Arithmetic Unitarity IVa

Counting Ghosts

Spinor Vacua, Saddles, the Weil Positivity Ladder, and the Prime Side

Thermodynamics of Arithmetic Action, Lemmas 48–65

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Grok Bot is an AI assistant; see the Statement of Intellectual Honesty.

Volume IV, first of two parts (companion: Volume IVb, Lemmas 66–85)
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Abstract

This is Volume IVa, the first of the two parts of Volume IV of the programme *Thermodynamics of Arithmetic Action*. It collects the standalone notes TAA Lemmas 48–65; the companion Volume IVb contains Lemmas 66–85. The two parts form one volume and are split only because of the archive's 5 MB file limit. Volume I (ai.viXra:2610.0045v2) recast the Riemann hypothesis (RH) as the absence of negative-norm ("ghost") directions for the Weil form; Volume II (ai.viXra:2610.0054) measured that obstruction; Volume III (ai.viXra:2610.0052) asked which structure could supply the positivity. Volume IVa tests three families of candidates. *Polarizations as vacua and complex structures* (Chs. 48–54): Kramers–Kronig causality and the Bode defect, the Betti–de Rham comparison, spinor vacua and Krein collisions, the Hardy-space vacuum and its ghost charge, and the complex structure that Frobenius chooses for itself. *Saddles and positivity ladders* (Chs. 55–59, 63): separatrices and on-line saddles of $|\zeta|$ and ξ', the Weil positivity ladder, the derivative tower of ξ, and ghosts as dents in a flat cone metric. *The prime side* (Chs. 60–62, 64–65): class numbers, the smooth/analytic visibility threshold, helicity and the Möbius function, local versus global positivity, and the field with one element. Certified computer-assisted results concern the Davenport–Heilbronn function (a zero-free box and the lower bound 0.230607 for its de Bruijn–Newman-type constant, Ch. 56) and truncated Weil forms (eigenvalue bounds, and certified indefiniteness of the Davenport–Heilbronn form for $A \geq 3.43$, Ch. 57). What we claim across both parts, and where, is stated precisely at the start. **RH is not proved.** No chapter supplies the missing positivity for ζ; most identify why a candidate cannot.

Statement of Intellectual Honesty

The Riemann Hypothesis is NOT proved in this volume (Volume IVa or Volume IVb).
Nothing here implies it, and no chapter reduces it to an easier statement.
Neither GRH nor the Birch–Swinnerton-Dyer conjecture is proved here.

1. No claim of proof. No statement in this volume is, or should be quoted as, a proof of RH, or as progress towards one beyond the modest items listed in "What this volume does and does not advance" below. Every RH-equivalent condition in the notes is labelled [Restatement]. Results about the Davenport–Heilbronn function f_{DH}, about function fields, or about codes are statements about *those* objects. They do not transfer to ζ unless a chapter says how, and the transfer is then labelled.

2. Authorship and AI assistance. The questions, hunches and direction of every chapter are J. W. McGreevy's, and each note records the request it answers (tags such as [t352u]). The written notes are a different matter. Their wording, proofs, literature checks and all computations (scripts in Python/mpmath, PARI/GP, FLINT/Arb) were produced by an AI assistant (Grok Bot) at his direction. That is why every note carries the byline "J. W. McGreevy; formalization by Grok Bot". The material has not been refereed or checked by an independent human expert. Read it as a research record, not as vetted mathematics.

3. Pasted texts were critiqued, not trusted. Many chapters respond to text pasted into the working conversation: excerpts from other AI systems, drafts, or the assistant's own earlier chat replies. These were treated as *claims to check*, never as sources. Each note ends with a "Corrections" section that records which claims were right, which were wrong, and how they were fixed (consolidated in Appendix A). Where a pasted text could not be verified, the note says so.

4. Numerics. Computations are [Numerical] unless they are explicitly labelled computer-assisted (Arb ball arithmetic, or exact arithmetic with certified root isolation, [EXACT-COMP]). The certified items are listed in §A below. Floating-point and multiprecision results are reported with their stated tolerances. They are not proofs, and checks over finite ranges say nothing beyond those ranges.

5. References. The notes distinguish [CITED] (checked at the source), [CITED-sec] (checked through a named secondary source) and [BIB] (bibliographic record verified, content not re-read). Each note's references are recorded in the project's per-lemma reference logs (`lit/lemmaNN/verified_refs.md`). A source that could not be verified was dropped or explicitly marked.

6. Facsimile chapters. Chapters 48–65 (this part) and Chapters 66–85 (Volume IVb) are the compiled notes themselves, reproduced page by page (see "Status of this volume"). A statement "Thm. $s.n$ of Ch. k" is Theorem $s.n$ of TAA Lemma k. Each page carries the note's own page number and the volume page number in its footer.

Claim labels

Every claim in the notes carries a label. The local tag variants map to seven volume classes:

Volume class	Meaning	Tags used in the chapters
Theorem	Proved in the note, or a published theorem used as stated.	[THEOREM]/[THM]; [CITED], [CITED-sec], [BIB] for cited results; [EXACT-COMP] and "[THM, computer-assisted]" for certified finite computations; [TAA-L] for a result of an earlier TAA lemma
Restatement	Equivalent to RH, GRH or a known statement; no new content.	[RESTATEMENT]
Analogy	A structural parallel; proves nothing.	[ANALOGY], [METAPHOR]
Conjecture	A stated expectation, with its evidence.	[CONJECTURE]/[CONJ]
Open	A question or gap explicitly left open.	[OPEN]; "Questions for J.W."
Numerical	Floating-point or multiprecision computation; not a proof.	[NUMERICAL]/[NUM]; [CROSS-CHECK]
Heuristic	A plausibility argument or estimate, weaker than a conjecture.	[HEURISTIC]

What this volume does and does not advance

Scope of this statement. This statement covers the whole of Volume IV, both parts: Volume IVa (Chapters 48–65) and Volume IVb (Chapters 66–85). It is printed identically in both parts. Locations in this part (IVa) give the note page and the volume page of this part; locations in the companion part are marked "in Volume IVb".

Summary. This volume does not advance RH for ζ. Its contributions are of four modest kinds. (A) A few certified computations, mostly about the Davenport–Heilbronn function, a fake with a functional equation but no Euler product. (B) Elementary theorems that we did not find stated in the literature we checked. The searches were not exhaustive, so each is "new to our knowledge" at most, and none bears directly on RH. (C) Re-packagings of known results, credited to their authors. (D) Negative results: proofs or computations showing that a proposed source of positivity cannot work. Items are labelled with the classes above. Locations give the chapter, the statement number inside the note, the note page and, for this part, the volume page.

A. Certified computer-assisted results [Theorem, computer-assisted]

A1. *No zeros of f_{DH} with $1 \le \sigma \le 1.6$, $0 \le t \le 1000$* (Arb ball arithmetic, winding number 0). Ch. 56, Thm. 7.5 (note p. 22, vol. p. 143). Davenport–Heilbronn proved that f_{DH} has infinitely many zeros with $\sigma > 1$; the certificate only says none lie in this box.

A2. *Lower bound $\Lambda_{\mathrm{DH}} \ge 0.230607$* for the de Bruijn–Newman-type constant of f_{DH}: H_t^{DH} has a non-real zero at $t = 0.230607$ (certified Laguerre sign). Ch. 56, Thm. 7.6 (note p. 23, vol. p. 144); the upper bound 1.6025 (de Bruijn's strip theorem, [BIB]) is in Prop. 7.4. The analogous statement for ζ is the Rodgers–Tao theorem $\Lambda \ge 0$, which we only cite.

A3. *Certified upper bounds for the smallest eigenvalue $\lambda^*(e^A)$ of the truncated Weil form* at $A = 1.8, 2, 2.2, 2.4, 3.5$, giving two-sided enclosures when combined with the Volume II lower bounds. Also certified: *the f_{DH} Weil form is indefinite for all $A \ge 3.43$*, and a *no-slack theorem* for a single local term. Ch. 57, Thms. 3.1, 4.1 (note p. 4, vol. p. 155); Thm. 5.1 on p. 6 (vol. p. 157). These are finite-window facts. They neither prove nor test RH.

A4. *Extremal weight enumerators of self-dual codes satisfy Duursma's RH*, with $\kappa = 0$ certified, for Types I, III, IV with $n \le 200$ and Type II with $n \le 1000$ [EXACT-COMP]. Also: exact RH sets on the Type II lines $n = 24, 32, 40$. Ch. 77, Answer (2) (note p. 1, in Volume IVb); Thm. 6.3 on p. 7 (in Volume IVb). Duursma's RH for codes is not RH for ζ.

B. Elementary theorems, new to our knowledge [Theorem]

Each is a short argument from standard facts. We claim the formulation, not depth, and we may have missed prior statements.

B1. *Multiplier no-go.* On a single copy of L^2 (scaling of multiplicity one), every positivity built from scaling-commuting operators is one of four things. It is blind (positive for fakes too), a restatement of Weil's criterion, useless, or dependent on knowing the zeros. Ch. 80, Thm. 3.2 (note p. 2, in Volume IVb). The commutant statement it rests on (Prop. 3.1: such operators are Fourier multipliers) is classical. The consequence that a candidate polarization needs a

space on which scaling has multiplicity is drawn in Ch. 78, §7(viii) (note p. 16, in Volume IVb) and Ch. 80, §§3–4.

B2. *Visibility criterion and strip visibility law.* A zero is visible to the scaling action on a weighted space if and only if an explicit weight integral converges. On Meyer's spaces S_I, the spectrum consists exactly of the zeros with real part in I, each one a Jordan block of size equal to its order, given Meyer's theorem. Ch. 61, Prop. 2.1 (note p. 3, vol. p. 219); Thm. 3.3 on p. 4 (vol. p. 220). The RH reading ("no ghosts $\iff$ nothing lost between the analytic and smooth classes") is [Restatement].

B3. *Zero sums in modular and gravitational partition functions.* Three expansions: the reflected half of the Maloney–Witten–Keller spin-0 density at high temperature equals an anomaly term plus a sum over ζ zeros (Ch. 72, Thm. 3.3 (note p. 6, in Volume IVb)); its Bianchi analogue (Ch. 73, Thm. 6.1 (note p. 10, in Volume IVb)); and a tropical limit and smoothed height count for $E(\tau, s)$ (Ch. 74, Thms. 5.2, 6.1 (note p. 9, in Volume IVb)). Each says where the zeros enter. The negativity of the density is shown to be RH-blind; RH enters only as a decay rate ([Restatement]).

B4. *Ghost charge and Krein-vacuum statements* (presentation-level, labelled as such in the notes). The ghost charge of the Hardy-space vacuum (ch $= -\#$ghosts) is in Ch. 53, Thm. 4.3 (note p. 5, vol. p. 76). The emergent complex structure J_F and the drift detector are in Ch. 54, Thm. 6.1 (note p. 11, vol. p. 93), with their prior art (Segal, Weinless, Kay; the matrix sign function) stated in the note. The exact count of ξ' zeros at admissible heights and the circle identity $\kappa_{\mathrm{FF}} = \#$ghosts over $\mathbb{F}_q$ are in Ch. 56, Thm. 4.2 (note p. 7, vol. p. 128) and Thm. 6.2 on p. 15 (vol. p. 136).

B5. *Forced zero at the Fricke fixed point.* For a newform f of level N with root number $\varepsilon = -1$, $f(i/\sqrt{N}) = 0$. Ch. 85, Thm. 10.2 (note p. 13, in Volume IVb). This is an immediate consequence of $f|W_N = wf$, $\varepsilon = -w$, and is almost certainly folklore.

C. Known results re-proved or re-packaged [Theorem / Restatement; exposition]

C1. *κ is the negative index of the Rosati/Weil trace form.* For a squarefree Weil polynomial, $\mathrm{Tr}(xx^\dagger)$ on $\mathbb{Q}[F]$, with $F^\dagger = qF^{-1}$, is positive definite iff all roots have absolute value $\sqrt{q}$. Its number of negative directions equals κ, the number of off-circle reciprocal pairs. Ch. 84, Thm. 3.2 (note p. 3, in Volume IVb). The finite Toeplitz form of the zero power sums has the same Krein inertia: Ch. 77, Thm. 4.1 (note p. 4, in Volume IVb), which Ch. 77 itself labels a restatement "of the kind Li's criterion is". These are Sylvester's law of inertia and Hermite's signature count for trace forms. The link to RH for curves is Weil's proof via Rosati positivity (Mumford, §21, [CITED-sec]). *Exposition, no novelty.* The accompanying computation is [Numerical] and exact in integer arithmetic. It is an exhaustive scan over $q \in \{2, 3, 5, 7, 11\}$ of all 13 135 squarefree genus-2 Weil-shaped polynomials, of which 785 satisfy RH, with 0 mismatches between "positive definite" and "RH" and $\#\mathrm{neg} = \kappa$ in every case (Ch. 84, p. 4, in Volume IVb). It confirms a proved theorem; it is a consistency check, not new evidence. The same κ shows up on the zero side. In a 25-Gaussian Weil Gram matrix there is one negative eigenvalue (-1.078) for f_{DH}, and none for ζ or $L(s, \chi)$ (Ch. 81, §5 (note p. 4, in Volume IVb); [Numerical]).

C2. *Every ghost is visible at finite Connes cutoff.* If a self-dual entire Λ has a zero off the line, the finite-support Weil form Q_L fails to be positive semidefinite for all $L > L^*$, with $L^* < \infty$. Ch. 82, Thm. 5.1 (note p. 3, in Volume IVb). The proof is Weil's criterion plus Paley–Wiener, so the statement is a *finite-support restatement of Weil's criterion*, in the spirit of Connes's cutoff (Connes 1999, [CITED]). What is new is numerical. For f_{DH}, Q_L becomes indefinite at $L^* \approx 3.73$ (cutoff $\Lambda^* \approx 6.4$, primes $n \le 41$) on the frequency band $[70, 102]$, with

$\lambda_{\min} = -1.340$ at $L = 4$. For ζ, $\lambda_{\min} \geq -9.7 \cdot 10^{-10}$ (the numerical floor) on the same band (Fig. 1, p. 3; [Numerical], not certified).

C3. *Epstein zeta functions have off-line zeros* (Davenport–Heilbronn 1936; Potter–Titchmarsh 1935; [BIB] in Ch. 83). New in Ch. 83 are a count for $x^2 + 5y^2$ and a channel theorem. The count: $N(200) = 253$ zeros, of which 183 are on the line and 70 off it, the first at $0.93297 + 15.66825i$ (Ch. 83, §5 (note p. 4, in Volume IVb); [Numerical]). The theorem: the closed-geodesic side of the Selberg trace formula for $\mathrm{PSL}_2(\mathcal{O}_K)$ sees only the Euler-product channels $\det \Phi = \pm \prod_\chi \Lambda(s,\chi)/\Lambda(1+s,\chi)$, so Epstein ghosts contribute nothing to it (Ch. 83, Thm. 5.1 (note p. 4, in Volume IVb); "channel blindness", an elementary consequence of the cited trace formula). The *transport gap* says that the elliptic terms at $\tau = i$ are zero-independent and cannot carry Rosati positivity at the elliptic point to the zeros (Ch. 83, Thm. 6.4 (note p. 6, in Volume IVb)). The Bianchi link remains [Open].

C4. *Mordell–Weil lattices, Shioda–Tate, Tate's conjecture and BSD over function fields* are cited (Shioda; Schütt–Shioda; Tate; Artin–Swinnerton-Dyer), as are modularity (Wiles; Taylor–Wiles; Breuil–Conrad–Diamond–Taylor) and Sato–Tate (Barnet-Lamb–Geraghty–Harris–Taylor and coauthors); see Ch. 85, §§2, 9, 10. Exposition, with explicit examples computed: a rational elliptic surface over $\mathbb{F}_5$ with Mordell–Weil lattice $A_3 \oplus A_1$ and a K3 family over $\mathbb{F}_7$ (§3); EDSAC-type products $\prod_{p \leq x} N_p/p$ for ranks 0–3 (§4). The *split-invariance detector*, i.e. independence of the Fricke-split Mellin integral from the split point (the classical device of Riemann's second proof), detects fakes with the wrong root number or altered coefficients. But it is *RH-blind*: f_{DH} passes it to 10^{-41} despite its off-line zero $0.808517\ldots + 85.699348\ldots i$, while the Ch. 82 cutoff detector rejects f_{DH}. See Ch. 85, §10 (note p. 14, in Volume IVb); [Theorem]/[Numerical].

C5. *Riemann's bilinear relations for the $(\chi, \bar{\chi})$ sector.* The first relation holds unconditionally (character orthogonality); the second is $\mathrm{GRH}(\chi)$, a restatement of Weil's criterion. Ch. 81, Thm. 3.1 (note p. 2, in Volume IVb).

C6. *Earlier chapters* state their own novelty assessments, and we adopt them. Ch. 48: every RH-equivalence is a restatement (Lagarias, Burnol, BSY), the remaining theorems are presentation-level and probably folklore, and the new item is numerical (an argument-principle count of all 904 f_{DH} zeros to height 1000: 28 ghosts with $\beta > \frac{1}{2}$, i.e. 56 off-line zeros). Chs. 49 and 50: classical results re-derived, with "no new mathematics". Ch. 55: the separatrix picture is correct and classical. Ch. 59: Laguerre–Pólya positivity is zero location in disguise. Ch. 63: Michałowski's PF_5 witness is reproduced (credited), and the cone-metric dictionary is standard. See each note's "What is new" or "Prior art" section.

D. Negative results (what cannot supply the positivity)

These are the volume's main content: precise reasons why a candidate source of positivity fails. They are labelled [Theorem] where proved, otherwise [Numerical].

D1. *Single-copy operators are multipliers, so multiplicity is needed* (B1 above; Ch. 78, §7(viii) (note p. 16, in Volume IVb); Ch. 80, Thm. 3.2 (note p. 2, in Volume IVb)).

D2. *Local positivity at every prime does not give global positivity.* One positive-definite binary form per good prime exists for every $E/\mathbb{Q}$ (Ch. 64, Prop. 6.5 (note p. 12, vol. p. 278)). In Ch. 85, §9, the normalized $a_{p^k}/p^{k/2}$ are spin-$k/2$ characters $U_k(\cos\theta_p)$ (Hecke relations; 1088 exact checks), and Sato–Tate is cited. "Local unitarity plus Sato–Tate" controls equidistribution, a density statement with no rate, whereas RH for $L(E,s)$ is a rate, $\sum_{p \leq x} 2\cos\theta_p \log p = O(x^{1/2+\varepsilon})$. So there is no route from local to global positivity this way (Ch. 85, §9 (note p. 12, in Volume IVb); [Theorem] for the identities, [Heuristic] for the moral).

D3. *A positivity counter and an algebraicity counter catch different ghosts.* In a rational elliptic surface over $\mathbb{F}_5$, a fake with an on-circle Frobenius eigenvalue that is not q times a root of unity has $\kappa = 0$ and a positive trace form. It is caught only by the algebraic count $\bar{\rho} = b_2$, which holds for rational surfaces (Ch. 85, §3.3, fake G3 (note p. 5, in Volume IVb); [Numerical], reading [Theorem]). For ζ there is no Néron–Severi group, so only the first kind of counter is conceivable.

D4. *Modularity sees the functional equation, Weil positivity sees the zeros* (C4 above; Ch. 85, §10 (note p. 14, in Volume IVb)).

D5. *Other no-go results*, each with a proof or computation in its note. Laguerre–Pólya positivity is of Krein type, not a polarization (Ch. 59, verdict). Small-support and archimedean positivity are blind (Ch. 82, §5(i)–(ii) (note p. 3, in Volume IVb)). Real-s positivity of the Mayer transfer operator cannot detect off-line zeros (Ch. 75, Prop. 7.2 (note p. 11, in Volume IVb)). No positivity in Ch. 76 sees off-line zeros (Ch. 76, Prop. 8.1 (note p. 12, in Volume IVb)). The MWK negativity is RH-blind (Ch. 72). Passivity and Kramers–Kronig causality of the modular scattering matrix are RH-blind (Ch. 78).

What remains open

The obstruction isolated in Chapters 84–85 (in Volume IVb) has three parts [Open]:

(1) *No rational lattice.* There is no finite-dimensional $\mathbb{Q}$-algebra $\mathbb{Q}[F]$ for ζ that contains a Frobenius (Ch. 84, §6, [Theorem]). An infinite-dimensional substitute would need a positive state, and that positivity is RH.

(2) *No ample class.* There is no known $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ carrying an ample class whose Hodge index theorem would give the sign. (Arithmetic Hodge index theorems, Faltings–Hriljac, give positivity of heights, not RH; Connes–Consani's scaling site is a candidate setting, not a construction.)

(3) *Zero global margin despite full local margin.* Every local positivity holds (Hasse at each prime, Sato–Tate), but no global margin is available.

Refining this, compare where the three kinds of data have a global algebraic framework over $\mathbb{Q}$:

- The Fourier coefficients of a newform do: the Hecke algebra acts on $X_0(N)$, a curve over $\mathbb{Q}$.
- The local zeros, i.e. the roots of the local factor at each prime p, have one at each prime: Frobenius acting on $X_0(N)$ mod p.
- The global zeros of $L(s)$ have none.

What is missing is a *global operator whose spectrum is the set of zeros, together with a positive form coming from a global ample class*, as Frobenius and the Hodge index theorem on $C \times C$ provide in Weil's proof. Nothing in this volume supplies it. The "Questions for J.W." sections of this part's chapters are collected in Appendix B; those of the other chapters are in the corresponding appendix of Volume IVb.

Status of this volume

Two parts. Volume IV is published as two files: Volume IVa (Chapters 48–65) and Volume IVb (Chapters 66–85). The split is made only because the archive accepts files of at most 5 MB; IVa and IVb together are one Volume IV. The Statement of Intellectual Honesty, the claim labels, "What this volume does and does not advance", "What remains open" and this status section cover the whole volume and appear in both parts. The introduction and the appendices cover only the chapters of the part they are in. Volume pages are counted separately in each part.

Assembly. This volume was assembled on 24 October 2026 from the latest compiled versions of TAA Lemmas 48–85. These are Lemma 54 v2, Lemma 56 v3, Lemma 81 with its two addenda (Möbius strip and Verschiebung; Cassini ovals and the de Bruijn–Newman flow), and Lemma 85 with Sections G–I. Chapter k is Lemma k.

Facsimile inclusion. Unlike Volumes II and III, the chapters are not re-typeset. Each note's compiled pages are reproduced at 93% scale, with its own title, numbering, figures and reference list. The reason is technical. Lemmas 60–85 are multi-file builds with conflicting local macro sets, and a faithful source conversion would have required editing them. Consequences:

- Statement numbers restart in each chapter.
- Each chapter keeps its own bibliography; there is no consolidated bibliography.
- Cross-references between chapters are by lemma number.

A re-typeset edition can be produced later without changing any content.

Raster figures. To keep the file size within the archive's limit, some heavy vector figures (dense scatter, contour and heat-map plots) are reproduced as raster images at 150 dpi, at the same size and position. All text and equations, and all other figures, remain vector. The vector originals are unchanged in the standalone notes.

Lemma 50. Volume III included only an eight-item condensation of Lemma 50 (an editorial section of its Introduction). The full note was not in Volume III: proofs, the sections on Riemann's θ and the metaplectic cover, Shale–Stinespring and the Krein–Fock construction, the half Tate twist and Sato–Tate, what $\operatorname{Spec}\mathbb{Z}$ would need, corrections and questions. It is therefore included in full as Chapter 50. The overlap with Volume III is that condensation only.

Lemma 63. Lemma 63 ("Ghosts as dents in a flat cone metric, the right-angle cross, zero-by-zero Weil positivity, the kernel-condition hierarchy, and DH Jensen polynomials to degree $2 \cdot 10^4$") was completed on 22 October 2026. Its compiled PDF remained in the build directory and was never copied to the project's top level, which is why it was briefly thought missing. It is included unchanged as Chapter 63.

Not included. The project's working notes (`STATE.md`, `NOTATION.md`), scripts and data are not part of the volume. Each note names its scripts and output files.

Place in the series

Volume	Title	Contents and status
I	Arithmetic Unitarity I: Where the Riemann Hypothesis Lives in the Scaling Flow	Lemmas 1–13; ai.viXra:2610.0045v2
II	Arithmetic Unitarity II: The Light Cone of the Weil Form	Lemmas 14–29; ai.viXra:2610.0054
III	Arithmetic Unitarity III: In Search of Positivity — G5, Polarization, and Hilbert versus Krein	Lemmas 30–47 (and a condensation of Lemma 50); ai.viXra:2610.0052
IVa	Arithmetic Unitarity IVa: Counting Ghosts — Spinor Vacua, Saddles, the Weil Positivity Ladder, and the Prime Side	Lemmas 48–65; this part
IVb	Arithmetic Unitarity IVb: Counting Ghosts — Eisenstein Anomalies, Hyperbolic Geometry, the Ample-Class Obstruction, and Mordell–Weil Lattices	Lemmas 66–85; companion part of Volume IV

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Introduction

The thread

Volume III ended with a slogan proved in several forms: *a reflection gives the functional equation; positivity needs a polarization.* Algebra gives Krein unitarity, an indefinite form with κ negative directions, one per pair of ghosts (off-line zeros). Hilbert unitarity ($\kappa = 0$) needs a positive polarization, which for curves over $\mathbb{F}_q$ comes from the Hodge index theorem on $C \times C$ through Rosati positivity. For ζ no such structure is known.

Volume IV pursues this along six lines, in roughly the order the notes were written. The first three are in Volume IVa (Chapters 48–65), the last three in Volume IVb (Chapters 66–85):

- polarizations as vacua and complex structures;
- saddles and zero counting;
- the prime side and $\mathbb{F}_1$;
- modular forms and Eisenstein anomalies;
- hyperbolic geometry, gravity, Farey fractions and codes;
- the final cluster on causality, the multiplier no-go, the Hodge–Riemann form, Connes's cutoff, Albert's classification and Mordell–Weil lattices.

Two diagnostics run throughout. One is the ghost counter κ; the other is the Davenport–Heilbronn function f_{DH}, which has a functional equation of Riemann type but no Euler product, and has off-line zeros (the first at $0.808517\ldots + 85.699348\ldots i$). A structure that f_{DH} also satisfies cannot prove RH. Most chapters apply this test explicitly.

Standing conventions

$s = \sigma + it$; $\rho = \beta + i\gamma$ is a nontrivial zero, a *ghost* if $\beta \neq \frac{1}{2}$; $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. The Weil form is Q (or W, W_A, Q_L for truncations, as defined per chapter). Over $\mathbb{F}_q$, F is Frobenius, $V = qF^{-1}$ the Verschiebung, and κ the number of off-circle reciprocal pairs of Frobenius eigenvalues. Objects such as h^1 (a motive, a categorical object), H^1 (a vector space, its realization), Hodge–Riemann relations (inequalities on $H^{1,0}$) and the Hodge index theorem (signature of the intersection form on Néron–Severi of a surface) are restated in each chapter that uses them. Appendix C lists the conventions that differ between chapters.

Chapter by chapter (this part)

Polarizations, vacua and complex structures (Chs. 48–54).

48: Kramers–Kronig, causality and passivity: inner–outer factorization of ζ, the Bode defect, all-pass lines. Every RH-equivalence is a restatement; a full count of f_{DH} zeros to height 1000.

49: The Betti–de Rham comparison: periods, Ramanujan's system as Gauss–Manin, E_2^* as the polarization term, the unit-root splitting.

50: Spinors and theta characteristics: the polarization as a positive fermionic vacuum (full note; only a condensation was in Vol. III).

51: Tracking that vacuum through a ghost birth: Krein collisions, the Siegel boundary, genus 2 through Spin(5).

52: A numerical probe of the Connes–Consani–Moscovici metaplectic/prolate model: no scaling-invariant positive vacuum, for structural reasons.
53: Bosons, fermions and the Hardy-space vacuum: Segal–Wilson Grassmannian, Beurling cyclicity, the ghost charge.
54: Emergent complex structure (v2): Frobenius chooses its own J; positive energy; a Siegel-orbit ghost detector.

Saddles, counts and positivity ladders (Chs. 55–59, 63).

55: Separatrices of $|\zeta|$: Cassinian curves, saddle geometry, Speiser's criterion (classical picture, no novelty claimed).
56: Orientation of on-line saddles (v3): an exact count for ξ', Cohn's theorem, the saddle form of RH, the whole-line f_{DH} heat flow, and two certified results.
57: The Weil positivity ladder: certified eigenvalue bounds, f_{DH} indefiniteness for $A \geq 3.43$, the no-slack theorem.
58: Inventory and synthesis: what the restatements are circling; a "via negativa" checklist for the missing object.
59: The derivative tower of ξ: Laguerre–Pólya positivity is localized zero location of Krein type, not a polarization.
63: Ghosts as dents in a flat cone metric, the right-angle cross at saddles, zero-by-zero Weil positivity, the kernel hierarchy, and f_{DH} Jensen polynomials to degree $2 \cdot 10^4$.

The prime side, class numbers and $\mathbb{F}_1$ (Chs. 60–62, 64–65).

60: Ghosts, class numbers and principalization: Deuring–Heilbronn, Epstein zeta functions as polarized lattices, Iwasawa towers.
61: The smooth/analytic threshold: Deninger's emission versus Connes' absorption; the visibility criterion and strip visibility law.
62: Helicity: the polarization as circular polarization; the Wick loop; signs, supertraces and the Möbius function.
64: Prime-side anatomy of Weil positivity: collective positivity, structure versus size, and local binary forms versus global positivity.
65: The field with one element and the single Frobenius.

Editorial remarks

- Chapters were not edited. Where a later chapter corrects an earlier one, the correction is recorded in the later chapter and listed in Appendix A.
- Chapter 59 states its own place in the series. J.W. asked for it to go into Volume IV with figures, and its figures are inside the note.
- The order of chapters is numerical. Chapter 63 follows Chapter 62 although it continues Chapters 55–59.

TAA LEMMA 48: KRAMERS–KRONIG, CAUSALITY, PASSIVITY AND RH. INNER–OUTER FACTORIZATION OF ζ, MINIMUM PHASE, THE BODE DEFECT, ALL-PASS LINES, AND THE SKIN-EFFECT DICTIONARY

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. **Statement of intellectual honesty.** RH is not proved here, and nothing below brings it closer except where explicitly said. Tags: [THM] proved here (elementary, proof given); [CITED] checked at the source (`lit/lemma48/verified_refs.md`); [BIB] bibliographic record verified (Crossref), content standard; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma48_*.py`, data `num/l48/`); [CONJ] conjecture; [OPEN] open. Lemmas 7–9, 25, 30, 33, 34, 36, 37, 40, 44–47 are cited by number and not redone.

Question (J.W., [t179u]). A pasted text on Kramers–Kronig (KK) relations, causality, passivity, Herglotz functions, the Bode gain–phase relation, all-pass/Blaschke/delay factors, and "the magnitude must drop to zero" asked whether this is the structure behind the 2π orbit of $(\frac{1}{2}, \frac{1}{2})$, and how it relates to RH.

Answer in brief. (1) *What TAA already has* (§1). Lagarias' positivity criterion, the Pick/Herglotz form, Foster reactance, the free KK relations, the de Branges threshold (Lemma 33); the KK/Fano sum rule and 2π integrality of the ghost deficit (Lemma 36); Nehari passivity and the exponential line (Lemma 37); the Balazard–Saias–Yor (BSY) and Burnol identities (Lemma 30). These are cited and not redone. (2) *Main theorem: ζ is "minimum phase" iff RH* (§3, Thm. 3.1, [THM]; the equivalence is [RESTATEMENT]). $F_\zeta(s) = \zeta(s)(s-1)/s$ lies in the Smirnov class N^+ of $\mathbb{H}_{1/2} = \{\Re s > \frac{1}{2}\}$, and its canonical factorization is exactly $F_\zeta = B_\zeta \cdot O_\zeta$. Here B_ζ is the Blaschke product over the off-line zeros (the "ghosts"), O_ζ is outer, and *there is no singular inner factor and no delay factor*. The pole of ζ is an *anti*-Blaschke factor $s/(s-1)$. The same holds for the Davenport–Heilbronn (DH) function. RH $\iff$ $B_\zeta \equiv 1$ $\iff$ F_ζ outer. BSY and Burnol are the single values $-\log|B_\zeta(1)|$ and $|B_\zeta(1)|$. (3) *Bode test, done correctly* (§4, §7). The magnitude on the line determines O_ζ. A ghost is *not* "invisible in magnitude and visible only in phase": it is visible in magnitude plus *one* interior value (the gain defect $\mathfrak{G}$), or in the phase (the phase defect $\mathfrak{F}$). [NUM]: for ζ, $|\mathfrak{G}| \le 8.3 \cdot 10^{-6}$ and $|\mathfrak{F}| \le 3.8 \cdot 10^{-6}$ at 29 interior points with $\tau \le 300.7$, and $\mathfrak{G}(1) = -8 \cdot 10^{-8}$ (BSY), with the line truncated at $|t| \le 1000$. For DH the measured $\mathfrak{G}, \mathfrak{F}$ match the Blaschke prediction from its 28 off-line zeros below height 1000 to $\le 1.1 \cdot 10^{-5}$, e.g. $\mathfrak{G} = 0.2952$ near the ghost $0.8085 + 85.6993i$. The count of all DH zeros, $904 = 848 + 2 \cdot 28$, was checked by the argument principle. (4) *Corrections to the pasted text* (§5). Blaschke factors and delays are causal and analytic. Passive responses need not vanish at infinity. "Flat magnitude $\Rightarrow$ zero phase" holds only for minimum-phase (outer) functions. The Hilbert transform is the complex structure J; positivity is a separate condition. (5) *Finite fields* (§6). The Bode/Jensen identity holds exactly, and the gain defect is log of the Mahler measure. The FE-symmetric factor $u^g = q^{-g(s-\frac{1}{2})}$ is a *true delay*, of length $g \log q$. Its ζ-analogue $e^{2i\theta}$ (Riemann–Siegel θ) is *not* a delay, and not inner or even of bounded type (Prop. 4.2). (6) *All-pass lines* (§8). Suzuki's symbols are pure Blaschke products (no delay, no singular part) whenever E_ω is HB; otherwise the ghosts give an anti-stable Blaschke denominator. The exponential line is a purely singular inner function under RH and is not of bounded type otherwise. Blaschke zeros coalesce into singular atoms as $\omega \to 0$. The group delay of the depth-h symbol is $2\Re\frac{\xi'}{\xi}(\frac{1}{2} + h + it)$, so Lagarias' criterion is "positive group delay". (7) *Skin-effect dictionary* (§9) folds in Thm. 3.1 of `notes/skin_effect_prior.md`, with an erratum on ai.viXra:2606.0029 and 2609.0007 as recorded there. *New versus restatement* (§11): every RH-equivalence here is [RESTATEMENT]. The new items are presentation-level theorems (the clean inner–outer statement with "no singular/no delay", the pole as anti-Blaschke, the θ non-bounded-type fact, the singular-inner form of the exponential line) and the numerics.

Date: October 22, 2026.

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1. Conventions, and what TAA already has

$s = \sigma + it$; $\mathbb{H}_{1/2} = \{\Re s > \frac{1}{2}\}$; $w = s - \frac{1}{2}$ (so $\mathbb{H}_{1/2}$ is the right half-plane $\Re w > 0$). In Lemma 33/34/37 coordinates, $s = \frac{1}{2} - iz$. Zeros $\rho = \beta + i\gamma$ with multiplicity m_ρ, and $\delta_\rho = \beta - \frac{1}{2}$. The *mirror* of ρ across the line is $\rho^* = 1 - \bar{\rho} = 1 - \beta + i\gamma$ (Lemma 30's $\sharp$). A *ghost* is a zero with $\beta > \frac{1}{2}$. $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. Hardy-space terms (Smirnov class N^+, inner, outer, Blaschke, singular inner, mean type) are as in Garnett [8] and Koosis [11] ([BIB]), transplanted to $\mathbb{H}_{1/2}$. The Poisson kernel of $\mathbb{H}_{1/2}$ at $s_0 = \frac{1}{2} + a + i\tau$ is $P_{s_0}(t) = \frac{1}{\pi}\frac{a}{a^2+(t-\tau)^2}$, and $\mathcal{P}[g](s_0) = \int_{\mathbb{R}} g(t)P_{s_0}(t)\,dt$.

Overlap table (checked by reading the cited lemmas; nothing in this table is redone):

Lemma	what it already contains
7–9	Pavlov–Faddeev scattering: $\zeta(1+2ik)/\zeta(1-2ik)$ as a scattering matrix; Blaschke-type factors in that matrix.
25, 34	Distortionless line; Suzuki's canonical systems; $\sup_t \|\mathcal{K}[t]\| < 1$ as the official G5 target; the escape count κ (Thm. 6.2).
30	BSY identity $\frac{1}{2\pi}\int \log\|\zeta(\frac{1}{2}+it)\|\frac{dt}{\|s\|^2} = \sum_{\beta>1/2}\log\|\rho/(1-\rho)\|$; Burnol's $\|P\mathbf{1}\| = \prod\|(1-\rho)/\rho\|$; the mirror as $\sharp$.
33	Lagarias [12]: RH $\iff$ $\Re\frac{\xi'}{\xi} > 0$ on $\mathbb{H}_{1/2}$ (Thm. 4.1); sign of the printed g (anti-Pick); Pick function $\mathfrak{m}$; Foster form, boundary absorption spectrum, "RH $\iff$ Z is the Herglotz completion of its own boundary data" (Thm. 5.1); de Branges E_h threshold (Thm. 6.2); Krein (Prop. 6.3); KK for ξ'/ξ "for free".
36	Absorption–phase sum rule (Thm. 4.3), definite-sign 2π deficit (Cor. 4.4), integrality (Thm. 7.1), the "+1" in $N(T)$ from $s(s-1)$ (Prop. 7.4), exact escape (Thm. 7.9).
37	Nehari: passive $\iff$ Schur (Thm. 2.2, Cor. 2.3); depth theorem and G5 $\iff$ RH (Thm. 3.2, Cor. 3.3); Sondow–Dumitrescu monotonicity [19] as step-2ω passivity; the exponential line $\Theta_c = e^{-2c\xi'/\xi}$ (Thm. 4.1); the strict pointwise form at Θ_c is open (Prop. 4.4).
39 §7(c), 40	J = Hilbert transform; Pick kernel = Krein Gram matrix.
44–47	Cavities; rapidities η; Krein index k; reflection $\Rightarrow$ FE ("stage, not actors").

So the Herglotz, Lagarias, Foster, KK-sum-rule, Nehari/Schur and de Branges material is already in TAA. This lemma adds the *multiplicative* side: the factorization of ζ itself into Blaschke $\times$ outer, the Bode gain/phase defects, the classification of the all-pass symbols, the archimedean phase θ, the finite-field Bode test, and the corrections.

2. Verification of the chat claims about Lagarias, Herglotz and de Branges

(V1) *Lagarias, Acta Arith. 89 (1999) 217–234* [CITED]: verified (Crossref; statement read in `lit/lemma33/la99.txt`). Thm. 1.1 there: RH $\iff$ $\Re\frac{\xi'}{\xi}(s) > 0$ for $\Re s > \frac{1}{2}$, stated more generally for admissible zero sets; Hinkkanen [9] had it earlier. The correction [13] (Acta Arith. 116 (2005) 293–294) repairs a lemma and a sign, and the theorems stand. *Correction to the chat*: the function Lagarias prints, $g(\tau) = i\frac{\xi'}{\xi}(\frac{1}{2}+i\tau)$, is *anti*-Pick under RH; the Pick function is $\mathfrak{m} = -g$ (Lemma 33 §4). This is a sign convention, but the chat's "g is Herglotz" is off by that sign.

(V2) *Herglotz representation* [THM] (given Lemma 33 Thm. 5.1): under RH, $\mathfrak{m}(z) = -i\frac{\xi'}{\xi}(\frac{1}{2}+iz) = \sum_\rho m_\rho \frac{1}{t_\rho - z}$ with $t_\rho = \gamma_\rho$ (symmetric summation), so in $\mathfrak{m}(z) = a + bz + \int(\frac{1}{t-z} - \frac{t}{1+t^2})\,d\mu(t)$ one has $b = 0$ and $\mu = \sum m_\rho \delta_{t_\rho}$, the zero-counting measure of the ordinates. The symmetric pairing $t_\rho \leftrightarrow -t_\rho$ makes the $\frac{t}{1+t^2}$ terms cancel, so $a = 0$.

(V3) *Weil/Bochner* [RESTATEMENT]: Weil's distribution is the Fourier transform of μ, and Weil positivity is Bochner positive-definiteness of that transform. This is Lemma 33's "one measure ν in every face"; nothing new.

(V4) *Sondow–Dumitrescu* [CITED]: the venue is *Period. Math. Hungar.* 60 (2010) 37–40, doi:10.1007/s10998-010-1037-3 (Crossref; arXiv:1005.1104 abstract read), *not* "Elem. Math. 2010" as the chat guessed. Content: $|\xi|$ is strictly increasing along horizontal half-lines in any zero-free open right half-plane, and RH $\iff$ it is increasing on $\sigma > \frac{1}{2}$. Since $\partial_\sigma \log|\xi| = \Re\frac{\xi'}{\xi}$, this is Lagarias in integrated form. Already used in Lemma 37 Remark after Cor. 3.3.

(V5) *Lagarias 2006 venue* [BIB]: "Hilbert spaces of entire functions and Dirichlet L-functions", in *Frontiers in Number Theory, Physics, and Geometry I*, Springer 2006, pp. 365–377, doi:10.1007/978-3-540-31347-2_10 (Crossref). Its content is not used here.

(V6) *de Branges / Hermite–Biehler* [RESTATEMENT]: $E_h(z) = \xi(\frac{1}{2} + h - iz)$ is HB for $h \geq \frac{1}{2}$ unconditionally ([14, Lemma 6.1], [CITED]) and for all $h > 0$ iff all zeros have $\beta \leq \frac{1}{2} + h$ (Lemma 33 Thm. 6.2). $E^{\#}/E$ is then a meromorphic inner function. Its structure (pure Blaschke, no delay) is Thm. 8.1 below.

3. The inner–outer factorization of ζ

Theorem 3.1 (canonical factorization of ζ on $\mathbb{H}_{1/2}$; [THM]; (c) is [RESTATEMENT])**.** *Let $F_\zeta(s) = \zeta(s)\,\frac{s-1}{s}$ (holomorphic on $\Re s > 0$, with $F_\zeta(1) = 1$).*

(a) $|F_\zeta(s)| \leq 1 + 2|s-1|$ on $\overline{\mathbb{H}_{1/2}}$, so $F_\zeta/(s+\frac{1}{2}) \in H^\infty(\mathbb{H}_{1/2})$ and $F_\zeta \in N^+(\mathbb{H}_{1/2})$. On the line, $|F_\zeta(\frac{1}{2}+it)| = |\zeta(\frac{1}{2}+it)|$.

(b) $F_\zeta = B_\zeta \cdot O_\zeta$ exactly, where

$$B_\zeta(s) = \prod_{\beta_\rho > 1/2} \Big(\frac{s-\rho}{s-\rho^*}\Big)^{m_\rho} \quad (\rho^* = 1-\bar\rho), \qquad \log|O_\zeta(s)| = \mathcal{P}\big[\log|\zeta(\tfrac{1}{2}+i\cdot)|\big](s).$$

The product converges locally uniformly on $\mathbb{C} \setminus \{\rho^*\}$ (conjugate zeros paired), $B_\zeta > 0$ on $(\frac{1}{2}, \infty)$, and $|B_\zeta| = 1$ on the line. In the canonical factorization cBSO of an N^+ function, the unimodular constant is $c = 1$, the singular inner factor is $S \equiv 1$, and there is no delay *$e^{-a(s-1/2)}$ ($a = 0$).*

(c) Consequently RH $\iff$ $B_\zeta \equiv 1$ $\iff$ F_ζ is outer $\iff$ for one (equivalently every) $s_0 \in \mathbb{H}_{1/2}$,

$$\log|F_\zeta(s_0)| = \mathcal{P}[\log|\zeta(\tfrac{1}{2}+i\cdot)|](s_0).$$

Proof. (a) For $\sigma > 0$, $\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \{x\}x^{-s-1}dx$, so $F_\zeta(s) = 1 - (s-1)I(s)$ with $|I(s)| \leq \int_1^\infty x^{-\sigma-1}dx = 1/\sigma$. For $\sigma \geq \frac{1}{2}$ this gives $|F_\zeta| \leq 1 + 2|s-1| \leq 6|s+\frac{1}{2}|$, using $|s+\frac{1}{2}| \geq 1$ and $|s-1| \leq |s+\frac{1}{2}| + \frac{3}{2}$. The function $s + \frac{1}{2} = w + 1$ is outer on $\Re w > 0$, and N^+ is an algebra containing H^∞ and the outer functions [8, Ch. II]. On the line, $|(s-1)/s| = |-\frac{1}{2}+it|/|\frac{1}{2}+it| = 1$.

(b) *Blaschke part.* The zeros of F_ζ in $\mathbb{H}_{1/2}$ are the ghosts. The Blaschke condition $\sum \delta_\rho/(1+|\rho|^2) < \infty$ holds since $\sum 1/|\rho|^2 < \infty$. For the pair $\rho, \bar\rho$ and real σ, $\frac{(\sigma-\rho)(\sigma-\bar\rho)}{(\sigma-\rho^*)(\sigma-\bar\rho^*)} = 1 - \frac{4\delta_\rho(\sigma-\frac{1}{2})}{|\sigma-\rho^*|^2} + O(\delta_\rho^2/\gamma^2)$. Hence the paired product converges absolutely and locally uniformly away from the ρ^*, with no convergence factors, and is positive on $(\frac{1}{2}, \infty)$. On the line, $|\frac{\frac{1}{2}+it-\rho}{\frac{1}{2}+it-\rho^*}| = 1$ because ρ^* is the reflection of ρ in the line. *No singular part.* Let $G = F_\zeta/B_\zeta = cSO$, zero-free on $\mathbb{H}_{1/2}$. B_ζ is holomorphic and unimodular near the line, and F_ζ is holomorphic across it. So G continues holomorphically across the line, with zeros only at the on-line zeros of ζ, which have finite order. Write $u = \mathcal{P}[\log|G|] - \log|G| = \mathcal{P}[\mu] + a(\sigma - \frac{1}{2}) \geq 0$, with $\mu \geq 0$ singular on $\mathbb{R}$ and $a \geq 0$. (i) A positive singular measure has Poisson integral with radial limit $+\infty$ at μ-a.e. point (its symmetric derivative is $+\infty$ μ-a.e.; [8, Ch. I], [BIB]). At such a point t_1, if $\zeta(\frac{1}{2}+it_1) \neq 0$, then $\log|G|$ is continuous on the line near t_1, so $\mathcal{P}[\log|G|]$ stays bounded radially, and $|G| \to 0$ radially. This contradicts $G(\frac{1}{2}+it_1) \neq 0$. So μ lives on the discrete set of on-line zero ordinates, i.e. $\mu = \sum c_j \delta_{t_j}$. (ii) At an on-line zero of order k, $\mathcal{P}[k\log|\cdot - t_j|](\frac{1}{2}+x+it_j) = k\log x$ exactly ($\log|w - it_j|$ is harmonic with these boundary values and grows only logarithmically). So $|G(\frac{1}{2}+x+it_j)| \leq Cx^k e^{-c_j/(\pi x)}$, while analyticity gives $|G| \geq c'x^k$. Hence $c_j = 0$. (iii) *No delay.* For real $\sigma \geq 2$, $-\log B_\zeta(\sigma) \leq \sum_\rho \frac{4m_\rho\delta_\rho(\sigma-\frac{1}{2})}{(\sigma-1)^2+\gamma^2} = O(\log\sigma)$ by $N(T) = O(T\log T)$. Also $\mathcal{P}[\log|G|](\sigma) \leq \mathcal{P}[\log(1+2|\cdot - 1|)](\sigma) =$

$O(\log\sigma)$ by (a), and $F_\zeta(\sigma)\to 1$. So $a(\sigma-\frac12)\le u(\sigma)=O(\log\sigma)$, which forces $a=0$. *Constant.* F_ζ, B_ζ and O_ζ are all positive on $(1,\infty)$: the last because $\log|\zeta(\frac12+it)|$ is even in t. So $c=1$.

(c) If $B_\zeta\not\equiv 1$ then $|B_\zeta(s_0)|<1$ at every $s_0\in\mathbb{H}_{1/2}$, and $\log|F_\zeta(s_0)|-\mathcal{P}[\log|\zeta|](s_0)=\log|B_\zeta(s_0)|<0$. □

Corollary 3.2 (gain defect, BSY and Burnol; [THM]; the RH clause is [RESTATEMENT]). *For $s_0=\frac12+a+i\tau\in\mathbb{H}_{1/2}$ define the* gain defect

$$\mathfrak{G}(s_0):=\mathcal{P}[\log|\zeta(\tfrac12+i\cdot)|](s_0)-\log|F_\zeta(s_0)|=\sum_{\beta_\rho>1/2}m_\rho\log\Big|\frac{s_0-\rho^*}{s_0-\rho}\Big|=\sum_{\beta_\rho>1/2}\frac{m_\rho}{2}\log\Big(1+\frac{4a\delta_\rho}{(a-\delta_\rho)^2+(\tau-\gamma_\rho)^2}\Big)\ge 0$$

RH $\iff$ $\mathfrak{G}(s_0)=0$ at one point $\iff$ at all points. At $s_0=1$ (with $F_\zeta(1)=1$ and $P_1(t)=\frac{1}{2\pi|\frac12+it|^2}$) this is the BSY identity of Lemma 30, and $|B_\zeta(1)|=\prod|(1-\rho)/\rho|$ is Burnol's $\|P\mathbf{1}\|$ [6, Thm. 1.2] *(Lemma 30). At $s_0=\frac32$ it reads $\frac1\pi\int_{\mathbb{R}}\log|\zeta(\frac12+it)|\frac{1}{1+t^2}dt=\log\frac{\zeta(3/2)}{3}+\mathfrak{G}(\frac32)$.*

Proof. $\mathfrak{G}=-\log|B_\zeta(s_0)|$ by Thm. 3.1(b). For the closed form, write $s_0-\rho=(a-\delta)+i(\tau-\gamma)$ and $s_0-\rho^*=(a+\delta)+i(\tau-\gamma)$. □

Corollary 3.3 (the pole is an anti-Blaschke factor; [THM], reading [ANALOGY]). *$\zeta=O_\zeta\cdot B_\zeta\cdot b_1^{-1}$ with $b_1(s)=\frac{s-1}{s}$, the Blaschke factor of the point $1\in\mathbb{H}_{1/2}$, whose mirror is $1^*=0$. So the pole of ζ is the one "gain" (non-passive) element: $|b_1^{-1}|=1$ on the line and >1 inside. Its phase on the line, $\arg\frac{\frac12+iT}{-\frac12+iT}=2\arctan 2T-\pi$, accounts for the "+1" in $N(T)=\frac{\theta}{\pi}+1+S(T)$ (compare Lemma 36 Prop. 7.4, where it appears as $\Delta\arg[s(s-1)]=\pi$). A ghost does the opposite. Its Blaschke factor on the line is $-e^{-2i\arctan((t-\gamma)/\delta)}$, which winds by -2π across a window of width $\sim\delta$ around $t=\gamma$. This is the multiplicative form of the definite 2π deficit of Lemma 36 Cor. 4.4.*

Remark 3.4 (DH and the FE; [THM]). Let $f=\frac12(1-i\kappa)L(s,\chi)+\frac12(1+i\kappa)L(s,\bar\chi)$ (χ mod 5, $\chi(2)=i$, $\kappa=\frac{\sqrt{10-2\sqrt5}-2}{\sqrt5-1}$; [7], [2]). It has real Dirichlet coefficients, $f(\sigma)\to1$, and polynomial growth on $\overline{\mathbb{H}_{1/2}}$. The proof above applies verbatim with $F_{\mathrm{DH}}:=f$ (no pole), so $F_{\mathrm{DH}}=B_{\mathrm{DH}}O_{\mathrm{DH}}$ with no singular part and no delay, and $B_{\mathrm{DH}}\not\equiv1$: the first ghost is $0.8085171824566+85.6993484853776i$ ([NUM], $|f|<10^{-14}$). DH satisfies the same functional equation shape. So *every* invariant built from the FE, the gamma factor, or $|f|$ on the line alone is blind to the difference between ζ and DH; the defect $\mathfrak{G}$ sees it (§7).

Remark 3.5 (why F_ζ and not ξ; [THM]). ξ is not of bounded type on $\mathbb{H}_{1/2}$: $\log\xi(\sigma)\sim\frac12\sigma\log\sigma$. The archimedean factor carries infinite exponential type (Prop. 4.2). The normalization $(s-1)/s$ is the minimal one that removes the pole without destroying $|\cdot|=|\zeta|$ on the line.

Prior art. The identity behind Cor. 3.2 at $s_0=1$ is BSY [1]; Nikolski [16] used Hardy-space distance methods to localize zeros of ζ (zero-free discs). We have not found Thm. 3.1(b) stated in exactly this form ("no singular factor, no delay"), but it is a short argument and is very likely folklore; we do *not* claim it as new mathematics.

4. Bode, minimum phase, and what the magnitude does and does not see

Proposition 4.1 (magnitude, anchor, phase; [THM]).
*(a) (*Magnitude alone is RH-blind.*) For every Blaschke product B' on $\mathbb{H}_{1/2}$ that is positive on $(\frac12,\infty)$, $O_\zeta B'$ has the same modulus $|\zeta|$ on the line as F_ζ. So $|\zeta(\frac12+it)|$ on the line, by itself, is consistent both with RH and with any ghost configuration.*
*(b) (*One interior anchor suffices.*) Magnitude on the line plus the single number $\log|F_\zeta(s_0)|$, at any $s_0\in\mathbb{H}_{1/2}$, decides RH, via Cor. 3.2. Taking $\Re s_0>1$, the anchor is computed from the Euler product.*

*(c) (*Phase.*) For $s_0, s_1 \in \mathbb{H}_{1/2}$ let $\mathcal{S}[g]$ be the Schwarz integral, i.e. the holomorphic function with $\Re\mathcal{S}[g] = \mathcal{P}[g]$, normalized by $\Im\mathcal{S}[g] = 0$ on $(\frac{1}{2}, \infty)$. Then*

$$\mathfrak{F}(s_0, s_1) := \Im\{\log F_\zeta(s_0) - \log F_\zeta(s_1) - \mathcal{S}[\log|\zeta|](s_0) + \mathcal{S}[\log|\zeta|](s_1)\} = \arg B_\zeta(s_0) - \arg B_\zeta(s_1),$$

with $\log F_\zeta$ continued along a path in $\mathbb{H}_{1/2}$ avoiding the ghosts, modulo 2π. RH $\iff$ $\mathfrak{F} \equiv 0$. On the line (boundary limit) this is the Bode gain–phase relation: under RH, $\arg\zeta(\frac{1}{2} + iT) - \arg\frac{s}{s-1} = \mathcal{C}[\log|\zeta|](T)$, the conjugate function.

Proof. (a) Blaschke products are unimodular a.e. on the boundary. (b) is Cor. 3.2. (c) $\log F_\zeta = \log B_\zeta + \log O_\zeta$ and $\log O_\zeta = \mathcal{S}[\log|\zeta|]$ (outer functions are $e^{\mathcal{S}[\log|f|]}$ up to a unimodular constant, which here is 1 by positivity on the real axis). The boundary statement is the boundary value of $\Im\log O_\zeta$ under RH. Here $\arg\zeta$ is continued from $+\infty$ horizontally, and $\arg(s/(s-1))$ is the pole factor of Cor. 3.3. □

Proposition 4.2 (the archimedean phase θ is not a delay; [THM])**.** *Let θ be the Riemann–Siegel theta function, so $Z(t) = e^{i\theta(t)}\zeta(\frac{1}{2} + it)$ is real.*

(a) $e^{-2i\theta(t)} = \chi(\frac{1}{2} + it)$, where $\zeta(s) = \chi(s)\zeta(1-s)$ and $\chi(s) = \pi^{s-1/2}\Gamma(\frac{1-s}{2})/\Gamma(\frac{s}{2})$.

(b) $\chi^{-1}(s) = \pi^{1/2-s}\Gamma(\frac{s}{2})/\Gamma(\frac{1-s}{2})$ is meromorphic, unimodular on the line, holomorphic on $\mathbb{H}_{1/2}$, with zeros at $s = 1, 3, 5, \ldots$. These violate the Blaschke condition: $\sum_k \frac{2k+1/2}{1+(2k+1)^2} = \infty$. Hence χ^{-1} is not *of bounded type on $\mathbb{H}_{1/2}$, and χ has poles in $\mathbb{H}_{1/2}$. Neither $e^{2i\theta}$ nor $e^{-2i\theta}$ is the boundary function of a bounded-type function on $\mathbb{H}_{1/2}$. In particular θ is not the phase of a delay $e^{-a(s-1/2)}$ (that phase would be linear, $-at$), nor of any Blaschke product or singular inner function.*

*(c) (*Unconditional bookkeeping.*) For $T > 0$ not an ordinate,*

$$\theta(T) = \pi N(T) - \mathcal{C}[\log|\zeta|](T) - \arg B_\zeta(\tfrac{1}{2} + iT) - 2\arctan 2T,$$

with $\arg B_\zeta$ continued horizontally from $+\infty$. Under RH the B_ζ term is absent: θ is the zero-counting phase minus the Bode phase of $|\zeta|$, minus the pole's Blaschke phase. [RESTATEMENT] (Riemann–von Mangoldt plus Prop. 4.1).

Proof. (a) $\zeta(\frac{1}{2} + it) = \chi(\frac{1}{2} + it)\overline{\zeta(\frac{1}{2} + it)}$, and Z real gives $\zeta/\bar{\zeta} = e^{-2i\theta}$. (b) Poles of $\Gamma(\frac{1-s}{2})$ are at $s = 1, 3, 5, \ldots$; $\Gamma(\frac{s}{2})$ has no zeros. A nonzero function of bounded type (a quotient of H^∞ functions with zero-free denominator, holomorphic on $\mathbb{H}_{1/2}$) has zeros satisfying the Blaschke condition [8, Ch. II]. (c) $N(T) = \frac{\theta}{\pi} + 1 + S(T)$ with $\pi S(T) = \arg\zeta(\frac{1}{2} + iT)$ continued along $2 \to 2 + iT \to \frac{1}{2} + iT$ (Riemann–von Mangoldt, [BIB]). Write $\arg\zeta = \arg O_\zeta + \arg B_\zeta + \arg\frac{s}{s-1}$ along that path: the first term is $\mathcal{C}[\log|\zeta|](T)$ and the last is $\arctan 2T - (\pi - \arctan 2T)$. □

The chat's claim (4), "θ is the delay/singular-inner part", is therefore false for ζ. It is *true* for curves over $\mathbb{F}_q$, where the analogous factor is a genuine delay of length $g\log q$ (Thm. 6.1(d)). The difference is the archimedean place: a Γ-factor has infinitely many "delay zeros" at $1, 3, 5, \ldots$ and is not of bounded type.

5. Corrections to the pasted KK/causality text

Each item is [THM] (standard Hardy-space facts; [8, 11]; Kronig [10], Bode [4], Bechhoefer [3], Nussenzveig [17]; non-uniqueness of phase from magnitude via Blaschke factors is also the point of [20]).

(K1) *"Blaschke/all-pass and delay factors are non-causal or non-analytic."* False. $b(w) = \frac{w-\lambda}{w+\lambda}$ ($\Re\lambda > 0$) and e^{-aw} ($a \geq 0$) are bounded holomorphic on $\Re w > 0$, i.e. causal transfer functions, and both satisfy KK. The delay e^{-aw} is entire, with its essential singularity only

at ∞ (as [3] notes). They are *non-minimum-phase*: same magnitude as 1, extra phase lag. Factorization: every $f \in N^+$ is *cBSO*, i.e. minimum phase $\times$ all-pass.

(K2) *"Inner functions are pure phase rotations."* Only on the boundary. Inside, $|B| < 1$ (so they *attenuate* at interior frequencies), and Blaschke factors have zeros inside. That interior attenuation is exactly the gain defect $\mathfrak{G}$.

(K3) *"Causality/passivity force $|H| \to 0$ at infinite frequency."* False. A resistor ($H \equiv R$), an inductor ($H = Lw$), every inner function ($|H| = 1$ on the axis), and Lagarias' ξ'/ξ itself (growth $\sim \frac{1}{2}\log|t|$ on vertical lines) are passive and causal. KK needs only a subtraction or a decay hypothesis in the *form* in which it is written. "Must drop to zero" is a hypothesis of the simplest KK formula, not a consequence of causality.

(K4) *"Flat magnitude ⇒ zero phase."* True only for outer (minimum-phase) functions. An all-pass has flat magnitude and nonzero phase. Bode's gain–phase *equality* holds exactly for minimum-phase systems [4, 3]; for the rest, the phase exceeds the Bode phase by the all-pass phase.

(K5) *"Hilbert transform = passivity."* The Hilbert transform (Fourier multiplier $-i\operatorname{sgn}\omega$) is the complex structure J splitting L^2 into $H^2_\pm$ (Lemma 39 §7(c)). Positivity is a separate condition. In geometric-quantization language, J gives the polarization and positivity makes it Kähler rather than pseudo-Kähler (§9). KK = causality = J-compatibility; passivity = positivity.

(K6) *"The phase reveals hidden zeros the magnitude cannot."* Half true: see Prop. 4.1. The magnitude on the line plus one interior value is equally decisive.

6. The finite-field Bode test

Let $C/\mathbb{F}_q$ have genus g and L-polynomial $P(T) = \prod_{i=1}^{2g}(1-\alpha_i T)$. Put $u = q^{1/2-s}$, so $|u| < 1 \iff s \in \mathbb{H}_{1/2}$, and $\mathcal{L}(u) = P(q^{-1/2}u) = \prod(1-\beta_i u)$ with $\beta_i = \alpha_i/\sqrt{q}$. A *ghost* is a β_i with $|\beta_i| > 1$, with rapidity $\eta_i = \log|\beta_i|$ (Lemma 45). For the purposes of the test, "fake" P (FE-shaped integer polynomials with ghosts) are allowed.

Theorem 6.1 (finite-field Bode; [THM]; (b) is [RESTATEMENT] of Weil's theorem as a statement about P)**.** *$\mathcal{L} = B \cdot O$ on the disc, where B is the finite Blaschke product over the zeros $1/\beta_i$ of $\mathcal{L}$ in the disc, i.e. the ghosts. There is no singular factor, and no "delay" u^k, since $\mathcal{L}(0) = 1$.*

*(b) (*Jensen = Bode gain defect.*) $\int_0^{2\pi} \log|\mathcal{L}(e^{i\phi})| \frac{d\phi}{2\pi} = \sum_{|\beta_i|>1} \log|\beta_i| = \sum_{\text{ghosts}} \eta_i = \log M(\mathcal{L})$, the Mahler measure. RH(C) $\iff$ this vanishes $\iff$ $\mathcal{L}$ is outer. This is Thm. 3.1(iii) of the skin-effect note, and the exact analogue of Cor. 3.2 at the centre $u = 0$, i.e. $s = +\infty$.*

*(c) (*Phase defect.*) If no zero lies on $|u| = 1$, the winding of $\arg\mathcal{L}(e^{i\phi}) - \mathcal{C}[\log|\mathcal{L}|](\phi)$ equals the number of ghosts, counted with multiplicity (the one-sided winding of the skin note, Thm. 3.1(iv)).*

*(d) (*The FE gauge is a true delay.*) The FE makes $\mathcal{L}$ self-reciprocal: $\mathcal{L}(u) = u^{2g}\mathcal{L}(1/u)$. So $\mathcal{L}(u) = u^g\, h(u)$ with $h(1/u) = h(u)$ real on $|u| = 1$. In s, $u^g = e^{-g\log q\,(s-1/2)}$ is* exactly *the half-plane delay factor of length $g\log q$. On the line, $\arg\mathcal{L}(q^{-it}) = -g\,t\log q$ up to jumps of π at the real zeros of h: a* linear-phase *filter (palindromic FIR). The analogue of $\zeta(\frac{1}{2}+it) = e^{-i\theta(t)}Z(t)$ is $\mathcal{L} = u^g h$, with $\theta(t) \leftrightarrow g\,t\log q$; the mean zero densities agree, since $\theta'(t)/\pi \leftrightarrow g\log q/\pi$. The delay is a factor of the FE split $\mathcal{L} = u^g \cdot h$,* not *of the inner–outer split of $\mathcal{L}$: h is unbounded on the disc.*

Proof. (a) Finite Blaschke products and zero-free polynomials on the closed disc are inner/outer; a polynomial with zeros only on $|u| = 1$ is outer. (b) Jensen's formula with $\mathcal{L}(0) = 1$: $\int \log|\mathcal{L}| = \sum_{|1/\beta_i|<1} \log|\beta_i|$. (c) Argument principle applied to B, together with $\log O = \mathcal{S}[\log|\mathcal{L}|]$. (d) The FE $q^g T^{2g} P(1/(qT)) = P(T)$ gives $\{\beta_i\} = \{1/\bar\beta_i\} = \{\bar\beta_i\}$ with $\prod \beta_i = 1$, so $\prod_i(1-\beta_i u) =$

$u^{2g}\prod_i(1-\beta_i/u)$. Hence $h = u^{-g}\mathcal{L}$ is invariant under $u \mapsto 1/u$, has real coefficients, and is real on $|u| = 1$. □

Numerics ([NUM]; `num/lemma48_ff.py` → `num/l48/ff.json`; 65536-point half-offset grid on $|u| = 1$, phase compared away from ± 0.01 rad of on-circle roots).

case	P	measured $\int \log\lvert\mathcal{L}\rvert$	$\sum \eta$	ghosts	phase winding
11a, $p = 5, 7, 13, 17, 19, 23$	e.g. $[1, -1, 5]$	$\lvert\cdot\rvert \le 2.2 \cdot 10^{-5}$	0	0	0
$y^2 = x^5 - x + 1$, $p = 11, 13$	$[1, 7, 31, 77, 121]$, $[1, 1, 9, 13, 169]$	$\le 3.5 \cdot 10^{-5}$	0	0	0
$y^2 = x^5 + x^2 + 3$, $p = 11, 13$	$[1, 0, 6, 0, 121]$, $[1, 2, 22, 26, 169]$	$\le 4.7 \cdot 10^{-5}$	0	0	0
fake, $q = 13$	$[1, -8, 13]$	0.463599	0.463599	1	1
fake, $q = 11$	$[1, -7, 11]$	0.331021	0.331021	1	1
fake, $q = 13$	$[1, 13, 13]$	1.194763	1.194763	1	1
fake g2 real pair, $q = 11$	$[1, 0, -25, 0, 121]$	0.516474	0.516474	2	2
fake g2 quadruple, $q = 11$	$[1, -13, 65, -143, 121]$	0.678812	0.678812	2	2
fake g2 mixed, $q = 11$	$[1, -10, 40, -110, 121]$	0.545711	0.545725	1	$1^\dagger$

For the real curves the small nonzero values come from discretizing $\log|\mathcal{L}|$ across zeros on the circle; the phase defect agrees with the (trivial) Blaschke prediction to $\le 8.9 \cdot 10^{-3}$ away from the roots. For fakes without circle roots the agreement is $\le 2.5 \cdot 10^{-14}$. †The mixed fake has two zeros on the circle, so the winding is defined only on the grid that avoids them; there it is 1, and the phase-defect error is $6.9 \cdot 10^{-3}$. The integer polynomials are those of the skin note (point counts by brute force over $\mathbb{F}_p, \mathbb{F}_{p^2}$, recorded there).

7. Numerical Bode test: ζ versus Davenport–Heilbronn

Method ([NUM]; `num/lemma48_core.py`, `lemma48_zeros.py`, `lemma48_bode.py`, `lemma48_bsy.py`; python-flint ball arithmetic at 96 bits for the function values). (1) On-line zeros up to $T = 1000$ from sign changes of the Hardy-type function $e^{i\theta_f} f(\frac{1}{2}+it)$ (step 0.02 with bisection), cross-checked by the argument principle for the completed function on $[-1, 2] \times [\cdot, 1000]$. ζ: 649 on-line zeros, argument-principle total 649.0000. DH: 848 on-line zeros, total 904.0000, and 28 ghosts with $\beta > \frac{1}{2}$ found by rectangle windings on $[\frac{1}{2}+\varepsilon, 2]$ and Newton: $848+2\cdot 28 = 904$. The first search used $\varepsilon = 0.02$ and missed one ghost, $0.5159183 + 520.9438010i$ ($\delta = 0.0159$). The deficit of 2 in the window $[500, 550]$ located it, and a search with $\varepsilon = 0.01$ found it (`num/l48/probe2.py`). Ghost list: $0.8085172+85.6993485i$, $0.6508301+114.1633427i$, $0.574356+166.479306i$, $0.724258+176.702461i$, …, $0.762499 + 983.752193i$ (all in `num/l48/zeros_T1000.json`). The first ghost agrees with the value recorded in the skin note (N5); for the DH zeros in the literature see [2] (record verified, table not compared). (2) $\mathfrak{G}(s_0)$: Gauss–Legendre panels with breakpoints at on-line zeros (smoothstep near the logarithmic singularities), panel width 0.2 ($a/3$ locally), on $|t| \le U = 1000$, using evenness. The phase defect $\mathfrak{F}(s_0, s_1)$ uses $s_1 = 2.5 + i\tau$ and continues $\log F$ along the horizontal segment. (3) Prediction: $\sum_{\text{ghosts}} \log|(s_0 - \rho^*)/(s_0 - \rho)|$ over the 28 ghosts and their conjugates, and the continuous Blaschke phase along the same path.

Results.

	a	τ	$\mathfrak{G}$ measured	$\mathfrak{G}$ predicted	$\mathfrak{F}$ measured	$\mathfrak{F}$ predicted
ζ (29 points)	0.05–1	0–300.7	$\lvert\mathfrak{G}\rvert \le 8.3\cdot10^{-6}$	0	$\lvert\mathfrak{F}\rvert \le 3.8\cdot10^{-6}$	0
ζ, $s_0=1$ (BSY)	0.5	0	$-8.1\cdot10^{-8}$	0	–	–
ζ, $s_0=\frac{3}{2}$	1	0	$-1.6\cdot10^{-7}$	0	–	–
DH	0.05	85.6	0.295162	0.295162	−2.48876	−2.48876
DH	0.05	85.8	0.294407	0.294407	2.48069	2.48069
DH	0.10	114.0	0.559581	0.559581	−1.28286	−1.28286
DH	0.30	85.6	1.822062	1.822063	−1.47888	−1.47888
DH	0.50	114.0	0.554834	0.554836	−0.17927	−0.17927
DH	1.00	300.7	0.002613	0.002624	−0.00028	−0.00028
DH, $s_0=1$ (BSY form)	0.5	0	$1.7428\cdot10^{-4}$	$1.7448\cdot10^{-4}$	–	–
DH, all 35 points				$\lvert\text{resid}\rvert \le 1.1\cdot10^{-5}$		$\lvert\text{resid}\rvert \le 5.4\cdot10^{-6}$

Profiles at $a=0.1, 0.4$ across $\tau\in[78,122]$ (89 points each) agree with the prediction to $\le 1.1\cdot10^{-5}$; the ζ profile at $a=0.1$ has $|\mathfrak{G}|\le 8.8\cdot10^{-6}$ (Fig. 1). Halving the panel width changes $\mathfrak{G},\mathfrak{F}$ by $\le 1.9\cdot10^{-6}$. The residuals are systematically negative and grow with a: they are the truncation $|t|>U$ (the Poisson tail is $\asymp a/U$) and the ghosts above 1000. Since the integral of $\log|\zeta|$ over the line is $O(\log T)$, the tail is small but not zero. The jump of $\mathfrak{F}$ from -2.49 to $+2.48$ between $\tau=85.6$ and 85.8 is the -2π Blaschke winding of Cor. 3.3, seen from $a=0.05$. *Reading:* the Bode test separates ζ from DH exactly as Thm. 3.1 says. It is a zero detector, equivalent to the argument principle, with Poisson resolution $\sim a$; it gives no information about ζ beyond the zeros it is near.

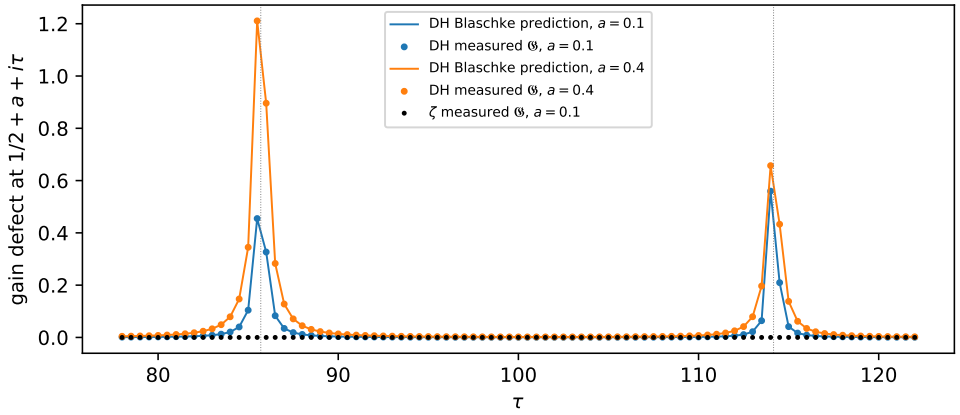

FIGURE 1. Gain defect $\mathfrak{G}(\frac{1}{2}+a+i\tau)$ for DH (dots) against the Blaschke prediction from its ghosts (lines; dotted verticals at the ghosts 85.699, 114.163), and for ζ (black, ≈ 0). Data `num/148/bode_profile.json`.

8. All-pass lines in inner–outer language

Theorem 8.1 (Suzuki's symbols in inner–outer form; [THM]; last clause [RESTATEMENT]). *For $\omega>0$, $\nu\in\mathbb{Z}_{>0}$, let $\Theta^{\omega,\nu}(s)=\big(\xi(s-\omega)/\xi(s+\omega)\big)^{\nu}$ (Lemma 34 §2, Lemma 37 §1).*

(a) $|\Theta^{\omega,\nu}|=1$ on the line, unconditionally.

(b) On $\mathbb{H}_{1/2}$, $\Theta^{\omega,\nu}$ has zeros at $\rho+\omega$ for every zero with $\delta_\rho>-\omega$, and poles at $\rho-\omega$ for every zero with $\delta_\rho>\omega$ (order νm_ρ each).

(c) If all $\delta_\rho\le\omega$ (so E_ω is HB by Lemma 33 Thm. 6.2(a)), then $\Theta^{\omega,\nu}=B_Z$ is a pure Blaschke product *over the zeros in (b): no singular factor, no delay, constant* 1. *In general $\Theta^{\omega,\nu}=$*

$(B_Z/B_P)\Psi$, with B_P the convergent Blaschke product over the poles in (b) (the anti-stable denominator, of degree $\nu \cdot \#\{\rho : \delta_\rho > \omega\}$, which is κ of Lemma 34 Thm. 6.2 / Lemma 36 Thm. 7.9 under their hypothesis), and Ψ zero-free, pole-free and unimodular on the line. We expect $\Psi \equiv 1$ but have not written out the growth estimate needed in the non-HB case; it is not claimed.

(d) *The group delay on the line is $\tau_g(t) = -\frac{d}{dt}\arg\Theta^{\omega,1}(\frac{1}{2}+it) = 2\Re\frac{\xi'}{\xi}(\frac{1}{2}+\omega+it)$. Hence Lagarias' criterion is:* every depth-ω symbol has positive group delay at every frequency. *[RESTATEMENT].*

Proof. (a) $\xi(\frac{1}{2}-\omega+it) = \xi(\frac{1}{2}+\omega-it) = \overline{\xi(\frac{1}{2}+\omega+it)}$. (b) Zeros of $\xi(s-\omega)$ are $\rho+\omega$, with real part $> \frac{1}{2}$ iff $\delta_\rho > -\omega$; zeros of $\xi(s+\omega)$ are $\rho-\omega$. (c) In the HB case, $|E_\omega^\#| < |E_\omega|$ on the upper half-plane, i.e. $|\Theta| < 1$ on $\mathbb{H}_{1/2}$; with (a), Θ is inner. Its canonical factors are: Blaschke over its zeros; no singular part, since Θ is holomorphic and unimodular across the line (argument (i)–(ii) of Thm. 3.1(b)); no delay, since $\log|\Theta(\sigma)| = -2\nu\omega\frac{\xi'}{\xi}(\sigma) + O(\nu\omega^3) \sim -\nu\omega\log\sigma = o(\sigma)$; constant 1 by positivity on $(\frac{1}{2}+\omega,\infty)$. The general decomposition just divides out the zeros and poles of (b), which satisfy the Blaschke condition. (d) $\frac{d}{dt}\arg\xi(\sigma+it) = \Re\frac{\xi'}{\xi}(\sigma+it)$ and $\arg\Theta = -2\arg\xi(\frac{1}{2}+\omega+it)$ on the line, by (a). □

Theorem 8.2 (the exponential line is a singular inner function; [THM]; equivalences [RESTATEMENT]). *Let $\Theta_c = \exp(-2c\,\xi'/\xi)$, $c > 0$ (Lemma 37 Thm. 4.1).*

(a) *Under RH, $\Theta_c(s) = \prod_\gamma \exp(-\frac{2c\,m_\gamma}{s-\frac{1}{2}-i\gamma})$ (symmetric product), a* purely singular inner function*: $\log|\Theta_c(s)| = -2\pi c\,\mathcal{P}[\nu](s)$ with $\nu = \sum m_\gamma\delta_\gamma$ the ordinate measure, so its singular measure is $2c$ times Lemma 33's boundary absorption spectrum $\pi\nu$ (Thm. 5.1(c) there). There is no Blaschke part and no delay. Equivalently, $\Theta_c = e^{2ic\,\mathfrak{m}}$ with $\mathfrak{m}$ Lagarias' Pick function (V2): $b = 0$ means no delay, and μ discrete means purely singular.*

(b) *If some $\beta_\rho > \frac{1}{2}$, then Θ_c has an essential singularity at $\rho \in \mathbb{H}_{1/2}$, so it is not meromorphic on $\mathbb{H}_{1/2}$ and in particular not of bounded type.*

(c) *Hence RH $\iff$ Θ_c is of bounded type on $\mathbb{H}_{1/2}$ $\iff$ Θ_c is inner $\iff$ Θ_c is Schur (the last is Lemma 37 Thm. 4.1(iv)).*

(d) *(*Blaschke $\to$ singular.*) Under RH, the depth-ω factor of a zero $\frac{1}{2}+i\gamma$ in $\Theta^{\omega,c/\omega}$ is $(\frac{w-\omega-i\gamma}{w+\omega-i\gamma})^{c/\omega}$ ($w = s-\frac{1}{2}$). As $\omega \to 0$ it converges to $\exp(-\frac{2c}{w-i\gamma})$ locally uniformly on $\Re w > 0$: c/ω Blaschke zeros at distance ω from the line merge into a singular atom of mass $2\pi c$ (Lemma 37 Thm. 4.1(iii) in inner–outer language).*

(e) *The Cayley transform $W_c = \frac{1+\Theta_c}{1-\Theta_c} = \coth(c\,\xi'/\xi)$ is positive-real on $\mathbb{H}_{1/2}$ iff RH (Cayley transform plus (c)).* Reading *[ANALOGY]: the input impedance of an open-circuited uniform line of electrical length c with propagation function ξ'/ξ; Lemma 37's slab remark in impedance form.*

Proof. (a) Under RH, Hadamard gives $\frac{\xi'}{\xi}(s) = \sum_\gamma \frac{m_\gamma}{s-\frac{1}{2}-i\gamma}$ (symmetric summation; the constant vanishes by $\xi'/\xi(\frac{1}{2}) = 0$). Each factor $\exp(-\frac{2cm}{w-i\gamma})$ has modulus $\exp(-2cm\frac{\Re w}{|w-i\gamma|^2}) \le 1$, equal to 1 on the line off $i\gamma$, and $\frac{\Re w}{|w-i\gamma|^2} = \pi P_w(\gamma)$. The sum converges because $\sum m/(1+\gamma^2) < \infty$. Then $\log|\Theta_c(\sigma)| = -2c\frac{\xi'}{\xi}(\sigma) \sim -c\log\sigma = o(\sigma)$, so there is no delay. The Pick form uses $\frac{\xi'}{\xi}(\frac{1}{2}-iz) = -i\mathfrak{m}(z)$. (b) Near ρ, $-2c\,\xi'/\xi = -2cm_\rho/(s-\rho)+$holomorphic, and exp of a simple pole is an essential singularity. Bounded-type functions are quotients of H^∞ functions, hence meromorphic. (c) combines (a), (b) and Lemma 37. (d) $\frac{c}{\omega}\log\frac{u-\omega}{u+\omega} = -\frac{2c}{u} + O(\omega^2/u^3)$. (e) The Cayley map takes the unit disc to the right half-plane. □

G5 link ([RESTATEMENT], all from Lemma 37 except the inner–outer reading). (i) Lagarias' PR condition on ξ'/ξ is the positive-real form; the Schur form is Θ_c Schur; Nehari makes Schur

= ultimate Hankel passivity $\sup_t \|\mathcal{K}[t]\| \le 1$ (Lemma 37 Thm. 2.2). (ii) By Sondow–Dumitrescu monotonicity, PR implies every Suzuki line is passive: $|\xi(s-\omega)| \le |\xi(s+\omega)|$ on $\mathbb{H}_{1/2}$ (Lemma 37 Thm. 3.2). (iii) Passivity along any one sequence $\omega_n \downarrow 0$ implies RH; a single ω is not enough, because of ladders (Lemma 37 Remark after Cor. 3.3). (iv) The strict pointwise form at the exponential line ($\|\mathcal{K}_c[t]\| < 1$ for each t under RH) is open (Lemma 37 Prop. 4.4). Thm. 8.2(a) shows why Suzuki's Prop. 5.1 does not apply: Θ_c is not $E^{\#}/E$ for an entire E. A purely singular inner symbol is a limit of Blaschke symbols, not one itself. Whether the strict bound survives the Blaschke → singular limit of (d) is [OPEN], equivalent to Lemma 37's M11.

9. The skin-effect dictionary (from notes/skin_effect_prior.md)

Let P satisfy the FE, $\tilde{P}(z) = \prod(1-\beta_i z) = \mathcal{L}(z)$ (monic and palindromic), and let $h(z) = z^{-g}\tilde{P}(z)$ be the Laurent symbol (Thm. 6.1(d)). $T_N(h)$ is the banded Toeplitz matrix (open boundary conditions, OBC) and the circulant gives periodic boundary conditions (PBC).

Theorem 9.1 (skin-effect dictionary for Weil polynomials; the note's Thm. 3.1)**.** *(i) [THM] h is real on $|z|=1$ and $T_N(h)$ is real symmetric. The PBC spectrum lies in $\mathbb{R}$, and the winding of $h(e^{ik}) - E$ is 0 for all $E \notin h(S^1)$.* No skin effect, for any P with FE.

(ii) [THM] For every E, the roots of $z^g(h(z)-E)$ come in pairs $\beta, 1/\beta$, so $|\beta_g||\beta_{g+1}| = 1$, and $|\beta_g| = |\beta_{g+1}|$ iff both equal 1. Hence the generalized Brillouin zone (GBZ) lies in the unit circle ($\sigma = \frac{1}{2}$), independent of RH.

(iii) [THM] (Jensen + (ii); Toeplitz limit cited [18, 5]) $\int \log|h(e^{ik})| \frac{dk}{2\pi} = \log M(\tilde{P}) = \sum_{|\beta_i|>1} \log|\beta_i| = \sum_{\text{ghosts}} \eta_i$, and $\lim \frac{1}{N} \log|\det T_N(h)|$ equals this. ***RH for $C/\mathbb{F}_q$ $\iff$ $M(\tilde{P}) = 1$ $\iff$ the OBC free energy vanishes.*** *This is Thm. 6.1(b): the skin-effect free energy* is *the Bode gain defect.*

(iv) [THM] (argument principle) For the one-sided symbol $\tilde{P}$: if no β_i lies on S^1, the PBC winding about 0 equals the number of β_i inside the disc, i.e. k (Lemma 46's index excess). If some $\beta_i \in S^1$, the point gap at 0 is closed and the winding is undefined. This is Thm. 6.1(c).

(v) [RESTATEMENT] (Lemma 46 Prop. 1.6) On the base-change tower, $s_m = \sum \beta_i^m$ is bounded for real curves and grows like $e^{m \max \eta}$ for ghosts. This is a skin mode on the tower that no single imaginary gauge removes.

Numerics recorded in the note ([NUM], N3/N4, $N \le 800$): real curves have all $|\beta| = 1$ to 10^{-15}, winding 0, and $\frac{1}{N}\log|\det T_N| \in [-0.0065, 0.0010]$ at $N = 800$. Fakes: $\sum \eta$ vs. $f_{800} = 0.4636/0.4642$, $0.3310/0.3319$, $1.1948/1.1949$, $0.5165/0.5176$, $0.6788/0.6814$, mixed $0.5457/0.5463$. These agree with the Bode values of §6 (computed here independently, by a different method). *Polarization* (the note, §4; [THM]/[RESTATEMENT]): a Kähler polarization is a complex Lagrangian P with $i\omega(X, \bar{X}) > 0$ on P. Half-dimensionality gives the FE/Krein structure; Kähler *positivity* gives Hilbert. This is (K5): J versus positivity. The ζ-version of (iii) is BSY (Cor. 3.2 at $s_0 = 1$). DH is the FE-blind control.

