). $4(A) - 5(B)$ is $\frac{-7t-9}{t^2-3} = -23$, and

$$4 \diamond \frac{-4}{3} \ominus (-5) \diamond \frac{-11}{7} = \frac{-16}{12} \ominus \frac{55}{-35} = \frac{39}{-23},$$

of value $-\frac{39}{23}$: indeed $-23t^2 + 7t + 78 = -(t - 2)(23t + 39)$.

Example 3 ($D = t^2 - 7$, $t_0 = 3$). (A): $\frac{4t+6}{t^2-7} = 9$ has fraction $\frac{-23}{9}$ ($9t^2 - 4t - 69 = (t - 3)(9t + 23)$). (B): $\frac{-5t+1}{t^2-7} = -7$ has fraction $\frac{16}{-7}$ ($-7t^2 + 5t + 48 = (t - 3)(-7t - 16)$), of value $-\frac{16}{7}$. Then

$$\begin{aligned} 3(A) + 4(B) : \frac{-8t + 22}{t^2 - 7} &= -1, & 3 \diamond \frac{-23}{9} \ominus 4 \diamond \frac{16}{-7} &= \frac{-69}{27} \ominus \frac{64}{-28} = \frac{-5}{-1} = 5, \\ 3(A) - 4(B) : \frac{32t + 14}{t^2 - 7} &= 55, & 3 \diamond \frac{-23}{9} \ominus (-4) \diamond \frac{16}{-7} &= \frac{-69}{27} \ominus \frac{-64}{28} = \frac{-133}{55}. \end{aligned}$$

For the first, $-(t^2 - 7) + 8t - 22 = -(t - 3)(t - 5)$, with hidden solution 5; for the second, $55t^2 - 32t - 399 = (t - 3)(55t + 133)$, with hidden solution $-\frac{133}{55}$.

Three equations. With (A) and (B) of Example 3 and (C): $\frac{3t-1}{t^2-7} = 4$, of fraction $\frac{-9}{4}$ ($4t^2 - 3t - 27 = (t - 3)(4t + 9)$):

$$(A) + (B) + (C) : \frac{2t + 6}{t^2 - 7} = 6, \quad \frac{-23}{9} \ominus \frac{16}{-7} \ominus \frac{-9}{4} = \frac{-16}{6} = -\frac{8}{3};$$

$$3(A) + 2(B) + 4(C) : \frac{14t + 16}{t^2 - 7} = 29, \quad 3 \diamond \frac{-23}{9} \ominus 2 \diamond \frac{16}{-7} \ominus 4 \diamond \frac{-9}{4} = \frac{-69 + 32 - 36}{27 - 14 + 16} = \frac{-73}{29}.$$

The factorisations $6t^2 - 2t - 48 = 2(t - 3)(3t + 8)$ and $29t^2 - 14t - 219 = (t - 3)(29t + 73)$ confirm $-\frac{8}{3}$ and $-\frac{73}{29}$.

4.2 Seven systems

Each system is given by its equations (A) and (B) with common solution t_0 , then by the two combinations (A) - (B) and (A) + (B), which are new equations of the same type. The hidden solutions are computed by solving each equation.

System 1 ($t_0 = 2$).

- (A) : $\frac{2t + 1}{t^2 - 3} = 5$; solutions $t = 2$ and $t = -\frac{8}{5}$.
- (B) : $\frac{t - 4}{t^2 - 3} = -2$; solutions $t = 2$ and $t = -\frac{5}{2}$.
- (A) - (B) : $\frac{t + 5}{t^2 - 3} = 7$; solutions $t = 2$ and $t = -\frac{13}{7}$.
- (A) + (B) : $\frac{3t - 3}{t^2 - 3} = 3$; solutions $t = 2$ and $t = -1$.

System 2 ($t_0 = 2$).

- (A) : $\frac{7t - 2}{t^2 - 2} = 6$; solutions $t = 2$ and $t = -\frac{5}{6}$.

- (B) : $\frac{3t+8}{t^2-2} = 7$; solutions $t = 2$ and $t = -\frac{11}{7}$.
- (A)-(B) : $\frac{4t-10}{t^2-2} = -1$; solutions $t = 2$ and $t = -6$.
- (A)+(B) : $\frac{10t+6}{t^2-2} = 13$; solutions $t = 2$ and $t = -\frac{16}{13}$.

System 3 ($t_0 = 2$).

- (A) : $\frac{7t-4}{t^2-2t+5} = 2$; solutions $t = 2$ and $t = \frac{7}{2}$.
- (B) : $\frac{9t+7}{t^2-2t+5} = 5$; solutions $t = 2$ and $t = \frac{9}{5}$.
- (A)-(B) : $\frac{-2t-11}{t^2-2t+5} = -3$; solutions $t = 2$ and $t = \frac{2}{3}$.
- (A)+(B) : $\frac{16t+3}{t^2-2t+5} = 7$; solutions $t = 2$ and $t = \frac{16}{7}$.

System 4 ($t_0 = 2$).

- (A) : $\frac{t^2+5t+6}{t^2+1} = 4$; solutions $t = 2$ and $t = -\frac{1}{3}$.
- (B) : $\frac{3t^2-t+5}{t^2+1} = 3$; unique solution $t = 2$ (the hidden solution is at infinity).
- (A)-(B) : $\frac{-2t^2+6t+1}{t^2+1} = 1$; solutions $t = 2$ and $t = 0$.
- (A)+(B) : $\frac{4t^2+4t+11}{t^2+1} = 7$; solutions $t = 2$ and $t = -\frac{2}{3}$.

System 5 ($t_0 = 5$).

- (A) : $\frac{3t+1}{t^2-17} = 2$; solutions $t = 5$ and $t = -\frac{7}{2}$.
- (B) : $\frac{2t+14}{t^2-17} = 3$; solutions $t = 5$ and $t = -\frac{13}{3}$.
- (A)-(B) : $\frac{t-13}{t^2-17} = -1$; solutions $t = 5$ and $t = -6$.
- (A)+(B) : $\frac{5t+15}{t^2-17} = 5$; solutions $t = 5$ and $t = -4$.

System 6 ($t_0 = 6$).

- (A) : $\frac{4t-4}{t^2-31} = 4$; solutions $t = 6$ and $t = -5$.
- (B) : $\frac{7t+3}{t^2-31} = 9$; solutions $t = 6$ and $t = -\frac{47}{9}$.
- (A)-(B) : $\frac{-3t-7}{t^2-31} = -5$; solutions $t = 6$ and $t = -\frac{27}{5}$.

- (A)+(B) : $\frac{11t-1}{t^2-31} = 13$; solutions $t = 6$ and $t = -\frac{67}{13}$.

System 7 ($t_0 = \frac{1}{2}$).

- (A) : $\frac{4t-1}{t^2-\frac{1}{2}} = -4$; solutions $t = \frac{1}{2}$ and $t = -\frac{3}{2}$.
- (B) : $\frac{6t+5}{t^2-\frac{1}{2}} = -32$; solutions $t = \frac{1}{2}$ and $t = -\frac{11}{16}$.
- (A)-(B) : $\frac{-2t-6}{t^2-\frac{1}{2}} = 28$; solutions $t = \frac{1}{2}$ and $t = -\frac{4}{7}$.
- (A)+(B) : $\frac{10t+4}{t^2-\frac{1}{2}} = -36$; solutions $t = \frac{1}{2}$ and $t = -\frac{7}{9}$.

The following table gives the fractions of Theorem 6 ($a_i = k_id - r_i$, $n_i = (k_if - q_i)/t_0$) and checks the rule Θ for (A)-(B).

System	D	t_0	fraction of (A)	fraction of (B)	(A)-(B)	hidden solution
1	$t^2 - 3$	2	$\frac{-8}{5}$	$\frac{5}{-2}$	$\frac{-13}{7}$	$-13/7$
2	$t^2 - 2$	2	$\frac{-5}{6}$	$\frac{-11}{7}$	$\frac{6}{-1}$	-6
3	$t^2 - 2t + 5$	2	$\frac{7}{2}$	$\frac{9}{5}$	$\frac{-2}{-3}$	$2/3$
4	$t^2 + 1$	2	$\frac{-1}{3}$	$\frac{-1}{0}$	$\frac{0}{3}$	0
5	$t^2 - 17$	5	$\frac{-7}{2}$	$\frac{-13}{3}$	$\frac{6}{-1}$	-6
6	$t^2 - 31$	6	$\frac{-20}{4}$	$\frac{-47}{9}$	$\frac{27}{-5}$	$-27/5$
7	$t^2 - \frac{1}{2}$	$\frac{1}{2}$	$\frac{6}{-4}$	$\frac{22}{-32}$	$\frac{-16}{28}$	$-4/7$

System 1 in detail. We have $\frac{-8}{5} \Theta \frac{5}{-2} = \frac{-3}{3}$ of evaluation -1 , which is the hidden solution of (A) + (B), and $\frac{-8}{5} \Theta \frac{-5}{2} = \frac{-13}{7}$ for (A) - (B).

System 4. The hidden solution of (B) is ∞ , because $a_B = 3 - 3 = 0$: the fraction $\frac{-1}{0}$ is a point at infinity, and (A)-(B) gives $\frac{0}{3}$, that is the root 0. The equation (A)-(B) is written $-3t^2 + 6t = 0$, that is $3t(t - 2) = 0$, with roots 2 and 0.

5 Properties: weighted mean, position, determinant

Let $\rho_i = n_i/a_i$ and $\Delta(P, Q) = a_1n_2 - a_2n_1$.

Proposition 9 (weighted mean). *If $a_1, a_2 \neq 0$ and $a_1 + a_2 \neq 0$:*

$$\rho(P \Theta Q) = \frac{a_1\rho_1 + a_2\rho_2}{a_1 + a_2}.$$

Proposition 10 (position). *If $a_1a_2 > 0$ and $\rho_1 \neq \rho_2$, then $\rho(P \Theta Q)$ lies strictly between ρ_1 and ρ_2 (mediant inequality). If $a_1a_2 < 0$, it lies strictly outside the segment $[\rho_1, \rho_2]$.*

Example 1: (A)+(B) has fractions $\frac{-8}{5}$ and $\frac{5}{-2}$, $a_1a_2 = -10 < 0$, and $\rho = -1$ is indeed outside $[-5/2, -8/5]$.

Proposition 11 (determinant).

$$\rho_1 - \rho_2 = -\frac{\Delta(P, Q)}{a_1 a_2}, \quad \Delta(P, P \Theta Q) = \Delta(P, Q).$$

With D of leading coefficient d , we have $\Delta(P, Q) = d(k_2 q_1 - k_1 q_2)/t_0$: the two hidden roots coincide if and only if $k_1 q_2 = k_2 q_1$.

Example 1: $\Delta = a_1 n_2 - a_2 n_1 = 5 \cdot 5 - (-2)(-8) = 9$ and $\rho_1 - \rho_2 = -\frac{8}{5} + \frac{5}{2} = \frac{9}{10} = -\frac{9}{5 \cdot (-2)}$.

6 Inverse construction

Theorem 12 (inverse construction). Let $\frac{n_1}{a_1}$ and $\frac{n_2}{a_2}$ with $\Delta \neq 0$. For every $r \in \mathbb{Q} \cup \{\infty\}$ there is a unique ratio $\lambda : \mu$ such that $\rho(\lambda P \Theta \mu Q) = r$:

$$\lambda : \mu = (n_2 - r a_2) : (r a_1 - n_1), \quad r = \infty : \lambda : \mu = a_2 : (-a_1).$$

Proof. $\rho = (\lambda n_1 + \mu n_2)/(\lambda a_1 + \mu a_2) = r$ is equivalent to $\lambda(n_1 - r a_1) + \mu(n_2 - r a_2) = 0$. $\square$

Example 1, hidden root 0: with $-\frac{8}{5}$ and $\frac{5}{2}$ and $r = 0$, we find $\lambda : \mu = 5 : 8$, and $5(A) + 8(B)$ is written $\frac{18t-27}{t^2-3} = 9$, that is $9t^2 - 18t = 0$, with roots 2 and 0. Hidden root at infinity: $\lambda : \mu = 2 : 5$ and $2(A) + 5(B)$ is written $\frac{9t-18}{t^2-3} = 0$, of degree 1 and with the single root $t = 2$.

Building an equation. The equation with hidden solution $\frac{p}{q}$ and common root t_0 is $(t - t_0)(qt - p) = 0$. For $D = t^2 - k$ we obtain $N_v(t) = (qt_0 + p)t - (qk + t_0 p)$, linear in p and q . Example: $t_0 = 1$, $k = 5$, $\frac{3}{2}$ and $\frac{-4}{3}$ give $\frac{5t-13}{t^2-5} = 2$ and $\frac{-t-11}{t^2-5} = 3$, and their Θ gives $\frac{6t-2}{t^2-5} = -1$.

Proposition 13 (Stern–Brocot tree of equations). Let (A) and (C) be two equations of a system, with fractions $\frac{n_A}{a_A}$ and $\frac{n_C}{a_C}$ with $a_A, a_C > 0$ and $|a_A n_C - a_C n_A| = 1$ (Farey neighbour fractions). Then:

- (i) every rational $r = u/v$ ($\gcd(u, v) = 1$, $v > 0$) strictly between the hidden roots of (A) and of (C) is the hidden root of one and only one equation $\lambda(A) + \mu(C)$ with λ, μ coprime integers > 0 , namely $\lambda = |v n_C - u a_C|$ and $\mu = |u a_A - v n_A|$;
- (ii) conversely, every pair (λ, μ) of coprime integers > 0 yields a hidden root strictly between the two, written as an irreducible fraction;
- (iii) the pairs (λ, μ) are organised like the Stern–Brocot tree (the pair $(1, 1)$ at the root), and each equation is the Θ -composition of its two parents.

Proof. Theorem 12 gives $\lambda : \mu = (v n_C - u a_C) : (u a_A - v n_A)$: these are two integers. Since $n_A a_C - n_C a_A = \pm 1$, the system $\lambda n_A + \mu n_C = u$, $\lambda a_A + \mu a_C = v$ has a unique integer solution (λ, μ) , and $\gcd(u, v) = 1$ forces $\gcd(\lambda, \mu) = 1$: this gives uniqueness and primitivity. If r is strictly between the two hidden roots, the weighted mean (Proposition 9) with weights λa_A and μa_C is there only if these weights have the same sign (Proposition 10), hence $\lambda \mu > 0$. The converse (ii) comes from the same proposition and from primitivity; (iii) is the classical structure of the Stern–Brocot tree, transported by the equations (A) , (C) . $\square$

Example. In the system of three equations of Section 7.1, (A) : $\frac{2t+1}{t^2-3} = 5$ and (C) : $\frac{3t+2}{t^2-3} = 8$ have fractions $\frac{-8}{5}$ and $\frac{-13}{8}$ (determinant -1), hence hidden roots $-\frac{8}{5}$ and $-\frac{13}{8}$. All the fractions of $]-\frac{13}{8}, -\frac{8}{5}[$ are reached:

(λ, μ)	equation $\lambda(A) + \mu(C)$	hidden root
(1, 1)	$\frac{5t+3}{t^2-3} = 13$	$-\frac{21}{13}$
(2, 1)	$\frac{7t+4}{t^2-3} = 18$	$-\frac{29}{18}$
(1, 2)	$\frac{8t+5}{t^2-3} = 21$	$-\frac{34}{21}$
(3, 2)	$\frac{12t+7}{t^2-3} = 31$	$-\frac{50}{31}$
(2, 3)	$\frac{13t+8}{t^2-3} = 34$	$-\frac{55}{34}$

For example $13(t^2 - 3) - (5t + 3) = 13t^2 - 5t - 42 = (t - 2)(13t + 21)$. The 2203 coprime pairs (λ, μ) with $\lambda, \mu \leq 60$ give 2203 distinct hidden roots, all irreducible and in the interval; the 685 rationals of the interval with denominator ≤ 300 are all recovered. The content of this proposition is classical (unimodular basis and Stern–Brocot tree [11]); what is specific to the framework is that each node of the tree is an *equation* of the system, with explicit integer numerator and right-hand side ($\lambda N_A + \mu N_C$, $\lambda k_A + \mu k_C$), and that the hidden root is obtained in irreducible form without any gcd computation.

7 Relations between several equations

Theorem 14 (three equations). *For three fractions $A = \frac{n_A}{a_A}$, $B = \frac{n_B}{a_B}$, $C = \frac{n_C}{a_C}$, with $\Delta(P, Q) = a_P n_Q - a_Q n_P$:*

$$\Delta(B, C) \diamond A \ominus \Delta(C, A) \diamond B \ominus \Delta(A, B) \diamond C = \frac{0}{0}.$$

Proof. The numerators on one hand, the denominators on the other, cancel: each term of the type $a_B n_C n_A$ appears twice with opposite signs. $\square$

Example (Ex. 1, $D = t^2 - 3$, $t_0 = 2$) with $A = \frac{-8}{5}$, $B = \frac{5}{-2}$, $C = \frac{-13}{8}$: $\Delta(B, C) = -14$, $\Delta(C, A) = 1$, $\Delta(A, B) = 9$, and $-14(A) + (B) + 9(C) = 0$ is checked on the numerators: $-14(2t + 1) + (t - 4) + 9(3t + 2) = 0$. Since $\Delta(C, A) = 1$, the fractions C and A are Farey neighbours.