Erratum on J.W.'s ai.viXra:2606.0029 and 2609.0007 (as recorded in the note). [A] ai.viXra:2606.0029v1 (2026-06-10, 5 pp). Nothing beyond definitions is proved. Specific errors: (i) the kinetic hopping of eq. (6) is Hermitian ($|t_R| = |t_L|$, $t_L = \overline{t_R}$), so there is no Hatano–Nelson asymmetry and no skin effect (note N2: $\|M - M^*\| = 0$); (ii) eq. (12) is false: on $\Re s = \frac{1}{2}$, $V(1-s) = \overline{V(s)}$, so (12) $\iff \Im\zeta'/\zeta(\frac{1}{2}+it) = 0$, which is false ($|V(s)-V(1-s)| = 0.397, 1.810, 26.1$ at $t = 5, 20, 30.5$). The true FE identity is $\zeta'/\zeta(s) + G(s) = -[\zeta'/\zeta(1-s) + G(1-s)]$ with $G = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\psi(\frac{s}{2})$ (residual $\le 3 \cdot 10^{-31}$); (iii) "Riemann–Roch ⇒ deg div = 0" is applied to a function of s, not to a section on a compact curve; (iv) the pseudospectral bound C/δ (eq. 7) and the vanishing theorem (13) are asserted, not proved; (v) eq. (9) mixes an argument-principle count around the pole with "691 torsion"; (vi) the argument is circular: the weight ω

and the "ghost-free" pseudospectrum presuppose the conclusion. The GBZ is never defined by $|\beta_M(E)| = |\beta_{M+1}(E)|$; it is asserted to collapse onto $\Re s = \frac{1}{2}$ via eqs. (11)–(12). Kähler polarization is not mentioned in [A].

[B] ai.viXra:2609.0007v1 (2026-09-03, 31 pp). Model eq. (6.7) is a genuine Hatano–Nelson chain on a basis $|p_n\rangle$ (primes) with $g = \ln 691$; boundary conditions are not stated. The claim "the critical strip is the literal GBZ" is asserted. (a) Eq. (6.9), "the $p^{m/2}$ in the explicit-formula prime sum is the Half-Dimension Kähler Polarization axis", is a correct *normalization* fact ($p^{-m/2}$ is the weight at $\sigma = \frac{1}{2}$) [RESTATEMENT]. (b) Eq. (9.3), "$\lambda \geq \frac{1}{4}$ and $\lambda = s(1-s) \Rightarrow \Re s = \frac{1}{2}$", is **circular**: $s(1-s)$ is real iff $\sigma = \frac{1}{2}$ or $\gamma = 0$, so "λ real $\geq \frac{1}{4}$" is RH itself. A Selberg-type argument needs self-adjointness, which is the unproved part. (c) 2609.0084 eq. (15.1) has a degree error ($H^{1,0} \oplus H^{0,1} = H^1$, not H^2) and omits the positivity condition that makes a polarization Kähler. $g = \ln 691$ is not connected to ζ by any computation, and under OBC it is pure gauge. [A] and [B] contain no explicit numerical computations. *Net*: the papers contain the vocabulary (Hatano–Nelson pump, GBZ, point gap, half-dimension Kähler axis) but no derivation. Thm. 9.1 is the correct version: FE $\Rightarrow$ Hermitian symbol, GBZ $= \{\sigma = \frac{1}{2}\}$ for real curves *and* fakes; the ghost invariant is the free energy (Mahler measure, = Bode gain defect, = BSY for ζ), not a winding.

10. Corrections to the chat claims

(C1) *Lagarias.* Reference correct. The printed g is anti-Pick; the Pick function is $-g$ (V1). "PR/Herglotz after $s = \frac{1}{2} - iz$": correct, with $\frac{\xi'}{\xi}(\frac{1}{2} - iz) = -i\mathfrak{m}(z)$. Herglotz data under RH: $a = b = 0$, $\mu =$ ordinate measure (V2). Weil/Bochner relation: correct and already in Lemma 33 (V3).

(C2) *Sondow–Dumitrescu.* Venue wrong: Period. Math. Hungar. 60 (2010) 37–40, not Elem. Math. (V4).

(C3) *"Ghosts are invisible in magnitude, visible only in phase."* False as stated (Prop. 4.1). The boundary magnitude plus one interior value (gain defect) decides RH. The phase decides it too. The magnitude alone does not.

(C4) *"RH $\iff$ no Blaschke part."* Correct for F_ζ (Thm. 3.1), with two additions: (i) one must use $F_\zeta = \zeta(s-1)/s$ (or any N^+ normalization), not ξ, which is not of bounded type; (ii) one must also rule out singular and delay factors, and Thm. 3.1 does so. The pole is an anti-Blaschke factor (Cor. 3.3).

(C5) *"Hilbert transform = J".* Correct (Lemma 39 §7(c)); positivity is separate (K5).

(C6) *Finite-field Bode test.* Proved and run (Thm. 6.1, table in §6): gain defect $= \sum \eta =$ log Mahler, phase winding = number of ghosts.

(C7) *Numerical ζ vs. DH Bode defect.* Run (§7): $\zeta \approx 0$ to 10^{-5}; DH matches its 28 ghosts to 10^{-5}.

(C8) *"θ is the delay/singular-inner part."* False for ζ (Prop. 4.2): $e^{2i\theta}$ is not of bounded type; F_ζ has no delay and no singular part. It is true in the function-field case, where u^g is a delay of length $g \log q$ (Thm. 6.1(d)). The analogy holds at the level of mean phase slope, not of inner functions.

(C9) *de Branges/HB, $E^\#/E$ inner.* Correct when E is HB (Thm. 8.1(c)). Lagarias' 2006 paper is in *Frontiers in Number Theory, Physics, and Geometry I* (Springer 2006) 365–377 (V5).

(C10) *G5 link.* Correct in outline, and already proved in Lemma 37. The inner–outer reading adds: Suzuki symbols are Blaschke products in the HB case (with the anti-stable ghost denominator otherwise), and the exponential line is singular inner (Thm. 8.2). The strict pointwise G5 at the exponential line remains open.

(C11) *Pasted text.* Corrected in (K1)–(K6).

11. What is new and what is restatement

- **Restatement ([RESTATEMENT]):** every RH-equivalence in this lemma: F_ζ outer, $\mathfrak{G} = 0$, $\mathfrak{F} = 0$, positive group delay, Θ_c bounded type, $\coth(c\,\xi'/\xi)$ PR. They are Lagarias/BSY/Burnol/Lemma 37 in Hardy-space dress. The finite-field statements are Jensen's formula plus Weil.
- **Presentation-level theorems, proved here, probably folklore:** Thm. 3.1 with "no singular factor, no delay"; the pole as anti-Blaschke (Cor. 3.3); Prop. 4.2(b) ($e^{2i\theta}$ not of bounded type: the archimedean place is not a delay); Thm. 6.1(d) (FE gauge = true delay $g \log q$); Thm. 8.1(c) in the HB case; Thm. 8.2(a),(b),(d) (exponential line singular inner, essential singularity at ghosts, Blaschke → singular). We found none of these stated in this form in the literature we checked, but each is a few lines from standard facts, and we claim no novelty.
- **Numerics ([NUM]):** the ζ/DH Bode comparison to height 1000, with an independent argument-principle count of all DH zeros (904, including 28 ghosts), and the finite-field Bode table.
- **Not new:** the skin-effect dictionary (from the note), and the corrections.
- **No lever on RH.** The Bode defect is a zero detector of Poisson resolution a. The FE-built and magnitude-only invariants are blind to DH, and the decisive invariants need either zeros or an interior anchor whose positivity is RH itself.

12. Questions for J.W.

Question 12.1. Your "2π orbit of $(\frac{1}{2}, \frac{1}{2})$": is it the -2π Blaschke phase winding of a ghost (Cor. 3.3, seen at $\tau = 85.6 \to 85.8$ in §7), the $+2\pi$ anti-Blaschke winding of the pole, or the 2π deficit of Lemma 36? These are three faces of one count, but the signs differ, and the lemma that comes next depends on which one you mean.

Question 12.2. Do you want the official G5 phrased in Bode form, "$\mathfrak{G}(s_0) = 0$ at one interior point" (Cor. 3.2; BSY is $s_0 = 1$), instead of Hankel passivity? It is RH-equivalent and needs no limit, but it is visibly a restatement. Lemma 37's Hankel form at least has a causal time filtration.

Question 12.3. Should θ be treated as an archimedean *non-bounded-type* all-pass (Prop. 4.2) in the stage/actors language of Lemma 47, i.e. part of the stage? In function fields it is the finite delay u^g, which is pure stage.

Question 12.4. For the erratum on 2606.0029/2609.0007: do you want it kept in this lemma, moved to the volume's errata appendix, or both?

Question 12.5. Is the open step "strict pointwise passivity survives the Blaschke → singular limit" (Thm. 8.2(d), Lemma 37 M11) worth a dedicated numerical lemma (Ritz values of $\mathcal{K}_c[t]$ against $\mathcal{K}^{\omega, c/\omega}[t]$ as $\omega \to 0$)?

Status

Proved ([THM]): Thm. 3.1, Cor. 3.2, Cor. 3.3, Prop. 4.1, Prop. 4.2, Thm. 6.1, Thm. 8.1(a),(b),(d) and (c) in the HB case, Thm. 8.2, (K1)–(K6). **Restatements of RH:** as listed in §11. **Numerical:** §6, §7. **Open:** $\Psi \equiv 1$ in Thm. 8.1(c) without HB (expected, not written); the strict pointwise G5 at Θ_c (Lemma 37 M11). **Not verified, not cited:** content of [15]; Kramers' 1927 paper (no Crossref record found); Suyama–Kapadia's content beyond the abstract. **RH:** not proved, and no step here brings it closer.

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TAA LEMMA 49: THE BETTI–DE RHAM COMPARISON AND THE MISSING POLARIZATION. PERIODS OF AN ELLIPTIC CURVE, RAMANUJAN AS GAUSS–MANIN, E_2^* AS THE POLARIZATION TERM, THE UNIT-ROOT SPLITTING, AND WHAT SURVIVES FROM THE 2026-10-22 MANUSCRIPT

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. **Statement of intellectual honesty.** RH is not proved here, and nothing below brings it closer. Everything in §§2–5 is classical mathematics about elliptic curves, stated carefully and checked; §6 says plainly that the rung ζ lacks is *not* supplied. Tags: [THM] proved here (elementary, proof given or computation recorded); [CITED] checked at the source (`lit/lemma49/verified_refs.md`); [BIB] bibliographic record verified (Crossref), content standard; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma49_*.py`, data `num/149/`); [CONJ] conjecture; [OPEN] open. Lemmas 14, 23, 42, 46–48 are cited by number and not redone.

Question (J.W., [t187u]). After dropping the undefined "Betti–de Rham plane" ([t186u]): start the genuine Betti–de Rham comparison for an elliptic curve—periods, Hodge filtration, Legendre, Riemann bilinear relations; Ramanujan's P, Q, R system as Gauss–Manin; E_2 and E_2^*; the crystalline analogue over $\mathbb{F}_p$—and say what $\operatorname{Spec}\mathbb{Z}$ lacks. Also: what of the 2026-10-22 manuscript survives, and "tell me more about this alternating sum of Betti numbers χ and how curvature squared becomes Betti".

Answer in brief. (1) *Periods* (§2). For $E = \mathbb{C}/\Lambda$ the comparison $H^1_{\mathrm{dR}} \cong H^1_{\mathrm{B}} \otimes \mathbb{C}$ has period matrix $\Pi = \left(\begin{smallmatrix} \omega_1 & \omega_2 \\ -\eta_1 & -\eta_2 \end{smallmatrix}\right)$ with $\det \Pi = \eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$ (Legendre, Thm. 2.2). The bilinear relations give $i\int \omega \wedge \bar{\omega} = 2|\omega_1|^2 \Im\tau > 0$ (Prop. 2.3); in genus 1 this positivity is *automatic up to orientation* (Rem. 2.4). (2) *Ramanujan = Gauss–Manin + Legendre* (Thm. 3.2, [THM], cf. [23, 29]). The GM connection of $y^2 = 4x^3 - ax - b$ (Prop. 3.1) along $\tau \mapsto \mathbb{Z} + \mathbb{Z}\tau$ gives exactly $\mathcal{D}E_2 = \frac{E_2^2 - E_4}{12}$, $\mathcal{D}E_4 = \frac{E_2E_4 - E_6}{3}$, $\mathcal{D}E_6 = \frac{E_2E_6 - E_4^2}{2}$ with $\mathcal{D} = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$; Legendre's $2\pi i$ fixes the constant. (3) E_2 *and the polarization* (§4). The E_2 anomaly is Legendre's relation (Prop. 4.1). In $H^1(E, \mathbb{C})$, $[\wp\, dz] = a[dz] + b[d\bar{z}]$ with $a = -\frac{\pi^2}{3}E_2^*/\omega_1^2$ and $b = -2\pi/(i\int dz \wedge d\bar{z})$ (Thm. 4.2; Katz [15] Lemma 1.3.6): the non-holomorphic term of $E_2^* = E_2 - \frac{3}{\pi\Im\tau}$ *is* -2π over the polarization norm. Holomorphic E_2 is the same coefficient for the Betti-rational "cusp" complement (Cor. 4.3). (4) *Crystalline* (§5). $FV = p$, $\mathrm{Fil}^1 = \ker \bar{F} = \operatorname{im} \bar{V}$ (Prop. 5.1); in the ordinary case the unit-root line U is a Frobenius-defined complement to Fil^1 and its slope is Katz's p-adic E_2 [15, 14]: the p-adic analogue of the $\Im\tau$ splitting, defined by dynamics, *not* by any positive form. It exists equally for fakes violating $|a| \le 2\sqrt{p}$ ([NUM]). Newton-above-Hodge is integrality, not positivity (Rem. 5.5). (5) $\operatorname{Spec}\mathbb{Z}$ (§6, [OPEN]): every rung of the comparison is missing or conjectural; Deninger's Hodge $*$ would give $\Theta = \frac{1}{2} + A$, A skew [8]. *This lemma does not supply the missing rung.* (6) *Manuscript* (§7): (4.1), (2.3), (5.4), $E_6(i) = E_4(\rho) = 0$, 691 and the sliver are correct; (5.1)–(5.3) are correct with $\mathcal{D}$ and off by $2\pi i$ as printed; the Nash identity $2|\mathrm{Ric}|^2 - R^2 = |C|^2 - \mathcal{E} - \frac{R^2}{3}$ and $\chi = \frac{1}{32\pi^2}\int \mathcal{E}$ are verified (Thm. 7.1, Thm. 7.2), with the explanation of $\chi = \sum(-1)^k b_k$ and of how curvature-squared "becomes Betti" (Rem. 7.4). Metaphors (heat flow $H^2 \to 0$, thermal pump, $R \cong T$ via the comparison) are listed as such.

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1. Conventions, and what TAA already has

$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ with $\Im(\omega_2/\omega_1) > 0$, $\tau = \omega_2/\omega_1$, $E = \mathbb{C}/\Lambda$, $\gamma_1, \gamma_2 \in H_1(E, \mathbb{Z})$ the images of $[0, \omega_1], [0, \omega_2]$ (so $\gamma_1 \cdot \gamma_2 = +1$ for the complex orientation). $\wp, \zeta$ are Weierstrass' functions; $x = \wp(z)$, $y = \wp'(z)$, $y^2 = 4x^3 - g_2 x - g_3$, so $dx/y = dz$ and $x\,dx/y = \wp\,dz$. Periods $\omega_j = \oint_{\gamma_j} dz$; quasi-periods $\eta_j = \zeta(z + \omega_j) - \zeta(z) = -\oint_{\gamma_j} \wp\,dz$. $q = e^{2\pi i \tau}$, $\mathcal{D} = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$. $E_2 = 1 - 24\sum \sigma_1(n) q^n$, $E_4 = 1 + 240\sum \sigma_3(n) q^n$, $E_6 = 1 - 504\sum \sigma_5(n) q^n$ (Ramanujan's P, Q, R). $E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \Im \tau}$. $\rho = e^{2\pi i/3}$. For $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$ [THM]/[NUM]:

$$\eta_1 = \tfrac{\pi^2}{3} E_2(\tau), \qquad g_2 = \tfrac{4\pi^4}{3} E_4(\tau), \qquad g_3 = \tfrac{8\pi^6}{27} E_6(\tau), \tag{1}$$

and in general $\eta_1(\Lambda) = \frac{\pi^2}{3} E_2(\tau)/\omega_1$, $g_2 = \frac{4\pi^4}{3} E_4/\omega_1^4$, $g_3 = \frac{8\pi^6}{27} E_6/\omega_1^6$ by homogeneity. These are classical; they were checked to $\le 5.6 \cdot 10^{-21}$ ([NUM], Rem. 2.6).

Overlap table (checked by reading the cited lemmas; nothing here is redone):

Lemma	what it already contains
14 §5–§7	Light cone of $\mathrm{NS}(E \times E)$ (Hodge index), Frobenius as a Lorentz boost, and $\operatorname{Spec}\mathbb{Z} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}$ (not a surface).
23 (tropical note)	"Bilinear relations need the phase i"; Deninger's conjectural Hodge $*$ gives $\Theta = \frac{1}{2}$+skew (recorded, not proved).
42 Thm. 8.2	Newton above Hodge (Mazur); fakes satisfy it.
46 §6, Prop. 6.9	Verschiebung, $FV = q$ integral; Dieudonné module M_a; fakes possess it; algebra gives Krein, positivity is extra.
47, 48	Stage vs. actors; J = Hilbert transform, positivity a separate condition.

So the light cone, the Verschiebung and Newton/Hodge are in TAA already. This lemma adds the *comparison* itself: the period matrix, Gauss–Manin, the two complex complements to F^1 and the p-adic one, and where (and how weakly) positivity enters each.

2. H^1 of an elliptic curve over $\mathbb{C}$

Definition 2.1. $H^1_{\mathrm{B}} = \mathrm{Hom}(H_1(E, \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}^2$ (Betti lattice). $H^1_{\mathrm{dR}} = H^1(E, \mathbb{C})$, with algebraic basis $\omega = [dx/y]$, $\xi = [x\,dx/y]$ (a form of the second kind). Hodge filtration $F^1 = H^0(E, \Omega^1) = \mathbb{C}\omega$;

Hodge decomposition $H^1(E,\mathbb{C}) = F^1 \oplus \overline{F^1}$, $\overline{F^1} = \mathbb{C}[d\bar z]$. Comparison: $H^1_{\mathrm{dR}} \xrightarrow{\sim} H^1_{\mathrm{B}} \otimes \mathbb{C}$, $c \mapsto (\oint_{\gamma_1} c, \oint_{\gamma_2} c)$, with matrix

$$\Pi = \begin{pmatrix} \oint_{\gamma_1}\omega & \oint_{\gamma_2}\omega \\ \oint_{\gamma_1}\xi & \oint_{\gamma_2}\xi \end{pmatrix} = \begin{pmatrix} \omega_1 & \omega_2 \\ -\eta_1 & -\eta_2 \end{pmatrix}.$$

Theorem 2.2 (Legendre; [THM], classical; [CITED] DLMF 23.2.14 [9], Katz [15] §1.2)**.** $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$. *Equivalently* $\det\Pi = 2\pi i$.

Proof. Integrate $\zeta(z)\,dz$ over the boundary of a translate P of the fundamental parallelogram with vertices $z_0, z_0+\omega_1, z_0+\omega_1+\omega_2, z_0+\omega_2$ (positively oriented since $\Im\tau > 0$) containing one lattice point. The residue theorem gives $2\pi i$. Pairing opposite edges gives $\int_{z_0}^{z_0+\omega_1}(\zeta(z)-\zeta(z+\omega_2))dz + \int_{z_0}^{z_0+\omega_2}(\zeta(z+\omega_1)-\zeta(z))dz = -\eta_2\omega_1 + \eta_1\omega_2$. Then $\det\Pi = -\omega_1\eta_2 + \omega_2\eta_1$. □

Proposition 2.3 (Riemann bilinear relations, $g=1$; [THM])**.** *For closed* 1*-forms,* $\int_E \alpha\wedge\beta = \oint_{\gamma_1}\alpha\oint_{\gamma_2}\beta - \oint_{\gamma_2}\alpha\oint_{\gamma_1}\beta$. *Hence (i)* $\int\omega\wedge\xi = 2\pi i$ *(the cup product of the de Rham basis is Legendre's relation); (ii)* $\int\omega\wedge\omega = 0$ *(F^1 is isotropic); (iii)* $i\int_E\omega\wedge\bar\omega = 2\Im(\bar\omega_1\omega_2) = 2|\omega_1|^2\Im\tau > 0$.

Proof. The first formula is the standard cut-open-the-torus computation. (i) is $\omega_1(-\eta_2) - \omega_2(-\eta_1)$ and Thm. 2.2. (ii) is trivial. (iii): $\int dz\wedge d\bar z = \omega_1\bar\omega_2 - \omega_2\bar\omega_1 = -2i\Im(\bar\omega_1\omega_2)$, and $\Im(\bar\omega_1\omega_2) = |\omega_1|^2\Im\tau$ is the area of E (also $dz\wedge d\bar z = -2i\,dx\wedge dy$). □

Remark 2.4 (how much positivity is in (iii); [THM])**.** $\Im\tau > 0$ is the Hodge–Riemann positivity in its smallest case. But in genus 1 it costs nothing: $\Im\tau \neq 0$ just says ω_1, ω_2 are $\mathbb{R}$-independent (Λ is a lattice), and the sign is fixed by orienting $\gamma_1\cdot\gamma_2 = +1$. Every one-dimensional complex torus is polarizable. Positivity becomes a real condition only for $g \geq 2$: a period matrix $(I\ \Omega)$ is polarized iff Ω is symmetric with $\Im\Omega$ positive definite (Riemann's conditions, standard), and a general complex torus of dimension ≥ 2 is not algebraic. For ζ (§6) the putative H^1 is infinite-dimensional, and there positivity is the whole problem. So the elliptic curve exhibits *where* the polarization sits in the comparison, but not *why* it would hold for ζ.

Proposition 2.5 (the τ-normalized period matrix; [THM])**.** *For* Λ_τ*:* $\Pi = \left(\begin{smallmatrix} 1 & \tau \\ -\eta_1 & -\eta_2 \end{smallmatrix}\right)$, $\eta_2 = \tau\eta_1 - 2\pi i$, $\eta_1 = \frac{\pi^2}{3}E_2(\tau)$. *The comparison is therefore determined by the single quasi-modular number* $E_2(\tau)$ *together with* τ *and* $2\pi i$.

Proof. Thm. 2.2 with $\omega_1 = 1$, $\omega_2 = \tau$, and (1). □

Remark 2.6 (numerics; [NUM])**.** `num/lemma49_periods.py`, `lemma49_rhombic.py` (mpmath, 40 digits; data `num/l49/periods.json`, `rhombic.json`) compute ω_j, η_j by direct integration of dx/y, $x\,dx/y$ along cycles for curves with real roots, $(g_2,g_3) = (7,1), (4,0), (12,-3)$, and with one real root, $(1,2)$ ($\tau = 0.5 + 0.939622i$) and $(-3,5)$ ($\tau = 0.5 + 0.759250i$). Legendre $-2\pi i$: $\leq 2.2\cdot 10^{-20}$; η_1 vs. E_2: $\leq 5.6\cdot 10^{-21}$; g_2, g_3 vs. E_4, E_6: $\leq 4.9\cdot 10^{-20}$. (A first Simpson-rule version with branch tracking had error $\sim 10^{-3}$; it was replaced by a continuous branch $y = 2i\sqrt{e_1-x}\sqrt{x-e_2}\sqrt{x-e_3}$.)

3. Ramanujan's system is Gauss–Manin plus Legendre

Proposition 3.1 (Gauss–Manin; [THM] by computation; [CITED] agrees with [23] Prop. 6)**.** *For* $y^2 = 4x^3 - ax - b$ *over* $\Delta_{\mathrm{m}} = a^3 - 27b^2 \neq 0$*, the GM connection* [16] *([BIB]) in the basis* $\omega = [dx/y]$, $\xi = [x\,dx/y]$ *is*

$$\Delta_{\mathrm{m}}\nabla_{\partial_a}\omega = -\tfrac{a^2}{4}\omega - \tfrac{9b}{2}\xi, \qquad \Delta_{\mathrm{m}}\nabla_{\partial_b}\omega = \tfrac{9b}{2}\omega + 3a\,\xi,$$
$$\Delta_{\mathrm{m}}\nabla_{\partial_a}\xi = \tfrac{3ab}{8}\omega + \tfrac{a^2}{4}\xi, \qquad \Delta_{\mathrm{m}}\nabla_{\partial_b}\xi = -\tfrac{a^2}{4}\omega - \tfrac{9b}{2}\xi.$$

Proof. Differentiate under the integral sign ($\partial_a(1/y) = x/(2y^3)$, $\partial_b(1/y) = 1/(2y^3)$) and reduce $x^k\,dx/y^3$ to the basis modulo exact forms $d(x^j/y)$. The linear algebra is done exactly in

`num/lemma49_gm.py` (sympy; output `num/l49/gm.json`). Movasati's Prop. 6 [23] prints the same matrix in variables t_2, t_3 with $\Delta = 27t_3^2 - t_2^3 = -\Delta_{\mathrm{m}}$; the entries agree. □

Theorem 3.2 (Ramanujan from GM + Legendre; [THM]; cf. [23] Prop. 7, [29] Prop. 15, [15])**.** *Along $\tau \mapsto E_\tau = \mathbb{C}/\Lambda_\tau$, with $(a,b) = (g_2(\tau), g_3(\tau))$,*

$$\tfrac{dg_2}{d\tau} = \tfrac{i}{\pi}(2g_2\eta_1 - 3g_3), \qquad \tfrac{dg_3}{d\tau} = -\tfrac{i}{6\pi}(g_2^2 - 18g_3\eta_1), \qquad \tfrac{d\eta_1}{d\tau} = -\tfrac{i}{24\pi}(g_2 - 12\eta_1^2),$$

and, by (1)*, exactly*

$$\mathcal{D}E_2 = \tfrac{E_2^2 - E_4}{12}, \qquad \mathcal{D}E_4 = \tfrac{E_2E_4 - E_6}{3}, \qquad \mathcal{D}E_6 = \tfrac{E_2E_6 - E_4^2}{2}, \qquad \mathcal{D} = q\tfrac{d}{dq} = \tfrac{1}{2\pi i}\tfrac{d}{d\tau}.$$

Proof. The cycles γ_j are locally constant, so $\frac{d}{d\tau}\oint_{\gamma_j} c = \oint_{\gamma_j} \nabla_{d/d\tau} c$, with $\nabla_{d/d\tau} = g_2'\nabla_{\partial_a} + g_3'\nabla_{\partial_b}$. Write $\nabla_{d/d\tau}\omega = u\,\omega + v\,\xi$. On γ_1 (periods $(1, -\eta_1)$): $0 = \frac{d}{d\tau}1 = u - v\eta_1$. On γ_2 (periods $(\tau, -\eta_2)$): $1 = u\tau - v\eta_2$. Subtracting τ times the first, $1 = v(\tau\eta_1 - \eta_2) = 2\pi i\, v$ *by Legendre*. So $v = \frac{1}{2\pi i}$ and $u = \frac{\eta_1}{2\pi i}$. By Prop. 3.1 these are two linear equations for (g_2', g_3'); solving gives the first two formulas. The ξ-row on γ_1 gives $-\eta_1' = \oint_{\gamma_1} \nabla_{d/d\tau}\xi$, i.e. the third. The ξ-row on γ_2 must then hold automatically, and does: the consistency residual is 0 (`gm.json`, "eta2_consistency"). Substituting (1) and $\frac{d}{d\tau} = 2\pi i\,\mathcal{D}$ gives the Eisenstein form. For example, $\frac{\pi^2}{3}2\pi i\,\mathcal{D}E_2 = -\frac{i}{24\pi}(\frac{4\pi^4}{3}E_4 - \frac{4\pi^4}{3}E_2^2)$, i.e. $\mathcal{D}E_2 = (E_2^2 - E_4)/12$. The symbolic output is `gm.json`: "DP $= P^2/12 - Q/12$", "DQ $= PQ/3 - R/3$", "DR $= PR/2 - Q^2/2$", checks True. □

Remark 3.3 (what "Ramanujan = Gauss–Manin" means precisely)**.** Three ingredients enter: (i) the flat GM connection on the (a,b)-family; (ii) the choice of section $\tau \mapsto (E_\tau, \text{period normalization } \oint_{\gamma_1}\omega = 1)$; (iii) Legendre's $2\pi i$, which is the *comparison determinant* $\det \Pi$. Without (iii), the $\frac{1}{2\pi i}$ in v would be an undetermined constant. So the Ramanujan system is the GM connection read in the Betti frame, with the de Rham/Betti pairing normalized. Movasati [23] makes this into a vector field on the moduli of (E, ω, ξ) with fixed $\langle\omega, \xi\rangle$ ([CITED] Prop. 6–7). Katz [15] Rem. 5.7.10 gives the p-adic counterpart, $12\eta = P\omega + \dots$ ([CITED]; see §5).

Remark 3.4 (numerics; [NUM])**.** At $\tau \in \{0.1 + 1.1i,\ -0.3 + 0.9i,\ 0.45 + 1.7i,\ i,\ \rho\}$, with q-series to 40 digits and $\mathcal{D}$ computed from the series, the Ramanujan residuals are $\le 2.3 \cdot 10^{-41}$. Replacing $\mathcal{D}$ by the literal $\frac{d}{d\tau}$, as printed in the manuscript's (5.1), gives residuals 0.0035–0.64 (`periods.json`, "literal_dtau_resid_5_1"). The fix is the factor $2\pi i$.

4. E_2 as a quasi-period, and E_2^* as the polarization term

Proposition 4.1 (the E_2 anomaly is Legendre's relation; [THM]; [CITED] [29] (19))**.** $E_2(-1/\tau) = \tau^2 E_2(\tau) + \frac{6\tau}{\pi i}$.

Proof. η is homogeneous of degree -1: $\eta(c\Lambda, c\gamma) = c^{-1}\eta(\Lambda, \gamma)$. Now $\Lambda_{-1/\tau} = \tau^{-1}\Lambda_\tau$, and its first basis vector $1 = \tau^{-1}\cdot\tau$ is the image of $\omega_2 = \tau$. Hence $\eta_1(\Lambda_{-1/\tau}) = \tau\,\eta_2(\Lambda_\tau) = \tau(\tau\eta_1 - 2\pi i)$ by Prop. 2.5. Dividing by $\pi^2/3$ gives $E_2(-1/\tau) = \tau^2 E_2(\tau) - \frac{6i\tau}{\pi}$. □

So the $2\pi i$ of the comparison determinant is exactly what stops E_2 from being modular. [NUM]: residual $\le 4.6 \cdot 10^{-41}$ at four τ; the naive $E_2(-1/\tau) - \tau^2 E_2(\tau)$ is 1.2–3.4 in absolute value (`periods.json`, "modular").

Theorem 4.2 (Hodge splitting of ξ; [THM]; [CITED] Katz [15] Lemma 1.3.6)**.** *In $H^1(E, \mathbb{C})$, with z the lattice coordinate on $\mathbb{C}/\Lambda$, write $[\wp\, dz] = a\,[dz] + b\,[d\bar z]$. Then*

$$a = -\frac{\pi^2}{3}\frac{E_2^*(\tau)}{\omega_1^2}, \qquad b = -\frac{\pi}{|\omega_1|^2\,\Im\tau} = -\frac{2\pi}{i\int_E dz\wedge d\bar z}.$$

Proof. Take periods: $a\omega_j + b\bar\omega_j = -\eta_j$ ($j = 1, 2$). Eliminating a: $b(\bar\omega_1\omega_2 - \bar\omega_2\omega_1) = -\eta_1\omega_2 + \eta_2\omega_1 = -2\pi i$ (Thm. 2.2). Since $\bar\omega_1\omega_2 - \bar\omega_2\omega_1 = 2i\Im(\bar\omega_1\omega_2) = 2iA$, $A = |\omega_1|^2\Im\tau$, we get $b = -\pi/A$, and $i\int dz \wedge d\bar z = 2A$ (Prop. 2.3(iii)). Then $a = (-\eta_1 - b\bar\omega_1)/\omega_1 = -\omega_1^{-2}(\eta_1\omega_1 - \pi|\omega_1|^2/A) = -\frac{\pi^2}{3\omega_1^2}(E_2 - \frac{3}{\pi\Im\tau})$, using $\eta_1\omega_1 = \frac{\pi^2}{3}E_2(\tau)$. □

Katz [15] Lemma 1.3.6 ([CITED], read) states the same thing in his normalization: the line $H^{0,1}$ has direction $-\frac{1}{12}S$ relative to (ω, η), with $S(L)$ the C^∞ weight-two form that Katz calls the "position of $H^{0,1}$" (the Hecke limit of the weight-two Eisenstein series, a multiple of E_2^*), and $\langle\bar\omega, \omega\rangle = 2i\,a(L)$ with $a(L)$ the area. [NUM]: for the three real-root curves of Rem. 2.6, a vs. E_2^* and b vs. $-2\pi/(i\int\omega\wedge\bar\omega)$ agree to $\le 2.1\cdot 10^{-21}$, and $b\cdot i\int\omega\wedge\bar\omega = -6.28318530717959 = -2\pi$ for each of them.

Corollary 4.3 (two complex complements to F^1; [THM])**.** *Let $L_{\gamma_1} = \{c \in H^1(E, \mathbb{C}) : \oint_{\gamma_1} c = 0\}$ (a $\mathbb{Q}$-rational line of $H^1_{\mathrm{B}} \otimes \mathbb{C}$). Then $H^1(E, \mathbb{C}) = F^1 \oplus \overline{F^1} = F^1 \oplus L_{\gamma_1}$, and the F^1-component of $[\wp\, dz]$ is*

- *along $\overline{F^1}$ (Hodge/polarization complement): $-\frac{\pi^2}{3}E_2^*(\tau)\,\omega_1^{-2}[dz]$ — modular of weight 2, not holomorphic;*
- *along L_{γ_1} (Betti-rational "cusp" complement): $-\eta_1\omega_1^{-1}[dz] = -\frac{\pi^2}{3}E_2(\tau)\,\omega_1^{-2}[dz]$ — holomorphic, but only invariant under the stabilizer $\pm\Gamma_\infty$ of γ_1.*

They differ by $\frac{\pi}{\omega_1^2\Im\tau}$.

Proof. The first is Thm. 4.2. For the second, if $[\wp\, dz] = a'[dz] + \ell$ with $\ell \in L_{\gamma_1}$, integrating over γ_1 gives $a'\omega_1 = -\eta_1$. L_{γ_1} changes when γ_1 does, and $\overline{F^1}$ is canonical. □

This is the cleanest statement of the chat claim "E_2^* uses $\Im\tau$ as the polarization": the canonical complement is $\overline{F^1}$, the $H^{0,1}$ line, which is the orthogonal complement of F^1 for the Hermitian form $h(c, c') = i\int c \wedge \bar c'$. The coefficient that measures it is -2π divided by $h(\omega, \omega)$.

Proposition 4.4 (Serre derivative vs. Maass–Shimura; [THM]; [CITED] [29] (53), (55); [BIB] [27])**.** *For f of weight k, $\vartheta_k f = \mathcal{D}f - \frac{k}{12}E_2 f$ (Serre) and $\partial_k f = \mathcal{D}f - \frac{k}{4\pi y}f$ (Maass–Shimura), $y = \Im\tau$. Then $\vartheta_k = \partial_k - \frac{k}{12}E_2^*$.*

Proof. $\partial_k f - \frac{k}{12}E_2^* f = \mathcal{D}f - \frac{k}{4\pi y}f - \frac{k}{12}E_2 f + \frac{k}{12}\cdot\frac{3}{\pi y}f = \vartheta_k f$. □

In the language of Cor. 4.3: ϑ_k is the holomorphic covariant derivative and ∂_k the one built from the Hodge splitting. They differ by $\frac{k}{12}$ times the difference of the splittings, and both are modular (the non-modularity of E_2 cancels against that of $\mathcal{D}$). The manuscript's (5.4), $\frac{1}{2\pi i}\frac{df}{d\tau} - \frac{k}{12}Pf$, is ϑ_k and is correct as printed. [NUM]: $E_2^*(-1/\tau) = \tau^2 E_2^*(\tau)$ to $\le 5.1\cdot 10^{-41}$.

Remark 4.5 (which part of "positivity" is used)**.** Thm. 4.2 uses (a) complex conjugation on $H^1_{\mathrm{B}}\otimes\mathbb{C}$, i.e. the real structure; (b) $h(\omega, \omega) \neq 0$. The *sign* of $h(\omega, \omega)$ plays no role in the formula, and in genus 1 it is fixed by orientation (Rem. 2.4). So "E_2^* uses $\Im\tau$ as the polarization" is correct. "E_2^* is where Hodge–Riemann *positivity* does work" is too strong: the work is done by the Hermitian form, and its definiteness is free in this case.

5. The crystalline analogue over $\mathbb{F}_p$

Let $E/\mathbb{F}_p$ be an elliptic curve, $a = p+1-\#E(\mathbb{F}_p)$, and $M = H^1_{\mathrm{crys}}(E/\mathbb{Z}_p)$ ($\cong H^1_{\mathrm{dR}}$ of any lift), free of rank 2, with $\mathbb{Z}_p$-linear Frobenius F (since the base is $\mathbb{F}_p$), characteristic polynomial $T^2 - aT + p$, and $V = pF^{-1}$, so $FV = VF = p$. Then $\bar M = M/pM = H^1_{\mathrm{dR}}(E/\mathbb{F}_p)$ with Hodge line $\mathrm{Fil}^1 = H^0(E, \Omega^1)$ [2, 18, 25] ([BIB]). As in Lemma 46 Prop. 6.9, the model is $M_a = \mathbb{Z}_p[T]/(T^2 - aT + p)$, $F = T$, $V = a - T$.

Proposition 5.1 (the Hodge line is $\operatorname{im}\bar V$; [THM] given $F(\mathrm{Fil}^1)\subset pM$)**.** *(a)* $\ker\bar F=\operatorname{im}\bar V$*, a line.*
(b) $\mathrm{Fil}^1=\ker\bar F=\operatorname{im}\bar V$*.*
*(c) (*ordinary*,* $p\nmid a$*)* T^2-aT+p *has a unique root* $u\in\mathbb{Z}_p^\times$*, and* $M=U\oplus U'$ *with* $F|_U=u$*,* $F|_{U'}=p/u$ *(so* $V|_{U'}=u$*). The reduction of* U' *is* Fil^1*, and* $\bar U$ *is a complement:* $\bar M=\mathrm{Fil}^1\oplus\bar U$*.* U *is the* unit-root subspace*.*
*(d) (*supersingular*,* $p\mid a$*)* $\bar F^2=0$*,* $\operatorname{im}\bar F=\ker\bar F=\mathrm{Fil}^1$*, and* $\bar M$ *has no* $\bar F$*-stable complement to* Fil^1*.*
(e) The Hasse invariant is $\equiv a \pmod p$*;* E *is ordinary iff* $\bar F$ *is not nilpotent.*

Proof. (a) $\det F=p$, so M/FM has length 1 and $\bar F$ has rank exactly 1. F is injective, so $Fm\in pM=FVM$ gives $m\in VM$, i.e. $\ker\bar F=\operatorname{im}\bar V$. (b) F is divisible by p on Fil^1 (Mazur's divisibility, [18]; used in this form by Katz [15] §5.7, [CITED]), so $\mathrm{Fil}^1\subset\ker\bar F$, and both are lines. (c) Hensel. $p/u\equiv 0$, so $\bar F$ kills $\bar U'$, and $\bar U'=\ker\bar F=\mathrm{Fil}^1$; $\bar F|_{\bar U}=\bar u\neq 0$. (d) $a\equiv 0$ gives $\bar F^2=a\bar F-p\equiv 0$. A nonzero nilpotent endomorphism of a plane has a unique invariant line, namely its kernel. (e) $\operatorname{tr}\bar F=\bar a$, and $\bar F$ is nilpotent iff $\bar a=0$. □

Remark 5.2 (numerics; [NUM])**.** `num/lemma49_dieud.py` (`num/l49/dieud.json`) for $E=11a$, with a_p re-checked by point count: $p=5$, $a=1$, $u\equiv 15346 \pmod{5^6}$; $p=7$, $a=-2$, $u\equiv 91117 \pmod{7^6}$; $p=13$, $a=4$, $u\equiv 1171720 \pmod{13^6}$. In each ordinary case $\bar U'=\mathrm{Fil}^1$ and $\bar U$ is complementary. For $p=19,29$ ($a=0$): $\bar F^2=0$ and $\operatorname{im}\bar F=\ker\bar F$. *Fakes* at $p=13$ (Lemma 42 sense): $a=8$ (violates $|a|\le 2\sqrt{13}\approx 7.21$) is ordinary with $u\equiv 1104020 \pmod{13^6}$ and a perfect unit-root splitting; $a=13$ is supersingular-type. The crystalline structure, Hodge line and unit-root splitting included, is blind to the Weil bound.

Theorem 5.3 (the p-adic E_2 is the slope of the unit-root complement; [CITED] Katz [15] (5.7.5), Lemma 5.7.8, Rem. 5.7.10; [14] App. 2, A2.3–A2.4)**.** *Over the ordinary locus there is a unique* F*-stable splitting* $H^1_{\mathrm{dR}}=H^0(\Omega^1)\oplus U$*, with* F *an automorphism on the unit-root part* U *(5.7.5). With* $\omega=dx/y$*,* $\eta=x\,dx/y$*,* $(\omega,\eta)_{\mathrm{DR}}=1$*, and* $u\in U$ *an invertible section, Katz defines* P *by the direction of* U *relative to* (ω,η) *"in complete analogy with 1.3.6", proves that* P *is a* p*-adic modular function of weight two whose* q*-expansion is Ramanujan's series* $1-24\sum\sigma_1(n)q^n$ *(Lemma 5.7.8, computation in* [14] *A2.4), and records* $12\eta=P\omega+12\frac{(\omega,\eta)}{(u,\omega)}u$ *(5.7.11, up to the OCR of the printed fraction). So the* $H^0(\Omega^1)$*-component of* 12η *along* U *is* $P\omega$*, the* p*-adic counterpart of Thm. 4.2, where the component along* $\overline{F^1}$ *is (up to normalization)* E_2^**.*

(We read both texts and give the statement in words; the precise normalizations are as printed there.)

Remark 5.4 (the dictionary; [ANALOGY] with [THM] pieces)**.**

	over $\mathbb{C}$	over $\mathbb{Z}_p$, ordinary
filtration	$F^1=\mathbb{C}\omega$	$\mathrm{Fil}^1=\operatorname{im}\bar V$ (lifted by U')
canonical complement	$\overline{F^1}=H^{0,1}$	U, unit-root line
defined by	conjugation + Hermitian form h	Frobenius dynamics ($F|_U$ a unit)
slope of η	E_2^* (Thm. 4.2)	p-adic $E_2=P$ (Thm. 5.3)
holomorphic but not canonical	L_{γ_1}, gives E_2	—
degenerate case	none (h nondegenerate)	supersingular: no complement

So the unit-root splitting is the p-adic analogue of the $\Im\tau$ splitting, as the chat said. But it is defined without any positive form: U is an eigenline. The positivity that forces $|a|\le 2\sqrt{p}$ lives elsewhere, in the Rosati/Hodge-index form on $\mathrm{NS}(E\times E)$ (Lemma 14 §5). In Lemma 14's notation (H, V there are the fibre classes, not the Verschiebung), $H^2=V^2=\Delta^2=0$, $H\cdot V=\Delta\cdot H=\Delta\cdot V=1$, and $\gamma=\Delta-H-V$ has $\gamma\cdot H=\gamma\cdot V=0$, $\gamma^2=-2$ [THM]. The transverse negativity on

$\langle H, V\rangle^{\perp}$ is the H^1 polarization tensored with itself (Rosati: $\deg\varphi = \varphi\varphi^{\dagger} > 0$; [RESTATEMENT], standard). Lemma 46 §6 made the same point from the Verschiebung side: $FV = p$ is integral, fakes possess it, and algebra alone gives a Krein form, not a Hilbert one.

Remark 5.5 (Mazur is integrality, not positivity; correction)**.** Newton-above-Hodge (Katz's conjecture, proved by Mazur [18, 19]; see also [2]) says here that the Frobenius slopes lie on or above the Hodge slopes $(0, 1)$, i.e. $u, p/u \in \mathbb{Z}_p$. This is an *integrality* statement, and the fakes of Rem. 5.2 satisfy it (Lemma 42 Thm. 8.2). The chat (t184) called it "the positivity-type constraint"; it is not one.

6. What $\operatorname{Spec}\mathbb{Z}$ lacks

For $E/\mathbb{C}$ (or $E/\mathbb{F}_p$), every rung of the comparison is present. For ζ, here is what is known, rung by rung.

rung	elliptic curve	$\operatorname{Spec}\mathbb{Z}$
lattice H^1_{B}	$\mathbb{Z}^2$	none known; Deninger's $H^1(\overline{\operatorname{Spec}\mathbb{Z}};\mathbb{R})$ is conjectural, infinite-dimensional [8] [CONJ]
filtration / complement	F^1, $\overline{F^1}$ (Thm. 4.2); Fil^1, U (§5)	none known
comparison $\det\Pi$	$2\pi i$ (Thm. 2.2)	none known
flow / Frobenius	GM along τ; F, V	Deninger's Θ, spectrum = zeros [8] [CONJ]; scaling action in Connes' approach [6] [BIB]
polarization	$i\int\omega\wedge\bar\omega > 0$ (automatic, Rem. 2.4); Rosati on $E\times E$	**missing**; archimedean local positivity only [5]
surface $X\times X$	$E\times E$, light cone (Lemma 14 §5)	$\operatorname{Spec}\mathbb{Z}\times_{\mathbb{Z}}\operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}$ (Lemma 14 §7); the arithmetic site square [4] lacks the positivity (Lemma 23)

Proposition 6.1 (Deninger's conditional argument; [CITED] [8] pp. 168–169; framework [CONJ]; implication [RESTATEMENT])**.** *Suppose there were a cohomology $H^1(\overline{\operatorname{Spec}\mathbb{Z}};\mathbb{R})$ with a flow $\phi^t = \exp(t\Theta)$ induced by a flow on an underlying space, so that Θ is a derivation for cup product and acts by 1 on $H^2 = \mathbb{R}(-1)$, and suppose a Hodge $*$-operator commuting with Θ makes $\langle f, f'\rangle = \operatorname{tr}(f\cup *f')$ positive definite. Then $\Theta = \frac{1}{2} + A$ with A skew-symmetric, so the spectrum of Θ, i.e. the nontrivial zeros, lies on $\Re s = \frac{1}{2}$.*

Sketch, as in [8]. $\Theta(f_1\cup *f_2) = \Theta f_1\cup *f_2 + f_1\cup *\Theta f_2$ and $\Theta = 1$ on H^2 give $\langle f_1, f_2\rangle = \langle \Theta f_1, f_2\rangle + \langle f_1, \Theta f_2\rangle$, so $\Theta - \frac{1}{2}$ is skew. □

This is precisely the polarization rung of the table, and it is the rung on which this lemma's elliptic-curve material has nothing to say. For $E/\mathbb{C}$ the analogous positivity is free (Rem. 2.4); over $\mathbb{F}_p$ it comes from $E\times E$ (Lemma 14), not from crystalline theory (Rem. 5.4, 5.5). Connes–Consani [5] ([CITED] in Lemma 46) prove the archimedean part of Weil positivity for test functions supported in $[2^{-1/2}, 2^{1/2}]$. That is a local statement and does not give the global form.

Question 6.2 ([OPEN])**.** Construct a positive-definite form on a space carrying the zeros of ζ as the spectrum of a derivation, compatible with a cup-product/Poincaré-duality structure (Prop. 6.1). **This lemma does not supply the missing rung.** It only shows, in the smallest case, where the rung sits in the Betti–de Rham comparison: as the Hermitian form that selects $\overline{F^1}$ (Thm. 4.2). Over $\mathbb{F}_p$, the analogous complement U is selected by dynamics alone and carries no positivity.

7. What survives from the 2026-10-22 manuscript

The manuscript ("Thermodynamics of Arithmetic Action: scale-free vacuum fields and the modularity isomorphism", 6 pp.) is read below equation by equation. We are plain about which

statements are mathematics and which are metaphor written in mathematical notation. That is not a judgement on the intuition behind it: several of the metaphors point at genuine structures, and we name them where they do.

7.1. **Mathematics that is correct as printed.**

(S1) *(4.1)*, the matrix $\left(\oint_{\gamma_i}\omega_j\right)$, is a genuine period matrix: it is Π of Def. 2.1 (with $\omega_1 = dx/y$, $\omega_2 = x\,dx/y$ there). The determinant is $2\pi i$ (Thm. 2.2). The "$\iff M_{R\cong T}$" attached to it has no mathematical content as an equivalence (§7.4).

(S2) *(5.4)*: the Serre derivative $\frac{1}{2\pi i}\frac{df}{d\tau} - \frac{k}{12}Pf$ is correct (Prop. 4.4).

(S3) *(2.3)*: $\Delta = (E_4^3 - E_6^2)/1728 = q\prod(1-q^n)^{24}$, residual $1.1\cdot 10^{-44}$ ([NUM], `periods.json`).

(S4) $E_6(i) = 0$, $E_4(\rho) = 0$: residuals $2.3\cdot 10^{-41}$, $3.4\cdot 10^{-41}$ ([NUM], `num/lemma49_misc.py`, `misc.json`). The manuscript's $\rho = e^{i\pi/3}$ is the translate of $e^{2\pi i/3}$ by 1; both are fine.

(S5) 691: $B_{12} = -691/2730$, and $691\,E_{12} = 441E_4^3 + 250E_6^2$ with $E_{12} = 1 + \frac{65520}{691}\sum\sigma_{11}(n)q^n$ (residual $\le 9\cdot 10^{-38}$ at three τ, [NUM]). So 691 is the numerator of B_{12} and the denominator of the normalized E_{12}: a normalization fact (and the source of Ramanujan's congruence $\tau(n)\equiv\sigma_{11}(n) \bmod 691$, standard). It does not "seal" anything.

(S6) *Elliptic points*: i and ρ are the only elliptic points of $PSL_2(\mathbb{Z})$ ("$PSS(2,\mathbb{Z})$" is a typo), with stabilizers of orders 2 and 3. The fundamental domain's angles there are π at i and $\pi/3$ at ρ. The 90° and 60° are the rotation angles of the automorphism groups μ_4, μ_6 of the square and hexagonal lattices $\mathbb{Z}[i]$, $\mathbb{Z}[\rho]$.

(S7) *The sliver* $\sim 1/\ln t$: this is the classical zero-free region $\sigma \ge 1 - \frac{c}{\log|t|}$ (de la Vallée Poussin [7], [BIB]). An explicit form is $\sigma \ge 1 - \frac{1}{5.573412\log|t|}$ for $|t|\ge 2$ (Mossinghoff–Trudgian [21], [CITED] abstract/Thm. 1). It is a region near $\Re s = 1$ that contains no zeros. It concentrates nothing on $\Re s = \frac12$.

7.2. **Correct after a fix.**

(S8) *(5.1)–(5.3)*: correct with $\frac{d}{d\tau}$ replaced by $\mathcal{D} = \frac{1}{2\pi i}\frac{d}{d\tau}$ (Thm. 3.2). This is the normalization the manuscript itself uses in (5.4). As printed they are false; the residuals are 0.0035–0.64 (Rem. 3.4).

(S9) *(1.1)*: $\frac{d}{d\theta}\tan\theta = 1 + \tan^2\theta = \sec^2\theta$ is correct calculus. "$2R_{\mu\nu}R^{\mu\nu} \leftrightarrow L^2$, $R^2 \leftrightarrow T^2$" is [ANALOGY] only: $\mathcal{L}_{\text{Nash}}$ is not constant and has dimension length^{-4}.

(S10) *(2.1)*: the Euler product converges absolutely for $\Re s > 1$. The ratio $90/60 = 3/2$ plays no role, and "$s > 3/2$" is not a threshold of anything here.

(S11) *§5.3, $\epsilon^3 = 0$*: there is a correct kernel. In deformation theory of Galois representations, lifts over $k[\epsilon]/\epsilon^2$ form the tangent space $H^1(\mathrm{ad})$, and the obstruction to lifting along a small extension such as $k[\epsilon]/\epsilon^3 \to k[\epsilon]/\epsilon^2$ lies in $H^2(\mathrm{ad})$ (standard). Taylor–Wiles [28] ([BIB]) prove that the relevant Hecke algebras are complete intersections. The topos language adds nothing to this.

7.3. **The Nash decomposition and Chern–Gauss–Bonnet.** Notation, in dimension 4 and any signature: $|\mathrm{Rm}|^2 = R_{abcd}R^{abcd}$, $|\mathrm{Ric}|^2 = R_{ab}R^{ab}$, C the Weyl tensor, $|C|^2 = |\mathrm{Rm}|^2 - 2|\mathrm{Ric}|^2 + \frac13 R^2$, and the Gauss–Bonnet density $\mathcal{E} = |\mathrm{Rm}|^2 - 4|\mathrm{Ric}|^2 + R^2$.

Theorem 7.1 (Nash decomposition; [THM], exact [NUM])**.** *Pointwise, in dimension 4 and any signature,*

$$2|\mathrm{Ric}|^2 - R^2 = |C|^2 - \mathcal{E} - \tfrac13 R^2.$$

So Nash's Lagrangian $\sqrt{|g|}\,(2R_{ab}R^{ab} - R^2)$ [24, 1] *is* $\sqrt{|g|}\,(|C|^2 - \mathcal{E} - \frac13 R^2)$.

Proof. $|C|^2 - \mathcal{E} - \frac13 R^2 = (-2+4)|\mathrm{Ric}|^2 + (\frac13 - 1 - \frac13)R^2$. [NUM]: `num/lemma49_curv.py` checks the Weyl decomposition, tracelessness, Bianchi and this identity exactly (rational arithmetic) on

random algebraic curvature tensors in signatures $(++++)$ and $(-+++)$ (`curv.json`, all residuals 0). □

Theorem 7.2 (Chern–Gauss–Bonnet and signature in dimension 4; [BIB] [3, 13, 10]; normalizations [NUM])**.** *For a closed oriented Riemannian 4-manifold M,*

$$\chi(M) = \frac{1}{32\pi^2}\int_M \mathcal{E}\,dV, \qquad \sigma(M) = \frac{1}{48\pi^2}\int_M \left(|W^+|^2 - |W^-|^2\right) dV,$$

with tensor norms ($|C|^2 = |W^+|^2 + |W^-|^2$). Hence

$$\int_M \mathcal{L}_{\mathrm{Nash}}\,dV = \int_M |C|^2\,dV - 32\pi^2\chi(M) - \tfrac{1}{3}\int_M R^2\,dV.$$

[NUM] (`curv.json`): unit S^4: $\mathcal{E} = 24$, Vol $= \frac{8\pi^2}{3}$, $\chi = 2$. Unit $S^2 \times S^2$: $\mathcal{E} = 8$, Vol $= 16\pi^2$, $\chi = 4$, $|W^\pm|^2 = \frac{8}{3}$, $\sigma = 0$. $\mathbb{C}P^2$ (Fubini–Study, holomorphic sectional curvature 4, Vol $= \pi^2/2$): $|\mathrm{Rm}|^2 = 192$, $|\mathrm{Ric}|^2 = 144$, $R = 24$, $\mathcal{E} = 192$, $|C|^2 = |W^+|^2 = 96$, $|W^-|^2 = 0$, so $\chi = 3$ and $\sigma = 1$. A non-round warped S^4 ($h = \sin r + 0.3\sin^3 r$) gives $\chi = 2.0$; there the integrand is $-24h''(1-h'^2) = 8\frac{d}{dr}(h'^3 - 3h')$, a total derivative, which is the mechanism in miniature.

Remark 7.3 (caveats; [THM]/standard)**.** (i) The variation of $\int \mathcal{E}$ vanishes identically in dimension 4 in *any* signature: $\mathcal{E}\,dV$ is a multiple of the Pfaffian of the curvature, and Chern–Weil theory makes its variation exact. So Nash's field equations are those of $\int(|C|^2 - \frac{1}{3}R^2)$, fourth order in g. They are *not* "first-order ($d\omega = 0$)" as the manuscript says: Nash's own note calls his equation "a 4th order covariant tensor partial differential equation" ([CITED] [24]; also [1]). (ii) Physical spacetime is Lorentzian and non-compact. There $\int \mathcal{E}$ is a boundary term, not $32\pi^2\chi$. A closed manifold admits a Lorentzian metric iff it has a nowhere-zero line field, iff $\chi = 0$ (standard), so on compact Lorentzian manifolds the χ-term is 0. (iii) $|C|^2$ is not topological; only $|W^+|^2 - |W^-|^2$ is. (iv) Every quadratic curvature action is scale invariant in dimension 4; "scale-free" does not single out Nash's combination. (v) What makes $\int \mathcal{E}$ topological is Chern–Weil (the Pfaffian), not "$N = 4$ Poincaré duality" (§6.3 of the manuscript). Poincaré duality gives $b_k = b_{4-k}$ and the unimodularity of the intersection form on H^2.

Remark 7.4 (for J.W.: χ as an alternating sum, and how curvature squared "becomes Betti")**.** *Betti numbers.* $b_k = \dim H^k(M;\mathbb{R})$ counts the independent k-dimensional "holes": b_0 components, b_1 independent loops, b_2 independent closed surfaces not bounding anything, and so on. $\chi(M) = \sum_k (-1)^k b_k$, which for a triangulation is also vertices − edges + triangles $-\cdots$. For a closed, connected, oriented 4-manifold, $b_0 = b_4 = 1$ and $b_1 = b_3$ (Poincaré duality), so

$$\chi = 2 - 2b_1 + b_2 : \qquad S^4 : 2, \quad S^2 \times S^2 : 4, \quad \mathbb{C}P^2 : 3, \quad T^4 : \ 2 - 8 + 6 = 0.$$

How curvature becomes Betti. There are three steps, each a theorem. (1) *Hodge*: on a compact Riemannian M, b_k is the dimension of the space of harmonic k-forms, the kernel of $\Delta_k = dd^* + d^*d$. (2) *McKean–Singer* [20] ([BIB]): $\chi(M) = \sum_k (-1)^k \operatorname{Tr} e^{-t\Delta_k}$ for *every* $t > 0$. The nonzero eigenvalues cancel between even and odd forms, because $d + d^*$ pairs them, so the alternating trace does not depend on t. As $t \to \infty$ only the harmonic forms survive, which gives $\sum(-1)^k b_k$. (3) *Patodi* [26] ([BIB]): as $t \to 0$ the alternating heat-kernel trace localizes pointwise to the Pfaffian of the curvature, which in dimension 4 is $\frac{1}{32\pi^2}\mathcal{E}\,dV$. The other t-dependent terms cancel, and the alternating sum is what forces the cancellation. So the curvature-squared density is the $t \to 0$ end of a quantity that is constant in t, and the Betti count is its $t \to \infty$ end. This is the honest form of the manuscript's "heat equation" passage (3.1)–(3.2): a heat flow on differential forms on M, with the Seeley–DeWitt coefficients as its small-t data. Those coefficients are curvature invariants. They are not "weights of the Eisenstein series", and the flow does not touch ζ.

The sibling statement, and the Lemma 14 link. The same mechanism with the signature operator gives $\sigma(M) = b_2^+ - b_2^-$, the signature of the intersection form on H^2, as $\frac{1}{48\pi^2}\int(|W^+|^2 - |W^-|^2)$

(Hirzebruch [13], [BIB]). For a complex algebraic surface the intersection form on $H^{1,1}_{\mathbb{R}}$ has signature $(1, h^{1,1}-1)$ (Hodge index, standard): this is the light cone of Lemma 14. This answers t183 (Hodge index vs. Atiyah–Singer) once more. Index theory computes the number σ from curvature, and it is blind to which classes are positive. The Hodge index theorem locates the positive direction, and that is the positivity Weil uses. Both are topological or Hodge-theoretic invariants of a *surface*, and as Lemma 14 §7 recalls, $\operatorname{Spec}\mathbb{Z}$ has no surface to apply them to.

7.4. **Metaphor written as mathematics.** Stated plainly, and with the genuine nearby object where there is one:

(M1) *"Heat flow $H^2 \to 0$", "spectral sequence collapses at E_2".* No heat equation acts on Galois cohomology, and no spectral sequence is specified. Nearby genuine object: Taylor–Wiles auxiliary primes are *chosen* (a discrete choice, not a flow) to kill a dual Selmer group, which by Poitou–Tate duality controls the obstructions [28] ([BIB], standard).

(M2) *"Taylor–Wiles is a non-Hermitian thermal pump".* No defined object.

(M3) *"$R \cong T$ via the comparison" / (4.1) $\iff$ $M_{R\cong T}$.* The Betti–de Rham comparison of one curve is not the isomorphism of a deformation ring with a Hecke algebra. Pointers to where periods genuinely meet $R = T$, not used here: congruence modules and adjoint L-values divided by periods (Hida [12], [BIB] title-level), and Flach's bound on the symmetric-square Selmer group [11] ([BIB]).

(M4) *Iwasawa "heat tower" (4.2).* $\varprojlim_n H^2(K_n, \cdot)$ is a genuine Iwasawa module. It is an inverse limit over a tower of fields, not a time evolution.

(M5) *"Maslov dequantization" turns R into T.* Tropicalization was tested in Lemma 23 and cannot force $|\alpha| = \sqrt{q}$.

(M6) *Kustaanheimo–Stiefel.* The KS map genuinely regularizes Kepler motion into a 4D harmonic oscillator [17] ([BIB]). The manuscript gives no map from it to modular forms or to H^2.

(M7) *"Poincaré duality forces R^2 to dequantize into the Betti cycles $2R_{\mu\nu}R^{\mu\nu}$" (§6.3).* The correct statement is Thm. 7.2 with Rem. 7.4: the *Euler combination* $\mathcal{E}$ integrates to $32\pi^2\chi$, while $2|\mathrm{Ric}|^2$ and R^2 separately are not topological.

(M8) *Critical line as shear-balance axis; Leech lattice; Lagrange points.* [ANALOGY] only. The genuine symmetry behind "$\frac{1}{2}$ from both walls" is the functional equation $s \leftrightarrow 1-s$, which is "stage" in Lemma 47's sense and is shared by Davenport–Heilbronn (Lemma 48).

Remark 7.5 (suggestion)**.** If the manuscript is revised: add a statement of intellectual honesty; replace $\frac{d}{d\tau}$ by $\mathcal{D}$ in (5.1)–(5.3); replace "first-order" by "fourth-order" for Nash; replace "Betti–de Rham plane" by the period matrix Π; and mark §§3, 4.2, 6 as heuristic.

8. Corrections to the chat claims

(C1) *"Ramanujan's P, Q, R system is the Gauss–Manin connection."* Correct, with a precision: it is GM *plus Legendre*, read in the Betti frame with $\oint_{\gamma_1}\omega = 1$ (Thm. 3.2, Rem. 3.3). The correct normalization uses $\mathcal{D} = q\frac{d}{dq}$; the manuscript's (5.1)–(5.3) with $\frac{d}{d\tau}$ are off by $2\pi i$.

(C2) *"E_2^* uses $\Im\tau$ as the polarization."* Correct (Thm. 4.2): the non-holomorphic term is $-2\pi/(i\int\omega\wedge\bar\omega)$, and Katz [15] Lemma 1.3.6 says exactly this. Precision (Rem. 2.4, 4.5): in genus 1 the definiteness is automatic up to orientation, so this is where the *Hermitian form* enters, not where positivity does nontrivial work.

(C3) *"The crystalline Hodge line is the image of V mod p."* Correct: $\mathrm{Fil}^1 = \operatorname{im}\bar V = \ker\bar F$ (Prop. 5.1). In the ordinary case the slope-1 eigenline U' lifts it and the unit-root line U is complementary.

(C4) *"The unit-root splitting is the p-adic analogue of the $\Im\tau$ splitting."* Correct as an analogy, and Katz [15] §5.7 builds the p-adic E_2 from it (Thm. 5.3). Difference: U is defined by dynamics, with no positive form, and it exists equally for fakes (Rem. 5.2).

(C5) *Legendre sign (t184).* With $\Im(\omega_2/\omega_1) > 0$ and $\eta_j = \zeta(z+\omega_j) - \zeta(z)$: $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$, so $\omega_1\eta_2 - \omega_2\eta_1 = -2\pi i$. The chat's "$\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$" has the wrong sign under this convention (DLMF 23.2.14, and [NUM]). Note: Movasati's Prop. 19 [23] appears, under our reading, to print the opposite sign as well. We did not reconcile this (probably an orientation convention), and we follow DLMF and the numerics.

(C6) *"Mazur's Newton-above-Hodge is the positivity-type constraint."* No: it is integrality, and fakes satisfy it (Rem. 5.5; Lemma 42 Thm. 8.2).

(C7) *Nash decomposition.* Correct (Thm. 7.1), exact in any signature.

(C8) $32\pi^2\chi$. Correct for closed oriented Riemannian 4-manifolds (Thm. 7.2), with the caveats of Rem. 7.3: compact Lorentzian manifolds have $\chi = 0$; on non-compact spacetimes the term is a boundary term; $|C|^2$ is not topological.

(C9) *Nash's equation.* Fourth order, not first order (Rem. 7.3(i)).

9. What is new and what is restatement

- **Classical, re-proved or re-derived here ([THM]):** Thm. 2.2, Prop. 2.3, Prop. 3.1, Thm. 3.2, Prop. 4.1, Thm. 4.2 (Katz 1.3.6), Prop. 4.4, Prop. 5.1, Thm. 7.1. None is new mathematics.
- **Presentation-level, probably folklore:** "the E_2 anomaly is the comparison determinant" (Prop. 4.1); the two complex complements and the p-adic one in one table (Cor. 4.3, Rem. 5.4); "in genus 1 polarization is free" as the honest limit of the elliptic model (Rem. 2.4); the fake-a_p unit-root computation showing the crystalline splitting is blind to the Weil bound (Rem. 5.2).
- **Restatement ([RESTATEMENT]):** Deninger's implication (Prop. 6.1).
- **Numerics ([NUM]):** periods and Legendre for five curves; Ramanujan, E_2, E_2^* checks to 10^{-40}; the Dieudonné table; exact curvature identities and the S^4, $S^2 \times S^2$, $\mathbb{C}P^2$ and warped-S^4 checks.
- **No lever on RH.** The comparison places the polarization exactly. It does not produce one for ζ (Question 6.2).

10. Questions for J.W.

Question 10.1. Should "polarization complement ($\overline{F^1}$, E_2^*) vs. dynamical complement (U, p-adic E_2)" (Rem. 5.4) become the official TAA dichotomy? It sharpens Lemma 46's "algebra gives Krein, positivity is extra": over $\mathbb{F}_p$ the canonical complement needs no positivity at all.

Question 10.2. The Nash/Chern–Gauss–Bonnet material (§7.3) is correct but is not about ζ. Keep it in this lemma, or move it to an appendix or a separate physics note?

Question 10.3. For the manuscript: do you want the fixes of Rem. 7.5 (statement of intellectual honesty; $\mathcal{D}$ in (5.1)–(5.3); Nash fourth order) drafted as a marked-up revision?

Question 10.4. Should Deninger's $\Theta = \frac{1}{2} + A$ (Prop. 6.1) be recorded as a stated G-target, "a positive Hodge $*$ commuting with the flow", next to the existing G5? It is a restatement-level target, not a lever.

Question 10.5. Follow-up on CM curves? For CM elliptic curves the comparison and the unit-root splitting interact through Damerell/Chowla–Selberg-type period relations, which is where Katz's paper goes next. It would be the first place in TAA where a period *value*, rather than a period *structure*, appears.

Status

Proved ([THM]): Thm. 2.2, Prop. 2.3, Rem. 2.4, Prop. 2.5, Prop. 3.1, Thm. 3.2, Prop. 4.1, Thm. 4.2, Cor. 4.3, Prop. 4.4, Prop. 5.1 (given Mazur's divisibility), Thm. 7.1. **Cited ([CITED]):** Katz 1.3.6, 5.7.5–5.7.11; Katz A2.3–A2.4; Movasati Prop. 6–7; Zagier (19), (21), (53)–(55); Deninger pp. 168–169; DLMF 23.2.14; MT15; Nash's note. **Bibliographic ([BIB]):** Thm. 7.2 (with [NUM] normalizations), McKean–Singer, Patodi, Hodge index, the remaining references. **Open ([OPEN]):** Question 6.2. **Not verified, not cited:** Ramanujan 1916 (no Crossref record found; used only via [29]); Titchmarsh Thm. 3.8 ([21] used instead); Wiles 1995 (record not returned by Crossref query; not cited); content of [25], [22] (records only). **Unreconciled:** the sign in [23] Prop. 19 (C5).

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TAA LEMMA 50: SPINORS, THETA CHARACTERISTICS, AND THE POLARIZATION AS A FERMIONIC VACUUM. SPIN STRUCTURES ON SURFACES, RIEMANN'S θ AND THE METAPLECTIC COVER, $\Lambda^\bullet H^1$ AS A SPINOR MODULE, SHALE–STINESPRING, THE HALF TATE TWIST AND SATO–TATE $SU(2) = \mathrm{Spin}(3)$

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

Date: October 22, 2026.

Abstract. **Statement of intellectual honesty.** RH is not proved here, and nothing below brings it closer. Sections 2–6 are classical mathematics, re-proved where elementary and checked numerically; the one RH-related equivalence (Thm. 4.4) is a *restatement* of Weil's theorem for curves; §7 says plainly that no spinor H^1 for ζ is constructed. Tags: [THM] proved here or classical with proof sketched; [CITED] checked at the source (`lit/lemma50/verified_refs.md`); [BIB] bibliographic record verified (Crossref), content standard; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma50_spin.py`, `num/lemma50_osc.py`; data `num/l50/`); [CONJ] conjecture; [OPEN] open. Lemmas 14, 45–49 are cited by number and not redone.

Question (J.W., [t188u]–[t189u]). "Spinors are the more fundamental object ... spinors might play front and center of this Betti–de Rham comparison and this search for the hypothetical lattice on $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$." Make the rigorous version and say what it does *not* give. Steering (parent, 06:52): add a section on the half Tate twist, $\sqrt{q}$, and Sato–Tate $SU(2)=\mathrm{Spin}(3)$, checking claims (a)–(e).

Answer in brief. (1) *Surfaces* (§2). Spin structures on a closed Riemann surface are the square roots L of K; $\deg L = g-1 = -\chi/2$; there are 2^{2g}, a torsor under $H^1(\Sigma;\mathbb{Z}/2)$, with $2^{g-1}(2^g\pm1)$ even/odd ones. The spinor bundle is $K^{1/2}\oplus K^{-1/2}$, of degrees ∓1 on S^2 (Prop. 2.2). Spinor holonomy is half the Levi-Civita angle (Prop. 2.4). $\Lambda^\bullet\otimes\mathbb{C}\cong S\otimes S^*$ and $\chi,\sigma,\hat{A}$ are indices of one Dirac-type operator with different gradings/twists (Thm. 2.5); but this identity holds on *non-spin* manifolds too, so Gauss–Bonnet does not need spinors. In dimension 4, χ even does not imply spin (Prop. 2.6). (2) *Riemann's θ* (§3). Hecke's dictionary for θ gives $L_\theta(u)=2\zeta(2u)$, symmetry $u\leftrightarrow\frac12-u$ with centre $k/2=\frac14$, and $s=2u$ (not $u=2s$) gives $s\leftrightarrow1-s$ with centre $\frac12$ (Prop. 3.2). The weight $\frac12$ of θ is the metaplectic weight of the *bosonic* oscillator vacuum, the zero-point $\frac12$ (Prop. 3.5); and $SL_2(\mathbb{R})$ is itself $\mathrm{Spin}(2,1)$, so Mp_2 is a cover *of* a spin group, not "the spin cover of SL_2" (Rem. 3.6). (3) *Curves* (§4). $\Lambda^\bullet H^1 = H^*(\operatorname{Jac}C)$ is the spinor module of $\mathrm{Cl}(H^1\oplus H^{1\vee})$; a symplectic H^1 alone gives a *Weyl* (bosonic) algebra. With the polarization J (Hodge $*$), the spinor module of $\mathrm{Cl}(H^1_{\mathbb{R}},g_J)$ is $H^{0,\bullet}(\operatorname{Jac}C)$ (Thm. 4.2). Over $\mathbb{F}_q$: RH for C $\iff$ normalized Frobenius fixes a point of Siegel space $\iff$ there is a Frobenius-invariant *positive* fermionic vacuum (Thm. 4.4, [RESTATEMENT]); with distinct eigenvalues it is unique and is selected by the Krein signs. Numerics for a counted genus-2 curve over $\mathbb{F}_{11}$ and two fakes (Rem. 4.5). (4) *Shale–Stinespring* (§5). Fock vacua are labelled by compatible complex structures J, unitarily equivalent iff $J-J'$ is Hilbert–Schmidt; a Fock space over a Krein space is Krein, with total signature $(2^{n-1},2^{n-1})$ (Prop. 5.2). *Spinors do not create positivity.* "Missing polarization = missing invariant positive fermionic vacuum" is a theorem-level restatement for curves and an analogy for ζ (Thesis 5.4). (5) *Half Tate twist* (§6). Claims (a), (c), (d), (e) are correct with precisions; (b) needs two corrections: the half twist over $\mathbb{F}_q$ is a choice of $\sqrt{q}$ (a $\mathbb{Z}/2$-torsor), over $\mathbb{Q}$ it does not exist at all (Prop. 6.2), and weight 0 is not unitarity. $H^1(\frac12)$ is the spin-$\frac12$ representation of $SU(2)=\mathrm{Spin}(3)$, $\mathrm{Sym}^2 H^1(1)$ the $SO(3)$ vector, and the $\sqrt{q}$ flip is $\ker(SU(2)\to SO(3))$. But a ghost lives in $SL_2(\mathbb{R})=\mathrm{Spin}(2,1)$, which is *also* a spin group: the spinor structure is ghost-blind, and what separates Weil from fake is the compact versus split real form, i.e. positivity (Prop. 6.5). (6) $\operatorname{Spec}\mathbb{Z}$ (§7, [OPEN]). A "spinor H^1 for ζ" needs a symplectic space, a conformally symplectic flow and an invariant positive J; given the first two, the third *is* RH (Prop. 7.1, [RESTATEMENT]). Prior work: the closest verified link is Connes–Consani–Moscovici's use of the metaplectic representation [7], on the bosonic side. *This lemma does not supply the missing positive structure.*

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1. Conventions, and what TAA already has

Σ is a closed oriented surface of genus g, $\chi = 2-2g$; on a Riemann surface $K = T^{*1,0}$ is the canonical bundle, $\deg K = 2g-2$. For a curve $C/\mathbb{F}_q$ of genus g, F is the geometric Frobenius on $H^1 = H^1_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ with characteristic polynomial $P(T) = \prod(T-\alpha_i)$, and $g_{\text{n}} = F/\sqrt{q}$ (once $\sqrt{q}$ is fixed). ω is the cup-product (Weil) pairing $H^1 \times H^1 \to H^2 = \mathbb{Q}_\ell(-1)$, so $\omega(Fx, Fy) = q\,\omega(x,y)$ [25, Thm. 1.30] ([CITED] in Lemma 45). A *compatible complex structure* on a real symplectic space (V,ω) is J with $J^2 = -1$, $\omega(Jx, Jy) = \omega(x,y)$; it is *positive* if $g_J(x,y) = \omega(x, Jy)$ is positive definite. Siegel space $\mathfrak{h}(V)$ is the set of positive compatible J. $\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}$, $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$.

Overlap table (checked by reading the cited lemmas; nothing here is redone):

Lemma	what it already contains
14 §5, §7	Light cone of NS($C \times C$); Frobenius as a boost; Spec $\mathbb{Z} \times_{\mathbb{Z}}$ Spec $\mathbb{Z}$ = Spec $\mathbb{Z}$.
45 Thm. 2.2, 2.4, 6.4, Cor. 6.5	Zeros as fermions in Deninger's super-determinant; fermion-bilinear sector = $H^1 \otimes H^1$; ghost = negative-norm two-fermion state; Weil $\iff$ some invariant h positive on Λ^2; in genus 1 the polarization is the pair state $\theta = e \wedge f$.
46 Thm. 6.1, §5	Abstract (F, V) pairs carry a Krein form, Hilbert iff no ghosts; $SU(2)$ vs. $SL_2(\mathbb{R})$ real forms, Wick rotation $\theta \mapsto i\eta$.
47 Thm. 2.3, 2.5	One prime: Dirac reflection and boson–fermion correspondence exist; no global Dirac reflection for ≥ 2 primes.
49 Prop. 6.1, Rem. 4.5, 7.4	Deninger: positive Hodge $* \Rightarrow \Theta = \frac{1}{2} + A$; E_2^* uses the Hermitian form, definiteness free in genus 1; Chern–Gauss–Bonnet, signature, index.

New here: spin structures and their counting; the Hecke normalization for θ; the metaplectic sign and the oscillator vacuum; $\Lambda^\bullet H^1$ as a spinor module and the Siegel-space form of Weil's theorem; Shale–Stinespring and Krein Fock spaces; the half Tate twist and Sato–Tate.

2. Surfaces: spin structures and theta characteristics

Definition 2.1. A *spin structure* on an oriented Riemannian surface Σ is a double cover of the oriented orthonormal frame bundle ($SO(2) = U(1)$) that restricts to the connected double cover $z \mapsto z^2$ on each fibre. On a Riemann surface it is equivalently a holomorphic line bundle L with an isomorphism $L^{\otimes 2} \cong K$, a *theta characteristic* $K^{1/2}$ [3] ([BIB]). Its *parity* is $h^0(L) \bmod 2$.

Proposition 2.2 (spin structures on surfaces; [THM], with [BIB] [3, 26] for (c))**.** *Let Σ be a closed Riemann surface of genus g.*

(a) Theta characteristics exist, $\deg K^{1/2} = g-1 = -\chi/2$*, and they form a torsor under* $\mathrm{Pic}(\Sigma)[2] \cong H^1(\Sigma; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{2g}$*; so there are exactly* 2^{2g}*.*

(b) The (complex) spinor bundle is $S = S^+ \oplus S^- = K^{1/2} \oplus K^{-1/2}$*, of degrees* $g-1$ *and* $1-g$*;* $c_1(S) = 0$*. On* S^2*:* $K = \mathcal{O}(-2)$*,* $S = \mathcal{O}(-1) \oplus \mathcal{O}(1)$*, while* $TS^2 = K^{-1} = \mathcal{O}(2)$ *has degree* $2 = \chi$*. So* $(K^{-1/2})^{\otimes 2} = TS^2$*, and* $K^{-1/2}$ *has degree* $\chi/2 = 1$*, the other half degree* -1*.*

(c) The parity $h^0(K^{1/2}) \bmod 2$ *is invariant under deformation (Atiyah* [3]*, Mumford* [26]*), and there are* $2^{g-1}(2^g+1)$ *even and* $2^{g-1}(2^g-1)$ *odd theta characteristics.*

Proof. (a) $\deg K = 2g-2$ is even, so K has a square root in Pic (the Jacobian is divisible); two roots differ by a 2-torsion line bundle, and $\mathrm{Pic}^0[2] = \mathrm{Jac}[2] \cong (\mathbb{Z}/2)^{2g}$. (b) On a Kähler manifold $S \cong \Lambda^{0,\bullet} \otimes K^{1/2}$ (standard, [16] [BIB]); on a curve $\Lambda^{0,1} \cong \bar{K} \cong K^{-1}$ via the metric, so $S^- = K^{-1} \otimes K^{1/2}$. (c) The counts are those of quadratic refinements of the intersection form on $H^1(\Sigma; \mathbb{Z}/2)$ with Arf invariant 0 and 1; under the Atiyah–Mumford correspondence parity is the Arf invariant ([BIB]). [NUM]: enumerating all quadratic refinements of the standard symplectic form on $\mathbb{F}_2^{2g}$ gives $(3,1), (10,6), (36,28), (136,120)$ for $g = 1, \dots, 4$, equal to $2^{g-1}(2^g \pm 1)$ (`spin.json`, "arf_counts"). □

Remark 2.3 (theta functions with characteristics; [THM], [NUM])**.** For $a, b \in \{0, \frac{1}{2}\}^g$ and Ω in Siegel's upper half space, $\theta[a;b](z,\Omega) = \sum_{n \in \mathbb{Z}^g} \exp\left(\pi i (n+a)^{\mathsf{T}} \Omega (n+a) + 2\pi i (n+a)^{\mathsf{T}} (z+b)\right)$ satisfies $\theta[a;b](-z) = (-1)^{4a \cdot b} \theta[a;b](z)$ (substitute $n \mapsto -n - 2a$). So the $2^{g-1}(2^g - 1)$ odd characteristics have $\theta[a;b](0,\Omega) \equiv 0$. For one genus-2 Ω (`spin.json`): the 6 odd constants are $\leq 8.9 \cdot 10^{-17}$ and the 10 even ones are ≥ 0.376. The identification of these 2^{2g} half-periods with the theta characteristics of Prop. 2.2 is classical ([26]; Mumford's Tata lectures [27] chapter on theta characteristics, [BIB]). (The chat said these are "the theta polarization we used in Lemma 47". That is inaccurate: Lemma 47 mentions the theta polarization only as a proposed next step in its closing questions, and uses no theta characteristics. The polarization is the class $[\Theta]$, Rem. 4.3; the characteristics are the 2^{2g} translates that make Θ symmetric.)

Proposition 2.4 (spinor holonomy is half the Gauss–Bonnet angle; [THM]; Berry phase [BIB] [5], [NUM])**.** *Let $D \subset \Sigma$ be a disc with boundary loop γ. Levi-Civita transport around γ rotates tangent vectors by $\int_D K\, dA$ (local Gauss–Bonnet). On $K^{\mp 1/2}$ the induced connection is half the Levi-Civita connection, so the holonomy is $e^{\pm \frac{i}{2} \int_D K\, dA}$. For a loop that bounds nothing, the spinor holonomy is* a *square root of the vector holonomy, and which one is exactly the spin structure: the 2^{2g} structures are the 2^{2g} sign assignments on a basis of $H_1(\Sigma; \mathbb{Z}/2)$.*

Proof. The lifted connection on the double cover $z \mapsto z^2$ of $U(1)$ has connection form $\frac{1}{2}$ of the original; $K^{-1/2}$ is the associated bundle of weight one. The last statement is the torsor structure of Prop. 2.2(a) read through holonomy. □

On S^2 with $K = 1$ and γ a circle of latitude enclosing solid angle Ω, this is Berry's spin-$\frac{1}{2}$ phase $-\frac{1}{2}\Omega$ [5]: the spin-$\frac{1}{2}$ coherent states form the degree-± 1 (Hopf) line bundle. [NUM]: discrete Berry phases of spin-s coherent states ($s = \frac{1}{2}, 1, \frac{3}{2}$; four latitudes; 4000 steps) equal $-s\Omega \bmod 2\pi$ to $\leq 7.5 \cdot 10^{-7}$ (`spin.json`, "berry_phase").

Theorem 2.5 (forms = spinors $\otimes$ spinors, and three indices; [BIB] [2, 23])**.** *Let M be a closed oriented Riemannian manifold of dimension $2m$.*

- *(a)* $\mathrm{Cl}(T^*M) \otimes \mathbb{C} \cong \Lambda^\bullet T^*M \otimes \mathbb{C}$ *as bundles (symbol map), and $d + d^*$ is the Dirac operator of the Clifford module* $\mathrm{Cl}(T^*M)$ *under left multiplication. If M is spin,* $\mathrm{Cl} \otimes \mathbb{C} \cong \mathrm{End}(S) = S \otimes S^*$ *($\cong S \otimes S$), and $d + d^*$ is the spin Dirac operator twisted by S^*.*
- *(b) With the grading even/odd,* $\mathrm{ind}(d + d^*) = \chi(M)$*; for $2m = 4k$, with the grading by the Hodge involution,* $\mathrm{ind} = \sigma(M)$*; for M spin,* $\mathrm{ind}(D : S^+ \to S^-) = \hat{A}(M)$.
- *(c) (a)–(b) for χ and σ hold whether or not M is spin:* $\mathrm{Cl}(T^*M)$ *always exists, and S exists only locally.*

So the chat's "forms $= S \otimes S$, Gauss–Bonnet is a twisted Dirac index" is correct as representation theory and locally. "Spinors come first" should not be read as a logical dependency: $\mathbb{C}P^2$ is not spin, yet $\chi = 3$ and $\sigma = 1$ are indices of $d + d^*$ (Lemma 49 Rem. 7.4 computed both by curvature integrals).

Proposition 2.6 (the obstruction is w_2, not χ; [THM] given standard facts [BIB] [23, 20])**.** *(a) An oriented manifold is spin iff $w_2 = 0$; spin structures then form a torsor under $H^1(M; \mathbb{Z}/2)$. (b)*

For a surface, $w_2[\Sigma] = \chi \bmod 2$ (top Stiefel–Whitney class = Euler class mod 2), so every closed oriented surface is spin because *χ is even. (c) In dimension 4 the parity of χ is irrelevant: by Wu's formula $w_2 \cup x = x \cup x$ on $H^2(M;\mathbb{Z}/2)$, a spin 4-manifold has even intersection form. $S^2 \times S^2$ ($\chi = 4$, form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$) is spin; $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ ($\chi = 4$, form $\mathrm{diag}(1,-1)$, odd) is not; $\mathbb{C}P^2$ ($\chi = 3$) is not. (d) (Rokhlin;* [20] *chapter "Rohlin's theorem",* [BIB]*.) A closed smooth spin 4-manifold has $\sigma \equiv 0 \pmod{16}$. In index language: $\hat{A} = -\sigma/8$ is the index of the Dirac operator, which is even because $S^{\pm}$ are quaternionic in dimension 4 (standard* [23]*). The K3 surface ($\sigma = -16$) attains the bound.*

So "spin exists because χ is even" is correct for surfaces and false as a general principle.

3. Riemann's θ, Hecke's dictionary, and the metaplectic cover

Theorem 3.1 (Riemann; [THM] classical, Poisson summation)**.** *$\theta(-1/\tau) = \sqrt{\tau/i}\,\theta(\tau)$ for $\Im\tau > 0$, with the branch of $\sqrt{\tau/i}$ that is positive on $\tau \in i\mathbb{R}_{>0}$. With $\psi(y) = \frac{1}{2}(\theta(iy) - 1)$,*

$$\Lambda(s) = \int_0^\infty \psi(y)\, y^{s/2}\tfrac{dy}{y}\ (\Re s > 1), \qquad \Lambda(s) = -\tfrac{1}{s} - \tfrac{1}{1-s} + \int_1^\infty \psi(y)\big(y^{s/2} + y^{(1-s)/2}\big)\tfrac{dy}{y},$$

and the right side is visibly invariant under $s \mapsto 1 - s$.