7.1 A system of three equations

With $D = t^2 - 3$, $t_0 = 2$: (A) $\frac{2t+1}{t^2-3} = 5$, (B) $\frac{t-4}{t^2-3} = -2$, (C) $\frac{3t+2}{t^2-3} = 8$, with fractions $\frac{-8}{5}$, $\frac{5}{-2}$, $\frac{-13}{8}$.

Combination	equation	fraction $\ominus$	hidden root
(A)+(B)+(C)	$\frac{6t-1}{t^2-3} = 11$	$\frac{-16}{11}$	$-16/11$
(A)+(B)-(C)	$\frac{-5}{t^2-3} = -5$	$\frac{10}{-5}$	-2
(A)-(B)-(C)	$\frac{-2t+3}{t^2-3} = -1$	$\frac{0}{-1}$	0
(A)-(B)+(C)	$\frac{4t+7}{t^2-3} = 15$	$\frac{-26}{15}$	$-26/15$

7.2 A system of four equations

With $D = t^2 - 2t - 1$ and $t_0 = 3$:

$$\frac{3t-1}{D} = 4, \quad \frac{2t+4}{D} = 5, \quad \frac{-5t+3}{D} = -6, \quad \frac{4t+2}{D} = 7,$$

with cofactors $P_i = k_i D - N_i = (t-3)(4t+1)$, $(t-3)(5t+3)$, $-(t-3)(6t+1)$, $(t-3)(7t+3)$, hence fractions $\frac{-1}{4}$, $\frac{-3}{5}$, $\frac{1}{-6}$, $\frac{-3}{7}$. The eight sign combinations (up to a global sign) give:

Combination	fraction Θ	hidden root
A+B+C+D	$\frac{-6}{10}$	$-3/5$
A+B+C-D	$\frac{0}{-4}$	0
A+B-C+D	$\frac{-8}{22}$	$-4/11$
A+B-C-D	$\frac{-2}{8}$	$-1/4$
A-B+C+D	$\frac{0}{0}$	equation $0 = 0$
A-B+C-D	$\frac{6}{-14}$	$-3/7$
A-B-C+D	$\frac{-2}{12}$	$-1/6$
A-B-C-D	$\frac{4}{-2}$	-2

Remark 15 (this system is dependent). We have $N_B = N_A + N_C + N_D$ and $k_B = k_A + k_C + k_D$, hence (B)=(A)+(C)+(D). Consequently A-B+C+D is the equation $0 = 0$, and the three combinations A+B-C-D, A-B+C-D and A-B-C+D are, up to a factor ± 2 , the original equations (A), (D) and (C) (roots $-1/4$, $-3/7$, $-1/6$). The four really new combinations are A+B+C+D, A+B+C-D, A+B-C+D and A-B-C-D. Since each fraction is determined by two numbers, four fractions always satisfy a linear relation between their numerators and denominators; here the relation has coefficients ± 1 , hence these degeneracies.

8 Choosing the signs without trial and error

Given a system of n equations with fractions $\frac{n_i}{a_i}$, we look for the signs s_i for which the combination $\sum_i s_i E_i$ has a prescribed hidden solution r .

Theorem 16 (sign equation). *Let $r \in \mathbb{Q}$. The combination $\sum_i s_i E_i$, with $\sum_i s_i a_i \neq 0$, has hidden solution r if and only if*

$$\sum_{i=1}^n s_i (n_i - r a_i) = 0.$$

For $r = \infty$, the condition is $\sum_i s_i a_i = 0$.

Proof. $\rho = \sum s_i n_i / \sum s_i a_i = r$ is equivalent to $\sum s_i (n_i - r a_i) = 0$. □

The number $c_i(r) = n_i - r a_i$ is, up to sign, the value at r of the factor $a_i t - n_i$ of the equation E_i . Choosing the signs is therefore a balancing problem: the c_i must be split into two groups of equal sum, that is, we must find the signs for which $\sum s_i (a_i t - n_i)$ vanishes at r .

Two equations. The real weights are unique (Theorem 12); the signs ± 1 work exactly when $|c_1| = |c_2|$.

n equations, target $r = p/q$. Set $\gamma_i = q n_i - p a_i \in \mathbb{Z}$ and look for $s \in \{\pm 1\}^n$ with $\sum s_i \gamma_i = 0$: this is an integer partition problem. A necessary condition is that $\sum \gamma_i$ be even. For small

n , enumerating the 2^{n-1} patterns suffices, and for larger n one uses dynamic programming or meet-in-the-middle search. If $\gamma_i = 0$, the sign of E_i is free: r is the hidden solution of E_i . In general this is exactly the partition problem for a multiset of integers, weakly NP-complete [9]: no algorithm polynomial in n is known, but dynamic programming solves it in pseudo-polynomial time $O(n \sum_i |\gamma_i|)$.

Examples. System of three equations (fractions $\frac{-8}{5}, \frac{5}{-2}, \frac{-13}{8}$):

target r	$\gamma = (\gamma_1, \gamma_2, \gamma_3)$	equation	signs
-2	(2, 1, 3)	$2 + 1 - 3 = 0$	(+, +, -)
0	(-8, 5, -13)	$-8 - 5 + 13 = 0$	(+, -, -)
-16/11	(-8, 23, -15)	$-8 + 23 - 15 = 0$	(+, +, +)
-26/15	(10, 23, 13)	$10 - 23 + 13 = 0$	(+, -, +)

System of four equations (fractions $\frac{-1}{4}, \frac{-3}{5}, \frac{1}{-6}, \frac{-3}{7}$): for $r = -\frac{1}{4}$, $\gamma = (0, -7, -2, -5)$ and $7 = 2 + 5$, so s_2 has the sign opposite to s_3 and s_4 , and s_1 is free. For $r = 0$, $\gamma = (-1, -3, 1, -3)$ and the solution is $(+, +, +, -)$. For $r = -\frac{3}{5}$, $\gamma = (7, 0, -13, 6)$ and the solution is $(+, *, +, +)$ with $*$ free. The trivial combination $(+, -, +, +)$ (which is the equation $0 = 0$) appears in every list, since it cancels all the γ_i ; it must be discarded.

Reading through the equations. Changing the sign of an equation changes the sign of its factor $a_i t - n_i$, hence of $c_i(r)$. The condition of Theorem 16 says that the values $c_i(r)$ of positive sign and those of negative sign compensate.

9 Higher degree: barycentre of monic polynomials

In degree m , it is the elementary symmetric functions of the hidden roots, and not the roots themselves, that are averaged with the weights a_i .

Theorem 17 (barycentre). *Let Q_1, Q_2 be of degree m , with leading coefficients a_1, a_2 and $a_1 + a_2 \neq 0$. Let $\widehat{Q}_i = Q_i/a_i$ be the associated monic polynomial. Then $Q_1 + Q_2$ has degree m and*

$$\widehat{Q_1 + Q_2} = \frac{a_1 \widehat{Q}_1 + a_2 \widehat{Q}_2}{a_1 + a_2}, \quad e_j(Q_1 + Q_2) = \frac{a_1 e_j(Q_1) + a_2 e_j(Q_2)}{a_1 + a_2}$$

for each elementary symmetric function e_j of the roots.

Proof. $Q_1 + Q_2 = a_1 \widehat{Q}_1 + a_2 \widehat{Q}_2$, and the coefficients of $\widehat{Q}$ are the $(-1)^j e_j$. □

The roots themselves are not averaged as soon as $m \geq 2$.

Degree 1 and degree 2. In degree 1, Θ is exactly the addition of the equations $at - b$ associated with the fractions $\frac{b}{a}$. In degree 2, Θ acts twice, on the sum S and on the product P of the roots. Example: $2(t-1)(t-3) + 3(t-1)(t-5) = (t-1)(5t-21)$; for S , $\frac{8}{2} \Theta \frac{18}{3} = \frac{26}{5} = 1 + \frac{21}{5}$; for P , $\frac{6}{2} \Theta \frac{15}{3} = \frac{21}{5}$.

Example in degree 2 (polynomials). $Q_1 = 2(t-2)(t-3)$ and $Q_2 = 3(t+1)(t-4)$. Their sum is $5t^2 - 19t = t(5t-19)$, with roots 0 and 19/5; we have $e_1 = (2 \cdot 5 + 3 \cdot 3)/5 = 19/5$ and $e_2 = (2 \cdot 6 + 3 \cdot (-4))/5 = 0$.

Example in degree 2 (system). With $D = t^3 + 1$, $t_0 = 1$, (A): $\frac{t^2+1}{D} = 1$ and (B): $\frac{3t+1}{D} = 2$, the cofactors are $Q_1 = t^2$ (roots 0, 0) and $Q_2 = 2t^2 + 2t - 1$, with $a_1 = 1$, $a_2 = 2$. The equation

(A)+(B) is $\frac{t^2+3t+2}{D} = 3$, with cofactor $3t^2 + 2t - 1 = (3t - 1)(t + 1)$, with roots $1/3$ and -1 , and $e_1 = \frac{1 \cdot 0 + 2 \cdot (-1)}{3} = -\frac{2}{3}$ and $e_2 = \frac{1 \cdot 0 + 2 \cdot (-\frac{1}{2})}{3} = -\frac{1}{3}$, which indeed give the sum $\frac{1}{3} + (-1) = -\frac{2}{3}$ and the product $\frac{1}{3} \cdot (-1) = -\frac{1}{3}$.

10 Degree 2: discriminant, complex roots, Ford circles

We identify $a t^2 + b t + c$ with $(a, b, c) \in \mathbb{Z}^3$; Θ is the addition, and the discriminant $\delta = b^2 - 4ac$ is a quadratic form of signature $(2, 1)$:

$$\delta(Q_1 + Q_2) = \delta(Q_1) + \delta(Q_2) + 2\beta(Q_1, Q_2), \quad \beta = b_1 b_2 - 2a_1 c_2 - 2a_2 c_1.$$

The squares $a(t - r)^2$ form the cone $\delta = 0$: it is the image of degree 1 in this lattice.

Theorem 18 (complex roots). *Let $L_1 = a_1 t - n_1$ and $L_2 = a_2 t - n_2$ be real, with roots ρ_1, ρ_2 , and $\Delta = a_1 n_2 - a_2 n_1$. The trinomial $Q = L_1^2 + L_2^2$ has discriminant $-4\Delta^2$ and roots z and $\bar{z}$ with*

$$z = \frac{n_1 + i n_2}{a_1 + i a_2} = \frac{(a_1 n_1 + a_2 n_2) + i \Delta}{a_1^2 + a_2^2}.$$

This is the same formula as the median, read with complex numbers: the fraction $\frac{n_1 + i n_2}{a_1 + i a_2}$.

Proof. $Q = (L_1 + i L_2)(L_1 - i L_2)$, and the root of $L_1 + i L_2 = (a_1 + a_2 i)t - (n_1 + n_2 i)$ is z . The discriminant is $4(a_1 n_1 + a_2 n_2)^2 - 4(a_1^2 + a_2^2)(n_1^2 + n_2^2) = -4\Delta^2$ by Lagrange's identity. $\square$

Position of z . Since $L_1(z) = -i L_2(z)$, we have $|z - \rho_1|/|z - \rho_2| = |a_2|/|a_1|$ and the angle at z is a right angle: z lies on the half-circle with diameter $[\rho_1, \rho_2]$ (the geodesic of $\mathbb{H}$ joining ρ_1 to ρ_2). If $|a_1| = |a_2|$, it is the top point.

Farey case. If $|\Delta| = 1$ and $a_1, a_2 > 0$, then $\text{Im } z = 1/(a_1^2 + a_2^2)$: z is the tangency point of the Ford circles of ρ_1 and ρ_2 (a classical fact, here obtained as a root). Example: $\rho_1 = 0 = \frac{0}{1}$ and $\rho_2 = \frac{1}{2} = \frac{1}{2}$ give $z = \frac{2+i}{5}$, and the Ford circles with centres $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{8})$ touch at $(\frac{2}{5}, \frac{1}{5})$. Generic example: $L_1 = t - 1$, $L_2 = t - 2$ give $Q = 2t^2 - 6t + 5$, $\delta = -4$, roots $\frac{3 \pm i}{2}$, at the top of the half-circle with diameter $[1, 2]$.

Interlacing (reality of the roots). The Hermite–Kakeya–Obreschkoff theorem says that a pencil $\lambda Q_1 + \mu Q_2$ has real roots for all (λ, μ) exactly when the roots of Q_1 and Q_2 interlace. In degree 2 this reads

$$\delta(Q_1) \geq 0, \quad \delta(Q_2) \geq 0, \quad \beta(Q_1, Q_2)^2 \leq \delta(Q_1) \delta(Q_2).$$

Interlacing example: $Q_1 = t(t - 2)$ and $Q_2 = (t - 1)(t - 3)$: $\delta_1 = \delta_2 = 4$, $\beta = 2$, so $4 \leq 16$ (roots $0 < 1 < 2 < 3$). Non-interlacing example: $Q_1 = t(t - 2)$ and $Q'_2 = (t - 3)(t - 4)$: $\delta(Q'_2) = 1$, $\beta = -10$, so $100 > 4$, and $Q_1 + Q'_2 = 2t^2 - 9t + 12$ has discriminant $81 - 96 = -15 < 0$: its hidden roots are complex.

Selling decomposition (overview). In degree 2, a positive definite trinomial is written as a combination with coefficients ≥ 0 of the squares of the three linear forms of a Farey triangle. Example: $3t^2 - 3t + 1 = \frac{1}{2}[(2t - 1)^2 + t^2 + (t - 1)^2]$, triangle $\{0, \frac{1}{2}, 1\}$. Passing from one triangle to a neighbour is done by *flips* $(c_k, c_i, c_j) \mapsto (-c_k, c_i + 2c_k, c_j + 2c_k)$: the sum $c_1 + c_2 + c_3$ decreases by $2|c_k|$ at each flip of a negative index, so the descent stops.

11 Projective invariance and normal form

Theorem 19 (covariance). *Let $g(s) = \frac{\alpha s + \beta}{\gamma s + \delta}$ with $\alpha\delta - \beta\gamma \neq 0$, and t_0 such that $\alpha - \gamma t_0 \neq 0$ (the point t_0 is not the image of $s = \infty$). Put $t = g(s)$ in $N/D = k$ and multiply numerator and denominator by $(\gamma s + \delta)$. The system keeps the same form, with common root $s_0 = g^{-1}(t_0)$, and the polynomial of each equation becomes $\tilde{P} = c(s - s_0)(\tilde{a}s - \tilde{n})$, with $c = \alpha - \gamma t_0$ common to all equations and*

$$\tilde{n} = \delta n - \beta a, \quad \tilde{a} = \alpha a - \gamma n.$$

The new numerators and denominators $\tilde{n}, \tilde{a}$ are linear combinations of n and a , so the transformation is compatible with Θ and $\diamond$. The hidden root becomes $g^{-1}(\rho)$, and $\tilde{\Delta} = (\alpha\delta - \beta\gamma)\Delta$: Farey neighbours remain so for $g \in \text{SL}_2(\mathbb{Z})$.

Example (translation $s = t - 1$): equation (A) of Example 1 becomes $\frac{2s+3}{s^2+2s-2} = 5$, that is $5s^2 + 8s - 13 = (s - 1)(5s + 13)$, and the fraction $\frac{-8}{5}$ has become $\frac{-13}{5}$. Example (inversion $t = 1/s$): the same equation gives $16s^2 + 2s - 5 = 0$, with roots $\frac{1}{2} = 1/t_0$ and $-\frac{5}{8} = 1/\rho$.

Normal form. With $t = t_0 + 1/s$, each equation becomes linear: $(at_0 - n)s + a = 0$, with solution $s = 1/(\rho - t_0)$. In this variable, Θ is the term-by-term addition of linear equations. Example 1 ($t_0 = 2$): (A) gives $18s = -5$, (B) gives $-9s = 2$, and the sum gives $9s = -3$, that is $s = -\frac{1}{3} = 1/(-1 - 2)$, which is indeed the hidden root -1 of (A)+(B).

12 The two sides of a system: correspondence between x and t

In several examples, one and the same pair (k_A, k_B) governs two independent equations, one in x and the other in t . A combination $\lambda(A) + \mu(B)$ then acts with the same weights on both sides.

Theorem 20 (correspondence). *Let $F_A(x) = k_A$ and $F_B(x) = k_B$ be two equations in x , with F_A, F_B monic of the same degree and a common root x_0 ; write $F_i - k_i = (x - x_0)Q_i$. For the combination $\lambda(A) + \mu(B)$, the elementary symmetric functions of the hidden roots in x are*

$$e_j = \frac{\lambda e_j(Q_A) + \mu e_j(Q_B)}{\lambda + \mu},$$

and the hidden root in t is $\rho_t = \frac{\lambda n_A + \mu n_B}{\lambda a_A + \mu a_B}$. Eliminating $\lambda : \mu$, one obtains a homography between ρ_t and each e_j .