[NUM] (`spin.json`, block A; 40 digits): the inversion holds to $\le 3.6 \cdot 10^{-42}$ at four τ, and with the other branch it fails by 2.0–2.4. Riemann's integral equals $\pi^{-s/2}\Gamma(s/2)\zeta(s)$ to $\le 5.7 \cdot 10^{-42}$ at $s = 2.5$, $0.3 + 5i$, $0.5 + 14.134725i$, $-1.2 + 2i$, and $\Lambda(s) = \Lambda(1-s)$ there to $\le 2.9 \cdot 10^{-42}$.

Proposition 3.2 (Hecke's dictionary for θ, with the normalization checked; [THM]; [BIB] [15])**.** *Hecke: if $f(\tau) = \sum_{m\ge 0} a_m e^{2\pi i m\tau/\lambda}$ satisfies $f(-1/\tau) = \gamma(\tau/i)^k f(\tau)$, then $\Lambda_f(u) = (2\pi/\lambda)^{-u}\Gamma(u)\sum_{m\ge 1} a_m m^{-u}$ continues meromorphically and $\Lambda_f(k-u) = \gamma\Lambda_f(u)$; the centre of symmetry is $u = k/2$. For $f = \theta$: $\lambda = 2$, $k = \frac{1}{2}$, $\gamma = 1$, $a_m = 2$ if m is a nonzero square and 0 otherwise, so*

$$L_\theta(u) = 2\zeta(2u), \qquad \Lambda_\theta(u) = 2\pi^{-u}\Gamma(u)\zeta(2u) = 2\Lambda(2u), \qquad \Lambda_\theta(\tfrac{1}{2} - u) = \Lambda_\theta(u).$$

The centre is $u = \frac{1}{4} = k/2$. In Riemann's variable $s = 2u$ this is $s \leftrightarrow 1 - s$ with centre $\frac{1}{2}$.

Proof. $e^{\pi i n^2\tau} = e^{2\pi i n^2\tau/2}$, so $m = n^2$ and $\pm n$ both contribute. The identities are substitution; the symmetry is Thm. 3.1 with $s = 2u$. □

[NUM]: $|\Lambda_\theta(u) - \Lambda_\theta(\frac{1}{2} - u)| \le 1.4 \cdot 10^{-42}$ at three u, whereas $|\Lambda_\theta(u) - \Lambda_\theta(1-u)| = 2.8 \cdot 10^{-5}$–$0.085$; $\Lambda_\theta(u) = 2\Lambda(2u)$ exactly.

Remark 3.3 (what "the $\frac{1}{2}$ of ζ is the weight $\frac{1}{2}$ of θ" means; correction)**.** The brief's "under $u = 2s$" has the substitution backwards: Hecke's variable is $u = s/2$. The precise statement is: *centre of $\zeta = \frac{1}{2} = 2 \cdot \frac{k}{2} = k$*, where the factor 2 is the degree of $n \mapsto n^2$. The numerical coincidence is not structural on its own. θ^2 has weight 1 and $L_{\theta^2}(u) = 4\zeta(u)L(u,\chi_{-4})$ has centre $\frac{1}{2} = k/2$ with no doubling ([NUM]: symmetry $u \leftrightarrow 1 - u$ to $\le 7.2 \cdot 10^{-42}$); in general θ^r has weight $r/2$ and Hecke centre $r/4$. What is structural is that the weight of θ is half-integral, which is the metaplectic phenomenon below.

Proposition 3.4 (weight $\frac{1}{2}$ lives on the double cover; [THM], [NUM]; [BIB] [33, 37])**.** *Put $j(\gamma,\tau) = (c\tau + d)^{1/2}$ (principal branch) for $\gamma = \left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in SL_2(\mathbb{R})$. Then $\sigma(\gamma_1,\gamma_2) = j(\gamma_1\gamma_2,\tau)/\big(j(\gamma_1,\gamma_2\tau)j(\gamma_2,\tau)\big) \in \{\pm 1\}$ is independent of τ, and it is not identically 1. So $\gamma \mapsto j(\gamma,\cdot)$ is a cocycle only on the double cover $\mathrm{Mp}_2(\mathbb{R}) = \{(\gamma,\phi) : \phi^2 = c\tau + d\}$, the unique connected double cover of $SL_2(\mathbb{R})$. Forms of weight $\frac{1}{2}$, θ among them, are forms on Mp_2 (Shimura* [33]*).*

Proof. Both sides square to $c_{12}\tau + d_{12}$ (cocycle of $c\tau + d$), so $\sigma = \pm 1$; it is continuous in τ on the connected half-plane, hence constant. [NUM] (`spin.json`, "Mp2_cocycle"): over 81 pairs of elements of $SL_2(\mathbb{Z})$ the values are exactly $\{+1, -1\}$; e.g. $\sigma(S, -I) = -1$ with $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$; τ-independence checked at three points. □

Proposition 3.5 (the $\frac{1}{2}$ is the oscillator's zero-point weight; [THM] classical (Mehler); [BIB] [24, 37]; [NUM])**.** *Let $H = \frac{1}{2}(x^2 - \frac{d^2}{dx^2})$ on $L^2(\mathbb{R})$ and $h_n = H_n(x)e^{-x^2/2}$ (Hermite). Then $e^{-i\phi H}h_n = e^{-i(n+\frac{1}{2})\phi}h_n$, so the rotation subgroup of the Weil (oscillator) representation acts through $\phi \in \mathbb{R}/4\pi\mathbb{Z}$, the double cover of $SO(2)$: at $\phi = 2\pi$ it is -1. The vacuum h_0 (the Gaussian) has weight $\frac{1}{2}$, the zero-point energy. Riemann's θ is the theta distribution $\sum_{n\in\mathbb{Z}} \delta_n$ paired with the Weil-representation translate of the Gaussian $e^{-\pi x^2}$, and its automorphy factor $\sqrt{\tau/i}$ is the $\widetilde{SO(2)}$-character of that vacuum.*

Proof. Mehler's kernel $K_\phi(x, y) = (2\pi i \sin\phi)^{-1/2} \exp\left(\frac{i}{2\sin\phi}((x^2+y^2)\cos\phi - 2xy)\right)$ represents $e^{-i\phi H}$ for $0 < \phi < \pi$; continuation in ϕ forces the branch of the square root to change across $\phi = \pi$, which is the metaplectic sign. For the last sentence: $\theta(x+iy) = y^{-1/4}\sum_n (\rho(g)\varphi_0)(n)$ with $\varphi_0 = e^{-\pi t^2}$, $g = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{smallmatrix}\right)$, where ρ multiplies by $e^{\pi i x t^2}$ and dilates unitarily by $y^{1/2}$; the modular behaviour comes from $\sum \delta_n$ being invariant under the Weil representation of the theta group (Poisson), and the weight from the K-type of φ_0 [24]. □

[NUM] (`num/lemma50_osc.py`, `osc.json`): with the principal branch of $(2\pi i \sin\phi)^{1/2}$, $e^{-i\phi H}h_n/h_n$ at $x = 0.37$ equals $e^{-i(n+\frac{1}{2})\phi}$ to $\le 4 \cdot 10^{-31}$ for $\phi = 0.7, 2.5$ ($n = 0, \ldots, 3$), and equals $-e^{-i(n+\frac{1}{2})\phi}$ for $\phi = 4.0$: past $\phi = \pi$ the principal branch is the *wrong* sheet. The Fourier transform ($\phi = \pi/2$ up to the phase $e^{-i\pi/4}$) has eigenvalues $(-i)^n$ exactly.

Remark 3.6 (Weyl vs. Clifford; [THM]/[CITED] [29] §1; "spin cover" [ANALOGY])**.**

	bosons	fermions
one-particle form	symplectic ω (CCR)	symmetric g (CAR)
algebra	Weyl algebra	Clifford algebra
Fock space over (V, J)	$\mathrm{Sym}^\bullet V^{1,0}$ (oscillator)	$\Lambda^\bullet V^{1,0}$ (spinors)
symmetry group, cover	$\mathrm{Sp}(V)$, metaplectic $\mathrm{Mp}(V)$	$SO(V)$, $\mathrm{Spin}(V)$
infinite dimensions	$\mathrm{Sp}_{\mathrm{res}}$: antilinear part HS	O_{res}: antilinear part HS

Naranjo-Alvarado–Várilly [29] ([CITED], §1 read) state this parallel: Shale [32] and Shale–Stinespring [35] showed that linear symmetries of free boson resp. fermion fields are implementable on Fock space iff their antilinear parts are Hilbert–Schmidt, and in finite dimensions the projective representations become true representations of the double covers Mp and Spin. So "θ lives on a spin cover" is right as an *analogy*: Mp_2 is to the Weyl algebra what Spin is to the Clifford algebra. Two precisions. (i) Mp_2 is on the *bosonic* side, so the $\frac{1}{2}$ of Riemann's proof is a bosonic zero-point weight (Prop. 3.5), not a spin-$\frac{1}{2}$; the fermionic vacuum energy of a Dirac sea comes with the opposite sign. (ii) $SL_2(\mathbb{R})$ is already a spin group: the adjoint action on $\mathfrak{sl}_2(\mathbb{R}) \cong \mathbb{R}^{2,1}$ (Killing form of signature $(2, 1)$) gives $SL_2(\mathbb{R})/\{\pm 1\} \cong SO(2, 1)_0$, so $SL_2(\mathbb{R}) = \mathrm{Spin}(2, 1)_0$ [THM], and $\mathrm{Mp}_2(\mathbb{R})$ is a double cover *of* that spin group. Unlike $\mathrm{Spin}(n)$, $\mathrm{Mp}_2(\mathbb{R})$ has no faithful finite-dimensional representation (standard): the metaplectic sign is visible only on the infinite-dimensional oscillator space.

4. Curves: $\Lambda^\bullet H^1$ as a spinor module, and the vacuum form of Weil's theorem

Proposition 4.1 (which Clifford algebra; [THM]; standard facts as marked)**.** *Let C be a smooth projective curve of genus g and $H^1 = H^1(C)$ (any Weil cohomology, or $H^1(C, \mathbb{R})$ over $\mathbb{C}$).*

(a) ω is alternating and non-degenerate, and $H^*(\mathrm{Jac}\, C) \cong \Lambda^\bullet H^1$ with $H^1(\mathrm{Jac}\, C) \cong H^1(C)$ (standard).

(b) For any finite-dimensional W, wedging $e(w)$ and contraction $\iota(\xi)$ satisfy $e(w)\iota(\xi)+\iota(\xi)e(w)=\xi(w)$; they make $\Lambda^\bullet W$ the irreducible module of $\mathrm{Cl}(W\oplus W^\vee)$ for the split symmetric form $\langle(w,\xi),(w',\xi')\rangle=\xi(w')+\xi'(w)$, and $\mathrm{Cl}(W\oplus W^\vee)\cong\mathrm{End}(\Lambda^\bullet W)$. With $W=H^1$: $H^*(\mathrm{Jac}\,C)$ is the spinor module of $\mathrm{Cl}(H^1\oplus H^{1\vee})$, of dimension 2^{2g}.

(c) (H^1,ω) by itself defines a Weyl *algebra ($[a(x),a(y)]=\omega(x,y)$), whose modules are bosonic, $\mathrm{Sym}^\bullet$ of a Lagrangian. A Clifford algebra needs a symmetric form. Choosing a Lagrangian splitting $H^1=L\oplus L'$, ω identifies $L'\cong L^\vee$ and $s_L(l+l',m+m')=\omega(l,m')+\omega(m,l')$ is symmetric and split; then $\mathrm{Cl}(H^1,s_L)\cong\mathrm{End}(\Lambda^\bullet L)$, a spinor module of dimension 2^g, and $\Lambda^\bullet H^1=\Lambda^\bullet L\otimes\Lambda^\bullet L'\cong S_L\otimes S_L^*$.*

Proof. (b) is the standard Fock construction; irreducibility: Λ^0 is cyclic and the only vectors killed by all $\iota(\xi)$ are in Λ^0. (c) $L\oplus L^\vee$ with the pairing is the form of (b), with $W=L$. □

So the brief's "Clifford algebra of H^1 with its symplectic form" should read "of H^1 with the symmetric form s_L built from ω and a Lagrangian", and its module is $\Lambda^\bullet L$, not $\Lambda^\bullet H^1$. Note that (c) is Thm. 2.5 again on the flat torus $\mathrm{Jac}\,C$: forms $=S\otimes S^*$.

Theorem 4.2 (the spinors of the Jacobian are the polarization's Fock space; [THM]; [BIB] [16] for harmonic spinors)**.** *Let $C/\mathbb{C}$ and $J=*$, the Hodge star on $H^1(C,\mathbb{R})$ (harmonic 1-forms). Then (i) $J^2=-1$, $\omega(J\alpha,J\beta)=\omega(\alpha,\beta)$ and $g_J(\alpha,\beta)=\int_C\alpha\wedge*\beta$ is positive definite (Riemann's bilinear relations); (ii) the $+i$-eigenspace of J on $H^1(C,\mathbb{C})$ is $H^{0,1}$ ($*dz=-i\,dz$, $*d\bar z=i\,d\bar z$); (iii) the spinor module of $\mathrm{Cl}(H^1(C,\mathbb{R}),g_J)$ is $\Lambda^\bullet H^{0,1}(C)\cong\bigoplus_q H^{0,q}(\mathrm{Jac}\,C)$, of dimension 2^g; these are exactly the harmonic spinors of the flat torus $\mathrm{Jac}\,C$, whose spinor bundle is $\Lambda^{0,\bullet}\otimes K_{\mathrm{Jac}}^{1/2}$ with K_{Jac} trivial; (iv) the Hilbert-space structure of this module is induced by g_J, i.e. by the polarization.*

Proof. (i) $*$ is conformally invariant on 1-forms with $*^2=-1$, and $\int\alpha\wedge*\alpha=\|\alpha\|^2$. (ii) $*dx=dy$, $*dy=-dx$. (iii) Fock construction $\Lambda^\bullet(V,J)=\Lambda^\bullet V^{1,0}$ (Rem. 3.6); $H^{0,q}(\mathrm{Jac})=\Lambda^qH^{0,1}(\mathrm{Jac})$ and $H^{0,1}(\mathrm{Jac})\cong H^{0,1}(C)$; on a flat Kähler torus harmonic spinors are the parallel sections of $\Lambda^{0,\bullet}$. (iv) Definition of the Fock inner product. □

This is the rigorous core of J.W.'s hunch over $\mathbb{C}$: *a choice of fermionic vacuum on H^1 is a choice of compatible J, and the vacuum is a Hilbert-space vacuum exactly when J is a polarization.* The Betti–de Rham comparison of Lemma 49 is what produces J (F^1 and its conjugate, Lemma 49 Thm. 4.2).

Remark 4.3 (the theta divisor and the pair state; [BIB] standard)**.** $[\Theta]\in H^2(\mathrm{Jac}\,C,\mathbb{Z})=\Lambda^2H^1(C,\mathbb{Z})$ is the class of ω (in a symplectic basis $\sum_i a_i\wedge b_i$), with $[\Theta]^g/g!=[\mathrm{pt}]$ (Poincaré's formula, standard; [27]). In Fock language $[\Theta]$ is the "Cooper pair" of Lemma 45 Cor. 6.5, and $L=[\Theta]\wedge\cdot$ together with its adjoint generates the Lefschetz $\mathfrak{sl}_2$, which commutes with $\mathrm{Sp}(H^1)$ on $\Lambda^\bullet H^1$ (the skew Howe duality [17], [BIB]). The theta characteristics of §2 are the 2^{2g} translates of $\Theta\subset\mathrm{Pic}^{g-1}$ into Jac that are symmetric.

Theorem 4.4 (Weil's theorem as an invariant positive vacuum; [RESTATEMENT] (equivalent to Weil's RH for curves), proof [THM])**.** *Let $P\in\mathbb{Z}[T]$ be monic of degree $2g$ with $P(T)=q^{-g}T^{2g}P(q/T)$, $P(\pm\sqrt{q})\neq0$. Put $V=\mathbb{R}[T]/(P)$, $g_{\mathrm{n}}=T/\sqrt{q}$, $x\mapsto x^\sharp$ the involution $T\mapsto q/T$ (Lemma 46 Thm. 6.1), $c=T-q/T$ and $\omega(x,y)=\mathrm{Tr}_{V/\mathbb{R}}(c\,x\,y^\sharp)$. Then ω is alternating, non-degenerate and g_{n}-invariant. Assume g_{n} semisimple. The following are equivalent:*

(i) every root of P has $|\alpha|=\sqrt{q}$;

(ii) there is a g_{n}-invariant positive compatible J, i.e. a Frobenius-invariant positive *fermionic Fock vacuum on $\Lambda^\bullet(V,J)$;*

(iii) g_{n} has a fixed point in Siegel space $\mathfrak{h}(V)$.

If the roots are distinct, the g_{n}-invariant compatible J are the J_ε, $\varepsilon\in\{\pm1\}^{\mathrm{pairs}}$ acting by $i\varepsilon_\lambda$ on E_λ, subject to $\varepsilon_{\bar\lambda}=-\varepsilon_\lambda$ and $\varepsilon_{1/\lambda}=-\varepsilon_\lambda$. In case (i) there are 2^g of them and exactly one is

positive, namely $\varepsilon_\lambda = \kappa_\lambda$, the Krein sign of E_λ for $h_K(x,y) = i\,\omega(\bar{x}, y)$. If some root is real and $\neq \pm\sqrt{q}$ (a real ghost), there is no invariant compatible J at all. If the off-line roots come in complex quadruples, invariant J exist but every one is indefinite.

Proof. ω: $(c\,y\,x^\sharp)^\sharp = c^\sharp y^\sharp x = -c\,x\,y^\sharp$ and $\sharp$ preserves the trace, so $\omega(y,x) = -\omega(x,y)$; non-degeneracy because the trace form of the étale algebra is, and c is a unit ($P(\pm\sqrt{q}) \neq 0$); invariance: $\omega(Tx, Ty) = \mathrm{Tr}(c\,Tx\,(q/T)y^\sharp) = q\,\omega(x,y)$. (ii) $\iff$ (iii): $\mathrm{Sp}(V)$ acts on $\mathfrak{h}(V)$ by $J \mapsto gJg^{-1}$. (ii)$\Rightarrow$(i): $g_J(g_\mathrm{n}x, g_\mathrm{n}y) = \omega(g_\mathrm{n}x, Jg_\mathrm{n}y) = \omega(g_\mathrm{n}x, g_\mathrm{n}Jy) = g_J(x,y)$, so g_n is orthogonal for a Euclidean form. (i)$\Rightarrow$(ii): decompose $V_\mathbb{C}$ into g_n-eigenspaces; ω pairs E_λ with $E_{1/\lambda} = E_{\bar\lambda}$, so h_K is non-degenerate on each E_λ. Split $E_\lambda = E_\lambda^+ \oplus E_\lambda^-$ h_K-orthogonally by sign, let $J = \pm i$ on $E_\lambda^\pm$ and extend by conjugation (on the real eigenvalues ±1 of g_n, if any, take any positive compatible J). Then J is real, commutes with g_n, is compatible, and $g_J(\bar{x}, x) = \omega(\bar{x}, Jx) = \pm i\,\omega(\bar{x}, x) = |h_K(x,x)| > 0$ on $E_\lambda^\pm$. Distinct roots: the commutant of g_n is diagonal, so $J = \mathrm{diag}(i\varepsilon)$; reality and compatibility give the two sign rules, and the computation just made shows $g_J > 0$ iff $\varepsilon_\lambda = \kappa_\lambda$ for all λ. A real eigenline cannot carry J with $J^2 = -1$. For a quadruple, E_λ is h_K-neutral (Lemma 45 Thm. 6.4(b)), so g_J restricted to $E_\lambda \oplus E_{1/\bar\lambda}$ is hyperbolic. □

Lemma 45 Thm. 6.4(c) already gave "(i) $\iff$ some invariant h is positive on Λ^2"; Lemma 46 Thm. 6.1 gave the signature $(D_\mathrm{on} + k, k)$ of the trace form. What Thm. 4.4 adds is the *vacuum* formulation and the fact that, generically, the positive vacuum is unique and is picked out by the Krein signs. *It is not a new proof*: over $\mathbb{F}_q$ there is no Hodge $*$, and Weil gets the positivity from the Rosati involution on $\mathrm{End}^0(\mathrm{Jac}\,C)$ (Lemma 46 Thm. 1.1). The vacuum J of (ii) is then a consequence, not an input.

Remark 4.5 (numerics; [NUM]). `num/lemma50_spin.py`, block C (`spin.json`). *Weil curve:* $y^2 = x^5 + 3x^3 + x + 7$ over $\mathbb{F}_{11}$ (f squarefree mod 11, checked), counted directly over $\mathbb{F}_{11}$ and $\mathbb{F}_{121}$: $N_1 = 10$, $N_2 = 134$, so $P = T^4 - 2T^3 + 8T^2 - 22T + 121$ and $|\alpha|/\sqrt{11} = 1$ to $2\cdot10^{-16}$. Model checks: ω alternating and invariant to $\leq 1.2\cdot10^{-11}$ (entries of size 10^2–10^3). There are $4 = 2^g$ invariant compatible J, with g_J of signature $(4,0)$, $(2,2)$, $(2,2)$, $(0,4)$; the positive one is unique, and its Fock space $\Lambda^\bullet V^{1,0}$ has signatures $(1,0), (2,0), (1,0)$ in degrees $0, 1, 2$. A positive invariant h on $H^1_\mathbb{C}$ induces $(6,0)$ on Λ^2 and $(16,0)$ on $\Lambda^\bullet$. *Real-ghost fake* $T^4 + 11T^3 - 13T^2 + 121T + 121$ (Lemma 46; $|\lambda| = 0.2609, 1, 1, 3.8329$): no invariant compatible J. 400 random non-degenerate invariant forms have signatures $(2,2), (3,1), (1,3)$ only, with Λ^2 signatures $(2,4), (3,3), (3,3)$. The ghost pair e, f has $|h_K(e,e)|, |h_K(f,f)| \leq 3.4\cdot10^{-13}$ and $\langle e \wedge f\rangle = -|h_K(e,f)|^2$ exactly (in the unnormalized basis, $-6.60\cdot10^6$), with $|\lambda_e/\bar\lambda_e| = 1$. *Complex-quadruple fake* $T^4 - 13T^3 + 65T^2 - 143T + 121$ ($|\lambda| = 0.7122^{\times2}, 1.4041^{\times2}$): two invariant compatible J, both with g_J of signature $(2,2)$; their Fock spaces have signatures $(1,0), (1,1), (0,1)$, total $(2,2)$: a Krein vacuum. All 400 invariant forms are $(2,2)$ with Λ^2 signature $(2,4)$; $n_-(\Lambda^2) = n_+n_-$ in every sample (asserted). *Krein form* h_K in all three cases: $(2,2)$ on $H^1_\mathbb{C}$, $(2,4)$ on Λ^2, $(8,8)$ on $\Lambda^\bullet$ (neutral, as in Lemma 45 Thm. 6.4(e)).

5. Shale–Stinespring, Segal, and Krein Fock spaces

Theorem 5.1 (Fock vacua are complex structures; [CITED] [29] §1 and §2 (statement of the conditions); [BIB] [32, 35, 1, 30]). *Let (V, g) be a real Hilbert space.*

- *(a) An orthogonal complex structure J ($J^2 = -1$, $g(Jx, Jy) = g(x,y)$) makes V a complex Hilbert space V_J, and the completed $\Lambda^\bullet V_J$ carries an irreducible Fock representation of the CAR algebra of (V, g), with vacuum Ω_J.*
- *(b) $T \in O(V)$ is implemented by a unitary on $\Lambda^\bullet V_J$ iff its J-antilinear part $\frac{1}{2}(T + JTJ)$ is Hilbert–Schmidt (Shale–Stinespring).*
- *(c) The Fock representations for J and J' are unitarily equivalent iff $J - J'$ is Hilbert–Schmidt.*

(d) *Bosonic version (Shale* [32]*):* (V,ω) *symplectic with a compatible* J *for which* $\omega(\cdot,J\cdot)$ *is a real inner product; symplectic* T *is implementable on* $\mathrm{Sym}^\bullet V_J$ *iff its antilinear part is Hilbert–Schmidt.*

In finite dimensions every operator is Hilbert–Schmidt, so all Fock vacua are equivalent and every orthogonal (resp. symplectic) map is implemented, projectively, by the spin (resp. metaplectic) representation [29].

Two inputs are positive *before* any spinor appears: the one-particle form g (a real *Hilbert* space), and in the bosonic case the positivity of $\omega(\cdot,J\cdot)$. The next proposition says what happens without them.

Proposition 5.2 (Fock spaces over Krein spaces are Krein; [THM], [NUM])**.** *Let* h *be a non-degenerate Hermitian form of signature* (n_+,n_-) *on a complex space* W*,* $n=n_++n_-$*. The induced form on* $\Lambda^k W$ *(Gram determinants) has signature*

$$\Big(\sum_{j\ \text{even}}\tbinom{n_+}{k-j}\tbinom{n_-}{j},\ \sum_{j\ \text{odd}}\tbinom{n_+}{k-j}\tbinom{n_-}{j}\Big),$$

and on $\Lambda^\bullet W$ *the total signature is* $(2^{n-1},2^{n-1})$ *whenever* $n_-\geq 1$*. On* $\mathrm{Sym}^k W$ *the form is indefinite for every* $k\geq 1$ *when* $n_\pm\geq 1$*, and on* $\mathrm{Sym}^2 W$ *one has* $n_-=n_+n_-$ *(Lemma 45 Thm. 6.4(d)).*

Proof. In an h-orthonormal basis the wedge monomials are orthogonal with sign $(-1)^{\#\{\text{negative factors}\}}$. The total is $\sum_S(-1)^{|S\cap N|}=2^{n_+}\cdot(1-1)^{n_-}=0$ for the difference, and the sum is 2^n. □

[NUM] (`spin.json`, block E): for $(n_+,n_-)=(1,1),(2,1),(3,2),(2,3)$ in random non-orthogonal bases, every degree matches the formula and the totals are $(2,2),(4,4),(16,16),(16,16)$.

Corollary 5.3 (spinors do not create positivity; [THM])**.** *Second quantization, fermionic or bosonic, transports the one-particle signature to the Fock space and never repairs it. A positive Fock space exists iff the one-particle form is definite; a* Frobenius-invariant *positive Fock vacuum exists iff Frobenius preserves some definite form (Thm. 4.4). Positivity must be put in through* J *(and* g*); it cannot come out of the Clifford or spinor structure.*

Thesis 5.4 (missing polarization = missing fermionic vacuum)**.** (a) *Curves over* $\mathbb{C}$ ([THM], Thm. 4.2): the polarization $J=*$ is the vacuum of $\mathrm{Cl}(H^1_{\mathbb{R}},g_J)$, and it is positive automatically (Riemann's bilinear relations).

(b) *Curves over* $\mathbb{F}_q$ ([RESTATEMENT], Thm. 4.4): RH for C $\iff$ a Frobenius-invariant positive vacuum exists; generically it is unique and selected by the Krein signs.

(c) ζ ([ANALOGY]; the precise version is [OPEN], §7): RH would follow from a positive vacuum J on a symplectic "$H^1(\overline{\mathrm{Spec}\,\mathbb{Z}})$" commuting with Deninger's flow (Prop. 7.1). By Cor. 5.3, the spinor language does not make constructing J any easier.

In infinite dimensions there is one genuinely new feature, absent for curves: inequivalent vacua exist (Thm. 5.1(c)). So for ζ one would have to choose a superselection sector as well as find positivity. That is a constraint on any construction, not a lever.

6. Half Tate twist, $\sqrt{q}$, and Sato–Tate $SU(2)=\mathrm{Spin}(3)$

Throughout, F is the *geometric* Frobenius. $\mathbb{Q}_\ell(1)=\varprojlim\mu_{\ell^m}$; the arithmetic Frobenius acts on it by q, so F acts by q^{-1}, and $\mathbb{Q}_\ell(n)=\mathbb{Q}_\ell(1)^{\otimes n}$.

Proposition 6.1 (claim (a): weights and integer twists; [THM] from the definitions; Deligne [11] [BIB])**.** *For* $X/\mathbb{F}_q$ *smooth projective, the eigenvalues of* F *on* H^k *are algebraic integers all of whose complex absolute values are* $q^{k/2}$ *(weight* k*; Weil for curves, Deligne in general).* $H^k(n)=H^k\otimes\mathbb{Q}_\ell(n)$ *has eigenvalues* αq^{-n} *and weight* $k-2n$*. For a curve,* $H^2=\mathbb{Q}_\ell(-1)$*, so* F *acts on* H^2 *by* q*.*

Claim (a) is correct for the geometric Frobenius (for the arithmetic one the factor is q^n).

Proposition 6.2 (claim (b): the half twist is a choice over $\mathbb{F}_q$ and impossible over $\mathbb{Q}$; [THM])**.** *(i) Over $\mathbb{F}_q$: $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$ is topologically generated by F, and the continuous characters ψ with $\psi^2 =$ (the character of $\mathbb{Q}_\ell(1)$) are exactly $\psi_\pm : F \mapsto \pm q^{-1/2}$, once $\sqrt{q} \in \bar{\mathbb{Q}}_\ell$ is fixed. They differ by the unramified quadratic character $F \mapsto -1$. So $H^1(\frac{1}{2}) := H^1 \otimes \psi_+$ has weight 0 and is defined up to this $\mathbb{Z}/2$-torsor; integer twists involve no choice. (ii) Over $\mathbb{Q}$ (or any number field with a real place) there is no continuous $\psi : G_\mathbb{Q} \to \bar{\mathbb{Q}}_\ell^\times$ with $\psi^2 = \chi_\ell$, the cyclotomic character: for complex conjugation c, $\chi_\ell(c) = -1$, while $c^2 = 1$ forces $\psi(c) = \pm 1$ and $\psi(c)^2 = 1$. (iii) Weight 0 is not unitarity. For a fake P (Lemma 46), the twisted eigenvalues $\alpha/\sqrt{q}$ are not unimodular. $H^1(\frac{1}{2})$ has unimodular eigenvalues iff RH holds for C, and it is unitary for some Hermitian form iff, in addition, F is semisimple.*

So claim (b) is right that the half twist needs a choice of $\sqrt{q}$ with a sign ambiguity, and that integer twists need none. Two corrections: globally over $\mathbb{Q}$ the half twist does not exist as a Galois twist at all, and the twist alone does not make H^1 unitary.

Proposition 6.3 (claim (c): $SU(2)$, spin $\frac{1}{2}$ and the $SO(3)$ vector; [THM]; Sato–Tate [CITED] [4] (abstract), [BIB] [18, 8, 36]; [NUM])**.** *Let $E/\mathbb{F}_q$ have $a = q + 1 - \#E(\mathbb{F}_q)$ and $|a| \le 2\sqrt{q}$ (Hasse), $a = 2\sqrt{q}\cos\theta$, $\theta \in [0, \pi]$.*

- *(a) $g_\mathrm{n} = F/\sqrt{q}$ on $H^1(\frac{1}{2})$ is conjugate in $SU(2)$ to $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$: the standard (spin-$\frac{1}{2}$) representation of $SU(2) = \mathrm{USp}(2) = \mathrm{Spin}(3)$.*
- *(b) $\mathrm{Sym}^2 H^1(1)$ needs only an integer twist and has eigenvalues α^2/q, $\alpha\bar{\alpha}/q = 1$, $\bar{\alpha}^2/q$, i.e. $e^{2i\theta}, 1, e^{-2i\theta}$: the adjoint (vector) representation, which factors through $SO(3) = SU(2)/\{\pm 1\}$ as a rotation by 2θ.*
- *(c) Replacing $\sqrt{q}$ by $-\sqrt{q}$ replaces g_n by $-g_\mathrm{n}$, the non-trivial element of $\ker(SU(2) \to SO(3))$; $\mathrm{Sym}^2 H^1(1)$ does not change.*
- *(d) $g_\mathrm{n} = -1$ on $H^1(\frac{1}{2})$ iff $\theta = \pi$ iff $a = -2\sqrt{q}$, which needs q to be a square (a supersingular case). For $q = p$ prime, $\theta_p \in (0, \pi)$.*
- *(e) As a statement about $SU(2)$: the one-parameter subgroup $\mathrm{diag}(e^{it}, e^{-it})$ returns to 1 at $t = 2\pi$, i.e. at rotation angle $4\pi = 720°$, and passes -1 at rotation angle $360°$.*

Proof. (a) Unimodular conjugate eigenvalues of determinant 1. (b) Sym^2 of the diagonal matrix. (c) $\mathrm{Sym}^2(-g) = \mathrm{Sym}^2(g)$. (d) $\cos\theta = -1$; $2\sqrt{p} \notin \mathbb{Z}$. (e) Direct. □

[NUM] (`spin.json`, block D). Curve 11a, a_p by point count: $a_5, a_7, a_{13}, a_{17}, a_{19} = 1, -2, 4, -2, 0$. Over the 2260 primes $3 \le p < 20000$, $p \ne 11$: $\max |a_p|/2\sqrt{p} = 0.9940$, and the moments of $2\cos\theta_p$ are $0.985, 1.945, 4.795, 13.24$ (orders $2, 4, 6, 8$) against the Sato–Tate (Catalan) values $1, 2, 5, 14$, with odd moments 0.0066 and 0.031. The Sato–Tate theorem holds for 11a: it is a non-CM curve whose newform has weight 2, and Barnet-Lamb–Geraghty–Harris–Taylor prove Sato–Tate for all non-CM holomorphic newforms of weight > 1 ([4], abstract read). Sym^2 eigenvalues match $e^{\pm 2i\theta}, 1$ to 10^{-12} at $p = 5, 7, 13$, and $\mathrm{Sym}^2(-g_\mathrm{n}) = \mathrm{Sym}^2(g_\mathrm{n})$ there.

Reading of (c). Items (a)–(c) and (e) are theorems. Calling the normalized Frobenius a "rotation" and the 720° return a "TAA phenomenon" is [ANALOGY]: Frobenius does not move along a continuous path; its powers F^n have angles $n\theta$ mod 2π, and $\theta = \pi$ never occurs for prime q. One more precision: $SU(2) = \mathrm{Spin}(3)$ and $\mathrm{USp}(4) \cong \mathrm{Spin}(5)$ are low-dimensional coincidences (types $A_1 = B_1 = C_1$ and $B_2 = C_2$). For $g \ge 3$ the group $\mathrm{USp}(2g)$ is not a spin group, so "H^1 is a spinor" holds literally only for $g \le 2$. What persists in every genus is the fermionic module $\Lambda^\bullet H^1$ of §4. (For curves over $\mathbb{F}_q$ in families, Frobenius classes in $\mathrm{USp}(2g)$ are the subject of Katz–Sarnak [22]; the genus-2 Sato–Tate groups are classified in [14]; Serre's book [31] sets up the general Sato–Tate group. All [BIB].)

Proposition 6.4 (claim (d): cup product, "spinor ⊗ spinor", and what sees ghosts; [THM]; [NUM])**.** *(i)* $\omega : H^1 \otimes H^1 \to H^2 = \mathbb{Q}_\ell(-1)$ *is alternating, non-degenerate and Frobenius-equivariant,* $\omega(Fx, Fy) = q\,\omega(x, y)$ [25, Thm. 1.30]*. For* $g = 1$*, as* $SU(2)$*-representations after the half twist,*

$$H^1(\tfrac{1}{2}) \otimes H^1(\tfrac{1}{2}) = \Lambda^2 \oplus \mathrm{Sym}^2 = \underbrace{\mathbb{Q}_\ell}_{\text{cup product to } H^2(1)} \oplus \underbrace{\mathrm{Sym}^2 H^1(1)}_{SO(3)\text{ vector}},$$

which is the Euclidean 3-dimensional identity $S \otimes S = \Lambda^0 \oplus \Lambda^1$ *(scalar ⊕ vector). (ii)* $FV = VF = q$ *and* $\omega(Fx, Fy) = q\,\omega(x, y)$ *live in* $\Lambda^2 = \det$ *and involve only the integer twist* $H^2 = \mathbb{Q}_\ell(-1)$*. They hold for every reciprocal* P*, fakes included (Lemma 46 Thm. 6.1), so they are ghost-blind. (iii) The individual angles see ghosts:* $|a| > 2\sqrt{q}$ *iff* $\theta = i\eta \notin \mathbb{R}$*. For the fake* $a = 8$*,* $q = 13$*:* $\cos\theta = 1.10940$*,* $\eta = 0.463599$*, normalized eigenvalues* $e^{\pm\eta} = 1.58978,\ 0.629016$*, and* $\mathrm{Sym}^2 H^1(1)$ *has eigenvalues* $e^{\pm 2\eta} = 2.52742,\ 0.395661$ *and* 1*: a boost in* $SO(2,1)_0$*, not a rotation.*

Proof. (i) Clebsch–Gordan for $SU(2)$: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$, with Λ^2 the singlet. (ii) Lemma 46 Thm. 6.1(i),(iii). (iii) Lemma 46 §5. The numbers are in `spin.json`. □

Proposition 6.5 (spinor structure is ghost-blind; real form is not; [THM])**.** *Weil Frobenius classes lie in the compact form* $SU(2) = \mathrm{Spin}(3)$*, which double covers* $SO(3)$ *acting on* $\mathrm{Sym}^2 H^1(1)$*. A ghost Frobenius (*$g = 1$*) is hyperbolic in the split form* $SL_2(\mathbb{R}) = \mathrm{Spin}(2,1)_0$*, which double covers* $SO(2,1)_0$ *acting on* $\mathrm{Sym}^2 H^1(1)$ *with its Lorentzian (Killing) form. Both are spin groups, with the same kernel* $\{\pm 1\}$ *and the same* $\sqrt{q}$ *sign. So the double cover, the half twist and the "*$720°$*" structure are present for Weil curves and fakes alike. What separates them is compact versus split real form, i.e. whether the invariant Hermitian form on* $H^1(\frac{1}{2})$ *is definite (Lemma 46 Prop. 5.1).*

Proof. Rem. 3.6(ii) for $SL_2(\mathbb{R}) = \mathrm{Spin}(2,1)_0$; Lemma 46 Prop. 5.1 for the invariant forms. □

Proposition 6.6 (claim (e): the motivic centre and the half twist; [THM] given the conjectural functional equation [CONJ], framework [BIB] [12])**.** *For a pure motive* M *of weight* w *over* $\mathbb{Q}$ *the completed* L*-function is expected to satisfy* $\Lambda(M, s) = \varepsilon\,\Lambda(M^\vee, 1 - s)$ *with* $M^\vee \cong M(w)$*, hence* $s \leftrightarrow w + 1 - s$*, with centre* $\frac{w+1}{2}$*. This is known for* ζ *(*$w = 0$*, centre* $\frac{1}{2}$*), for elliptic curves over* $\mathbb{Q}$ *(*$w = 1$*, centre* 1*) and for* Δ *(*$w = 11$*, centre* 6*). Twisting gives* $L(M(n), s) = L(M, s + n)$ *and weight* $w - 2n$*;* $n = w/2$ *moves the centre to* $\frac{1}{2}$*, the "analytic normalization". For odd* w*,* $M(w/2)$ *is not a motive over* $\mathbb{Q}$ *(Prop. 6.2(ii)), so over* $\mathbb{Q}$ *the half twist is a change of variable in* s*, not an object.*

Remark 6.7 (over $\mathbb{F}_q$ the centre is $w/2$; precision to (e); [THM])**.** Over a finite field $L(H^w, s) = \det(1 - Fq^{-s} | H^w)^{-1}$ has its poles on $\Re s = w/2$ (Deligne), and its functional equation is centred at $w/2$, not $(w+1)/2$. The extra $\frac{1}{2}$ appears over a global field: for $K = \mathbb{F}_q(C)$, $\zeta_K(s) = Z(C, q^{-s})$ is the L-function of the weight-0 motive over the 1-dimensional base C, with centre $\frac{0+1}{2} = \frac{1}{2}$, and its zeros come from $H^1(C)$, of weight 1, placed at $\frac{w}{2} = \frac{1}{2}$. For $\zeta = \zeta_{\mathbb{Q}}$ the same bookkeeping says the $\frac{1}{2}$ is half the weight of the H^1 of the *base* $\overline{\mathrm{Spec}\,\mathbb{Z}}$ in Deninger's picture ([CONJ], [13]). So the half twist that recentres ζ is a half twist of the base's own H^1, the object that does not exist yet, and not of any motive over $\mathbb{Q}$.

7. $\mathrm{Spec}\,\mathbb{Z}$: what a spinor H^1 for ζ would need

Recall. Lemma 47 Thm. 2.3: on $\ell^2(\mathbb{Q}_S)$ with $|S| \geq 2$ there is no (anti)unitary $\mathcal{J}$ with $\mathcal{J} U_r \mathcal{J}^{-1} = U_r^*$ that swaps $\ell^2(\mathbb{N}_S)$ with its complement; for one prime the Dirac reflection exists, and so does the semi-infinite-wedge (boson–fermion) picture (Lemma 47 Thm. 2.5). Lemma 49 Prop. 6.1 (Deninger): a positive Hodge $*$ commuting with the flow gives $\Theta = \frac{1}{2} + A$, A skew.

Proposition 7.1 (Deninger's argument in vacuum form; [THM] for bounded Θ on a Hilbert space; implication [RESTATEMENT])**.** *Let* (H, ω) *be a real symplectic Hilbert space and* Θ *a bounded*

operator with $\omega(\Theta x, y) + \omega(x, \Theta y) = \omega(x, y)$ *(a derivation of the cup product acting by* 1 *on* H^2*). If* J *is a positive compatible complex structure commuting with* Θ*, then* $A = \Theta - \frac{1}{2}$ *is skew for* g_J*, so* $\operatorname{spec}\Theta \subset \frac{1}{2} + i\mathbb{R}$*.*

Proof. $\omega(Ax, y) + \omega(x, Ay) = 0$, so $A \in \mathfrak{sp}(\omega)$; then $g_J(Ax, y) = \omega(Ax, Jy) = -\omega(x, AJy) = -\omega(x, JAy) = -g_J(x, Ay)$. □

This is Lemma 49 Prop. 6.1 with Deninger's $*$ written as a vacuum J. Conversely, for finite-dimensional semisimple Θ the argument of Thm. 4.4 produces such a J whenever the spectrum is on the line. So, given the symplectic space and the flow, *constructing the positive vacuum is equivalent to RH.* The spinor language reorganizes the problem; it does not shorten it. (Unbounded Θ, which is what ζ needs, requires domain conditions not addressed here.)

Question 7.2 ([OPEN]: a "spinor H^1 for ζ")**.** Construct:

- (S1) a real Hilbert (or Pontryagin) space H with a symplectic form $\omega : H \times H \to \mathbb{R}(-1)$ playing the cup product, the conjectural $H^1(\overline{\operatorname{Spec}\mathbb{Z}})$ of [13] ([CONJ]);
- (S2) a flow Θ whose spectrum is the nontrivial zeros and which satisfies the derivation identity of Prop. 7.1. Spectral realizations of the zeros exist in Connes' framework [10] ([BIB]), but without any cup product or symplectic form;
- (S3) a compatible positive J commuting with Θ. Given (S1)–(S2) this is RH (Prop. 7.1);
- (S4) if H is assembled from local data at primes, as in the semilocal approach [7], a compatibility between the global J and local vacua J_p. Shale–Stinespring (Thm. 5.1) is the criterion for two vacua to define equivalent Fock spaces; whether "$J - J_p$ Hilbert–Schmidt in the p-component" is the right requirement cannot be decided until (S1) exists. The one candidate TAA has tested, the Dirac reflection on $\ell^2(\mathbb{Q}_S)$, already fails for two primes (Lemma 47 Thm. 2.3).

None of (S1)–(S4) is constructed here.

Remark 7.3 ($\sqrt{q}$ and spin structures; [ANALOGY], with the [THM] facts marked)**.** Three square-root choices have the same shape. A spin structure on Σ is a square root of K, a torsor under $H^1(\Sigma; \mathbb{Z}/2)$ (Prop. 2.2, [THM]). The half twist over $\mathbb{F}_q$ is a square root of $\mathbb{Q}_\ell(1)$, a torsor under $\operatorname{Hom}(\hat{\mathbb{Z}}, \pm 1) = \mathbb{Z}/2$ (Prop. 6.2(i), [THM]). The weight-$\frac{1}{2}$ factor $\sqrt{\tau/i}$ is a point of Mp_2 (Prop. 3.4, [THM]). Over $\mathbb{Q}$ the half twist is obstructed by the real place (Prop. 6.2(ii), [THM]), much as w_2 obstructs spin structures (Prop. 2.6). "$\operatorname{Spec}\mathbb{Z}$ is not spin in this naive sense" is a [ANALOGY] only. The analytic normalization uses the positive real $p^{-1/2}$, which always exists; the obstruction is Galois-theoretic, and nothing here shows it bears on RH.

Prior work (searched; only verified items cited).

- *Weil 1964* [37] ([BIB]): the paper that constructs the metaplectic (Weil) representation and uses it for theta series. It is not about RH or Weil positivity (title-level and standard content).
- *Connes–Consani–Moscovici 2024* [7] ([CITED], abstract and introduction read): in the semilocal trace-formula framework built around Weil's positivity criterion, they relate the prolate operator to the metaplectic representation of the double cover of $SL_2(\mathbb{R})$. They announce, without giving it in that paper, a second candidate using the Weil representation of the metaplectic cover of $SL_2(\mathbb{A}_S)$. *This is the closest verified link between a metaplectic/spin-type structure and Weil positivity*, and it sits on the bosonic (Weyl) side.
- *Connes–Moscovici 2022* [9] ([CITED], abstract read): the negative spectrum of the prolate operator off a finite interval reproduces the UV behaviour of the squares of zeros, and they build an isospectral family of Dirac operators with that UV behaviour. These are Dirac operators in the spectral-triple sense; no spin structure on an arithmetic space and no positivity proof are claimed.

- *Connes 1999* [10] and *Connes–Consani 2021* [6] ([BIB], verified in Lemmas 46–47): the trace formula and archimedean Weil positivity.
- *Fermionic primon gas*: Julia [19] ([BIB]) named the primon gas. Its fermionic version sums over squarefree n, with partition function $\zeta(s)/\zeta(2s)$ (standard; the curve analogue $Z(T)/Z(T^2)$ is Lemma 45 Thm. 2.2(A)). Spector [34] ([CITED] abstract, Lemma 45): the supersymmetric gas, with $(-1)^F = \mu$ and Witten index 1. Neither gives a positivity mechanism.
- *Kalauni–Milton 2023* [21] ([CITED] abstract): a supersymmetric quantum-mechanical model whose ground-state energies vanish at the zeros of ζ. The abstract claims no positivity argument.
- *Neretin 2004* [28] ([CITED] abstract) and Naranjo-Alvarado–Várilly [29] ([CITED]): Weil and spinor representations of infinite-dimensional groups on Fock spaces. These are background and not about ζ.
- *Bost–Connes fermionic variants*: our Crossref searches ("Bost–Connes fermionic", "Bost–Connes free fermions") returned only unrefereed 2026 SSRN preprints, which we do not cite. We found no verified fermionic or CAR version of the BC system bearing on RH.

Finding. We found no verified prior work deriving Weil positivity, or any RH-equivalent, from a spin or Clifford structure. The verified metaplectic link (CCM) is bosonic and does not supply positivity: Connes–Consani prove the archimedean positivity by other means [6].

8. Corrections to the chat claims

Chat [t188]:

(C1) *"The spinor bundle on S^2 has twist $\chi/2 = 1$."* Correct for the half $K^{-1/2} = \mathcal{O}(1)$, the square root of $TS^2 = \mathcal{O}(2)$. The full spinor bundle is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ with $c_1 = 0$ (Prop. 2.2(b)).

(C2) *"Spinor holonomy is half the angle" (Berry $-\frac{1}{2}\Omega$).* Correct for loops bounding a disc. For non-bounding loops the spinor holonomy is a square root of the vector holonomy, and the sign is the spin structure (Prop. 2.4). Berry phase checked numerically for $s = \frac{1}{2}, 1, \frac{3}{2}$.

(C3) *"Forms $= S \otimes S$, so Gauss–Bonnet is a twisted Dirac index."* Correct as $\Lambda^\bullet \otimes \mathbb{C} \cong S \otimes S^*$ ($\cong S \otimes S$), with χ and σ from *different gradings* of one operator (Thm. 2.5). But it holds on non-spin manifolds too ($\mathbb{C}P^2$), so "spinors come first" is a statement about local and representation-theoretic structure, not a logical dependency.

(C4) *"Spin exists because χ is even; 2^{2g} theta characteristics."* Correct for surfaces ($w_2[\Sigma] = \chi \bmod 2$), with $2^{g-1}(2^g \pm 1)$ even/odd ones. False as a general principle: in dimension 4, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ has $\chi = 4$ and is not spin (Prop. 2.6).

(C5) *Rokhlin's 16.* Correct for closed *smooth* spin 4-manifolds (Prop. 2.6(d)). Smoothness is essential.

(C6) *"The $\sqrt{q}$ choice $\sim$ choice of spin structure."* A good [ANALOGY], now sharpened: both are $\mathbb{Z}/2$-torsors of square roots, and the global version over $\mathbb{Q}$ is obstructed (Prop. 6.2, Rem. 7.3). The sign flip is exactly $\ker(\mathrm{Spin}(3) \to SO(3))$ (Prop. 6.3(c)). But the same structure is present for fakes (Prop. 6.5).

(C7) *"The theta polarization we used in Lemma 47."* Inaccurate: Lemma 47 only proposes the theta polarization as a next step, and uses no theta characteristics (Rem. 2.3).

(C8) *"Spinors are the more fundamental object"; "the $\frac{1}{2}$ is a spinor-type choice."* For the $\frac{1}{2}$ of Riemann's proof, the relevant double cover is *metaplectic* (bosonic), and the $\frac{1}{2}$ is the oscillator's zero-point weight (Prop. 3.5). $SL_2(\mathbb{R})$ is itself $\mathrm{Spin}(2,1)_0$ (Rem. 3.6).

Brief and steering (parent):

(C9) *"The centre $\frac{1}{2}$ corresponds to weight $\frac{1}{2}$ under $u = 2s$."* The substitution is $s = 2u$. Hecke's centre is $k/2 = \frac{1}{4}$, and $\frac{1}{2} = 2 \cdot \frac{k}{2}$; θ^2 (weight 1, centre $\frac{1}{2}$, no doubling) shows the coincidence is not structural on its own (Prop. 3.2, Rem. 3.3).

(C10) *"Clifford algebra of H^1 with its symplectic form."* A symplectic form gives a Weyl algebra. Use $H^1 \oplus H^{1\vee}$ (module $\Lambda^\bullet H^1$) or the symmetric s_L from a Lagrangian (module $\Lambda^\bullet L$) (Prop. 4.1).

(C11) *"Mp_2 is a spin cover of SL_2."* [ANALOGY] only: Mp_2 is the bosonic analogue of Spin, and a double cover of the spin group $SL_2(\mathbb{R}) = \mathrm{Spin}(2,1)_0$.

(C12) *(a)* Correct, for the geometric Frobenius. *(b)* Correct that a choice of $\sqrt{q}$ is needed and integer twists need none. Corrections: over $\mathbb{Q}$ no half twist exists, and weight 0 is not unitarity (unitarity is RH plus semisimplicity) (Prop. 6.2). *(c)* Correct (Prop. 6.3); $\theta = \pi$ occurs only for $a = -2\sqrt{q}$ with q a square, and the "rotation/720°" reading of Frobenius is [ANALOGY]. *(d)* Correct: $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$ is exactly $S \otimes S = \Lambda^0 \oplus \Lambda^1$ in 3D; $FV = q$ is ghost-blind, and ghosts are imaginary angles (Prop. 6.4). *(e)* Correct over $\mathbb{Q}$: centre $(w+1)/2$, recentred to $\frac{1}{2}$ by the twist $w/2$ (a change of variable for odd w). Over $\mathbb{F}_q$ the centre is $w/2$ (Prop. 6.6, Rem. 6.7).

9. What is new and what is restatement

- **Classical, re-proved or checked ([THM]/[NUM]):** Prop. 2.2, 2.4, Thm. 3.1, Prop. 3.2, 3.4, 3.5, 4.1, Thm. 4.2, Prop. 5.2, 6.1, 6.2, 6.3, 6.4, 6.5. None is new mathematics.
- **Presentation-level, probably folklore:** the vacuum/Siegel-space form of Weil's theorem with Krein-sign uniqueness (Thm. 4.4); "spinor structure is ghost-blind, real form is not" (Prop. 6.5); "the $\frac{1}{2}$ of θ is bosonic" (Prop. 3.5); the half twist as a $\mathbb{Z}/2$-torsor obstructed over $\mathbb{Q}$ (Prop. 6.2); centre $w/2$ vs. $(w+1)/2$ (Rem. 6.7).
- **Restatement ([RESTATEMENT]):** Thm. 4.4 (Weil's RH for curves), Prop. 7.1 (Deninger), Thesis 5.4(b).
- **Analogy ([ANALOGY]):** "Mp_2 is a spin cover", the 720° reading of Frobenius, "$\mathrm{Spec}\,\mathbb{Z}$ is not spin" (Rem. 7.3), Thesis 5.4(c).
- **Open ([OPEN]):** Question 7.2.
- **No lever on RH.** The spinor reading places the polarization precisely, as a choice of positive invariant vacuum. It does not produce one for ζ, and spinors cannot manufacture the positivity (Cor. 5.3).

10. Questions for J.W.

Question 10.1. Adopt Thesis 5.4 ("the missing polarization is the missing invariant positive vacuum") as TAA language, with its labels: [THM] over $\mathbb{C}$, [RESTATEMENT] over $\mathbb{F}_q$, [ANALOGY] for ζ?

Question 10.2. Prop. 3.5 says the $\frac{1}{2}$ in Riemann's proof is a bosonic zero-point weight on Mp_2, not a spin $\frac{1}{2}$. Should the Volume 3 polarization chapter present the $\frac{1}{2}$ through the oscillator vacuum rather than through spinors? Or do you want both, side by side as in Rem. 3.6?

Question 10.3. Thm. 4.4 selects the positive vacuum by the Krein signs κ_λ. A ghost is born when two eigenvalues of opposite Krein sign collide (Volume 2 Ch. 27, Lemma 48). Do you want a short follow-up lemma tracking the invariant vacua J_ε through such a collision: a positive vacuum turning into a Krein vacuum, and then into none (real ghost)?

Question 10.4. CCM [7] announce a semilocal operator built from the Weil representation of $\mathrm{Mp}_2(\mathbb{A}_S)$. Should TAA run an exploratory numerical lemma asking whether that bosonic setup has a natural compatible J (a vacuum), and how it behaves as S grows? This is item (S4) of Question 7.2. It would be exploratory, with no promise of positivity.

Question 10.5. In genus 2, $\mathrm{USp}(4) \cong \mathrm{Spin}(5)$, so "$H^1(\frac{1}{2})$ is a spinor" is still literal; it stops at $g = 3$. Is a genus-2 section worth having, or is the uniform $\Lambda^\bullet H^1$ picture enough?

Question 10.6. Where should the surface material (§2) live: as is, folded into Lemma 49 Rem. 7.4, or in the Volume 3 polarization chapter?

Status

Proved ([THM]): Prop. 2.2(a),(b), 2.4, 2.6(b),(c) (given Wu), Thm. 3.1, Prop. 3.2, 3.4, 3.5 (Mehler), 4.1, Thm. 4.2, Thm. 4.4 (proof), Prop. 5.2, Cor. 5.3, Prop. 6.1, 6.2, 6.3, 6.4, 6.5, 7.1. **Cited ([CITED]):** NV22 §1–2 (Shale–Stinespring conditions; Weyl/Clifford parallel); BLGHT abstract (Sato–Tate); CCM24 abstract and introduction; CM22, KM23, Ner04 abstracts; MilpRH Thm. 1.30 (via Lemma 45); Spector abstract (via Lemma 45). **Bibliographic ([BIB]):** At71, Mum71, Mum07, LM89, AS68, Kir89 (Rokhlin), Hit74, Ber84, Hec36, Shi73, Wei64, LV80, Sha62, SS65, Ara70, Ott95, How89, Del74, Del79, HSBT, CHT, Tay08, FKRS, KS99, Ser12, Den98, Con99, CC21, Jul90. **Open ([OPEN]):** Question 7.2. **Not verified, not cited:** Rokhlin's 1952 Doklady note (no Crossref record; used only via [20]); Riemann 1859 (no DOI; content via Thm. 3.1); any "fermionic Bost–Connes" paper (none found in refereed sources); the content of Atiyah 1971 and Mumford 1971 beyond the standard statements (the Numdam PDFs could not be fetched from the box; the counts are checked numerically instead).

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TAA LEMMA 51: TRACKING THE INVARIANT FERMIONIC VACUUM THROUGH GHOST BIRTH. KREIN COLLISIONS, THE SIEGEL BOUNDARY, $SU(2) = \mathrm{Spin}(3) \to SL_2(\mathbb{R}) = \mathrm{Spin}(2,1)_0$, AND GENUS 2 THROUGH $\mathrm{USp}(4) \cong \mathrm{Spin}(5)$, $\mathrm{Sp}(4,\mathbb{R}) \cong \mathrm{Spin}(3,2)_0$

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Date: October 22, 2026.

Abstract. **Statement of intellectual honesty.** RH is not proved here, and nothing below brings it closer. Everything is finite-dimensional linear algebra about symplectic matrices whose characteristic polynomials are (fake or honest) L-polynomials, plus classical Lie theory and point counts. Tags: [THM] proved here (or classical with proof given); [CITED] checked at the source (`lit/lemma51/verified_refs.md`); [BIB] bibliographic record verified (Crossref), content standard; [RESTATEMENT] restatement of a known result or of an earlier TAA lemma; [ANALOGY] analogy only; [NUM] numerical (`num/lemma51_collide.py`, `num/lemma51_spin5.py`, `num/l51/chartdet.py`; data `num/l51/`); [CONJ] conjecture; [OPEN] open. Lemmas 27 (Vol. 2 Ch. 27), 45–50 are cited by number and not redone.

Question (J.W., [t196u], answers to Lemma 50's questions). "1. Yes. 2. I like spinors better. 3. Yes. 4. Yes. 5. Yes. 6. Volume 3." Answer 3 asks for the follow-up announced in Lemma 50 Question 9.3: track the invariant vacua J_ε through a Weil→fake collision. Answer 5 asks for the genus-2 section ($\mathrm{USp}(4) \cong \mathrm{Spin}(5)$). Answer 2 fixes the presentation: spinor language wherever it is literal. The approved TAA sentence (answer 1) is "*the missing polarization = the missing invariant positive (fermionic) vacuum*".

Answer in brief. (1) *The positive vacuum does not turn into a ghost; it goes to infinity.* Whenever a path of normalized Frobenius-type matrices $g_t \in \mathrm{Sp}(V)$ reaches a matrix g_* not conjugate into a unitary group, every positive invariant vacuum J_t leaves every compact subset of Siegel space (Thm. 2.1). Its distance to a base point diverges, its pure-spinor transversality $\nu(J) = \prod_i \operatorname{sech} d_i$ tends to 0, and so does its bosonic fidelity $\prod_i \operatorname{sech}(d_i/2)$ (Thm. 2.2). It reaches the Siegel boundary, and *which* boundary stratum it reaches records how the ghost is born (Thm. 4.2, Prop. 4.3). (2) *Genus 1* (§3). In the fixed chart $(1, T/\sqrt{q})$ the positive vacuum is $J_+ = (g - \frac{t}{2})/\sin\varphi$: the normalized *rotation axis* of $\mathrm{Ad}(g) \in \mathrm{SO}(2,1)_0$ acting on $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}^{2,1}$, i.e. the vacuum is the vector of the $\mathrm{Spin}(2,1)_0$ element. It has $\cosh d = 1/\sin\varphi$ and transversality $\nu = \sin\varphi$, which equals the Krein norm of the unit eigenvector. As $a \uparrow 2\sqrt{q}$ the axis becomes null and J_+ blows up like $2/\varphi$. Past $2\sqrt{q}$ the axis is spacelike and no invariant J exists. Under $\varphi = i\eta$ the Krein norm continues into Lemma 48's Bode gain defect η, with matching leading constants at symmetric offsets (Prop. 3.1). The endpoint $a = \pm 2\sqrt{q}$ is parabolic in every family chart, but for a real curve it is semisimple ($\pm\sqrt{q} \cdot 1$), and then every J is invariant (Prop. 3.2). (3) *Genus 2, opposite-sign collision* (§4). Lemma 50's trace model has Krein pattern $(+,+)$ for every Weil P, so it cannot host a Krein collision. Its form ω_P degenerates exactly at the collision, $\det\omega_P = (-1)^g \operatorname{disc}(P) P(\sqrt{q})P(-\sqrt{q})/q^g$ (Thm. 2.5). In a fixed chart the same event is a mixed-sign (Hamiltonian–Hopf) collision. These are Ch. 27's two stages, *coinciding* (Cor. 2.6). Along an explicit Hamiltonian–Hopf path from the Weil polynomial to the quadruple fake of Lemma 50, the *positive* and *negative* vacua diverge equally and go to the Shilov boundary. The two *mixed* vacua converge to $J_* = \pm i$ on the generalized eigenspace and continue smoothly as the two Krein vacua of the quadruple. The count of invariant J is $4 \to 2 \to 2$ (Thm. 4.2). Along a path to the real-ghost fake, one rapidity diverges and the vacuum reaches a corank-one boundary component (Prop. 4.3). In both cases the Krein norm before equals the gain defect after, to leading order. (4) *Spinors* (§5). $\Lambda^2 H^1 = \mathbf{1} \oplus \mathbf{5}$, and the wedge form on $\mathbf{5} = \Lambda^2_0$ has signature $(5,0)$, $(4,1)$ or $(2,3)$ for $\mathrm{USp}(4)$, $\mathrm{Sp}(1,1)$ and $\mathrm{Sp}(4,\mathbb{R})$ ([THM], checked [NUM]). The positive vacuum is a Frobenius-invariant definite 2-plane in $\mathbf{5}$: the plane Frobenius rotates by $\theta_1 \pm \theta_2$ (Prop. 5.3). The isomorphism $B_2 = C_2$ *swaps* the two kinds of ghost birth: a Hamiltonian–Hopf collision on the spinor H^1 is the genus-1 picture (elliptic→parabolic→hyperbolic in an $\mathrm{SO}(2,1)$) on the vector $\mathbf{5}$, and conversely (Thm. 5.4). Quadruple fakes live in $\mathrm{Spin}(4,1) = \mathrm{Sp}(1,1)$; real-ghost fakes do not (Prop. 5.5). For $y^2 = x^5 + 3x^3 + x + 7$ the $\mathrm{USp}(4)$ moments are matched numerically: trace moments $0.988, 2.92, 13.56, 81.7$ against $1, 3, 14, 84$, and $\chi_{\mathbf{5}}$ moments $-0.003, 0.923, 0.013, 3.013$ against $0, 1, 0, 3$ ([NUM]; Sato–Tate for this curve is [CONJ]). For $g \geq 3$, H^1 is no longer a spinor and only $\Lambda^\bullet H^1$ survives (Prop. 5.7). (5) *What it is not.* No step here produces a positive vacuum where there is none. The lemma locates where and how the vacuum is lost.

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1. Conventions, and what TAA already has

(V,ω) is a real symplectic space of dimension $2g$. A *compatible complex structure* is $J \in \mathrm{Sp}(V)$ with $J^2 = -1$. It is *positive* if $g_J(x,y) = \omega(x, Jy)$ is positive definite. Siegel space $\mathfrak{h}(V)$ is the set of positive compatible J [12] ([BIB]). For $g \in \mathrm{Sp}(V)$ the *Krein form* is $h_K(x,y) = i\,\omega(\bar{x}, y)$ on $V_{\mathbb{C}}$, Hermitian of signature (g,g), and $h_K(gx, gy) = h_K(x,y)$. For an eigenvalue λ with $|\lambda| = 1$ the *Krein sign* is the sign of h_K on the eigenspace E_λ when that sign is definite (Lemma 50, Thm. 2.1 below). We fix a reference $J_0 \in \mathfrak{h}(V)$ and a g_{J_0}-orthonormal symplectic basis; "unit vector" means g_{J_0}-unit. For $x \in E_\lambda$ a unit eigenvector, $\hat{\kappa}(x) = h_K(x,x)$ is its *normalized Krein norm*.

Vacuum dictionary (Lemma 50, §§3–4; adopted TAA language per J.W. [t196u]). A compatible J is a fermionic vacuum: the Fock module $\Lambda^\bullet L_J$, $L_J = \ker(J - i) \subset V_{\mathbb{C}}$, a Lagrangian. Its *pure spinor* is the decomposable g-vector $\phi_J = b_1 \wedge \cdots \wedge b_g \in \Lambda^g V_{\mathbb{C}}$ for a basis (b_i) of L_J. This is the pure spinor of the maximal isotropic subspace $L_J \oplus L_J^\perp$ of $V_{\mathbb{C}} \oplus V_{\mathbb{C}}^\vee$ in the spinor module $\Lambda^\bullet V_{\mathbb{C}}$ of Lemma 50 Prop. 3.1(b) ([1], [BIB]). The vacuum is *invariant* if $Jg = gJ$, and it is a Hilbert-space vacuum iff $J \in \mathfrak{h}(V)$. Lemma 50 Thm. 4.4: RH for $C/\mathbb{F}_q \iff g_{\mathrm{n}}$ has an invariant positive vacuum. For distinct roots, the invariant vacua are the J_ε, and the positive one is $\varepsilon = \kappa$.

Source	what is used, not redone
Vol. 2 Ch. 27 (L27)	Two-stage mechanism: stage 1 is a metric degeneration (Sylvester) and gives ghost birth; stage 2 is an opposite-sign Krein collision and gives the boost. Definite-type stability (Krein–Gelfand–Lidskii, via Mehl Thm. 4.7 [10], [CITED] in L27). L41's note: "one event in two charts".
L46 Thm. 3.1, 6.1	Isotropy bound: for a h_K-type form of signature (p,q), the span of generalized eigenvectors with $\lvert\lambda\rvert > 1$ is isotropic, so has dimension $\le \min(p,q)$. Trace form $b(x,y) = \mathrm{Tr}(xy^\sharp)$ on $\mathbb{R}[T]/(P)$ has signature $(D_{\mathrm{on}} + k, k)$.
L48 Thm. 3.4	Finite-field Bode: gain defect $\int \log\lvert\mathcal{L}\rvert = \sum_{\mathrm{ghosts}} \eta_i = \log M(\mathcal{L})$; ghosts = Blaschke zeros $1/\beta_i$.
L50 Thm. 4.4, Rem. 4.5, Prop. 5.4–5.5	Trace model $V = \mathbb{R}[T]/(P)$, $\omega_P(x,y) = \mathrm{Tr}(c\,xy^\sharp)$, $c = T - q/T$; invariant J_ε; the Weil curve $y^2 = x^5 + 3x^3 + x + 7$ over $\mathbb{F}_{11}$ with $P_W = T^4 - 2T^3 + 8T^2 - 22T + 121$; the fakes $P_R = T^4 + 11T^3 - 13T^2 + 121T + 121$ (real ghost) and $P_Q = T^4 - 13T^3 + 65T^2 - 143T + 121$ (complex quadruple); $SU(2) = \mathrm{Spin}(3)$ vs. $SL_2(\mathbb{R}) = \mathrm{Spin}(2,1)_0$.

2. General theorems: how an invariant vacuum is lost

For $J \in \mathfrak{h}(V)$ there are $k \in U(J_0)$ and $a = \mathrm{diag}(e^{d_i/2}, e^{-d_i/2})$ (in the fixed basis) with $J = kaJ_0a^{-1}k^{-1}$. The *rapidities* $d_1, \ldots, d_g \ge 0$ are well defined: $e^{\pm d_i}$ are the eigenvalues of $-J_0J$. The invariant Riemannian distance is $d(J_0, J) = (\sum d_i^2)^{1/2}$, and for $g = 1$ it is the hyperbolic distance on $\mathfrak{h} = \mathbb{H}$. The Siegel-disc (Harish-Chandra) image of J has singular values $\tanh(d_i/2)$. The *Shilov*

boundary is where all of them equal 1 (real Lagrangians); a *corank-r boundary component* is where exactly r of them equal 1 ([12, 5], [BIB]).

Theorem 2.1 (the vacuum escapes; [THM]). *Let $g_t \in \mathrm{Sp}(V)$ be continuous with $g_t \to g_*$ as $t \to t_*$.*

(a) If g_* is not conjugate in $\mathrm{Sp}(V)$ into a compact subgroup (equivalently: g_* is not semisimple with all $|\lambda| = 1$), then any positive invariant J_t leaves every compact subset of $\mathfrak{h}(V)$: $\max_i d_i(J_t) \to \infty$.

(b) If g_* has no invariant compatible J at all, then every invariant compatible J_t, of any signature, is unbounded.

(c) If $J_t \to J_*$ along a sequence, then J_* is an invariant compatible complex structure for g_*, of the same signature (if it is non-degenerate there).

Proof. (c) $J^2 = -1$, $J^T \Omega J = \Omega$ and $Jg = gJ$ are closed conditions. (a) If J_{t_n} stayed in a compact set $K \subset \mathfrak{h}$, a subsequence would converge to some $J_* \in K$ with $J_* g_* = g_* J_*$ by (c). Then g_* preserves the Euclidean form g_{J_*}, i.e. lies in the compact $U(J_*)$, a contradiction. Conversely, an element of a compact subgroup of $\mathrm{Sp}(V)$ fixes a point of $\mathfrak{h}$ (Cartan's fixed point theorem), and the stated spectral condition is equivalent (Lemma 50 Thm. 4.4). (b) A bounded sequence has a convergent subsequence; apply (c). □

Theorem 2.2 (transversality of the pure spinor, fidelity, and the Krein norms; [THM], checked [NUM]). *Define $\nu(J) = |\det[B, \bar{B}]| / \det(B^* B)$, where B is any $2g \times g$ matrix whose columns span L_J (in the g_{J_0}-orthonormal symplectic basis). Equivalently, $\nu(J) = |\phi_J \wedge \bar{\phi}_J| / \|\phi_J\|^2$, normalized so that $\nu(J_0) = 1$.*

(a) ν does not depend on B, is $U(J_0)$-invariant, and $\nu(J) = \prod_{i=1}^{g} \operatorname{sech} d_i$ for $J \in \mathfrak{h}(V)$. So $\nu \to 0$ exactly when some $d_i \to \infty$, i.e. when L_J meets its conjugate in the limit (the boundary). In particular, under Thm. 2.1(a), $\nu(J_t) \to 0$.

*(b) (*Krein volume.*) Let g have $2g$ distinct eigenvalues on the unit circle and let J be invariant, with L_J spanned by unit eigenvectors $b_1, \dots, b_g$. Then*

$$\nu(J) = \frac{\prod_{i=1}^{g} |\hat{\kappa}(b_i)|}{\det(B^* B)} \;\geq\; \prod_i |\hat{\kappa}(b_i)|,$$

with equality iff the b_i are g_{J_0}-orthogonal. J is positive iff every $\hat{\kappa}(b_i) > 0$.

*(c) (*Bosonic fidelity.*) The metaplectic (Gaussian) vacua Ω_J satisfy $|\langle \Omega_{J_0}, \Omega_J \rangle|^2 = \prod_i \operatorname{sech}(d_i/2) = 2^g \det(1 - J_0 J)^{-1/2}$.*

Proof. (a) Replacing B by BA multiplies the numerator and the denominator by $|\det A|^2$. For $k \in U(J_0)$ (real orthogonal and symplectic), $|\det k| = 1$ and $(kB)^*(kB) = B^* B$. Reduce to $J = a J_0 a^{-1}$, a product of g planar blocks. In one block L_J is spanned by $(e^{d/2}, -ie^{-d/2})$, so $|\det[B, \bar{B}]| = 2$ and $B^* B = 2 \cosh d$, and the ratio is $\operatorname{sech} d$. (b) Let $C = [B, \bar{B}]$ and $M = C^* \Omega C$, so $|\det M| = |\det C|^2$ ($\det \Omega = 1$). The columns of C are eigenvectors for the $2g$ distinct eigenvalues. Invariance gives $\omega(\bar{u}, v)(\bar{\mu}_u \mu_v - 1) = 0$ for eigenvectors with eigenvalues μ_u, μ_v, and for unimodular μ this means $\mu_u = \mu_v$. So M is diagonal with entries $\omega(\bar{u}, u) = -i h_K(u, u)$, and $|\det C| = \prod_i |\hat{\kappa}(b_i)|$ (each b_i and $\bar{b}_i$ contribute $|\hat{\kappa}(b_i)|$). Hadamard's inequality gives $\det(B^* B) \leq 1$. Positivity: $g_J(\bar{x}, x) = \omega(\bar{x}, ix) = h_K(x, x)$ on L_J, as in Lemma 50 Thm. 4.4. (c) In one block the normalized vacua are $\Omega_{J_0} \propto e^{-x^2/2}$ and $\Omega_J \propto e^{-e^{-d} x^2/2}$ (up to a $U(1)$ rotation k, which does not change the modulus). Gaussian integration gives $|\langle \Omega_{J_0}, \Omega_J \rangle|^2 = 2e^{-d/2}/(1 + e^{-d}) = \operatorname{sech}(d/2)$. Since $-J_0 J$ has eigenvalues $e^{\pm d_i}$, $\det(1 - J_0 J) = \prod_i (1 + e^{d_i})(1 + e^{-d_i}) = \prod_i 4 \cosh^2(d_i/2)$. □

[NUM]: (a) and (c) hold to 10^{-50} along all paths of §§3–4 (`collide.json`, fields `pure_spinor_volume` = `sech_prod`, `fidelity_bosonic` = `fidelity_gauss_formula`). (b) is checked on 20 random $\mathrm{Sp}(4, \mathbb{R})$-conjugates of elliptic elements, all four J_ε each, with Gram determinants down to 0.23: maximal error $3 \cdot 10^{-16}$ (`num/l51/kreinvol.py`). *Why only bosonic*

fidelity: the fermionic Fock spaces of J_0 and J are built on different Euclidean forms $g_{J_0} \neq g_J$, so there is no common Clifford algebra and no overlap $\langle 0_{J_0}|0_J\rangle_{\mathrm{ferm}}$. The transversality ν is the spinor-side substitute: it measures how far the pure spinor ϕ_J is from being "real" ($\phi_J \wedge \bar{\phi}_J = 0$).

Proposition 2.3 (Krein–Pontryagin count; [THM], [RESTATEMENT] of L45 Thm. 6.4/L46 Thm. 3.1 in h_K language)**.** *For $g \in \mathrm{Sp}(V)$ let m be the number of eigenvalues with $|\lambda| > 1$ (with multiplicity), and let (n_+, n_-) be the signature of h_K on $\bigoplus_{|\lambda|=1} E_\lambda$ (generalized eigenspaces). Then $n_+ = n_- = g - m$. So every eigenvalue that leaves the circle removes one positive and one negative direction.*

Proof. $h_K(E_\lambda, E_\mu) = 0$ unless $\bar{\lambda}\mu = 1$, so $V_{\mathbb{C}}$ is the h_K-orthogonal sum of $\bigoplus_{|\lambda|=1} E_\lambda$ and the pairs $E_\lambda \oplus E_{1/\bar{\lambda}}$, $|\lambda| > 1$. Each pair is neutral, of signature $(\dim E_\lambda, \dim E_\lambda)$. The total signature is (g, g). □

Corollary 2.4 (definite type cannot leave; [THM], [CITED] [10, Thm. 4.7])**.** *If h_K is definite on the generalized eigenspace of a unimodular λ_* of g_*, then for g near g_* all eigenvalues near λ_* are unimodular and semisimple. A unimodular eigenvalue leaves the circle only after colliding with one of the opposite Krein sign.*

Proof. Let P_g be the Riesz projector of g for a small circle around λ_*. It is continuous in g, so h_K stays definite on $\operatorname{im} P_g$. An eigenvector with $|\lambda| \neq 1$ is h_K-isotropic, since $h_K(x,x)(|\lambda|^2 - 1) = 0$, which is impossible on a definite space. A Jordan chain x_0, x_1 with $(g-\lambda)x_1 = x_0$ gives $h_K(x_0, x_0) = 0$ in the same way. □

Theorem 2.5 (the two charts and their determinants; [THM], checked [NUM])**.** *Let P be monic, reciprocal ($P(T) = q^{-g}T^{2g}P(q/T)$), of degree $2g$ with $P(\pm\sqrt{q}) \neq 0$. In the power basis of $V = \mathbb{R}[T]/(P)$:*

$$\det b = (-1)^g \operatorname{disc}(P), \qquad \det \omega_P = (-1)^g \operatorname{disc}(P)\, \frac{P(\sqrt{q})P(-\sqrt{q})}{q^g}.$$

So L50's trace chart degenerates exactly *at root collisions. For $g = 1$, $P = T^2 - aT + q$: $\omega_P(1, T) = 4q - a^2 = -\operatorname{disc} P$.*

Proof. $b = \mathrm{Tr}(x \cdot \sharp y)$ is the trace form composed with the algebra involution $\sharp$, and the trace form has determinant $\operatorname{disc}(P)$ in the power basis. The fixed algebra of $\sharp$ is $\mathbb{R}[T + q/T]$, of dimension g, so $\sharp$ has eigenvalues $+1$ and -1 each g times and $\det \sharp = (-1)^g$. Next, $\omega_P = b(c\,\cdot, \cdot)$ and $\det(\text{mult by } c) = N(c) = \prod_\alpha (\alpha - q/\alpha) = \prod_\alpha (\alpha^2 - q)/\prod \alpha = P(\sqrt{q})P(-\sqrt{q})/q^g$. □

[NUM]: exact symbolic check for $g = 1$, and for random integer reciprocal P with $g = 2$ (4 cases) and $g = 3$ (2 cases) (`num/l51/chartdet.json`).

Corollary 2.6 (Ch. 27's two stages coincide for honest L-families; [THM])**.** *For Weil P with distinct roots, h_K in L50's trace model has Krein sign* $\operatorname{sign} \sin \varphi_\alpha = +$ *on every upper eigenvalue $\alpha = \sqrt{q}e^{i\varphi_\alpha}$. The pattern is $(+, \ldots, +)$, so by Cor. 2.4 no collision is possible* inside *the trace chart. A path P_t that produces a ghost must pass through* $\operatorname{disc} P_t = 0$*, where ω_{P_t} degenerates (Thm. 2.5): this is Ch. 27's stage 1 (metric degeneration, Sylvester). In any fixed symplectic chart that realizes the same path continuously, the same instant is an opposite-sign collision (stage 2). So for honest L-polynomial families the stages occur at the same parameter, "one event in two charts" (Lemma 41, as recorded in Ch. 27).*

Proof. Write $V_{\mathbb{C}} = \bigoplus_\alpha \mathbb{C}e_\alpha$ (idempotents at the roots). Then $\bar{e}_\alpha = e_{\bar{\alpha}}$, $e_\alpha^\sharp = e_{q/\alpha}$, and $q/\alpha = \bar{\alpha}$ on the circle. So $\omega_P(\bar{e}_\alpha, e_\alpha) = \mathrm{Tr}(c\, e_{\bar{\alpha}} e_{\bar{\alpha}}) = c(\bar{\alpha}) = \bar{\alpha} - \alpha = -2i\sqrt{q}\sin\varphi$, and $h_K(e_\alpha, e_\alpha) = 2\sqrt{q}\sin\varphi > 0$ for $\sin\varphi > 0$. (This is also the content of the uniqueness clause of Lemma 50 Thm. 4.4: the positive J is $+i$ on all upper eigenspaces.) The rest is Thm. 2.5 and Cor. 2.4. □

3. Genus 1: the vacuum is the rotation axis of $\mathrm{Spin}(2,1)_0$

Let $P = T^2 - aT + q$, $t = a/\sqrt{q}$. In the basis $(1, T/\sqrt{q})$ of $\mathbb{R}[T]/(P)$, with the *fixed* standard form $\hat{\omega}$ ($\hat{\omega}(e_0, e_1) = 1$), the normalized Frobenius is

$$g = \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix} \in SL_2(\mathbb{R}) = \mathrm{Sp}(\hat{\omega}), \qquad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ (the } a = 0 \text{ vacuum)}.$$

This chart is defined for all real a, including $|a| \geq 2\sqrt{q}$, and g is continuous in a. L50's trace chart, by contrast, degenerates at $a = \pm 2\sqrt{q}$ (Thm. 2.5).

Proposition 3.1 (genus-1 vacuum; [THM], checked [NUM]). *Let $|t| < 2$, $t = 2\cos\varphi$, $\varphi \in (0, \pi)$, $s = \sin\varphi$.*

- *(a)* $A := g - \frac{t}{2} = \left(\begin{smallmatrix} -t/2 & -1 \\ 1 & t/2 \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{R})$ *spans the axis of* $\mathrm{Ad}(g) \in \mathrm{SO}(2,1)_0$ *acting on* $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}^{2,1}$ *(form* $\frac{1}{2}\operatorname{Tr} XY$*).* $\operatorname{Tr} A^2 = (t^2 - 4)/2$*, so the axis is timelike for $|t| < 2$, null for $|t| = 2$, spacelike for $|t| > 2$.*
- *(b) The two invariant compatible J are $\pm A/s$. The positive one is $J_+ = A/s$, the* unit timelike axis. *Its fixed point in $\mathbb{H}$ is the fixed point $\tau = -\cos\varphi + i\sin\varphi$ of $z \mapsto -1/(z+t)$.*
- *(c)* $\cosh d(J_0, J_+) = 1/\sin\varphi$, $\|J_+\|_F^2 = 4/s^2 - 2$, $\|J_+ - J_0\|_F^2 = 4(1-s)/s^2$, $\nu(J_+) = s$*, and the bosonic fidelity is* $\mathrm{sech}(d/2)$.
- *(d) The unit eigenvector for $e^{i\varphi}$ has normalized Krein norm $\hat{\kappa} = +s$, and that for $e^{-i\varphi}$ has $-s$. So $\nu(J_+) = \hat{\kappa}$ (Thm. 2.2(b) with* $\det B^* B = 1$*).*
- *(e) As $a \uparrow a_c = 2\sqrt{q}$, with $\Delta = a_c - a$: $\varphi = \sqrt{\Delta/\sqrt{q}}\,(1 + O(\Delta))$ and $d = \frac{1}{2}\log(1/\Delta) + \log 2 + \frac{1}{4}\log q + o(1)$. J_+ blows up like $2/\varphi$, and $\nu, \hat{\kappa} \to 0$.*
- *(f) For $a > a_c$ write $t = 2\cosh\eta$. There is no invariant compatible J. The continuation $\varphi = i\eta$ of the same analytic function* $\arccos(t/2)$ *gives $\eta = \sqrt{\Delta/\sqrt{q}}\,(1 + O(\Delta))$ at $a = a_c + \Delta$, and η is Lemma 48's Bode gain defect (the Blaschke zero is $e^{-\eta}$). So the* Krein norm before the collision and the gain defect after it agree to leading order at symmetric offsets*: $\hat{\kappa}(a_c - \Delta)/\eta(a_c + \Delta) \to 1$.*

Proof. (a) $A^2 = (t^2/4 - 1) \cdot 1$; g commutes with A, so $\mathrm{Ad}(g)A = A$. (b) $J^2 = -1$ and $[J, g] = 0$ force $J \in \mathbb{R}1 + \mathbb{R}A$ with $J^2 = -1$, hence $J = \pm A/s$. $\Omega J_+ = \frac{1}{s}\left(\begin{smallmatrix} 1 & t/2 \\ t/2 & 1 \end{smallmatrix}\right)$ is positive definite. The Möbius fixed point solves $z^2 + tz + 1 = 0$. (c) $\cosh d(i, \tau) = 1 + |\tau - i|^2/(2\operatorname{Im}\tau) = 1/s$. The norms are direct ($J_+ = \frac{1}{s}\left(\begin{smallmatrix} -c & -1 \\ 1 & c \end{smallmatrix}\right)$, $c = \cos\varphi$), and the rest is Thm. 2.2 with $g = 1$. (d) $x = (1, -\lambda)/\sqrt{2}$ is a unit eigenvector, and $\hat{\omega}(\bar{x}, x) = -(\lambda - \bar{\lambda})/2 = -is$, so $h_K = s$. (e) $2 - t = \Delta/\sqrt{q}$ and $2 - 2\cos\varphi = \varphi^2(1 + O(\varphi^2))$; $d = \mathrm{arccosh}(1/s) = \log(2/s) + O(s^2)$. (f) $2\cosh\eta - 2 = \Delta/\sqrt{q}$. The normalized roots are $e^{\pm\eta}$, so $\log M = \eta$ (Lemma 48 Thm. 3.4(b)). A real eigenline admits no J. □

[NUM] (`collide.json`, block `A1_genus1`, $q = 13$, $a_c = 7.21110255\ldots$, mpmath at 60 digits). Below a_c there are 2 invariant J, with signatures $(2,0)$ and $(0,2)$; above a_c there are none:

a	φ	d	$\|J_+\|_F$	$\nu=\hat\kappa$	fidelity sech $\frac{d}{2}$	axis
0	$\pi/2$	0	1.414	1	1	timelike
3	1.14173	0.44288	1.684	0.90935	0.97597	timelike
6	0.58800	1.19476	3.317	0.55470	0.84474	timelike
7	0.24256	2.10472	8.206	0.24019	0.62237	timelike
a_c-10^{-4}	$5.2664\cdot10^{-3}$	5.93955	379.8	$5.2664\cdot10^{-3}$	0.10236	timelike
a_c-10^{-8}	$5.2664\cdot10^{-5}$	10.54472	$3.798\cdot10^{4}$	$5.2664\cdot10^{-5}$	$1.0263\cdot10^{-2}$	timelike
a_c-10^{-12}	$5.2664\cdot10^{-7}$	15.14990	$3.798\cdot10^{6}$	$5.2664\cdot10^{-7}$	$1.0263\cdot10^{-3}$	timelike

a	gain defect η	Blaschke zero $e^{-\eta}$	# inv. J	axis
a_c+10^{-12}	$5.2664\cdot10^{-7}$	0.99999947	0	spacelike
a_c+10^{-8}	$5.2664\cdot10^{-5}$	0.99994734	0	spacelike
a_c+10^{-4}	$5.26640\cdot10^{-3}$	0.99474745	0	spacelike
8	0.463599	0.629016	0	spacelike

The predicted d at a_c-10^{-12} is $\frac12\log10^{12}+\log2+\frac14\log13=15.1498$, against 15.14990 measured. $\|J_+\|_F^2$ agrees with $4/s^2-2$ to all printed digits; the $a=8$ row is Lemma 50 Prop. 5.5(iii) and Lemma 48's table. The bosonic fidelity computed from τ agrees with $\operatorname{sech}(d/2)$ and with $2\det(1-J_0J)^{-1/2}$.

Reading. In spinor language (J.W. [t196u], answer 2): $SU(2)=\mathrm{Spin}(3)$ and $SL_2(\mathbb{R})=\mathrm{Spin}(2,1)_0$ act on the same 3-space $\mathfrak{sl}_2$ by their vector representations. The Weil vacuum *is* the unit timelike rotation axis of the vector image of $g_{\mathrm n}$. Ghost birth is the axis going null and then spacelike, i.e. the conjugacy class going elliptic → parabolic → hyperbolic. This is a path of *conjugacy classes* in one group, $SL_2(\mathbb{R})$, not a path from the group $SU(2)$ to the group $SL_2(\mathbb{R})$: a Weil element of $SL_2(\mathbb{R})$ is conjugate into $SO(2)$, the compact torus shared by both real forms. (Correction to the "$SU(2)\to SL_2(\mathbb{R})$" phrasing; §7.) Lemma 48's picture: the same parameter that measured the vacuum's *Krein norm* (= its transversality ν) before the collision measures the *Blaschke zero's depth* η after it.

Proposition 3.2 (the endpoint $a=\pm2\sqrt{q}$; [THM]; example [NUM])**.** *(a) In the family chart, g at $t=2$ is unipotent and $\neq1$. Its commutant is $\mathbb{R}[N]$, $N=g-1$, $N^2=0$, which contains no J with $J^2=-1$. So there is no invariant vacuum, and J_+ leaves every compact set (Thm. 2.1). (b) For an* actual *elliptic curve with $a=\pm2\sqrt{q}$ (q a square), Frobenius is semisimple:* End^0 *is semisimple (Tate* [13]*, [BIB]), so $F=\pm\sqrt{q}$ and $g_{\mathrm n}=\pm1$. Then* every *J is invariant: the invariant vacua form all of $\mathfrak{h}$, not the empty set. (c) Example: $E: y^2=x^3-x$. Over $\mathbb{F}_3$: $N=4$, $a_3=0$. Over $\mathbb{F}_9$: $N=16$, $a_9=a_3^2-2\cdot3=-6=-2\sqrt{9}$, and $F_9=F_3^2=-3$ since $F_3^2+3=0$ (supersingular; such curves exist in general* [15]*, [BIB]).*

Proof. (a) Direct. (b) $g_{\mathrm n}$ is semisimple with the single eigenvalue ±1, so it is ±1. (c) Count: over $\mathbb{F}_3$ the only affine points are $(0,0),(1,0),(2,0)$, since $x^3-x=0$ for all $x\in\mathbb{F}_3$. Over $\mathbb{F}_9$, $N=9+1-a_9=16$ (direct count in `collide.json`, `endpoint_real_curve`). □

So the endpoint of an honest L-family is *not* where a real Frobenius degenerates. The parabolic matrix is an artefact of continuing through the companion class. Hasse's bound is $|a|\le2\sqrt{q}$, with equality possible (§7).

4. Genus 2: the opposite-sign collision, and the real ghost

4.1. **A fixed chart that can host a Krein collision.** By Cor. 2.6, L50's trace model has pattern $(+,+)$. A Krein collision needs pattern $(+,-)$, which is realized by the Hamiltonian–Hopf normal form ([14], [BIB]; our own form, verified below). On $\mathbb{R}^4=(x_1,x_2,y_1,y_2)$ with $\Omega=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\otimes1_2$, let

$$H_{w,\nu}=w(x_2y_1-x_1y_2)+\tfrac12(x_1^2+x_2^2)+\tfrac{\nu}{2}(y_1^2+y_2^2),\qquad g(w,\nu)=\exp(-\Omega S_{w,\nu})\in\mathrm{Sp}(4,\mathbb{R}),$$

where $S_{w,\nu}$ is the Hessian of $H_{w,\nu}$.

Lemma 4.1 (Hamiltonian–Hopf normal form; [THM], checked [NUM])**.** *The angular momentum $x_2y_1 - x_1y_2$ commutes with the isotropic oscillator, so the eigenvalues of $g(w,\nu)$ are $e^{\pm i(w\pm\sqrt{\nu})}$ for $\nu > 0$ and $e^{\pm\sqrt{-\nu}\pm iw}$ for $\nu < 0$. For $\nu > 0$ the two upper eigenvalues $e^{i(w+\sqrt{\nu})}, e^{i(w-\sqrt{\nu})}$ have* opposite *Krein signs. At $\nu = 0$, g has two 2×2 Jordan blocks.*

Proof. The rotation generator and the oscillator commute, so their spectra add. The oscillator has frequency $\sqrt{\nu}$, and the rotation contributes $\pm iw$. The Krein sign of a mode is the sign of its energy times its frequency: the co- and counter-rotating modes have opposite signs. At $\nu = 0$ the oscillator part $\frac{1}{2}|x|^2$ is nilpotent. □

The path $P_W \to P_Q$ (`A2_HH`). Let $P_t = (1-t)P_W + tP_Q$, coefficientwise ($q = 11$). It is realized by $g_t = g(w_t, \nu_t)$, where $w = (\varphi_1 + \varphi_2)/2$ and $\nu = ((\varphi_1 - \varphi_2)/2)^2$ while the normalized roots are $e^{\pm i\varphi_j}$, and $w = \theta$, $\nu = -\eta^2$ once they are $e^{\pm\eta\pm i\theta}$. So g_t is continuous, and $\det(T - \sqrt{11}\, g_t) = P_t$ to $\le 5\cdot 10^{-60}$ (symplecticity error $\le 10^{-61}$). The roots collide on the circle at $t_* = 0.47356443681161$ ($\nu(t_*) < 10^{-61}$, $\operatorname{rank}(g_* - \lambda_*) = 3$), and form a complex quadruple for $t > t_*$, ending at L50's quadruple fake ($|\lambda| = 0.712193, 1.404113$).

Theorem 4.2 (vacua through a Hamiltonian–Hopf collision; [THM], checked [NUM])**.** *Let g_t pass through a collision of λ_1, λ_2 ($\operatorname{Im} > 0$, Krein signs $+,-$) at λ_*, as above. Label the invariant vacua by $\varepsilon = (\varepsilon_1, \varepsilon_2)$, $J = i\varepsilon_j$ on E_{λ_j}.*

- *(a)* Split-sign vacua. *$\varepsilon = (+,-)$ (positive) and $(-,+)$ (negative) leave every compact set as $t \uparrow t_*$ (Thm. 2.1). Their limits lie in the Shilov boundary: both rapidities diverge at the same rate.*
- *(b)* Same-sign vacua. *$\varepsilon = \pm(+,+)$ converge to $J_* = \pm(i$ on $E^{\mathrm{gen}}_{\lambda_*}$, $-i$ on $E^{\mathrm{gen}}_{\bar\lambda_*})$. They are the only invariant vacua of g_* and have g_J-signature $(2,2)$. For $t > t_*$ they continue analytically as the two invariant vacua of the quadruple ($+i$ on $E_\lambda \oplus E_{1/\bar\lambda}$, the eigenvalues near λ_*), which are Krein vacua (Lemma 50 Thm. 4.4).*
- *(c) The count of invariant compatible J is $4 \to 2 \to 2$. Fock signatures in degrees $0,1,2$: the positive vacuum $(1,0),(2,0),(1,0)$ disappears; the Krein vacuum $(1,0),(1,1),(0,1)$ passes through the collision unchanged.*

Proof. (b) Let Γ be a small circle around λ_* and $P_t = \frac{1}{2\pi i}\oint_\Gamma (z - g_t)^{-1} dz$. It is analytic in t near t_* because no eigenvalue crosses Γ (the cluster near λ_* is $\{\lambda_1, \lambda_2\}$ before t_* and $\{\lambda, 1/\bar\lambda\}$ after). $\operatorname{im} P_t$ is ω-isotropic: $\omega(E_\mu, E_{\mu'}) = 0$ unless $\mu\mu' = 1$, and $\mu\mu'$ is near $\lambda_*^2 \ne 1$. It has dimension $2 = g$, so it is Lagrangian. Hence $J_t = i(P_t - \bar P_t)$ is real, compatible, invariant and analytic, and equals $J_{(+,+)}$ before and the quadruple vacuum after. At t_* every invariant J preserves the generalized eigenspaces. On the Jordan block $E^{\mathrm{gen}}_{\lambda_*}$ the commutant is $\{\alpha + \beta N\}$, and $(\alpha + \beta N)^2 = -1$ forces $\beta = 0$, $\alpha = \pm i$. So $J = \pm J_*$. Its signature is $(2,2)$, because g_J is non-degenerate for every compatible J, so the signature is constant along the continuous family J_t, and it is $(2,2)$ for $t < t_*$ (Lemma 50 Thm. 4.4, mixed ε). (a) g_* is not semisimple, so Thm. 2.1(a) applies to $J_{(+,-)}$, and to $J_{(-,+)}$ by $J \mapsto -J$. Equal rapidities: the simultaneous rotation R_β of the (x_1, x_2)- and (y_1, y_2)-planes is orthogonal and symplectic, and it commutes with J_0 and with $g(w,\nu)$. So it fixes the unique positive invariant vacuum, and $-J_0 J_{(+,-)}$ commutes with the complex structure $R_{\pi/2}$. Its eigenvalues $e^{\pm d_1}, e^{\pm d_2}$ therefore have even multiplicities, which forces $d_1 = d_2$. Hence both Harish-Chandra values tend to 1, i.e. to the Shilov boundary. (c) Counting: (a), (b) and Lemma 50 Thm. 4.4. The Fock signatures are Λ^k of $(2,0)$ and of $(1,1)$. □

[NUM] (`collide.json`, `A2_HH`; $\hat\kappa$ is the normalized Krein norm of the colliding eigenvectors, which is equal for both in this chart; $d_1 = d_2$ are the rapidities of $J_{(+,-)}$):

$t-t_*$	$\lvert\hat\kappa\rvert$	$d_1=d_2$	$\nu(J_+)$	fidelity	$\lVert J_+\rVert_F$	HC values
-0.4	0.85568	0.57293	0.73219	0.92223	2.632	0.27888
-0.1	0.465639	1.39826	0.21682	0.63541	5.736	0.60382
-10^{-2}	0.149423	2.58849	$2.2327\cdot10^{-2}$	0.26000	18.82	0.86023
-10^{-4}	$1.4960\cdot10^{-2}$	4.89545	$2.2381\cdot10^{-4}$	$2.9479\cdot10^{-2}$	189.1	0.98515
-10^{-6}	$1.4960\cdot10^{-3}$	7.19808	$2.2381\cdot10^{-6}$	$2.9876\cdot10^{-3}$	1891	0.99851
-10^{-8}	$1.4960\cdot10^{-4}$	9.50067	$2.2381\cdot10^{-8}$	$2.9916\cdot10^{-4}$	$1.891\cdot10^{4}$	0.99985
-10^{-10}	$1.4960\cdot10^{-5}$	11.80325	$2.2381\cdot10^{-10}$	$2.9920\cdot10^{-5}$	$1.891\cdot10^{5}$	0.99999

$t-t_*$	$\lvert\lambda\rvert$ of quadruple	gain defect 2η	# inv. J	Λ_0^2 real pair
10^{-10}	$1\pm7\cdot10^{-6}$	$1.49604\cdot10^{-5}$	2	$\approx 1\pm1\cdot10^{-5}$
10^{-8}	1.000075, 0.999925	$1.49604\cdot10^{-4}$	2	1.00015, 0.99985
10^{-6}	1.000748, 0.999252	$1.49604\cdot10^{-3}$	2	1.0015, 0.99851
10^{-2}	1.077313, 0.928235	0.148941	2	1.1606, 0.86162
0.526 (P_Q)	1.404113, 0.712193	0.678812	2	1.97153, 0.50722

Readings: $|\hat\kappa| = 1.4960\sqrt{|t-t_*|}$ before and gain defect $= 1.49604\sqrt{t-t_*}$ after, the genus-1 law of Prop. 3.1(f) again. $\nu(J_+)=\hat\kappa^2$ exactly (Thm. 2.2(b); the eigenvectors are orthogonal in this chart). $d_j = \frac12\log(1/|t-t_*|)+0.29$, with the same slope $\frac12$ as in genus 1. The same-side vacua $\pm J_{(+,+)}$ have $\|J\|_F=2$ and $\nu=1$ at every sampled t, and coincide with $\pm J_*$ to $\le 4\cdot10^{-58}$ for $|t-t_*|\le10^{-4}$. The J count is 4 for $t<t_*$ and 2 for $t>t_*$. The final gain $0.678812 = 2\log 1.404113$ is Lemma 48's $\sum\eta$ for P_Q.

4.2. **The real ghost: a genus-1 event inside genus 2.** *The path* $P_W\to P_R$ (`A2_realghost`). Interpolate the coefficients of P_W and P_R linearly, and write $P_t(T)=\prod_{j=1,2}(T^2-z_j(t)\sqrt{q}\,T+q)$, with z_j the (real) roots of the trace polynomial, and realize $g_t=\operatorname{diag}\left(\left(\begin{smallmatrix}0&-1\\1&z_1\end{smallmatrix}\right),\left(\begin{smallmatrix}0&-1\\1&z_2\end{smallmatrix}\right)\right)\in SL_2(\mathbb{R})^2\subset\mathrm{Sp}(4,\mathbb{R})$. The discriminant of the trace polynomial stays ≥ 5.45 (no collision $z_1=z_2$, no Hamiltonian–Hopf). z_2 passes through -2 at $t_*^R = 0.403484039183$, where $g_{\mathrm n}$ has eigenvalue -1 with a Jordan block. For $t>t_*^R$ the eigenvalue pair is real and negative, ending at $|\lambda| = 0.260896,\ 3.832949$.

Proposition 4.3 (the real-ghost vacuum hits a corank-one boundary component; [THM], checked [NUM])**.** *(a) At t_*^R there is no invariant compatible J. So* all four J_ε *are unbounded as $t\uparrow t_*^R$ (Thm. 2.1(b)), and none exists afterwards (Lemma 50 Thm. 4.4). (b) The positive vacuum is $J_+ = J_+^{(1)}\oplus J_+^{(2)}$ (Prop. 3.1 in each block). Its first rapidity stays bounded and its second diverges, so it tends to a corank-one boundary component $\mathbb{H}\times\partial\mathbb{H}$ of the Siegel space,* not *to the Shilov boundary.*

Proof. (a) As in Prop. 3.2(a), on the z_2 block. (b) Block-diagonal J; Prop. 3.1(e) in block 2 and continuity in block 1. □

[NUM]: $d = (0.56375,\ 9.31781)$ at $t-t_*^R=-10^{-8}$, with Harish-Chandra values $(0.27464,\ 0.99982)$. $\nu(J_+) = 1.5442\cdot10^{-4} = \hat\kappa_1\hat\kappa_2 = 0.859727\times1.796\cdot10^{-4}$. The colliding pair has $\hat\kappa = 1.796\sqrt{|t-t_*^R|}$, and the gain defect after is $1.7962\cdot10^{-4}$ at $+10^{-8}$. The endpoint gain is $\log 3.832949 = 1.34363$ (L48). The mixed vacua blow up too: $\|J\|_F = 1.11\cdot10^4$ at -10^{-8}.

Remark 4.4 (boundary strata as a ghost-type detector; [THM] for these paths, [CONJ] in general)**.** In both paths, the stratum of the Siegel boundary reached by the positive vacuum is determined by how the ghost is born. A real ghost (one pair through ±1) reaches a corank-1 component, and the remaining $\mathbb{H}$ factor carries the surviving Weil pair's vacuum. A Hamiltonian–Hopf quadruple reaches the corank-2 (Shilov) stratum, because a quadruple uses up two Krein-positive directions at once (Prop. 2.3: $m=2$). [CONJ]: for a generic one-parameter family in $\mathrm{Sp}(2g,\mathbb{R})$, the corank

of the limiting boundary component equals the number m of eigenvalues that leave the circle at t_* (outside the disc).

5. Genus 2 in spinor language: $\mathrm{USp}(4) \cong \mathrm{Spin}(5)$, $\mathrm{Sp}(4,\mathbb{R}) \cong \mathrm{Spin}(3,2)_0$

Let $H = H^1$ with $\dim H = 4$ and symplectic form ω, let $\pi \in \Lambda^2 H$ be the dual bivector (the class $[\Theta]$ of Lemma 50 Rem. 4.3), and let $B(\alpha,\beta) = \alpha \wedge \beta \in \Lambda^4 H \cong \mathbb{R}$, oriented by $\omega \wedge \omega > 0$ (Liouville). Put $\Lambda^2_0 = \pi^{\perp_B}$, the primitive part: $\Lambda^2 H = \mathbb{R}\pi \oplus \Lambda^2_0$, $\dim \Lambda^2_0 = 5$ ([2], [BIB]).

Proposition 5.1 ($\Lambda^2 H^1 = \mathbf{1} \oplus \mathbf{5}$ and the three real forms; [THM] (classical), checked [NUM])**.** *(a) B is symmetric of signature $(3,3)$ on $\Lambda^2 H$, $B(\pi,\pi) > 0$, and $\mathrm{Sp}(H)$ preserves π and B. The induced map $\mathrm{Sp}(4,\mathbb{R}) \to \mathrm{SO}(\Lambda^2_0, B) = \mathrm{SO}(2,3)$ has kernel $\{\pm 1\}$ and image $\mathrm{SO}(2,3)_0$. So $\mathrm{Sp}(4,\mathbb{R}) \cong \mathrm{Spin}(3,2)_0$ and H is its 4-dimensional spinor representation. (b) For the compact form $\mathrm{USp}(4)$ (resp. $\mathrm{Sp}(1,1)$), acting on $\mathbb{C}^4 = \mathbb{H}^2$ with its quaternionic structure j, the real points of $\Lambda^2\mathbb{C}^4$ under $\alpha\wedge\beta \mapsto j\alpha\wedge j\beta$ form a 6-dimensional real space, on which B (up to an overall sign) is definite on Λ^2_0, of signature $(5,0)$, resp. $(4,1)$. Hence $\mathrm{USp}(4) \cong \mathrm{Spin}(5)$ and $\mathrm{Sp}(1,1) \cong \mathrm{Spin}(4,1)$, and H^1 is the spinor in each case* [9, 3] *([BIB]; the $\mathrm{SO}(5)$ quotient is used in* [3, proof of Lemma 3.7]*, [CITED]). (c) For a complex abelian surface, $\Lambda^2_0 H^1(\mathbb{R}) = H^2_{\mathrm{prim}}$. B is the cup product, and its signature $(2,3)$ is the Hodge–Riemann signature: positive on $(H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$, negative on $H^{1,1}_{\mathrm{prim},\mathbb{R}}$.*

Proof. (a) Λ^2 of a symplectic basis: B pairs $e_i \wedge e_j$ with the complementary pair, so it is split, $(3,3)$. Sp preserves ω, hence π and the orientation. $\mathfrak{sp}_4 \to \mathfrak{so}(\Lambda^2_0)$ is injective between 10-dimensional algebras, hence an isomorphism. The kernel on the group is $\{g : \Lambda^2 g = 1\} = \{\pm 1\}$. Since $B(\pi,\pi) > 0$, Λ^2_0 has signature $(2,3)$. (b) Same computation with the real structure induced by j. (c) Hodge–Riemann bilinear relations on a surface. □

[NUM] (`spin5.json`, `real_forms`; random group elements generated by exponentials): $\mathrm{Sp}(4,\mathbb{R})$: B is $(3,3)$ on Λ^2 and $(2,3)$ on Λ^2_0, with invariance error $9 \cdot 10^{-16}$. $\mathrm{USp}(4)$: $(1,5)$ on real Λ^2 and $(0,5)$ on Λ^2_0. $\mathrm{Sp}(1,1)$: $(1,5)$ and $(1,4)$. π is fixed to $\le 3 \cdot 10^{-16}$, and g commutes with j to $\le 2 \cdot 10^{-16}$.

Proposition 5.2 (spinor → vector dictionary; [THM]; [CITED] [3, fn. 10])**.** *If g_{n} has eigenvalues $\lambda_1^{\pm 1}, \lambda_2^{\pm 1}$ on H, then $\Lambda^2 g_{\mathrm{n}}$ has eigenvalues 1 (on π) and, on Λ^2_0,*

$$\lambda_1\lambda_2,\ (\lambda_1\lambda_2)^{-1},\ \lambda_1/\lambda_2,\ \lambda_2/\lambda_1,\ 1.$$

For a Weil class ($\lambda_j = e^{i\theta_j}$) these are the rotation angles $\theta_1+\theta_2$, $\theta_1-\theta_2$ of an element of $\mathrm{SO}(5)$. Its character is $\chi_{\mathbf{5}} = 1 + 2\cos(\theta_1+\theta_2) + 2\cos(\theta_1-\theta_2) = a_2 - 1$, where $a_2 = e_2(\lambda)$ is the normalized middle coefficient. This is Serre's relation as quoted in FKRS: $a_2 = 2(1+\cos u+\cos v)$, $a_1^2 = 4(1+\cos u)(1+\cos v)$, with u, v the $\mathrm{SO}(5)$ angles.

Proof. The eigenvalues of Λ^2 are the pairwise products. One product, $\lambda_1\lambda_1^{-1} = 1$, belongs to π, and the other $1 = \lambda_2\lambda_2^{-1}$ lies in Λ^2_0. $2 + 2\cos(\theta_1+\theta_2) + 2\cos(\theta_1-\theta_2) = 2 + 4\cos\theta_1\cos\theta_2 = e_2$. □

[NUM]: for the Weil curve over $\mathbb{F}_{11}$, the angles θ_1, θ_2 give vector angles 2.764421 and 1.273071, equal to $\theta_1+\theta_2$ and $|\theta_1-\theta_2|$ to 10^{-15}.

Proposition 5.3 (the vacuum is a Frobenius-invariant period plane; [THM])**.** *For a compatible J, $\phi_J \in \Lambda^2 L_J$ (the pure spinor of §1) lies in $\Lambda^2_0 \otimes \mathbb{C}$ and satisfies $\phi_J \wedge \phi_J = 0$ and $\phi_J \wedge \bar\phi_J \neq 0$. $\Pi_J = \mathrm{span}(\mathrm{Re}\,\phi_J, \mathrm{Im}\,\phi_J)$ is a B-definite 2-plane. It is positive iff g_J is definite ($\pm J \in \mathfrak{h}$), and negative iff g_J has signature $(2,2)$. J and $-J$ give the same plane with opposite orientations. The map $J \mapsto \Pi_J$ (oriented) identifies Siegel space of genus 2 with the type-IV domain $\mathrm{SO}(2,3)_0/\mathrm{SO}(2)\times\mathrm{SO}(3)$ of positive 2-planes in Λ^2_0 ([5], [BIB]). For a complex abelian surface, $\Pi_J = (H^{2,0} \oplus H^{0,2})_{\mathbb{R}}$ and ϕ_J is the holomorphic 2-form. If J is g_{n}-invariant, g_{n} preserves Π_J and rotates it by $\arg\det(g_{\mathrm{n}}|_{L_J})$. For the positive vacuum this is $\theta_1+\theta_2$ in Krein pattern $(+,+)$ (L50's trace model) and $\theta_1-\theta_2$ in pattern $(+,-)$. The transversality $\nu(J)$ of Thm. 2.2 is $|\phi_J \wedge \bar\phi_J|/\|\phi_J\|^2$, which tends to 0 exactly when Π_J becomes degenerate (touches the light cone of B).*

Proof. L_J is Lagrangian, so $\pi \wedge \phi_J = \omega(L_J, L_J) \cdot \mathrm{vol} = 0$. ϕ_J is decomposable, so $\phi_J \wedge \phi_J = 0$, and $L_J \oplus \bar{L}_J = V_{\mathbb{C}}$ gives $\phi_J \wedge \bar{\phi}_J \neq 0$. Writing $\phi = u + iv$, $\phi \wedge \phi = 0$ gives $B(u,u) = B(v,v)$ and $B(u,v) = 0$, and $\phi \wedge \bar{\phi} = 2B(u,u)$. With $\phi_J = b_1 \wedge b_2$, $\phi_J \wedge \bar{\phi}_J$ is a fixed multiple of $\det(\omega(\bar{b}_i, b_j)) = -\det(h_K(b_i, b_j))$. So its sign is constant on definite g_J (both signs of J) and opposite on signature $(2,2)$; the constant is fixed at J_0, where Hodge–Riemann gives a positive plane. g_{n} acts on $\Lambda^2 L_J$ by $\det(g_{\mathrm{n}}|_{L_J})$, which is $\lambda_1\lambda_2$ or $\lambda_1\bar{\lambda}_2$ according to which eigenlines span L_J (Thm. 2.2(b)). □

Theorem 5.4 ($B_2 = C_2$ swaps the two kinds of ghost birth; [THM], checked [NUM])**.** *(a) A Hamiltonian–Hopf collision on the spinor H ($\lambda_1, \lambda_2 \to \lambda_*$, pattern $(+,-)$) is, on the vector Λ^2_0, the* genus-1 event*. The eigenvalues $\lambda_1/\lambda_2, \lambda_2/\lambda_1$ meet the eigenvalue* 1*, and on the* 3*-dimensional B-signature-$(2,1)$ sum of these eigenspaces the element moves in $\mathrm{SO}(2,1)_0$ from elliptic (rotation by $\theta_1 - \theta_2$) through parabolic (one Jordan block of size* 3*) to hyperbolic (boost $e^{\pm 2\eta}$). The positive vacuum plane Π_{J_+} is the B-orthogonal complement of the rotation axis inside this $\mathbb{R}^{2,1}$, exactly as in Prop. 3.1(b). In pattern $(+,+)$ (L50's trace model) the $\theta_1 - \theta_2$ plane is* negative*, the* 3*-space is* $(0,3)$ *and definite, and the collision is impossible. This is Cor. 2.6 seen on the vector side. (b) A real-ghost birth on H ($\lambda_2 \to -1$) is, on Λ^2_0, a* Hamiltonian–Hopf *collision. The eigenvalues $\lambda_1\lambda_2$ and λ_1/λ_2 meet at $-\lambda_1$ and leave as the quadruple $-\lambda_1 r^{\pm 1}$, $-\bar{\lambda}_1 r^{\pm 1}$ ($r > 1$). Their rotation planes have opposite B-signs, as Cor. 2.4 requires.*

Proof. (a) Prop. 5.2. At $\lambda_1 = \lambda_2$ the vector eigenvalue 1 has algebraic multiplicity 3 on Λ^2_0. The Λ^2 of a 2×2 Jordan block pair (Lemma 4.1) has a size-3 block, and the rest is semisimple. The signature of the 3-space follows from $(2,3)$ minus the B-negative rotation plane of $\theta_1 + \theta_2$. The vacuum plane is the rotation plane of $\theta_1 - \theta_2$ (Prop. 5.3, pattern $(+,-)$), which is the orthogonal complement of the fixed line inside the 3-space. (b) Prop. 5.2 with $\lambda_2 = e^{i\theta_2}$, $\theta_2 \to \pi$, then $\lambda_2 = -r$. In $(2,3)$ minus the negative fixed line, the two rotation planes are $(2,0)$ and $(0,2)$. □

[NUM] (`collide.json`). HH path at t_*: the ranks of $(\Lambda^2 g_* - 1)^k$ on Λ^2 are $4, 3, 2$ for $k = 1, 2, 3$, i.e. a size-3 block in Λ^2_0 plus π. After t_*, Λ^2_0 has a real pair, e.g. 1.1606, 0.86162 at $t - t_* = 10^{-2}$ ($= e^{\pm 2\eta}$, with $\eta = 0.07447$), and at P_Q it has 1.97153, 0.50722 ($= 1.404113^{\pm 2}$). Real-ghost path: the two complex pairs of Λ^2_0 approach each other, and at P_R they are $-1.48952 \pm 3.53169i$, $-0.10139 \pm 0.24039i$, of moduli 3.83295 and 0.26090: a quadruple on the vector side.

Proposition 5.5 (which real form hosts which ghost; [THM], checked [NUM])**.** *Let g_{n} be semisimple and reciprocal of degree* 4 *with m eigenvalues outside the unit disc. Its image in the vector representation has $m_{\mathbf{5}}$ eigenvalues outside, with $m_{\mathbf{5}} = 0, 1, 2$ for Weil, quadruple and real-ghost classes respectively. (a) An element of $\mathrm{SO}(p,q)$ has at most $\min(p,q)$ eigenvalues outside the unit disc (L46 Thm. 3.1, the isotropy bound for $B(\bar{x}, y)$). (b) Hence Weil classes lie in $\mathrm{Spin}(5) = \mathrm{USp}(4)$, as they must. Quadruple classes ($e^{\pm\eta \pm i\theta}$) are realized in $\mathrm{Spin}(4,1) = \mathrm{Sp}(1,1)$: lift the boost of rapidity 2η times a rotation by 2θ in the orthogonal $\mathbb{R}^3$. Real-ghost classes are* not *in $\mathrm{Sp}(1,1)$, and live only in the split form $\mathrm{Spin}(3,2)_0 = \mathrm{Sp}(4,\mathbb{R})$, which hosts both.*

Proof. (a) If $\Lambda x = \mu x$ and $\Lambda y = \mu' y$, then $B(\bar{x}, y)(\bar{\mu}\mu' - 1) = 0$. So the span of generalized eigenvectors with $|\mu| > 1$ is isotropic for the Hermitian extension of B, of signature (p,q). (b) Count with Prop. 5.2: quadruple $e^{\pm 2\eta}$ (one outside), real ghost $re^{\pm i\theta}$ (two outside). The lift: the spinor weights of a boost $\oplus$ rotation $(2\eta, 2\theta)$ are $\pm\eta \pm i\theta$. Alternatively, j maps E_λ to $E_{\bar{\lambda}}$, so in $\mathrm{Sp}(1,1)$ a real eigenvalue has even multiplicity. □

[NUM] (`spin5.json`, `classes`). The vector moduli are $1^{\times 5}$ for P_W; $0.50722, 1, 1, 1, 1.97153$ for P_Q (with arguments $0, \pm 0.77393, 0, 0$); and $0.26090^{\times 2}, 1, 3.83295^{\times 2}$ for P_R (arguments ± 1.96992). These give $m_{\mathbf{5}} = 0, 1, 2$.

Remark 5.6 (the USp(4) moments for a curve we counted; [NUM]; equidistribution [CONJ])**.** $C : y^2 = x^5 + 3x^3 + x + 7$ over $\mathbb{Q}$, the Weil curve of Lemma 50 at $p = 11$. $\mathrm{disc}(f) = 7892781 =$

$3 \cdot 13 \cdot 439 \cdot 461$, and we skip $p \mid 2\,\mathrm{disc}(f)$. We counted N_1 over $\mathbb{F}_p$ for the 9587 good primes $3 < p < 10^5$, and N_2 over $\mathbb{F}_{p^2}$ (norm trick $\chi_{p^2}(z) = \chi_p(Nz)$) for the 447 good primes $p < 3200$. All L-polynomials satisfy $|\alpha| = \sqrt{p}$ to $3 \cdot 10^{-15}$, and $\max |a_1|/4\sqrt{p} = 0.856$. The reference values come from Weyl integration with density $\propto (\cos\theta_1 - \cos\theta_2)^2 \sin^2\theta_1 \sin^2\theta_2$ on a 1200^2 midpoint grid:

moment k	1	2	3	4	5	6	7	8
$E[(a_p/\sqrt{p})^k]$, 9587 primes	−0.012	0.988	−0.052	2.919	−0.42	13.56	−3.6	81.7
USp(4) (Weyl)	0	1	0	3	0	14	0	84
$E[\chi_{\mathbf{5}}^k]$, 447 primes	−0.003	0.923	0.013	3.013				
SO(5) vector (Weyl)	0	1	0	3				

The observed range of $\chi_{\mathbf{5}}$ is $[-2.64, 3.88] \subset [-3, 5]$. The match is what Katz–Sarnak/Sato–Tate predict for USp(4) [8, 7, 3]. Equidistribution is [CONJ]: we have not proved $\mathrm{End}(\mathrm{Jac}\, C_{\bar{\mathbb{Q}}}) = \mathbb{Z}$, and for ST = USp(4) "equidistribution is known in no cases" (FKRS §1.7, [CITED]; Kedlaya's 2014 slides [6], pointer). The SO(5) moments $0, 1, 0, 3$ say that **5** is irreducible and real orthogonal, with $E[\chi^4] = 3$ as for $\mathrm{SO}(N)$, $N \geq 5$.

Proposition 5.7 (why the spinor picture stops at genus 3, and what survives; [THM])**.** *(a) H^1 (dimension $2g$) is a spinor representation of a spin group via an isomorphism with Sp_{2g} only for $g \leq 2$. The (half-)spin representations of B_n, D_n have dimension 2^n, 2^{n-1}, and C_g is isomorphic to a B or D algebra only for $C_1 = B_1$ $(= A_1)$ and $C_2 = B_2$. Equivalently, $2^g = 2g$ only for $g = 1, 2$. For $g = 3$: $\mathfrak{sp}_6 \ncong \mathfrak{so}_7$ (same dimension 21, different Dynkin diagrams), and the spin representation of $\mathfrak{so}_7$ is 8-dimensional, not 6. (b) What survives in every genus is the fermionic picture of Lemma 50: $\Lambda^\bullet H^1$ is the spinor module of $\mathrm{Cl}(H^1 \oplus H^{1\vee})$, and a vacuum J gives the 2^g-dimensional spinor module $\Lambda^\bullet L_J$ of $\mathrm{Cl}(H^1_{\mathbb{R}}, g_J)$, with pure spinor $\phi_J \in \Lambda^g L_J$. All of §2 is genus-independent. Only the "H^1 is itself the spinor of $\mathrm{Spin}(2g+1)$" reading of §5 is special to $g \leq 2$.*

Proof. Classification of simple Lie algebras and their minuscule representations ([2], [BIB]). $2^g = 2g$ is checked directly for $g \leq 3$, and $2^g > 2g$ for $g \geq 3$. □

6. Dictionary: Lemma 48's Blaschke picture and Ch. 27

before t_* (Weil side)	at t_*	after t_* (ghost side)
positive invariant vacuum J_+, unique (L50)	J_+ escapes; Siegel boundary (Thm. 2.1)	none (L50 Thm. 4.4)
Krein norm $\hat{\kappa}$ of colliding eigenvectors	0	gain defect $\sum \eta = \log M$, Blaschke zero $e^{-\eta}$ (L48 Thm. 3.4)
transversality $\nu(J_+) = \prod \lvert\hat{\kappa}\rvert$/Gram (Thm. 2.2)	0	—
SO(2, 1) axis timelike, rotation (g= 1; vector side of HH)	null, parabolic, Jordan block	spacelike, boost $e^{\pm 2\eta}$ on the vector side
L50 trace chart non-degenerate, pattern $(+, \ldots, +)$	$\det \omega_P = 0$: Ch. 27 stage 1	trace chart non-degenerate again, signature $(D_{\mathrm{on}} + k, k)$ (L46)
fixed chart: opposite-sign pair	mixed Krein collision: Ch. 27 stage 2	quadruple, or real pair (g= 1-type)
mixed vacua $J_{(+,+)}$ (HH)	converge to $\pm J_*$	Krein vacua of the quadruple

Rate law. In every path the colliding Krein norm and the gain defect scale like $\sqrt{|t - t_*|}$, as the Puiseux expansion of a double eigenvalue requires, and the rapidities like $\frac{1}{2}\log(1/|t - t_*|)$. In genus 1 (and in the block-diagonal real-ghost chart) the two constants are *equal*, Krein norm before = gain defect after (Prop. 3.1(f), [THM]). In the Hamiltonian–Hopf chart they are 1.4960 and 1.49604

([NUM]). The Krein norm depends on the chart's Euclidean normalization while the gain defect is intrinsic, so in genus 2 this equality is a property of the normal-form chart, not an invariant ([OPEN] whether a canonical normalization makes it one).

Ch. 27 in one sentence. For honest L-polynomial families the metric degeneration (stage 1) and the mixed-sign collision (stage 2) occur at the same parameter, in two charts (Cor. 2.6). This is the finite-field instance of Lemma 41's "one event in two charts". The window pencil of Ch. 27 separates the stages because its Krein space is truncated, which has no analogue here.

7. Corrections to the chat claims and to the premise

- **[t192] "$a_p^2 < 4q$ ... That's Hasse's theorem."** Hasse's bound is $|a| \le 2\sqrt{q}$ [11]. Strict inequality holds for $q = p$ prime ($2\sqrt{p} \notin \mathbb{Z}$), but equality occurs for square q: $y^2 = x^3 - x$ has $a_9 = -6$ over $\mathbb{F}_9$ (Prop. 3.2). At equality Frobenius is the scalar $\pm\sqrt{q}$, and every vacuum is invariant.
- **Premise (L50 Question 9.3): "a positive vacuum turning into a Krein vacuum, and then into none (real ghost)".** The positive vacuum does not turn into a Krein vacuum: it escapes to the Siegel boundary and disappears (Thm. 4.2(a)). The Krein vacuum that survives the quadruple collision was already present before it, as the same-sign $J_{(+,+)}$ (Thm. 4.2(b)). A real ghost is a different birth, not a later stage: one pair through ± 1, a corank-one boundary, and no invariant J afterwards (Prop. 4.3).
- **Premise: track J_ε "in L50's model".** L50's trace model has Krein pattern $(+,+)$ and cannot host an opposite-sign collision. Its form degenerates at the collision instead (Thm. 2.5, Cor. 2.6). We used fixed charts (normal forms) for the paths.
- **"$SU(2) = \mathrm{Spin}(3) \to SL_2(\mathbb{R}) = \mathrm{Spin}(2,1)_0$".** This is not a path between groups. It is a path of conjugacy classes inside the split group (elliptic $\to$ parabolic $\to$ hyperbolic). Weil classes lie in the compact torus $SO(2)$, which is shared by both real forms (§3, Reading).
- **Endpoint.** The parabolic endpoint is a feature of the family chart. A real curve at $|a| = 2\sqrt{q}$ has semisimple Frobenius (Prop. 3.2).
- **Lemma 50 Rem. 5.4 "**$\mathrm{USp}(4) \cong \mathrm{Spin}(5)$**".** Correct. The precise genus-3 failure is $2^g \neq 2g$ (Prop. 5.7).

8. What is new and what is restatement

Restatement / classical: Prop. 2.3 (L45/L46 in h_K language); Cor. 2.4 (Krein–Gelfand–Lidskii, [10, 4]); Lemma 4.1 (Hamiltonian–Hopf normal form, [14]); Prop. 5.1, 5.2 (classical $B_2 = C_2$, Serre's relation in [3]); Prop. 5.7 (classification); the identification of Siegel space with the type-IV domain in Prop. 5.3. **New in TAA (elementary, but not found stated in this form in the sources we checked):** Thm. 2.1 as a vacuum statement; Thm. 2.2(b), the Krein-volume formula for the pure-spinor transversality; Thm. 2.5 with Cor. 2.6 (the two Ch. 27 stages coincide for L-families); Prop. 3.1 (vacuum = unit $\mathrm{Spin}(2,1)_0$ axis, and Krein norm = continued gain defect); Thm. 4.2 and Prop. 4.3 (vacua through HH and real-ghost births, boundary strata); Thm. 5.4 (the $B_2 = C_2$ swap of birth types); Prop. 5.5 (quadruples in $\mathrm{Sp}(1,1)$, real ghosts not). None of this touches RH for ζ.

9. Questions for J.W.

Question 9.1. Adopt "*a ghost is born where the positive vacuum reaches the Siegel boundary; the boundary stratum records the kind of ghost*" as TAA language? (Proved for the paths here; Rem. 4.4 is [CONJ] in general. Should a follow-up lemma prove it for generic families in $\mathrm{Sp}(2g, \mathbb{R})$?)

Question 9.2. For the spinor-side measure of how far a vacuum is from being lost: the pure-spinor transversality $\nu(J) = \prod \operatorname{sech} d_i = \prod |\hat{\kappa}|/\mathrm{Gram}$ (Thm. 2.2), or the bosonic fidelity $\prod \operatorname{sech}(d_i/2)$? You prefer spinors (answer 2), so we propose ν, with the fidelity as a remark.

Question 9.3. Prop. 5.3: in genus 2 the vacuum is the Frobenius-invariant period plane (the real span of the holomorphic 2-form) in $H^2_{\mathrm{prim}}(\mathrm{Jac})$, and Weil positivity is Hodge–Riemann $(2,3)$ with Frobenius rotating that plane. Should the Volume 3 polarization chapter (answer 6) present genus 2 this way?

Question 9.4. Prop. 5.5: quadruple ghosts fit $\mathrm{Sp}(1,1) = \mathrm{Spin}(4,1)$, but real ghosts do not. Off-line zeros of ζ come in quadruples (L27). Do you want an exploratory lemma asking whether L27's window ghosts carry the quaternionic structure of $\mathrm{Sp}(1,1)$? It is exploratory, with no claim of a lever on RH.

Question 9.5. The rate law "Krein norm before = gain defect after" is exact in genus 1 and chart-dependent in genus 2 (§6). Is a canonical normalization worth pursuing ([OPEN]), or should TAA state only the $\sqrt{|t-t_*|}$ law?

Question 9.6. For $g \geq 3$ we keep only $\Lambda^\bullet H^1$ (Prop. 5.7). Is that acceptable for Volume 3, or do you want the $\mathrm{Spin}(2g+1) \supset ?$ embeddings (the spin representation of $\mathrm{USp}(2g)$ through $\Lambda^\bullet$) worked out?

Status

Proved ([THM]): Thm. 2.1, 2.2, Prop. 2.3, Cor. 2.4, Thm. 2.5, Cor. 2.6, Prop. 3.1, 3.2, Lemma 4.1, Thm. 4.2, Prop. 4.3, 5.1, 5.2, 5.3, Thm. 5.4, Prop. 5.5, 5.7. **Numerical ([NUM]):** the tables of §§3–4; the real-form signatures; the class dictionary; Rem. 5.6; the HH rate constant. **Cited ([CITED]):** FKRS §1.2, §1.7, Lemma 3.7 proof, fn. 10 (arXiv full text read); Mehl Thm. 4.7 (via Lemma 27). **Bibliographic ([BIB]):** Sie43, Tat66, Wat69, vdM85, KS99, KS09, LM89, Che54, FH91, Hel, GLR, MilpRH. **Conjectural ([CONJ]):** Rem. 4.4 in general; Sato–Tate USp(4) for C. **Open ([OPEN]):** canonical normalization of the rate law. **Not verified, not cited:** Kedlaya–Sutherland arXiv abstract page (fetch timed out; Crossref record used); any source for Prop. 5.5 (proved here instead); post-2018 literature on USp(4) Sato–Tate was not searched exhaustively (Kedlaya's 2014 slides are used only as a pointer).

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TAA LEMMA 52: IS THERE A SCALING-INVARIANT POSITIVE VACUUM IN THE CONNES–CONSANI–MOSCOVICI SEMILOCAL PROLATE MODEL? A NUMERICAL PROBE OF LEMMA 50'S REQUIREMENT (S4)

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

Date: October 22, 2026.

ABSTRACT. **Statement of intellectual honesty.** RH is not proved here, and nothing below brings it closer. This is an exploratory, mostly numerical lemma about finite truncations of objects defined by Connes, Consani and Moscovici (CCM). Every positivity statement about ζ that appears is either a restatement of Weil's criterion or a finite-precision computation, never a certificate. Tags: [THM] proved here (elementary, proof given); [CITED] checked at the source (`lit/lemma52/verified_refs.md`; arXiv full texts in `lit/lemma22/`); [BIB] bibliographic record verified (Crossref), content standard; [RESTATEMENT] restatement of a known result or of an earlier TAA lemma; [ANALOGY] analogy only; [NUM] numerical (scripts `num/lemma52_*.py`, data `num/l52/`); [HEUR] heuristic derivation, not a proof; [CONJ] conjecture; [OPEN] open.

Question (J.W., [t196u], answer 4 to Lemma 50's questions). "4. Yes": run the exploratory numerical lemma on whether the CCM bosonic (metaplectic/prolate) set-up has a natural compatible complex structure J, a *vacuum*, invariant under Frobenius or scaling, and how it behaves as the set S of places grows. This is item (S4) of Lemma 50's requirement list: local–global compatibility of vacua.

Answer in brief. (1) *Scaling is a boost, at every finite S.* In CCM's archimedean cyclic pair the vacuum $\xi_\infty = 2^{1/4}e^{-\pi x^2}$ is fixed by the Fourier transform (a metaplectic *rotation*) but moved by scaling: $|\langle \xi_\infty, U(t)\xi_\infty\rangle| = (\cosh t)^{-1/2}$, which is Lemma 51's bosonic fidelity at Siegel distance $d = 2|t|$. The scaling orbit of the vacuum is a geodesic in $\mathfrak{h}_1 = \mathbb{H}$. For every finite S the scaling group has Lebesgue spectrum, so no vector is scaling-invariant and every vacuum fidelity tends to 0 as $|t| \to \infty$ (Thm. 2.1, [THM]). (2) *Growing S: almost invariant, then gone.* The semilocal vacua $\xi_S = \eta_S(\xi_\infty)$ have closed-form norms and overlaps, $\|\xi_S\|^2 = \sqrt{2}\sum_{m,n}(m^2+n^2)^{-1/2}$ over S-units (Prop. 2.3, [THM], matched by the data). Normalized, their scaling spectral measures concentrate at $s = 0$ with variance $\simeq 1/K_X$, $K_X = \frac{1}{2}\psi'(\frac{1}{4}) + \sum_{p<X} 2p^{-1/2}\log^2 p/(1-p^{-1/2})^2$ (ratio 1.0032 at $X = 2\cdot 10^5$, [NUM]/[HEUR]). So the fidelity at fixed t tends to 1: the vacua become *almost* invariant. But they converge weakly to 0 (Prop. 2.4, [THM] given the variance limit). Their overlap with ξ_∞ falls to $4.9\cdot 10^{-15}$ at $S = \{p \le 19997\}$. There is no invariant limit vacuum, only a Weyl sequence for the point 0 of the continuous spectrum. This is the bosonic, infinite-dimensional version of Lemma 51's "the vacuum escapes to the boundary" ([ANALOGY]). (3) *The metaplectic structure does not survive semilocalization.* CCM's Theorem 5.2 says the $\mathfrak{sl}(2,\mathbb{R})$ relations force the archimedean Jacobi coefficients $a_n = \frac{1}{2}\sqrt{(2n+1)(2n+2)}$. Our Stieltjes computation of the semilocal coefficients, which CCM defer to a forthcoming paper, violates them already at $n = 0$ for every $S \neq \emptyset$ tested (Prop. 3.1, [NUM], with the logic [THM]). The semilocal grading differs from the archimedean one by the phase $e^{-2i\Phi_S}$, and $\|F_S - F_\infty\| = 2.000$ already for $S = \{2,3,5\}$ (Prop. 3.2). (4) *The prolate polarization needs a boundary condition.* For CCM's Jacobi matrices $W_\lambda^\pm$ (Prop. 3.5) both solutions of $W_\lambda^\pm u = 0$ are numerically in ℓ^2, with summands $\sim n^{-3/2}$ (12 of 12 cases; $\lambda = 0$ control diverges like $\sqrt{n}$). That is the limit-circle case, so a circle of self-adjoint extensions exists and $J = \operatorname{sign} W_\lambda^\pm$ is not canonical ([NUM]; consistent with CM's selection of one extension, [CITED]). The number of positive eigenvalues of the $M \times M$ truncation is $(4/\sqrt{\pi})\lambda\sqrt{M}$ within 1.92 in all 42 cases ([NUM]/[HEUR]). Low positive eigenvalues match the classical prolate values $\chi_n(2\pi\lambda^2)$, while one truncation-dependent eigenvalue wanders. In the extension family used, n_+ takes two adjacent values: the polarization is ambiguous by one boundary mode (§4). (5) *Weil positivity on windows, ζ versus the Davenport–Heilbronn fake.* With CCM's truncated Weil matrices QW_λ^N, ζ is positive at every window computed in multiprecision ($\lambda^2 \le 13$; $\epsilon_N = 9.5\cdot 10^{-54}$ at $\lambda^2 = 13, N = 40$). The ground-state zeros reproduce $\gamma_1, \dots$ to 10^{-14}. For f_{DH} the same pipeline is positive up to $\lambda^2 = 20$ and reproduces *the fake's* on-line zeros just as well ($2\cdot 10^{-14}$). It is indefinite from $\lambda^2 = 60$ on, with $8, 22, 42$ negative eigenvalues at $\lambda^2 = 60, 150, 400$. At $\lambda^2 = 60$ the minimizer's negativity is the ghost contribution: explicit side -0.49183; zero side $+0.21653$ (on-line) -0.70839 (off-line) $+0.00009$ (tail) $= -0.49177$ ([NUM]). So the CCM real-zero mechanism cannot tell ζ from a fake below the ghost, and only window positivity can. That is Lemma 22's conclusion, now with a genuine fake ([RESTATEMENT]). (6) *Verdict on (S4).* None of the three CCM candidates is a scaling-invariant positive vacuum: the cyclic vector, the prolate polarization, and the window Weil form. The first two cannot be invariant: scaling is hyperbolic on the metaplectic phase plane. The third is G5 itself (Weil's criterion, [RESTATEMENT]). What the probe adds is where each candidate fails and by how much. No step here supplies the missing positivity.

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1. Conventions; what CCM prove; what TAA already has

1.1. **Vacuum dictionary (Lemmas 50, 51, 53).** A *vacuum* for a real symplectic space (V,ω) is a compatible complex structure J. It is *positive* if $\omega(\cdot, J\cdot) > 0$, i.e. a point of Siegel space $\mathfrak{h}(V)$, and on the bosonic side it corresponds to a Gaussian, the metaplectic (Fock) vacuum Ω_J (Lemma 53 Prop. 2.1). It is *invariant* under a flow g_t if $Jg_t = g_t J$. Lemma 51 Thm. 2.1: if g is not conjugate into a compact subgroup, no positive invariant vacuum exists, and invariant vacua along a path escape to the boundary. Lemma 51 Thm. 2.2(c): $|\langle\Omega_{J_0}, \Omega_J\rangle|^2 = \prod_i \operatorname{sech}(d_i/2)$ for rapidities d_i. Lemma 53: *ghost charge* $= -\#$ghosts for the Hardy-space vacuum $W_\zeta = \tilde{B}H_+$. Lemma 50 (S1)–(S4): (S3) a positive J commuting with the arithmetic flow is RH given (S1)–(S2); (S4) asks for compatibility of a global vacuum with local vacua (Shale–Stinespring type, Lemma 50 Thm. 5.1). Lemma 54 (STATE §63, finished while this lemma was being completed; J.W. [t201u]–[t203u]) constructs the Frobenius-built $J_F = (F-V)/\sqrt{4q-(F+V)^2}$ and the Siegel-orbit drift detector, with translation length $\ell = 2(\sum_{|\lambda|>1}\sigma^2)^{1/2}$. We use only its Prop. 4.2 and Thm. 6.1, as stated in STATE §63, in Rem. 2.2.

1.2. **What CCM prove (all [CITED], read in the arXiv texts).**

(C1) [4, Thm. 2.1] A *cyclic pair* (D,ξ) is a self-adjoint D with a cyclic unit vector ξ. It is determined by the spectral measure μ of ξ; orthogonal polynomials give a Jacobi matrix for D. It is *even* if a grading γ ($\gamma D = -D\gamma$, $\gamma\xi = \xi$) exists, iff μ is symmetric (Prop. 2.1). Selfadjointness of the formal prolate operator of a cyclic pair is "delicate" (remark after Def. 2.2).

(C2) [4, Prop. 3.3] For the archimedean pair (scaling S, $\xi_\infty = 2^{1/4}e^{-\pi x^2}$) the measure is $(2\pi)^{-3/2}|\Gamma(\frac{1}{4} - \frac{is}{2})|^2\,ds$ and the Jacobi matrix has 0 diagonal and off-diagonal $\alpha_n = \frac{i}{2}\sqrt{(2n+1)(2n+2)}$.

(C3) [4, Prop. 3.5, (27),(28)] The prolate operator W_λ splits as Jacobi matrices $W_\lambda^\pm$ with $a_n = \frac{1}{4}\sqrt{(4n+1)(4n+2)(4n+3)(4n+4)}$, $b_n = -8n^2 - 2n - \frac{3}{4} + 2\pi\lambda^2(8n+1)$ (even) and $a_n = \frac{1}{4}\sqrt{(4n+3)\cdots(4n+6)}$, $b_n = -8n^2 - 10n - \frac{15}{4} + 2\pi\lambda^2(8n+5)$ (odd).

(C4) [4, Thm. 1 = Props. 4.1, 4.2, (47)] For a finite set $S \ni \infty$ of places, $\xi_S := \eta_S(\xi_\infty)$ (class of $1_{R_S} \otimes \xi_\infty$). A unitary U_S to $L^2(\mathbb{R})$ takes scaling to multiplication by s and ξ_S to $\prod_{v \in S} L_v(\frac{1}{2} - is)$. The pair (S, ξ_S) is even, with grading the semilocal Fourier transform F_S, which fixes ξ_S and becomes $s \mapsto -s$ under $V_S = M_S \circ U_S$. The general-S Jacobi coefficients are "deferred to a forthcoming paper". A second candidate built from the Weil representation of $\mathrm{Mp}(\mathbb{A}_S)$ is announced.

(C5) [4, Thm. 5.2] The a_n are specified uniquely by requiring that CCM's (74) define a representation of $\mathfrak{sl}(2, \mathbb{R})$: $a_n = \frac{1}{2}\sqrt{(2n+1)(2n+2)}$. The moment problem is determinate, the measure is $\propto |\Gamma(\frac{1}{4} + \frac{is}{2})|^2 ds$, and the representation is the even part of the metaplectic representation.

(C6) [5, §5.1, (5.1), Lemma 5.1; Def. 5.3; Thm. 1.1] The truncated Weil form QW_λ^N in the basis V_n, $|n| \le N$, of $L^2([\lambda^{-1}, \lambda], d^*u)$, $L = 2\log\lambda$, is $\tau_{ii} = a_i$, $\tau_{ij} = (b_i - b_j)/(i-j)$ with $a_n = 2\int_0^L (1 - \frac{y}{L}) \cos\frac{2\pi n y}{L}\, D(y)$, $b_n = -\frac{1}{\pi}\int_0^L \sin\frac{2\pi n y}{L}\, D(y)$. If the smallest eigenvalue ϵ_N is simple with even eigenvector ξ ("even-simple"), then $\hat{\xi}$ is entire with only real zeros, the spectrum of a self-adjoint rank-one perturbation of $D_{\log}$. Convergence to the zeros of ζ is numerical in CCM, and §8 lists the missing steps.

CCM24 is the closest verified link between the metaplectic representation and Weil positivity (Lemma 50, §7), and it is on the bosonic side. Lemma 22 (§31 of STATE) already showed that the prolate objects do not detect a *synthetic* ghost before full window positivity does, and that Thm. 1.1 does not use positivity. We do not redo that.

1.3. **What the scripts actually compute.**

script → data	content
`lemma52_jacobi.py` →`jacobi.json`	discretized Stieltjes for $d\mu_S \propto \prod_{p\in S} \lvert L_p(\frac{1}{2} - is)\rvert^2\, (2\pi)^{-3/2} \lvert\Gamma(\frac{1}{4} + \frac{is}{2})\rvert^2 ds$ (long double, two grids $[-1500, 1500]/0.02$, $[-2000, 2000]/0.025$), S = first k primes, $k \in \{0, 1, 2, 3, 4, 6, 10\}$, $n < 260$; records a_n, $\lvert b_n\rvert_{\max}$, $\delta_n = a_n^2 - a_{n-1}^2 - (2n + \frac{1}{2})$
`lemma52_vacuum.py` →`vacuum.json`	in the Mellin picture $U_S\xi_S = m_S \cdot U_\infty\xi_\infty$, $m_S(s) = \prod_{p\in S}(1 - p^{-1/2+is})^{-1}$: norms, scaling fidelities at $t = \log q$, nested overlaps up to $S = \{p \le 19997\}$, grading phases Φ_S
`lemma52_vacuum2.py` →`vacuum2.json`	normalized spectral measure μ_X of ξ_S, $S = \{p < X\}$: variance, mass in $\lvert s\rvert < 0.1$, fidelity at $\log 2, \log 3$
`lemma52_prolate.py`	dense version of the prolate block; was still running (pipe to `tail`, no output file) after 40 min, killed by the finishing worker and replaced by
`lemma52_prolate_tri.py` →`prolate_tri.json`	identical tests with LAPACK tridiagonal solvers: truncation spectra of $W_\lambda^\pm$ ($M \le 3200$), $\chi_n(2\pi\lambda^2)$ (`scipy pro_cv`), $z = 0$ solutions to $n = 6000$ (60 digits), boundary-condition extensions, compressed HS distances of positive projections; $\lambda \in \{1, \sqrt{2}, 2\}$, $\lambda = 0$ control
`lemma52_prolate_ext.py` →`prolate_ext.json`	(new) 48-point extension scan, tail test, semiclassical count
`lemma52_weil.py` →`weil_{extra,hp,scan}.json`	QW_λ^N by (C6) for ζ and f_{DH} (Lemma 48: $\chi \bmod 5$ odd, $Q = 5$, $a = 1$), mpmath (dps 30–60); eigen-decomposition in mp for $N \le 40$, float64 for $N > 40$; ground-state zeros vs. zeros to $T = 1000$ (`num/l48/zeros_T1000.json`)
`lemma52_weil_decomp.py`, `_decomp2.py` →`weil_decomp{,2}.json`	for the minimizer x: $x^T\tau x$ vs. $\sum_\rho x^T K(\gamma_\rho) x$ split on-line / off-line, with a tail estimate past $T = 1000$
`lemma52_checks.py` →`checks.json`	(new) closed forms of Prop. 2.3, bounds, variance law

All runs used `nice`, with single-threaded BLAS for the new ones. The status of each output file is recorded in §9.

2. The archimedean and semilocal vacua under scaling

Theorem 2.1 (scaling is a boost; [THM], checked [NUM]). *(a) With $(U(t)f)(x) = e^{t/2}f(e^t x)$ on $L^2(\mathbb{R})_{\mathrm{ev}}$, $\langle \xi_\infty, U(t)\xi_\infty\rangle = (\cosh t)^{-1/2}$. Equivalently, $\int e^{its}\,\mathrm{d}m(s) = (\cosh t)^{-1/2}$ for CCM's probability measure $\mathrm{d}m$ of (C2).*

(b) $U(t)$ maps the Gaussian with parameter $\tau = i$ (in $e^{i\pi\tau x^2}$) to the one with $\tau = ie^{2t}$: the scaling orbit of the vacuum is a geodesic of $\mathfrak{h}_1 = \mathbb{H}$, at distance $d = 2|t|$ after time t. The identity in (a) is Lemma 51 Thm. 2.2(c), $|\langle \Omega_{J_0}, \Omega_J\rangle|^2 = \operatorname{sech}(d/2)$, at $d = 2t$. On the phase plane scaling is the hyperbolic element $\operatorname{diag}(e^t, e^{-t})$ of $\mathrm{SL}_2(\mathbb{R})$, which fixes no point of $\mathbb{H}$. The Fourier transform is the elliptic element (rotation by $\pi/2$) and fixes ξ_∞.

(c) For every finite S, in the picture U_S of (C4) scaling acts on $L^2(\mathbb{R}, ds)$ by multiplication by e^{its}, which has Lebesgue spectrum. Hence no nonzero vector of $L^2(X_S)^{K_S}$ is scaling-invariant. For every vector ξ, $\langle \xi, U(t)\xi\rangle = \int e^{its} d\mu_\xi \to 0$ as $|t| \to \infty$ (Riemann–Lebesgue, $\mu_\xi \ll ds$).

Proof. (a) $\langle \xi_\infty, U(t)\xi_\infty\rangle = \sqrt{2}\, e^{t/2}\int e^{-\pi(1+e^{2t})x^2}dx = \sqrt{2}e^{t/2}(1+e^{2t})^{-1/2} = (\cosh t)^{-1/2}$. The Mellin form follows because U_∞ is unitary and intertwines $U(t)$ with multiplication by e^{its} (C4, $S = \{\infty\}$). (b) $e^{t/2}e^{-\pi(e^t x)^2} = e^{t/2}e^{i\pi(ie^{2t})x^2}$; the hyperbolic distance from i to ie^{2t} is $2|t|$, and $\operatorname{sech}(d/2) = \operatorname{sech} t$. A hyperbolic element of $\mathrm{SL}_2(\mathbb{R})$ has no fixed point in $\mathbb{H}$. Fourier fixes ξ_∞ by [4, Prop. 3.1/4.1] ([CITED]). (c) U_S is unitary (C4), and multiplication by e^{its} on $L^2(\mathbb{R}, ds)$ has no eigenvectors. □

[NUM]: the fidelities in `vacuum.json` reproduce (a) to $2\cdot 10^{-16}$ at $t = \log 2, \log 3, \log 5, \log 7$: 0.894427, 0.774597, 0.620174, 0.529150.

Remark 2.2 (rotation × boost, and Lemma 54's drift detector; [ANALOGY], with [RESTATEMENT] of Lemma 54). In J.W.'s decomposition ([t202u]) Frobenius is a legitimate boost $\sqrt{q}$ times a rotation, and a ghost is an extra boost. Lemma 54 (Prop. 4.2) makes this precise. The factor $\sqrt{q}$ is central and acts *trivially* on Siegel space, and only the ghost boost moves the vacuum, with rapidity 2σ. Its drift detector (Lemma 54 Thm. 6.1) has translation length $2|\sigma|$ for a real pair. In the CCM metaplectic picture scaling by e^t is *not* central. On the phase plane it is the hyperbolic element $\operatorname{diag}(e^t, e^{-t})$ and moves the vacuum with translation length $2|t|$ (Thm. 2.1(b)). On Siegel space it behaves like Lemma 54's ghost boost with $\sigma = t$, not like the harmless $\sqrt{q}$. The CCM vacuum is invariant only under the rotation (Fourier, the grading). Consequences: (i) an invariant positive vacuum for scaling on L^2 of functions is excluded for a structural reason, independent of the zeros; (ii) Lemma 54's drift detector, applied to the scaling flow on this space, would "detect a ghost" for ζ as well. So it must be applied on a space where the flow can be elliptic, the spectral or H^1-type space in which Connes' framework places the zeros [7].

Proposition 2.3 (closed forms for the semilocal vacua; [THM], checked [NUM]). *In the normalization $U_S\xi_S = m_S\, U_\infty\xi_\infty$ of (C4), let N_S be the set of positive integers whose prime factors lie in S. Then*

$$\|\xi_S\|^2 = \sqrt{2}\sum_{m,n\in N_S}\frac{1}{\sqrt{m^2+n^2}}, \qquad \langle \xi_\infty, \xi_S\rangle = \sqrt{2}\sum_{n\in N_S}\frac{1}{\sqrt{n^2+1}},$$

$$\prod_{p\in S}(1-p^{-1})^{-1} \;\le\; \|\xi_S\|^2 \;\le\; \prod_{p\in S}(1-p^{-1/2})^{-2}.$$

For one prime, $\|\xi_{\{p\}}\|^2 = (1-\frac{1}{p})^{-1}\big(1+2\sum_{k\ge1}\sqrt{2}\,p^{-k}(1+p^{-2k})^{-1/2}\big)$. In particular $\|\xi_S\| \to \infty$ as S exhausts the primes, by Mertens.

Proof. $m_S(s) = \sum_{n\in N_S} n^{-1/2}n^{is}$, absolutely convergent because $\sum_{N_S} n^{-1/2} = \prod(1-p^{-1/2})^{-1} < \infty$. By Thm. 2.1(a), $\int (m/n)^{is}\mathrm{d}m = (\cosh\log(m/n))^{-1/2} = (2mn/(m^2+n^2))^{1/2}$. Hence $\int |m_S|^2\mathrm{d}m = \sum_{m,n}(mn)^{-1/2}(2mn)^{1/2}(m^2+n^2)^{-1/2}$, and similarly for the overlap with $n' = 1$. Bounds: the

diagonal $m = n$ gives $\sum_{N_S} \frac{1}{n}$, and $(m^2+n^2)^{-1/2} \le (2mn)^{-1/2}$. The one-prime formula is the sum over $d = j - k$. □

[NUM] (`checks.json`): the series, truncated at 10^9 (10^8), give 7.3097767, 31.68851, 87.7217 for $S = \{2\}, \{2,3\}, \{2,3,5\}$, against the quadrature values 7.3097769, 31.68854, 87.7283 (series tails explain the differences). The overlaps agree to $\le 6 \cdot 10^{-10}$ relative. All 13 nested norms in `vacuum.json` satisfy the bounds; e.g. $S = \{p \le 19997\}$: $17.65 \le 6.48 \cdot 10^{32} \le 2.11 \cdot 10^{34}$.

2.1. **Growing S: almost-invariance without a limit.** Let μ_X be the scaling spectral measure of $\xi_S/\|\xi_S\|$, $S = \{p < X\}$, i.e. $|m_S|^2 \mathrm{d}m/\|\xi_S\|^2$.

$S = \{p < X\}$, X	$\mathrm{Var}\,\mu_X$	$1/K_X$	ratio	$\mu_X(\lvert s\rvert < 0.1)$	fidelity $t = \log 2$	fidelity $t = \log 3$
2 ($S = \{\infty\}$)	0.50000	0.1163	4.30	0.1646	0.894427	0.774597
13	0.02991	0.02155	1.388	0.4695	0.992861	0.982221
97	$6.915 \cdot 10^{-3}$	$6.238 \cdot 10^{-3}$	1.109	0.7775	0.998340	0.995837
997	$1.435 \cdot 10^{-3}$	$1.390 \cdot 10^{-3}$	1.033	0.9910	0.999655	0.999134
9973	$3.325 \cdot 10^{-4}$	$3.287 \cdot 10^{-4}$	1.012	$1 - 9 \cdot 10^{-8}$	0.999920	0.999799
99991	$8.191 \cdot 10^{-5}$	$8.156 \cdot 10^{-5}$	1.0043	1.0000	0.999980	0.999951
199999	$5.421 \cdot 10^{-5}$	$5.403 \cdot 10^{-5}$	1.0032	1.0000	0.999987	0.999967

[NUM] (`vacuum2.json`, grid $[-12, 12]$, step $5 \cdot 10^{-4}$; $1/K_X$ from `checks.json`). The label X means primes $p < X$ (see §7). [HEUR]: $\log|m_S(s)|^2 + \log|\Gamma(\frac{1}{4} + \frac{is}{2})|^2 = \mathrm{const} - \frac{K_X}{2}s^2 + O(s^4)$ with $K_X = \frac{1}{2}\psi'(\frac{1}{4}) + \sum_{p<X} \frac{2p^{-1/2}\log^2 p}{(1-p^{-1/2})^2}$, by the second derivative of $-\log|1 - re^{i\theta}|^2$ at $\theta = 0$, which is $2r/(1-r)^2$. The Gaussian approximation gives $\mathrm{Var} \approx 1/K_X \approx (4\sqrt{X}(\log X - 2))^{-1}$ (prime number theorem). The concentration at $s = 0$ is the Euler product at the centre $\frac{1}{2}$, where all local factors $|L_p(\frac{1}{2})|$ are maximal at once. It is *not* a statement about zeros.

Proposition 2.4 (weak escape; [THM] given $\mathrm{Var}\,\mu_X \to 0$)**.** *Identify all spaces with $L^2(\mathbb{R}, ds)$ by U_S (C4). If $\mathrm{Var}\,\mu_X \to 0$ ([NUM]), then (a) $\langle \hat\xi_X, U(t)\hat\xi_X\rangle \to 1$ for each fixed t, where $\hat\xi_X = \xi_S/\|\xi_S\|$; (b) $\hat\xi_X \to 0$ weakly; (c) no subsequence converges in norm.*

Proof. (a) $|1 - \int e^{its} d\mu_X| \le \int |ts|\, d\mu_X \le |t|\,\mathrm{Var}(\mu_X)^{1/2}$ (μ_X is even, so its mean is 0). (b) For fixed w and $\varepsilon > 0$, write $w = E(|s| < \varepsilon)w + E(|s| \ge \varepsilon)w$ with spectral projections of the multiplication operator. Then $|\langle E(|s| \ge \varepsilon)w, \hat\xi_X\rangle| \le \|w\|\mu_X(|s| \ge \varepsilon)^{1/2} \to 0$ by Chebyshev, and $\|E(|s| < \varepsilon)w\| \to 0$ as $\varepsilon \to 0$ since $\{0\}$ has Lebesgue measure 0. (c) A norm limit would be a unit vector, contradicting (b). □

[NUM] (`vacuum.json`): the normalized overlap $|\langle \xi_\infty, \hat\xi_S\rangle|$ for $S = \{p \le P\}$ is $0.861, 0.660, 0.509, 0.399, 0.324, 0.266$ ($P = 2, \ldots, 13$), then 0.137 (29), 0.0264 (97), $4.0 \cdot 10^{-4}$ (541), $1.5 \cdot 10^{-5}$ (1223), $1.1 \cdot 10^{-7}$ (3001), $4.7 \cdot 10^{-11}$ (7919), $4.9 \cdot 10^{-15}$ (19997). Successive overlaps $|\langle \hat\xi_{P_1}, \hat\xi_{P_2}\rangle|$ are $0.934, 0.966, 0.979, 0.987, 0.990$ for consecutive small primes, but $0.915, 0.698, 0.297, 0.641, 0.488, 0.308, 0.213$ along the cuts $13 \to 29 \to 97 \to 541 \to 1223 \to 3001 \to 7919 \to 19997$. The script's "Kakutani" column $\prod_{p \le P}(1 - 1/p)^{1/2}$ (0.238 at 19997) does *not* describe the decay (§7). The vacua are not Gaussian (they are m_S·Gaussian), so Kakutani's product criterion [11] does not apply as such.

Remark 2.5 (reading in TAA language; [ANALOGY])**.** Prop. 2.4 is the infinite-dimensional, bosonic shadow of Lemma 51 Thm. 2.1. A sequence of vacua becomes more and more invariant under the flow but leaves every compact set, here weakly, and has no invariant limit. The would-be invariant limit is a spectral point mass at $s = 0$, which is not a vector. In Lemma 51 the escape signals a ghost. Here it signals only that scaling has continuous spectrum on L^2, with no reference to zeros. The analogy should not be read as ghost detection.

3. The metaplectic structure and the grading under semilocalization

Proposition 3.1 (the $\mathfrak{sl}(2,\mathbb{R})$ relations fail for $S \neq \emptyset$; logic [THM] from [CITED] (C5), values [NUM]). *For an even cyclic pair with Jacobi coefficients $a_n > 0$ put $\delta_n = a_n^2 - a_{n-1}^2 - (2n+\frac{1}{2})$ ($a_{-1} = 0$). Then $\delta \equiv 0$ iff $a_n = \frac{1}{2}\sqrt{(2n+1)(2n+2)}$ for all n, by telescoping. By* [4, Thm. 5.2] *this is the only Jacobi sequence for which CCM's (74) defines a representation of $\mathfrak{sl}(2,\mathbb{R})$. The discretized Stieltjes coefficients of (S,ξ_S) give:*

$S\setminus\{\infty\}$	a_0	a_1	δ_0	δ_1	$a_{259}/a_{259}^{\mathrm{arch}}$
$\emptyset$	0.707107	1.732051	0 ($<10^{-13}$)	0	1 ($8.9\cdot10^{-16}$)
$\{2\}$	0.396868	1.061680	−0.3425	−1.5303	0.963
$\{2,3\}$	0.280246	0.716588	−0.4215	−2.0650	0.983
$\{2,3,5\}$	0.225638	0.540679	−0.4491	−2.2586	0.960
$\{2,3,5,7\}$	0.193687	0.442686	−0.4625	−2.3415	0.966
$\{2,\dots,13\}$	0.157734	0.327449	−0.4751	−2.4177	0.953
$\{2,\dots,29\}$	0.123249	0.226405	−0.4848	−2.4639	0.947

So for every tested $S \neq \emptyset$ the semilocal cyclic pair does not carry CCM's metaplectic $\mathfrak{sl}(2,\mathbb{R})$ structure of §5. Already $a_0^2 = \int s^2 d\mu_S \neq \frac{1}{2}$.

[NUM] (`jacobi.json`): both grids agree to all printed digits up to $n = 259$. The odd moments vanish ($|b_n| \le 6\cdot10^{-13}$, evenness). $\log\|\xi_{\{2\}}\|^2 = 1.989213$ reproduces Prop. 2.3 ($e^{1.989213} = 7.30978$). For $n \gtrsim 20$ the δ_n are large and irregular, e.g. $\delta_{259} \in [-8.0, -0.46]\cdot10^3$. We report only the low coefficients as reliable. As S grows, $a_0 \to 0$, which is the variance collapse of the table above (the μ_S here are normalized over the whole line, without the cut-off of `vacuum2`).

Proposition 3.2 (the semilocal grading drifts away from the archimedean one; [THM], values [NUM]). *In $L^2(\mathbb{R},\mathrm{d}m)$ (via $f \mapsto m_S f$ from $L^2(\mathrm{d}m_S)$) the grading F_S is $g(s) \mapsto e^{-2i\Phi_S(s)}g(-s)$ with $\Phi_S(s) = \sum_{p\in S}\arg L_p(\frac{1}{2}+is)$, while F_∞ is $g(s)\mapsto g(-s)$. Hence $\|F_S - F_\infty\| = \sup_s|e^{2i\Phi_S(s)}-1|$, and $|\Phi_S| \le \sum_{p\in S}\arcsin p^{-1/2}$.*

Proof. By (C4), F_S is $s\mapsto -s$ on $L^2(\mathrm{d}m_S)$ and $|m_S|^2\mathrm{d}m = \mathrm{d}m_S$. Conjugating by multiplication with m_S gives $g \mapsto \frac{m_S(s)}{m_S(-s)}g(-s)$, and $m_S(-s) = \overline{m_S(s)}$, so the factor is $e^{2i\arg m_S(s)} = e^{-2i\Phi_S(s)}$. The norm of a multiplication operator is the sup. Finally $|\arg(1-z)^{-1}| \le \arcsin|z|$ for $|z|<1$. □

[NUM] (`vacuum.json`, $s\in[0,3000]$, step 0.01): $\sup|\Phi_S| = 0.785398$ ($= \arcsin 2^{-1/2} = \pi/4$), 1.400865 (bound 1.400878), 1.862835 (bound 1.864525) and $\|F_S - F_\infty\| = 1.414214$, 1.971193, 2.000000 for $S = \{2\},\{2,3\},\{2,3,5\}$. Pointwise, $\Phi_{\{p\le X\}}(s)$ does not settle as X grows. At $s = 0.5$ it is $-1.97, -5.09, -6.29, 6.51, 39.5, 25.1, -235.3$ for $X = 10,\dots,10^7$; at $s = 14.134725$ it is $0.11, 0.15, -0.19, 0.05, 1.76, 4.59, -1.07$. This is the expected non-convergence of the Euler product on the critical line. *Reading* ([ANALOGY]): the $\mathbb{Z}/2$ gradings, the compact (rotation) part of the vacuum data, have no norm limit as S grows. This is a second way in which the local vacua fail to assemble.

4. The prolate polarization in the Jacobi picture

A natural polarization attached to W_λ is $J = \mathrm{sign}(W_\lambda)$, i.e. the positive spectral projection P_+. CCM relate the positive spectrum to low-lying zeros and the negative (Sonin) spectrum to the UV ([4], abstract, [CITED]; Lemma 22). We test whether P_+ is well defined and stable in truncations of the Jacobi matrices $W_\lambda^{\pm}$ of (C3).

Proposition 4.1 (limit circle, numerically; [NUM], with [HEUR] continuum model). *Let P, Q be the real solutions of $(W_\lambda^{\pm}u)_n = 0$ with $P_0 = 1$ (and the $n=0$ equation) and $Q_0 = 0, Q_1 = 1$,*

computed to $n = 6000$ *with 60 digits. For* $\lambda \in \{1, \sqrt{2}, 2\}$*, both parities, and both* P *and* Q *(12 cases), the partial sums* $\Sigma(n) = \sum_{j<n} u_j^2$ *satisfy*

$$C_k = \frac{\Sigma(n_{k+1}) - \Sigma(n_k)}{n_k^{-1/2} - n_{k+1}^{-1/2}} \quad \text{constant to } \le 4\% \quad (n_k = 100, 1000, 3000, 5999),$$

i.e. $u_n^2 \asymp n^{-3/2}$*, and both solutions are in* ℓ^2 *(*$\lambda = 1$*, even:* $C = 18766, 18498, 18606$ *for* P *and* $931, 918, 923$ *for* Q*). For the control* $\lambda = 0$*,* $C_k = 252, 1382, 3385$ *grows, and* $\Sigma(n)/\sqrt{n} = 0.832, 0.809, 0.804, 0.802$*:* $P \notin \ell^2$*.* [HEUR]*: since* $a_{n-1} + a_n + b_n = 16\pi\lambda^2 n + O(1)$*, the continuum model is* $(4x^2u')' + 16\pi\lambda^2 x\, u = 0$*. Its solutions are* $x^{-1/2}J_1, x^{-1/2}Y_1$ *of argument* $4\lambda\sqrt{\pi x}$*, so* $|u| \sim x^{-3/4}$*, matching the data. At* $\lambda = 0$ *the model is of Euler type with* $|u| \sim x^{-1/4}$*; the computed exponent is* -0.2499925*.*

Consequence ([BIB] standard theory [13, 16], applied to [NUM]). In the limit-circle case the minimal Jacobi operator has deficiency indices $(1, 1)$, and its self-adjoint extensions form a circle, parametrized by a boundary condition at $n = \infty$. The spectra of different extensions interlace, every real number is an eigenvalue of exactly one extension, and P_+ *depends on the extension.* This is consistent with CCM's remark that selfadjointness is delicate (C1), and with CM's choice of the unique extension commuting with $P_\lambda, \hat{P}_\lambda$ ([6, Thm. 2.6], [CITED] via Lemma 22). A proof that $W_\lambda^\pm$ is limit circle for all $\lambda > 0$ is [OPEN]. We found no statement of it in the sources read.

Proposition 4.2 (truncations; [NUM], [HEUR])**.** *(a) The lowest positive eigenvalues of the naive* $M \times M$ *truncations (*$M = 50, \ldots, 3200$*) are stable and equal the classical prolate values* $\chi_n(c)$*,* $c = 2\pi\lambda^2$*, with* $n \equiv 0 \bmod 4$ *for* W^+ *and* $n \equiv 2 \bmod 4$ *for* W^-*. For* $\lambda = \sqrt{2}$*:* $11.7997 = \chi_0$*,* $100.85 = \chi_4$ *(ref.* 100.8541*; to* 10^{-2} *for* $M \le 1600$*,* 100.77 *at* $M = 3200$*), and* $58.8026 = \chi_2$*. For* $\lambda = 2$*:* $24.3749 = \chi_0$*,* $214.847 = \chi_4$*,* $121.7903 = \chi_2$*. For* $\lambda = 1$ *only* $\chi_0 = 5.4941$ *is stable (to* $3 \cdot 10^{-3}$*). This is consistent with CM's description of the positive spectrum as the classical prolate spectrum (Lemma 22,* [CITED]*).*

(b) Every truncation has, in addition, eigenvalues that move with M*. For* $\lambda = 1$*,* W^+*, the one of the two lowest positive eigenvalues that is not* χ_0 *is* $22.30, 4.17, 22.14, 30.84, 19.39, 34.93, 12.39$ *for* $M = 50, \ldots, 3200$*; at* $M = 100$ *it lies* below χ_0*. This is the signature of a boundary condition at* $n = M$ *that changes with* M*.*

(c) The number of positive eigenvalues is $n_+(M) = (4/\sqrt{\pi})\,\lambda\sqrt{M}$ *within* 1.92 *in all 42 cases (*$\lambda \in \{1, \sqrt{2}, 2\}$*,* $\pm$*, seven* M*); e.g.* $128, 181, 255$ *at* $M = 3200$ *against* $127.7, 180.5, 255.3$*.* [HEUR]*: in the continuum model the classically allowed region is* $4x^2k^2 < 16\pi\lambda^2 x$*, i.e.* $|k| < 2\lambda\sqrt{\pi/x}$*, whose phase-space volume over* $[0, M]$ *divided by* 2π *is* $(4/\sqrt{\pi})\lambda\sqrt{M}$*.*

(d) Naive P_+ *does not converge: the compressed HS distance* $\|(P_+^{(800)} - P_+^{(1600)})_{[0,K)}\|_{HS}$ *is* 0.45–0.78 *(*$\lambda = 1$*,* W^+*,* $K = 25$–400*),* 0.54–0.88 *(*$\lambda = 1$*,* W^-*),* 0.45–0.84 *and* 0.34–0.69 *(*$\lambda = \sqrt{2}$*), and* 0.022–0.080 *(*$\lambda = 2$*, both).*

So the positive spectrum is infinite, as it must be for an operator whose positive part replicates the classical prolate spectrum. Low-lying positive eigenvalues are stable. What the truncation does not fix is the boundary at infinity.

Proposition 4.3 (extension scan: the polarization is ambiguous by one boundary mode; [NUM])**.** *Truncate with the boundary condition of the extension defined by* $v = \cos\theta\, P + \sin\theta\, Q$*: last diagonal entry* $b_{M-1} + a_{M-1}v_M/v_{M-1}$*. For* θ *on a 48-point grid in* $[0, \pi)$*:*

(a) n_+ *takes exactly two adjacent values for* $\lambda = 1, \sqrt{2}$*, namely* $\{128, 129\}$*,* $\{127, 128\}$*,* $\{181, 182\}$*,* $\{180, 181\}$ *at* $M = 3200$ *and* $\{90, 91\}$*,* $\{90, 91\}$*,* $\{128, 129\}$*,* $\{127, 128\}$ *at* $M = 1600$*. One eigenvalue near* 0 *changes sign with* θ*: its range is* $[-3.2, 1.7] \cdot 10^{-3}$*,* $[-2.1, 3.6] \cdot 10^{-2}$*,* $[-7.4, 10.1] \cdot 10^{-9}$*,* $[-4.9, 1.4] \cdot 10^{-7}$*.*

(b) $\theta = 0$ *is the extension in which* $P(0)$ *is an eigenvector, so* 0 *is an eigenvalue there. In the limit-circle case such an extension exists because* $P \in \ell^2$ *([BIB] standard). For that extension* $\operatorname{sign} W$ *is undefined on one mode.*

(c) The tails of P *and* Q *are proportional to high accuracy:* $P_M/Q_M = -4.4900$, -5.75, -9.6359885, -12.42167946 *for* $\lambda = 1, \sqrt{2}$ *(±), and* -19.902732988036433, -25.686078248388778 *to all 17 digits for* $\lambda = 2$. *So in double precision every* θ *produces the* $\theta = 0$ *boundary condition for* $\lambda = 2$ *(*n_+ *constant; eigenvalue* $\approx -1.4 \cdot 10^{-11}$*). The extension parameter is invisible at* $M \le 3200$ *there, and the scan is* inconclusive *for* $\lambda = 2$.

(d) Positive projections of the extensions $\theta = 0$ *and* $\theta = 1.5$, *compressed to the first* K *coordinates (*$M = 1600$*), differ by* $\le 2.4 \cdot 10^{-5}$ *(*$\lambda = 1$, W^+*),* $\le 1.8 \cdot 10^{-10}$, $\le 1.6 \cdot 10^{-8}$ *(*$\lambda = \sqrt{2}$*), and* $\le 2.6 \cdot 10^{-14}$ *(*$\lambda = 2$*) when the boundary mode has the same sign. They differ by* 0.79–0.99 *(*$\lambda = 1$, W^-*) when it does not, i.e. by one mode.*

Reading ([ANALOGY]/[CONJ]). The prolate polarization is well defined up to one boundary mode once a point of the extension circle is chosen. In the tested family, extensions differ in P_+ by a rank-≤ 1 piece, so their Fock vacua would be Shale–Stinespring equivalent. [CONJ]: for any two self-adjoint extensions of $W_\lambda^\pm$, $P_+(\theta) - P_+(\theta')$ is Hilbert–Schmidt. Nothing here makes P_+ scaling-invariant: W_λ commutes with Fourier, not with scaling. The comparison at fixed extension across M (`hs_same_ext_800_1600`) is dominated by the sign of the zero mode for $\theta = 0$ (0.54–0.98 in four cases, 0.007–0.05 in two) and is not used.

5. Weil positivity on windows: ζ versus the Davenport–Heilbronn fake

5.1. **Checks of the implementation.** [NUM] (`weil_*.out`). (i) The general archimedean term with $(Q, a) = (1, 0)$ equals CCM's (3.15) [5] to all 20 printed digits on three test functions (2.2560896685586849864, -0.31216045808874923028, 0.75835903657777475968). (ii) Explicit side vs. the sum over zeros to $T = 1000$, on the tent and sin test functions at $\lambda^2 = 3, 8$: ζ: 0.122329 vs. 0.118463, -0.131611 vs. -0.127745, 0.046423 vs. 0.044520, -0.047210 vs. -0.045307; f_{DH}: 0.273890 vs. 0.269568, -0.192528 vs. -0.188207, 0.190609 vs. 0.188296, -0.248031 vs. -0.245717. The differences, $2\text{–}4 \cdot 10^{-3}$, are the expected tails past $T = 1000$ for test functions with $1/\gamma^2$ decay.