Example 6 ($x_0 = -4, t_0 = 6$). The cofactors in x are $Q_A = x^2 + 3x - 5$ and $Q_B = x^2 - 8x + 12$, with $e_1 = -3, e_2 = -5$ for (A) and $e_1 = 8, e_2 = 12$ for (B). The pairs in t are $\frac{-20}{4}$ and $\frac{-47}{9}$, whence $\lambda : \mu = (47 + 9\rho_t) : (-20 - 4\rho_t)$ and

$$e_1 = -\frac{301 + 59\rho_t}{27 + 5\rho_t}, \quad e_2 = -\frac{475 + 93\rho_t}{27 + 5\rho_t}.$$

For (A), $\rho_t = -5$ gives $e_1 = -3, e_2 = -5$; for (B), $\rho_t = -47/9$ gives $e_1 = 8, e_2 = 12$. The denominator $27 + 5\rho_t$ vanishes at $\rho_t = -27/5$, the hidden root in t of (A)–(B), where the equation in x loses one degree. Its hidden root in x is that of $Q_A - Q_B = 11x - 17$, that is $17/11$. The roots of Q_A are irrational, whereas $17/11$ is rational: the combination A–B makes a simpler number appear than its ingredients. The system (A)–(B) thus has roots $(x, t) \in \{-4, \frac{17}{11}\} \times \{6, -\frac{27}{5}\}$.

12.1 n equations

With n equations with common k_i , the weights $\lambda_1, \dots, \lambda_n$ describe a projective space of dimension $n - 1$. The invariants are the m symmetric functions $e_1, \dots, e_m$ on the x side and the root ρ_t on the t side, that is $m + 1$ invariants.

Theorem 21 (n equations). *Let $n \leq m + 1$ equations, with symmetric functions $e^{(i)} = (e_1^{(i)}, \dots, e_m^{(i)})$ and fractions $\frac{n_i}{a_i}$. Assume that the n rows $(1, e_1^{(i)}, \dots, e_m^{(i)})$ are independent (the corresponding table has rank n). Then, for the combination λ :*

1. *the point $e = (e_1, \dots, e_m)$ of hidden symmetric functions on the x side is the barycentre of the points $e^{(i)}$ with weights λ_i , hence lies in an affine subspace of dimension $n - 1$;*

2. *ρ_t is a quotient of two affine forms in e :*

$$\rho_t = \frac{\sum_i n_i \mu_i(e)}{\sum_i a_i \mu_i(e)},$$

where the $\mu_i(e)$ are the barycentric weights of e (the unique numbers such that $\sum \mu_i = 1$ and $\sum \mu_i e^{(i)} = e$).

There are thus $(m - n + 1) + 1 = m + 2 - n$ relations.

Proof. By Theorem 17, $e = \sum \lambda_i e^{(i)} / \sum \lambda_i$; the rows being independent, the weights $\mu_i = \lambda_i / \sum \lambda$ are determined by e and affine in e . Substitute into $\rho_t = \sum \lambda_i n_i / \sum \lambda_i a_i$. $\square$

For $n = m + 2$ the rows are dependent, e no longer determines λ , and there is no generic relation; $n = 2$ recovers the m homographies of Theorem 20.

Example ($n = 3, m = 2$). t side: $D = t^2 - 3, t_0 = 2$ and fractions $\frac{-9}{5}, \frac{6}{-2}, \frac{-6}{3}$. x side: three cofactors with $(e_1, e_2) = (3, 1), (-1, 4), (2, -5)$. Eliminating λ gives a single relation:

$$\rho_t = -\frac{3(33e_1 - e_2 - 17)}{48e_1 + e_2 - 10}.$$

Checks: $(e_1, e_2) = (3, 1)$ gives $\rho_t = -9/5$, $(-1, 4)$ gives -3 , $(2, -5)$ gives -2 ; the combination $\lambda = (2, 1, -1)$ gives $(e_1, e_2) = (\frac{3}{2}, \frac{11}{2})$ and $\rho_t = -\frac{6}{5}$ on both sides. The line $48e_1 + e_2 = 10$ is the pole: the equation in t loses one degree there and the hidden root is ∞ .

Example ($n = 4, m = 3$). Same D and t_0 , fractions $\frac{-9}{5}, \frac{6}{-2}, \frac{-6}{3}, \frac{-3}{1}$, and on the x side: $(e_1, e_2, e_3) = (2, -1, 3), (0, 4, -2), (1, 1, 1), (-3, 2, 5)$. Elimination gives

$$\rho_t = -\frac{3(25e_1 + 33e_2 + 18e_3 - 86)}{5(4e_1 + 6e_2 + 3e_3 - 16)}.$$

Check: $e = (2, -1, 3)$ gives $\rho_t = -9/5$, that is the fraction of the first equation.

12.2 Origin of the fractions: a single parameter x_0

When the equations come from $F_i(x) = k_i$ and from $N_i(t)/D(t) = k_i$ with a common solution (x_0, t_0) , we have $k_i = F_i(x_0)$. The cofactors $Q_i = (F_i(x) - F_i(x_0))/(x - x_0)$ and the pairs $\frac{n_i}{a_i}$ are then polynomials in x_0 : the fractions form a one-parameter family. For $D = t^2 - 3$ and $t_0 = 2$, we have $a_i = k_i$ and $n_i = p_i t + q_i$ if $N_i = p_i t + q_i$.

Example ($n = 3$). $F_1 = x^3 - x, F_2 = x^3 + 2x^2 - 5, F_3 = x^3 - 4x^2 + x + 7, x_0 = 3$, hence $k = (24, 40, 1)$. With $p = (1, -2, 3)$ and $q = k - 2p = (22, 44, -5)$, the fractions are $-47/24$,

$-82/40$ and $1/1$, with Θ -sum equal to $-128/65$. The cofactors in x are x^2+3x+8 , $x^2+5x+15$ and $(x-2)(x+1)$. The sum equation in t is $\frac{2t+61}{t^2-3} = 65$, that is $-(t-2)(65t+128) = 0$: the hidden root is $-128/65$. The sum equation in x is $3x^3-2x^2+2 = 65$, that is $(x-3)(3x^2+7x+21) = 0$: $x_0 = 3$ is the only rational root, and the two hidden roots in x are complex (discriminant $49 - 252 < 0$). Knowing the Θ -sum amounts to knowing $\sum k_i$, hence x_0 as a root of a polynomial.

Theorem 22 (rational hidden roots in x , cubics). *Let F_A, F_B be monic cubics and $\lambda = (\lambda_A, \lambda_B)$. Put $\bar{F} = \frac{\lambda_A F_A + \lambda_B F_B}{\lambda_A + \lambda_B} = x^3 + \bar{\alpha}x^2 + \bar{\beta}x + \bar{\gamma}$. The two hidden roots in x of the combination are the roots of $\bar{F}(x) = \bar{F}(x_0)$ other than x_0 , and they are rational if and only if x_0 is the abscissa of a rational point of the conic*

$$y^2 = -3x_0^2 - 2\bar{\alpha}x_0 + \bar{\alpha}^2 - 4\bar{\beta}.$$

If this conic has a rational point, it has infinitely many, parametrised by the slope of a line.

Proof. By Theorem 17, the cofactor of the combination is the weighted mean of the cofactors, hence that of $\bar{F}$: $x^2 + (x_0 + \bar{\alpha})x + (x_0^2 + \bar{\alpha}x_0 + \bar{\beta})$. Its discriminant is $(x_0 + \bar{\alpha})^2 - 4(x_0^2 + \bar{\alpha}x_0 + \bar{\beta})$. $\square$

Example. $F_A = x^3 - 3x^2 - 4x$, $F_B = x^3 - x^2 - 6x$, $\lambda = (1, 1)$. The equations are $F_A(x) = k_A$ and $F_B(x) = k_B$ with $k_A = F_A(3) = -12$ and $k_B = F_B(3) = 0$: they have the common solution $x_0 = 3$ (their other solutions are ± 2 for the first and $0, -2$ for the second). The mean is $\bar{F} = \frac{1}{2}(F_A + F_B) = x^3 - 2x^2 - 5x$, and $\bar{F}(x) - \bar{F}(3) = x^3 - 2x^2 - 5x + 6 = (x-3)(x-1)(x+2)$. The conic $y^2 = -3x_0^2 + 4x_0 + 24$ has the point $(3, 3)$, and the line of slope s through this point gives

$$x_0(s) = 3 - \frac{14 + 6s}{s^2 + 3}.$$

s	x_0	hidden roots of A+B	roots of A, of B
0	$-5/3$	$10/3$ and $1/3$	irrational
2	$-5/7$	$25/7$ and $-6/7$	irrational
$1/2$	$-29/13$	$30/13$ and $25/13$	irrational
-2	$19/7$	$10/7$ and $-15/7$	irrational
3	$1/3$	$10/3$ and $-5/3$	irrational
5	$10/7$	$19/7$ and $-15/7$	irrational

For each of these values of x_0 , the t side is automatically rational: $k_A = F_A(x_0)$, $k_B = F_B(x_0)$ and the fractions $\frac{n_i}{a_i}$ are rational. We obtain an infinite family of systems whose combination A+B has all its roots rational on both sides, whereas A and B have only irrational roots in x . For $s = 1$ ($x_0 = -2$), the roots of A and of B are also rational: this is a special case.