5.2. **Results.**

f	λ^2	N	arithmetic	ϵ_N	$\#\{<0\}$	even-simple	$\lvert\hat{\xi}$-zero$-$zero$\rvert$, first k
ζ	2	6	mp40	$1.50 \cdot 10^{-3}$	0	yes	$0.23, 1.7, 0.93$
ζ	3	12	mp60	$8.46 \cdot 10^{-8}$	0	yes	$7 \cdot 10^{-5}, 6 \cdot 10^{-3}, 5 \cdot 10^{-2}$
ζ	5	18	mp60	$1.74 \cdot 10^{-17}$	0	yes	$3 \cdot 10^{-14}, 2 \cdot 10^{-11}, 8 \cdot 10^{-10}, 10^{-7}$
ζ	13	30	mp60	$4.92 \cdot 10^{-48}$	0	flag no*	$\le 4 \cdot 10^{-14}$ (first 12)
ζ	13	40	mp60	$9.46 \cdot 10^{-54}$	0	flag no*	$\le 4 \cdot 10^{-14}$ (first 12)
ζ	60/150/400	56/80/110	float64	$\approx -3 \cdot 10^{-15}$	27, 44, 73	—	round-off; inconclusive
f_{DH}	2	6/8	mp	0.657	0	yes	$1.6, 9 \cdot 10^{-3}, 0.3$
f_{DH}	5	16	mp30	$6.08 \cdot 10^{-3}$	0	yes	$0.05, 0.27, 0.93$
f_{DH}	10	24	mp30	$9.46 \cdot 10^{-8}$	0	yes	$10^{-6}, 3 \cdot 10^{-5}, 6 \cdot 10^{-4}$
f_{DH}	13	30	mp60	$1.04 \cdot 10^{-10}$	0	yes	$2 \cdot 10^{-9}, 5 \cdot 10^{-8}, 10^{-6}$
f_{DH}	20	36	mp30	$4.88 \cdot 10^{-18}$	0	yes	$2 \cdot 10^{-14}, \ldots, 6 \cdot 10^{-5}$ (first 12)
f_{DH}	60	56	float64	-0.4918	8	no	—
f_{DH}	150	80	float64	-1.8685	22	no	—
f_{DH}	400	110	float64	-3.6513	42	yes	—

*The script's evenness test uses tolerance 10^{-30} on the eigenvector, and at $\epsilon_N \sim 10^{-48}$ the next eigenvalue is only a factor $2.4 \cdot 10^3$ ($N = 30$) or $3.7 \cdot 10^3$ ($N = 40$) larger. The "no" is a statement about the computed vector at that tolerance, not evidence against CCM's assumption. Zeros are compared with the double-precision list `num/l48/zeros_T1000.json`, so 10^{-14} is the floor. For the first ghost of f_{DH} ($\beta = 0.80852$, $\gamma = 85.69935$), $2\pi N/L$ exceeds γ first at $\lambda^2 = 60$ (85.94).

Proposition 5.1 (the negativity at $\lambda^2 = 60$ is the ghosts'; [NUM])**.** *For the minimizing eigenvector x of $\tau = QW^{56}$ (f_{DH}, $\lambda^2 = 60$), $x^T \tau x = -0.491828$ (explicit side). The zero side gives on-line zeros $+0.216531$, off-line zeros (all 28 ghost quadruples with $\gamma < 1000$) -0.708388, and tail estimate $+8.8 \cdot 10^{-5}$, total -0.491769, a relative difference of $1.2 \cdot 10^{-4}$. At $\lambda^2 = 8$, $N = 8$ the same split gives $1.30697 \cdot 10^{-5}$ (explicit) against $1.21228 \cdot 10^{-5} + 3.97 \cdot 10^{-7} + 5.5 \cdot 10^{-7}$: the ghosts contribute positively and below 3.1%. For ζ ($\lambda^2 = 8$) and f_{DH} ($\lambda^2 = 20$) in float64 both sides are at the 10^{-15} round-off level, so no split is meaningful.*

Theorem 5.2 (what the window test can and cannot see; [RESTATEMENT] of Weil/Bombieri and of Lemma 22)**.** *(a) $QW_\lambda \succeq 0$ for all λ iff RH (Weil;* [1]*, [BIB]). A negative eigenvalue of QW_λ^N certifies a negative direction of QW_λ (min-max; Lemma 22 Level 2). (b) Thm. 1.1 of* [5] *applies to $QW_\lambda^N - \epsilon_N$ whatever the sign of ϵ_N, so the ground-state zeros are real for a fake too (Lemma 22). The data confirm this for a genuine fake. Up to $\lambda^2 = 20$, $\hat{\xi}$ reproduces the* on-line zeros of f_{DH} *with the same accuracy as for ζ, while f_{DH} violates RH. Below the ghost height the CCM construction is blind to RH, and only the sign of ϵ_N carries information.*

Remark 5.3 (relation to Lemmas 15, 24, 51, 53; [ANALOGY]/[RESTATEMENT])**.** (i) Lemma 15's signatures and Lemma 24's Thm. 3.2 are interval-arithmetic certificates (Arb). Nothing in this section is certified. The ζ windows are mp computations at dps ≤ 60, and at $\lambda^2 \geq 60$ float64 cannot resolve eigenvalues below $\sim 10^{-15}\|\tau\|$. (ii) In vacuum language $QW_\lambda \succeq 0$ is a positive vacuum on the window (Lemma 15: Lorentzian index ≤ 1 after adding the pole plane). f_{DH} has it on small windows and loses it once the window resolves its ghosts. This is the finite-truncation counterpart of Lemma 51's escape, with the window size playing the role of the path parameter. (iii) The counts $8, 22, 42$ are lower bounds for the Pontryagin index of QW_λ for f_{DH}, which is infinite because f_{DH} has infinitely many off-line zeros (Lemma 53, "no charge"). We make no claim that these counts equal a number of ghosts below a height.

6. Answer to (S4)

candidate vacuum	positive?	invariant under	fate in truncation / as S grows
cyclic vector ξ_S (Gaussian $\times m_S$)	yes (a vector in a Hilbert space)	Fourier F_S (rotation); *not* scaling (Thm. 2.1)	almost invariant, weakly $\to 0$; overlap with $\xi_\infty \to 0$; $\mathfrak{sl}_2$ lost; gradings drift
$\operatorname{sign} W_\lambda$	by construction	Fourier; not scaling	needs a boundary condition (limit circle, [NUM]); rank-one ambiguity; naive truncations do not converge
$QW_\lambda^N \succeq 0$	ζ: yes (mp, $\lambda^2 \leq 13$); f_{DH}: up to $\lambda^2 = 20$	window symmetry $u \mapsto u^{-1}$	f_{DH} loses it at $\lambda^2 = 60$, by the ghosts; for ζ this is RH ([RESTATEMENT])

So the CCM model satisfies (S4)'s "local vacua" only in the weak sense that each finite S has a canonical Fourier-invariant vacuum. These local vacua do not converge as S grows: the norm goes to infinity, the normalized vectors converge weakly to 0, and the gradings drift. None of them is invariant under the arithmetic flow. The one positivity that is actually tied to the zeros is the window Weil form, which is axiom G5 itself.

7. Corrections

- **Premise "a Frobenius- or scaling-invariant positive vacuum might survive in finite truncations".** In the CCM metaplectic picture this is impossible for structural reasons. Scaling

is hyperbolic on the phase plane at every finite S (Thm. 2.1). The only invariance the CCM vacuum has is under Fourier, a rotation.

- **Script labels (`vacuum2.py`).** Row "X" records the measure *before* the prime X is multiplied in, so it is $S = \{p < X\}$. The first row "$X = 2$" is the archimedean measure ($\mathrm{Var} = 0.5 = a_0^2$), and the last row is written twice.
- **Script heuristic (`vacuum.py`, "`kakutani_heur`").** $\prod(1-1/p)^{1/2}$ does not describe the overlap decay (0.238 vs. $4.9\cdot10^{-15}$ at 19997), and Kakutani's theorem concerns product measures, which these vacua are not.
- **Script field `ghosts_below` (`weil.py`).** It lists f_{DH}'s ghosts also in the ζ rows. That is cosmetic; ζ has no ghosts below $T = 1000$.
- **Script flag `even_simple` in float64 runs.** It uses a 10^{-8} test on nearly degenerate eigenvectors and is meaningless when ϵ_N is at round-off. The float64 "negative eigenvalues" of ζ ($27, 44, 73$, all $\approx -3\cdot10^{-15}$) are round-off, not negativity.
- **Script `prolate.py`, two-point exponent `Pexp`.** It is unreliable for oscillating solutions (-0.075, -2.09, -1.24, $+0.37$, -2.38, -0.001). The tail test of Prop. 4.1 replaces it. The dense script ran > 40 min with its output piped to `tail`. It was killed and re-implemented with tridiagonal solvers (same tests and parameters). Since the dense run never wrote output, no field-by-field comparison with it was possible.
- **Lemma 50, §7 (S4) wording "local vacua J_p".** In CCM the local data enter as multipliers $L_p(\frac{1}{2} - is)$ on one archimedean vacuum, not as separate local vacua. A Shale–Stinespring comparison "$J - J_p$ HS" has no direct meaning here. Prop. 2.4 is the closest substitute.

8. Prior art; what is new and what is restatement

Prior art (verified). CCM24 [4]: semilocal cyclic pairs, metaplectic $\mathfrak{sl}_2$ uniqueness (Thm. 5.2), prolate Jacobi matrices; the general-S coefficients and the $\mathrm{Mp}(\mathbb{A}_S)$ candidate are deferred. CCM-ZST [5]: truncated Weil matrices and real ground-state zeros, with real-zero mechanism from [8]. CM22 [6]: UV prolate spectrum; a unique self-adjoint extension singled out by commutation. CC [3]: tiny eigenvalues of QW_λ. Connes 1999 [7]: semilocal trace formula. Davenport–Heilbronn [9, 10], zeros [2]. Weil representation [17], Shale [12], Shale–Stinespring [15]. Limit-point/limit-circle theory [13, 16]. Classical prolate functions [14]. Weil positivity: Weil, Bombieri [1], Yoshida [18] (small support).

Restatement / classical: Thm. 2.1(c) (Riemann–Lebesgue); Thm. 5.2 (Weil, Bombieri, Lemma 22); the consequences of the limit-circle case [13]. **New in TAA (elementary, not found stated in this form in the sources read):** Thm. 2.1(a)–(b) as a vacuum/Siegel statement, with the fidelity equal to Lemma 51's at $d = 2t$; Prop. 2.3 (closed forms and bounds); Prop. 2.4; Prop. 3.2. **New numerics:** semilocal Jacobi coefficients and δ_n (Prop. 3.1); the variance law $1/K_X$; the limit-circle evidence, the count $(4/\sqrt{\pi})\lambda\sqrt{M}$, and the extension scan for $W_\lambda^{\pm}$; QW_λ^N for f_{DH} with the on-line/ghost split (Prop. 5.1).

9. Questions for J.W.

Question 9.1. Adopt the sentence "*in the CCM metaplectic model the vacuum is invariant under the rotation (Fourier) and moved by the boost (scaling); an invariant positive vacuum for scaling is excluded for structural reasons, independent of the zeros*" (Thm. 2.1, Rem. 2.2) as TAA language? It sharpens what (S3) must mean. The flow has to act on a space where it can be elliptic, an H^1 or a cokernel, not on L^2 of functions.

Question 9.2. Should (S4) be reworded? In CCM the local data are multipliers on one vacuum, not separate local vacua (§7). Proposal: (S4′) "the semilocal vacua $\hat{\xi}_S$ have a nonzero weak limit". Prop. 2.4 shows (S4′) fails.

Question 9.3. Prop. 4.1 gives numerical evidence that CCM's Jacobi matrices $W_\lambda^\pm$ are limit circle for $\lambda > 0$ (limit point at $\lambda = 0$). Is a proof worth a short follow-up lemma (continuum model: Bessel J_1, Y_1 at argument $4\lambda\sqrt{\pi x}$), together with the conjecture that all extensions give HS-equivalent polarizations?

Question 9.4. Prop. 5.1 attributes f_{DH}'s first window negativity to its off-line zeros collectively. Do you want the split *per ghost* (the 85.70 quadruple vs. the rest) and the first-negativity window located between $\lambda^2 = 20$ and 60 in multiprecision? This is cheap and would calibrate Lemma 54's drift detector against a genuine fake.

Question 9.5. Volume 3 is submitted. Should this lemma go into Volume 4's "Hilbert versus Krein" material as the bosonic counterpart of Lemma 51, or remain a working note? It adds no lever on RH.

Files and status of the inherited outputs

`jacobi.json` complete (14 runs, $n < 260$). `vacuum.json`, `vacuum2.json` complete. `weil_extra.json` (2 runs), `weil_hp.json` (5 runs), `weil_scan.json` complete (10 runs; the process writing it finished at ζ, $\lambda^2 = 400$, after the hand-over). `weil_decomp.json` (2 runs) complete; `weil_decomp2.json` new. `prolate.json` never written (dense job killed); replaced by `prolate_tri.json`. New: `prolate_ext.json`, `checks.json`.

Status

Proved ([THM]): Thm. 2.1, Prop. 2.3, Prop. 2.4 (given the variance limit), Prop. 3.2, the logic of Prop. 3.1. **Numerical ([NUM]):** the tables of §§2–5; Props. 3.1, 4.1, 4.2, 4.3, 5.1. **Heuristic ([HEUR]):** variance law $1/K_X$; continuum Bessel model; count $(4/\sqrt{\pi})\lambda\sqrt{M}$. **Cited ([CITED]):** CCM24 Thm. 2.1, Def. 2.2 remark, Props. 2.1, 3.3, 3.5, 4.1, 4.2, (47), Thm. 5.2, the deferrals; CCM-ZST (5.1), Lemma 5.1, Def. 5.3, Thm. 1.1, (3.15), §8; CM22 Thm. 2.6 (via Lemma 22). **Restatement ([RESTATEMENT]):** Thm. 5.2. **Conjectural ([CONJ]):** HS-equivalence of all extension polarizations. **Open ([OPEN]):** limit-circle proof for $W_\lambda^\pm$; the variance limit $\mathrm{Var}\,\mu_X \to 0$ as a theorem. **Not verified, not cited:** CCM's announced forthcoming paper (general-S coefficients; $\mathrm{Mp}(\mathbb{A}_S)$ operator); any published limit-circle statement for $W_\lambda^\pm$.

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TAA LEMMA 53: BOSONS, FERMIONS, AND THE HARDY-SPACE VACUUM. METAPLECTIC VERSUS SPIN, SIEGEL POSITIVITY, THE SEGAL–WILSON GRASSMANNIAN, BEURLING CYCLICITY, AND THE GHOST CHARGE

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. **Statement of intellectual honesty.** RH is not proved here, and nothing below brings it closer. The one RH-equivalence (Thm. 4.3(d)) is a *restatement* of the Nyman–Beurling criterion in its Hardy-space (outer-function) form, which is in the literature (§7). Tags: [THM] proved here or classical with proof given; [CITED] checked at the source (`lit/lemma53/verified_refs.md`); [BIB] bibliographic record verified, content standard; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma53_charge.py`, data `num/l53/charge.json`); [CONJ] conjecture; [OPEN] open. Lemmas 45–50 and `notes/skin_effect_prior.md` are cited by number and not redone.

Question (J.W., [t197u]). "What's a bosonic metaplectic twin of a spinor? I think the positivity condition might be similar, or perhaps a pure mathematical translation of how fermionic spinors emerge from bosons." TAA language (Lemma 50, Thesis 5.4): *missing polarization = missing invariant positive vacuum.*

Answer in brief. (1) *The twin* (§2). The oscillator (metaplectic) representation of $\mathrm{Mp}(V) \to \mathrm{Sp}(V)$ on $\mathrm{Sym}^\bullet V^{1,0}$ is the bosonic twin of the spin representation of $\mathrm{Spin}(V) \to SO(V)$ on $\Lambda^\bullet V^{1,0}$. Both vacua are complex structures J. The difference is *where positivity sits.* For bosons it is a condition on J: $\omega(v, Jv) > 0$, which is exactly Siegel's upper half space, i.e. Riemann's bilinear relations $\Omega = \Omega^{\mathsf{T}}$, $\Im\Omega > 0$ (Prop. 2.1). For fermions it is a condition on the one-particle form g, and then every orthogonal J is admissible. The "2-out-of-3" rule $g = \omega(\cdot, J\cdot)$ turns one into the other (Prop. 2.2). This is the precise sense in which fermionic positivity *is* a translation of the bosonic Siegel condition. Zero-point weights are $+\frac{1}{2}$ (boson) and $-\frac{1}{2}$ (fermion) per mode and cancel (Prop. 2.3). (2) *Segal–Wilson* (§3). Verified at the source: the vacuum is the Hardy space $H_+ = \overline{\mathrm{span}}\{z^k\}_{k\geq 0}$; the charge is the virtual dimension = the Fredholm index of $W \to H_+$; the fermionic Fock space has the Plücker coordinates as orthonormal basis, and charge-0 Plücker coordinates restrict to Schur functions (boson–fermion). Krichever's subspace has virtual dimension $\chi(X, \mathcal{L}) - 1 = \deg \mathcal{L} - g$ (an Euler characteristic, §3), and τ is a theta function. (3) *The TAA link* (§4). For $F_\zeta = \zeta(s)(s-1)/s$ on $\mathbb{H}_{1/2} = \{\Re s > \frac{1}{2}\}$, the closed invariant subspace W_ζ generated by $F_\zeta/(s+\frac{1}{2})^2$ is exactly $B_\zeta H^2$ (Lemma 48). Transported to the disc it lies in H_+. It is a point of Gr iff there are finitely many ghosts, and then its *charge is $-$#ghosts* (Thm. 4.3). RH $\iff W_\zeta = H_+ \iff$ charge 0 $\iff F_\zeta$ cyclic: a restatement of Nyman–Beurling (Kapustin's outer form, Burnol's $B(s)H^2$). Each ghost is a *normalizable hole* $\psi(a)|0\rangle$ with $\|\psi(a)|0\rangle\|^2 = (1-|a|^2)^{-1}$, which diverges as the ghost reaches the line (Thm. 4.7). Multiplication by the Blaschke factor is Shale–Stinespring implementable iff the ghosts are finite in number, with $\|[\varepsilon, M_B]\|^2_{HS} \propto$ #ghosts (Thm. 4.6). *Windings reconciled* (Thm. 4.8): the skin-effect Toeplitz winding is the index of the FE-symmetric symbol $h - E$ off its spectrum and is 0 for every curve and fake, because the FE puts exactly g zeros inside and the delay u^{-g} removes g; Lemma 48's phase winding is the winding of the inner factor, $= -$charge of the one-sided point; at $E = 0$ the FE-symmetric point has charge $g - k$. For curves over $\mathbb{F}_q$ everything is exact. Numerically, $\mathrm{dist}^2(1, W_\zeta) = 1 - M(\mathcal{L})^{-2}$ to 10^{-15} for fakes, and $\approx n_{\mathrm{circle}}/N \to 0$ for real curves. (4) *Fermions from bosons do not give positivity* (§5). The boson–fermion correspondence is an isometry between two positive Hilbert spaces (Plücker $\leftrightarrow$ Schur, orthonormal to orthonormal), so it *transports* positivity and never creates it. The bosonic positivity $\langle \alpha_n \alpha_{-n} \rangle = n > 0$ is the Hilbert-transform J of the loop algebra, i.e. the choice H_+ rather than H_-. In the Hardy picture positivity is put in at the start, so ghosts cannot be negative-norm states; they appear as *charge.* In the cohomological picture of Lemmas 45–50 they appear as negative norm. The two are exchanged by $B \leftrightarrow 1/B$ (Prop. 4.10).

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1. Conventions, and what TAA already has

$s = \sigma + it$, $\mathbb{H}_{1/2} = \{\Re s > \frac{1}{2}\}$, $w = s - \frac{1}{2}$. Ghosts are zeros ρ of ζ (or of a comparison function) with $\Re\rho > \frac{1}{2}$, *both* ρ and $\bar\rho$ counted, with multiplicity; k denotes their number (finite or ∞). F_ζ, B_ζ, $\mathfrak{G}$ are as in Lemma 48 Thm. 3.1 and Cor. 3.2. $\mathbb{D}$ is the unit disc, $\mathbb{T} = \partial\mathbb{D}$, $H = L^2(\mathbb{T})$, $H_+ = H^2(\mathbb{D})$ (boundary values), $H_- = H_+^\perp$, $\varepsilon = P_+ - P_-$. For a curve or fake over $\mathbb{F}_q$: $\mathcal{L}(u) = P(u/\sqrt{q}) = \prod_i(1-\beta_i u)$ (Lemma 48 §6), $|u| < 1 \iff s \in \mathbb{H}_{1/2}$, and the ghosts are the zeros $1/\beta_i$ of $\mathcal{L}$ inside $\mathbb{D}$; $M(\mathcal{L})$ is the Mahler measure. The Cayley map is $z = (w-1)/(w+1)$, so $s = \frac{3}{2} \mapsto z = 0$, the line $\Re s = \frac{1}{2}$ maps to $\mathbb{T}$, and $s = \infty$ maps to $z = 1$. The map $(Uf)(z) = \sqrt{2}\, f(w(z))/(1-z)$ is unitary $H^2(\mathbb{H}_{1/2}) \to H^2(\mathbb{D})$ for the normalized boundary measures dt and $d\theta$ [THM] ($dw = 2\,dz/(1-z)^2$). Because $1/(1-z)$ is outer, U maps inner factors to inner factors by composition and outer factors to outer factors [12] [BIB].

Overlap table (checked by reading the cited lemmas; nothing here is redone):

source	what it already contains
L45 Thm. 6.4	Ghost = negative-norm two-fermion state; Weil $\iff$ an invariant h positive on Λ^2.
L46 Thm. 6.1	Abstract (F, V) Krein form of signature $(D_{\text{on}} + k, k)$, k = # distinct roots with $\lvert\alpha\rvert > \sqrt{q}$.
L47 Thm. 2.3, 2.5	No global Dirac reflection on $\ell^2(\mathbb{Q}_S)$ for $\lvert S\rvert \geq 2$; one-prime semi-infinite wedge and boson–fermion.
L48 Thm. 3.1, 6.1, 9.1	$F_\zeta = B_\zeta O_\zeta$ with no singular factor and no delay; RH $\iff$ F_ζ outer; finite-field Bode: gain defect = $\log M$, phase winding = k, FE gauge u^g is a true delay; skin dictionary.
L50 Thm. 4.4, 5.1, Prop. 5.2, Cor. 5.3	Invariant positive vacuum $\iff$ Weil; Shale–Stinespring (via [21]); Fock spaces over Krein spaces are Krein; spinors do not create positivity.
skin note Thm. 3.1	FE $\Rightarrow$ Hermitian symbol h, winding of $h - E$ is 0 for all $E \notin h(\mathbb{T})$; one-sided winding = k when the gap is open; ghost invariant = Mahler measure.

New here: the Siegel/2-out-of-3 form of the boson–fermion twin; the Segal–Wilson verification; the ghost *charge* of W_ζ and the Shale–Stinespring and Plücker forms of it; the reconciliation of the three windings; the $B \leftrightarrow 1/B$ bridge between charge and Krein index.

2. The bosonic twin of a spinor: metaplectic versus spin

Let $V = \mathbb{R}^{2g}$ with coordinates (a, b), symplectic form $\omega(x, y) = x^{\mathsf{T}}\Omega_0 y$ where $\Omega_0 = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, and let $V_{\mathbb{C}} = V \otimes \mathbb{C}$.

Proposition 2.1 (Siegel positivity is positivity of the bosonic vacuum; [THM]; [NUM]). *Let $\Omega = X + iY$ be a complex symmetric $g \times g$ matrix with $\det Y \neq 0$, and let $L_\Omega = \{(\Omega c, c) : c \in \mathbb{C}^g\} \subset V_{\mathbb{C}}$.*

- *(a) L_Ω is Lagrangian, $L_\Omega \cap \overline{L_\Omega} = 0$, and $J := +i$ on L_Ω, $-i$ on $\overline{L_\Omega}$ is a real symplectic complex structure: $J^2 = -1$, $J \in \mathrm{Sp}(V, \omega)$.*
- *(b) For $v = x + \bar{x}$ with $x = (\Omega c, c)$: $g_J(v, v) := \omega(v, Jv) = 4\, c^{\mathsf{T}} Y \bar{c}$. Hence the signature of g_J is twice that of Y, and $g_J > 0 \iff Y > 0 \iff \Omega$ lies in Siegel's upper half space $\mathfrak{h}_g$ [32].*
- *(c) (*The twin of a pure spinor.*) In the Schrödinger model on $L^2(\mathbb{R}^g)$, the vacuum annihilated by the J-antilinear combinations is the Gaussian $\psi_\Omega(x) = e^{i\pi x^{\mathsf{T}}\Omega x}$, and $\int |\psi_\Omega|^2 = \int e^{-2\pi x^{\mathsf{T}} Y x} dx < \infty \iff Y > 0$. Summing the same Gaussian over $\mathbb{Z}^g$ gives Riemann's $\theta(z, \Omega) = \sum_n e^{i\pi n^{\mathsf{T}}\Omega n + 2\pi i n^{\mathsf{T}} z}$, which converges for exactly the same reason.*
- *(d) (*Riemann bilinear relations*; [BIB]* [13].*) For a compact Riemann surface with a symplectic basis of H_1, the normalized period matrix satisfies $\Omega = \Omega^{\mathsf{T}}$, $\Im\Omega > 0$. Under (b) this is the statement that the Hodge complex structure on $H^1(X, \mathbb{R})$ is ω-positive (Hodge–Riemann for curves).*

Proof. (a) $\omega((\Omega c, c), (\Omega c', c')) = c^{\mathsf{T}}\Omega^{\mathsf{T}} c' - c^{\mathsf{T}}\Omega c' = 0$ by symmetry. $L \cap \bar{L} = 0$ iff Y is invertible, since $(\Omega c, c) = (\bar{\Omega} c', c')$ forces $c = c'$ and $Yc = 0$. J commutes with conjugation, so it is real. It preserves ω because both eigenspaces are Lagrangian and $\omega(x, \bar{y})$ pairs L with $\bar{L}$, where J acts by $i \cdot (-i) = 1$. (b) $\omega(v, Jv) = \omega(x + \bar{x}, ix - i\bar{x}) = -2i\,\omega(x, \bar{x})$ and $\omega(x, \bar{x}) = c^{\mathsf{T}}(\Omega - \bar{\Omega})\bar{c} = 2i\, c^{\mathsf{T}} Y \bar{c}$. As a real form on $\mathbb{C}^g = \mathbb{R}^{2g}$, $c^{\mathsf{T}} Y \bar{c}$ has signature $2\,\mathrm{sig}(Y)$. (c) is a Gaussian integral. (d) is classical. □

[NUM] 120 random Ω ($g = 1, 2, 3$, half with indefinite Y): $J^2 = -1$ and $J^{\mathsf{T}}\Omega_0 J = \Omega_0$ to 6×10^{-13}, and $\mathrm{sig}(g_J) = 2\,\mathrm{sig}(Y)$ in 120 of 120 cases.

Proposition 2.2 (2-out-of-3: where positivity sits; [THM]; [NUM]). *On a real vector space with a complex structure J, a symplectic form ω and a symmetric form g, any two compatible structures determine the third by $g(x, y) = \omega(x, Jy)$, and $U(n) = O(2n) \cap \mathrm{Sp}(2n, \mathbb{R}) = O(2n) \cap GL(n, \mathbb{C}) = \mathrm{Sp}(2n, \mathbb{R}) \cap GL(n, \mathbb{C})$.*

- *(a)* Bosons (CCR, Mp → Sp). *ω is given. The vacua are the compatible J with $\omega(\cdot, J\cdot) > 0$, i.e. $\mathrm{Sp}(2n, \mathbb{R})/U(n) \cong \mathfrak{h}_n$, which is non-compact. Positivity is a* condition on the vacuum*: an open condition that cuts the compatible J down to one signature component.*
- *(b)* Fermions (CAR, Spin → SO). *$g > 0$ is given. The vacua are all orthogonal J, i.e. $O(2n)/U(n)$, which is compact. Positivity sits in the* one-particle space*, and every vacuum is admissible.*
- *(c)* Translation. *The dictionary $g = \omega(\cdot, J\cdot)$ turns the bosonic positivity condition on J into the fermionic one-particle positivity. This is the precise sense of J.W.'s "pure mathematical translation". It is a translation of the* data*, not a source of positivity: one of the two pairs (ω, J) or (g, J) must already be positive.*

[NUM] A random $R \in U(3) \subset GL(6, \mathbb{R})$ is orthogonal, symplectic, and commutes with J_0 to 7×10^{-16}.

Proposition 2.3 (zero-point weights; [THM]). *For a mode of frequency $\epsilon > 0$, the symmetric (Weyl) quantization of $\frac{\epsilon}{2}(p^2 + x^2)$ is $\epsilon(a^\dagger a + \frac{1}{2})$ and the antisymmetric quantization of $\frac{\epsilon}{2}[b^\dagger, b]$ is $\epsilon(b^\dagger b - \frac{1}{2})$. In the super-Fock space $\mathrm{Sym}^\bullet \otimes \Lambda^\bullet$ of a mode and its fermionic partner, the vacuum energies cancel. This matches the $+\frac{1}{2}$ weight of Lemma 50 Prop. 3.5 (oscillator) and Rem. 3.6 (Weyl versus Clifford).*

Remark 2.4 (Shale and Shale–Stinespring; [CITED] via Lemma 50 Thm. 5.1 and [21]; [BIB] [30, 31])**.** In infinite dimensions the twin persists only after restriction. A symplectic map is implemented in the bosonic Fock space of J iff its J-antilinear part is Hilbert–Schmidt (Shale), and an orthogonal map is implemented in the fermionic one under the same condition (Shale–Stinespring). Two vacua J, J' give equivalent representations iff $J - J'$ is Hilbert–Schmidt. The resulting groups are $\mathrm{Sp}_{\mathrm{res}}$ (bosonic) and O_{res}, U_{res} (fermionic). The fermionic U_{res} and its Grassmannian are the setting of §3.

Answer to "what is the bosonic metaplectic twin of a spinor" [THM]. A spinor is a vector in $\Lambda^\bullet L$ for a maximal isotropic L in $(V_{\mathbb{C}}, g)$; the pure spinors are the vacua, and there is one for every orthogonal J. Its bosonic twin is a vector in $\mathrm{Sym}^\bullet L$ for a Lagrangian L in $(V_{\mathbb{C}}, \omega)$, realized in $L^2(\mathbb{R}^g)$ by the oscillator representation of Mp. The twin of a pure spinor is the Gaussian ψ_Ω, and it exists (is normalizable) *iff* $\Omega \in \mathfrak{h}_g$. The fermionic vacuum always exists, while the bosonic one exists only under Siegel positivity. That asymmetry is J.W.'s "positivity condition". On H^1 of a curve it is the Riemann bilinear relations, and Lemma 50 Thm. 4.4 showed that what fails over $\mathbb{Q}$ is *invariance* of such a vacuum, not its existence.

3. The boson–fermion correspondence and the Segal–Wilson Grassmannian

Everything in this section is [CITED] from [29] (full text read; section numbers refer to it) unless tagged otherwise.

Definition 3.1 ([29] §1–§2)**.** $H = L^2(S^1)$, $H_+ =$ closed span of $\{z^k\}_{k\geq 0}$ (boundary values of holomorphic functions in $|z| < 1$), $H_- =$ closed span of $\{z^k\}_{k<0}$. $\mathrm{Gr}(H)$ is the set of closed $W \subset H$ such that $\mathrm{pr}_+ : W \to H_+$ is Fredholm and $\mathrm{pr}_- : W \to H_-$ is compact. The *virtual dimension* of W is $\mathrm{ind}(\mathrm{pr}_+)$; the connected components of Gr are indexed by it. The vacuum is $W = H_+$, of virtual dimension 0. For a loop $\gamma : S^1 \to \mathbb{C}^\times$ acting by multiplication, "the winding number of γ is minus the virtual dimension of $W_\gamma = \gamma H_+$". In §10 the Hilbert–Schmidt version (pr_- Hilbert–Schmidt) is used.

We call $\mathrm{vdim}\, W$ the *charge* of W (fermion number relative to the vacuum). By the quoted identity, the charge of γH_+ is $-\mathrm{wind}\,\gamma$.

Theorem 3.2 (fermionic Fock space and boson–fermion; [29] §10, Prop. 10.1)**.** *The Plücker embedding takes the Hilbert–Schmidt Grassmannian into the projectivization of $\mathcal{H} = \bigoplus_k \mathcal{H}_k \cong \Lambda(H_+ \oplus \bar{H}_-)$ (the fermionic Fock space), which has an orthonormal basis of Plücker coordinates indexed by the admissible sets $S \subset \mathbb{Z}$ (equivalently, charge and a partition). The CAR field operators act on it. The Plücker coordinates restricted to* Gr_0 *are the Schur functions, and $\mathcal{H}_0$ is the completion of the ring of symmetric polynomials with its standard inner product. This is the boson–fermion correspondence ([BIB]* [10, 14, 11, 16, 15, 27]*).*

Theorem 3.3 (Krichever's construction; [29] Prop. 6.1 and Thm. 9.11; [BIB] [18])**.** *For a compact Riemann surface X of genus g with a point x_∞, a local parameter, and a line bundle $\mathcal{L}$ with trivialization near x_∞, the space W of boundary values of sections of $\mathcal{L}$ holomorphic off x_∞ lies in* Gr*. The kernel and cokernel of* $\mathrm{pr}_+ : W \to H_+$ *are $H^0(X, \mathcal{L}(-x_\infty))$ and $H^1(X, \mathcal{L}(-x_\infty))$, so*

$$\mathrm{vdim}\, W = \chi(\mathcal{L}(-x_\infty)) = \deg \mathcal{L} - g.$$

(The text is OCR-garbled at this point. The displayed formula is Riemann–Roch applied to the quoted kernel and cokernel, [THM].) The τ-function of W is, up to an exponential of a quadratic and a linear form, Riemann's θ function of X (Thm. 9.11).

Remark 3.4 (Siegel positivity reappears; [THM] from Prop. 2.1(c))**.** In Thm. 3.3 the KP flows move $\mathcal{L}$ along the Jacobian, and $\tau = \theta$ converges because $\Im\Omega > 0$. The Jacobian (a bosonic Gaussian datum) and the fermionic charge-0 sector meet in the *same* positivity: Riemann's bilinear relations.

Proposition 3.5 (the loop-algebra cocycle; [THM]; [NUM]; [BIB] [28, 26])**.** *On real loops $f(\theta) = \sum f_n e^{in\theta}$ with $f_0 = 0$, let $\omega(f,h) = \frac{1}{2\pi}\int_0^{2\pi} f\, h'\, d\theta$ and let J be the Hilbert transform ($Je^{in\theta} = -i\,\mathrm{sgn}(n)e^{in\theta}$). Then $\omega(f, Jf) = \sum_n |n||f_n|^2 > 0$. The choice H_- in place of H_+ replaces J by $-J$ and flips the sign. [NUM]:* 218.2692 *on both sides for a random loop.*

So the bosonic current positivity $\langle \alpha_n \alpha_{-n}\rangle = n$ is Siegel positivity (Prop. 2.1) for the infinite-dimensional symplectic space of loops, and its J is the fermionic choice of H_+.

4. The TAA core: ghosts as charge

4.1. **The zeta subspace.** Let $G_\zeta(s) := F_\zeta(s)/(s+\frac12)^2$. Let $U : H^2(\mathbb{H}_{1/2}) \to H^2(\mathbb{D}) = H_+$ be the Cayley unitary of §1. Write $\tilde B := B_\zeta \circ s(z)$, a Blaschke product on $\mathbb{D}$ with zeros $z_\rho = (\rho - \frac32)/(\rho + \frac12)$.

Lemma 4.1 ([THM])**.** *$G_\zeta \in H^2(\mathbb{H}_{1/2})$, its inner factor is exactly B_ζ (no singular factor, constant 1), and $\zeta(\sigma) < 0$ for $0 < \sigma < 1$. In particular every ghost is non-real, the ghosts come in conjugate pairs, and k is even or infinite.*

Proof. Lemma 48 Thm. 3.1(a) gives $|F_\zeta| \le 6|s+\frac12|$ on $\overline{\mathbb{H}_{1/2}}$. So $G_\zeta = [F_\zeta/(w+1)]\cdot(w+1)^{-1}$, with the first factor in H^∞ and the second in $H^2(\Re w > 0)$; hence $G_\zeta \in H^2$. The factor $(w+1)^{-2}$ is outer, so the inner factor of G_ζ is that of F_ζ, which is B_ζ by Lemma 48 Thm. 3.1(b). For $0 < \sigma < 1$, the formula in the proof of Lemma 48 Thm. 3.1(a) gives $\zeta(\sigma) = \frac{\sigma}{\sigma-1} - \sigma\int_1^\infty \{x\} x^{-\sigma-1}dx < 0$, since both terms are negative. Coefficients are real, so zeros are symmetric under conjugation. □

Definition 4.2. $W_\zeta :=$ the smallest closed subspace of H_+ that contains UG_ζ and is invariant under multiplication by z.

Theorem 4.3 (the ghost charge; [THM]; (d) is [RESTATEMENT])**.** *(a) $W_\zeta = \tilde B H_+$. Equivalently, $U^{-1}W_\zeta = B_\zeta H^2(\mathbb{H}_{1/2})$, the Lax subspace generated by G_ζ under the semigroup $e^{-\lambda w}$, $\lambda \ge 0$. By* [8] *this is the Mellin image of the Nyman–Beurling space ([CITED]: "the Mellin transform of NB exactly coincides with $B(s)H^2$").*

(b) $\mathrm{pr}_-|_{W_\zeta} = 0$*, and* $\mathrm{pr}_+|_{W_\zeta}$ *is injective with closed range and cokernel $K_{\tilde B} = H_+ \ominus \tilde B H_+$, of dimension k.*

(c) $W_\zeta \in \mathrm{Gr}$ (in either the compact or the Hilbert–Schmidt version) iff $k < \infty$. In that case

$$\mathrm{ch}(W_\zeta) := \mathrm{vdim}\, W_\zeta = -k = -\#\{\textit{ghosts}\} = -\,\mathrm{wind}(\tilde B|_{\mathbb{T}}),$$

in agreement with the Segal–Wilson identity "winding = −virtual dimension" (Def. 3.1). By Lemma 4.1, $\mathrm{ch}(W_\zeta) \in \{0, -2, -4, \dots\}$ *whenever it is defined.*

(d) RH $\iff$ $W_\zeta = H_+$ $\iff$ $W_\zeta \in \mathrm{Gr}_0$ (charge 0) $\iff$ G_ζ is cyclic for z $\iff$ F_ζ is outer on $\mathbb{H}_{1/2}$.

(e) For the Davenport–Heilbronn function f (Lemma 48 Rem. 3.4), with $G_{\mathrm{DH}} := f/(s+\frac12)^2$, the subspace W_{DH} has infinitely many ghosts ([CITED] [25]*, Introduction: f "has infinitely many zeros for $\sigma > 1$"). So $W_{\mathrm{DH}} \notin \mathrm{Gr}$: it has no charge.*

Proof. (a) Beurling's theorem [5] ([BIB]): $[h]_z = \theta_h H_+$ for $h \in H_+$ with inner factor θ_h. Here $\theta_{UG} = \tilde B$, because U multiplies by the outer function $\sqrt{2}/(1-z)$ and composes with the Cayley map. The half-plane statement is Lax's theorem [19] ([BIB]). (b) $\tilde B H_+ \subset H_+$ and multiplication by $\tilde B$ is an isometry. $\dim K_\theta < \infty$ iff θ is a finite Blaschke product, and then $\dim K_\theta =$ the number of zeros [22, 12] ([BIB]; also Thm. 4.6 below). (c) follows from (b) and the definition. For a finite Blaschke product, wind = number of zeros. (d) $\tilde B \equiv 1 \iff k = 0$. "Outer $\iff$ cyclic" is Beurling. That RH is equivalent to F_ζ being outer is Lemma 48 Thm. 3.1(c). (e) Lemma 48 Rem. 3.4 gives polynomial growth with no singular factor and no delay, so the same argument applies. $\{\sigma > 1\} \subset \mathbb{H}_{1/2}$. □

Remark 4.4 (status)**.** (d) is the Nyman–Beurling criterion in its Hardy-space form. That form is in the literature: Burnol's "NB $= B(s)H^2$" [8], and Kapustin's "RH $\iff$ $(s-1)\zeta(s)/s^2$ is outer in $H^2(\Re s > \frac{1}{2})$" ([CITED] [17], abstract). What §4 adds is the Segal–Wilson *reading*: the ghosts are exactly the obstruction to W_ζ being the vacuum, and they enter as *negative charge*, not negative norm.

Corollary 4.5 (distance = gain defect; [THM]; RH clause [RESTATEMENT])**.** *Let $\hat{k}_{s_0}$ be the normalized reproducing kernel of $H^2(\mathbb{H}_{1/2})$ at s_0. Then*

$$\operatorname{dist}^2(\hat{k}_{s_0}, B_\zeta H^2) = 1 - |B_\zeta(s_0)|^2 = 1 - e^{-2\mathfrak{G}(s_0)},$$

with $\mathfrak{G}$ the gain defect of Lemma 48 Cor. 3.2. At $s_0 = 1$ (where $k_1 \propto 1/s$) this is Burnol's Thm. 1.2 and Lemma 2.4 ([CITED] [8]): the projection of $1/s$ onto BH^2 is $\overline{B(1)}B(s)/s$.

Proof. For an inner θ, $P_{\theta H^2} k_{s_0} = \overline{\theta(s_0)}\,\theta k_{s_0}$. So the projected norm is $|\theta(s_0)|\,\|k_{s_0}\|$. □

[NUM] (data of Lemma 48, `num/148/bsy.json`). For ζ, $\mathfrak{G}(1) = -8.06 \times 10^{-8}$, which is 0 at the accuracy of the quadrature. For DH, the 28 ghosts with $0 < \gamma \le 1000$ (56 zeros counting conjugates) give $\mathfrak{G}(1) \ge 1.7448 \times 10^{-4}$, since every term is positive, while the quadrature gives 1.7428×10^{-4}. Hence $\operatorname{dist}^2(\hat{k}_1, W_{\mathrm{DH}}) \ge 3.489 \times 10^{-4}$.

4.2. **Shale–Stinespring form of the charge.**

Theorem 4.6 (Hilbert–Schmidt norm counts ghosts; [THM]; [NUM])**.** *Let θ be inner on $\mathbb{D}$, M_θ multiplication on $H = L^2(\mathbb{T})$, $\varepsilon = P_+ - P_-$, and $X = P_+ M_\theta P_-$.*

- *(a) $P_- M_\theta P_+ = 0$ and $XX^* = P_{K_\theta}$.*
- *(b) $\|X\|_{HS}^2 = \dim K_\theta = \sum_{n\ge1} n|\hat{\theta}(n)|^2 =: \mathcal{D}(\theta)$ (the Dirichlet integral). So $\|[\varepsilon, M_\theta]\|_{HS}^2 = 4 \dim K_\theta$.*
- *(c) Hence $M_\theta \in U_{\mathrm{res}}$, and is Shale–Stinespring implementable on $\Lambda(H_+ \oplus \bar{H}_-)$ (Lemma 50 Thm. 5.1(b) with $J = i\varepsilon$), iff θ is a finite Blaschke product. The implementer maps the vacuum $|H_+\rangle$ to $|\theta H_+\rangle$ in the charge $-k$ sector.*
- *(d) For ζ: $M_{\tilde{B}}$ is implementable iff there are finitely many ghosts; $\|[\varepsilon, M_{\tilde{B}}]\|_{HS}^2 = 4\,\#\text{ghosts}$; and under RH the implementer is the identity.*

Proof. (a) $\theta H_+ \subset H_+$. $XX^* = P_+ M_\theta (1 - P_+) M_{\bar\theta} P_+ = P_+ - T_\theta T_\theta^*$, and $T_\theta T_\theta^*$ is the projection onto θH_+ because T_θ is an isometry. (b) $\operatorname{Tr} XX^* = \dim K_\theta$. Also $\|X\|_{HS}^2 = \sum_{m\ge1} \|P_+(\theta z^{-m})\|^2 = \sum_{m\ge1}\sum_{n\ge m} |\hat{\theta}(n)|^2 = \sum_n n|\hat{\theta}(n)|^2$. The commutator $[\varepsilon, M_\theta] = -2X$ here, since the other off-diagonal block vanishes. (c) The $i\varepsilon$-antilinear part of M_θ is its off-diagonal part. $\dim K_\theta < \infty$ iff θ is a finite Blaschke product. (d) is (c) with Thm. 4.3. □

[NUM] For the five pure fakes of §4.4, the FFT (2^{15} points) gives $\mathcal{D}(B) = 1.000000, 1.000000, 1.000000, 2.000000, 2.000000 = k$.

4.3. **Ghost states are normalizable holes.** For $a \in \mathbb{D}$ let $\psi(a) := \sum_{n\ge0} a^n \psi_n$, where ψ_n removes the mode z^n from the Dirac sea. Equivalently, $\psi(a)$ is the contraction with the Szegő kernel $k_a = \sum \bar{a}^n z^n$.

Theorem 4.7 (Plücker vector of a ghost subspace; [THM]; [NUM])**.** *Let $a_1, \dots, a_k \in \mathbb{D}$ be distinct and $\theta = \prod_j \frac{z - a_j}{1 - \bar{a}_j z}$. Then:*

- *(a) $\theta H_+ = \{f \in H_+ : f(a_j) = 0\ \forall j\}$, and its Plücker vector is $|\theta H_+\rangle = \psi(a_1)\cdots\psi(a_k)|0\rangle$ (up to a phase). Its coefficient on the state with holes at $n_1 > \cdots > n_k$ is $\det[a_j^{n_i}] = \Delta(a)\, s_\lambda(a)$, with $\lambda_i = n_i - (k - i)$ (bialternant).*

*(b) (*Cauchy–Pick*)* $\big\|\psi(a_1)\cdots\psi(a_k)|0\rangle\big\|^2 = \det\left[\frac{1}{1-a_i\bar a_j}\right] = \frac{\prod_{i<j}|a_i-a_j|^2}{\prod_{i,j}(1-a_i\bar a_j)} = |\Delta(a)|^2\exp\Big(\sum_{n\ge1}\frac{|p_n(a)|^2}{n}\Big)$. *The last expression is the bosonic norm. Under Thm. 3.2 the Schur expansion* $\sum_\lambda s_\lambda(a)s_\lambda = \exp(\sum_n p_n(a)p_n/n)$ *is a coherent state of the Heisenberg algebra.*

(c) For one ghost the norm squared is $(1-|a|^2)^{-1}$. *It is finite for every ghost strictly inside and diverges as the ghost approaches the critical line (*$|a|\to1$*). A zero* on *the line produces no state at all: it is absorbed into the outer factor (Thm. 4.8(a)).*

Proof. (a) $f(a) = \langle f, k_a\rangle$, so $\theta H_+ = H_+ \ominus \operatorname{span}\{k_{a_j}\}$, and removing a vector from the sea is the contraction $\psi(a)$ (antilinear in k_a, hence linear in a). Expanding the wedge gives the minors $\det[a_j^{n_i}]$. (b) Cauchy–Binet: $\sum_{n_1>\cdots>n_k}|\det[a_j^{n_i}]|^2 = \det[\langle k_{a_j}, k_{a_i}\rangle]$. The closed form is Cauchy's determinant. For the bosonic form use $\sum_\lambda|s_\lambda(a)|^2 = \prod_{i,j}(1-a_i\bar a_j)^{-1}$ (Cauchy identity, [20], [BIB]) and $-\sum_{i,j}\log(1-a_i\bar a_j) = \sum_n|p_n(a)|^2/n$. □

[NUM] Cauchy determinant, fermionic sum over $n\le N$, and bosonic exponential agree to $\le 2\times10^{-14}$: 1.654701 (fake $a=8, q=13$), 1.100925 ($a=13$), 5.753443 ($g=2$ real pair), 2.237571 ($g=2$ quadruple), 0.434929 (three generic points).

4.4. **The finite-field case, exactly.** Let $\mathcal{L}(u) = 1 + c_1u + \cdots + c_{2g}u^{2g}$ be a normalized L-polynomial with real coefficients satisfying the functional equation $c_{2g-i} = c_i$ (curves and the fakes of Lemmas 46–50). Let k be the number of its zeros in $\mathbb{D}$ (the ghosts) and $n_\circ$ the number on $\mathbb{T}$. Put $h(u) := u^{-g}\mathcal{L}(u)$.

Theorem 4.8 (finite-field Grassmannian; [THM]; [NUM])**.** *(a)* $W_{\mathcal{L}} := [\mathcal{L}H_+] = B_{\mathcal{L}}H_+$, *where* $B_{\mathcal{L}}$ *is the Blaschke product of the* k *ghosts. Zeros on* $\mathbb{T}$ *are absorbed into the outer factor. So* $W_{\mathcal{L}}\in\mathrm{Gr}$ *with* $\mathrm{ch}(W_{\mathcal{L}}) = -k$. *Weil (RH for the curve)* $\iff W_{\mathcal{L}} = H_+$.

(b) $\operatorname{dist}^2(1, W_{\mathcal{L}}) = 1-|B_{\mathcal{L}}(0)|^2 = 1-M(\mathcal{L})^{-2}$, *where* M *is the Mahler measure (the ghost invariant of the skin note and of Lemma 48 Thm. 6.1).*

(c) If $n_\circ = 0$, *the Toeplitz operator* $T(\mathcal{L})$ *on* H_+ *is Fredholm of index* $-k = -\operatorname{wind}\mathcal{L}$. *Lemma 48's phase winding* k *equals* $\operatorname{wind}B_{\mathcal{L}} = -\mathrm{ch}(W_{\mathcal{L}})$.

*(d) (*FE-symmetric point.*)* h *is real on* $\mathbb{T}$ *and* $h(1/u) = h(u)$. *Then* $[hH_+] := u^{-g}W_{\mathcal{L}}\in\mathrm{Gr}$ *has charge* $g-k$. *For* $E\notin h(\mathbb{T})$, $[(h-E)H_+]$ *has charge* $0 = -\operatorname{wind}(h-E)$, *which is the skin-effect winding of the skin note Thm. 3.1.*

(e) If $0\notin h(\mathbb{T})$ *(gap open at* $E=0$*), then* $k=g$. *So whenever the skin winding at* $E=0$ *is defined, the FE-symmetric charge* $g-k$ *there is* 0. *For a curve satisfying Weil,* $0\in h(\mathbb{T})$ *and the* $E=0$ *skin winding is undefined, while the Hardy charge* $g-k=g$ *still is.*

Proof. (a) Factor $\mathcal{L} = B_{\mathcal{L}}\cdot O$, where O (zeros on or outside $\mathbb{T}$) is outer; $[OH_+] = H_+$. (b) As in Cor. 4.5 with $\hat k_0 = 1$: $|B_{\mathcal{L}}(0)| = \prod_{\text{ghosts}}|a_j| = \prod_i\max(1,|\beta_i|)^{-1} = M^{-1}$, since $\mathcal{L}(0)=1$. (c) $T(\mathcal{L})$ is injective with cokernel $K_{B_{\mathcal{L}}}$ ([7], [BIB]). (d) Realness follows from $c_{2g-i}=c_i\in\mathbb{R}$. Then $u^{-g}H_+$ has charge $+g$, and multiplying by u^{-g} shifts the charge by $+g$. For real $E\notin h(\mathbb{T})$, $h-E$ is real and of constant sign on $\mathbb{T}$, so $\operatorname{wind}(h-E)=0$. Hence $\mathcal{L}-Eu^g$ has exactly g zeros in $\mathbb{D}$, and the charge is $g-g=0$. (e) is (d) at $E=0$. □

Remark 4.9 (the three windings reconciled; [THM])**.** The "winding" in TAA has meant three different subspaces: (i) the skin-effect Toeplitz winding, $-\mathrm{ch}[(h-E)H_+]=0$ for every curve and fake (FE plus the delay u^{-g}); (ii) the Lemma 48 phase winding, $-\mathrm{ch}[\mathcal{L}H_+] = k$, which counts ghosts; (iii) the FE-symmetric charge at $E=0$, $g-k$, which vanishes exactly when the gap is open. They are consistent. The skin winding is zero because the FE forces h to be real, not because the ghosts are absent. The ghost count lives in (ii). For ζ the analogue of u^{-g} would be $e^{-i\theta(t)}$, which by Lemma 48 Prop. 4.2 is not of bounded type. So there is *no* FE-symmetric point of Gr for ζ, and only (ii) has a ζ analogue (Thm. 4.3).

[NUM] (`num/l53/charge.json`). $\mathrm{dist}^2(1, [\mathcal{L} \cdot \mathrm{Poly}_{<N}])$ computed by QR. The "exact" column uses the projector onto $\mathrm{span}\{k_{a_j}\}$. $\sigma_{\min}$ lists the smallest singular values of the 200×200 section $T_{200}(\mathcal{L})$.

case	k	$n_\circ$	$1-M^{-2}$	dist^2, $N=2000$	FE charge $E=0$	$\sigma_{\min}(T_{200})$
11a, $p=5,7,13,17,19,23$	0	2	0	0.000999 ($\approx 2/N$)	1	≈ 1.3–$1.6\cdot10^{-2}$
$g=2$ curves (4 cases), L50 Weil curve	0	4	0	0.001993–0.001996	2	$\geq 1.0\cdot10^{-2}$
fake $a=8, q=13$	1	0	0.604339	0.604339	0	$2\cdot10^{-17}$, then 0.22
fake $a=7, q=11$	1	0	0.484203	0.484203	0	$3\cdot10^{-17}$, then 0.11
fake $a=13, q=13$	1	0	0.908327	0.908327	0	$1\cdot10^{-16}$, then 1.6
fake $g=2$, real pair	2	0	0.644044	0.644044	0	two $\leq 2\cdot10^{-16}$, then 0.27
fake $g=2$, quadruple	2	0	0.742728	0.742728	0	two $\leq 3\cdot10^{-16}$, then 0.07
fake $g=2$, mixed	1	2	0.664271	0.664606	1	$9\cdot10^{-18}$, then 0.023
L50 real-ghost fake	1	2	0.931933	0.932001	1	$8\cdot10^{-17}$, then 0.070

For pure fakes the QR distance matches the exact projector to $\leq 3\times10^{-15}$, and the residual Gram matrix has rank k. With circle zeros present, the convergence is $O(n_\circ/N)$: for curves $\mathrm{dist}^2 \to 0$ like $n_\circ/N$, and the mixed fakes approach $1-M^{-2}$ from above with error $\approx 1\times10^{-3}$ in the Gram matrix at $N=2000$. Off-spectrum ($E>\max h$, $E<\min h$, $E=0.5i$) the FE charge is 0 in all 18 cases, and $|\Im h|$ on $\mathbb{T}$ is at rounding level. The "$E=0$" column equals $g-k$ (for the mixed and real-ghost fakes, $g=2$, $k=1$). The FFT winding of B equals k.

4.5. **The bridge to the Krein picture.**

Proposition 4.10 (charge = Pontryagin index; [THM]; reading [ANALOGY])**.** *For a finite Blaschke product B with k zeros, let $K_B(z,w)=\frac{1-B(z)\overline{B(w)}}{1-z\bar w}$, the reproducing kernel of K_B, which has rank k and is positive. Then*

$$K_{1/B}(z,w) := \frac{1-B(z)^{-1}\overline{B(w)}^{-1}}{1-z\bar w} = -\frac{K_B(z,w)}{B(z)\overline{B(w)}},$$

so $1/B$ is a generalized Schur function with exactly k negative squares and no positive ones on $\mathbb{D}\setminus B^{-1}(0)$. The Hardy-space codimension of BH_+ (the charge $-k$) is the Pontryagin index of $1/B$.

Proof. The identity is algebra. The negative-square count equals the rank of K_B on finite point sets, which is at most $\dim K_B = k$, with equality on generic k-point sets. □

Reading ([ANALOGY]). Lemma 48 Cor. 3.3 wrote $\zeta = O_\zeta B_\zeta b_1^{-1}$. The pole is one anti-Blaschke factor ($\kappa=1$), and a ghost enters as B_ζ (contractive). In the Hardy picture (this lemma) positivity is imposed, and ghosts are *charge* (codimension). In the Krein picture of Lemmas 45–46 they are *negative norms*: signature $(D_{\mathrm{on}}+k, k)$, k counted on distinct roots. The two pictures are exchanged by $B \leftrightarrow 1/B$, and the counts agree when the ghosts are simple. Whether L46's Krein form *is* the de Branges–Pontryagin space of $1/B_{\mathcal{L}}$ is Q2.

Remark 4.11 (consistency with Lemma 47; [THM])**.** The vacuum H_+ used here exists for the *one-parameter* scaling group $\mathbb{R}_{>0}$: $f(t)\mapsto t^{-1}f(1/t)$ is unitary on $L^2(0,\infty)$, exchanges $L^2(0,1)$ and $L^2(1,\infty)$, and acts on Mellin transforms by $\hat f(s)\mapsto \hat f(1-s)$. The Mellin transform maps $L^2(0,1)$ onto $H^2(\mathbb{H}_{1/2})$ [8]. Lemma 47 Thm. 2.3 is about the rank ≥ 2 group $\mathbb{Q}_S^\times$ and is not contradicted.

5. Does "fermions from bosons" give positivity?

Theorem 5.1 (transport, not creation; [THM] from [CITED] inputs)**.** *(a) The boson–fermion correspondence $\mathcal{H}_0 \cong \overline{\Lambda}_{\mathrm{sym}}$ (Thm. 3.2) is an isometry between two* positive *Hilbert spaces: Plücker coordinates (orthonormal) go to Schur functions (orthonormal). An isometry carries positive vectors to positive vectors and Krein spaces to Krein spaces (Lemma 50 Prop. 5.2). It cannot turn an indefinite form into a definite one.*

(b) *The bosonic positivity* $\langle \alpha_n \alpha_{-n} \rangle = n > 0$ *is Prop. 3.5, i.e. the fermionic choice of* H_+. *Choosing* H_- *flips every sign. The positivity on either side is the same datum, the polarization.*

(c) *In the Hardy picture of §4 positivity is put in at the start, through* H^2, *so a ghost cannot have negative norm. It shows up as charge (Thm. 4.3) or as a normalizable hole (Thm. 4.7). In the cohomological picture of Lemmas 45–50 the polarization is not known to be invariant, and the ghost shows up as negative norm (Lemma 45 Thm. 6.4, Lemma 46 Thm. 6.1). The RH content is the same in both pictures and is not removed by passing between them (Prop. 4.10).*

TAA reading ([ANALOGY]). "Missing polarization = missing invariant positive vacuum" (Lemma 50 Thesis 5.4) becomes, in the Hardy picture: *the vacuum* H_+ *exists (one-parameter scaling), and the question is whether the* ζ*-state* W_ζ *is the vacuum or sits in a charged sector.* The "fermions from bosons" mechanism relates the two pictures but supplies neither the invariance (Lemma 50) nor the vanishing of the charge (RH).

6. Corrections to the planned chat claims

(1) *"The metaplectic/oscillator representation is the bosonic twin of the spin representation."* Correct, with refinements. The twin of a *pure spinor* (vacuum) is the Gaussian $e^{i\pi x^{\mathsf{T}}\Omega x}$, which exists only for $\Omega \in \mathfrak{h}_g$. The fermionic vacuum exists for every orthogonal J. The oscillator representation is infinite-dimensional already for $\mathrm{Sp}(2n, \mathbb{R})$, where the spin representation has dimension 2^n. In infinite dimensions both twins survive only for the restricted groups (Shale; Shale–Stinespring).

(2) *"Bosonic positivity is Siegel/*$\Im\tau > 0$*."* Correct for $g = 1$. In general it is $\Im\Omega > 0$ *positive definite*, the signature of $\omega(\cdot, J\cdot)$ being $2\,\mathrm{sig}(\Im\Omega)$. It is a condition on the vacuum J, not on a one-particle metric (the fermionic positivity sits in g). It is *not* the missing ingredient over $\mathbb{Q}$: Siegel positivity at one point holds for every curve, and what Lemma 50 Thm. 4.4 lacks is *invariance*.

(3) *"The Segal–Wilson vacuum is the Hardy space."* Correct, verified at the source: $H_+ = \overline{\mathrm{span}}\{z^k\}_{k\geq 0}$, boundary values of holomorphic functions in $|z| < 1$. Caveats: Gr requires pr_+ Fredholm and pr_- compact (Hilbert–Schmidt in §10). The TAA transfer to $H^2(\Re s > \frac{1}{2})$ needs the Cayley and Mellin unitaries (§1, [8]).

(4) *"Ghosts as Blaschke factors change the charge."* Correct only after correction. Each ghost *lowers* the charge by one (ch $= -k$, i.e. winding $+k$), and only when there are finitely many ghosts. For ζ the ghosts are non-real and paired, so a nonzero charge would be ≤ -2 and even. Infinitely many ghosts (as for Davenport–Heilbronn) put the subspace outside Gr, so there is no charge at all. The charge is *invisible* to the skin-effect Toeplitz winding (always 0 by the FE) and is seen by the Lemma 48 phase winding. "RH $\iff$ charge 0" is a restatement of Nyman–Beurling/Beurling–Lax (Kapustin, Burnol), not a new criterion.

(5) *"The correspondence transports but does not create positivity."* Correct (Thm. 5.1). One refinement: the bosonic current positivity *is* the fermionic choice H_+ (Prop. 3.5), so the two sides have one positivity, not two.

7. Prior art; what is new and what is restatement

Prior art found. Nyman (1950 thesis, cited through [8, 23]) and Beurling [6]: RH $\iff$ density of the NB space. Báez-Duarte [2]: integer dilations suffice. Bercovici–Foias [4]; Burnol [8, 9]: NB $= B(s)H^2$, $\|P1\| = \prod |(1-\rho)/\rho|$. Bagchi [1]: Hilbert-space reformulation. Nikolski [22, 23]: distance formulae and cyclicity in Hardy spaces on the multidisc. Kapustin [17]: RH $\iff$ $(s-1)\zeta(s)/s^2$ outer. Noor [24]: Hardy-space ($H^2(\mathbb{D})$) form of Báez-Duarte. BSY [3]. The Grassmannian side: Sato [27], DJKM [10], Jimbo–Miwa [14], Segal–Wilson [29], Pressley–Segal [26], Krichever [18]. A

limited web search found *no* work placing ζ or RH in the Sato/Segal–Wilson Grassmannian or reading ghosts as charge. This is not proof of novelty.

Restatement ([RESTATEMENT]). Thm. 4.3(d); Cor. 4.5 at $s_0 = 1$ (Burnol).
Classical ([THM]/[BIB]). Prop. 2.1, 2.2, 2.3; Thm. 3.2, 3.3; Thm. 4.6(a)–(b) (standard Hankel identities); Thm. 4.7(b) (Cauchy, Pick).
New in presentation, elementary in proof ([THM]). The ghost charge $\mathrm{ch}(W_\zeta) = -\#\text{ghosts}$ and "$W_{\mathrm{DH}} \notin \mathrm{Gr}$" (Thm. 4.3(c),(e)); implementability of $M_{\tilde{B}}$ iff finitely many ghosts, with $\|[\varepsilon, M_{\tilde{B}}]\|_{HS}^2 = 4\#\text{ghosts}$ (Thm. 4.6(c)–(d)); ghost states as normalizable holes (Thm. 4.7(c)); the three-winding reconciliation and "gap open $\Rightarrow k = g$" (Thm. 4.8(d)–(e), Rem. 4.9); charge = Pontryagin index of $1/B$ (Prop. 4.10). None of these brings RH closer.

8. Questions for J.W.

Question 8.1. Adopt "ghost charge" ($\mathrm{ch}(W_\zeta) = -\#\text{ghosts}$, with no charge for infinitely many) as TAA vocabulary alongside "negative norm"? Both are correct, and they are the two sides of $B \leftrightarrow 1/B$.

Question 8.2. [OPEN] Is Lemma 46's Krein form of signature $(D_{\mathrm{on}} + k, k)$ isometric to the de Branges–Pontryagin space of $1/B_{\mathcal{L}}$ (Prop. 4.10)? If so, Lemmas 45–46 and this lemma are one statement.

Question 8.3. Over $\mathbb{F}_q$, $W_{\mathcal{L}} = B_{\mathcal{L}} H_+$ is a "Krichever-type" point of virtual dimension $-k$. Is there a curve-and-line-bundle realization, as in Thm. 3.3, with charge $\deg \mathcal{L} - g$? [OPEN] (We have not constructed one.)

Question 8.4. For Volume 3: present "bosons versus fermions" through the 2-out-of-3 rule (Prop. 2.2), i.e. positivity on J versus on g, rather than as a mechanism that creates positivity?

Question 8.5. Numerics for ζ: the charge is not computable (it is 0 or ≤ -2 or undefined). The Burnol distance $1 - e^{-2\mathfrak{G}(1)}$ is the computable shadow. Is a higher-precision $\mathfrak{G}(s_0)$ at several s_0 worth a run, given that it can only confirm RH-consistency to finite height?

Status

Proved: §§2–5 as tagged. Restatement: Thm. 4.3(d). Open: Q2, Q3. RH: not addressed beyond restatement. Numerics: `num/lemma53_charge.py` (under `nice`, under 1 GB) → `num/l53/charge.json`. References: `lit/lemma53/verified_refs.md`. Unverified and therefore not cited: Balazard–Saias (Expo. Math. 2000), Nyman's thesis, Krein–Langer, and the content of Pressley–Segal.

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TAA LEMMA 54: EMERGENT COMPLEX STRUCTURE. FROBENIUS CHOOSES ITS OWN J. THE SIGN OF THE FREQUENCY, ROTATION $\times$ BOOST, POSITIVE ENERGY, CARTAN'S CENTRE, AND THE SIEGEL-ORBIT GHOST DETECTOR. V2: INVARIANTS THAT SEE REAL-PAIR GHOSTS

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ABSTRACT. **Statement of intellectual honesty.** RH is not proved here, and nothing below brings it closer. Everything is finite-dimensional linear algebra and classical geometry of Siegel space, applied to (honest or fake) L-polynomials and to finite truncations of zero sets. The one statement equivalent to RH for curves (Thm. 2.4(c)) is a *restatement* of Lemma 50 Thm. 4.4; for ζ nothing is constructed (§7, [OPEN]). Tags: [THM] proved here (or classical with proof given); [CITED] checked at the source (`lit/lemma54/verified_refs.md`); [BIB] bibliographic record verified (Crossref), content standard; [RESTATEMENT] restatement of a known result or of an earlier TAA lemma; [ANALOGY] analogy only; [NUM] numerical (`num/lemma54_emergent.py`, `num/lemma54_zeta.py`; data `num/l54/`); [CONJ] conjecture; [OPEN] open. Lemmas 14, 46, 48, 50, 51, 53 are cited by number and not redone.

Question (J.W., [t201u]–[t203u]). "Is there any way to make choosing a complex structure a natural emergent selection rather than an arbitrary choice?" Then: "If there is this $\cos\theta$ term when the root is [real] and J is an honest rotation by 90°, shouldn't this have some extension to the cosh function, that is essentially a Wick rotation?" Then: "Yes to Lemma 54." Scope (from the chat answer): the formula $J = (F-V)(4q-(F+V)^2)^{-1/2}$, the rotation×boost decomposition, the positive-energy, Cartan and KMS routes, the Siegel-orbit drift as a ghost detector, and a check of six chat claims.

Answer in brief. (1) *Frobenius does choose a J, and the choice is the sign of the frequency* (§2). With $V = qF^{-1}$, $D = 4q-(F+V)^2$ and the principal square root, $J_F = (F-V)D^{-1/2}$ is defined iff F has no real eigenvalue. It is then real, $J_F^2 = -1$, it commutes with F, it is ω-compatible for *every* ω that F scales by q, and $J_F = i\,\mathrm{sgn}(\Im\alpha)$ on the generalized eigenspace of α (Thm. 2.2). So $J_F = i\,\mathrm{sign}(-iF)$, a matrix sign function. It depends on F only, not on ω and not on RH. For an elliptic curve, $J_F = (2F-a)/\sqrt{4q-a^2}$, which *is* Lemma 51's positive vacuum $J_+ = (g-\frac{t}{2})/\sin\varphi$ (difference 0 to 60 digits). (2) *Reality is not RH; positivity is* (Thm. 2.4). J_F is positive iff all roots lie on $|\alpha| = \sqrt{q}$, F is semisimple, and every upper eigenvalue has Krein sign $+$. Otherwise J_F differs from the positive vacuum by the Krein sign of each Frobenius plane. In Lemma 50's trace model the pattern is $(+,\dots,+)$ and $\omega_P(x, J_F x) = \mathrm{Tr}(D^{1/2} x\, x^\sharp)$, so there J_F *is* the positive vacuum (Prop. 2.5). For the quadruple fake P_Q it is real but of signature $(2,2)$; for the real-ghost fake P_R it is not real. Along Lemma 51's Hamiltonian–Hopf path, J_F is Lemma 51's surviving Krein vacuum $J_{(+,+)}$ and passes through the collision analytically: Frobenius's own J does not see a quadruple being born. (3) *Uniqueness* (Thm. 3.1). Under RH (semisimple), the positive invariant J form $\prod_{\Im\alpha>0} U(p_\alpha, q_\alpha)/U(p_\alpha) \times U(q_\alpha) \times \prod_\pm \mathfrak{h}(E_{\pm\sqrt{q}})$. They are unique iff every eigenspace is Krein-definite and $\pm\sqrt{q}$ is not an eigenvalue, e.g. for simple frequencies. (4) *Rotation × boost* (§4). $\alpha + q/\alpha = 2\sqrt{q}\cos(\theta - i\sigma)$ and $D(\alpha) = 4q\sin^2(\theta - i\sigma)$. A real ghost gives a split structure ($E^2 = +1$). $\sqrt{q}$ is central in GSp and acts trivially on Siegel space. The ghost's "second boost" is the hyperbolic Jordan factor of g_n, whose rapidity is 2σ per step in Lemma 51's metric. Quadruples are the loxodromics of $\mathrm{SL}_2(\mathbb{C}) = \mathrm{Spin}(3,1)_0 \subset \mathrm{Sp}(4,\mathbb{R})$, and $2\cos(\theta - i\sigma)$ is their complex trace. Wick rotation is analytic continuation and decides nothing. (5) *Other routes* (§5): positive energy $J_A = A(-A^2)^{-1/2}$ (Segal–Weinless–Kay); Cartan's circumcentre; KMS states. Each yields a positive J exactly when an RH-type positivity holds. Emergence turns "choose J" into "J exists", not into a proof. Cartan's route is the one that matches RH exactly, independently of Krein patterns, but it is not unique when the patterns are mixed. (6) *Drift detector* (§6). $\ell(g_\mathrm{n}) = \lim d(J_0, g_\mathrm{n}^n J_0)/n = 2\big(\sum_{|\lambda|>1}(\log|\lambda|)^2\big)^{1/2}$ (Thm. 6.1). That is $2|\sigma|$ for a real ghost pair and $2\sqrt{2}|\sigma|$ for a quadruple; the combination is quadratic, not additive. Jordan blocks on the circle give $O(\log n)$ drift. Measured rates agree with the prediction (genus 1, Lemma 51's path, the trace model, and a 216-dimensional Davenport–Heilbronn truncation with its 28 ghosts). (7) ζ (§7). On every finite truncation of a zero set with no real zeros, J_A exists, and the negative index of $\omega(\cdot, J_A\cdot)$ is exactly 2×(number of ghost quadruples): 56 for Davenport–Heilbronn to height 1000, 0 for ζ's first 50 zeros. For ζ itself, no real space with ω and a flow is known ([OPEN]).

v2 ([t218u], J.W.'s answer 5 to the Lemma 56 v2 questions: "Yes"). The task: redefine the Lemma 54 invariants (index and drift) so that they also detect *real-pair* ghosts. Test them on ζ (no ghosts may be reported), on DH, and on polynomial toys with real-pair ghosts.

v2 in brief (§8). (1) *Frequency-form index* $\nu(F) := n_-(h_F)$, with $h_F(x,y) = \omega(x,(F-Ve)y)$. It is symmetric, F-conformally invariant, needs no square root, and is defined for *every* F; its kernel is $\ker(F^2-q)$. Every ghost block is neutral: a real pair has signature $(1,1)$ and a quadruple $(2,2)$. Hence $\nu = k + 2m_-$, where k is the number of eigenvalues outside $|\alpha| = \sqrt{q}$ and m_- the number of Krein-negative Frobenius planes (Thm. 8.2) [THM]. For flows this is the Hamiltonian–Krein index theorem (Rem. 8.3, [RESTATEMENT]). In Lemma 50's trace model $h_F(x,x) = \mathrm{Tr}(D\,xx^\sharp)$, and Hermite's signature theorem gives $\nu = k$ for every $P = T^g Q(T + q/T)$ with Q squarefree and $Q(\pm 2\sqrt{q}) \neq 0$, real pairs included (Thm. 8.4) [THM] from [CITED]. Where v1's $n_-(g_{J_F})$ is defined, it equals ν. (2) *Drift rank* $r(F)$: the number of linearly growing Siegel rapidities, $= \#\{|\lambda(g_n)| > 1\} = k$. It is chart-free and ω-free (Thm. 8.5) [THM]. ℓ is kept as the *size*: $\ell = 2\|(\sigma_1, \ldots, \sigma_r)\|$. (3) *Tests* [NUM].

- ζ truncations (100 and all 649 zeros below 1000): $\nu = r = 0$.
- DH truncations (28 ghosts with 52 or with all 848 on-line zeros): $\nu = r = 56$.
- Proxies with real pairs added: $\nu = r = k$.
- 14 named trace-model cases and Lemma 56's 19 FF rows: $\nu = r = k$.
- 1480 random non-degenerate trace models: $\nu = k$ in 1480/1480, including 779/779 with real pairs; the orbit estimate gives $r = k$ in 295/295.

In a mixed Krein chart ν is 2 with $k = 0$, while $r = 0$. So r is the chart-free ghost count, and ν also sees Krein signs.

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1. Conventions, and what TAA already has

(V,ω) is a real symplectic space of dimension $2g$. $F \in \mathrm{GSp}(V,\omega)$ has multiplier $q > 0$: $\omega(Fx,Fy) = q\,\omega(x,y)$. Its *Verschiebung* is $Ve := qF^{-1}$. Then

$$\omega(Fx,y) = \omega(x,Vey), \qquad FVe = VeF = q, \tag{1}$$

so Ve is the ω-adjoint of F (Lemma 46's $V = F^\dagger$). We write V for the space and Ve for the operator. $g_{\mathrm{n}} = F/\sqrt{q} \in \mathrm{Sp}(V)$. Eigenvalues of F are $\alpha = \sqrt{q}\,e^{\sigma+i\theta}$, so $\sigma = \log(|\alpha|/\sqrt{q})$; $E_\alpha \subset V_{\mathbb{C}}$ is the generalized eigenspace. A *compatible complex structure* is $J \in \mathrm{Sp}(V)$ with $J^2 = -1$; it is *positive* if $g_J(x,y) = \omega(x,Jy)$ is positive definite; $\mathfrak{h}(V)$ is Siegel space. The Krein form is $h_K(x,y) = i\,\omega(\bar{x},y)$ on $V_{\mathbb{C}}$, and κ_α is its sign on E_α when definite. Rapidities and the distance $d(J_0,J) = (\sum d_i^2)^{1/2}$ are those of Lemma 51 §2 ($e^{\pm d_i}$ are the eigenvalues of $-J_0J$; for $g = 1$ this is the hyperbolic metric of curvature -1). In a g_{J_0}-orthonormal symplectic basis with $\Omega = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \otimes 1_g$, $J_0 = -\Omega$ and $G_J := \Omega J$ is the Gram matrix of g_J.

Source	what is used, not redone
L14 §(g)	Frobenius as a boost of rapidity $\frac{1}{2}\log q$ *in the Lorentzian space* $\mathrm{NS}(C\times C)_{\mathbb{R}}$ (Hodge index).
L46 Thm. 6.1, Rem. (unitary trick)	Abstract (F,Ve) pairs: algebra gives Krein, Hilbert iff no ghosts; $SU(2)$ vs. $SL_2(\mathbb{R})$; "Wick rotation $\theta \mapsto i\eta$ carries it to a boost" ([ANALOGY] there).
L50 Thm. 4.4, Prop. 7.1	Trace model $V = \mathbb{R}[T]/(P)$, $\omega_P = \mathrm{Tr}(c\,xy^\sharp)$, $c = T - q/T$; invariant J_ε, the positive one is $\varepsilon = \kappa$; RH $\iff$ invariant positive vacuum ([RESTATEMENT]); Deninger in vacuum form. P_W, P_Q, P_R $(q = 11)$.
L51 Thm. 2.1, Prop. 3.1–3.2, Thm. 4.2	The vacuum escapes; genus-1 chart, $J_+ = (g - \frac{t}{2})/\sin\varphi$; endpoint $\lvert a\rvert = 2\sqrt{q}$ semisimple for real curves; Hamiltonian–Hopf (HH) path $P_W \to P_Q$, $t_* = 0.47356443681161$, surviving Krein vacuum $J_{(+,+)}$.
L48, L53	Gain defect $= \sum\eta$, Blaschke zeros; DH zeros to height 1000 (`num/l48/zeros_T1000.json`: 848 on-line, 28 ghosts); Segal–Wilson vacuum H_+, Hilbert transform J, ghost charge $-k$.

2. The canonical complex structure of Frobenius

Definition 2.1. $D := 4q - (F + Ve)^2$. If D has no eigenvalue in $(-\infty, 0]$, let $D^{1/2}$ be its principal square root (holomorphic functional calculus with the branch $\Re > 0$; real for real D [9], [BIB]) and

$$J_F := (F - Ve)\,D^{-1/2}.$$

Theorem 2.2 (Frobenius chooses its own J; [THM], checked [NUM])**.** *Let $F \in \mathrm{GSp}(V,\omega)$ with multiplier q.*

(a) F, Ve, D commute, D is ω-self-adjoint, $F - Ve$ is ω-skew, and $(F - Ve)^2 = -D$. On E_α, D has the single eigenvalue $D(\alpha) = -(\alpha - q/\alpha)^2$.

(b) D has an eigenvalue in $(-\infty, 0]$ iff F has a real eigenvalue. (Real eigenvalues are $\pm\sqrt{q}$, where $D(\alpha) = 0$, and real ghosts, where $D(\alpha) < 0$.)

(c) If F has no real eigenvalue, then J_F is real, $J_F^2 = -1$, $J_F F = F J_F$, and $J_F \in \mathrm{Sp}(V,\omega)$. Moreover

$$J_F = i\,\mathrm{sgn}(\Im\alpha) \quad \text{on } E_\alpha, \qquad \text{i.e.} \qquad J_F = i(P_+ - P_-) = i\,\mathrm{sign}(-iF),$$

where $P_\pm$ are the Riesz projectors onto the spectrum in $\{\pm\Im z > 0\}$ and $\mathrm{sign}(M) = M(M^2)^{-1/2}$ is the matrix sign function.

(d) J_F depends only on F: not on q (it equals $i\,\mathrm{sign}(-ig_{\mathrm{n}})$), and not on which ω is used. It is compatible with every *symplectic form that F scales.*

(e) $F \mapsto J_F$ is real-analytic on the open set of $F \in \mathrm{GSp}$ without real eigenvalues. In particular it is analytic through collisions of eigenvalues off the real axis.

Proof. (a) $FVe = q$ gives commutation. By (1) the ω-adjoint of F is Ve and that of Ve is F. So $F+Ve$ is self-adjoint, $F-Ve$ is skew, and D is self-adjoint. $(F-Ve)^2-(F+Ve)^2 = -4FVe = -4q$, so $(F-Ve)^2 = -D$. On E_α, Ve has eigenvalue q/α. (b) $D(\alpha) \in (-\infty, 0]$ iff $c := \alpha - q/\alpha \in \mathbb{R}$. With $\alpha = re^{i\theta}$, $\Im c = (r + q/r)\sin\theta$, which vanishes iff $\alpha \in \mathbb{R}$. For real α, $D(\alpha) = 4q - (\alpha + q/\alpha)^2$, and $|\alpha + q/\alpha| \geq 2\sqrt{q}$ with equality iff $\alpha = \pm\sqrt{q}$ (AM–GM). (c) $J_F = f(F)$ for the function $f(z) = (z - q/z)\big(-(z - q/z)^2\big)^{-1/2}$ (principal branch), holomorphic near $\operatorname{spec} F$. For $c = z - q/z$ the principal root of $-c^2$ is the one of $\pm ic$ with positive real part: $\Re(-ic) = \Im c$. So if $\Im c > 0$ the root is $-ic$ and $f = c/(-ic) = i$; if $\Im c < 0$ then $f = -i$. Since $\operatorname{sgn}\Im c = \operatorname{sgn}\sin\theta = \operatorname{sgn}\Im z$, $f(z) = i\operatorname{sgn}\Im z$, a locally constant function, and $f(F) = i(P_+ - P_-)$. This gives $J_F^2 = -1$ and $[J_F, F] = 0$. Reality: $f(\bar z) = \overline{f(z)}$ and F is real. Compatibility: the adjoint of $f(F)$ is $f(Ve)$, and $f(q/z) = -f(z)$ because $\Im(q/z) = -q\Im z/|z|^2$. So J_F is ω-skew, and with $J_F^2 = -1$ this gives $\omega(J_F x, J_F y) = -\omega(x, J_F^2 y) = \omega(x, y)$. The sign-function form: $-i\alpha$ has real part $\Im\alpha$. (d) Riesz projectors depend only on F; $\operatorname{sgn}\Im\alpha = \operatorname{sgn}\Im(\alpha/\sqrt{q})$. The adjointness (1) holds for every ω scaled by F. (e) Riesz projectors are analytic in F as long as no eigenvalue crosses the contour, which can be taken as a pair of curves enclosing the upper and lower spectrum and avoiding $\mathbb{R}$. □

Corollary 2.3 (elliptic curves; [THM], checked [NUM])**.** *On H^1 of an elliptic curve (or any 2-dimensional model with $\operatorname{Tr} F = a$, $\det F = q$), $F + Ve = a$, so*

$$J_F = \frac{2F - a}{\sqrt{4q - a^2}},$$

defined and real iff $|a| < 2\sqrt{q}$. In Lemma 51's chart $(1, T/\sqrt{q})$, $J_F = (g_{\mathrm{n}} - \frac{t}{2})/\sin\varphi = J_+$, Lemma 51's positive *vacuum, exactly. At $|a| = 2\sqrt{q}$ the formula is undefined. In the family chart g_{n} is then parabolic (no invariant J), and for a real curve $F = \pm\sqrt{q}$ is scalar (Lemma 51 Prop. 3.2), every J is invariant, and Frobenius chooses none of them.*

Proof. Cayley–Hamilton: $F^2 - aF + q = 0$, so $F + qF^{-1} = a$, $F - Ve = 2F - a$ and $D = 4q - a^2$. With $F = \sqrt{q}\, g_{\mathrm{n}}$ and $t = 2\cos\varphi$: $(2\sqrt{q}g_{\mathrm{n}} - \sqrt{q}t)/(\sqrt{q}\cdot 2\sin\varphi) = (g_{\mathrm{n}} - \frac{t}{2})/\sin\varphi$. □

[NUM] (`emergent.json`, `B_genus1`; $q = 13$, 60 digits). For $a \in \{0, 3, 6, 7, a_c - 10^{-4}, a_c - 10^{-8}\}$: $\|J_F - J_+\|_{\max} \leq 3.4 \cdot 10^{-49}$, imaginary part 0, and J_F agrees with $(2F - a)/\sqrt{4q - a^2}$ to $3 \cdot 10^{-61}$.

Theorem 2.4 (when is Frobenius's own J the positive vacuum; [THM]; (c) [RESTATEMENT])**.** *Assume F has no real eigenvalue, and let $H(x, y) := g_{J_F}(\bar x, y) = \omega(\bar x, J_F y)$ on $V_{\mathbb{C}}$.*

- *(a) For $x \in E_\alpha$ with $\Im\alpha > 0$, $H(x, x) = h_K(x, x)$. E_α and E_β are H-orthogonal unless $\beta = q/\bar\alpha$.*
- *(b) J_F is positive iff (i) every root has $|\alpha| = \sqrt{q}$, (ii) F is semisimple, and (iii) h_K is positive definite on every E_α with $\Im\alpha > 0$ (Krein pattern $(+, \ldots, +)$).*
- *(c) ([RESTATEMENT] of L50 Thm. 4.4.) If (i)–(ii) hold and the roots are distinct, the unique positive invariant vacuum is $J_+ = J_\kappa$, and $J_+ = \kappa_\alpha J_F$ on $E_\alpha \oplus E_{\bar\alpha}$: J_F and J_+ differ* exactly by the Krein sign of each Frobenius plane. *Replacing ω by $-\omega$ flips every κ_α and fixes J_F.*
- *(d) If some root is off the circle, J_F is real and invariant (when no root is real) but indefinite: H vanishes on each E_α with $|\alpha| \neq \sqrt{q}$.*

Proof. (a) $H(x, x) = \omega(\bar x, ix) = i\omega(\bar x, x) = h_K(x, x)$. Invariance of ω up to q gives $\omega(\bar x, y)(\bar\alpha\beta - q) = 0$ for eigenvectors, and by induction along Jordan chains $\omega(E_{\bar\alpha}, E_\beta) = 0$ unless $\bar\alpha\beta = q$. (b) g_{J_F} is positive iff H is positive definite on $V_{\mathbb{C}}$. If some $|\alpha| \neq \sqrt{q}$, then $q/\bar\alpha \neq \alpha$ and E_α is H-isotropic by (a). If F has a Jordan chain $(F - \alpha)x_1 = x_0 \neq 0$ with $|\alpha| = \sqrt{q}$, then $h_K(x_0, x_0) = 0$ (Lemma 51 Cor. 2.4, proof), so $H(x_0, x_0) = \pm h_K(x_0, x_0) = 0$. Conversely, under (i)–(ii) the E_α are H-orthogonal, $H = h_K$ on upper eigenspaces, and on lower ones $H(\bar x, \bar x) = \omega(x, -i\bar x) = h_K(x, x)$, so $H > 0$ iff (iii). (c) By L50 Thm. 4.4 the invariant J are J_ε ($i\varepsilon_\alpha$ on E_α), the positive one is $\varepsilon = \kappa$, and $J_F = J_{\operatorname{sgn}\Im}$. (d) is (a). □

Proposition 2.5 (in Lemma 50's trace model J_F *is* the positive vacuum; [THM], checked [NUM])**.** *Let $V = \mathbb{R}[T]/(P)$, $\omega_P(x,y) = \mathrm{Tr}(c\,xy^\sharp)$, $c = T - q/T$ (L50 Thm. 4.4), $F = T$, and let P have no real root. Then $Ve = q/T$, $c = F - Ve$, $D = -c^2$, $J_F^\sharp = -J_F$, and*

$$\omega_P(x, J_F x) = \mathrm{Tr}\,(D^{1/2}\, x\, x^\sharp).$$

If P is a Weil polynomial with distinct roots, $\omega_P(x, J_F x) = \sum_\alpha 2\sqrt{q}\,|\sin\theta_\alpha|\,|x(\alpha)|^2 > 0$, so $J_F = J_\kappa$ is L50's positive vacuum. (This is the vacuum form of L51 Cor. 2.6: the trace model has pattern $(+,\dots,+)$.)