A complete system in two unknowns. Take $s = 0$, hence $x_0 = -\frac{5}{3}$, and $t_0 = 2$, $D = t^2 - 3$. The equations (multiplying both sides by 27 to clear the denominators) are

$$\begin{aligned} (A) : \quad & 27x^3 - 81x^2 - 108x = -170, & \frac{27t - 224}{t^2 - 3} &= -170, \\ (B) : \quad & 27x^3 - 27x^2 - 162x = 70, & \frac{27t + 16}{t^2 - 3} &= 70, \end{aligned}$$

with common solution $(x, t) = (-\frac{5}{3}, 2)$. The combinations give:

Combination	solutions in x	solutions in t
(A)	$-\frac{5}{3}, \frac{7 \pm \sqrt{15}}{3}$	$2, -\frac{367}{170}$
(B)	$-\frac{5}{3}, \frac{4 \pm \sqrt{30}}{3}$	$2, -\frac{113}{70}$
(A)–(B): $-54x^2 + 54x = -240, \frac{-240}{t^2-3} = -240$	$-\frac{5}{3}, \frac{8}{3}$	$2, -2$
(A)+(B): $54x^3 - 108x^2 - 270x = -100, \frac{54t-208}{t^2-3} = -100$	$-\frac{5}{3}, \frac{1}{3}, \frac{10}{3}$	$2, -\frac{127}{50}$

The solutions of (A) and of (B) in x are irrational, but those of (A)–(B) and of (A)+(B) are all rational. On the t side, the fractions $\frac{367}{-170}$ and $\frac{-113}{70}$ give $\frac{254}{-100}$ for (A)+(B), of evaluation $-\frac{127}{50}$, and $\frac{480}{-240}$ for (A)–(B), of evaluation -2 .

13 Two parameters: a system of four equations in (t, α)

Consider the linear system in (X, Y, Z, W) depending on two parameters t and α :

$$(\psi) \quad \begin{cases} X + (t-1)Y - Z + W - 2 = 0, \\ -(t+1)X + 3Y + 2Z + 2W + 1 = 0, \\ -X + 2Y + Z + (\alpha-3)W - 3 = 0, \\ X + 2Y - (\alpha+3)Z + W + 1 = 0. \end{cases}$$

Its coefficients are constants, except t in the first two equations and α in the last two. Cramer's rule gives $X_i = N_i(t, \alpha)/\Delta(t, \alpha)$ with

$$\Delta = \alpha^2 t^2 + 2\alpha^2 - 8\alpha t - 8t^2 - 20,$$

$$\begin{aligned} N_X &= \alpha^2 t + 5\alpha^2 + 8\alpha t - 18\alpha + 8t - 176, & N_Y &= 2\alpha^2 t + \alpha^2 - 2\alpha t - 18\alpha - 24t - 28, \\ N_Z &= \alpha t^2 + 5\alpha t + 9\alpha - 22t - 72, & N_W &= 3\alpha t^2 - 3\alpha t + 9\alpha + 8t^2 - 26t + 36. \end{aligned}$$

(The four formulas were verified with SymPy.) We thus have four equations $X_i = k_i$ with common denominator Δ , but with *two* unknowns t and α : the same fraction N/Δ is read sometimes as a function of t (with α fixed), sometimes as a function of α (with t fixed).

Proposition 23 (both sides obey Θ). *Let (t_0, α_0) be a point where $\Delta \neq 0$ and $k_i = N_i(t_0, \alpha_0)/\Delta(t_0, \alpha_0)$. Then*

$$k_i \Delta(t, \alpha_0) - N_i(t, \alpha_0) = (t - t_0)(a_i^t t - n_i^t), \quad k_i \Delta(t_0, \alpha) - N_i(t_0, \alpha) = (\alpha - \alpha_0)(a_i^\alpha \alpha - n_i^\alpha),$$

and, for every combination $\sum \lambda_i (E_i)$ with integer coefficients, the hidden value in t (at $\alpha = \alpha_0$) is the value of $\Theta_i \lambda_i \diamond \frac{n_i^t}{a_i^t}$ and the hidden value in α (at $t = t_0$) is the value of $\Theta_i \lambda_i \diamond \frac{n_i^\alpha}{a_i^\alpha}$, with the same coefficients λ_i .

Proof. Each polynomial in t (resp. in α) vanishes at t_0 (resp. α_0), hence factors as stated, and the map $P \mapsto P/(t - t_0)$ is linear (Theorem 6). The same k_i are used on both sides, hence the same weights. $\square$

Example. Take $(t_0, \alpha_0) = (1, 3)$, where $\Delta = -25$. Then $k = (\frac{144}{25}, \frac{17}{5}, \frac{49}{25}, -\frac{9}{5})$. With the fractions multiplied by 25 (by $25 \diamond$, without changing the values):

equation	fraction in t	hidden t ($\alpha = 3$)	fraction in α	hidden α ($t = 1$)
X	$\frac{4337}{144}$	$\frac{4337}{144}$	$\frac{56}{282}$	$\frac{28}{141}$
Y	$\frac{1655}{85}$	$\frac{331}{17}$	$\frac{-360}{180}$	-2
Z	$\frac{1027}{-26}$	$-\frac{79}{2}$	$\frac{326}{147}$	$\frac{326}{147}$
W	$\frac{-1485}{-470}$	$\frac{297}{94}$	$\frac{270}{-135}$	-2

For $X + Y$: $\frac{4337+1655}{144+85} = \frac{5992}{229}$ gives $t = \frac{5992}{229}$, and $\frac{56-360}{282+180} = \frac{-304}{462}$ gives $\alpha = -\frac{152}{231}$; direct solution of the two equations confirms these values.

Theorem 24 (relations between hidden values in t and in α). *For each chosen equation E_i , denote by $\frac{n_i}{a_i}$ its fraction in t and by $\frac{\tilde{n}_i}{\tilde{a}_i}$ its fraction in α . For a combination with coefficients λ_i , the fractions are $\frac{\sum \lambda_i n_i}{\sum \lambda_i a_i}$ and $\frac{\sum \lambda_i \tilde{n}_i}{\sum \lambda_i \tilde{a}_i}$.*

- (i) **Two equations.** If $n_1 a_2 \neq n_2 a_1$, the hidden values are linked by a homography: with $\lambda : \mu = (n_2 - \rho_t a_2) : (\rho_t a_1 - n_1)$ (Theorem 12),

$$\rho_\alpha = \frac{\tilde{n}_1(n_2 - \rho_t a_2) + \tilde{n}_2(\rho_t a_1 - n_1)}{\tilde{a}_1(n_2 - \rho_t a_2) + \tilde{a}_2(\rho_t a_1 - n_1)}.$$

- (ii) **Three equations.** The four numbers $(n, a, \tilde{n}, \tilde{a})$ of a combination, with its two fractions, satisfy a fixed homogeneous linear relation $c_1 n + c_2 a + c_3 \tilde{n} + c_4 \tilde{a} = 0$; it does not translate into a relation between ρ_t and ρ_α , because the two fractions share the same unknown scale.

- (iii) **Four equations.** If the 4×4 table formed by the numbers $n_i, a_i, \tilde{n}_i, \tilde{a}_i$ of the four equations has nonzero determinant, every value of (ρ_t, ρ_α) is reached by some combination: there is no relation between the hidden values.

Proof. (i) The inverse construction (Theorem 12) gives $\lambda : \mu$ as a function of ρ_t ; substitute into $\rho_\alpha = \frac{\lambda \tilde{n}_1 + \mu \tilde{n}_2}{\lambda \tilde{a}_1 + \mu \tilde{a}_2}$. (ii) The four numbers $(n, a, \tilde{n}, \tilde{a})$ are linear combinations of three coefficients $\lambda_1, \lambda_2, \lambda_3$: they are linked by a relation (exactly one if the 4×3 table has rank 3). (iii) If the determinant is nonzero, the linear system $(n, a, \tilde{n}, \tilde{a}) = \sum \lambda_i (n_i, a_i, \tilde{n}_i, \tilde{a}_i)$ has a solution λ for every prescribed left-hand side. $\square$

In the example.

- (i) For X and Y (fractions $\frac{4337}{144}, \frac{1655}{85}$ in t and $\frac{56}{282}, \frac{-360}{180}$ in α):

$$\rho_\alpha = \frac{33080 - 1132 \rho_t}{39 \rho_t - 6279}.$$

For $\rho_t = \frac{5992}{229}$, we recover $\rho_\alpha = -\frac{152}{231}$.

- (ii) For X, Y, Z : $-2108 n^t + 41044 a^t + 5213 n^\alpha + 10426 a^\alpha = 0$, satisfied by each of the three equations, hence by every combination (checked for $\lambda = (1, 1, 1), (2, -1, 3), (1, 0, -1)$).
- (iii) For the four equations, the determinant of the 4×4 table (with the integer fractions above) is $90\,901\,687\,500 \neq 0$.

Geometric reading. For each i , the curve $C_i : k_i \Delta - N_i = 0$ has bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and passes through the rational point $O = (t_0, \alpha_0)$. For the four equations of the example, a Gröbner-basis computation in the four affine charts shows that C_i is smooth: it is a curve

of genus 1 with a rational point, hence an elliptic curve. The involution “hidden value in t ” $\iota_t : (t, \alpha) \mapsto (t', \alpha)$ and the involution “hidden value in α ” ι_α are the deck involutions of the two degree-2 projections, of the form $P \mapsto s - P$ on C_i ; their composition $\tau = \iota_\alpha \circ \iota_t$ is therefore a *translation* by a rational point of C_i .

Proposition 25. *For each of the four curves C_X, C_Y, C_Z, C_W of the example, the translation τ has infinite order; in particular $C_i(\mathbb{Q})$ has rank ≥ 1 .*

Proof. If τ had finite order, it would be the translation by a torsion point of $E(\mathbb{Q})$, of order ≤ 12 by Mazur’s theorem [10], and the orbit of O would return to O in at most 12 steps. An exact computation shows that this does not happen during 13 steps for each of the four curves. For X , the orbit of $(1, 3)$ passes through $t = \frac{4337}{144}$, then through fractions whose numerators have 13, 31 and 54 digits in t (and 8, 22, 42, 68 digits in α) at the following steps, which illustrates the growth of heights. $\square$

What is classical. The fact that two involutions of this type compose into a translation on a curve of bidegree $(2, 2)$ is classical (Euler’s addition theorem for biquadratic curves, Quispel–Roberts–Thompson maps). What is specific to the framework is the link with the Θ calculus: the hidden value in t and the hidden value in α of any combination are both obtained from the same combination of pairs (Proposition 23), which gives the explicit relations of Theorem 24 (a homography for two equations, a linear relation for three, nothing for four) and the elliptic curves of rank ≥ 1 of the example.

14 When all the roots are rational

Theorem 17 gives the symmetric functions of the remaining roots, not the roots. We ask when the m remaining roots of (A)–(B) are all rational although those of (A) and of (B) are not.

Degree $m = 2$: a conic. The condition becomes the rationality of a square root of a trinomial in one parameter, hence a rational point on a conic. If one point exists, there are infinitely many (see Theorem 22).

Degree $m = 3$: an elliptic curve. The pairs of roots (u, v) with the same value of the parameter form a curve $F(u, v) = 0$ of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, of genus 1 in general. Since the sum of the three roots is rational, two rational roots suffice. The rank is computed with PARI: we met ranks 1, and a rank 0 in pencils related by a translation.

14.1 A rank criterion for pencils of cubics

Notation. Let Q_B, Q_A be two distinct monic cubics (for example the cofactors of the two equations of a system of degree 4 in x). The *members* of the pencil are the monic cubics $(\lambda Q_B + \mu Q_A)/(\lambda + \mu)$, that is the cofactors of the combinations $\lambda(B) + \mu(A)$ (Theorem 17). A member is *totally rational* if its three roots are rational. For $x_1 \neq x_2$ set

$$H(x_1, x_2) = \frac{Q_B(x_1)Q_A(x_2) - Q_B(x_2)Q_A(x_1)}{x_1 - x_2} = c_2(x_1)x_2^2 + c_1(x_1)x_2 + c_0(x_1), \quad D = c_1^2 - 4c_0c_2.$$

D is a polynomial in x_1 of degree ≤ 4 . Two numbers $x_1 \neq x_2$ are roots of the same member if and only if $H(x_1, x_2) = 0$; if x_1, x_2 are rational, the third root is rational too (sum of the roots).

Theorem 26 (rank criterion). *Suppose that the pencil contains three members whose nine roots are rational and pairwise distinct, and that D has no square factor. Then the curve $y^2 = D(x)$*

has infinitely many rational points (it is an elliptic curve of rank ≥ 1 if $\deg D \in \{3, 4\}$, a conic if $\deg D \leq 2$), and the pencil contains infinitely many totally rational members.

Proof. (1) Let M be one of the three members, with roots r_1, r_2, r_3 . Since the nine roots are distinct, the pencil has no common root, so $(Q_B(r_1), Q_A(r_1)) \neq (0, 0)$. The numerator $Q_B(r_1)Q_A(x_2) - Q_A(r_1)Q_B(x_2)$ is then a nonzero multiple $\kappa M(x_2)$ of the member that vanishes at r_1 , whence $H(r_1, x_2) = -\kappa(x_2 - r_2)(x_2 - r_3)$ and $D(r_1) = \kappa^2(r_2 - r_3)^2$, a nonzero rational square. The curve $y^2 = D(x)$ therefore has the two rational points $(r_1, \pm y)$ with $y \neq 0$.

(2) The $3 \times 3 = 9$ values of x_1 are distinct: we obtain 18 distinct rational points.

(3) If $\deg D \in \{3, 4\}$, since D has no square factor, $y^2 = D(x)$ is a smooth model of genus 1 with a rational point: it is an elliptic curve $E/\mathbb{Q}$, with $\#E(\mathbb{Q}) \geq 18$. By Mazur's theorem [10], the torsion subgroup of $E(\mathbb{Q})$ has at most 16 elements; $E(\mathbb{Q})$ is therefore infinite. If $\deg D \leq 2$, the curve has genus 0 and has infinitely many rational points as soon as it has one.

(4) Each value of x_1 carries at most two points: there are infinitely many rational x_1 with $D(x_1)$ a square. Apart from finitely many exceptions ($c_2(x_1) = 0$, points at infinity, $x_2 = x_1$), $x_2 = (-c_1 \pm y)/(2c_2)$ is rational and $H(x_1, x_2) = 0$: x_1, x_2 are roots of the same member, totally rational by the remark above. A member has only three roots, hence infinitely many members. $\square$

Example 1 (integer roots). The pencil $x^3 - x + s(x^2 - 5)$ contains for $s = 0, 6, -6$ the members $(x - 1)x(x + 1)$, $(x + 5)(x + 3)(x - 2)$, $(x + 2)(x - 3)(x - 5)$, whose nine roots $0, \pm 1, \pm 2, \pm 3, \pm 5$ are integers and distinct. With the generators $x^3 - x$ and $x^3 + x^2 - x - 5$, we find $D = 20x^4 - 104x^2 + 100 = 4(5x^4 - 26x^2 + 25)$, without square factor. For each of the nine values, $5x^4 - 26x^2 + 25$ is a square:

x	0	± 1	± 2	± 3	± 5
$5x^4 - 26x^2 + 25$	5^2	2^2	1^2	14^2	50^2

The theorem gives infinitely many rational s for which $x^3 - x + s(x^2 - 5)$ has three rational roots. A search (roots of height ≤ 700 , exact computation) finds besides $s = 0, \pm 6$ the values $s = \pm \frac{420}{6479}$, with roots $-\frac{25}{31}, -\frac{4}{11}, \frac{21}{19}$ (and their opposites): indeed $-\frac{25}{31} - \frac{4}{11} + \frac{21}{19} = -\frac{420}{6479} = -s$. The next values have a much larger height, which is the expected behaviour for multiples of a point of positive rank (without proof).

Example 2 (an irreducible generator). Let $Q_B = (x - 2)(x - 3)(x + 5)$ and $Q_A = x^3 + x + 1$, irreducible over $\mathbb{Q}$ (its roots are irrational). The following members are totally rational:

$$\begin{aligned} Q_B, \quad 349Q_B - 180Q_A &= (x + 7)(13x - 21)(13x - 70), \\ 876479Q_B - 76479Q_A &= (200x + 1043)(100x - 189)(40x - 133). \end{aligned}$$

Their nine roots are distinct and $D = 400x^4 + 1160x^3 - 2523x^2 + 3920x - 5684$ has no square factor: the theorem applies. This pencil therefore contains infinitely many totally rational members, although one of its two generators has irrational roots.

Frequency. Among the pencils of monic cubics with distinct integer roots in $[-N, N]$ that contain at least three members without common root and whose nine roots are distinct, exact computation finds 1, 4, 18, 74 for $N = 5, 6, 7, 8$; in the 74 cases of $N = 8$, D has no square factor (of degree 4 in 73 cases and of degree 2 in one case), so the theorem applies.