Proof. $\sharp$ is the algebra involution $T \mapsto q/T$, which swaps F and Ve and preserves the trace. So $c^\sharp = -c$, $D^\sharp = D$ and $J_F^\sharp = f(Ve) = -J_F$ (proof of Thm. 2.2(c)). Then $\omega_P(x, J_F x) = \mathrm{Tr}(c\,x\,J_F^\sharp x^\sharp) = -\mathrm{Tr}(cJ_F\,xx^\sharp) = -\mathrm{Tr}(c^2 D^{-1/2} xx^\sharp) = \mathrm{Tr}(D^{1/2} xx^\sharp)$. For étale V, $\mathrm{Tr}(y) = \sum_\alpha y(\alpha)$. Under RH $q/\alpha = \bar\alpha$, so $(xx^\sharp)(\alpha) = x(\alpha)x(\bar\alpha) = |x(\alpha)|^2$ for real x, and $D^{1/2}(\alpha) = (4q\sin^2\theta_\alpha)^{1/2} = 2\sqrt{q}|\sin\theta_\alpha|$. □

[NUM] (`emergent.json`, `A_trace_model`; 60 digits). The sign-function and the square-root formula agree, and for the two real cases:

P	$\lvert\alpha\rvert/\sqrt{q}$	$\Im J_F$	$\lVert J_F^2+1\rVert$	compat. err	sig. g_{J_F}	which J_ε
P_W (Weil)	$1^{\times 4}$	0	$1\cdot 10^{-60}$	$1\cdot 10^{-57}$	$(4,0)$	the unique positive one (of $(4,0),(2,2),(2,2),(0,4)$)
P_Q (quadruple)	$0.712, 1.404$	0	$1\cdot 10^{-58}$	$4\cdot 10^{-54}$	$(2,2)$	one of the two Krein vacua
P_R (real ghost)	$0.261, 3.833$	8.58	—	—	—	none (no invariant J, L50)

The identity $\omega_P(x, J_F x) = \mathrm{Tr}(D^{1/2} xx^\sharp)$ holds to $3\cdot 10^{-57}$ on random x for P_W ($3\cdot 10^{-53}$ for P_Q). For P_R, D has eigenvalues -140.355 ($\times 2$) and 37.355 ($\times 2$): two positive (the Weil pair) and two negative (the real ghost).

Remark 2.6 (Lemma 51's paths in this light; [THM], checked [NUM])**.** In Lemma 51's Hamiltonian–Hopf chart the colliding upper eigenvalues have Krein signs $(+,-)$. So $J_F = J_{(+,+)}$, the same-sign vacuum of signature $(2,2)$, and *not* the positive $J_{(+,-)}$. By Thm. 2.2(e), J_F is analytic through t_* (the eigenvalues collide at $e^{iw} \notin \mathbb{R}$), and $J_F(t_*) = +J_*$, Lemma 51's limit. Frobenius's own J passes through the birth of a quadruple without noticing it, because it is the Krein vacuum, which survives (L51 Thm. 4.2(b)). Along the real-ghost path, J_F ceases to exist exactly when an eigenvalue reaches $-1 \in \mathbb{R}$.

[NUM] (`C_HH`). For $t - t_* \in \{-0.1, -10^{-4}, -10^{-8}, 0, 10^{-8}, 10^{-4}, 10^{-2}\}$, J_F is real, with $\lVert J_F^2+1\rVert$, compatibility and commutator errors $\le 3\cdot 10^{-61}$, and signature $(2,2)$ at every point. $\lVert J_F(t_* + 10^{-8}) - J_F(t_* - 10^{-8})\rVert_{\max} = 8\cdot 10^{-62}$, and $\lVert J_F(t_*) - J_*\rVert_{\max} = 3\cdot 10^{-61}$.

3. Uniqueness: the space of invariant vacua

Theorem 3.1 (structure of the invariant positive vacua; [THM], checked [NUM])**.** *Let $F \in \mathrm{GSp}(V,\omega)$ be semisimple with all $|\alpha| = \sqrt{q}$. For $\Im\alpha > 0$ let (p_α, q_α) be the signature of h_K on the eigenspace E_α, and let $E_\pm := \ker(F \mp \sqrt{q}) \cap V$ (real, ω-non-degenerate). Then $J \mapsto \big((\ker(J-i)\cap E_\alpha)_{\Im\alpha>0},\ (J|_{E_\pm})\big)$ is a bijection*

$$\{J \in \mathfrak{h}(V) : JF = FJ\} \xrightarrow{\ \sim\ } \prod_{\Im\alpha>0} \frac{U(p_\alpha,q_\alpha)}{U(p_\alpha)\times U(q_\alpha)} \ \times\ \mathfrak{h}(E_+)\times\mathfrak{h}(E_-),$$

where the first factors are the spaces of maximal h_K-positive subspaces of E_α. Hence:

- *(a) the positive invariant J is unique iff every upper eigenspace is Krein-definite ($p_\alpha q_\alpha = 0$) and $\pm\sqrt{q}$ are not eigenvalues; then $J_+ = i\kappa_\alpha$ on E_α;*
- *(b) in particular it is unique when all frequencies are simple and $\pm\sqrt{q} \notin \mathrm{spec}\,F$;*

(c) a mixed-type repeated eigenvalue gives a non-compact family of complex dimension $p_\alpha q_\alpha$, and an eigenvalue $\pm\sqrt{q}$ of multiplicity $2m$ gives a whole Siegel space $\mathfrak{h}_m$.

The invariant compatible J of all signatures are obtained by replacing "maximal positive" with "h_K-non-degenerate, with h_K-orthogonal complement" (L50 Thm. 4.4 for distinct roots: 2^g points).

Proof. J commutes with semisimple F, so it preserves each E_α, and by reality $J|_{E_{\bar\alpha}} = \overline{J|_{E_\alpha}}$. On E_α, $J^2 = -1$ gives $E_\alpha = W_+ \oplus W_-$ ($\pm i$). Compatibility is equivalent to $h_K(Jx, Jy) = h_K(x, y)$ on E_α, because $h_K(Jx, Jy) = i\omega(J\bar{x}, Jy)$. That holds iff $W_+ \perp_{h_K} W_-$. By Thm. 2.4(a) (with J in place of J_F), $g_J(\bar{x}, x) = \pm h_K(x, x)$ on $W_\pm$, and the E_α are g_J-orthogonal. So J is positive iff W_+ is positive and W_- negative, i.e. iff W_+ is maximal positive and $W_- = W_+^\perp$. On $E_\pm$, F is scalar and any positive compatible J of $(E_\pm, \omega)$ commutes with it. Conversely, every such choice assembles into a real compatible invariant J. The space of maximal positive subspaces of a Hermitian form of signature (p, q) is $U(p, q)/U(p) \times U(q)$, a point iff $pq = 0$ [8] ([BIB]). □

[NUM] (`E_uniqueness`). For $g_{\mathrm{n}} = R_\varphi \oplus R_{\varphi'}$ on $(x_1, y_1) \oplus (x_2, y_2)$ ($\varphi = 0.7$), the commutant of g_{n} in $\mathfrak{sp}_4$ and its non-compact (symmetric) part have dimensions: definite type $\varphi' = \varphi$: 4 and 0; mixed type $\varphi' = -\varphi$: 4 and 2; simple $\varphi' = 1.9$: 2 and 0. In the mixed case, $J_s = e^{sX} J_0 e^{-sX}$ (X symmetric in the commutant) is invariant and positive (signature $(4, 0)$, errors $\le 8 \cdot 10^{-30}$) with $d(J_0, J_s) = 2, 6, 20$ for $s = 1, 3, 10$. Uniqueness genuinely fails.

4. Rotation × boost: the cosine of a complex angle

Proposition 4.1 (the Wick decomposition; [THM])**.** *Write $\alpha = \sqrt{q}\, e^{\sigma + i\theta}$. Then*

$$\alpha + \frac{q}{\alpha} = 2\sqrt{q}\cos(\theta - i\sigma) = 2\sqrt{q}(\cos\theta\cosh\sigma - i\sin\theta\sinh\sigma),$$

$$\alpha - \frac{q}{\alpha} = 2i\sqrt{q}\sin(\theta - i\sigma), \qquad D(\alpha) = 4q\sin^2(\theta - i\sigma).$$

(a) Weil *($\sigma = 0$, $\sin\theta \neq 0$): $D(\alpha) = 4q\sin^2\theta > 0$ and $J_F = i\,\mathrm{sgn}\sin\theta$ on E_α. g_{n} is a rotation by θ in the plane $(E_\alpha \oplus E_{\bar\alpha}) \cap V$, and J_F is the rotation by $90°$ in the same sense.*

(b) Real ghost *($\theta \in \{0, \pi\}$, $\sigma \neq 0$): $D(\alpha) = -4q\sinh^2\sigma < 0$. No real J commutes with F on the plane $E_\alpha \oplus E_{q/\alpha}$, but $E := (F - Ve)|D|^{-1/2}$ is real, commutes with F, is ω-skew, and satisfies $E^2 = +1$ there: the split (hyperbolic) complex structure, ± 1 on the two real eigenlines. Globally $E^2 = -\,\mathrm{sgn}\, D$ when D is diagonalizable with real spectrum.*

(c) Quadruple *($\sigma \neq 0$, $\sin\theta\cos\theta \neq 0$): $D(\alpha) \notin \mathbb{R}$, and $J_F = i\,\mathrm{sgn}\sin\theta$ on E_α is still real and compatible, but indefinite (Thm. 2.4(d)). The case $\theta = \pm\pi/2$ gives $D(\alpha) = 4q\cosh^2\sigma > 0$.*

Proof. $\alpha/\sqrt{q} = e^{i(\theta - i\sigma)}$, so $\alpha/\sqrt{q} + \sqrt{q}/\alpha = 2\cos(\theta - i\sigma)$ and the difference is $2i\sin(\theta - i\sigma)$. $D = -(\alpha - q/\alpha)^2$. (a), (c): Thm. 2.2(c). (b) On E_α, $(\alpha - q/\alpha)/|2\sqrt{q}\sinh\sigma| = \pm 1$ with opposite signs on E_α and $E_{q/\alpha}$. E is a function of F that is odd under $z \mapsto q/z$, hence ω-skew. A real J commuting with $\mathrm{diag}(\alpha, q/\alpha)$ ($\alpha \neq q/\alpha$ real) is diagonal and cannot square to -1. □

[NUM] (`A_trace_model`, `P_R`). E^2 has eigenvalues $-1, -1, +1, +1$, $\|Eg_{\mathrm{n}} - g_{\mathrm{n}}E\| = 5 \cdot 10^{-60}$, and $\|E^{\mathsf{T}}W + WE\| = 1 \cdot 10^{-53}$ (W the matrix of ω_P).

Proposition 4.2 (multiplicative Jordan decomposition; [THM] (classical))**.** *(a) $g_{\mathrm{n}} = e\,h\,u$ with $e, h, u \in \mathrm{Sp}(V)$ commuting polynomials in g_{n}: e elliptic (eigenvalues $e^{i\theta}$), h hyperbolic (eigenvalues $e^{\sigma} > 0$, diagonalizable), u unipotent. RH for a semisimple F is $h = 1$; a ghost is $h \neq 1$; a collision on the circle is $u \neq 1$. (b) The scalar $\sqrt{q}$ is central in $\mathrm{GSp}(V)$. The group $\mathrm{GSp}^+(V)$ acts on $\mathfrak{h}(V)$ by $J \mapsto kJk^{-1}$, and the centre acts trivially. So $F = \sqrt{q} \cdot e\,h\,u$, and the factor $\sqrt{q}$ (rapidity $\frac{1}{2}\log q$) moves no vacuum. The only boost that acts on Siegel space is h. In Lemma 51's normalization, h moves J_0 (in an adapted basis) by rapidities $2|\sigma|$, one for each eigenvalue with $|\lambda| > 1$.*

Proof. (a) Let s be the semisimple part of g_{n} (a polynomial in g_{n}), $u = s^{-1}g_{\mathrm{n}}$, and $e = \phi(s)$, $h = \psi(s)$ with $\phi(z) = z/|z|$ and $\psi(z) = |z|$ on $\operatorname{spec} s$ (Lagrange interpolation). ϕ and ψ satisfy $f(1/z) = 1/f(z)$ and $f(\bar{z}) = \overline{f(z)}$. The ω-adjoint of g_{n} is g_{n}^{-1}, so the adjoint of $f(s)$ is $f(s^{-1}) = f(s)^{-1}$, i.e. $f(s) \in \mathrm{Sp}$. Unipotence and Sp for u follow. (b) For k of multiplier $\mu > 0$: $\omega(kJk^{-1}x, kJk^{-1}y) = \mu\,\omega(Jk^{-1}x, Jk^{-1}y) = \omega(x, y)$, and $\omega(x, kJk^{-1}x) = \mu\,\omega(k^{-1}x, Jk^{-1}x)$. A scalar commutes with J. For $h = \operatorname{diag}(e^{\sigma}, e^{-\sigma})$ in a symplectic g_{J_0}-orthonormal pair, $hJ_0h^{-1} = aJ_0a^{-1}$ with $a = \operatorname{diag}(e^{d/2}, e^{-d/2})$, $d = 2\sigma$ (Lemma 51 §2). □

Proposition 4.3 (the SL_2 trichotomy, and quadruples as loxodromics; [THM] (classical), checked [NUM])**.** *(a) $g \in SL_2(\mathbb{R})$ acting on $\mathbb{H}$ (curvature -1) is elliptic for $|\operatorname{Tr} g| < 2$: a rotation by 2φ about its fixed point, $\operatorname{Tr} g = 2\cos\varphi$. It is parabolic or ± 1 for $|\operatorname{Tr} g| = 2$. It is hyperbolic for $|\operatorname{Tr} g| > 2$: a translation of length $\ell = 2\eta$ along its axis, $|\operatorname{Tr} g| = 2\cosh\eta$* [5] *([BIB]). These are the cases θ real, the boundary, and $\theta = i\eta$ of $2\cos\theta$. (b) The complex symplectic form $\omega_{\mathbb{C}}(z, w) = z_1w_2 - z_2w_1$ on $\mathbb{C}^2 = \mathbb{R}^4$ has a non-degenerate real part, and $SL_2(\mathbb{C}) = \mathrm{Sp}(2, \mathbb{C}) \subset \mathrm{Sp}(\mathbb{R}^4, \Re\omega_{\mathbb{C}}) \cong \mathrm{Sp}(4, \mathbb{R})$. This is* $\mathrm{Spin}(3,1)_0 \subset \mathrm{Spin}(3,2)_0$*, beside Lemma 51's* $\mathrm{Spin}(4,1) = \mathrm{Sp}(1,1)$*. An element with eigenvalues $e^{\pm(\sigma+i\theta)}$ has real eigenvalues $e^{\pm\sigma\pm i\theta}$, a ghost quadruple, and complex trace $2\cosh(\sigma+i\theta) = 2\cos(\theta - i\sigma)$. On $\mathbb{H}^3$ it is loxodromic: translation by $2|\sigma|$ along an axis combined with rotation by 2θ about it. (c) At $\sigma = 0$ the eigenvalue $e^{i\theta}$ of the realification is double, semisimple, and of* mixed *Krein type* $(1,1)$*. So $SL_2(\mathbb{C})$-elliptic elements are* semisimple *1:−1 resonances. Lemma 51's Hamiltonian–Hopf point is the non-semisimple one. By Thm. 3.1(c) their positive invariant vacua form a disc $U(1,1)/U(1)^2$.*

Proof. (a) Normal forms R_{φ}, $\pm\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, $\operatorname{diag}(e^{\eta}, e^{-\eta})$. The last is $z \mapsto e^{2\eta}z$, with $d(i, e^{2\eta}i) = 2\eta$, and R_{φ} acts on the tangent space at i by $e^{2i\varphi}$. (b) $\Re\omega_{\mathbb{C}}$ is preserved by $\mathrm{Sp}(2, \mathbb{C})$. Realifying $\operatorname{diag}(\mu, \mu^{-1})$ gives eigenvalues $\mu, \bar{\mu}, \mu^{-1}, \bar{\mu}^{-1}$. On $\mathbb{H}^3$, $\operatorname{diag}(e^{\sigma+i\theta}, e^{-\sigma-i\theta})$ acts by $(z, t) \mapsto (e^{2(\sigma+i\theta)}z, e^{2\sigma}t)$. (c) A small σ inside $SL_2(\mathbb{C})$ pushes the eigenvalue off the circle. A Krein-definite eigenvalue cannot leave the circle (L51 Cor. 2.4), so the type is mixed; the signature is $(1,1)$ because the total is $(2,2)$ on $E_{e^{i\theta}} \oplus E_{e^{-i\theta}}$. □

[NUM] (`zeta.json`, `SL2C_in_Sp4R`; random conjugate). Symplecticity error $\leq 7\cdot 10^{-16}$. At $(\sigma, \theta) = (0.3, 0.9)$ the moduli are $0.74082^{\times 2}, 1.34986^{\times 2} = e^{\mp 0.3}$, and the complex trace $1.29959 + 0.47708i$ equals $2\cos(\theta - i\sigma)$. At $\sigma = 0$ the Krein eigenvalues on $E_{e^{0.9i}}$ are -0.323, 0.475, signature $(1,1)$.

Remark 4.4 (what Wick rotation does and does not do; [THM] for (a), [ANALOGY] for (b))**.** (a) The substitution $\theta \mapsto \theta - i\sigma$ is analytic continuation of one polynomial identity. It maps the elliptic family to the hyperbolic one and leaves the characteristic polynomial's *form* unchanged. So it cannot decide on which side a given Frobenius lies. That is the content of Lemma 46's remark and of L51 Prop. 3.1(f): the same function $\arccos(t/2)$ is a Krein norm on one side and a gain defect on the other. (b) In Euclidean field theory the step that produces a Hilbert space from analytically continued correlation functions is a positivity axiom: Osterwalder–Schrader reflection positivity [14] ([BIB]). Its finite-dimensional counterpart here is positivity of an invariant form: $g_J > 0$ for an invariant J (L50 Thm. 4.4, [RESTATEMENT]), or equivalently positive energy (Prop. 5.1). "OS positivity decides" is therefore an *analogy* whose precise TAA form is L50 Thm. 4.4. No OS-type reflection on a space attached to ζ is constructed here.

5. Other emergence routes, and what each one needs

5.1. **Positive energy: $J_A = A(-A^2)^{-1/2}$.** Let $A \in \mathfrak{sp}(V, \omega)$, i.e. $\omega(Ax, y) + \omega(x, Ay) = 0$, generate the flow $e^{\tau A}$. Then $h_A(x, y) := \omega(x, Ay)$ is symmetric, and $E(x) = \frac{1}{2}h_A(x, x)$ is the quadratic Hamiltonian of the flow (for the sign convention $\dot{x} = Ax$).

Proposition 5.1 (positive energy selects a unique vacuum; [THM] in finite dimension; [BIB] [15, 16]; [CITED] [12] (abstract))**.** *Suppose h_A is positive definite. Then:*

(a) A is semisimple with spectrum in $i\mathbb{R}\setminus 0$, and $h_K>0$ on every eigenspace $E_{i\nu}$ with $\nu>0$.
(b) $J_A := A(-A^2)^{-1/2} = i\,\operatorname{sgn}\nu$ on $E_{i\nu}$ is a positive compatible complex structure commuting with the flow, and $-J_A A = (-A^2)^{1/2} > 0$ (positive one-particle Hamiltonian).
(c) J_A is the unique positive compatible J commuting with A. It is also the unique compatible J commuting with A for which $-JA$ is g_J-positive.
(d) If $g_{\mathrm{n}} = e^A$ with $\operatorname{spec} A \subset i(-\pi,\pi)$, then $J_A = J_F$.

Conversely, if A is semisimple with $\operatorname{spec} A \subset i\mathbb{R}\setminus 0$ and $h_K>0$ on every $E_{i\nu}$, $\nu>0$, then $h_A>0$. So in finite dimension "positive energy" is exactly "RH, semisimple, Krein pattern $(+,\dots,+)$" for the generator.

Proof. (a) $h_A(Ax,y)=\omega(Ax,Ay)=-h_A(x,Ay)$, so A is skew for a Euclidean form: semisimple, imaginary spectrum, and invertible since h_A is non-degenerate. For $x\in E_{i\nu}$, $h_A(\bar{x},x)=\omega(\bar{x},i\nu x)=\nu\,h_K(x,x)$, so $h_K>0$ if $\nu>0$. (b) $-A^2$ is h_A-symmetric and positive, and $(-A^2)^{-1/2}$ is a polynomial in A^2, hence ω-self-adjoint. So J_A is ω-skew with $J_A^2=-1$, i.e. compatible. $g_{J_A}(x,x)=\omega(x,A(-A^2)^{-1/2}x)=h_A(x,(-A^2)^{-1/2}x)>0$. (c) By (a) every $E_{i\nu}$ is Krein-definite and A has no eigenvalue 0, so the positive invariant J is unique (the proof of Thm. 3.1 with A in place of F). For the second statement: on $E_{i\nu}$ a commuting J has eigenvalues $\pm i$, and $-JA=\pm\nu$ there; positivity forces $J=i\operatorname{sgn}\nu$. (d) $\operatorname{sgn}\Im e^{i\nu}=\operatorname{sgn}\sin\nu=\operatorname{sgn}\nu$ for $|\nu|<\pi$. Converse: $h_A(\bar{x},x)=\nu h_K(x,x)>0$ on each $E_{i\nu}$ (for $\nu<0$ use $\bar{x}\in E_{-i\nu}$), and the $E_{i\nu}$ are h_A-orthogonal. □

Infinite dimension. Segal [15] and Weinless [16] construct and characterize the vacuum of a linear field from positivity of the energy ([BIB]: titles and records verified; the papers were not read). The abstract of Kay [12] (read, [CITED]) states the uniqueness that we prove here in finite dimension: a one-parameter symplectic group, intertwined with a unitary group of *strictly positive energy* by a symplectic map with dense range, determines that map up to unitary equivalence. Kay–Wald [13] ([BIB]) concerns the uniqueness and thermality of quasi-free states on spacetimes with a bifurcate Killing horizon. It is a different theorem from the one used in the chat.

Proposition 5.2 (Lemma 53's Hilbert-transform vacuum is a positive-energy J_A; [THM])**.** *On real loops $f=\sum_{n\neq 0} f_n e^{in\theta}$ with Lemma 53's form $\omega(f,h)=\frac{1}{2\pi}\int_0^{2\pi} f\,h'\,d\theta$, the rotation generator $A=-\frac{d}{d\theta}$ lies in $\mathfrak{sp}(\omega)$ and has positive energy $h_A(f,f)=\frac{1}{2\pi}\int f'^2\,d\theta$. Its J_A is $J_A e^{in\theta}=-i\,\operatorname{sgn}(n)e^{in\theta}$, which is exactly Lemma 53's Hilbert transform J, with $\omega(f,J_Af)=\sum|n||f_n|^2$. In particular $J_A=-i$ on $H_+=\overline{\operatorname{span}}\{e^{in\theta}\}_{n>0}$.*

Proof. $\omega(Af,h)+\omega(f,Ah)=-\frac{1}{2\pi}\int(f'h'+fh'')=-\frac{1}{2\pi}\int(fh')'=0$. $h_A(f,f)=-\frac{1}{2\pi}\int ff''=\frac{1}{2\pi}\int f'^2$. $Ae^{in\theta}=-in\,e^{in\theta}$ and $-A^2=n^2$, so $J_A=-in/|n|$. The last formula is L53 §3. □

5.2. **Relaxation: Cartan's circumcentre.** $\mathfrak{h}(V)$ with Lemma 51's metric is a Riemannian symmetric space of non-compact type, $\mathrm{Sp}(2g,\mathbb{R})/U(g)$. It is complete, simply connected and of non-positive curvature, hence CAT(0) ([3], chapter on symmetric spaces, [BIB]). In a complete CAT(0) space every non-empty bounded set has a unique circumcentre, and a group of isometries with a bounded orbit fixes the circumcentre of that orbit ([3, II.2.7–2.8]; statement and numbering [CITED] as quoted in [4]).

Proposition 5.3 (the relaxed vacuum; [THM])**.** *For $g_{\mathrm{n}}\in\mathrm{Sp}(V)$ and any $J_0\in\mathfrak{h}(V)$, the following are equivalent: (i) $\{g_{\mathrm{n}}^n J_0 g_{\mathrm{n}}^{-n}\}_{n\in\mathbb{Z}}$ is bounded; (ii) g_{n} fixes a point of $\mathfrak{h}$; (iii) g_{n} is semisimple with all eigenvalues of modulus* 1*. Then the circumcentre $c(J_0)$ of the orbit is a positive invariant vacuum. It equals the unique positive invariant J whenever the latter is unique (Thm. 3.1(a)), for every starting J_0. When the fixed set is larger (mixed Krein type, or $\pm\sqrt{q}$), $c(J_0)$ depends on J_0.*

Proof. (i)⇒(ii) by the circumcentre theorem; (ii)⇒(iii): g_{n} is g_J-orthogonal; (iii)⇒(i): g_{n} is conjugate into $U(g)$, the stabilizer of a point J_1, so the orbit lies in the sphere of radius $d(J_0,J_1)$ about

J_1. If the fixed point is unique, the circumcentre, being fixed, is it. If not, take J_0 in the fixed set: its orbit is $\{J_0\}$ and $c(J_0) = J_0$. □

[NUM] (`D_drift`, `P_W_centre`). For P_W in a Darboux basis of L50's trace model, $\sup_{n\le 1000} d(J_0, g_{\mathrm{n}}^n J_0) = 17.12329 \le 2d(J_0, J_F) = 17.12396$. From the first 200 orbit points, after 3000 Bădoiu–Clarkson geodesic steps the approximate circumcentre is at distance $1.6 \cdot 10^{-3}$, with radius 8.5634 against 8.5620 about J_F. (The Bădoiu–Clarkson iteration converges slowly, like $1/\sqrt{k}$. A Karcher-mean fixed-point iteration from J_0 did not converge on this widely spread orbit, ending at distance 6.46, and is not used.) Starting from an arbitrary J_0, the orbit relaxes to Frobenius's own $J_F = J_+$.

5.3. **Thermal states: KMS.**

Proposition 5.4 (the thermal route needs the same positivity; [THM] (classical); [BIB] [6])**.** *For the quadratic Hamiltonian $E(x) = \frac{1}{2}h_A(x,x)$ on (V,ω):*

(a) The classical Gibbs measure $e^{-\beta E(x)}dx$ ($\beta > 0$) has finite mass iff h_A is positive definite.

(b) If $h_A > 0$, a symplectic normal form gives frequencies $\nu_j > 0$. The quantized $\hat{E}$ (metaplectic/oscillator representation) has $\operatorname{Tr} e^{-\beta\hat{E}} = \prod_j (2\sinh(\beta\nu_j/2))^{-1}$, and as $\beta \to \infty$ the Gibbs state tends to the Fock vacuum of J_A.

Proof. (a) Gaussian integral; along a direction with $h_A(x,x) \le 0$ the integrand does not decay. (b) Williamson normal form $E = \sum \frac{\nu_j}{2}(x_j^2 + y_j^2)$ ([BIB]), the spectrum $\nu_j(n + \frac{1}{2})$ of each oscillator, and the ground state, which is annihilated by the J_A-antiholomorphic combinations. □

Bost–Connes ([CITED], [2] Thm. 5, read at the source). The BC system has a unique KMS_β state for $0 < \beta \le 1$. For $\beta > 1$ the extremal KMS_β states are parametrized by the complex embeddings of $\mathbb{Q}^{\mathrm{cycl}}$, and the partition function is $\zeta(\beta)$. Its partition function is evaluated at real $\beta > 1$, where ζ has no zeros. The zeros appear only after continuation to complex β, which is not a temperature. So the thermal route fixes a state from the dynamics, as advertised, but the known arithmetic example does not see the zeros ([RESTATEMENT] of the chat remark). Reading the zeros as zeros of a partition function at complex β (Lee–Yang/Fisher style) is [ANALOGY] here; we did not look for a source.

Theorem 5.5 (emergence converts "choose" into "exists"; [THM])**.** *For $F \in \mathrm{GSp}(V)$ in finite dimension,* without real eigenvalues*:*

(a) (sign of frequency) J_F exists, and it is positive iff RH + semisimple + Krein pattern $(+,\dots,+)$ (Thm. 2.4);

(b) (energy) $g_{\mathrm{n}} = e^A$ for some $A \in \mathfrak{sp}$ with $\operatorname{spec} A \subset i(-\pi,\pi)$ and positive energy iff RH + semisimple + pattern $(+,\dots,+)$, and then $J_A = J_F$ (Prop. 5.1);

(c) (relaxation) the circumcentre is a positive vacuum iff RH + semisimple, with no condition on Krein patterns (Prop. 5.3);

(d) (thermal) the Gibbs state of such a generator is normalizable iff (b) holds (Prop. 5.4).

Each route produces a positive J exactly when an RH-type positivity holds, and the J is then canonical: unique in (a), (b), (d), and in (c) iff Thm. 3.1(a) holds. None of them produces the positivity.

Proof. Collect the cited statements. For (b): if $h_A > 0$, then A is semisimple with $\operatorname{spec} A \subset i\mathbb{R}$ (Prop. 5.1(a)), so g_{n} is semisimple and unimodular, and the Krein signs of g_{n} on $E_{e^{i\nu}}$ are those of A on $E_{i\nu}$ (same eigenspaces, since $|\nu| < \pi$). Conversely, under RH and semisimplicity with pattern $(+)$, $\operatorname{spec} g_{\mathrm{n}} \cap \mathbb{R} = \emptyset$, and the principal logarithm A (a polynomial in g_{n}, odd under $z \mapsto 1/z$, hence in $\mathfrak{sp}$) has spectrum in $i(-\pi,\pi) \setminus 0$ and $h_A > 0$ by the converse in Prop. 5.1. □

6. The Siegel-orbit drift: a ghost detector

Theorem 6.1 (drift rate; [THM]; Yamamoto's theorem [CITED] (abstract) [17])**.** *Let $g_{\mathrm{n}} \in \mathrm{Sp}(2g,\mathbb{R})$ have eigenvalue moduli $|\lambda_1| \geq \cdots \geq |\lambda_{2g}|$, and let d be Lemma 51's distance.*

(a) In a g_{J_0}-orthonormal symplectic basis, $d(J_0, g_{\mathrm{n}}^n J_0 g_{\mathrm{n}}^{-n}) = 2\big(\sum_{i=1}^{g}(\log s_i(g_{\mathrm{n}}^n))^2\big)^{1/2}$, where $s_1 \geq \cdots \geq s_g \geq 1$ are the top singular values.

(b) The translation length $\ell(g_{\mathrm{n}}) := \lim_{n\to\infty} \frac{1}{n} d(J_0, g_{\mathrm{n}}^n J_0)$ exists, does not depend on J_0, and

$$\ell(g_{\mathrm{n}}) = 2\Big(\sum_{i=1}^{g}(\log|\lambda_i|)^2\Big)^{1/2} = 2\Big(\sum_{|\lambda|>1}\sigma_\lambda^2\Big)^{1/2}, \qquad \sigma_\lambda = \log|\lambda|,$$

with the sum over eigenvalues outside the unit circle, counted with multiplicity. So $\ell = 2|\sigma|$ for one real ghost pair $\pm e^{\pm\sigma}$, $\ell = 2\sqrt{2}|\sigma|$ for one quadruple $e^{\pm\sigma\pm i\theta}$, and $\ell = 2(2\sum_j \sigma_j^2)^{1/2}$ for quadruples $\sigma_1,\dots,\sigma_k$.

(c) The orbit is bounded iff g_{n} is elliptic (Prop. 5.3). If all $|\lambda| = 1$ but g_{n} has a Jordan block of size $k \geq 2$, the orbit is unbounded and $d(J_0, g_{\mathrm{n}}^n J_0) = O(\log n)$. More precisely $2(k-1)\log n - O(1) \leq d \leq 2\sqrt{g}(k-1)\log n + O(1)$. For the genus-1 parabolic, $d = 2\log n + O(1)$. At Lemma 51's Hamiltonian–Hopf point, $d = 2\sqrt{2}\log n + O(1)$.

So the detector has three outcomes: bounded (RH, semisimple), logarithmic (collision on the circle), linear with slope ℓ (ghosts, and ℓ measures $\big(\sum\sigma^2\big)^{1/2}$).

Proof. (a) Cartan decomposition $g_{\mathrm{n}}^n = k_1 a k_2$ with $k_i \in U(J_0) = O(2g) \cap \mathrm{Sp}$ and $a = \mathrm{diag}(s_1,\dots,s_g,s_1^{-1},\dots,s_g^{-1})$. Since k_2 commutes with J_0, $g_{\mathrm{n}}^n J_0 g_{\mathrm{n}}^{-n} = k_1 a J_0 a^{-1} k_1^{-1}$, and in Lemma 51's normalization a with entries $e^{d_i/2}$ produces rapidities d_i, so $d_i = 2\log s_i$. (b) Yamamoto: $\lim_n s_i(M^n)^{1/n} = |\lambda_i(M)|$ for every i [17]. Symplectic spectra and singular values come in pairs x, x^{-1}, so the top g moduli are ≥ 1. Independence of J_0: $|d(x, g_{\mathrm{n}}^n x) - d(y, g_{\mathrm{n}}^n y)| \leq 2d(x,y)$. (c) Bounded iff elliptic is Prop. 5.3. In Jordan form, $\|g_{\mathrm{n}}^n\| \asymp n^{k-1}$ (binomial coefficients), so $\log s_1 = (k-1)\log n + O(1)$, and $\log s_1 \leq \big(\sum_{i\leq g}\log^2 s_i\big)^{1/2} \leq \sqrt{g}\log s_1$. Genus-1 parabolic: g_{n}^n is conjugate to $\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$, so $s_1 \sim n$. HH point: $g_{\mathrm{n}} = R\,e^N$ with R orthogonal-symplectic and commuting with J_0 (L51 Thm. 4.2 proof) and N the nilpotent part of the oscillator, $e^{nN} = \left(\begin{smallmatrix}1 & 0\\ -n & 1\end{smallmatrix}\right) \otimes 1_2$ in L51's ordering. So $s_1 = s_2 \sim n$ and $d = 2\sqrt{2}\log n + O(1)$. □

Remark 6.2 (normalization)**.** With ℓ per Frobenius step in Lemma 51's metric (curvature -1 in genus 1), a real ghost pair drifts at 2σ, twice $\log(|\alpha|/\sqrt{q})$. The "2" is the familiar factor in $d(i, e^{2\sigma}i) = 2\sigma$. For a flow $e^{\tau A}$ the rate per unit time is $2\big(\sum_{\Re\mu>0}(\Re\mu)^2\big)^{1/2}$. Several ghosts combine in ℓ^2, not in ℓ. By contrast Lemma 48's gain defect $\sum_{|\lambda|>1}\sigma_\lambda = \log M$ is additive. The two agree, up to the factor 2, exactly when one σ is non-zero.

[NUM] (`emergent.json`, `B_genus1`, `D_drift`; mpmath, 120 digits for increments and 700 digits for the $n = 200, 400$ slopes):

case	predicted ℓ	measured	note
genus 1, $q = 13$, $a = 6$ (Weil)	0; $\sup d \leq 2r = 2.389526$	$\sup_{n\leq 200} d = 2.389478$	$\cosh d_n = \cosh^2 r - \sinh^2 r\cos 2n\varphi$ to $3\cdot 10^{-59}$
genus 1, $a = 2\sqrt{13}$ (parabolic)	0, $d - 2\log n \to$ const	$d - 2\log n = 1.3913,\ 1.38634,\ 1.386295,\ 1.3862944\ (\to 2\log 2)$	$n = 10, 10^2, 10^3, 10^4$
genus 1, $a = 8$ (real ghost, $\eta = 0.463599$)	$2\eta = 0.9271974$	$d_{n+1} - d_n = 0.93675,\ 0.92719740,\ 0.92719739$	$n = 5, 20, 50$
L51 HH path, $t_* + 10^{-2}$	$2\sqrt{2}\eta = 0.2106342$	$0.31771,\ 0.21163,\ 0.2106343$	$n = 10, 40, 100$
L51 HH path at P_Q	$2\sqrt{2}\log 1.404113 = 0.9599846$	$0.96097,\ 0.9599846,\ 0.9599846$	$n = 10, 40, 100$
L51 HH point t_*	$d/(2\sqrt{2}\log n) \to 1$	$1.0043,\ 1.000022,\ 1.00000014,\ 1.000000001$	$n = 10, 10^2, 10^3, 10^4$
L50 trace model, P_Q	0.9599846	$\mathrm{slope}_{200\to 400} = 0.95917$	non-orthonormal chart
L50 trace model, P_R	$2\log 3.832949 = 2.687269$	$\mathrm{slope}_{200\to 400} = 2.68676$	
L50 trace model, P_W	0	$\sup_{n\leq 1000} d = 17.12329$	bounded, $\leq 2d(J_0, J_F) = 17.12396$

In the orthonormal normal-form charts (genus 1, HH) the single-step increments converge to ℓ to 8–9 digits by $n = 50$–100. In the trace model's Darboux basis the bounded part of d_n oscillates with amplitude $O(1)$, and only the long-baseline slope converges.

7. Finite truncations of zero sets, and ζ

7.1. **Truncations.** Given finitely many zeros $\rho = \beta + i\gamma$ ($\gamma > 0$) of a function with a functional equation $s \leftrightarrow 1 - s$ and real coefficients, put $\mu = \rho - \frac{1}{2} = \sigma + i\gamma$. Let A be the real block-diagonal Hamiltonian matrix with eigenvalues $\pm i\gamma$ (one 2×2 rotation block per on-line zero) and $\pm\sigma \pm i\gamma$ (one 4×4 block $\mathrm{diag}(M, -M^{\mathsf{T}})$, $M = \left(\begin{smallmatrix} \sigma & -\gamma \\ \gamma & \sigma \end{smallmatrix}\right)$, per ghost quadruple). Conjugate it by a random symplectic matrix so that J_0 is not adapted. This is a *proxy*: a finite-dimensional symplectic flow with the right spectrum, not a construction from the arithmetic.

Proposition 7.1 (ghost count = negative index of the canonical vacuum; [THM], checked [NUM])**.** *If $A \in \mathfrak{sp}$ is semisimple with no real eigenvalue, then $J_A := i\,\mathrm{sign}(-iA)$ (= $i\,\mathrm{sgn}\,\Im\mu$ on E_μ) is a real, invariant, compatible complex structure, and g_{J_A} has negative index*

$$n_-(g_{J_A}) = \#\{\textit{on-line pairs with Krein sign } -\} + 2 \cdot \#\{\textit{ghost quadruples}\}.$$

In the truncation above every on-line block has sign $+$, so $n_- = 2k$ with k the number of quadruples (that is, of ghosts ρ with $\beta > \frac{1}{2}$, $\gamma > 0$), and J_A is positive iff $k = 0$.

Proof. As in Thm. 2.2(c), with $\mathrm{sign}(-iA)$ in place of the Riesz projectors (no eigenvalue of $-iA$ on $i\mathbb{R}$ because A has no real eigenvalue). By Thm. 2.4(a),(d), g_{J_A} is the orthogonal sum of $\pm h_K$ on the unimodular eigenspaces and of a neutral form of signature $(2,2)$ on each quadruple block $E_\mu \oplus E_{\bar\mu} \oplus E_{-\mu} \oplus E_{-\bar\mu}$: the pairs $E_\mu, E_{-\bar\mu}$ are H-isotropic and H-dual. For a rotation block γJ_0, $J_A = J_0$ is positive. □

[NUM] (`zeta.json`; DH data of L48: 28 ghosts and 52 on-line zeros below height 100; ζ: first 50 zeros). Errors: J_A imaginary part $\le 7 \cdot 10^{-16}$, $\|J_A^2 + 1\| \le 4 \cdot 10^{-15}$, compatibility $\le 8 \cdot 10^{-15}$, commutator $\le 8 \cdot 10^{-12}$.

truncation	dim	sig. g_{J_A}	predicted n_-	L53 charge of the finite Blaschke product
DH, 28 ghosts + 52 on-line	216	$(160, 56)$	56	-56
DH, 28 ghosts only	112	$(56, 56)$	56	-56
ζ, first 50 zeros	100	$(100, 0)$	0	0

The last column is Lemma 53 Thm. 4.8's charge $-k$, which counts *all* zeros in $\Re s > \frac{1}{2}$ (here ρ and $\bar\rho$ for each of the 28 ghosts). On truncations, then, the negative index of Frobenius's own vacuum equals minus Lemma 53's ghost charge ([THM] for truncations; for DH itself the charge does not exist, L53 Thm. 4.3, since DH has infinitely many ghosts).

Drift of the DH proxy ([NUM], `emergent.json`, `F_proxies`; time step $\tau = 1$, float64, $n \le 60$). $\sum_{j=1}^{28} \sigma_j^2 = 1.651561$ and $\max \sigma_j = 0.369531$. The prediction from Thm. 6.1(b) is $\ell = 2(2\sum \sigma_j^2)^{1/2} = 3.634898$ per unit time.

truncation	dim	predicted ℓ	slope $(d_{60} - d_{30})/30$	$d_{60} - d_{59}$
DH, 28 ghosts	112	3.634898	3.634402	3.633602
DH, 28 ghosts + 52 on-line	216	3.634898	3.632551	3.634886
ζ, first 50 zeros	100	0	0.0031	$\sup_{n \le 400} d = 7.600 \le 2r = 10.029$

The on-line zeros do not change the slope (they are elliptic). The approach to ℓ is slow because the ghost rapidities range from $\sigma = 0.016$ to 0.37, so the smaller ones take $n \gg 1/\sigma$ steps to dominate their $O(1)$ offsets.

7.2. **What ζ would need.** Lemma 50 Q. 7.2 listed (S1) a real symplectic "H^1", (S2) a flow with the derivation identity, (S3) an invariant positive J. In the present language:

- Given (S1)–(S2) with generator $A = \Theta - \frac{1}{2}$ (spectrum $\rho - \frac{1}{2}$), the canonical $J_A = i\,\mathrm{sign}(-iA)$ exists as soon as ζ has no real zero in $(0, 1)$. That is true: $\zeta(\sigma) < 0$ there (as used in L53). On any finite truncation it is real, invariant and compatible *whether or not RH holds* (Prop. 7.1).

So, for ζ, Frobenius's own J would exist unconditionally, and *the whole of RH would sit in its positivity* ([THM] on truncations; [OPEN] for ζ, since (S1)–(S2) are not constructed).

- If A is skew for a positive g_J commuting with it, the spectrum is on $i\mathbb{R}$ (L50 Prop. 7.1, Deninger, [RESTATEMENT]). Spectrally, RH $\iff$ $\operatorname{spec}(A^2) \subset (-\infty, 0]$, because $(\sigma + i\gamma)^2$ is real and ≤ 0 iff $\sigma = 0$. That is a trivial restatement. As an *operator* inequality $A^2 \le 0$ needs the Hilbert structure that is missing ([RESTATEMENT]).
- Lemma 53's Segal–Wilson vacuum H_+ is J_A for the rotation generator (Prop. 5.2), so it is "Frobenius's own J" of the circle. Lemma 48's "RH $\iff$ outer" and Lemma 53's charge are, on truncations, the statement $n_-(g_{J_A}) = 0$ (Prop. 7.1). This is a dictionary entry ([RESTATEMENT]), not a new criterion.
- The relaxation route needs bounded Siegel orbits of the flow, which on truncations means exactly RH (Thm. 6.1). For ζ the flow is unknown ([OPEN]).

8. v2: invariants that see every ghost (J.W. answer 5 to Lemma 56 v2)

Throughout, $k = k(F)$ is the number of eigenvalues of F with $|\alpha| > \sqrt{q}$, counted with multiplicity: 1 for each real pair $\{\alpha, q/\alpha\}$ and 2 for each quadruple $\{\alpha, \bar\alpha, q/\alpha, q/\bar\alpha\}$. This is Lemma 48's ghost count. v1's index $n_-(g_{J_F})$ needs J_F, which does not exist when F has a real eigenvalue (Thm. 2.2(b)), i.e. exactly for real-pair ghosts. v1's ℓ is defined always, but it is a size, not a count.

8.1. The frequency form.

Definition 8.1 (frequency form and its index)**.** For $F \in \mathrm{GSp}(V, \omega)$ with multiplier q, put

$$h_F(x, y) := \omega(x, (F - Ve)y), \qquad \nu(F) := n_-(h_F), \quad \nu_+(F) := n_+(h_F), \quad \nu_0(F) := \dim\ker h_F,$$

and $w(F) := \min(\nu, \nu_+)$, the Witt index. For a flow generator $A \in \mathfrak{sp}(V, \omega)$, put $h_A(x, y) = \omega(x, Ay)$, the energy form, and define $\nu(A)$ in the same way.

Theorem 8.2 (structure of h_F; [THM], checked [NUM])**.** *(a) h_F is symmetric and F-conformal: $h_F(Fx, Fy) = q\, h_F(x, y)$. It is defined for every F (no square root), and $\ker h_F = \ker(F^2 - q)$, so $\nu_0 = 0$ iff $\pm\sqrt{q} \notin \operatorname{spec} F$.*

(b) In $V_{\mathbb{C}} = \bigoplus E_\alpha$, $h_F(E_\alpha, E_\beta) = 0$ unless $\alpha\beta = q$. On every ghost block (a real pair $E_\alpha \oplus E_{q/\alpha}$, or a quadruple), h_F is non-degenerate with a Lagrangian-type isotropic half, so it is neutral*: $(1, 1)$ for a real pair and $(2, 2)$ for a quadruple. This holds also for non-semisimple blocks.*

(c) On the real plane of a semisimple unimodular pair $\alpha = \sqrt{q}e^{i\theta}$, $\theta \in (0, \pi)$, $h_F(x + \bar x, x + \bar x) = 4\sqrt{q}\sin\theta\, h_K(x, x)$ for $x \in E_\alpha$. So the plane contributes $2n_\mp(h_K|_{E_\alpha})$ to $\nu_\pm$ (sign of h_K per Krein type).

(d) Hence, if the unimodular part is semisimple and $\nu_0 = 0$:

$$\nu = k + 2m_-, \qquad \nu_+ = k + 2m_+, \qquad w = k + 2\min(m_+, m_-),$$

where $m_\pm = \sum_{\Im\alpha > 0, |\alpha| = \sqrt{q}} n_\pm(h_K|_{E_\alpha})$. In particular $\nu = k$ iff every Frobenius plane is Krein-positive, and $w \ge k$ always. Replacing ω by $-\omega$ swaps ν and ν_+ and fixes k and w.

(e) If F has no real eigenvalue, then $h_F(x, y) = g_{J_F}(x, D^{1/2}y)$ and $\nu(F) = n_-(g_{J_F})$: v1's index is the special case of ν.

The flow version is the same. For $A \in \mathfrak{sp}$, $h_A(E_\mu, E_{\mu'}) = 0$ unless $\mu + \mu' = 0$; ghost blocks are neutral; an on-line pair $\pm i\gamma$ ($\gamma > 0$) contributes $2n_\mp(h_K)$. So $\nu(A) = \#\{\operatorname{Re}\mu > 0\} + 2m_-$.

Proof. (a) $M := F - Ve$ is ω-skew (§2, (a) of Thm. 2.2). So $h(y, x) = \omega(y, Mx) = -\omega(Mx, y) = \omega(x, My) = h(x, y)$. M commutes with F, so $h(Fx, Fy) = \omega(Fx, FMy) = q\, h(x, y)$. $h(x, \cdot) = 0$ iff $\omega(Mx, \cdot) = 0$ iff $Mx = 0$ iff $F^2x = qx$. (b) M preserves each E_β, and $\omega(E_\alpha, E_\beta) = 0$ unless $\alpha\beta = q$ (proof of Thm. 2.4(a)). For a ghost block $W = U \oplus U'$, take U the real part of E_α (or $E_\alpha \oplus E_{\bar\alpha}$) and U' that of $E_{q/\alpha}$ (or $E_{q/\alpha} \oplus E_{q/\bar\alpha}$). U and U' are h-isotropic, because $\alpha^2 \ne q$ and $|\alpha|^2 \ne q$. M is invertible on W and ω pairs U with U', so $h|_W$ is non-degenerate with an isotropic subspace of

half the dimension. Hence it has signature $(\frac{1}{2}\dim W, \frac{1}{2}\dim W)$. The number of eigenvalues outside the circle in W is $\frac{1}{2}\dim W$. (c) For $x \in E_\alpha$, $h(\bar{x}, x) = (\alpha - q/\alpha)\omega(\bar{x}, x) = 2i\sqrt{q}\sin\theta\,\omega(\bar{x}, x) = 2\sqrt{q}\sin\theta\, h_K(x, x)$, and $h(x, x) = 0$ ($\alpha^2 \neq q$). So $h(x + \bar{x}, x + \bar{x}) = 2h(\bar{x}, x)$. (d) This is (b)+(c), since the blocks are h-orthogonal. $-\omega$ flips h and every Krein sign. (e) $M = J_F D^{1/2}$ (Def. 2.1). D is ω-self-adjoint and commutes with J_F, so $D^{s/2}$ ($0 \le s \le 1$, principal branch, defined because D has no spectrum in $(-\infty, 0]$ by Thm. 2.2(b)) is g_{J_F}-self-adjoint and invertible. Hence $g_{J_F}(\cdot, D^{s/2}\cdot)$ is a continuous path of non-degenerate symmetric forms. Its signature is constant in s; it equals that of g_{J_F} at $s = 0$ and that of h_F at $s = 1$. (This covers Jordan blocks too.) Flow version: $\omega(Ax, y) = -\omega(x, Ay)$ gives $(\mu + \mu')\omega(x, y) = 0$ for eigenvectors; the rest is as above, with $\mu = i\gamma$ in place of $\alpha - q/\alpha$. □

Remark 8.3 (prior art: the flow version is the Hamiltonian–Krein index theorem; [RESTATEMENT], [CITED])**.** The flow statement $\nu(A) = \#\{\operatorname{Re}\mu > 0\} + 2m_-$ is, in finite dimension and with $\ker A = 0$, the count $k_r + 2k_c + 2k_i^- = n(L)$ of Kapitula–Kevrekidis–Sandstede [11, Thm. 3.3, Def. 2.4] (text read). There $A = JL$, L is the energy (our h_A), k_r is the number of positive real eigenvalues, k_c the number of quadruples, and k_i^- the number of negative Krein squares over $\operatorname{Im}\mu > 0$. KKS attribute the method to Grillakis and coworkers. So the flow half of Thm. 8.2 is a *restatement*, and the map half (h_F, multiplier q) is its discrete-time version. We did not find the map version stated with $h_F = \omega(\cdot, (F - Ve)\cdot)$, but the search was short and we claim no novelty for it.

Theorem 8.4 ($\nu = k$ in Lemma 50's trace model, real pairs included; [THM] from Hermite [CITED])**.** *Let $V = \mathbb{R}[T]/(P)$ with $\omega_P(x, y) = \operatorname{Tr}(c\,xy^\sharp)$, $c = T - q/T$, $F = T$ (L50 Thm. 4.4). Write $P(T) = T^g Q(T + q/T)$ with $Q \in \mathbb{R}[u]$ of degree g, and assume that Q has distinct roots, none equal to $\pm 2\sqrt{q}$. Then*

$$h_F(x, x) = \operatorname{Tr}(D\,xx^\sharp), \qquad D = 4q - (F + Ve)^2, \qquad \text{and} \qquad \nu(F) = k(F).$$

This holds whether the ghosts are quadruples or real pairs; no square root and no real-eigenvalue condition is needed.

Proof. $h(x, x) = \omega_P(x, cx) = \operatorname{Tr}(c\,x\,c^\sharp x^\sharp) = -\operatorname{Tr}(c^2 xx^\sharp) = \operatorname{Tr}(Dxx^\sharp)$, since $c^\sharp = -c$ and $c^2 = u^2 - 4q = -D$ with $u = T + q/T$. Let $A^\sharp = \mathbb{R}[u]/(Q)$ be the $\sharp$-invariant subalgebra. Then $A = A^\sharp \oplus A^\sharp c$ ($T = \frac{1}{2}(u + c)$), $\operatorname{Tr}_A(a) = 2\operatorname{Tr}_{A^\sharp}(a)$ for $a \in A^\sharp$, and $\operatorname{Tr}_A(A^\sharp c) = 0$ (since $c + c^\sharp = 0$). For $x = a + bc$, $xx^\sharp = a^2 + Db^2$, so

$$h(x, x) = 2\operatorname{Tr}_{A^\sharp}(D\,a^2) + 2\operatorname{Tr}_{A^\sharp}(D^2 b^2) = 2\operatorname{Her}(Q, D)(a) + 2\operatorname{Her}(Q, D^2)(b),$$

a sum of two Hermite forms. By Hermite's theorem [1, Thm. 4.57] (read at the source: rank $= \#\{x \in \mathbb{C} : Q(x) = 0,\ w(x) \neq 0\}$ and signature $= \sum_{x\in\mathbb{R},\, Q(x)=0} \operatorname{sgn} w(x)$ for the form $\operatorname{Her}(Q, w)$):

- $\operatorname{Her}(Q, D)$ has rank g and signature $\#\{\text{real } u : |u| < 2\sqrt{q}\} - \#\{\text{real } u : |u| > 2\sqrt{q}\}$, hence $n_- = R + C$, where R is the number of real roots with $|u| > 2\sqrt{q}$ and C the number of conjugate pairs of non-real roots;
- $\operatorname{Her}(Q, D^2)$ has rank g and signature $\#\{\text{real } u\}$, hence $n_- = C$.

Under $T^2 - uT + q = 0$: a real u with $|u| < 2\sqrt{q}$ is a Weil pair; a real u with $|u| > 2\sqrt{q}$ is a real pair; a non-real u is a quadruple (a unimodular α has $u = 2\sqrt{q}\cos\theta \in \mathbb{R}$). So $\nu = R + 2C = k$. □

In Thm. 8.2(d) terms, the trace model is a Krein-positive chart (Prop. 2.5: pattern $(+, \dots, +)$), and Thm. 8.4 is the case $m_- = 0$ *without* assuming that J_F exists.

8.2. **The drift rank.**

Theorem 8.5 (drift rank; [THM] from Yamamoto [CITED])**.** *Let $r(F) := \#\{\lambda \in \operatorname{spec} g_{\mathrm{n}} : |\lambda| > 1\}$ (with multiplicity), and for flows $r(A) = \#\{\operatorname{Re}\mu > 0\}$.*

(a) $r = k$.

(b) $r = \#\{i \le g : \lim_n \frac{1}{n} d_i(g_{\rm n}^n) > 0\}$, *the number of* linearly growing rapidities *of the Siegel orbit* $g_{\rm n}^n J_0$ *(Thm. 6.1(a),* $d_i = 2\log s_i$*).*
(c) r *does not depend on* J_0*, on* ω*, or on the chart, and it is invariant under conjugation in* $GL(V)$*.*
(d) Jordan blocks on the circle contribute 0 *(their rapidities grow like* $\log n$*).*
(e) $\ell = 2(\sum_{i\le r}\sigma_i^2)^{1/2}$*, so* $\ell > 0$ *iff* $r > 0$*, and* $\ell \le 2\sqrt{r}\max\sigma$*.*

Proof. (a) is the definition of k with $g_{\rm n} = F/\sqrt{q}$. (b) Yamamoto: $s_i(g_{\rm n}^n)^{1/n} \to |\lambda_i|$ [17], and the top g moduli are ≥ 1; the number of them > 1 is $r \le g$. (c) Spectral. (d),(e) are Thm. 6.1(b),(c). □

Corollary 8.6 (truncations: $\nu = r = -$charge; [THM])**.** *On the finite truncations of §7, enlarged by real-pair blocks* $\mathrm{diag}(s,-s)$ *for real zeros* $\frac{1}{2} \pm s$*, every on-line block has Krein sign* $+$*. So*

$$\nu(A) = r(A) = 2\#\{\textit{quadruples}\} + \#\{\textit{real pairs}\} = \#\{\textit{zeros of the truncation with } \mathrm{Re}\, s > \tfrac{1}{2}\},$$

which is minus Lemma 53's charge of the finite Blaschke product. In Lemma 56's notation this is $2\kappa + R$ *on full symmetric windows. For* ζ *truncations,* $\nu = r = 0$ *whatever the number of zeros, because no ghost is input.*

Remark 8.7 (chart dependence and which invariant to use)**.** ν depends on the Krein chart through m_-, and r does not. In a mixed chart such as Lemma 51's Hamiltonian–Hopf chart, $\nu = 2$ with $k = 0$. So ν, like J_F, does not register an HH birth inside a fixed mixed chart: it already reads 2 before the birth. r registers the birth, jumping $0 \to 2$ when the quadruple forms. Recommendation: use r as the chart-free ghost count, ℓ as its size, and ν (or w) where a Krein-positive chart is given, as in the trace model and the truncations, where it equals k without any spectral computation.

8.3. **Numerics.** [NUM] (`num/lemma54v2_invariants.py`: mpmath trace model, 30–50 digits, 30-digit for the random cases; `num/lemma54v2_proxies.py`: float64 truncations conjugated by a random symplectic matrix $e^{\Omega^{-1}S}$, $\|S\| = 0.3$–0.5; `num/lemma54v2_ffrows.py`; data `num/l54v2/{invariants_trace,proxies,ffrows}.json`). $r_{\rm orb}$ is the finite-time estimate: the number of QR-Lyapunov exponents above $3\log n/n$ after $n = 300$ steps. "undef" means J_F does not exist.

trace-model case (q)	k	ν	ν_+	r	$r_{\rm orb}$	v1 n_-	trace id.
L50 P_W, Weil, $g = 2$ (11)	0	0	4	0	0	0	0
L50 P_Q, quadruple (11)	2	2	2	2	2	2	$5\cdot10^{-47}$
L50 P_R, real pair + Weil (11)	1	1	3	1	1	undef	$1\cdot10^{-46}$
$g = 1$ curve $a = 6$ (13)	0	0	2	0	0	0	0
$g = 1$ fakes $a = 8$ (13), 7 (11), 13 (13)	1	1	1	1	1	undef	0
$g = 2$, two real pairs (11)	2	2	2	2	2	undef	$5\cdot10^{-17}$
$g = 2$, real pair + Weil (11)	1	1	3	1	1	undef	$8\cdot10^{-16}$
$g = 3$, real pair + quadruple (11)	3	3	3	3	3	undef	$4\cdot10^{-15}$
$g = 3$, real pair + 2 Weil (11)	1	1	5	1	1	undef	$4\cdot10^{-17}$
$g = 4$, 2 real pairs + quadruple (11)	4	4	4	4	4	undef	$1\cdot10^{-13}$
L56 $g = 2$, $(c_1, c_0) = (-5, 9)$ (1)	2	2	2	2	2	2	0
L56 $g = 3$, $1 + 10u^2 + 10u^4 + u^6$ (1)	2	2	4	2	2	2	0
L56 FF rows (19: 10 curve, 6 fake, 3 extra)	k	k		k	k	k/undef	$\le 6\cdot10^{-37}$

Random trace models (1500, genus 1–5; each u is Weil, a real pair or part of a quadruple with probabilities 0.45/0.25/0.30). 20 (all of genus 4 or 5) were flagged $\nu_0 > 0$ by the numerical rank tolerance although none has an eigenvalue at $\pm\sqrt{q}$. This is an ill-conditioning artefact at 30 digits, and these cases were excluded, not counted as successes. Of the other 1480: $\nu = k$ and $w = k$ in all 1480, including all 779 with real pairs; $r_{\rm orb} = k$ in all 295 tested (the first 300, minus 5 degenerate). v1's n_- was defined in 701 cases and equal to k in 700. By Thm. 8.2(e) the remaining case must be a numerical failure of the v1 eigenvector-based J_F (an ill-conditioned eigenbasis is the likely cause);

it was not diagnosed further. *Chart test* (Lemma 51-type chart, two Weil planes, $k = 0$). Krein signs $(+,+)$: $\nu = 0$, $w = 0$. $(+,-)$: $\nu = 2$, $w = 2$. $(-,-)$: $\nu = 4$, $w = 0$. $r_{\mathrm{orb}} = 0$ in all three.

truncation (random symplectic conjugation)	dim	k	ν	ν_+	r_{orb}	v1 n_-
ζ, first 100 zeros	200	0	0	200	0	0
ζ, all 649 zeros to height 1000	2000	0	0	2000	—	0
ζ, 50 zeros + real pair $\beta = 0.7$	102	1	1	101	1	undef
DH, 28 ghosts + 52 on-line	216	56	56	160	54*	56
DH, 28 ghosts + all 848 on-line	1808	56	56	1752	—	56
DH, 28 + 52 + real pair $\beta = 0.8$	218	57	57	161	55*	undef
toy: quadruple ($\beta - \frac{1}{2} = 0.3$), real pair (0.4), 3 on-line	12	3	3	9	3	undef
toy: real pairs $\beta = 0.55, 0.9$ + 5 on-line	14	2	2	12	1*	undef
toy: three real pairs ($s = 0.01, 0.1, 0.45$)	6	3	3	3	2*	undef

Here $k = \#\{\mathrm{Re}\,\mu > 0\}$. *The finite-time estimate misses exponents below the threshold $3\log 300/300 = 0.057$: the smallest-δ DH ghost ($\beta - \frac{1}{2} = 0.0159$, two exponents), and the real pairs with $s = 0.05$ and $s = 0.01$. The spectral r equals k in every row. ζ is reported ghost-free by ν, r and v1 n_- alike.

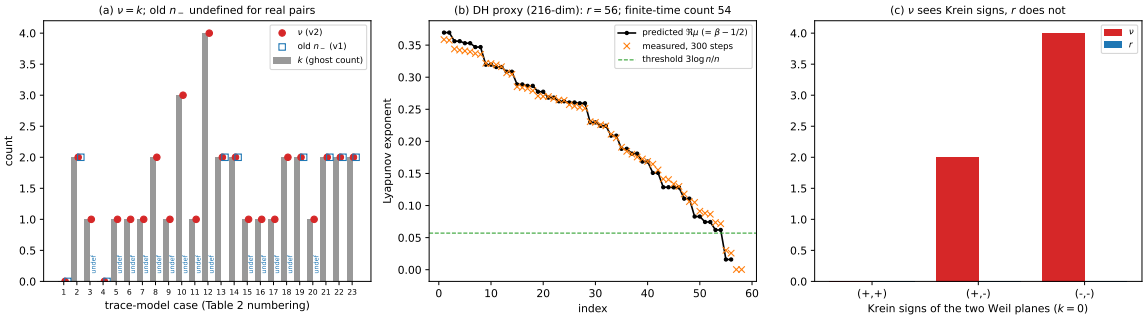

FIGURE 1. Lemma 54 v2 invariants [NUM]. (a) The 14 named trace-model cases (1–14 of the first table) followed by the 9 L56 FF rows with $k > 0$: ghost count k (bars), ν (dots), and v1's n_- (squares; "undef" for real pairs). (b) Lyapunov spectrum of the 216-dimensional DH truncation after 300 steps against the predicted $\mathrm{Re}\,\mu = \beta - \frac{1}{2}$; the threshold line explains $r_{\mathrm{orb}} = 54$. (c) Chart dependence at $k = 0$: ν counts Krein-negative planes, r does not.

9. v2 corrections to Grok Bot's earlier claims

These corrections concern claims made by Grok Bot (in v1 of this lemma and in the Lemma 56 v2 chat), not by J.W.

(C1) *v1 Q3 and abstract (7) presented $n_-(g_{J_A})$ as "the" standard ghost index.* It is undefined, not zero, whenever a real-pair ghost is present (Thm. 2.2(b)), so as a diagnostic it was incomplete. v2 replaces it by ν, which agrees with it wherever it is defined (Thm. 8.2(e)), and by r.

(C2) *The Lemma 56 v2 question (Q5) said the v1 invariants "do not see real pairs".* This was half wrong. ℓ *does* see them ($\ell = 2|\sigma|$, Thm. 6.1(b)), and only the index was blind. What ℓ cannot do is count, since several ghosts combine quadratically. r supplies the count.

(C3) *Novelty.* The flow form of v2's index theorem is the Hamiltonian–Krein index theorem of KKS (Rem. 8.3), and the trace-model computation is Hermite's theorem. Neither is new.

What is new here is only the assembly: the map form via h_F, $\nu = k$ in Lemma 50's model, and the use of r as the chart-free count.

(C4) *Finite-time orbit counts.* The r_{orb} estimates undercount when some σ is below $3 \log n/n$ (DH: 54 instead of 56). r itself is spectral and exact. Earlier drift tables in v1 were slopes, not counts, and they are unaffected.

10. Corrections to the chat claims (made to J.W. on 2026-10-22)

- **(i) "$J = (F - V)/\sqrt{4q - (F + V)^2}$ is real exactly when $|a| < 2\sqrt{q}$ (Hasse)."** *Correct in genus 1, wrong as a general criterion.* In genus 1, $J_F = (2F - a)/\sqrt{4q - a^2}$ is real iff $|a| < 2\sqrt{q}$ (Cor. 2.3). Hasse's bound itself is $|a| \leq 2\sqrt{q}$, with equality for square q, where $D = 0$, J_F is undefined, and a real curve has scalar Frobenius (L51 Prop. 3.2). In genus ≥ 2, with the principal root, J_F is real iff F has *no real eigenvalue* (Thm. 2.2(b)). For the quadruple fake P_Q it is real, invariant and compatible, of signature $(2, 2)$. RH is detected by the *positivity* of J_F, in a chart with Krein pattern $(+, \ldots, +)$ such as L50's trace model (Thm. 2.4, Prop. 2.5), not by its reality. The chat's "the root goes imaginary for a ghost" holds only for real ghosts. For quadruples $D(\alpha)$ is complex and the root is still a legitimate (principal) root.
- **(ii) "The invariant compatible J is unique up to Krein signs if it exists."** *Correct for simple frequencies, false in general.* With distinct roots on the circle, the invariant compatible J are the 2^g J_ε and exactly one is positive (L50 Thm. 4.4). If a unimodular eigenvalue is repeated with mixed Krein type, the positive invariant J form $U(p, q)/U(p) \times U(q)$, a non-compact family (Thm. 3.1; numerically a family with $d(J_0, J_s) = 2, 6, 20$). If $\pm\sqrt{q}$ is an eigenvalue (real curves at Hasse equality), they form a whole Siegel space. "Unique up to Krein signs" should read "unique when every eigenspace is Krein-definite and $\pm\sqrt{q} \notin \operatorname{spec} F$".
- **(iii) "A ghost's Siegel orbit drifts at a rate set by $\log(|\alpha|/\sqrt{q})$."** *Correct in substance; the precise rate is* $\ell = 2\big(\sum_{|\lambda|>1} \sigma_\lambda^2\big)^{1/2}$ per step in Lemma 51's metric (Thm. 6.1): $2|\sigma|$ for a real pair and $2\sqrt{2}|\sigma|$ for a quadruple. Several ghosts combine quadratically. Collisions on the circle drift like $\log n$. Measured: genus 1 0.92719739 vs. 0.92719739; HH at P_Q 0.9599846 vs. 0.9599846; DH proxy 3.6349 vs. 3.6349.
- **(iv) "Positive Hamiltonian $\Rightarrow$ unique invariant compatible $J = A/|A|$ (Segal, Kay–Wald)."** *Correct in finite dimension* (Prop. 5.1; here $|A| = (-A^2)^{1/2}$). Positivity of the Hamiltonian is itself the RH-type condition: it is equivalent to RH + semisimple + Krein pattern $(+)$ for the generator. In infinite dimension one needs strictly positive energy and a density condition (Kay 1979, abstract read). The attribution should be Segal–Weinless–Kay [15, 16, 12]. Kay–Wald [13] is about quasi-free states on spacetimes with a bifurcate Killing horizon.
- **(v) "Frobenius = boost$(\frac{1}{2}\log q)\times$rotation$(\theta)$; a ghost is a second boost of rapidity σ."** *Imprecise.* $\sqrt{q}$ is central in GSp and acts *trivially* on Siegel space (Prop. 4.2(b)). It is a genuine boost only in Lemma 14's Lorentzian $\mathrm{NS}(C \times C)_{\mathbb{R}}$, a different space. On H^1, $g_{\mathrm{n}} = e\,h\,u$ (Jordan). RH is $h = u = 1$. A ghost is $h \neq 1$, the only boost that moves vacua, and its rapidity is 2σ per step in Lemma 51's metric (or σ in eigenvalue units). There is also a third factor, u: a collision on the circle is neither rotation nor boost.
- **(vi) "Wick rotation names the ghost but does not exclude it; OS positivity decides."** *First half correct* ([THM], Rem. 4.4(a)). *Second half an analogy* ([ANALOGY]). Its precise TAA form is that an invariant positive form decides (L50 Thm. 4.4, [RESTATEMENT]), equivalently positive energy (Prop. 5.1). No reflection-positivity structure for ζ is known.
- **(vii) Sign convention, Hilbert transform (chat, [t200]: "i on H_+").** In Lemma 53 the Hilbert transform is $Je^{in\theta} = -i\,\mathrm{sgn}(n)e^{in\theta}$, i.e. $-i$ on H_+, and that sign is the one that makes $\omega(f, Jf) = \sum |n||f_n|^2 > 0$ for L53's ω (Prop. 5.2). With "$+i$ on H_+" the form is negative unless ω is replaced by $-\omega$.

- **(viii) "Frobenius builds its own J; it can do so precisely when RH holds."** *Corrected.* Frobenius builds J_F whenever it has no real eigenvalue, RH or not. What holds precisely under RH (with semisimplicity) is that *some* positive invariant J exists. Frobenius's own J_F is that J iff the Krein pattern is $(+,\dots,+)$ (Thm. 2.4). The relaxation route matches RH exactly (Thm. 5.5(c)).

11. What is new, what is restatement, and prior art

Restatement / classical: Thm. 2.4(c) (L50 Thm. 4.4); Prop. 4.2 (Jordan decomposition); Prop. 4.3(a) (classical SL_2 trichotomy); Prop. 5.1 (Segal–Weinless–Kay in finite dimension); Prop. 5.3 (Cartan/Bruhat–Tits fixed point); Prop. 5.4 (Gaussian/oscillator); Thm. 6.1(b) is the translation length of a symmetric-space isometry, proved via Yamamoto; the Wick remark (L46).

New in TAA (elementary; not found stated in this form in the sources we checked): the formula $J_F = (F-Ve)(4q-(F+Ve)^2)^{-1/2} = i\,\mathrm{sign}(-iF)$ and Thm. 2.2; $J_F = J_+$ in Lemma 51's genus-1 chart and in L50's trace model, via $\omega_P(x, J_F x) = \mathrm{Tr}(D^{1/2} x x^\sharp)$ (Prop. 2.5); J_F is L51's surviving Krein vacuum and is blind to HH births (Rem. 2.6); Thm. 3.1 as a vacuum statement; quadruples as $SL_2(\mathbb{C})$ loxodromics with semisimple mixed-type limit (Prop. 4.3(b,c)); Prop. 5.2; the drift detector with its three outcomes and rates (Thm. 6.1); ghost count $= n_-(g_{J_A})$ on truncations (Prop. 7.1).

Prior art (searched: Crossref, and web searches on Bridson–Haefliger, Bost–Connes, Weinless/Kay, Yamamoto, CM polarizations; not exhaustive). (1) $J = A|A|^{-1}$ for a positive-energy generator is the standard one-particle structure of linear quantum fields (Segal 1962, Weinless 1969, Kay 1979). Our J_F is its discrete-time version, and the finite-dimensional proofs here are routine. (2) The matrix sign function $\mathrm{sign}(M) = M(M^2)^{-1/2}$ is standard [9]. (3) For abelian varieties over $\mathbb{F}_q$, polarizations of Deligne modules are described by totally imaginary elements $\lambda \in \mathbb{Q}(\pi)$ that are positive for a CM type (Howe [10], after Deligne [7]; [BIB], content seen only in a secondary source, a Leiden thesis quoting [10, Notation 4.6], where the CM type is defined p-adically). L50's trace model $\omega_P = \mathrm{Tr}(c\,xy^\sharp)$ with $c = \pi - q/\pi$ has this shape, and J_F is the complex structure of the archimedean CM type $\{\varphi : \Im\varphi(\pi) > 0\}$. We did not check whether this observation is stated in that literature, so we claim no novelty for it. (4) Translation lengths in symmetric spaces and the classification elliptic/parabolic/hyperbolic are classical ([3], [BIB]). We found no source applying them to L-polynomials as a ghost detector, but the application is immediate. None of this touches RH for ζ.

12. Questions for J.W.

v2 questions (the five v1 questions remain in `archive/lemma54.v1-20261023-044042.tex`, §"Questions for J.W.").

Question 12.1 (Q1)**.** Make the drift rank r (chart-free, ω-free) TAA's primary ghost *count*, with ℓ as its *size* and ν as the Krein-sensitive companion? Or should ν be primary because it is a quadratic-form index, computable without iterating?

Question 12.2 (Q2)**.** Record in the dictionary (NOTATION.md and Lemma 53) the identity "$\nu = r =$ minus the L53 charge" on truncations, real zeros included (Cor. 8.6)?

Question 12.3 (Q3)**.** Test ν and r on Lemma 52's CCM truncations, if they can be put in the form of a real Hamiltonian matrix? (This needs a symplectic form there; at present none is constructed, [OPEN].)

Question 12.4 (Q4)**.** Use the Witt index $w = \min(\nu, \nu_+)$, which is invariant under $\omega \to -\omega$ and satisfies $w \geq k$, as the chart-symmetric version, or is ν with a fixed orientation preferable?

Question 12.5 (Q5)**.** Volume placement: queue Lemma 54 (v1+v2) for Volume 4 as a chapter with figures, beside Lemmas 51 and 53, or keep it as a standalone note? (Nothing is queued without your answer.)

Status

v2. Proved ([THM]): Thm. 8.2, 8.4, 8.5, Cor. 8.6, Rem. 8.7. Restatement ([RESTATEMENT]): the flow half of Thm. 8.2 (KKS04 Thm. 3.3, Rem. 8.3). Cited ([CITED]): BPR06 Thm. 4.57 (text read), KKS04 Thm. 3.3 and Def. 2.4 (text read). Numerical ([NUM]): §8.3, `num/l54v2/`. RH is not proved, and v2 does not change that.

Proved ([THM]): Thm. 2.2, Cor. 2.3, Thm. 2.4(a),(b),(d), Prop. 2.5, Rem. 2.6, Thm. 3.1, Prop. 4.1, 4.2, 4.3, Rem. 4.4(a), Prop. 5.1, 5.2, 5.3, 5.4, Thm. 5.5, Thm. 6.1, Prop. 7.1. **Restatement ([RESTATEMENT]):** Thm. 2.4(c); the ζ bullets of §7 (Deninger, spectral $A^2 \leq 0$, outer/charge dictionary); the BC remark. **Analogy ([ANALOGY]):** OS positivity (Rem. 4.4(b)); Fisher-zero reading of BC. **Numerical ([NUM]):** all tables; `num/l54/emergent.json`, `num/l54/zeta.json`. **Cited ([CITED]):** BC95 Thm. 5 (text read); Kay79 abstract; Yam67 abstract and Thm. 1 statement; BH99 II.2.7–2.8 as quoted in Bri10. **Bibliographic ([BIB]):** Seg62, Wei69, KW91, OS73, BH99, Hig08, Hel, Bea83, BR97, Del69, How95. **Open ([OPEN]):** a real symplectic space with a flow for ζ (S1–S2), hence J_A for ζ. **Not verified, not cited:** Ashtekar–Magnon 1975 (record found, content not read; not needed); Kay 1993 JMP uniqueness theorem (record only); any explicit source for J_F in the CM-polarization literature (Howe's paper itself not read). No content of Segal 1962, Weinless 1969 or Kay–Wald 1991 beyond titles was read.

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TAA LEMMA 55: SEPARATRICES OF $|\zeta|$. CASSINIAN CURVES, SADDLE GEOMETRY, PHASE PRECESSION, AND SPEISER'S CRITERION

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Date: October 22, 2026.

Abstract. **Statement of intellectual honesty.** RH is not proved here, and nothing below brings it closer. The saddle criteria that are equivalent to RH (Speiser's theorem, Thm. 5.2, and the "saddle positivity" of §9) are *in the literature* (Speiser 1934/35, Levinson–Montgomery 1974, Lagarias 1999). Every equivalence stated here is tagged [RESTATEMENT]. Tags: [THM] proved here, or classical with proof given; [CITED] checked at the source (`lit/lemma55/verified_refs.md`); [BIB] bibliographic record verified, content standard or quoted from a source that was read; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma55_*.py`, data in `num/l55/`); [CONJ] conjecture; [OPEN] open. Earlier TAA lemmas are cited by number and not redone.

Question (J.W., [t205u]). Level sets of the magnitude define a phase precession around poles. A *separatrix* is an asymptotic boundary that behaves as a saddle. It should be definable *naive to the zeros*, and it should naturally produce the reflective symmetry about the zeros on the line. *Steering (J.W.):* read $|\xi|$ and $|\zeta|$ as Needham's modular surfaces, their level curves as *Cassinian curves of* ζ with the zeros as foci, and the separatrices as generalized lemniscates through the saddles.

Answer in brief. (1) *The picture is right and classical* (§2–3). By Hadamard, $|\xi|$ is a Cassinian function with infinitely many foci (the zeros). Its saddles are the zeros of ξ'. The level curves through them are the lemniscates (Morse separatrices of the level-set foliation). The phase lines through them are the separatrices of the gradient flow of $\log|\xi|$. Along every level curve the phase $\arg\xi$ precesses monotonically, by $2\pi\times$(foci enclosed). The precession period diverges exactly on a lemniscate (Prop. 3.3). That divergence is J.W.'s "asymptotic boundary behaving as a saddle" [THM]. A bounded sublevel component with m foci contains exactly $m-1$ saddles. This is the Morse merging count, and it is zero-naive (Prop. 3.2) [THM]. (2) *Symmetry is free; saddle type is not* (§4). $|\xi(s)| = |\xi(1-\bar{s})|$ for every s, ghosts or not. Gauss–Lucas and Laguerre–Pólya closure give *RH* $\Rightarrow$ *every saddle of* $|\xi|$ *lies on the line*, with exactly one saddle between consecutive zeros [THM]/[CITED]. The converse fails. In the toy model of Fig. 1(b) every saddle lies on the line although a ghost pair is present [NUM]. What does see the ghost is the zero-naive *type* of an on-line saddle: *longitudinal* (the lemniscate is a vertical figure eight) or *transverse* (a horizontal figure eight, i.e. a minimum of $|\Xi|$ with no sign change). Under RH every saddle is longitudinal (Laguerre inequality). A transverse saddle at height t_0 forces a ghost with $|\operatorname{Im}\rho - t_0| < \operatorname{Re}\rho - \frac{1}{2}$, and a near-line ghost forces a transverse or degenerate saddle (Thm. 4.2) [THM]. For polynomials, all foci lie on the line iff every saddle lies on the line and is longitudinal, and #transverse = #ghost pairs [THM]. For ξ the converse is open, and it would be a [RESTATEMENT]. Davenport–Heilbronn: each of the 28 ghosts below height 1000 produces a transverse on-line saddle of $|\xi_{\mathrm{DH}}|$ within δ of its height, and no off-line saddle was found [NUM]. (3) *Speiser's tilt is the point* (§5). $\zeta'/\zeta = \xi'/\xi - A$, and $\operatorname{Re}A(\frac{1}{2}+it) = \theta'(t)$ (the pole term cancels exactly on the line). So $\operatorname{Re}\zeta'/\zeta(\frac{1}{2}+it) = -\theta'(t) < 0$ for $t > 6.2898$ [THM]/[NUM]. The Γ-factor tilts the symmetric landscape so that $|\zeta|$ decreases across the line. Speiser's theorem follows: RH $\iff$ $\zeta' \neq 0$ in $0 < \sigma < \frac{1}{2}$ [CITED][RESTATEMENT]. Its proof uses the functional equation and $\theta' > 0$, not the Euler product. For Davenport–Heilbronn ($\theta'_{\mathrm{DH}} > 0$ for $t > 1.2116$) the left-strip zeros of f'_{DH} count the ghosts *exactly*: $28 = 28$ for $10 < t \le 1000$, window by window [NUM]. (4) *Phase and pole* (§6). On the line the pole contributes nothing to $|\xi|$, nothing to the tilt and nothing to the phase. In the $|\zeta|$ landscape it is an anti-focus of winding -1, and it is the $+1$ in $N(T)$ [THM]. Gram points are where Speiser's "Lamellen" ($\operatorname{Im}\zeta = 0$) cross the line [CITED]. (5) *Finite fields* (§8). Speiser's theorem is Cohn's 1922 theorem: a self-inversive P has all zeros on $|u| = 1$ iff P' has none outside [CITED]. For all 16 curves and fakes of Lemma 48, $\#\{L' = 0,\ |u| > 1\}$ = #ghosts [NUM]. In contrast, the zeros of the symmetric derivative (ξ-analogue) lie on the circle in *every* case, fakes included [NUM]. (6) *Mechanism* (§9). The "saddle positivity" that would make the separatrix picture force RH is $\operatorname{Re}\xi'/\xi > 0$ on $\sigma > \frac{1}{2}$. That is Lagarias's criterion: a [RESTATEMENT] of RH and the same Herglotz positivity as Lemmas 33/36. It is another face of the missing G5 positivity, not a new source of it.

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1. Conventions, and what TAA already has

$s = \sigma + it$, $\mathbb{H}_{1/2} = \{\sigma > \frac{1}{2}\}$. $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and $\Xi(t) = \xi(\frac{1}{2}+it)$, which is real and even. A *ghost* is a zero $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$ of ζ or of a comparison function; its *mirror* is $1-\bar{\rho}$, and $\delta = \beta - \frac{1}{2}$. For a meromorphic f, the completed function is $\Lambda = \gamma_f f$, with $\Lambda'/\Lambda = f'/f + A_f$. For ζ:

$$A(s) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\psi\left(\frac{s}{2}\right),$$

where $\psi = \Gamma'/\Gamma$. Davenport–Heilbronn (Lemma 48 §5): $f_{\mathrm{DH}} = \frac{1}{2}(1-i\kappa)L(s,\chi) + \frac{1}{2}(1+i\kappa)L(s,\bar{\chi})$, with χ the character mod 5 with $\chi(2) = i$ and $\kappa = (\sqrt{10-2\sqrt{5}}-2)/(\sqrt{5}-1)$. Its completion is $\xi_{\mathrm{DH}} = (5/\pi)^{s/2}\Gamma(\frac{s+1}{2})f_{\mathrm{DH}}$, so $A_{\mathrm{DH}} = \frac{1}{2}\log\frac{5}{\pi} + \frac{1}{2}\psi(\frac{s+1}{2})$. θ is the Riemann–Siegel theta function: $\zeta(\frac{1}{2}+it) = e^{-i\theta(t)}Z(t)$, with Z real. Finite fields: $\mathcal{L}(u) = P(u/\sqrt{q}) = \prod_i(1-\beta_i u)$ with $u = q^{1/2-s}$, so $|u| < 1 \iff s \in \mathbb{H}_{1/2}$ (Lemma 48 §6).

Overlap table (checked by reading STATE.md and the cited lemmas; nothing here is redone):

Lemma 9, 17	Lagarias's form $W^{(\sigma)}$ with weight $2\,\mathrm{Re}\,\xi'/\xi(\sigma+it)$. RH $\iff$ $-\Xi'/\Xi$ is Herglotz (Lemma 17, = Lagarias 1999). Used in §9 as the "saddle positivity".
Lemma 33, 37	Sondow–Dumitrescu monotonicity of $\lvert\xi\rvert$ in σ. Lemma 37 Thm 3.2: passivity at $\omega \iff \lvert\xi(s-\omega)\rvert \le \lvert\xi(s+\omega)\rvert$ on $\mathbb{H}_{1/2}$, and $G5_{\text{line}} \iff$ RH [RESTATEMENT]. In landscape language: the modular surface of ξ is a valley along the line.
Lemma 43	Holomorphic flows $\dot w = i\Xi(w)$, Broughan's separatrix criterion for $\dot z = \xi(z)$ (prior art), and the Newton field $i\Xi/\Xi'$. These are *different flows* from the gradient flow of $\log\lvert\xi\rvert$ used here. The Newton field's trajectories are our phase lines, but its equilibria are the zeros, not the saddles (Remark 3.5).
Lemma 48	Davenport–Heilbronn data: 28 ghosts to height 1000 (`num/l48/zeros_T1000.json`). Pole as anti-Blaschke factor $b_1 = (s-1)/s$, giving the $+1$ in $N(T)$. Phase winding $k = \#\text{ghosts}$. Finite-field fakes and their coefficient lists, all reused here.
Lemma 53	Ghosts as charge $-k$ in the Segal–Wilson Grassmannian. In §6 the left-strip saddles of $\lvert\zeta\rvert$ are compared with this charge.
Lemma 30, 46, 50, 54	G5: the missing geometric positivity (polarization). Lemma 54: the canonical J_F is positive iff RH (+ semisimple + Krein pattern), and its Wick rotation $\alpha + q/\alpha = 2\sqrt{q}\cos(\theta - i\sigma)$ turns a ghost into a rapidity σ. In §9 the saddle positivity is identified as one more face of G5. In §8, Lemma 54's cos → cosh is exactly the factor that turns a longitudinal saddle into a transverse one.

2. Modular surfaces and the Cassinian curves of ζ

2.1. **Needham's picture.** Everything in this subsection is [CITED] from [21, Ch. 2, §II.3 "Cassinian Curves", pp. 60–63]. Only those pages were read (page images supplied by J.W.; edition identified from the 1997 table of contents, §11). Given two points a_1, a_2 (the *foci*) and distances $r_j = |z-a_j|$, the curves $r_1 r_2 = k^2$ are the *Cassinian curves* (Fig. [8]b there). $Q(z) = (z-a_1)(z-a_2)$ maps each of them to a circle $|w| = k^2$. As k grows, two egg-shaped components around the foci meet at the midpoint of the foci in a figure eight, the *lemniscate* of Bernoulli (1694). The curves then pass through hourglass and ellipse-like shapes towards circles (Fig. [9]: z^2 maps the lemniscate $r^2 = 2\cos 2\theta$ to the circle $|w-1| = 1$). The Cassinian curves are the contour map of the *modular surface* $|Q(z)|$ over the plane, which has a cone-like leg at each focus (Fig. [10], adapted by Needham from Brieskorn–Knörrer). For n foci the Cassinian curves are the preimages of the circles $|w|$ =const under a degree-n polynomial P_n. The modular surface then has n cone-like legs and looks like $|z^n|$ far away. (Needham's Fig. [11] shows Perseus' spiric sections of a torus, not an n-focus picture.)

Definition 2.1 (J.W.'s vocabulary made precise)**.** Let f be meromorphic and $\not\equiv 0$ on a domain U. Its *modular surface* is the graph of $|f|$. Its *Cassinian curves* are the level sets $\{|f| = k\}$. Its *foci* are the zeros of f; its *anti-foci* are the poles. A *saddle* is a point s_0 with $f(s_0) \neq 0, \infty$ and $f'(s_0) = 0$, i.e. a critical point of $|f|$ off the foci. A *lemniscate* is a level curve $\{|f| = |f(s_0)|\}$ through a saddle. The *gradient separatrices* at s_0 are the $2(m+1)$ arcs of the phase curve $\{\arg f = \arg f(s_0)\}$ that leave s_0, where m is the order of the zero of f'.

Proposition 2.2 (Hadamard: $|\xi|$ is a Cassinian function with infinitely many foci)**.** $\xi(s) = \xi(\frac12)\prod_\rho(1-(s-\frac12)^2/(\rho-\frac12)^2)$*, the product running over one representative of each pair* $\{\rho, 1-\rho\}$ *of nontrivial zeros and converging absolutely. So* $\log|\xi(s)| = \log|\xi(\frac12)| + \sum_\rho \log|1 - \frac{s-1/2}{\rho-1/2}|$ *(paired). The level curves of* $|\xi|$ *are Cassinian curves with the zeros as foci, normalized focus by focus. They are invariant under* $s \mapsto 1-\bar s$ and *under* $s \mapsto \bar s$ *[THM].*

Proof. $\xi(\frac12 + iz) = \Xi(z)$ is even, real entire of order 1, so it is a function of z^2 of order $\frac12$. Hadamard's theorem gives $\Xi(z) = \Xi(0)\prod(1 - z^2/z_\rho^2)$ with $\sum |z_\rho|^{-2} < \infty$. The two symmetries are $\xi(1-s) = \xi(s)$ and $\overline{\xi(\bar s)} = \xi(s)$, so $|\xi|$ is invariant under their composition $s \mapsto 1-\bar s$. □

For ζ itself the foci are the nontrivial zeros together with the trivial zeros $-2n$, and there is one anti-focus at $s = 1$. The Γ-factor tilts the surface; there is no mirror symmetry (§5).

3. Topology of the modular surface: saddles, separatrices, precession

Proposition 3.1 (gradient and Hamiltonian flows)**.** *Let $g = f'/f$. [THM]*

(a) $\nabla \log|f| = \overline{g}$, and $\nabla \arg f = i\,\overline{g}$ (as complex numbers). So the phase curves $\arg f = \text{const}$ are the gradient lines of $\log|f|$, and the Cassinian curves are the orbits of the Hamiltonian field $i\overline{g}$, with Hamiltonian $\log|f|$.

(b) $\log|f|$ is harmonic off the foci and anti-foci. So it has no local maxima or minima there, and every critical point is a saddle. A zero of order m of f' (with $f \neq 0$) is a saddle of index $-m$ with $2(m+1)$ separatrices.

(c) As vector fields, $\overline{g}$ has index $+1$ at each simple focus and at each simple anti-focus (in general index = multiplicity), and index $-m$ at an order-m saddle.

(d) (Poincaré–Hopf = argument principle) For a Jordan domain R with no foci, anti-foci or saddles on ∂R: $S(R) = Z(R) + P(R) + \mathrm{wind}(g, \partial R)$, where S, Z and P count saddles, zeros and poles with multiplicity.

Proof. (a) Cauchy–Riemann for $\log f = \log|f| + i\arg f$. (b) The local form at s_0 is $\log|f| = \log|f(s_0)| + \mathrm{Re}(c(s-s_0)^{m+1}) + \dots$ with $c \neq 0$. (c) Near a zero or pole of order n, $g \sim \pm n/(s-a)$, so $\overline{g} \sim \pm n\,\overline{(s-a)}^{-1} = \pm n(s-a)/|s-a|^2$ (index $+1$). Near a saddle, $\overline{g} \sim \overline{c'(s-s_0)^m}$ (index $-m$). (d) g is meromorphic with simple poles at the foci and anti-foci and zeros at the saddles, so $\mathrm{wind}(g) = S - Z - P$. □

Proposition 3.2 (Morse merging count; zero-naive)**.** *Let f be entire, $k > 0$, and let Ω be a* bounded *component of $\{|f| < k\}$. Then Ω is simply connected, and if it contains m foci (with multiplicity) it contains exactly $m-1$ saddles (with multiplicity). As k increases, the topology of the sublevel sets changes only at saddle heights. At a simple saddle two components (with m_1 and m_2 foci) merge into one, with $m_1 + m_2$ foci, through a figure-eight lemniscate [THM].*

Proof. A bounded component of $\{|f| \geq k\}$ enclosed by Ω would, by the maximum principle, carry $|f| \leq k$ inside, hence $|f| \equiv k$, which is impossible. So Ω is simply connected. $f : \Omega \to \{|w| < k\}$ is proper, so it is a branched covering of degree m. Riemann–Hurwitz gives $\chi(\Omega) = m\chi(\mathrm{disc}) - \sum(e_p - 1)$, i.e. $1 = m - \#\{\text{critical points}\}$. Between saddle heights the normalized gradient flow $\overline{g}/|g|^2$ of $\log|f|$ carries level sets to level sets (standard Morse theory; $\log|f|$ has no critical points there). At a non-degenerate saddle the local model is $\mathrm{Re}\, w^2$. □

This is the precise "Cassinian" statement: one saddle per merger, $m-1$ saddles per m foci. It makes no reference to *where* the foci are.

Proposition 3.3 (phase precession and the separatrix as an asymptotic boundary)**.** *Let C be a smooth component of $\{|f| = k\}$, a closed curve enclosing Z foci and P anti-foci. [THM]*

(a) $\arg f$ is strictly monotone along C, and $\oint_C d\arg f = 2\pi(Z - P)$ (counter-clockwise).

(b) Parametrize C by the Hamiltonian time τ of $\dot{s} = i\overline{g(s)}$. The phase velocity is $d\arg f/d\tau = |g|^2$, and the period is $T(k) = \oint_C |ds|/|g|$.

(c) If $k \to k_c = |f(s_0)|$ with s_0 a non-degenerate saddle on the limit lemniscate, then $T(k) \to \infty$, like $c\log(1/|k-k_c|)$.

Proof. (a) $d\arg f = \partial_n \log|f|\,|ds|$ by Cauchy–Riemann, and $\partial_n \log|f| = |g| \neq 0$ on a smooth level curve. The total is the argument principle. (b) $|\dot{s}| = |g|$ and $d\arg f = |g|\,|ds|$. (c) $|g(s)| \asymp |s - s_0|$ near s_0, so $\int |ds|/|g|$ diverges logarithmically along the hyperbolas of $\mathrm{Re}\, c(s-s_0)^2 = \pm\epsilon$. □

Remark 3.4 (reading J.W.'s sentence)**.** "Level sets of magnitude define a phase precession around poles (and zeros)" is Prop. 3.3(a). Around an anti-focus the phase runs backwards (P enters

with sign $-$). "A separatrix is an asymptotic boundary behaving as a saddle" is Prop. 3.3(c). The lemniscate is the level at which the precession period diverges, i.e. the separatrix of the Hamiltonian flow. Its partner, the gradient separatrix (Def. 2.1), is a phase curve through the same saddle. Both notions use only f, not its zeros, so they are zero-naive. They do *not* by themselves produce symmetry: for ζ there is none (§5), and for ξ the symmetry is imposed by the functional equation.

Remark 3.5 (Lemma 43 and Broughan)**.** Broughan's flows $\dot{z} = \xi(z)$ and Lemma 43's $\dot{w} = i\Xi(w)$ are holomorphic flows whose equilibria are the zeros (centers, under RH and simplicity), with separatrices coming from infinity. The Newton field $\dot{s} = f/f'$ (which equals $1/g$) has the same trajectories as our gradient flow, $\overline{g} = |g|^2/g$, up to a positive time change. Its equilibria are the foci, and the saddles are *poles* of f/f'. So "separatrix" in Lemma 43 and here refer to different objects. Here they are the curves through zeros of f' [THM].

3.1. Gauss–Lucas, Laguerre–Pólya, Rolle.

Theorem 3.6 (foci on a line ⇒ saddles on the line)**.** *(a) (Gauss–Lucas* [20], [17, §1]*) The zeros of P' lie in the convex hull of the zeros of a polynomial P. If all foci lie on a line, all saddles lie on it [BIB].*

(b) (Laguerre–Pólya closure) For real entire functions of order < 2, having only real zeros is preserved under differentiation [12, §2] *[CITED]. Hence* RH ⇒ every zero of Ξ' is real*, i.e. every saddle of $|\xi|$ lies on $\sigma = \frac{1}{2}$. More generally RH ⇒ all zeros of every $\xi^{(m)}$ lie on the line* [8] *[CITED].*

(c) (Rolle, interlacing) If RH holds and $\gamma < \gamma'$ are consecutive zeros of Ξ, then Ξ' has exactly one zero in (γ, γ'), and it is simple. So the saddles of $|\xi|$ interlace the zeros: under RH the critical line is a chain focus–saddle–focus–$\cdots$ joined by gradient separatrices [THM].

(d) (Converse fails) Saddles on the line do not imply foci on the line. $P(x) = (x^2+1)(x^2-4)$ has non-real zeros $\pm i$, while $P'(x) = 2x(2x^2-3)$ has only real zeros. In $s = \frac{1}{2} + ix$ this is a ghost pair at $\frac{1}{2} \pm 1$ with all saddles on the line [THM]. A symmetric toy with a near-line ghost pair is in Fig. 1(b) and Table 1.

Proof. (c) Under RH, $(\Xi'/\Xi)'(t) = -\sum_\gamma (t-\gamma)^{-2} < 0$ (the sum is from Prop. 2.2, taken over all zeros $\pm\gamma$). So Ξ'/Ξ decreases strictly from $+\infty$ to $-\infty$ on (γ, γ') and has exactly one simple zero there. By (b) there are no other saddles. (d) Direct computation. □

Unconditionally, Conrey proved that the proportion of zeros of $\xi^{(m)}$ on the line is $1 - O(m^{-2})$ [8] [CITED]. Kim proved that for a real entire function of order < 2 with zeros in a horizontal strip, $f^{(n)}$ has only real zeros in any fixed disc once n is large [15] (statement via [12]) [BIB]. Ki proved that suitably rescaled $\Xi^{(2n)}$ converge to cos [14] (via [12, Thm. 2.1]) [BIB]. Kim's and Ki's theorems apply to every such function, ghosts or not. Farmer notes that the analogue of Ki's theorem holds in the extended Selberg class, which has no Euler product. So these results are blind to RH (Farmer's point) [CITED].

4. On-line saddles: longitudinal versus transverse

On $\sigma = \frac{1}{2}$ write $\xi(\frac{1}{2} + it) = \Xi(t)$, which is real. A real zero t_0 of Ξ' with $\Xi(t_0) \neq 0$ is a saddle $s_0 = \frac{1}{2} + it_0$ of $|\xi|$. Since $\log|\xi|$ is harmonic, $\partial_\sigma^2 \log|\xi| = -\partial_t^2 \log|\xi| = -\Xi''(t_0)/\Xi(t_0)$ at s_0.

Definition 4.1. The on-line saddle s_0 is *longitudinal* if $\Xi''(t_0)/\Xi(t_0) < 0$. Then $|\Xi|$ has a strict local maximum at t_0 along the line, the descending separatrices run *along* the line to the neighbouring foci, and $|\xi|$ increases horizontally. It is *transverse* if $\Xi''(t_0)/\Xi(t_0) > 0$. Then $|\Xi|$ has a local minimum along the line *without a sign change*, and the descending separatrices leave the line horizontally, symmetric under $s \mapsto 1 - \bar{s}$. Numerically, the type is the sign of $\mathrm{Re}(\Lambda''/\Lambda)(s_0)$ (positive means longitudinal).

Theorem 4.2 (Laguerre criterion; transverse saddles detect ghosts)**.** *Let F be Ξ, or more generally a real entire function of the form $F(t) = c\prod(1 - t^2/z_j^2)$ with $\sum |z_j|^{-2} < \infty$. For example, $F(t) = \xi_{\mathrm{DH}}(\frac{1}{2} + it)$ suitably normalized (the same argument applies).*

(a) $-(\log F)''(t) = \sum_j (t - z_j)^{-2}$*, summed over all zeros $\pm z_j$. A conjugate pair $\gamma \pm i\delta$ (a ghost $\rho = \frac{1}{2} + \delta + i\gamma$ together with its mirror) contributes $2((t-\gamma)^2 - \delta^2)/((t-\gamma)^2 + \delta^2)^2$. This is negative exactly for $|t - \gamma| < \delta$, with minimum $-2/\delta^2$ at $t = \gamma$ [THM].*

(b) Under RH (all z_j real), every on-line saddle is longitudinal: at $\Xi' = 0$, $-\Xi''/\Xi = \sum(t - \gamma)^{-2} > 0$. This is the Laguerre inequality $\Xi'^2 - \Xi\Xi'' > 0$ [THM].

(c) Detector. *If $|\xi|$ has a transverse on-line saddle at height t_0, there is a ghost ρ with $|\operatorname{Im}\rho - t_0| < \operatorname{Re}\rho - \frac{1}{2}$ [THM].*

(d) Converse, local. *Let $\rho = \frac{1}{2} + \delta + i\gamma$ be a ghost, and let $R(t) = F'/F(t) - 2(t-\gamma)/((t-\gamma)^2 + \delta^2)$ be the rest. If $|R| < 1/\delta$ on $[\gamma - \delta, \gamma + \delta]$, then F'/F changes sign from $-$ to $+$ there. So $|\xi|$ has an on-line saddle in $(\gamma - \delta, \gamma + \delta)$ that is transverse or degenerate [THM].*

(e) Polynomials (finitely many foci). *Let p be a real polynomial with n zeros, the real ones simple, all of whose $n-1$ critical points are real and non-degenerate. Then $\#\{\text{transverse}\} = \#\{\text{non-real conjugate pairs of zeros}\}$. In particular,* all foci lie on the line iff every saddle lies on the line and is longitudinal. *The same holds on the circle for a real trigonometric polynomial (the finite-field case, §8) [THM].*

Proof. (a) Hadamard (Prop. 2.2), differentiated twice. (b) Immediate from (a). (c) At a transverse saddle the sum in (a) is negative. Real zeros and pairs with $|t_0 - \gamma| \geq \delta$ contribute ≥ 0, so some pair has $|t_0 - \gamma| < \delta$. (d) At $t = \gamma \mp \delta$ the pair term equals $\mp 1/\delta$, so $F'/F(\gamma - \delta) < 0 < F'/F(\gamma + \delta)$. Some zero of F'/F in between is crossed upwards, i.e. $(F'/F)' \geq 0$ there, and at a zero of F' this is $F''/F \geq 0$. (e) Between consecutive real zeros, $|p|$ goes $0 \nearrow \cdots \searrow 0$, so its critical points alternate max/min with one more max: $\#\text{long} = \#\text{trans} + 1$ on each of the $Z - 1$ bounded gaps. On each unbounded end, $|p|$ goes $0 \nearrow \infty$, so the two kinds are equal in number. (If $Z = 0$, $|p|$ goes $\infty \searrow \cdots \nearrow \infty$ and $\#\text{trans} = \#\text{long} + 1$.) So $\#\text{trans} = (n - 1 - (Z - 1))/2 = (n - Z)/2$, or $n/2$ when $Z = 0$. On the circle the same alternation applies with $2g$ critical points, and $\#\text{trans} = (2g - Z)/2 = k$. □

So "symmetry cannot exclude ghosts" (chat claim (i)) is right about the *landscape* but wrong about *saddle types*. The level-set topology alone is FE-symmetric for every function. The longitudinal/transverse split is equally symmetric and zero-naive, and it does see ghosts, at least for polynomials. For ξ itself:

$$\mathrm{RH} \implies \text{every saddle of } |\xi| \text{ lies on } \sigma = \tfrac{1}{2} \text{ and is longitudinal [THM] (Thm. 3.6(b), Thm. 4.2(b)).}$$

The converse would follow from (e) together with a counting statement: the number of zeros of Ξ' in $[0, T]$ is at least $N(T) - C$ along a sequence $T_j \to \infty$. We have neither proved nor found this [OPEN]. In any case it would be a [RESTATEMENT].

4.1. **Davenport–Heilbronn.** ξ_{DH} satisfies the functional equation, and its on-line restriction is real up to a constant phase. f_{DH} has no Euler product and has ghosts (Lemma 48). Table 2 shows that *each of the 28 ghosts with $\gamma \leq 1000$ produces a transverse on-line saddle of $|\xi_{\mathrm{DH}}|$ inside $|t_0 - \gamma| < \delta$*, as Thm. 4.2(d) leads one to expect. No off-line saddle of $|\xi_{\mathrm{DH}}|$ was found in any ghost window, and every non-ghost on-line saddle in these windows is longitudinal [NUM]. The four small-δ ghosts ($\gamma \approx 166.48, 520.94, 595.02, 847.47$; $\delta < 0.085$) were missed by the 2D grid search and found by the 1D on-line scan (step 10^{-3}; marked *). Fig. 2(d) shows the transverse saddle at 85.7472 and its horizontal figure-eight lemniscate around the ghost $0.8085 + 85.6993i$ and its mirror.

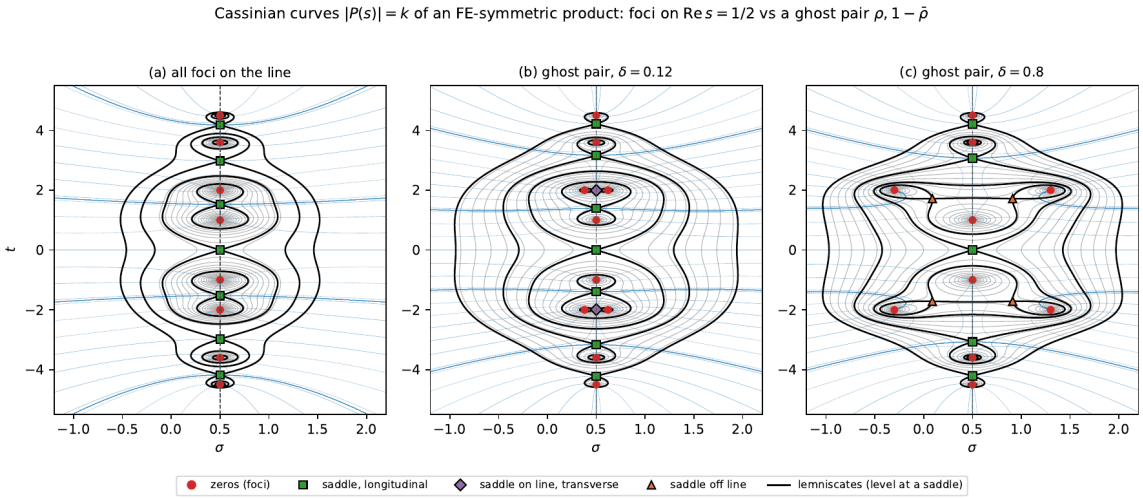

FIGURE 1. Needham-style Cassinian curves of $P(s) = \prod(s-\rho)$ with foci $\frac{1}{2} \pm i\{1, 2, 3.6, 4.5\}$. (a) All foci on the line: every saddle is longitudinal (green squares), and the lemniscates (black) are figure eights centred on the line. (b) The pair at $\pm 2i$ replaced by the ghost quartet $\frac{1}{2} \pm 0.12 \pm 2i$. *All saddles still lie on the line*, but those at $\pm 1.994i$ are transverse (purple diamonds), and the lemniscate through them is a horizontal figure eight enclosing the ghost and its mirror. (c) $\delta = 0.8$: the ghosts generate off-line saddles (orange) in symmetric pairs. Thin blue lines are phase lines. Data `num/l55/toy.json`; PNG `num/l55/lemma55_toy.png`.

TABLE 1. Toy of Fig. 1: saddle types as the ghost pair at height 2 moves off the line (8 or 10 foci, 7 or 9 saddles) [NUM]. For $\delta \le 0.4$ every saddle is on the line, and #transverse = 2 = #conjugate pairs, as Thm. 4.2(e) predicts. For $\delta \ge 0.6$ the pair is seen through off-line saddles instead.

δ	long.	transv.	off-line	saddles near $\lvert t\rvert = 2$ (upper half)
0	7	0	0	$0.5000 + 1.5276i$ (longi)
0.05	7	2	0	$0.5000 + 1.9986i$ (trans); $0.5000 + 1.3830i$ (longi)
0.1	7	2	0	$0.5000 + 1.9942i$ (trans); $0.5000 + 1.3880i$ (longi)
0.2	7	2	0	$0.5000 + 1.9760i$ (trans); $0.5000 + 1.4088i$ (longi)
0.3	7	2	0	$0.5000 + 1.9417i$ (trans); $0.5000 + 1.4472i$ (longi)
0.4	7	2	0	$0.5000 + 1.8808i$ (trans); $0.5000 + 1.5139i$ (longi)
0.6	5	0	4	$0.7186 + 1.7055i$ (off-l); $0.2814 + 1.7055i$ (off-l)
0.8	5	0	4	$0.9113 + 1.7164i$ (off-l); $0.0887 + 1.7164i$ (off-l)
1	5	0	4	$1.0809 + 1.7291i$ (off-l); $-0.0809 + 1.7291i$ (off-l)
1.5	5	0	4	$0.5000 + 2.8920i$ (longi); $1.5076 + 1.7594i$ (off-l); $-0.5076 + 1.7594i$ (off-l)

5. THE Γ-TILT: WHY SPEISER USES ζ' AND NOT ξ'

Proposition 5.1 (tilt decomposition)**.** *(a) $\zeta'/\zeta = \xi'/\xi - A$. So the saddles of $|\zeta|$ (zeros of ζ' with $\zeta \neq 0$) are the solutions of $\xi'/\xi = A$ [THM].*

(b) On the line, $\xi'/\xi(\frac{1}{2}+it) = -i\Xi'(t)/\Xi(t) \in i\mathbb{R}$. Also $\operatorname{Re}(\frac{1}{s}+\frac{1}{s-1}) = 0$, since $s(s-1) = -\frac{1}{4}-t^2$ is real there. So $\operatorname{Re} A(\frac{1}{2}+it) = \frac{1}{2}\operatorname{Re}\psi(\frac{1}{4}+\frac{it}{2}) - \frac{1}{2}\log\pi = \theta'(t)$ [THM].

TABLE 2. Davenport–Heilbronn ghosts $\rho = \beta + i\gamma$, $\gamma \le 1000$ (Lemma 48 data), the transverse on-line saddles t_0 of $|\xi_{\mathrm{DH}}|$, and the zero of f'_{DH} in the left half-strip near the mirror $1 - \bar{\rho}$ [NUM]. *: found by the 1D scan (`num/l55/dh_fix.json`); otherwise found by 2D grid+Newton with an argument-principle check (`num/l55/dh_ghost_saddles.json`). In window 646.99 the 2D winding check gave 2 against 3 saddles found, but the 1D scan confirms the 3 on-line saddles listed in the JSON.

γ	β	δ	transverse saddle t_0	$\lvert t_0-\gamma\rvert<\delta$	left zero of f'_{DH}
85.6993	0.8085	0.3085	85.7472	yes	$0.4054 + 85.7051i$
114.1633	0.6508	0.1508	114.1696	yes	$0.4748 + 114.1608i$
166.4793	0.5744	0.0744	166.4820*	yes	$0.4933 + 166.4798i$
176.7025	0.7243	0.2243	176.7302	yes	$0.4403 + 176.7080i$
240.4047	0.8695	0.3695	240.4398	yes	$0.3402 + 240.3918i$
320.8765	0.8195	0.3195	320.9364	yes	$0.3725 + 320.8849i$
331.0503	0.7682	0.2682	331.0964	yes	$0.4063 + 331.0600i$
366.6409	0.6285	0.1285	366.6465	yes	$0.4770 + 366.6400i$
411.7967	0.8159	0.3159	411.8762	yes	$0.3686 + 411.8138i$
440.4845	0.7089	0.2089	440.4902	yes	$0.4389 + 440.4752i$
520.9438	0.5159	0.0159	520.9439*	yes	$0.4996 + 520.9438i$*
531.2797	0.8470	0.3470	531.3104	yes	$0.3418 + 531.2689i$
548.9068	0.7295	0.2295	548.9186	yes	$0.4246 + 548.8997i$
566.5097	0.7866	0.2866	566.5381	yes	$0.3859 + 566.5055i$
595.0234	0.5829	0.0829	595.0245*	yes	$0.4895 + 595.0218i$
611.7751	0.6283	0.1283	611.7923	yes	$0.4744 + 611.7842i$
646.9868	0.6108	0.1108	646.9929	yes	$0.4812 + 646.9879i$
657.1084	0.7606	0.2606	657.1598	yes	$0.4013 + 657.1209i$
692.8925	0.7887	0.2887	692.9206	yes	$0.3818 + 692.8880i$
737.7670	0.7774	0.2774	737.8415	yes	$0.3865 + 737.7872i$
783.6530	0.8530	0.3530	783.7204	yes	$0.3288 + 783.6564i$
811.7658	0.6686	0.1686	811.7745	yes	$0.4558 + 811.7635i$
847.4657	0.5619	0.0619	847.4660*	yes	$0.4938 + 847.4645i$
857.2959	0.8561	0.3561	857.3708	yes	$0.3246 + 857.3011i$
864.1181	0.6809	0.1809	864.1352	yes	$0.4492 + 864.1210i$
892.1490	0.6884	0.1884	892.1876	yes	$0.4425 + 892.1656i$
921.1727	0.7594	0.2594	921.1749	yes	$0.3982 + 921.1569i$
983.7522	0.7625	0.2625	983.7427	yes	$0.3906 + 983.7304i$

(c) Hence $\operatorname{Re}\zeta'/\zeta(\frac{1}{2}+it) = -\theta'(t)$ *exactly. Equivalently, from* $\zeta(\frac{1}{2}+it) = e^{-i\theta}Z(t)$, $\zeta'/\zeta(\frac{1}{2}+it) = -\theta'(t) - iZ'(t)/Z(t)$ *[THM]. Numerically* $\theta'(t) > 0$ *for* $t > t_* = 6.289836$, *and* $\theta'_{\mathrm{DH}}(t) = \frac{1}{2}\log\frac{5}{\pi} + \frac{1}{2}\operatorname{Re}\psi(\frac{3}{4}+\frac{it}{2}) > 0$ *for* $t > 1.211636$ *[NUM] (`num/l55/theta.json`; identity (c) checked to* 10^{-16} *at* $t = 10, \dots, 7005$*).*

(d) $\operatorname{Re} A > 0$ *on* $0 \le \sigma \le \frac{1}{2}$ *for* $6.5 < t \le 200$, *and* $\operatorname{Re} A_{\mathrm{DH}} > 0$ *there for* $1.4 < t \le 200$ *[NUM] (grid, `num/l55/tilt.json`). Beyond* 200, $\operatorname{Re}\psi(z) = \log|z| + O(|z|^{-1})$ *makes this clear [THM].*

So $|\zeta| = |\xi|/|\gamma_\zeta|$ is the symmetric landscape with the factor $|\gamma_\zeta(s)|^{-1}$, which on the line decreases in σ at rate $\theta'(t)$. This *tilt* is what breaks the mirror symmetry. It drags the saddles of $|\zeta|$ off the line. Heuristically, a longitudinal saddle of $|\xi|$ moves to the right (all ζ' zeros in Table 4 have $\sigma > \frac{1}{2}$) [ANALOGY]. Near a ghost pair, the mirror ghost (left of the line) gets a ζ' zero to its right but still in the left strip (Fig. 2(c)).

Theorem 5.2 (Speiser 1934/35; Levinson–Montgomery 1974)**.** *(a)* [27, p. 520, Satz] *RH holds iff every non-real zero of ζ' lies to the right of the critical line, or on it, the latter only at multiple zeros of ζ [CITED][RESTATEMENT]. Speiser's proof (pp. 519–520) is topological. Through a branch point z_0 (a zero of ζ') run lines of constant amplitude ($\arg\zeta$, i.e. gradient separatrices). The descending ones bound a region the ascending one cannot leave. The input is that $|\zeta|$ decreases across the line from left to right, because $\arg\zeta(\frac{1}{2}+it)$ decreases (our Prop. 5.1(c)), together with the functional equation. The Euler product is not used. Arias de Reyna's account* [1, Thm. 4] *remarks that Speiser's methods are "between the proved and the acceptable" and points to* [19] *for a flawless proof [CITED].*

(b) [19, Thm. 1] *Let $N^-(T)$ and $N_1^-(T)$ count the zeros of ζ and of ζ' in $0<\sigma<\frac{1}{2}$, $0<t<T$. Then $N_1^-(T)=N^-(T)+O(\log T)$, and unless $N^-(T)>T/2$ for all large T, there is a sequence $T_j\to\infty$ with equality [CITED].* Corollary [19]*: RH $\iff$ $\zeta'(s)\neq 0$ for $0<\sigma<\frac{1}{2}$, "essentially due to Speiser", with one half rediscovered by Spira* [29] *[CITED][RESTATEMENT].*

(c) Further results [19] *[CITED]: for $k\geq 1$, RH $\Rightarrow$ $\zeta^{(k)}$ has only finitely many non-real zeros in $\sigma<\frac{1}{2}$ (Thm. 7). For $n\geq 2$, ζ' has exactly one zero in $(-2n,-2n+2)$ and no other zeros in $\sigma<0$ (Thm. 9, using Spira's region $|s|>165$, $\sigma\leq 0$). On average the zeros of ζ' satisfy $2\pi\sum_{T<\gamma'\leq T+U}(\beta'-\frac{1}{2})=U\log\log\frac{T}{2\pi}+U(\frac{1}{2}\log 2-\log\log 2)+O(U^2/(T\log T))+O(\log T)$ (Thm. 3, $k=1$). Berndt's count* [2] *is $N_1(T)=\frac{T}{2\pi}\log\frac{T}{4\pi}-\frac{T}{2\pi}+O(\log T)$ (as quoted in* [19, (1.3)]*).*

Proposition 5.3 (RH $\Rightarrow$ Speiser, via the tilt)**.** *Assume RH. Then ζ' has no zero with $0\leq\sigma<\frac{1}{2}$ and $t>6.5$ [THM] (with [NUM] for $6.5<t\leq 200$, Prop. 5.1(d)).*

Proof. Under RH, for $\sigma<\frac{1}{2}$: $\operatorname{Re}\xi'/\xi(s)=\sum_\rho \operatorname{Re}(s-\rho)^{-1}=\sum_\gamma(\sigma-\frac{1}{2})/|s-\rho|^2<0$ (paired, absolutely convergent). This is the monotonicity of Lemma 33 / [25]. A zero of ζ' would need $\xi'/\xi=A$, so $\operatorname{Re}A<0$, contradicting Prop. 5.1(d). □

Proposition 5.3 uses only the symmetric half ($\operatorname{Re}\xi'/\xi$ has the sign of $\sigma-\frac{1}{2}$) and the tilt $\operatorname{Re}A>0$. The converse (ghost $\Rightarrow$ a ζ' zero on the left) is Speiser's topological argument, made quantitative by Levinson–Montgomery. *Its hypotheses hold verbatim for f_{DH}* (functional equation, $\theta'_{\mathrm{DH}}>0$, no pole), and the numerics agree exactly:

Which directions are equivalences.

statement	status
$\zeta'\neq 0$ in $0<\sigma<\frac{1}{2}$	$\iff$ RH [27, 19] [CITED][RESTATEMENT]; holds for $t\leq 1000$ [NUM]
all zeros of ξ' (all $\xi^{(m)}$) on the line	$\Leftarrow$ RH [8]; $\Rightarrow$ RH unknown; false for general symmetric functions (Thm. 3.6(d), Table 1)
all saddles of $\lvert\xi\rvert$ on the line *and longitudinal*	$\Leftarrow$ RH [THM]; $\Rightarrow$ RH for polynomials [THM]; for ξ [OPEN] (would be [RESTATEMENT])
$\Xi^{(n)}$ real-rooted in $\lvert z\rvert<R$, $n\gg_R 1$	unconditional [15]; RH-blind
$\Lambda\leq 0$ (de Bruijn–Newman)	$\iff$ RH; $\Lambda\geq 0$ [24]; $\Lambda\leq 0.22$ [23] [CITED][RESTATEMENT]
$\operatorname{Re}\xi'/\xi>0$ on $\sigma>\frac{1}{2}$	$\iff$ RH (Lagarias; Lemma 17) [RESTATEMENT]

On de Bruijn–Newman [24] [CITED]: $H_t(z)=\int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du$ has only real zeros iff $t\geq\Lambda$, and H_0 is a rescaling of Ξ. The real zeros move by $\partial_t x_k=2\sum_{j\neq k}(x_k-x_j)^{-1}$. In landscape language, the backward heat flow pushes on-line foci together; a collision of two foci produces a ghost pair, and our transverse saddle is born at such a collision [ANALOGY].

TABLE 3. Left-strip zeros of f'_{DH} (winding of f'_{DH} around $[0,\frac{1}{2}]\times[a,a+30]$) against Lemma 48's ghosts, $10<t\le 1000$ [NUM] (`num/l55/dh_count.json`). The Speiser–Levinson–Montgomery count $N_1^- = N^-$ holds window by window, here with no $O(\log T)$ defect. The same computation for ζ gives $N_1^- = 0$ on $[0,\frac{1}{2}]\times[10,1000]$ (`num/l55/zeta_saddles.json`), in agreement with [19], whose proof uses $\beta=\frac{1}{2}$ for $|\gamma|<1000$.

t	10–40	40–70	70–100	100–130	130–160	160–190	190–220	220–250	250–280	280–310
N_1^-	0	0	1	1	0	2	0	1	0	0
ghosts	0	0	1	1	0	2	0	1	0	0
t	310–340	340–370	370–400	400–430	430–460	460–490	490–520	520–550	550–580	580–610
N_1^-	2	1	0	1	1	0	0	3	1	1
ghosts	2	1	0	1	1	0	0	3	1	1
t	610–640	640–670	670–700	700–730	730–760	760–790	790–820	820–850	850–880	880–910
N_1^-	1	2	1	0	1	1	1	1	2	1
ghosts	1	2	1	0	1	1	1	1	2	1
t	910–940	940–970	970–1000							
N_1^-	1	0	1							
ghosts	1	0	1							
total: $N_1^- = 28$, ghosts $= 28$										

6. Phase precession, Gram points and the pole

Proposition 6.1 (what the pole contributes)**.** *[THM]*

(a) To $|\xi|$: *nothing. The factor $s-1$ removes the pole, and $\xi(1)=\frac{1}{2}\neq 0$.*

(b) To the tilt on the line: *nothing.* $\mathrm{Re}(\frac{1}{s-1}+\frac{1}{s})=0$ *on $\sigma=\frac{1}{2}$ (Prop. 5.1(b)), so Speiser's sign* $\mathrm{Re}\,\zeta'/\zeta=-\theta'$ *comes from* Γ *and* $\pi^{-s/2}$ *alone.*

(c) To the phase along the line: *nothing. $s(s-1)=-\frac{1}{4}-t^2<0$ on the line.*

(d) To the $|\zeta|$ landscape: *an anti-focus. Near $s=1$, $|\zeta|\approx 1/|s-1|$, the Cassinian curves are small near-circles, and* $\arg\zeta$ *precesses* backwards *once around them (index of $\overline{g}$ is $+1$, but Prop. 3.3(a) counts it with sign -1). In Prop. 3.1(d) it counts as $P=1$. Fig. 2(a) shows it at the bottom ($\times$).*

(e) To counting: *the $+1$ in $N(T)=\frac{1}{\pi}\theta(T)+1+S(T)$. This comes from the argument change of $s(s-1)$ along the* off-line *parts of the contour. In Lemma 48's language (Cor. pole there) it is the anti-Blaschke factor $b_1=(s-1)/s$, which is unimodular on the line. A ghost's Blaschke factor instead winds -2π (Lemma 48).*

Gram points and Speiser's Lamellen [CITED] [1]. Speiser and Utzinger drew the curves $\mathrm{Im}\,\zeta=0$ (thick in the X-ray) and $\mathrm{Re}\,\zeta=0$. Speiser's first criterion [27, p. 519] is: RH iff every *Lamelle* (region of the ζ-Riemann surface bounded by curves on which ζ is real, coming in from the left) reaches the critical line. At a Gram point g_n ($\theta(g_n)=n\pi$), $\zeta(\frac{1}{2}+ig_n)=(-1)^nZ(g_n)$ is real, so a curve $\mathrm{Im}\,\zeta=0$ crosses the line there. "Gram's law" (Hutchinson's name) is the statement that consecutive Gram points enclose a zero. Since $\arg\zeta(\frac{1}{2}+it)=-\theta(t)\bmod\pi$, the phase *precesses uniformly at rate* $\theta'(t)\approx\frac{1}{2}\log\frac{t}{2\pi}$ along the line. That is the on-line face of Prop. 5.1(c). Along a lemniscate of $|\xi|$ the phase still advances monotonically (Prop. 3.3(a)), and the precession stalls at the saddle, where $|g|=0$. The gradient separatrices through an on-line saddle are the critical line itself and the horizontal curve $\mathrm{Im}\,\xi=0$ through it (blue in Fig. 2(b)). Their counterparts in the $|\zeta|$ landscape are Speiser's Lamellen boundaries.

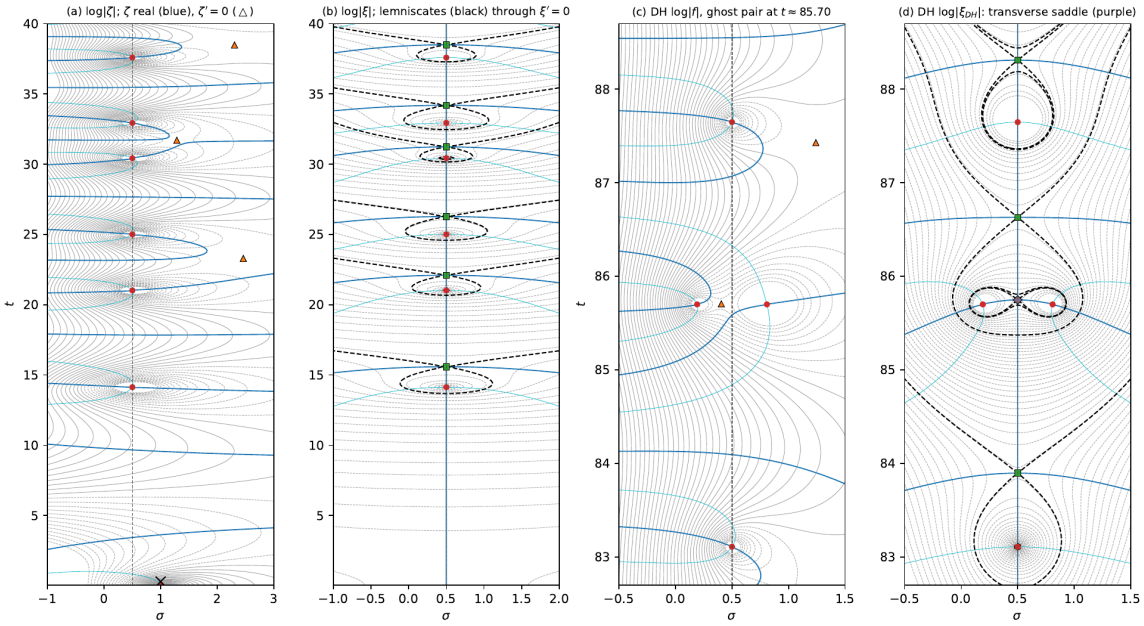

FIGURE 2. (a) $\log|\zeta|$ for $-1 \le \sigma \le 3$, $0 < t \le 40$. Grey: level curves (Cassinian curves). Blue: $\operatorname{Im}\zeta = 0$ (Speiser's Lamellen boundaries). Cyan: $\operatorname{Re}\zeta = 0$. Red: zeros. Orange: zeros of ζ', all with $\sigma > \frac{1}{2}$. $\times$: the pole. (b) $\log|\xi|$ in the same window. Every saddle (green) lies on the line and is longitudinal. The dashed black curves are the lemniscates through the saddles, i.e. the levels at which consecutive foci merge (dashed only because the level is negative). (c) $\log|f_{\mathrm{DH}}|$ around the ghost $0.8085 + 85.6993i$ and its mirror. The f'_{DH} zero at $0.4054 + 85.7051i$ lies *left* of the line (Speiser). (d) $\log|\xi_{\mathrm{DH}}|$: the transverse saddle at $\frac{1}{2} + 85.7472i$ (purple) and its horizontal figure-eight lemniscate around the ghost pair. Data `num/l55/zeta_saddles.json`, `fig_dh_window.json`; PNG `num/l55/lemma55_zeta_vs_dh.png`.

Link to Lemma 48's winding k and Lemma 53's charge [ANALOGY]/[NUM]. Lemma 48: the inner factor of F_ζ winds once per ghost. Lemma 53: charge $= -k$. Here are two zero-naive saddle counters for the same k:

(S) *symmetric*: the number of transverse on-line saddles of $|\xi|$. It equals k for polynomials and trigonometric polynomials with all saddles on the line (Thm. 4.2(e)), and it equals the 28 ghosts of f_{DH} [NUM].

(T) *tilted*: the number of saddles of $|\zeta|$ in the left strip, $N_1^-(T) = N^-(T) + O(\log T)$ [19] [CITED], exactly $28 = 28$ for f_{DH} [NUM]. For curves over $\mathbb{F}_q$ it is exact (§8).

Both count ghosts *without being told where the zeros are*. Neither *forces* $k = 0$.

7. Numerics for ζ: windows 0–60, the Lehmer pair, and $t \approx 10^5$

Method [NUM] (`num/lemma55_core.py`, `lemma55_saddles.py`, `lemma55_online.py`, `lemma55_zp_windows.py`; python-flint ball arithmetic at 96 bits, values taken at ball midpoints):

- Zeros of ζ' by grid minima of $|\zeta'|$ plus Newton, checked by the winding of ζ' around the rectangle.

- On-line saddles of $|\xi|$ by a 1D scan of the real function $\Xi'/\Xi = i\,\xi'/\xi(\frac{1}{2}+it)$ (step 0.002–0.004) with bisection. The type is the sign of $\mathrm{Re}(\Lambda''/\Lambda)$.
- The total number of saddles of $|\xi|$ in $0 \le \sigma \le 1$ from Prop. 3.1(d).

On-line zeros are checked against `mpmath.siegelz`. Table 4: in all three windows *every saddle of* $|\xi|$ *is on the line and longitudinal*, and the saddles interlace the zeros. Near the Lehmer pair 7005.062866, 7005.100565 the saddle sits at 7005.081716, and ζ' has a zero at $0.5006235 + 7005.0818556i$, only $6.2 \cdot 10^{-4}$ right of the line. In the landscape, the two foci are so close that the tilt barely separates the ζ' saddle from the ξ' saddle. Speiser check: ζ' has no zero in $[0, \frac{1}{2}] \times [10, 1000]$ (winding in 33 windows of height 30).

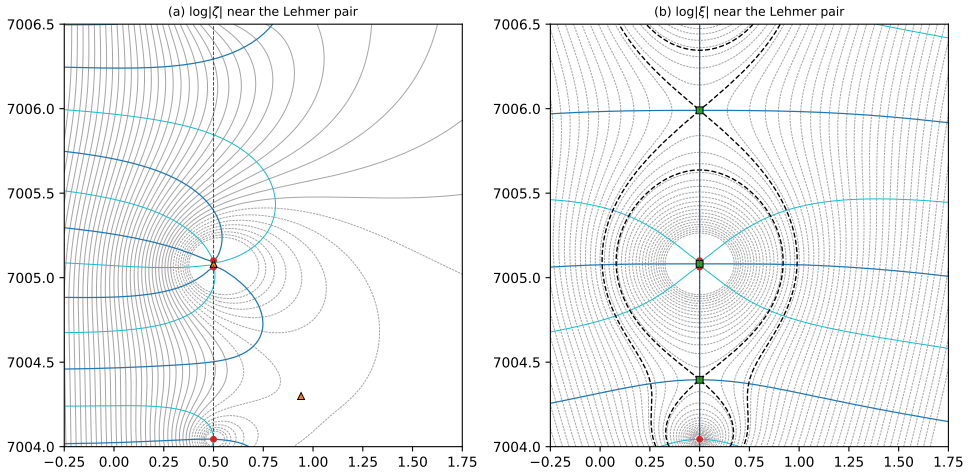

FIGURE 3. The Lehmer pair: (a) $\log|\zeta|$ with the ζ' zero at $0.50062 + 7005.08186i$ squeezed between the two foci; (b) $\log|\xi|$ with three longitudinal saddles and the lemniscate enclosing the pair. PNG `num/l55/lemma55_lehmer.png`.

TABLE 4. Zeros, saddles of $|\xi|$ and zeros of ζ' in three windows [NUM] (`num/l55/online.json`, `zeta_saddles.json`, `zp_windows.json`).

window	zeros of ζ on the line	saddles of $\lvert\xi\rvert$ ($\sigma = \frac{1}{2}$; type)	zeros of ζ' ($\sigma > 0.45$)
$1 \le t \le 60$	13 zeros, 14.134725, …, 59.347044	13, 15.585709, …, 59.930699; all long.; total in $0 \le \sigma \le 1$: 13	$2.4631619 + 23.29832i$, $1.2864968 + 31.70825i$, $2.3075701 + 38.48998i$, $1.3827636 + 42.29096i$, $0.9646856 + 48.84716i$, $2.1016999 + 52.43216i$, $1.8959598 + 57.13475i$
$7004 \le t \le 7006.5$	7004.043723, 7005.062866, 7005.100565	7004.395870, 7005.081716, 7005.990185; all long.; total in $0 \le \sigma \le 1$: 3	$0.9400235 + 7004.29940i$, $0.5006235 + 7005.08186i$
$10^5 \le t \le 10^5 + 3$	100000.743723, 100001.180558, 100002.093723, 100002.517282	100000.145556, 100000.939597, 100001.568905, 100002.285162, 100002.889838; all long.; total in $0 \le \sigma \le 1$: 5	$0.6500253 + 100000.94430i$, $1.1002387 + 100001.09311i$, $0.6353707 + 100002.29338i$, $1.2323473 + 100002.55187i$

8. FINITE FIELDS: SPEISER = COHN

Let $\mathcal{L}(u) = \prod_{i=1}^{2g}(1-\beta_i u)$ be self-inversive ($u^{2g}\mathcal{L}(1/u) = \mathcal{L}(u)$) with real coefficients, $u = q^{1/2-s}$, and $d/ds = -\log q \cdot u\, d/du$. Then: [THM]

- the *tilted* saddles (analogue of ζ') are the zeros of $\mathcal{L}'(u)$;
- the *symmetric* function is $h(\varphi) = e^{-ig\varphi}\mathcal{L}(e^{i\varphi})$, which is real (checked to 10^{-16}) and is the analogue of Ξ;
- the symmetric saddles are the zeros of $u\mathcal{L}' - g\mathcal{L}$, since $\frac{d}{du}(u^{-g}\mathcal{L}) = u^{-g-1}(u\mathcal{L}' - g\mathcal{L})$;
- the delay u^{-g} plays the role of the Γ-tilt: $\partial_\sigma \log|u^g| = -g\log q$.

Theorem 8.1 (Cohn 1922)**.** *A polynomial has all its zeros on $|u| = 1$ iff it is self-inversive and its derivative has all its zeros in $|u| \le 1$* [6]*, as stated in* [17, §1] *[CITED]. Since $|u| \le 1 \iff \sigma \ge \frac{1}{2}$, this is* exactly Speiser's theorem *for curves over $\mathbb{F}_q$: RH for $\mathcal{L} \iff$ no tilted saddle with $\sigma < \frac{1}{2}$ [RESTATEMENT] (here RH is Weil's theorem).*

Lalín–Smyth's identity [17, (6)], $P(z) = \frac{z}{d}P'(z) + \varepsilon\frac{z^{d-1}}{d}\overline{P'}(1/z)$, is the polynomial counterpart of the Levinson–Montgomery remark that $\zeta'(s)$ and $J(1-s)$, with $J = \zeta + \zeta'[h'/h(s) + h'/h(1-s)]^{-1}$, have the same zeros in $0 < \sigma < 1$ [19, §2] [ANALOGY]. A variant of the latter identity is used by Conrey for the derivatives of ξ [8] [CITED].

TABLE 5. Curves and fakes of Lemma 48 [NUM] (`num/l55/toy.json`). "ghosts" = #zeros of $\mathcal{L}$ in $|u| < 1$. Tilted saddles: #zeros of $\mathcal{L}'$ with $|u| > 1$ (i.e. $\sigma < \frac{1}{2}$), and the moduli of all zeros of $\mathcal{L}'$. Symmetric saddles: zeros of $u\mathcal{L}' - g\mathcal{L}$ on $|u| = 1$, with types L (longitudinal) or **T** (transverse).

case	ghosts	$\#\{\lvert u\rvert > 1\}$ of $\mathcal{L}'$	$\lvert u\rvert$ of zeros of $\mathcal{L}'$	sym. on $\lvert u\rvert = 1$	types (sym.)
11a p=5	0	0	0.2236	2/2	L L
11a p=7	0	0	0.3780	2/2	L L
11a p=13	0	0	0.5547	2/2	L L
11a p=17	0	0	0.2425	2/2	L L
11a p=19	0	0	0.0000	2/2	L L
11a p=23	0	0	0.1043	2/2	L L
g2 x^5-x+1 p=11	0	0	0.6595, 0.8945, 0.8945	4/4	L L L L
g2 x^5-x+1 p=13	0	0	0.2011, 0.5872, 0.5872	4/4	L L L L
g2 x^5+x^2+3 p=11	0	0	0.0000, 0.5222, 0.5222	4/4	L L L L
g2 x^5+x^2+3 p=13	0	0	0.1724, 0.8967, 0.8967	4/4	L L L L
fake a=8 q=13	1	1	1.1094	2/2	L **T**
fake a=7 q=11	1	1	1.0553	2/2	L **T**
fake a=13 q=13	1	1	1.8028	2/2	**T** L
fake g2 real pair q=11	2	2	0.0000, 1.0660, 1.0660	4/4	**T** L L **T**
fake g2 quadruple q=11	2	2	0.7326, 1.1566, 1.1566	4/4	L **T** **T** L
fake g2 mixed q=11	1	1	0.7573, 0.7573, 1.3143	4/4	L L L **T**

Table 5 shows:

(i) $\#\{\mathcal{L}' = 0, |u| > 1\}$ = #ghosts in all 16 cases. This is Cohn's theorem plus an exact count. In a random test of 20000 self-inversive polynomials of degree 2–8, the only mismatches (7) had a critical point within 10^{-7} of the circle, a tolerance artefact [NUM]. We conjecture the exact count for self-inversive P with simple zeros on the circle [CONJ]. Cohn's paper (title: number of roots in a circle) is the likely source, but it was not read.

(ii) The symmetric saddles lie on the circle *in every case, fakes included.* In Lemma 54's Wick language, a unimodular pair contributes the factor $2\cos(\varphi - \theta)$ to h (up to sign), while a ghost pair contributes $\pm 2\cosh(\sigma - i(\varphi - \theta))$-type factors, whose modulus has a positive minimum on the circle. That minimum is where the transverse saddle sits [ANALOGY]. So "all symmetric saddles on the line" is blind to ghosts here, as in Table 1. But the number of transverse ones equals the number of ghosts, as Thm. 4.2(e) (circle version) predicts [THM][NUM].

9. Mechanism: is there a zero-naive principle?

What is zero-naive and true.

- Saddles, lemniscates, separatrices, the Morse count $m-1$ and the divergence of the precession period (§3).
- The FE symmetry of the $|\xi|$ landscape.
- The saddle type (longitudinal/transverse).
- Speiser's tilted saddles.

All of these are defined from f alone. Two of them count ghosts: (S) and (T) of §6.

What would force RH. Each criterion needs one positivity statement:

(a) (T) needs: *no saddle of $|\zeta|$ in the left strip.* By Speiser and LM this is *equivalent* to RH [RESTATEMENT].

(b) (S) needs: *every saddle of $|\xi|$ is on the line and longitudinal*, i.e. $\Xi''/\Xi < 0$ wherever $\Xi' = 0$. This follows from the Laguerre inequality, which follows from RH.

(c) The common strengthening is the *saddle positivity*

$$\operatorname{Re}\frac{\xi'}{\xi}(s) > 0 \quad (\sigma > \tfrac{1}{2}),$$

which says that $|\xi|$ increases away from the line. It is Lagarias's criterion (Lemma 17; Lemma 9's weight $2\operatorname{Re}\xi'/\xi$), equivalently Sondow–Dumitrescu monotonicity (Lemma 33), equivalently Lemma 37's $G5_{\text{line}}$ [RESTATEMENT]. Being equivalent to RH, it implies (a) (Prop. 5.3) and (b) (Thm. 4.2(b)). Geometrically, the modular surface is then a valley along the line.

So the "saddle positivity" that would make J.W.'s separatrix picture prove RH is exactly the Herglotz/passivity positivity already identified as a face of G5 in Lemmas 36–37. In Farmer's taxonomy [12, §4.1] it is a *repackaging* (B): it concentrates the information locally (each ghost shows up as a saddle within δ of its height, Thm. 4.2(c)–(d)), which is a mild point in its favour, but it supplies no positivity of its own. For curves over $\mathbb{F}_q$ the positivity comes from the polarization (Lemmas 46, 50: Hodge index / $\operatorname{Tr}(\alpha\alpha^\dagger) > 0$). It forces the zeros onto the circle, hence by Cohn the tilted saddles inside, hence all symmetric saddles longitudinal. In that chain *saddle geometry is the last link, not the first* [ANALOGY].

Does the separatrix "naturally produce reflective symmetry about the zeros on the line"? No. The symmetry $s \mapsto 1-\bar{s}$ is put in by the functional equation and holds for f_{DH} as well [THM]. What the separatrices do produce is *local*: under RH, the lemniscate through the saddle between two consecutive zeros is a figure eight symmetric about the line, with the two foci on the line (Fig. 2(b)). With a ghost, the transverse figure eight is *horizontal*, and the ghost and its mirror sit in its two lobes (Fig. 2(d)). The orientation of the figure eight at each saddle (vertical versus horizontal) is the zero-naive local invariant that distinguishes the two [THM].

10. Corrections to the chat claims

(i) "$|\xi(s)| = |\xi(1-\bar{s})|$, so the $|\xi|$ landscape is mirror-symmetric for free. ξ is real on the line, so the line is a zero-phase curve. Ghost pairs $\rho, 1-\bar{\rho}$ are symmetric too, so symmetry cannot exclude them." **Correct, with two refinements.** First, the line is a curve of phase 0 *or* π ($\operatorname{Im}\xi = 0$), and the sign changes at each zero. Second, symmetry of the *landscape* cannot exclude ghosts, but the zero-naive *type* of the on-line saddles can detect them (Thm. 4.2(c)–(e); 28/28 for Davenport–Heilbronn).

(ii) "Saddles of $|f|$ are exactly the zeros of f', and separatrices pass through them." **Correct for zeros of f' with $f \neq 0$.** A zero of f' at a multiple zero of f is a focus, not a saddle. An order-m zero of f' is a monkey saddle of index $-m$ with $2(m+1)$ separatrices. Both

the gradient separatrices (phase curves) and the lemniscates (level curves) pass through it (Prop. 3.1).

(iii) "Speiser 1934: RH $\iff$ ζ' has no zeros in $0 < \operatorname{Re} s < \frac{1}{2}$." **Correct in substance.** The paper was received on 6.4.1934 and published in Math. Ann. 110 (1935). Speiser's own wording is "all non-trivial zeros of ζ' lie right of the critical line or on it" (on it only at multiple zeros of ζ). The "no zeros in $0 < \sigma < \frac{1}{2}$" wording is Levinson–Montgomery's Corollary, which calls it essentially due to Speiser and credits one half to Spira. Speiser's argument is topological; the rigorous proof is in [19] (cf. [1]).

(iv) "Levinson–Montgomery 1974 made it quantitative, and Levinson's method grew out of it." **First half correct**: $N_1^-(T) = N^-(T) + O(\log T)$, with equality along a sequence unless $N^- > T/2$ for all large T [19, Thm. 1]. **Second half not verified**, and LM74 does not say so. What we did verify: Levinson's 1/3 paper [18] appeared the same year, and LM's §2 Remark gives the identity $\zeta'(s) \leftrightarrow J(1-s)$, a variant of which Conrey uses [8]. The genealogy of Levinson's method is left unasserted.

(v) "For DH, ghosts should drag saddles left of the line." **Correct for the tilted landscape** $|f_{\mathrm{DH}}|$: each of the 28 ghosts has a zero of f'_{DH} in the left strip near its mirror, and $N_1^- = N^- = 28$ window by window (Tables 2, 3). **False for the symmetric landscape** $|\xi_{\mathrm{DH}}|$: there the ghosts leave all saddles on the line and create a *transverse* on-line saddle instead (28/28; no off-line saddle found). Which hypothesis is used: Speiser's argument needs the functional equation, $\theta'_{\mathrm{DH}} > 0$ ($t > 1.2116$) and the absence of a pole in the region, but not the Euler product. So it applies to f_{DH}, and its conclusion ("RH $\iff$ no left saddles") is borne out: f_{DH} violates both sides.

(vi) (Steering (4), J.W.) "A symmetric ghost pair can produce a saddle of $|\xi|$ on the line (a real local extremum of Ξ with no sign change)." **Confirmed**: these are exactly the transverse saddles (Fig. 1(b), Table 1, Fig. 2(d)). This is one reason Speiser uses the tilted ζ'. The other reason is that his topological argument needs a function whose modulus decreases across the line.

(vii) (Literature) "$\Xi^{(n)}$ has only real zeros for large n" is **not** a theorem in that form. Kim's theorem is local (in $|z| < R$ for $n \geq n(R)$) [15], and Ki's is a rescaled cosine limit [14]. Both are RH-blind [12].

(viii) (Steering (1), detail) Needham's Fig. [11] is Perseus' spiric section of a torus, not an n-focus Cassinian picture. The n-focus statement is in the text on p. 63.

11. Prior art; what is new and what is restatement

- *Modular surfaces and Cassinian curves*: [21, Ch. 2 §II.3], crediting Brieskorn–Knörrer (not read). Edition check: the 1997 Clarendon Press/OUP edition (ISBN 0-19-853447-7) has "Cassinian Curves" at p. 60 in Ch. 2 §II "Polynomials" (archive.org table of contents). This matches the running heads on J.W.'s page images (pp. 61–63). The 2023 25th-anniversary edition has it at §2.2.3, p. 67.
- *Speiser's geometric picture* [27]; Utzinger's dissertation (via [1]; not read); Arias de Reyna's X-ray [1]. *Flows*: Broughan–Barnett [5] and Broughan [4] (prior art in Lemma 43).
- ζ' *zeros*: [19, 28, 29, 2]. The horizontal distribution of ζ' zeros was studied by Soundararajan [26] and Zhang [30] (records only; content not used).
- $\xi^{(m)}$ *zeros*: [7, 8, 15, 14, 12]. Laguerre-type inequalities and RH: [9] (record only). De Bruijn–Newman: [11, 22, 13, 23, 24].
- *Self-inversive polynomials*: [6, 3, 17, 20].
- *Positivity*: Lagarias [16], Sondow–Dumitrescu [25] (Lemmas 9, 17, 33, 37).

Restatements [RESTATEMENT]: Speiser/LM (Thm. 5.2), Cohn (Thm. 8.1), saddle positivity = Lagarias (§9). **Not found in the sources read, possibly classical** (no novelty claimed):

- the longitudinal/transverse vocabulary and Thm. 4.2(c)–(e) (elementary; counts of this kind for real polynomials are plausibly classical);
- the phase-precession/period reading of the lemniscate (Prop. 3.3);
- the Morse $m-1$ count for ξ-sublevel components (Prop. 3.2, a Riemann–Hurwitz/Walsh-type statement).

New computations [NUM] (as far as we know): the Davenport–Heilbronn transverse-saddle table (28/28), the $N_1^- = N^-$ window count for f_{DH} ($28 = 28$), and the finite-field tilted/symmetric saddle table.

12. Questions for J.W.

Question 12.1. Shall TAA adopt the *orientation of the figure-eight lemniscate at each on-line saddle* (vertical = longitudinal, horizontal = transverse) as the official zero-naive separatrix invariant? It is FE-symmetric, defined from ξ alone, and counts ghosts for (trigonometric) polynomials.

Question 12.2. The converse "all saddles of $|\xi|$ on the line and longitudinal $\Rightarrow$ RH" reduces to a lower bound $\#\{\Xi' = 0 \text{ in } [0,T]\} \geq N(T) - C$ along a sequence (§4). Should a lemma attempt this counting step? It would be a [RESTATEMENT], with the honest expectation that it is as hard as RH.

Question 12.3. For the Lemma 48/53 dictionary (winding k, charge $-k$), which saddle counter should be official: (S) symmetric transverse saddles, or (T) Speiser's left-strip saddles? (T) is exact over $\mathbb{F}_q$ by Cohn (conjecturally with exact count), and (S) is exact whenever all symmetric saddles lie on the line.

Question 12.4. Should we read Cohn 1922 (Math. Z. 14) to settle the conjectured exact count $\#\{\mathcal{L}' = 0,\ |u| > 1\} = \#\text{ghosts}$ (§8(i))?

Question 12.5. Under the de Bruijn–Newman heat flow H_t, two colliding foci become a ghost pair, and a transverse saddle is born. Is a short numerical study (toy and H_t for $t < 0$) of this saddle birth useful as a picture of "where ghosts come from", given that $\Lambda \geq 0$ [24] makes it RH-neutral?

Status

Done:

- Needham framing (verified edition and pages).
- Topology: saddles, separatrices, Morse count, precession period.
- Gauss–Lucas, Laguerre–Pólya, Rolle interlacing; converse fails.
- Longitudinal/transverse criterion and ghost detector.
- Tilt decomposition and the exact on-line identity $\mathrm{Re}\,\zeta'/\zeta = -\theta'$.
- Speiser/LM at the source.
- ζ numerics in three windows; Speiser check to $t = 1000$.
- Davenport–Heilbronn: 28/28 transverse saddles and $N_1^- = N^- = 28$.
- Finite-field Cohn table.
- Mechanism (saddle positivity = Lagarias = $\mathrm{G5}_{\mathrm{line}}$, [RESTATEMENT]).

Not done:

- the counting step for the converse ([OPEN]);
- Cohn 1922, Utzinger and Brieskorn–Knörrer not read;
- no DH window beyond $t = 1000$.

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TAA LEMMA 56 (V3): THE ORIENTATION OF ON-LINE SADDLES. TRANSVERSE SADDLES, JENSEN DISCS, AN EXACT COUNT FOR ξ', COHN'S THEOREM, THE SADDLE FORM OF RH, AND THE WHOLE-LINE DAVENPORT–HEILBRONN FLOW

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

Date: October 22, 2026 (v3; v2 archived as `archive/lemma56.v2-20261023-044042.{tex,pdf}`, v1 as `archive/lemma56.v1-*`).

ABSTRACT. **Statement of intellectual honesty.** *RH is not proved here.* Nothing below brings it closer. v2 proves that RH is *equivalent* to a statement (H) about the zeros of ξ' (Thm. 4.10); that is a restatement [RESTATEMENT], not progress on RH. The main structural results are in the literature: the localization of critical points in Jensen discs (Jensen 1913, Pólya 1943) and the exact window count of critical zeros (Y.-O. Kim, Proc. AMS 1996, Thm. 1). We re-derive them for Ξ, apply them to ξ, Davenport–Heilbronn, finite fields and the heat flow, and check them numerically. Every equivalence with RH is tagged [RESTATEMENT]. Tags: [THM] proved here, or classical with proof given; [CITED] checked at the source (`lit/lemma56/verified_refs.md`); [BIB] record verified, content standard or quoted from a source that was read; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma56_*.py`, data in `num/l56/`); [CONJ] conjecture; [OPEN] open.

J.W.'s answers to the Lemma 55 questions ([t207u]). "1. Yes. 2. Yes. 3. Transverse saddle count. 4. Yes. 5. Yes." That is: (1) adopt the saddle-lemniscate orientation as the official zero-naive invariant; (2) attempt the counting step for the converse; (3) use the transverse-saddle count as the ghost counter; (4) read Cohn 1922; (5) study the de Bruijn–Newman flow.

v2 in brief (J.W.'s answers to the v1 questions; details in "Corrections and updates in v2"). (i) $\kappa = \tau + S^+$ is official. (ii) The converse is *not* open, contrary to v1. An unconditional count $|N_{\xi'}(T) - N(T)| \le C \log T$ (Thm. 4.8) gives (H)⇒RH, so RH ⟺ (H) (Thm. 4.10) [THM][RESTATEMENT]. The effective barrier: every $T \in [305, 10^{116.7}]$ has an admissible height in $[T+1, 2T-1]$ (Thm. 4.13). $c < \frac{1}{2}$ holds asymptotically (Conrey: $c = 0.1612 + o(1)$) but is *not* available with verified explicit constants (Rem. 4.15). (iii) A Volume 3 patch has been drafted and test-built (not applied, not submitted). (iv) DH forward flow: all 28 ghosts below 1000 land at $t_c \le 0.2306 \approx 2\delta^2$, each as one zero collision with no saddle collision. So $0.2306 \le \Lambda_{\mathrm{DH}} \le 1.6025$ (§7.5). (v) Diagnostics table, Table 7.

Answer in brief (v1, kept; item (2) is corrected by (ii) above). (1) *Orientation* (Def. 2.1). An on-line critical point t_0 of Ξ is *longitudinal* if $\Xi''/\Xi(t_0) < 0$ and *transverse* if $\Xi''/\Xi(t_0) > 0$. Notation: $\mathfrak{o}(t_0) = -\operatorname{sgn}(\Xi''/\Xi)(t_0) \in \{+1, -1, 0\}$. The invariant is zero-naive and FE-symmetric [THM]. Every transverse saddle and every off-line saddle lies in a Jensen disc of a ghost (Thm. 3.1; Jensen/Pólya, proof given) [THM][BIB]. (2) *Counting.* At every height T outside the closed Jensen intervals we prove the exact count $N_{\xi'}(T) = N(T) - \frac{1+\varepsilon(T)}{2}$, with $\varepsilon = \operatorname{sgn} \Xi'/\Xi(T)$ (Thm. 4.2) [THM]. On windows with admissible ends,

$$\#\{\text{ghosts}\} = \#\{\text{transverse saddles}\} + \#\{\text{off-line saddles in } \sigma > \tfrac{1}{2}\}$$

(Thm. 4.3). This is Kim's 1996 theorem specialized to Ξ [THM][CITED]. Consequences: with finitely many ghosts, RH ⟺ (H) (all saddles on the line and longitudinal) [THM][RESTATEMENT]. In general, (H)∧¬RH forces the ghost Jensen intervals to *cover a half-line*, so $\sum_{T<\gamma\le 2T}(\beta - \frac{1}{2}) \ge \frac{T}{2} - O(1)$ (Prop. 4.5) [THM]. Excluding this needs an explicit Selberg-type bound with constant $< \frac{1}{2}$, which we do not have. *This is where the argument breaks* [OPEN]. Numerics: ζ to height 1000 and DH to height 1000 satisfy the exact count at every tested admissible height (0 violations). DH has 28 ghosts, 28 transverse saddles and 0 off-line saddles, window by window [NUM]. (3) *Ghost counter.* The transverse count alone is *not* the ghost count in general. The exact counter is $\kappa = \tau + S^+$ (transverse plus off-line saddles) [THM]. It agrees with τ for all 16 curves/fakes of Lemma 48 and for DH up to 1000 [NUM]. It differs from τ in explicit fakes (genus 2 with $|a_1| > 4\sqrt{q}$; a genus-3 quadruple with $\tau = 0$), in 1510 of 3000 random self-inversive tests, and along the heat flow [NUM]. We recommend κ as the official counter, with τ as its on-line part (Correction C1). (4) *Cohn 1922, read at the source.* "Fall IV" (self-inversive) gives: f and $x^{n-1}f'(1/x)$ have equally many roots inside the unit circle [CITED]. Hence, for every real self-inversive L-polynomial, $\#\{P' = 0,\ |T| > q^{-1/2}\} = \#\{\text{ghosts}\}$ exactly. This proves the Lemma 55 count [THM]/[CITED]. On the circle, $k = \tau + \frac{1}{2} Z_{\mathrm{off}}(uL' - gL)$ [THM]. In genus 2, $\tau = k$ iff $|a_1| \le 4\sqrt{q}$ [THM]. 19/19 table cases and 3000/3000 random cases agree [NUM]. (5) *Heat flow.* An orientation can only change at (i) a *zero collision*, where Ξ''/Ξ has a universal pole $1/(t_c - t)$ and the lemniscate rotates by 90°, or (ii) a *saddle collision* (degenerate monkey saddle), where a transverse and a longitudinal saddle leave the line as an off-line pair [THM]. κ is constant through both events, while τ is not. Numerics for H_t, $z \le 125$, $-16 \le t \le 0.3$: 5 zero collisions and 4 saddle collisions; $(t_c - t)H''/H \to 1$ to $3 \cdot 10^{-4}$; the identity held at all 327 time steps [NUM]. Rodgers–Tao ($\Lambda \ge 0$) gives a ghost, hence a transverse or off-line saddle, in H_t for every $t < 0$ [CITED][THM]. The polynomial analogue behaves the same way [THM][NUM].

J.W.'s answers to the v2 questions ([t218u]). "1. Yes..and can you elaborate to me on what (H) as saddle for RH means? 2. What does pushing the explicit range get us? What benefit and con does it provide? 3. Lets include the Xi-side results. Can you also explain these results to me. 4. Yes. 5. Yes."

v3 in brief (details in "Corrections and updates in v3"). (i) *(H) is the official saddle form of RH* (Def. 5.1, §5) [RESTATEMENT]. What it means, in pictures, is explained in §5.1. The converse is no longer a research target. (ii) *Pushing the explicit range* (Rem. 5.2): it would extend an unconditional "no covering" statement from $10^{116.7}$ to about 10^{232}, or to all large T. It would not change any logical status, because RH $\iff$ (H) is already a theorem. It would cost heavy explicit analysis. Following J.W., it is *not* done. (iii) The ξ-side results are added to the Volume 3 addendum (test build only; not submitted). (iv) *Whole-line DH flow* (§7.6). On $0 < z < 2000$ (heights below 1000), for $0 \le t \le 0.25$: $K^+ = \tau + S^+$ held at all 126 steps; $S^+ \equiv 0$; the number of real critical points was constant (903); no saddle collision occurred and no new transverse saddle appeared; there were exactly 28 zero collisions, one per ghost. There are no DH zeros with $\beta \ge 1$ up to height 1000; this is *certified* with ball arithmetic (Thm. 7.5). Numerically there are none with $1 < \beta < 1.425$ up to height 10^4 either. The new bounds are $0.230607 \le \Lambda_{\mathrm{DH}} \le 1.6025$, and the lower bound is *certified* by a rigorous Laguerre-inequality violation (Thm. 7.6). The ghosts with large δ between heights 1000 and 10^4 land *earlier* than $2\delta^2$ predicts (ratio 0.44–0.78; all 23 with $\beta > 0.839$ computed, the largest $t_c = 0.2232$ at $\gamma = 2442.53$), so none of them raises the bound. (v) The diagnostics table (Table 7) now carries Lemma 54 v2's invariants ν (frequency-form index) and r (drift rank). Both equal k in every row where they are defined, real-pair ghosts included.

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Corrections and updates in v3

(V1) *Official saddle form.* (H) is recorded as the official "saddle form" of RH in the TAA RESTATEMENT ledger (Def. 5.1) [RESTATEMENT]. No novelty is claimed. The converse (H)⇒RH is Thm. 4.10, and it is not pursued further as a research target. STATE.md, NOTATION.md and the Lemma 58 inventory notes are updated in text.

(V2) *Explicit range.* J.W. asked what pushing the explicit range buys, and did not approve doing it. The answer is Rem. 5.2; nothing was computed.

(V3) *Volume 3.* The addendum now also has the ξ-side: the global count, Kim's identity, RH $\iff$ (H) with a proof sketch, the admissible-height range, the Conrey remark, the ζ/DH census, the DH flow, and the figures. It is a test build only (`volume3/volume3_l56test.tex`). `volume3.tex` and `volume3_submit/` were not touched, and nothing was submitted.

(V4) *Whole-line DH flow* (§7.6). This replaces v2's per-window picture with a global count on $0 < z < 2000$, for all $t \in [0, 0.25]$. The landing times t_c were recomputed by an independent method (contour-shifted FFT). They agree with v2 to $4.2 \cdot 10^{-5}$.

(V5) *Certified parts* (Thms. 7.5, 7.6). (a) No zero of f_{DH} in $1 \le \sigma \le 1.6$, $0 \le t \le 1000$. (b) $H^{\mathrm{DH}}_{0.230607}$ has a non-real zero, so $\Lambda_{\mathrm{DH}} \ge 0.230607$. Both use Arb ball arithmetic [11] and are computer-assisted proofs [THM]. All other flow statements are [NUM].

(V6) *Diagnostics* (Table 7): κ first, then k, τ, S^+, ν, r, ℓ and the L53 charge. ν and r are defined in Lemma 54 v2.

Corrections and updates in v2

J.W.'s answers to the v1 questions. "1. $\kappa = \tau + S^+$ is the official counter. 2. Yes, make Rem. 4.6 effective. 3. Yes, add Cohn and the genus-2 criterion to Volume 3. 4. Run DH forward under the heat flow. 5. Yes, the side-by-side diagnostics table with κ as the main column."

(U1) *Official counter.* $\kappa = \tau + S^+$ is the official TAA ghost counter (Def. 2.1); τ is its on-line part. Over $\mathbb{F}_q$ it is $\kappa_{\mathrm{FF}} = \tau + \frac{1}{2} Z_{\mathrm{off}}$. Earlier notes that wrote "transverse count" or "(S)" for the counter are not edited; read them with this convention (NOTATION.md, STATE §§64–65).

(U2) *Correction to v1: the converse is not open.* v1 Cor. 4.4(d), Prop. 4.5 and Rem. 4.6 said "(H)⇒RH is [OPEN]" and placed the gap in excluding the covering scenario. That was wrong. An unconditional Backlund-type count $|N_{\xi'}(T) - N(T)| \le C \log T$ (Thm. 4.8) shows that under (H) there are only $O(\log T)$ ghosts up to T. So their Jensen intervals cannot cover, and *(H)⇒RH* (Thm. 4.10) [THM]. Hence RH $\iff$ (H) [THM][RESTATEMENT]. **This is an equivalent form of RH, not a proof of RH.** (H) is a statement about all zeros of ξ', and it is no easier to verify than RH. We expect it to be known to specialists (it is close to Kim 1996 and Ki–Kim 2000, which we read only in excerpt), and we claim no novelty.

(U3) *Rem. 4.6 made effective* (Thm. 4.13). With Littlewood's lemma, Simonič's explicit mean square (2020, Cor. 5, read at the source), an explicit Euler–Maclaurin bound and Backlund's argument for $\arg\zeta$, every $T \in [305, 10^{116.7}]$ has an admissible height in $[T+1, 2T-1]$ [THM][NUM]. Unconditionally, then, the covering of Prop. 4.5 cannot happen at those heights. The value 10^{232} in v1 used the asymptotic mean square on $[T, 2T]$; the rigorous one-sided constant gives $10^{116.7}$ (Correction C9).

(U4) *Is $c < \frac{1}{2}$ achievable? Asymptotically yes:* Conrey 1989, Thm. 3 (read at the source) gives $\sum_{0<\gamma\le T}(\beta - \frac{1}{2}) \le (0.0806 + o(1))T$, so $c = 0.1612 + o(1) < \frac{1}{2}$ on $(T, 2T]$. That alone also excludes covering for all large T, hence also gives (H)⇒RH (Prop. 4.14) [THM] from [CITED]. *With explicit constants for all $T \ge T_0$: not achievable from what we verified.* The only explicit Selberg-type density we found (Simonič 2020, Thm. 1, $a = 10395.2$) gives $c \approx 4 \cdot 10^4$, five orders too large. Kadiri–Lumley–Ng (2018) is explicit only for $\sigma \ge \frac{3}{5}$ (record and statement via Simonič) (Rem. 4.15). The explicit T_0 is therefore the range in (U3), together with the trivial $3 \cdot 10^{12}$ of [19] (no ghosts below it).

(U5) *DH forward flow* (§7.5). We define H_t^{DH} by e^{tu^2} on the Fourier side, with Φ_{DH} even and doubly exponentially decaying (checked). All 28 ghosts below height 1000 land on the line at $0 < t_c < 0.231$. At each landing exactly one transverse saddle turns longitudinal, $K^+ = \kappa$ drops by one, and S^+ stays 0. Hence $\Lambda_{\mathrm{DH}} \ge 0.2306$, which is outside ζ's $[0, 0.2]$. We also have $\Lambda_{\mathrm{DH}} \le 1.603$ by de Bruijn's strip theorem. The landing times satisfy $t_c \approx 2\delta^2$ (ratio 0.80–1.00) [NUM][THM].

(U6) *Diagnostics table* (§8, Table 7), with κ first, over FF curves and fakes, DH windows, ζ windows, toys and heat-flow snapshots [NUM].

(U7) *Volume 3.* A patch `volume3/addendum_l56.tex` (Cohn Fall IV, the circle identity Thm. 6.2, the genus-2 criterion, κ) has been drafted and test-built. It is queued in `pending_replacements.md`. It is not applied and not submitted, and `volume3_submit/` was not touched.

1. Conventions, and what TAA already has

$s = \sigma + it$, $\mathbb{H}_{1/2} = \{\sigma > \frac{1}{2}\}$. $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and $\Xi(t) = \xi(\frac{1}{2} + it)$, which is real, even, of order 1 and genus 1. A *ghost* is a zero $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$. Its *mirror* is $1 - \bar{\rho}$, and $\delta = \beta - \frac{1}{2}$. In the t-variable the pair $\{\rho, 1 - \bar{\rho}\}$ is the conjugate couple $\gamma \mp i\delta$ of zeros of Ξ. Its *Jensen disc* is $D_\rho = \{|t - \gamma| \le \delta\}$ (equivalently $\{|s - \frac{1}{2} - i\gamma| \le \delta\}$), and its *Jensen interval* is $I_\rho = (\gamma - \delta, \gamma + \delta)$. $N(T)$ counts zeros of ξ with $0 < \gamma \le T$, and $N_{\xi'}(T)$ counts zeros of ξ' with $0 < \operatorname{Im} s \le T$ (all σ). Davenport–Heilbronn (DH), θ, A and the finite-field conventions $\mathcal{L}(u) = P(u/\sqrt{q})$, $u = q^{1/2-s}$ are as in Lemma 55 §1. A *saddle* of $|\xi|$ is a zero of ξ' that is not a zero of ξ. It is *on-line* if $\sigma = \frac{1}{2}$ and *off-line* otherwise.

Overlap table (checked against STATE.md §§57–64; nothing is redone):

Lemma 48	ghost count k (FF: zeros of P with $\lvert T\rvert < q^{-1/2}$); DH ghosts below 1000: 28 (`num/l48/zeros_T1000.json`).
Lemma 50	Krein signature, trace-model positivity $\iff$ RH (Thm. 4.4). Used here only through Lemma 54.
Lemma 51	tracking invariants through Weil→fake collisions. §7 is the ξ-analogue for the heat flow [ANALOGY].
Lemma 53	charge $-k$.
Lemma 54	J_F (exists iff no real Frobenius eigenvalue), $n_- = 2\#$quadruples, drift rate $\ell = 2(\sum_{\lvert\lambda\rvert>1}\sigma^2)^{1/2}$, Wick continuation cos → cosh.
Lemma 55	saddles of $\lvert\xi\rvert$ = zeros of ξ'; longitudinal/transverse; Thm. 4.2 (Laguerre detector); tilt $\operatorname{Re}\zeta'/\zeta(\frac{1}{2} + it) = -\theta'(t)$; Speiser/LM74; Cohn via LS12; DH 28 transverse saddles. The counting step was left [OPEN].

2. The orientation invariant (J.W. answer 1)

Definition 2.1 (Orientation; official TAA notation)**.** Let F be a real entire function of a real variable ($F = \Xi$, Ξ_{DH}, H_t, or a real polynomial). Let $t_0 \in \mathbb{R}$ with $F'(t_0) = 0 \ne F(t_0)$. Its

orientation is

$$\mathfrak{o}(t_0) = -\operatorname{sgn}\frac{F''(t_0)}{F(t_0)} \in \{+1, -1, 0\}.$$

The point is *longitudinal* if $\mathfrak{o} = +1$ ($|F|$ has a local maximum along the line) and *transverse* if $\mathfrak{o} = -1$ ($|F|$ has a local minimum along the line, with no sign change). It is *degenerate* if $\mathfrak{o} = 0$. For a window (a, b), write

$$\tau(a,b) = \#\{\text{transverse critical points in } (a,b)\},$$

$$S^+(a,b) = \#\{\xi'(s) = 0 :\ \sigma > \tfrac{1}{2},\ a < \operatorname{Im} s < b\},$$

$K^+(a,b) = \#\{\text{ghosts with } a < \gamma < b\}$, and

$$\kappa(a,b) = \tau(a,b) + S^+(a,b) \qquad (\text{the } \textit{ghost counter}\text{; official since v2, J.W.}).$$

Multiple critical points are counted with Kim's convention for "critical zeros" [14, 13]. A zero of F' of multiplicity m with $F \neq 0$ carries $\lceil m/2 \rceil$ or $\lfloor m/2 \rfloor$ critical zeros according to the sign of $F\,F^{(m+1)}$, and $m/2$ if m is even. For simple points this is "transverse". On the unit circle (finite fields), τ, Z_{off} and $\kappa_{\text{FF}} = \tau + \frac{1}{2}Z_{\text{off}}$ are defined in §6.

Lemma 2.2 (Geometry; zero-naivety)**.** *Let t_0 be a non-degenerate on-line saddle.*

(a) $\log|\xi|$ is harmonic near $s_0 = \frac{1}{2} + it_0$. Its Hessian there is $\pm\operatorname{diag}(\Xi''/\Xi, -\Xi''/\Xi)$ in the (σ, t) frame. So the lemniscate $\{|\xi| = |\xi(s_0)|\}$ is a figure eight with lobes along the line in the longitudinal case and across it in the transverse case (Lemma 55) [THM].

(b) $\mathfrak{o}$ is computed from Ξ on the line alone. It needs no zero locations (it is zero-naive) and it is invariant under $s \mapsto 1 - \bar{s}$ [THM].

(c) At a critical point the Laguerre expression is $\Xi'^2 - \Xi\Xi'' = -\Xi\Xi''$. So "longitudinal" is the first Laguerre inequality restricted to critical points [THM]. The full Laguerre-type inequalities for all orders are equivalent to RH (Csordas–Varga [7]; record only) [BIB].

Proof. (a) $\partial_\sigma^2 + \partial_t^2 = 0$ for $\log|\xi|$. On the line $\log|\xi| = \log|\Xi(t)|$, whose second t-derivative at t_0 is Ξ''/Ξ. The mixed derivative vanishes by the reflection symmetry $|\xi(s)| = |\xi(1-\bar{s})|$. (b) is the definition. (c) is immediate. □

2.1. **Relation to Lemmas 48, 50, 51, 53, 54.**

(i) *Lemma 48 (ghost count).* On windows with admissible ends, $K^+ = \kappa$ exactly (Thm. 4.3) [THM]. $K^+ = \tau$ holds iff $S^+ = 0$. This is true for DH up to 1000 ($28 = 28$, Table 1) and for all 16 FF cases (Table 2) [NUM], but false in general (§6, §7).