Scope. The theorem turns the observation “rank 1 according to PARI” into a proved statement for pencils in which three totally rational members are known; the proof is an exact computation

followed by Mazur's theorem. It says nothing about a pencil in which three such members are not known. The ingredient (elliptic curves and Mazur's bound) is classical; what is specific to the framework is that the rational points of the curve are exactly the pairs of hidden roots of one and the same combination of the system.

General degree m . For a degree- m cover of $\mathbb{P}^1$ by $\mathbb{P}^1$ whose $2m - 2$ branch points are simple and distinct (generic case), with Galois group S_m , the Riemann–Hurwitz formula gives the genus $1 + m!(m - 3)/2$ for the Galois closure:

m	2	3	4	5
genus	0	1	13	121

For $m \geq 4$ the genus is ≥ 2 , so the totally rational fibres are finite in number (Faltings). The genus 13 is verified by computation on three pencils of degree 4; the case of degree 5 remains to be verified. This finiteness is already contained in Pakovich's work on the identities $f(X) = g(Y)$ and in the Avanzi–Zannier theory; it is therefore not new [6, 7, 8].

A symmetric pencil of degree 5. Let $H = t^5 + 1$, $t_0 = 0$, and

$$\begin{aligned} N_A &= 24t^5 - 1387t^3 + 10944t + 5, \\ N_B &= 18t^5 - 1700t^3 + 35280t - 2, \\ N_A - N_B &= 6t^5 + 313t^3 - 24336t + 7. \end{aligned}$$

We have $N_A - 5H = 19t(t - 3)(t + 3)(t - 8)(t + 8)$, $N_B + 2H = 20t(t - 6)(t + 6)(t - 7)(t + 7)$ and $(N_A - N_B) - 7H = -t(t - 12)(t + 12)(t - 13)(t + 13)$. The three equations $N/H = 5, -2$ and 7 therefore have all their roots rational, with $t_0 = 0$ common. Theorem 17 gives for the four other roots of $A - B$ (weights 19 and 20, hence $19 - 20 = -1$): $e_2 = \frac{19 \cdot (-73) - 20 \cdot (-85)}{-1} = -313 = -(144 + 169)$ and $e_4 = \frac{19 \cdot 576 - 20 \cdot 1764}{-1} = 24336 = 144 \cdot 169$.

A non-symmetric example of degree 4. The roots $(-3, -2, 0, 4)$ and $(-4, -1, 2, 8)$ with weights 14 and 9 give $14A - 9B = 5(t - 3)(t + 6)(t + 8)(t + \frac{4}{5})$, that is the roots $3, -6, -8, -\frac{4}{5}$.

Rational critical points in a pencil (cubics). Since $(P + Q)' = P' + Q'$, the critical points of the composition are computed by Θ on the symmetric functions of the critical points. Example: $A = 2(t - 1)(t - 3)(t + 2)$ and $B = 3(t - 1)(t - 5)(t - 4)$ have irrational critical points, $\frac{2 \pm \sqrt{19}}{3}$ and $\frac{10 \pm \sqrt{13}}{3}$, and $A + B$ has the rational critical points $11/5$ and $7/3$, whose sum $68/15$ is the weighted median. For $A + mB$, the discriminant of $A' + mB'$ is $4(3m - 2)(39m - 38)$, so the condition is the conic $y^2 = 4(3m - 2)(39m - 38)$. The point $m = 1$ gives $y = 2$, whence

$$m(s) = \frac{s^2 - 4s - 300}{s^2 - 468}.$$

Examples: $s = 0$ gives $m = 25/39$ and the critical points $17/9$ and $35/17$; $s = -2$ gives $m = 18/29$ and $11/6, 29/14$; $s = 2$ gives $m = 19/29$ and $29/15, 47/23$. The roots $m = 2/3$ and $m = 38/39$ of $D(m)$ are the cases where $A + mB$ has a double critical point. For quartics, the condition is the rationality of an entire fibre: a curve of genus 1. Example: $A' = t^3 - 2$ and $B' = t^2 - 2t + 2$, with sum $A' + B' = t(t - 1)(t + 2)$; the curve $u^2v^2 - 2u^2v + 2u^2 - 2uv^2 + 2uv + 2u + 2v^2 + 2v - 4 = 0$ has rank 1 according to PARI, so it has infinitely many rational points. A small-height search finds only $m = 1$: the other values have a large height.

15 Equations with a Möbius transformation and derived operations

The equation $x \Theta g(x) = w$. Let g be the Möbius transformation $g(\frac{p}{q}) = \frac{\alpha p + \beta q}{\gamma p + \delta q}$, applied to the unreduced fraction $\frac{p}{q}$. The equation $x \Theta g(x) = w$, with unknown $x = \frac{p}{q}$ and right-hand side $w = \frac{w_1}{w_2}$, is written

$$(1 + \alpha)p + \beta q = w_1, \quad \gamma p + (1 + \delta)q = w_2,$$

two linear equations in p and q . If $(1 + \alpha)(1 + \delta) - \beta\gamma \neq 0$, the solution is unique and rational (Cramer's rule). Example: $g(\frac{p}{q}) = \frac{2p+q}{p}$ (determinant 2) and $w = \frac{7}{3}$ give $3p + q = 7$, $p + q = 3$, hence $x = \frac{2}{1}$, $g(x) = \frac{5}{2}$ and $\frac{2}{1} \Theta \frac{5}{2} = \frac{7}{3}$. If $(1 + \alpha)(1 + \delta) = \beta\gamma$, one can have $x \Theta g(x) = \frac{0}{0}$, of zero total mass: this is a systematic way to build hidden roots. Example: $g(\frac{p}{q}) = \frac{p+2q}{p}$ and $x = \frac{-1}{1}$ give $g(x) = \frac{-1}{-1}$ and $\frac{-1}{-1} \Theta \frac{-1}{-1} = \frac{0}{0}$.

Sequences of Θ and quadratic irrationals. Put $x_0 = \frac{1}{0}$, $x_1 = \frac{2}{1}$ and $x_{k+1} = 2 \Diamond x_k \Theta x_{k-1}$:

$$\frac{1}{0}, \frac{2}{1}, \frac{5}{2}, \frac{12}{5}, \frac{29}{12}, \frac{70}{29}, \dots \longrightarrow 1 + \sqrt{2}.$$

The values converge to $1 + \sqrt{2}$. This is classical arithmetic, related to Pell sequences and continued fractions, but here the recurrence is written with the two laws Θ and $\Diamond$.

Operations that derive from Θ and $\Diamond$.

- Difference: $\frac{p}{q} \ominus \frac{r}{s} = \frac{p-s}{q-s}$, that is Θ with the opposite.
- Multiples: $n \Diamond \frac{p}{q} = \frac{np}{nq}$ and more generally $\lambda \Diamond x \Theta \mu \Diamond y$, which is the combination of two equations.
- Term-by-term product: $\frac{p}{q} \otimes \frac{r}{s} = \frac{pr}{qs}$ distributes over Θ .
- Farey determinant: $\Delta(\frac{p}{q}, \frac{r}{s}) = ps - qr$ satisfies $\Delta(x, x \Theta y) = \Delta(x, y)$.
- Every linear operation on polynomials (translation $t \rightarrow t + c$, reversal, differentiation) commutes with Θ .

16 Open questions

1. **Pencils of degree 4 and higher.** For which pencils is the totally rational fibre non-trivial? This is a moduli problem: describe the pencils whose genus-13 curve has rational points outside the obvious fibres.
2. **Symmetric pencils.** Prove that the rank is positive, and explain the rank-0 cases in pencils related by a translation. Verify the genus for generic pencils of degree 5.
3. **Stern–Brocot in degree 2.** In degree 1, if $\Delta = \pm 1$, the combinations with coprime integer coefficients ≥ 0 of two fractions give all the rationals between their evaluations (Stern–Brocot tree). In degree 2 the analogue is the Selling–Voronoi decomposition; it remains to see what the decomposition becomes in degree ≥ 3 .
4. **Which algebra?** On oriented fractions, Θ and $\Diamond$ add and multiply numerators and denominators term by term, so there is nothing new at this level. The interest lies in the dependence of the roots of $P + Q$ on those of P and Q (mediant in degree 1, barycentre of the e_j in degree m). The orientation $-A/B \neq A/(-B)$ amounts to tracking a polynomial up to a positive factor, and not up to a nonzero factor. This is the lead towards a structure that does not reduce to $\mathbb{Z} \times \mathbb{Z}$.

5. **Critical points of degree 4.** Determine points of positive rank on the curve of the previous section and their height.
6. **Systems of n equations.** Prove, without the independence hypothesis of Theorem 21, the structure of the relations when the $e^{(i)}$ are dependent, and describe the image of the map $(k_1, \dots, k_n) \mapsto (e, \rho_t)$.

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