(ii) *Lemma 53 (charge).* The L53 charge $-k$ equals $-\kappa_{\text{FF}}$ by Thm. 6.2 [THM]. It equals $-\tau$ only when $Z_{\text{off}} = 0$.

(iii) *Lemma 54.* Where J_F is real (no real ghost), $n_- = 2\#\{\text{quadruples}\}$. For quadruple-only fakes this equals $k = \kappa_{\text{FF}}$ (*fake g2 quadruple*: $n_- = k = \tau = 2$; Table 2) [NUM]. Real-pair ghosts make J_F non-real, yet τ and κ are still defined. So κ is the more robust invariant, and n_- detects only part of it [THM]. Genus 1, Wick link: write $\mathcal{L}(e^{i\varphi}) = e^{i\varphi}h(\varphi)$, $h = 2\cos\varphi - c$, $c = a/\sqrt{q}$. For a fake, $c = 2\cosh\sigma > 2$, so $h(0) = -4\sinh^2(\sigma/2)$ and $h''/h(0) = 1/(2\sinh^2(\sigma/2)) > 0$: the point $\varphi = 0$ is transverse. This is the continuation $\varphi_a \mapsto i\sigma$ of $h''/h(0) = -1/(2\sin^2(\varphi_a/2))$ for a curve with $c = 2\cos\varphi_a$ [THM] (NUM check to 10^{-15}, `num/l56/ff.out`). This is L54's cos → cosh in orientation form.

(iv) *Lemmas 50/51.* Krein signature (L50) and the collision tracking of L51 are *analogies* for $\mathfrak{o}$ and for §7 [ANALOGY]. No theorem connects them.

3. Jensen localization

Theorem 3.1 (Jensen discs; Jensen 1913 for polynomials, Pólya 1943 Thm. I for genus 1^*)**.** *Let $s \notin \bigcup_\rho D_\rho$ (union over ghosts) with $s \neq$ zero of ξ. Then $\operatorname{Re}\frac{\xi'}{\xi}(s)$ has the sign of $\sigma - \frac{1}{2}$ (and vanishes for $\sigma = \frac{1}{2}$). Consequently:*

(a) every off-line zero of ξ' lies in the interior of a Jensen disc of a ghost;

(b) every transverse or degenerate on-line saddle lies in a Jensen interval I_ρ. This recovers and unifies Lemma 55 Thm. 4.2(c);

(c) under RH, every zero of ξ' lies on the line and every saddle is longitudinal (Lemma 55) [THM].

The same holds verbatim for ξ_{DH} and for H_t (any real even entire function of genus ≤ 1 in t with zeros symmetric under $t \mapsto \bar{t}$) [THM].

Proof. By Hadamard and the functional equation, $\operatorname{Re}\xi'/\xi(s) = \sum_\rho(\sigma-\beta)/|s-\rho|^2$, absolutely convergent. Group each ghost $\rho = \frac{1}{2}+\delta+i\gamma$ with its mirror. With $x = \sigma - \frac{1}{2}$ and $w = s - \frac{1}{2} - i\gamma$, the two terms add to

$$\frac{x-\delta}{|w-\delta|^2} + \frac{x+\delta}{|w+\delta|^2} = \frac{2x\,(|w|^2-\delta^2)}{|w^2-\delta^2|^2}.$$

This has the sign of x when $|w| > \delta$. On-line zeros contribute $x/|s-\rho|^2$. (a) If $\sigma \neq \frac{1}{2}$ and s is outside all discs, then $\operatorname{Re}\xi'/\xi \neq 0$. (b) At an on-line saddle $\xi'/\xi = 0$. In the t-variable, $-(\log\Xi)''(t_0) = \sum_j (t_0 - z_j)^{-2}$, and the couple $\gamma \pm i\delta$ contributes $2((t_0-\gamma)^2-\delta^2)/((t_0-\gamma)^2+\delta^2)^2$, which is ≥ 0 unless $|t_0-\gamma| < \delta$. So $\Xi''/\Xi(t_0) \geq 0$ forces some $|t_0-\gamma| < \delta$. (c) follows. For DH and H_t the same Hadamard argument applies, because the zeros are symmetric and the exponential factor is e^{A+Bs} with $\operatorname{Re} B = -\sum \operatorname{Re} 1/\rho$ (FE). For DH this is Lemma 55's normalization. □

Prior art. For polynomials this is Jensen's theorem [12]. For real entire functions of genus 1^*, "all nonreal zeros and critical zeros of f' are in $\mathcal{J}(f)$" is quoted by Kim [14] as [Pólya, Theorem I] [20] [BIB] (Kim's abstract and introduction, read through the publisher page text; Pólya not read).

Corollary 3.2 (ζ' in the left strip; tilt version)**.** *For $0 \leq \sigma < \frac{1}{2}$ and $t \geq 6.53$, every zero of ζ' lies in a Jensen disc of a ghost [THM][NUM].*

Proof. $\zeta'/\zeta = \xi'/\xi - A$. Outside the discs, with $\sigma < \frac{1}{2}$, we have $\operatorname{Re}\xi'/\xi < 0$. Also $\operatorname{Re} A(\sigma+it) > 0$ for $0 \leq \sigma \leq \frac{1}{2}$, $t \geq 6.53$ ([NUM]: the zero of $\operatorname{Re} A(it)$ is at $t = 6.52638$, that of $\operatorname{Re} A(\frac{1}{2}+it)$ at $6.28984 =$ Lemma 55's θ' threshold; checked on a grid to $t = 1000$). So $\operatorname{Re}\zeta'/\zeta < 0$ and $\zeta' \neq 0$. □

This sharpens the easy half of Speiser/LM74 (Lemma 55 §5): a left-strip zero of ζ' is not only evidence of a ghost but lies within δ of one. DH example [NUM]: ghost $0.8085 + 85.6993i$; f'_{DH} zero $0.4054 + 85.7051i$ at distance $0.0948 < \delta = 0.3085$. We did not find this corollary stated in the sources we read, but it is an immediate combination of known facts. No novelty is claimed.

4. The counting step (J.W. answer 2)

Definition 4.1 (Admissible heights; zero-naive)**.** $T > 0$ is *admissible* if $\Xi(T)\Xi'(T) \neq 0$ and $\operatorname{Re}\frac{\xi'}{\xi}(\sigma+iT) < 0$ for $-\frac{1}{2} \leq \sigma < \frac{1}{2}$ (equivalently, $\operatorname{Im}(\Xi'/\Xi)(T+iy) < 0$ for $0 < y \leq 1$). Every T outside all closed Jensen intervals is admissible (Thm. 3.1). Admissibility itself is checked from values of ξ alone [THM].

Theorem 4.2 (Exact count of ξ' zeros)**.** *If T is admissible, then*

$$N_{\xi'}(T) = N(T) - \frac{1+\varepsilon(T)}{2}, \qquad \varepsilon(T) = \operatorname{sgn}\frac{\Xi'}{\Xi}(T) \in \{\pm 1\}.$$

[THM]

Proof. Let $F = \Xi$ and $F'/F(t) = i\,\xi'/\xi(\frac{1}{2}+it)$. For $t = x+iy$ we have $\operatorname{Im} F'/F = \operatorname{Re} \xi'/\xi(\frac{1}{2}-y+ix)$. By Thm. 3.1 this is < 0 for $y > 0$ and > 0 for $y < 0$ outside the discs. *Step 1 (rectangle lemma).* Let $a < b$ be admissible and $R = [a, b] \times [-1, 1]$. All zeros of F and F' with $a < \operatorname{Re} t < b$ lie in R. Then

$$(*) \qquad \#Z(F', R) - \#Z(F, R) = \frac{\varepsilon(a) - \varepsilon(b)}{2}.$$

Indeed, write $\vartheta \in (-\pi, 0)$ for the argument of $w = F'/F$ on the lower half-plane and $\vartheta \in (0, \pi)$ on the upper. On the top side w stays in the lower half-plane and on the bottom side in the upper half-plane. On the right side, going up, w crosses $\mathbb{R}$ once, at $w(b)$, from the upper to the lower half-plane. Its continuous argument changes by $\vartheta(b + i) - \vartheta(b - i)$, plus 2π if the crossing is through the negative axis ($\varepsilon(b) < 0$). The left side is the same with -2π if $\varepsilon(a) < 0$. The four sides telescope to $2\pi([\varepsilon(b) < 0] - [\varepsilon(a) < 0])$, which is $(*)$. *Step 2.* Apply $(*)$ to $[-T, T]$. F'/F is odd, so $\varepsilon(-T) = -\varepsilon(T)$ and the difference is $-\varepsilon(T)$. By evenness, $\#Z(F) = 2N(T)$ and $\#Z(F') = 2N_{\xi'}(T) + 1$ (the $+1$ is $t = 0$). So $2N_{\xi'} + 1 - 2N = -\varepsilon$. □

Theorem 4.3 (Window identity; Kim 1996, Thm. 1, specialized to Ξ)**.** *Let $0 \le a < b$ be admissible, or $a = 0$ when $t = 0$ is a simple longitudinal critical point. Then*

$$K^+(a, b) = \tau(a, b) + S^+(a, b), \qquad \textit{i.e.} \qquad \#\textit{ghosts} = \kappa.$$

[THM][CITED]

Proof (ours). Assume simple critical points; the general case follows with Kim's multiplicity convention. Let n be the number of real zeros of $F = \Xi$ in (a, b) and C the number of real critical points. The zeros cut $[a, b]$ into $n - 1$ bounded gaps and two end pieces. In a bounded gap $|F|$ rises from 0 and returns to 0, so its maxima exceed its minima by one. On the end piece $(z_n, b]$, $|F|$ rises from 0 and is increasing or decreasing at b according as $\varepsilon(b) = \pm 1$, because $(\log |F|)' = F'/F$. Similarly at a. Counting minima (the transverse points) gives

$$C = 2\tau + (n - 1) + [\varepsilon(a) > 0] + [\varepsilon(b) < 0],$$

which also holds for $n = 0$. The non-real zeros of F and of F' in the strip come in conjugate couples: $2K^+$ and $2S^+$ (a ghost ρ with its mirror is the couple $\gamma \mp i\delta$, and an off-line saddle with $\sigma > \frac{1}{2}$ with its mirror is a couple of zeros of F'). Insert $\#Z(F) = n + 2K^+$ and $\#Z(F') = C + 2S^+$ into $(*)$. Since $[\varepsilon(a) > 0] + [\varepsilon(a) < 0] = 1$, this leaves $2\tau + 2S^+ - 2K^+ = 0$. For $a = 0$ with $t = 0$ a simple longitudinal critical point, apply the argument to $[-b, b]$ and halve. □

Attribution. Kim [14], Theorem 1 (wording from the publisher's text of the paper): "Let $f(z)$ be a nonconstant real entire function of genus 1^* and suppose that a, b are real numbers with $a < b$ and $a, b \notin \mathcal{J}(f)$. Then $f(z)$ and $f'(z)$ have finitely many zeros in the region $a \le \operatorname{Re} z \le b$. Moreover, if f has $2J$ nonreal zeros in $a \le \operatorname{Re} z \le b$ and f' has $2J'$ nonreal zeros there, then f has exactly $J - J'$ critical zeros in $[a, b]$." Ξ is real entire of genus 1, so Theorem 4.3 is a special case [CITED]. Our admissibility (Def. 4.1) is a slightly weaker, zero-naive endpoint condition, and the proof above covers it. Kim's paper was read only through its abstract and the opening of §1 (the full PDF was not accessible). *Lemma 55 Q2/§4 called this counting step "neither proved nor found". It is in the literature* (Correction C3).

Corollary 4.4 (Consequences)**.** *Let (H) be the statement "every zero of ξ' is on the line and every on-line saddle is longitudinal". Let (A_∞) be "there are admissible heights above every bound".*

(a) RH $\Rightarrow$ (H)$\wedge$(A_∞) [THM].

(b) (H)$\wedge$(A_∞) $\Rightarrow$ RH. So RH $\iff$ (H)$\wedge$(A_∞) [THM][RESTATEMENT].

(c) If ξ has only finitely many ghosts, then RH $\iff$ (H) [THM][RESTATEMENT].

(d) (H) alone is zero-naive. v1 said: whether (H)$\Rightarrow$RH is [OPEN]. Corrected in v2: *(H)$\Rightarrow$RH is a theorem (Thm. 4.10), so RH $\iff$ (H) [THM][RESTATEMENT] (update U2).*

Proof. (a) Thm. 3.1(c). Every T is admissible when there are no ghosts. (b) Given (H), $\tau = S^+ = 0$ on $(0,T)$ for each admissible T. Thm. 4.3 with $a = 0$ (the point $t = 0$ is a longitudinal maximum under (H)) gives $K^+(0,T) = 0$, and $T \to \infty$ along admissible heights. (c) Above the last ghost every T outside its finitely many closed intervals is admissible, so (A_∞) holds. Apply (b). □

Proposition 4.5 (The covering scenario; v1: "where the converse breaks", excluded in v2)**.** *Assume (H) and ¬RH. Then every $T \geq \gamma_0$ (some γ_0) lies in a closed Jensen interval: the ghost intervals cover $[\gamma_0, \infty)$. In particular, for all large T,*

$$\sum_{\substack{T-1<\gamma\leq 2T+1 \\ \beta>1/2}} \Big(\beta - \frac{1}{2}\Big) \;\geq\; \frac{T}{2}.$$

[THM]

Proof. If some T' above a ghost γ_1 were admissible, Thm. 4.3 on $(0,T')$ would give $K^+ \geq 1$, while (H) gives $\kappa = 0$. That is a contradiction. So no $T > \gamma_1$ is admissible, and every such T lies in some $\overline{I_\rho}$. Intervals of total length $\sum 2\delta$ cover $[T, 2T]$, and only ghosts with $\delta < \frac{1}{2}$ and $|\gamma - [T,2T]| < \frac{1}{2}$ can meet it. □

v2 note. Under (H) the covering scenario is impossible (Thm. 4.10), and unconditionally it cannot occur at heights in $[305, 10^{116.7}]$ (Thm. 4.13). Prop. 4.5 remains true as stated, since its hypothesis is now known to be contradictory.

Remark 4.6 (What is known, and the gap; v1 sketch, superseded by §§4.1–4.2)**.** *Littlewood's lemma* [26, Ch. 9]: $2\pi \sum_{T_1<\gamma\leq T_2, \beta>1/2}(\beta - \frac{1}{2}) = \int_{T_1}^{T_2} \log|\zeta(\frac{1}{2}+it)|\,dt + O(\log T_2)$ [BIB]. By Jensen's inequality and the mean square $\int_T^{2T} |\zeta(\frac{1}{2}+it)|^2 dt \sim T\log T$ [26, Ch. 7] [BIB], the right side is $\leq \frac{T}{2}\log\log T\,(1+o(1))$. So

$$\sum_{T<\gamma\leq 2T} (\beta - \tfrac{1}{2}) \leq \frac{T}{4\pi}\log\log T\,(1+o(1)).$$

This is $< T/2$ only while $\log\log T < 2\pi$, i.e. $T \lesssim \exp(e^{2\pi}) \approx 10^{232}$. So under (H) the covering, and hence the first ghost, would have to start beyond roughly that height. This is a sketch: the $o(1)$ terms are effective, but we have not made them explicit, so the number is heuristic [OPEN]. In general, Selberg's bound $\sum_{\gamma\leq T}(\beta - \frac{1}{2}) \ll T$ has an implied constant that is not explicit in the sources we read. To exclude covering one would need $\sum_{T<\gamma\leq 2T}(\beta - \frac{1}{2}) \leq (\frac{1}{2} - \eta)T$ along a sequence $T \to \infty$. That is an explicit statement about how close zeros stay to the line on average. It is not known (as far as we found) and is plausibly as hard as known open density problems. **Classification (v1):** the converse "(H)⇒RH" is [OPEN] — *withdrawn in v2*: it is Thm. 4.10. The effective version of this sketch is Thm. 4.13, where the rigorous one-sided mean square gives $10^{116.7}$, not 10^{232} (Correction C9).

Remark 4.7 (Circularity and metaphor check)**.** Corollary 4.4(b) is a restatement, not progress. (A_∞) is a zero-naive repackaging of "ghost Jensen intervals do not cover". Prop. 4.5 shows that the content of the converse sits entirely in (A_∞). Calling transverse saddles "ghost detectors" is literal only through Thm. 3.1(b) and Thm. 4.3. The words "orientation flip" and "rotation of the lemniscate" in §7 describe the Hessian sign, nothing more. (v2: (A_∞) follows from (H) by Thm. 4.8, so RH ⟺ (H) is also a restatement.)

4.1. **v2: the global count, and the converse.** Notation: $F = \Xi$, $R_T = [-T,T] \times [-1,1]$ in the t-plane, and $\#Z(G,R)$ is the number of zeros of G in R with multiplicity.

Theorem 4.8 (Global count; unconditional, Backlund-type)**.** *There are absolute constants C, T_1 such that for all $T \geq T_1$*

$$|N_{\xi'}(T) - N(T)| \leq C\log T.$$

The same holds for ξ_{DH}, with other constants. [THM]

Proof. Location. $\operatorname{Re}\xi'/\xi(s) = \sum_\rho(\sigma-\beta)/|s-\rho|^2$ (Hadamard, Thm. 3.1) has the sign of $\sigma-\frac12$ when $\sigma\le 0$ or $\sigma\ge 1$. So every zero of F and F' with $|\operatorname{Re}t|<T$ lies in $|\operatorname{Im}t|<\frac12$, inside R_T. Take T such that F and F' have no zeros on ∂R_T; the general T follows, since N and $N_{\xi'}$ jump by $O(\log T)$ at a single height. By evenness, and since $t=0$ is a zero of F', $\#Z(F',R_T)-\#Z(F,R_T) = 2(N_{\xi'}(T)-N(T))+1$. By the argument principle this equals $\frac{1}{2\pi}\Delta_{\partial R_T}\arg(F'/F)$. *Horizontal sides.* For $t=x+iy$, $\operatorname{Im}F'/F(t) = \operatorname{Re}\xi'/\xi(\frac12-y+ix)$. On $y=1$ ($\sigma=-\frac12$) this is <0, and on $y=-1$ ($\sigma=\frac32$) it is >0. So on each horizontal side F'/F stays in an open half-plane, and the change of argument is $<\pi$. *Vertical sides.* On $t=T+iw$, $w\in[-1,1]$, put

$$\Psi(w) = \tfrac12\big[F'(T+iw)F(T-iw)+F'(T-iw)F(T+iw)\big].$$

Ψ is entire in w. For real w, $F(T-iw)=\overline{F(T+iw)}$ (F is real), so $\Psi(w)=\operatorname{Re}(F'\bar F)(T+iw)$. Between consecutive zeros of $\operatorname{Re}(F'/F)$ the argument changes by at most π, so $|\Delta\arg|\le\pi(1+q)$, where q is the number of zeros of Ψ in a disc containing $[-1,1]$. Take the centre $w_0=-\frac32$, i.e. $s=2+iT$ on one factor and $s=-1+iT$ on the other. Using $\xi(s)=\xi(1-s)$, $\xi'(s)=-\xi'(1-s)$ and $F'(t)=i\xi'(\frac12+it)$, one finds

$$\Psi(w_0) = -|\xi(2+iT)|^2\,\operatorname{Im}\frac{\xi'}{\xi}(2+iT).$$

Here $\frac{\xi'}{\xi}=\frac1s+\frac{1}{s-1}-\frac12\log\pi+\frac12\psi(\frac s2)+\frac{\zeta'}{\zeta}$, so $\operatorname{Im}\frac{\xi'}{\xi}(2+iT)=\frac12\arg(iT/2)+O(1/T)+\operatorname{Im}\frac{\zeta'}{\zeta}(2+iT)\ge\frac{\pi}{4}-0.5700-O(1/T)>0.2$ for $T\ge T_1$, since $|\zeta'/\zeta(2+it)|\le|\zeta'/\zeta(2)|=0.5700$. So $|\Psi(w_0)|\ge 0.2|\xi(2+iT)|^2$. On $|w-w_0|\le R=4$ both factors are values of ξ,ξ' at heights $T+O(1)$ with $-5\le\sigma\le 6$. By Stirling and the polynomial bound $|\zeta(\sigma+it)|\le|t|^A$ there ((1) below for $\sigma\ge-\frac12$, the functional equation for $\sigma<-\frac12$, Cauchy's estimate for ξ'), $\max|\Psi|\le|\xi(2+iT)|^2\,T^B$ for an absolute B. Jensen's formula, with $r=2.6$ and $R=4$, gives $q\le\log(T^B/0.2)/\log(4/2.6)=O(\log T)$. *DH.* The same argument with centre $\sigma=3$, where $|f_{\mathrm{DH}}-1|\le\sum_{n\ge2}|a_n|n^{-3}=0.0716$ and $|f'_{\mathrm{DH}}/f_{\mathrm{DH}}(3+it)|\le 0.0791/0.9284<0.086<\frac{\pi}{4}$ (sums computed to $n=10^6$; tail $<10^{-11}$). Here R_T uses $|\operatorname{Im}t|\le1$, which contains all zeros since $\beta_{\mathrm{DH}}<1.3952$ (below). □

Corollary 4.9 (The counter counts globally)**.** *Unconditionally, $|\kappa(0,T)-K^+(0,T)|\le C'\log T$ for all large T, for ξ and for ξ_{DH}. At admissible T the difference is* 0 *(Thm. 4.3) [THM]. Numerically [NUM]: $N_{\xi'}(T)-N(T)\in\{-2,-1,0,1\}$ for* all *$T\le1000$ for DH (-2 on* 27 *stretches of length <0.08 between a ghost's height and its transverse saddle just above it; $+1$ once, on $(983.743, 983.752)$, where the saddle lies just below the ghost), and $\in\{-1,0\}$ for ζ (`num/l56/dh_globalcount.json`).*

Proof. The real-line gap count in the proof of Thm. 4.3 holds at every T, not only at admissible ones: $\#\{\text{real critical points}\}=2\tau+(n-1)+O(1)$. So $\#Z(F')-\#Z(F)=4(\kappa-K^+)(0,T)+O(1)$. Apply Thm. 4.8. □

Theorem 4.10 ((H) implies RH)**.** *If every zero of ξ' lies on the critical line and every on-line saddle is longitudinal, then RH holds. Hence RH $\iff$ (H). [THM][RESTATEMENT]*

Proof. Assume (H). Then F' has only real zeros. Each zero of F' with $F\ne0$ is a strict local maximum of $|F|$ (it is longitudinal, so $F''/F<0$). A real function on an interval whose critical points are all strict local maxima has at most one of them. So each of the $n-1$ gaps between consecutive distinct real zeros of F in $(-T,T)$ has exactly one, and each of the two end pieces has at most one. A real zero of multiplicity m is a zero of F' of multiplicity $m-1$. So $\#Z(F',R_T)=n_{\mathbb R}-1+e$ with $e\in\{0,1,2\}$, where $n_{\mathbb R}$ counts real zeros of F with multiplicity. Also $\#Z(F,R_T)=n_{\mathbb R}+4K^+(0,T)$, since each ghost gives four zeros of F. Thm. 4.8 now gives

$$K^+(0,T)\le\tfrac12 C\log T+1.$$

Every ghost has $\delta = \beta - \frac{1}{2} < \frac{1}{2}$. So only ghosts with $T < \gamma \le 2T$ have Jensen intervals meeting $[T+1, 2T-1]$, and these intervals have total length $< K^+(0, 2T) = O(\log T)$. For large T this is $< T - 2$, so $[T+1, 2T-1]$ contains a point outside all closed Jensen intervals, with $\Xi\Xi' \neq 0$. That point is an admissible height T^* (Def. 4.1). By Thm. 4.3 on $(0, T^*)$, where $t = 0$ is a longitudinal maximum under (H), $K^+(0, T^*) = \tau + S^+ = 0$. Let $T \to \infty$. □

Remark 4.11 (What this does and does not do)**.** Thm. 4.10 replaces v1's [OPEN] by a theorem, but the theorem is an *equivalence*: it moves RH to (H), and (H) is a statement about infinitely many zeros of ξ'. **RH is not proved.** The proof uses only the Hadamard product, the functional equation, Stirling and Jensen's formula. These are tools that hold equally for DH, where (H) fails (28 transverse saddles below 1000). Related equivalences are recorded, not used: Lagarias 1999 (RH $\iff$ $\operatorname{Re}\xi'/\xi > 0$ on $\sigma > \frac{1}{2}$) [15], Sondow–Dumitrescu 2010 (monotonicity of $|\xi|$ in σ) [23], Spira 1973 [25] [BIB]. Prior-art search (Crossref and web, 22 Oct 2026; not exhaustive): we found no explicit statement of Thm. 4.10. It is a short combination of Kim's window theorem with a Backlund count, and we expect it to be known; no novelty is claimed.

4.2. **v2: the effective barrier (Rem. 4.6 made explicit).**

Lemma 4.12 (Inputs)**.** *(a) (Littlewood's lemma, as stated in* [24, §4.2]*.) For f holomorphic near $[\frac{1}{2}, a] \times [T, 2T]$, $a > 1$, with no zeros on the boundary, $2\pi \int_{1/2}^{a} n_f(\tau)\, d\tau = \int_T^{2T} \big(\log|f(\frac{1}{2}+it)| - \log|f(a+it)|\big)dt + \int_{1/2}^{a} \big(\arg f(\tau+2iT) - \arg f(\tau+iT)\big)d\tau$. For $f = \zeta$ and $a = 2$, the left side is $2\pi \sum_{T<\gamma<2T} (\beta - \frac{1}{2})^+$ [CITED].*

(b) (Simonič 2020, Cor. 5, read at the source [24]*.) $\int_0^T |\zeta(\frac{1}{2}+it)|^2 dt = T\log T - (1 + \log 2\pi - 2\gamma_E)T + \mathcal{E}(T)$, with $\mathcal{E}(T) \le 13.803\, T^{3/4}\sqrt{\log\frac{T}{2\pi}} + 83.964\sqrt{T}\log\frac{T}{2\pi} + 2\cdot 10^3 \log T + 3691.24$ for $T \ge 2\pi$ [CITED].*

(c) (Euler–Maclaurin; verified numerically.) For $\sigma > -1$, $s \neq 1$:

$$\zeta(s) = \tfrac{1}{s-1} + \tfrac{1}{2} + \tfrac{s}{12} - \tfrac{s(s+1)}{2}\int_1^\infty \tilde{B}_2(x) x^{-s-2} dx, \qquad |\tilde{B}_2| \le \tfrac{1}{6}. \tag{1}$$

So $|\zeta(s)| \le 1 + \frac{1}{2} + \frac{|s|}{12} + \frac{|s||s+1|}{6}$ for $\sigma \ge -\frac{1}{2}$, $|t| \ge 2$ [THM].

(d) (Backlund.) If $\operatorname{Re}\zeta(x+iT)$ has q zeros on $[\tau, 2]$, then $|\arg\zeta(\tau+iT)| \le \pi(q+1) + 0.71$, because $|\zeta(2+iT) - 1| \le \zeta(2) - 1$ gives $|\arg\zeta(2+iT)| \le \arcsin 0.645 < 0.71$. Jensen for $g(z) = \frac{1}{2}(\zeta(z+iT) + \zeta(z-iT))$ on $|z-2| \le \frac{3}{2}$ inside $|z-2| \le \frac{5}{2}$, with $|g(2)| \ge 1 - 0.645$, gives $q \le \log(M/0.355)/\log\frac{5}{3}$, where M is bounded by (c) [THM].

Theorem 4.13 (Effective barrier)**.** *For every $T \in [305,\ 10^{116.7}]$,*

$$\sum_{T<\gamma\le 2T} \Big(\beta - \frac{1}{2}\Big)^+ \ \le\ \frac{1}{2\pi}\Big[\frac{T}{2}\log\frac{I(2T)}{T} + 1.074 + 3B(2T)\Big] \ <\ \frac{T}{2} - 1,$$

where $I(X) = X\log X - 1.6834X + \mathcal{E}(X)$ and $B(X) = \pi(1 + \log(M_X/0.355)/\log\frac{5}{3}) + 0.71$, with $M_X = 1.5 + \frac{X+5}{12} + \frac{(X+5)(X+6)}{6}$. Consequently $[T+1, 2T-1]$ contains an admissible height for every such T, unconditionally. So under (H)$\wedge\neg$RH the covering of Prop. 4.5 could not start below $10^{116.7}$ [THM][NUM].

Proof. Apply Lemma 4.12(a) with $f = \zeta$, $a = 2$. The first term: by Jensen's inequality for the concave log, $\int_T^{2T} \log|\zeta(\frac{1}{2}+it)|dt \le \frac{T}{2}\log(\frac{1}{T}\int_T^{2T}|\zeta|^2) \le \frac{T}{2}\log\frac{I(2T)}{T}$ by (b). The second term: $\log\zeta(2+it) = \sum_{p,k} p^{-k(2+it)}/k$, and $|\int_T^{2T} p^{-ikt}dt| \le 2/(k\log p)$, so it is at most $\sum_{p,k} 2/(k^2 p^{2k}\log p) = 1.0731$ (computed). The arg terms are at most $\frac{3}{2}(B(T) + B(2T)) \le 3B(2T)$, by (c) and (d) with $|s| \le X+5$ on the Jensen disc. The ghost intervals meeting $[T+1, 2T-1]$ belong to ghosts with $T < \gamma \le 2T$ ($\delta < \frac{1}{2}$), and covering would need $\sum 2\delta \ge T - 2$. The final inequality

was checked in multiprecision at the endpoints and on a grid of 11421 points with $\log_{10}$-step 0.01, with 0 failures (`num/lemma56v2_barrier.py`, `num/l56/barrier.json`). The ratio of the bound to T is 0.320 at $T = 3 \cdot 10^{12}$ and 0.432 at 10^{50}. The upper end is where $\log(2 \log 2T - 3.37)$ reaches 2π, i.e. about $10^{116.7}$. □

Proposition 4.14 (Asymptotic $c < \frac{1}{2}$; Conrey)**.** [3, Thm. 3]*:* $\int_{1/2}^{1} N(\sigma, T)\, d\sigma \le (0.0806 + o(1))T$ *[CITED]. Since* $\int_{1/2}^{1} N(\sigma,T)d\sigma = \sum_{0<\gamma\le T}(\beta - \frac{1}{2})^{+}$*, we get* $\sum_{T<\gamma\le 2T}(\beta - \frac{1}{2}) \le (0.1612 + o(1))T$*. So for all* $T \ge T_2$ *(with* T_2 not *explicit)* $[T + 1, 2T - 1]$ *contains an admissible height, and Prop. 4.5 gives a second proof of Thm. 4.10 [THM] from [CITED].*

Remark 4.15 (Explicit $c < \frac{1}{2}$ for all $T \ge T_0$: not achievable from verified sources)**.** Conrey's $o(1)$ (via Balasubramanian–Conrey–Heath-Brown [2]) is not explicit in the sources we read. Simonič's explicit Selberg-type bound [24, Thm. 1], $N(\sigma, 2T) - N(\sigma, T) \le aT^{1-\frac{1}{4}(\sigma-\frac{1}{2})} \log T + \dots$ with $a = 10395.2$ for $\sigma \in [\frac{1}{2}, 0.831]$, $T \ge H_0$, integrates to $\sum(\beta - \frac{1}{2}) \lesssim 4aT \approx 4 \cdot 10^4 T$. That is far from $T/2$. Kadiri–Lumley–Ng [10] is explicit only for $\sigma \ge \frac{3}{5}$ (statement as quoted in [24]) [BIB]. **So with verified explicit constants,** $c < \frac{1}{2}$ **is not available for all large** T**.** The effective statement is Thm. 4.13, and the converse is settled by Thm. 4.10 without it.

4.3. **Numerics for** ζ **and DH.** Method: a 1D scan of $\Xi'/\Xi(t)$ on $(0, 1000]$, step 0.002 (`num/lemma56_scan.py`, python-flint 96-bit), with bisection and the type from $\Xi''/\Xi = -\operatorname{Re} G'(\frac{1}{2} + it)$, $G = \Lambda'/\Lambda$. Off-line saddles were counted only inside Jensen discs (Thm. 3.1), via the winding of G on a circle of radius $\delta + 0.004$ about each ghost centre (`num/lemma56_dh.py`). Theorem 4.2 was checked at every admissible $T \in \{1, 1.5, \dots, 999.5\}$.

TABLE 1. DH, $0 < t \le 1000$ [NUM] (`num/l56/count_dh.json`). Window ends are admissible. 848 on-line zeros (L48: 848), 903 on-line saddles, of which 28 are transverse. Theorem 4.2: 0 violations at 1966 admissible grid heights. Each of the 28 Jensen discs contains exactly one zero of Λ': winding of $G = -1.000$, 2 zeros of Λ, 1 on-line transverse saddle, 0 off-line. Central point $t = 0$: $\Xi''/\Xi(0) = -0.1961$ (longitudinal).

window	on-line zeros	on-line saddles	K^+	τ	S^+	$\tau = K^+ - S^+$
$(0, 100]$	52	53	1	1	0	yes
$(100, 200]$	70	76	3	3	0	yes
$(200, 300]$	82	84	1	1	0	yes
$(300, 400]$	83	90	3	3	0	yes
$(400, 500]$	90	94	2	2	0	yes
$(500, 600]$	87	96	5	5	0	yes
$(600, 700]$	91	100	4	4	0	yes
$(700, 800]$	98	101	2	2	0	yes
$(800, 900]$	93	104	5	5	0	yes
$(900, 999.9]$	102	105	2	2	0	yes
total	848	903	28	28	0	

ζ [NUM] (`num/l56/count_zeta.json`). 649 on-line zeros in $(0, 1000]$ (L48: 649) and 648 on-line saddles, all longitudinal. Theorem 4.2: 0 violations at 1992 admissible grid heights (all heights are admissible here, since there are no ghosts in this range). Central point: $\Xi''/\Xi(0) = -0.0462$. With RH verified far beyond this height [19], this is only a consistency check of Theorem 4.2 and of the scan (a missed pair of saddles would show up as a violation).

DH (Table 1). Every ghost below 1000 has its own transverse saddle and no off-line saddle. So in this range $K^+ = \tau$, and DH sits in the regime where J.W.'s "transverse count" is exact. The heat flow (§7) shows that this regime is not stable.

5. v3: the saddle form of RH (official; J.W. answers 1 and 2)

Definition 5.1 (Saddle form of RH; official TAA restatement)**. (H)**: every zero of ξ' lies on the critical line, and every on-line saddle is longitudinal; that is, $\Xi''/\Xi < 0$ at every real zero of Ξ' where $\Xi \neq 0$. Equivalently, in the official counter notation: $S^+ = 0$ and $\tau = 0$ on every window, i.e. $\kappa \equiv 0$. By Thm. 4.10, RH $\iff$ (H) [THM][RESTATEMENT]. TAA records (H) in the RESTATEMENT ledger as the *saddle form* of RH. It sits beside Lagarias's form $\operatorname{Re}\xi'/\xi > 0$ on $\sigma > \frac{1}{2}$ [15] and the Sondow–Dumitrescu monotonicity form [23]. **This is an equivalent form of RH. It is not progress on RH, and RH is not proved.**

5.1. **What (H) means.** *The landscape.* Picture the surface $s \mapsto |\xi(s)|$ over the complex plane, or equivalently $t \mapsto |\Xi(t)|$ with $s = \frac{1}{2} + it$. Away from zeros $\log|\xi|$ is harmonic, so the surface has no peaks and no pits other than the zeros, which are holes going down to $-\infty$. Its only other special points are *saddles*, the critical points where $\xi' = 0$ and $\xi \neq 0$. Near a saddle the surface looks like a mountain pass. The level curve through the pass is a figure-eight (a lemniscate), and its two lobes enclose the lower ground on either side of the pass. Both sides contain zeros (Jensen, Thm. 3.1).

The line. On the critical line $\Xi(t)$ is real. Between two consecutive real zeros Ξ rises from 0 and returns to 0, so it has a turning point there, which is a saddle of $|\xi|$ sitting on the line. A saddle on the line can be oriented in two ways (Def. 2.1):

- *Longitudinal* ($\Xi''/\Xi < 0$). Along the line $|\Xi|$ has a *maximum*, the top of the hump between two zeros, while across the line the surface rises. The pass is crossed walking along the line. The two lobes of the lemniscate enclose the two neighbouring zeros *on* the line.
- *Transverse* ($\Xi''/\Xi > 0$). Along the line $|\Xi|$ has a *minimum* that does not reach 0, a dip without a zero. The pass is crossed walking *across* the line. The two lobes now enclose low ground *above and below* the line. By the symmetry $\xi(s) = \xi(1-s) = \overline{\xi(\bar{s})}$, that low ground is a pair of zeros placed symmetrically off the line: a ghost.

A saddle *off* the line (S^+) is a pass whose lobes again point at off-line zeros.

The statement. (H) says that the whole landscape is a single chain along the critical line: pit, hump, pit, hump, with exactly one longitudinal pass between consecutive zeros and no pass anywhere else. RH says that all the pits are on the line. The theorem RH $\iff$ (H) says that these two descriptions of the same landscape agree.

- RH$\Rightarrow$(H) is classical: zeros of ξ' interlace the zeros of ξ on the line, and each is a maximum of $|\Xi|$.
- (H)$\Rightarrow$RH is the counting argument. Kim's window identity $K^+ = \tau + S^+$ (Thm. 4.3) shows that each ghost costs one transverse saddle or one off-line saddle between admissible heights. The global count $|N_{\xi'}(T) - N(T)| \leq C \log T$ (Thm. 4.8) shows that, under (H), the ghosts cannot hide by making every height inadmissible.

DH is the cautionary example. Its landscape has 28 transverse passes below height 1000, one beside each ghost (Table 1), so (H) fails for DH exactly as RH does.

Why record it. (a) It is *zero-naive*: it is phrased through ξ' and the sign of Ξ''/Ξ, not through zero positions. (b) It is *local and checkable*: on any window with admissible ends it is a finite count (Thm. 4.2). (c) It *transfers*: over $\mathbb{F}_q$ it becomes the circle identity $k = \tau + \frac{1}{2}Z_{\text{off}}$ with Cohn's theorem (Thms. 6.1, 6.2). It also applies to DH, and to the heat flow, where H_t satisfies (H) for every $t \geq \Lambda$. *What it does not do.* It is no easier to verify than RH, since it concerns infinitely many zeros of ξ'. It gives no new route to RH.

5.2. **What pushing the explicit range would buy (J.W. answer 2; not done).**

Remark 5.2 (Benefit and cost)**.** *What it is.* Thm. 4.13 gives, unconditionally, an admissible height in $[T+1, 2T-1]$ for every $T \in [305, 10^{116.7}]$. Pushing the range means proving the same for larger T. Two ways are possible: an explicit *two-sided* mean square of ζ on $[T, 2T]$, which would move the end towards $\exp(e^{2\pi}) \approx 10^{232}$ (Correction C9); or an explicit version of Conrey's $o(1)$, i.e. an explicit $c < \frac{1}{2}$ for all $T \geq T_0$, which would cover *all* large T (Prop. 4.14, Rem. 4.15).

Benefit.

- A fully explicit, unconditional statement that the ghosts' Jensen intervals cannot cover $[T, 2T]$ in the new range, i.e. "admissible heights exist everywhere up to X".
- With an explicit $c < \frac{1}{2}$, a second proof of (H)⇒RH that is effective and does not use the Backlund-type count of Thm. 4.8.
- An explicit Selberg-type constant would also be of independent interest.

Cost.

- It changes no logical status: RH $\iff$ (H) is already a theorem (Thm. 4.10).
- It says nothing about whether (H) holds.
- Below $3 \cdot 10^{12}$ there are no ghosts at all [19], and $10^{116.7}$ is already far beyond any height relevant for numerics.
- The work is long, explicit analytic number theory: an explicit mollified mean value in the style of Balasubramanian–Conrey–Heath-Brown [2], or explicit two-sided mean squares, with error-prone constants. The only verified explicit density we have (Simonič) is five orders of magnitude too weak (Rem. 4.15).

Verdict. A presentational gain at a high cost. Following J.W. ([t218u]), it is not pursued [OPEN].

6. Finite fields: Cohn 1922 read at the source (J.W. answer 4)

6.1. **What Cohn proves.** Bibliography ([CITED], scan from GDZ, PPN314393552, 44 pp.): A. Cohn (Berlin), *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise*, Math. Z. 14 (1922) 110–148, received 15 August 1921. Cohn writes $f(x) = a_0 x^n + \cdots + a_n$ and $f^*(x) = x^n \bar{f}(1/x)$ (p. 112, Fälle I–III). *Fall IV* (p. 113) is $a_\nu = \varepsilon \bar{a}_{n-\nu}$, $|\varepsilon| = 1$; footnote: "Bei reellen Koeffizienten: Reziproke Gleichung". The statement read on p. 113 is:

> In Fall IV the equation $f_1(x) = x^{n-1} f'(1/x) = a_{n-1}x^{n-1} + 2a_{n-2}x^{n-2} + \cdots + na_0 = 0$ has *"ebenso viele Wurzeln im Innern des Einheitskreises wie die Gleichung (1)".*

p. 118 restates this: "Im Fall IV haben $f(x) = 0$ und $f'^*(x) = 0$ im Innern des Einheitskreises die gleiche Anzahl Wurzeln" (proof via a Rouché-type Hilfssatz, p. 114). §3 (p. 113): the number of roots on the circle is $t_1 = s - 2l$ [CITED]. (Formulas were read from the page images `lit/lemma56/cohn_pg/p-07,08,13.png`. The OCR text is only an aid.)

Theorem 6.1 (Exact FF count; Cohn Fall IV)**.** *Let $P(T)$ be a real L-polynomial of degree $2g$ with $q^g T^{2g} P(1/(qT)) = P(T)$, and let $k = \#\{P = 0,\ |T| < q^{-1/2}\}$ be the number of ghosts (Lemma 48). Then*

$$\#\{P'(T) = 0,\ |T| > q^{-1/2}\} = k.$$

The zeros of P' on $|T| = q^{-1/2}$ are exactly the multiple zeros of P there, each of multiplicity one less. [THM][CITED]

Proof. With $u = \sqrt{q}\,T$, $\mathcal{L}(u) = P(u/\sqrt{q})$ is real self-inversive (Cohn's Fall IV with $\varepsilon = 1$). Cohn's statement says that $x^{2g-1}\mathcal{L}'(1/x)$ has as many roots in $|x| < 1$ as $\mathcal{L}$, namely k. So $\mathcal{L}'$ has k roots in $|u| > 1$ (the leading coefficient of $\mathcal{L}'$ is $2g \neq 0$, so there are no roots at ∞) [CITED]. Circle part (ours): on $|u| = 1$ write $\mathcal{L}(e^{i\varphi}) = e^{ig\varphi}h(\varphi)$ with h real. Then $u\mathcal{L}'/\mathcal{L} = g - ih'/h$, so $\mathrm{Re}(u\mathcal{L}'/\mathcal{L}) = g \neq 0$ wherever $\mathcal{L} \neq 0$ (the FF tilt identity, Lemma 55 §9). Hence $\mathcal{L}'$ vanishes on the circle only at multiple zeros of $\mathcal{L}$, with multiplicity one less [THM]. □

This settles the Lemma 55 observation ($\#\{L' = 0, |u| > 1\} = \#$ghosts in all 16 cases): it is Cohn's theorem, not a coincidence (Correction C4).

6.2. **Transverse saddles on the circle.** Write $\mathcal{L}(e^{i\varphi}) = e^{ig\varphi}h(\varphi)$ with h a real trigonometric polynomial of degree g. The *symmetric saddles* (the ξ'-analogue, Lemma 55 §9) are the zeros of $S(u) = u\mathcal{L}'(u) - g\mathcal{L}(u)$; on the circle they are the critical points of h. S is anti-self-inversive: $u^{2g}S(1/u) = -S(u)$. Let τ be the number of transverse critical points of h on the circle ($h''/h > 0$) and Z_{off} the number of zeros of S off the circle.

Theorem 6.2 (Circle identity)**.** $k = \tau + \frac{1}{2}Z_{\text{off}}$*, i.e.* κ_{FF} = *number of ghosts. In particular* $\tau = k$ *iff all symmetric saddles lie on the circle [THM].*

Proof. h has $2g$ complex zeros per period strip: $2g-2k$ real ones ($|u|=1$) and $2k$ others. h' has $2g$ as well: C real and Z_{off} non-real. If h has real zeros, it has $2g-2k$ cyclic gaps. In each gap the maxima of $|h|$ exceed the minima by one, so $C = (2g-2k) + 2\tau$. Then $2g = C + Z_{\text{off}}$ gives $2k = 2\tau + Z_{\text{off}}$. If h has no real zero ($k = g$), maxima and minima are equal in number, so $C = 2\tau$ and $2g = 2\tau + Z_{\text{off}}$. □

This is the compact (trigonometric) analogue of Theorem 4.3: no admissibility is needed.

Proposition 6.3 (Genus 1 and 2)**.**
(a) $g = 1$*:* $h = 2\cos\varphi - c$ *and* $h' = -2\sin\varphi$ *have real critical points* $0, \pi$ *only. So* $Z_{\text{off}} = 0$ *and* $\tau = k$ *always.* $\varphi = 0$ *is transverse iff* $c > 2$*, and* $\varphi = \pi$ *iff* $c < -2$ *[THM].*
(b) $g = 2$*:* $h = 2\cos 2\varphi + 2c_1\cos\varphi + c_0$ *with* $c_1 = a_1/\sqrt{q}$*, and* $h' = -2\sin\varphi\,(4\cos\varphi + c_1)$*. So* $Z_{\text{off}} = 0$ *iff* $|c_1| \le 4$*, i.e.* $|a_1| \le 4\sqrt{q}$*. In that case* $\tau = k$*. If* $|a_1| > 4\sqrt{q}$*, then* $Z_{\text{off}} = 2$ *and* $\tau = k - 1$ *[THM].*

Example 6.4 (Transverse count ≠ ghosts)**.** (i) $\mathcal{L} = 1 - 5u + 9u^2 - 5u^3 + u^4$ ($c_1 = -5$, $c_0 = 9$): $k = 2$, $\tau = 1$, $Z_{\text{off}} = 2$. (ii) $\mathcal{L} = 1 + 10u^2 + 10u^4 + u^6$ ($h = 2(\cos 3\varphi + 10\cos\varphi)$, genus 3, one ghost quadruple): $k = 2$, $\tau = 0$, $Z_{\text{off}} = 4$. The ghosts are seen *only* by off-circle saddles. (iii) $1 - 4.5u + 8u^2 - 4.5u^3 + u^4$: $k = 2$, $\tau = 1$, $Z_{\text{off}} = 2$ [NUM][THM].

7. The de Bruijn–Newman flow (J.W. answer 5)

7.1. **Set-up (cited).** $H_t(z) = \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du$, with $\Phi(u) = \sum_{n\ge1}(2\pi^2n^4e^{9u} - 3\pi n^2e^{5u})e^{-\pi n^2e^{4u}}$, so $H_0(z) = \frac{1}{8}\xi(\frac{1}{2} + \frac{iz}{2}) = \frac{1}{8}\Xi(z/2)$. Each H_t is even and real entire, and solves the backwards heat equation $\partial_t H_t = -\partial_z^2 H_t$. H_t has only real zeros iff $t \ge \Lambda$ (Newman), and RH $\iff \Lambda \le 0$ [22, §1] [CITED]. $\Lambda \ge 0$ (Rodgers–Tao [22], Thm. 1) [CITED]. $\Lambda \le 0.22$ (Polymath [21]) [CITED]. $\Lambda \le 0.2$ (Platt–Trudgian [19], Cor. 2, using the Polymath method and RH to $3 \cdot 10^{12}$) [CITED]. Zeros of H_t move by $\partial_t z_k = 2\sum_{j\ne k}(z_k - z_j)^{-1}$ (formally; made rigorous in [21, Prop. 3.1]) [CITED]. At a double zero, the pair splits like the Hermite polynomial H_2, as $z_0 \pm \sqrt{2}\,(t-t_0)^{1/2}$ [21, Prop. 3.1(ii)] [CITED].

7.2. **How an orientation can change.**

Theorem 7.1 (Orientation events)**.** *Let* $x_s(t)$ *be a real critical point of* H_t *that depends continuously on* t*. Then* $\mathfrak{o}(x_s(t))$ *can change only at*
(i) a zero collision*:* $H_t(x_s) = H_t'(x_s) = 0$ *(a double real zero), or*
(ii) a saddle collision*:* $H_t'(x_s) = H_t''(x_s) = 0 \ne H_t(x_s)$ *(a double critical point).*
Normal forms (generic case). *(i)* $H_t(x) \approx A[(x-x_c)^2 - 2(t-t_c)]$*. There are real zeros* $x_c \pm \sqrt{2(t-t_c)}$ *for* $t > t_c$ *and a couple* $x_c \pm i\sqrt{2(t_c-t)}$ *(a ghost with* $\delta = \sqrt{2(t_c-t)}$ *in* z*-units) for* $t < t_c$*. The saddle* x_s *persists smoothly through* t_c*, and*

$$\frac{H_t''}{H_t}(x_s) = \frac{1}{t_c - t} + O(1):$$

a universal pole of residue -1 *in* t*. The orientation flips from longitudinal* ($t > t_c$) *to transverse* ($t < t_c$)*. At* $t = t_c$ *the lemniscate collapses to the double zero, and it reopens rotated by* 90° *(the*

TABLE 2. FF table [NUM] (`num/lemma56_ff.py` → `num/l56/ff.json`). k = ghosts (L48); Cohn = $\#\{P' = 0,\ |T| > q^{-1/2}\}$; τ, Z_{off} as in Thm. 6.2; L53 charge = $-k$; L54: J_F real? and n_- (only when J_F is real); ℓ = L54 drift rate. The last three rows are the extra fakes of Ex. 6.4 ($q = 1$ normalization). Random test: 3000 random real self-inversive polynomials of genus 1–5. Thm. 6.2 held in 3000/3000, Cohn in 3000/3000, and the genus-2 criterion in 601/601; $\tau = k$ held in only 1490/3000.

case	q	g	k	Cohn	τ	Z_{off}	κ_{FF}	J_F real (n_-)	ℓ
11a p=5	5	1	0	0	0	0	0	yes (0)	0.0000
11a p=7	7	1	0	0	0	0	0	yes (0)	0.0000
11a p=13	13	1	0	0	0	0	0	yes (0)	0.0000
11a p=17	17	1	0	0	0	0	0	yes (0)	0.0000
11a p=19	19	1	0	0	0	0	0	yes (0)	0.0000
11a p=23	23	1	0	0	0	0	0	yes (0)	0.0000
g2 x^5-x+1 p=11	11	2	0	0	0	0	0	yes (0)	0.0000
g2 x^5-x+1 p=13	13	2	0	0	0	0	0	yes (0)	0.0000
g2 x^5+x^2+3 p=11	11	2	0	0	0	0	0	yes (0)	0.0000
g2 x^5+x^2+3 p=13	13	2	0	0	0	0	0	yes (0)	0.0000
fake a=8 q=13	13	1	1	1	1	0	1	no	0.9272
fake a=7 q=11	11	1	1	1	1	0	1	no	0.6620
fake a=13 q=13	13	1	1	1	1	0	1	no	2.3895
fake g2 real pair q=11	11	2	2	2	2	0	2	no	0.7304
fake g2 quadruple q=11	11	2	2	2	2	0	2	yes (2)	0.9600
fake g2 mixed q=11	11	2	1	1	1	0	1	no	1.0915
g2 c1=-5,c0=9	1	2	2	2	1	2	2	yes (2)	2.4397
g3 cos3+10cos	1	3	2	2	0	4	2	yes (2)	3.0896
g2 c1=-4.5,c0=8	1	2	2	2	1	2	2	yes (2)	2.2700

figure eight's lobes go from along the line to across it: $a \mapsto i\delta$, the "Wick rotation" of Lemma 54 in pictures). (ii) $H_t - H_t(x_*) \approx c(x - x_*)^3$ at the event: a monkey saddle *of $|H_t|$. A transverse and a longitudinal saddle annihilate on the line and leave as a conjugate couple of off-line saddles.* Invariance. *$\kappa = \tau + S^+$ is unchanged by (i) and by (ii). Running t downwards, τ goes up by one at (i), together with K^+, but it goes down by one at (ii) while K^+ is unchanged. So τ alone is not a flow invariant of the ghost number [THM].*

Proof. $\mathfrak{o} = -\operatorname{sgn}(H''/H)$ is continuous along $x_s(t)$ unless H or H'' vanishes there; these are (i) and (ii). (i) $\partial_t H = -H''$ forces the normal form $A[(x - x_c)^2 - 2(t - t_c)]$ to second order. At $x_s = x_c$, $H''/H = 2A/(-2A(t - t_c))$. (ii) Expand H' to second order. κ is invariant because it equals K^+ on windows with admissible ends (Thm. 4.3), and K^+ jumps exactly by the ghosts born in (i). In (ii) one transverse point becomes one off-line saddle with $\sigma > \frac{1}{2}$ (the couple counts once in S^+). □

Corollary 7.2 (Rodgers–Tao forces transverse or off-line saddles)**.** *For every $t < 0$, H_t has a non-real zero ($\Lambda \geq 0$ [22]). Take any window $[a, b]$ with admissible ends that contains it, and assume the zeros of H_t lie in a horizontal strip (Kim's standing hypothesis; true for $t \geq 0$ by de Bruijn, while for $t < 0$ we did not find a reference). Then $\tau(a, b) + S^+(a, b) \geq 1$: H_t has a transverse on-line saddle or an off-line saddle [CITED][THM].*

7.3. **Numerics for H_t.** *Method* [NUM] (`num/lemma56_heatcore.py`, `lemma56_heat.py`, `lemma56_heatfig.py`). Write $H_t = \frac{1}{2}\int_{\mathbb{R}} e^{tu^2}\Phi(u)e^{izu}du$ and shift the contour to $\operatorname{Im} u = \frac{\pi}{8} - 0.07$; Φ is even and analytic in $|\operatorname{Im} u| < \pi/8$. This removes the $e^{-\pi|x|/8}$ cancellation. The trapezoid rule ($h = 0.003$, $|v| \le 1.6$) then gives float64 accuracy: relative error $\le 6 \cdot 10^{-12}$ against $\frac{1}{8}\xi(\frac{1}{2} + \frac{iz}{2})$ (mpmath, 30 digits) for $z \in \{0, 10, 28.27, 50, 100, 140, 30 + 0.8i, 80 + 1.5i\}$. Grid: $z \in (0, 128)$ step 0.02; t from 0.3 down to -16 in steps of 0.05 (327 steps). Real zeros and critical points were found by bisection, and orientation from H''/H. Off-line zeros and saddles were tracked by Newton continuation from birth points, and checked against argument-principle counts of H_t and H_t' on $[-X_t, X_t] \times [-Y_t, Y_t]$ with adaptive phase steps (≤ 0.4 rad; maximum observed 0.21). Here X_t is midway between a longitudinal critical point and the next zero ($\approx$ 111–122).

Results [NUM].

- Identity $\tau = J - J'$ on $[-X_t, X_t]$ (full window, J, J' = couples): **327/327** time steps. Tracked off-line counts agree with the windings: 327/327.
- Events in $0 < z < 124$ (Table 3): 5 zero collisions and 4 saddle collisions. Each ghost created at a zero collision carries a transverse saddle for a t-interval of length 1.5–3. It then loses it at a saddle collision, where the transverse saddle and a neighbouring longitudinal saddle leave as an off-line pair. After that, the ghost is seen *only* by S^+.
- Residue: $(t_c - t)H''/H(x_s) = 1 \pm 2.6 \cdot 10^{-4}$ at $|t - t_c| = 10^{-3}$, at all five zero collisions, with deviation linear in $t - t_c$ (Fig. 3).
- Snapshot at $t = 0$: $\tau = 0$ and $S^+ = 0$ (consistent with RH in this range). At $t = -5$: $J = 6$, $J' = 2$, $\tau = 4$. At $t = -10$: $J = 6$, $J' = 6$, $\tau = 0$. At $t = -15.8$: $J = 8$, $J' = 8$, $\tau = 0$. So late in the backward flow, *all* ghosts are invisible to the transverse count.

TABLE 3. Orientation events of H_t in $0 < z < 124$, $-16 \le t \le 0.3$ [NUM] (`num/l56/heat_events.json`). The z-coordinate is 2× the Ξ-height. "zero" = zero collision (birth of a ghost and of a transverse saddle); "saddle" = saddle collision (transverse saddle → off-line pair). t_c is bisected to 10^{-10} on the t-grid bracket.

event	z	t_c	partner
zero	120.98	−1.28527	saddle event beyond $z = 124$
zero	99.33	−1.87791	saddle at $z \approx 102.19$, $t = -3.40533$
saddle	102.19	−3.40533	
zero	66.20	−3.95479	saddle at $z \approx 69.93$, $t = -6.33989$
zero	86.81	−4.03382	saddle at $z \approx 90.58$, $t = -6.80581$
saddle	69.93	−6.33989	
saddle	90.58	−6.80581	
zero	54.64	−12.49451	saddle at $z \approx 59.04$, $t = -15.45058$
saddle	59.04	−15.45058	

The first collision in our window ($z \approx 99.3$, $t_c = -1.878$) is the pair of Ξ-zeros 48.005, 49.774. The collision at $z \approx 121$, $t_c = -1.285$, involves the Ξ-zeros 59.347, 60.832 ($z = 118.69, 121.66$ at $t = 0$). We make no claim about collisions above $z = 124$. The lower bounds on Λ listed in [22, Table 1] (down to $-1.15 \cdot 10^{-11}$) come from Lehmer pairs [6] at far greater heights, which collide much closer to $t = 0$ [CITED].

7.4. **Polynomial analogue (Csordas–Smith–Varga setting).** For a real polynomial p, set $p_t = e^{-tD^2}p = \sum_k (-t)^k p^{(2k)}/k!$, so $\partial_t p_t = -p_t''$. Since $e^{-tD^2}x^n = (2t)^{n/2}\mathrm{He}_n(x/\sqrt{2t})$ for $t > 0$, p_t is real-rooted for all large t (Hurwitz). e^{-tD^2} ($t \ge 0$) preserves real-rootedness (e^{-tx^2} is in the Laguerre–Pólya class; classical, cf. [6, 4]) [THM][BIB]. Zeros move by $\dot{x}_k = 2\sum_{j \ne k}(x_k - x_j)^{-1}$ [THM]. Numerics [NUM] (`num/lemma56_poly.py` → `num/l56/poly.json`), for two degree-10 toys (zeros $\pm 1, \pm 2 \pm 0.3i, \pm 3.6, \pm 4.5$, and the same with $2 \pm 0.3i \to 2, 2.4$), $t \in [-1, 0.6]$, step 0.002:

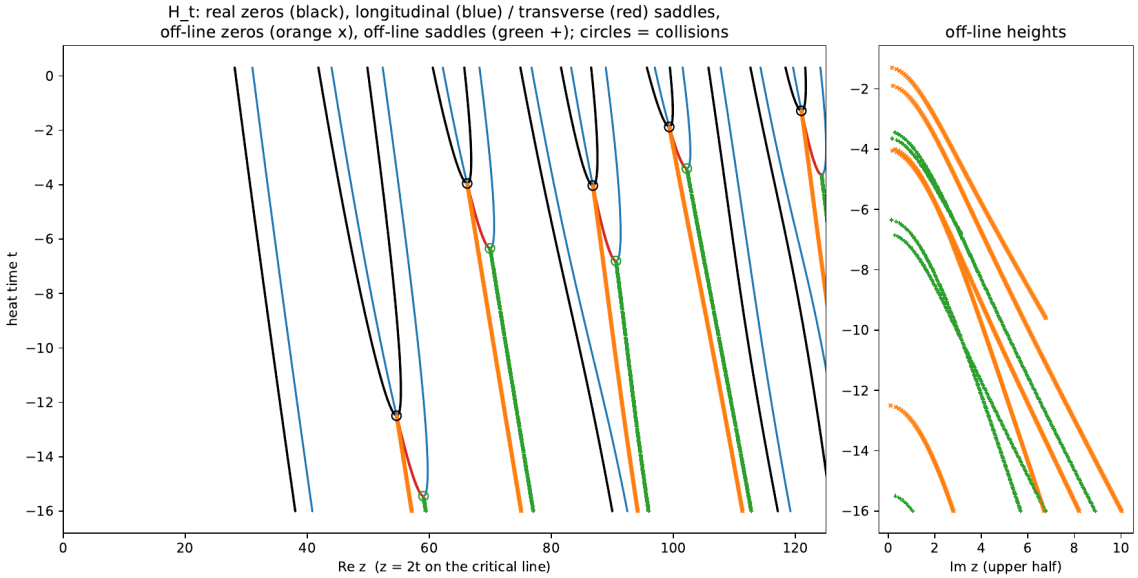

FIGURE 1. Backward heat flow of H_t [NUM]. Left: real zeros (black), longitudinal (blue) and transverse (red) critical points, real parts of off-line zeros (orange) and off-line saddles (green). Circles mark zero collisions (black) and saddle collisions (green). Right: imaginary parts. Each red segment runs from a zero collision to a saddle collision.

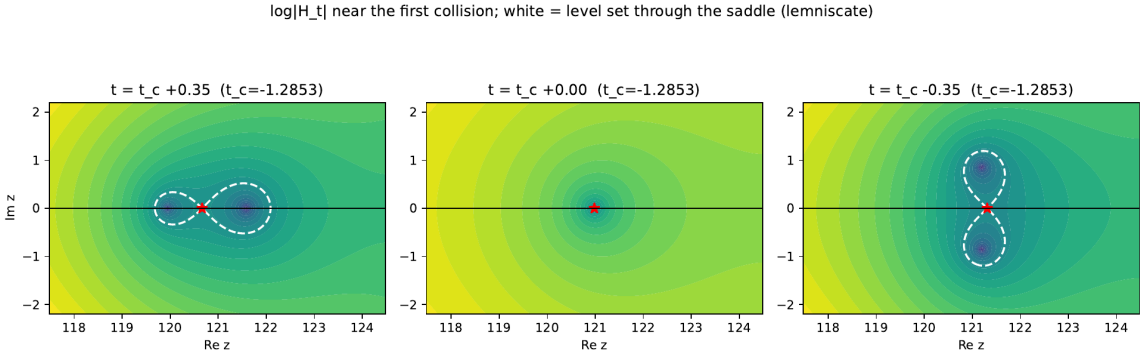

FIGURE 2. $\log|H_t|$ near the collision at $z \approx 121$ ($t_c = -1.2853$), at $t = t_c + 0.35$, t_c, $t_c - 0.35$ [NUM]. White: the level set through the saddle (red star). Before the collision it is a longitudinal figure eight around two real zeros. At t_c it collapses onto the double zero. After the collision it is a transverse figure eight around the ghost couple: the lemniscate has rotated by 90°.

- $\tau = J - J'$ at all $801 + 801$ steps; 4000/4000 random even toys (up to 8 symmetric pairs, 30% ghost quadruples). $\tau = J$ held in only 1378/4000.
- The ghost $2 \pm 0.3i$ lands on the axis at $t_c = +0.03933$ (normal form: $\delta^2/2 = 0.045$). Backward collisions occur at $t_c = -0.11933$ and -0.91373 (ghost toy), and at -0.02151, -0.12702, -0.98994 (real toy).

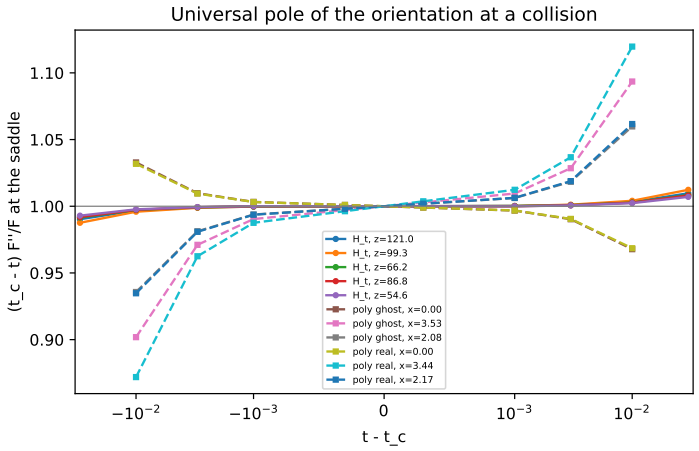

FIGURE 3. $(t_c - t)F''/F$ at the saddle near each zero collision, for H_t (circles) and the polynomial flow (squares) [NUM]. All curves tend to 1, the universal residue of Thm. 7.1(i).

- $(t_c - t)p''/p \to 1$: within $3.8 \cdot 10^{-3}$ of 1 at $|t - t_c| = 3 \cdot 10^{-4}$, for all six collisions, with deviation linear in $t - t_c$.
- The L55 toy (ghost pair at height 2, δ scan): $\tau = 2$, $J' = 0$ for $\delta \le 0.4$, and $\tau = 0$, $J' = 2$ for $\delta \ge 0.6$. A ghost pair with *no* transverse saddle, already in the Lemma 55 toy.

7.5. **v2: the DH forward flow (J.W. answer 4).**

Definition 7.3 (DH heat deformation)**.** With $\Lambda_{\mathrm{DH}}(s) = (5/\pi)^{s/2}\Gamma(\frac{s+1}{2})f_{\mathrm{DH}}(s)$ (Lemma 55; $\Lambda_{\mathrm{DH}}(s) = \Lambda_{\mathrm{DH}}(1-s)$, real on $\mathbb{R}$), put $H_0^{\mathrm{DH}}(z) = \Lambda_{\mathrm{DH}}(\frac{1}{2} + \frac{iz}{2})$ and

$$H_t^{\mathrm{DH}}(z) = \int_{\mathbb{R}} e^{tu^2}\,\Phi_{\mathrm{DH}}(u)e^{izu}\,du, \qquad \Phi_{\mathrm{DH}}(u) = 4\sqrt{\tfrac{\pi}{5}}\,e^{3u}\sum_{n\ge 1} n\,a_n\,e^{-\pi n^2 e^{4u}/5},$$

where $a_n = 1, \kappa_{\mathrm{DH}}, -\kappa_{\mathrm{DH}}, -1, 0$ for $n \equiv 1, 2, 3, 4, 0 \pmod 5$ and $\kappa_{\mathrm{DH}} = (\sqrt{10 - 2\sqrt{5}} - 2)/(\sqrt{5} - 1) = 0.284079$. (This is the DH constant of Lemma 55. It is unrelated to the counter κ.)

Justification. (i) $f_{\mathrm{DH}} = \sum a_n n^{-s}$ with $a_n = \mathrm{Re}((1 - i\kappa_{\mathrm{DH}})\chi(n))$ for the odd character $\chi = \chi_5(\cdot)$ with $\chi(2) = i$, and $\theta_f(x) = \sum n a_n e^{-\pi n^2 x/5}$ is the odd theta series. The Mellin transform $\Lambda_{\mathrm{DH}}(s) = (\pi/5)^{1/2}\int_0^\infty \theta_f(x)x^{(s+1)/2}\frac{dx}{x}$, with $x = e^{4u}$, gives the displayed Φ_{DH} with $H_0^{\mathrm{DH}}(z) = \int \Phi_{\mathrm{DH}}e^{izu}du$ [THM]. (ii) Φ_{DH} is even: this follows from the functional equation, and it was checked to 12 digits at $u = \pm 0.05, \ldots, \pm 0.8$. It decays like $\exp(-\frac{\pi}{5}e^{4|u|})$ [THM][NUM]. (iii) Hence H_t^{DH} is even, real entire of order 1, $\partial_t H_t^{\mathrm{DH}} = -\partial_z^2 H_t^{\mathrm{DH}}$, and $H_t^{\mathrm{DH}} = e^{-tD^2}H_0^{\mathrm{DH}}$, exactly as for ζ (Polymath normalization up to the factor $\frac{1}{8}$) [THM]. (iv) A ghost $\rho = \frac{1}{2} + \delta + i\gamma$ is the couple $2\gamma \pm 2i\delta$ in z. An isolated couple obeys $\dot{y} = -1/y$, so it lands at $t_c = y_0^2/2 = 2\delta^2$. Real neighbours at distances d_k give $\frac{d}{dt}y^2 = -2 - 4y^2\sum_k (d_k^2 + y^2)^{-1}$, so $t_c < 2\delta^2$ (formal zero dynamics; rigorous for ζ in [21, Prop. 3.1], for DH by analogy) [THM][ANALOGY].

Proposition 7.4 (Bounds for Λ_{DH})**.** *Let* $\Lambda_{\mathrm{DH}} = \inf\{t : H_t^{\mathrm{DH}}$ *has only real zeros*$\}$*. Then* $\Lambda_{\mathrm{DH}} \ge \max_\rho t_c(\rho)$ *over any set of ghosts, and* $\Lambda_{\mathrm{DH}} \le \frac{1}{2}(2(\sigma_0 - \frac{1}{2}))^2 = 1.6025$*. Here* $\sigma_0 = 1.39514$ *solves* $\sum_{n\ge 2}|a_n|n^{-\sigma} = 1$*, so every zero of* f_{DH} *has* $\beta < \sigma_0$ *and every zero of* H_0^{DH} *has* $|\mathrm{Im}\, z| < 1.7903$ *[THM] (with de Bruijn's strip theorem* [9, Thm. 13]*, as stated in* [21, Thm. 3.2] *[CITED] for* ζ *and applied here to* Φ_{DH}*, which satisfies the same decay and evenness hypotheses [BIB]).*

Method [NUM] (`num/lemma56v2_lines.py`, `lemma56v2_conv.py`, `lemma56v2_dhflow.py`, `lemma56v2_dhcheck.py`). Float64 Fourier quadrature is infeasible at $z \sim 2000$ (cancellation $e^{-\pi z/4}$). We use instead the exact identity

$$H_t(x+iy) = \int_{\mathbb{R}} H_0(x+iy')\,\frac{e^{-(y-y')^2/4t}}{2\sqrt{\pi t}}\,dy' \qquad (t>0),$$

a Gaussian smoothing along each vertical line, which is local in x. H_0 was evaluated with python-flint (64-bit balls; completed Λ_{DH}, normalized by $|\Lambda_{\mathrm{DH}}(\frac{3}{2}+iT)|$) on 121 vertical lines $x \in [2\gamma - 1.2, 2\gamma + 1.2]$ per ghost, with $|y| \le Y = 1.2 + 6.5\sqrt{2t_{\max}}$ and step $h_y = \min(0.025, 0.55\delta)$. That is $1.37 \cdot 10^6$ evaluations in three niced background processes. z-derivatives come from derivatives of the kernel, and points between lines from a degree-7 Taylor expansion. *Validation:* (a) for ζ at $z \approx 99$ the convolution agrees with the contour-shifted Fourier integral of §7.3 to $5 \cdot 10^{-15}$ ($t = 0.02, 0.05$) and $8 \cdot 10^{-7}$ ($t = 0.15$, Y truncation; hence $6.5\sqrt{2t}$ above); (b) for DH at $z \approx 171$ it agrees with a 110-digit mpmath Fourier integral of $e^{tu^2}\Phi_{\mathrm{DH}}$ to $1.5 \cdot 10^{-14}$, $1.2 \cdot 10^{-14}$, $2.8 \cdot 10^{-14}$ at $(t, z) = (0.1, 171.7)$, $(0.1, 171.7 + 0.5i)$, $(0.17, 171.2 + 0.2i)$ (`num/l56/dhflow_check.json`). Per ghost: t_c by 45 bisections on the number of real zeros in $[2\gamma - 0.9, 2\gamma + 0.9]$; the trajectory by Newton (residual $\le 1.2 \cdot 10^{-13}$); and counts at $t = t_{\min}, \frac{1}{2}t_c, 0.98t_c, 1.02t_c, 1.5t_c$, from real-axis sign changes and from the argument principle for H_t and H_t' on a rectangle $[x_a, x_b] \times [-1.2, 1.2]$ (boundary step 0.005, maximal phase step 0.19 rad, all windings integral to 10^{-6}).

Results [NUM] (`num/l56/dhflow.json`, Table 4, Fig. 4).

- All 28 ghosts land on the line at a zero collision, at $0.000507 \le t_c \le 0.230608$. No ghost survives to $t = 0.231$. At each landing, in its window: (K^+, τ, S^+, κ) goes from $(1, 1, 0, 1)$ to $(0, 0, 0, 0)$. The transverse saddle becomes the longitudinal maximum between the two new real zeros, and the number of real critical points is unchanged. So the identity $K^+ = \kappa$ holds at all 5×28 snapshots. **No saddle collision occurs in the forward flow** of any window, and $S^+ = 0$ throughout.
- $t_c/(2\delta^2) \in [0.798, 0.9995]$, so $t_c \approx 2\delta^2 = \frac{1}{2}(2\delta)^2$; the ratio decreases with δ ($\mathrm{corr}(\delta, \mathrm{ratio}) = -0.92$; the smallest ghost, $\delta = 0.0159$, has ratio 0.9995). $\frac{d}{dt}y^2$ at $t \approx 0$ lies in $[-3.04, -2.00]$, against -2 for an isolated couple, in line with the neighbour term in (iv).
- $\Lambda_{\mathrm{DH}} \ge 0.2306$ (ghost $0.8695 + 240.4047i$, $\delta = 0.3695$). This is *outside* ζ's interval $[0, 0.2]$ ($\Lambda_\zeta \ge 0$ [22], ≤ 0.2 [19]) [NUM][CITED]. With Prop. 7.4: $0.2306 \le \Lambda_{\mathrm{DH}} \le 1.6025$. DH ghosts above 1000 with larger δ would raise the lower bound; we make no claim about them.
- Caveat: only windows around the 28 ghosts were followed. The forward flow elsewhere on $0 < z < 2000$ was not monitored, and we claim nothing about saddle collisions there.

7.6. **v3: the whole-line DH flow (J.W. answer 4).** *Engine* [NUM] (`num/lemma56v3_fft.py`). Shift the contour to $u = v + ic$, $c = \frac{\pi}{8} - \varepsilon$, where Φ_{DH} is analytic for $|\operatorname{Im} u| < \frac{\pi}{8}$. Then $e^{izu} = e^{izv}e^{-zc}$, and the cancellation $e^{-\pi z/8}$ is removed. Write $B_k(z) = e^{zc}H_t^{(k)}(z)$, which is real for real z. It is computed from coarse-grid FFTs G_m of $(iv)^m e^{tu^2}\Phi_{\mathrm{DH}}(u)$, a Taylor expansion of order 12 between grid points, and $B_k = \sum_m \binom{k}{m}(-c)^{k-m}G_m$. Horizontal lines $z = x + iy$ use the factor e^{-yv}. *Validation:*

- against python-flint values of Λ_{DH}: relative error $\le 6.6 \cdot 10^{-13}$ for $z \le 2000$, and $2 \cdot 10^{-13}$ on the line $y = -1.5$;
- against v2's 110-digit Fourier values: $1.4 \cdot 10^{-15}$, $5.5 \cdot 10^{-15}$ and $2.6 \cdot 10^{-14}$;
- with $\varepsilon = 5 \cdot 10^{-4}$ and step $1.3 \cdot 10^{-5}$: $\le 3 \cdot 10^{-11}$ up to $z = 2 \cdot 10^4$.

Global count [NUM] (`num/lemma56v3_global.py`; data `num/l56v3/global_t.jsonl`). For $t = 0, 0.002, \ldots, 0.25$ (126 steps) we take the rectangle $[-X, X] \times [-2.2, 2.2]$ in the z-plane, i.e. $\sigma \in (-0.6, 1.6)$. Its end $X \approx 2000$ is chosen midway between a longitudinal critical point and the next zero. At each step we compute:

TABLE 4. DH forward flow: landing times of the 28 ghosts below height 1000 [NUM]. Height γ, β, $\delta = \beta - \frac{1}{2}$, landing time t_c (in the Polymath $z = 2t$ normalization), and $t_c/(2\delta^2)$.

γ	β	δ	t_c	$\frac{t_c}{2\delta^2}$	γ	β	δ	t_c	$\frac{t_c}{2\delta^2}$
85.6993	0.8085	0.3085	0.176512	0.9272	595.0234	0.5829	0.0829	0.013578	0.9890
114.1633	0.6508	0.1508	0.044528	0.9786	611.7751	0.6283	0.1283	0.031171	0.9475
166.4793	0.5744	0.0744	0.011034	0.9978	646.9868	0.6108	0.1108	0.024004	0.9783
176.7025	0.7243	0.2243	0.094662	0.9411	657.1084	0.7606	0.2606	0.118737	0.8742
240.4047	0.8695	0.3695	0.230608	0.8444	692.8925	0.7887	0.2887	0.143360	0.8600
320.8765	0.8195	0.3195	0.178727	0.8752	737.7670	0.7774	0.2774	0.132117	0.8587
331.0503	0.7682	0.2682	0.128718	0.8946	783.6530	0.8530	0.3530	0.199898	0.8021
366.6409	0.6285	0.1285	0.032123	0.9726	811.7658	0.6686	0.1686	0.053291	0.9379
411.7967	0.8159	0.3159	0.170627	0.8550	847.4657	0.5619	0.0619	0.007601	0.9903
440.4845	0.7089	0.2089	0.081097	0.9293	857.2959	0.8561	0.3561	0.202485	0.7984
520.9438	0.5159	0.0159	0.000507	0.9995	864.1181	0.6809	0.1809	0.061670	0.9423
531.2797	0.8470	0.3470	0.201276	0.8360	892.1490	0.6884	0.1884	0.064469	0.9079
548.9068	0.7295	0.2295	0.095486	0.9062	921.1727	0.7594	0.2594	0.118342	0.8797
566.5097	0.7866	0.2866	0.141454	0.8613	983.7522	0.7625	0.2625	0.114637	0.8318

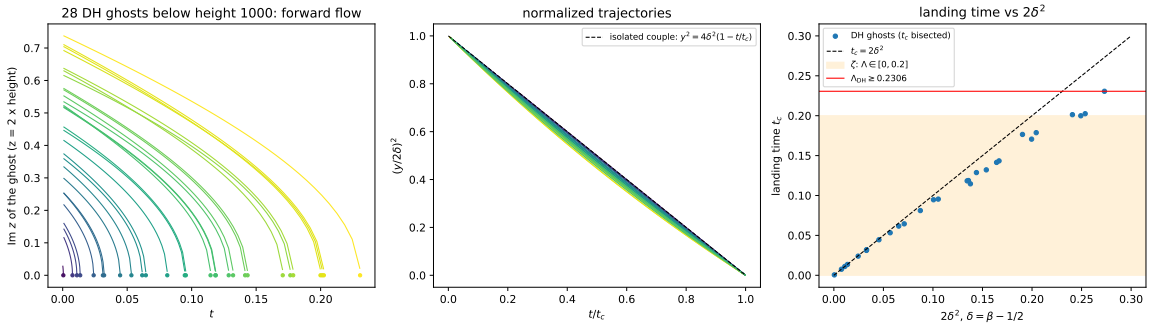

FIGURE 4. DH forward heat flow [NUM]. Left: $\operatorname{Im} z$ of each ghost against t (colour by δ), with dots at the landing times. Middle: $(y/2\delta)^2$ against t/t_c, nearly the isolated-couple line. Right: t_c against $2\delta^2$, with ζ's band $\Lambda \in [0, 0.2]$ and the bound $\Lambda_{\mathrm{DH}} \geq 0.2306$.

- the argument principle for H_t and H_t' (boundary step 0.0025, largest phase step 0.0995 rad, windings integral to $5 \cdot 10^{-13}$);
- all real zeros and all real critical points with their orientation.

Then $K^+ = (Z_H - n_{\mathbb{R}})/4$ and $S^+ = (Z_{H'} - C_{\mathbb{R}})/4$, where $C_{\mathbb{R}} = 2\#\{\text{critical points in } (0, X)\} + 1$.

- $t = 0$: 848 real zeros, 904 critical points, $\tau = 28$, wind $H = 1808 = 2 \cdot 848 + 4 \cdot 28$, wind $H' = 1809$, so $K^+ = 28$ and $S^+ = 0$. Hence *there are no other zeros* of f_{DH} with $-0.6 < \sigma < 1.6$ and $|t| < 1000.35$ besides L48's $848 + 28$. In particular none has $\beta \geq 1$ (certified in Thm. 7.5), and none is real (no real-pair ghosts).
- K^+ falls from 28 to 0 in single steps, and the last drop is between $t = 0.230$ and 0.232. Twice two ghosts land within one step: $0.094 \to 0.096$ (heights 176.70, 548.91) and $0.118 \to 0.120$ (657.11, 921.17). The staircase equals $28 - \#\{\rho : t_c(\rho) \leq t\}$ at all 126 steps, with the t_c of Table 5 (0 mismatches).
- $K^+ = \tau + S^+$ at all 126 steps (deviation $\leq 2 \cdot 10^{-13}$). $|S^+| \leq 6 \cdot 10^{-14}$ throughout, so no saddle ever leaves the line.

- The number of real critical points in the fixed window $0 < z < 1999.5$ is 903 at every step. The smallest gap between consecutive critical points over all steps is 0.73. So, at this resolution, *no saddle collision* occurs (it would change the count by 2), and τ never increases, so *no new transverse saddle* is born. v2 saw a change of the window end at $t = 0.188$ (X: $2000.49 \to 2002.30$); it is a boundary effect and not an event.
- *Events:* exactly 28 zero collisions, one per ghost, each with $(K^+, \tau, S^+) \mapsto (K^+ - 1, \tau - 1, 0)$. A transverse saddle turns into the longitudinal maximum between the two new real zeros. There are no other events.

Landing times [NUM] (`num/lemma56v3_tc.py`, `num/l56v3/tc_fft.json`). These come from bisection on the number of real zeros in $[2\gamma - 0.9, 2\gamma + 0.9]$ with the FFT engine. They are independent of v2's vertical-line convolution. The two agree to $4.2 \cdot 10^{-5}$ (median $4.8 \cdot 10^{-6}$); the FFT value is always the smaller, for a reason not diagnosed. $t_c \in [0.000507, 0.230607]$ and $t_c/2\delta^2 \in [0.798, 0.9995]$.

TABLE 5. Whole-line DH flow: the 28 ghosts below height 1000 [NUM]. t_c from the FFT engine; $\Delta = 10^6(t_c - t_c^{\mathrm{v2}})$; the step of the global run in which K^+ drops.

γ	δ	t_c	Δ	$\frac{t_c}{2\delta^2}$	step	γ	δ	t_c	Δ	$\frac{t_c}{2\delta^2}$	step
85.6993	0.3085	0.176505	-6.7	0.9272	0.176–0.178	595.0234	0.0829	0.013569	-9.6	0.9882	0.012–0.014
114.1633	0.1508	0.044518	-9.4	0.9784	0.044–0.046	611.7751	0.1283	0.031171	-0.6	0.9475	0.030–0.032
166.4793	0.0744	0.010991	-42.2	0.9940	0.010–0.012	646.9868	0.1108	0.023999	-5.5	0.9780	0.022–0.024
176.7025	0.2243	0.094659	-3.0	0.9411	0.094–0.096	657.1084	0.2606	0.118695	-42.1	0.8739	0.118–0.120
240.4047	0.3695	0.230607	-0.4	0.8444	0.230–0.232	692.8925	0.2887	0.143331	-29.5	0.8598	0.142–0.144
320.8765	0.3195	0.178726	-0.8	0.8751	0.178–0.180	737.7670	0.2774	0.132115	-1.7	0.8587	0.132–0.134
331.0503	0.2682	0.128713	-4.8	0.8945	0.128–0.130	783.6530	0.3530	0.199893	-5.1	0.8021	0.198–0.200
366.6409	0.1285	0.032122	-1.2	0.9725	0.032–0.034	811.7658	0.1686	0.053259	-31.9	0.9373	0.052–0.054
411.7967	0.3159	0.170613	-13.8	0.8550	0.170–0.172	847.4657	0.0619	0.007600	-0.6	0.9903	0.006–0.008
440.4845	0.2089	0.081092	-5.3	0.9293	0.080–0.082	857.2959	0.3561	0.202481	-4.3	0.7983	0.202–0.204
520.9438	0.0159	0.000507	-0.0	0.9995	0.000–0.002	864.1181	0.1809	0.061669	-1.1	0.9422	0.060–0.062
531.2797	0.3470	0.201276	-0.1	0.8360	0.200–0.202	892.1490	0.1884	0.064467	-1.6	0.9078	0.064–0.066
548.9068	0.2295	0.095479	-6.8	0.9061	0.094–0.096	921.1727	0.2594	0.118334	-8.8	0.8796	0.118–0.120
566.5097	0.2866	0.141422	-32.5	0.8611	0.140–0.142	983.7522	0.2625	0.114636	-1.1	0.8318	0.114–0.116

$\beta > 1$. DH has infinitely many zeros with $\sigma > 1$ (Davenport–Heilbronn [8], as quoted in [18]; Lemma 53) [BIB]. The question is where they start.

Theorem 7.5 (No DH zeros with $\beta \geq 1$ below height 1000; computer-assisted)**.** *$f_{\mathrm{DH}}(s) \neq 0$ for $1 \leq \sigma \leq 1.6$, $0 \leq t \leq 1000$. Together with $\beta < \sigma_0 = 1.39514$ (Prop. 7.4), DH has no zero with $\beta \geq 1$ and $|\gamma| \leq 1000$. [THM]*

Proof. Argument principle with Arb balls [11] (`num/lemma56v3_beta1cert.py`, 80-bit, `num/l56v3/beta1_cert.json`). $f_{\mathrm{DH}} = \frac{1-i\kappa_{\mathrm{DH}}}{2} L(s,\chi) + \frac{1+i\kappa_{\mathrm{DH}}}{2} L(s,\bar{\chi})$ with κ_{DH} in ball arithmetic. *Contour.* The boundary of $[1, 1.6] \times [0, 1000]$ minus the square $Q_0 = [1, 1.05] \times [0, 0.05]$ is split into segments. Each segment is enclosed in a box B. If $f(B) \subset D(c, r)$ with $r \leq 0.8|c|$, then $\arg f$ stays in a cone of half-angle $< 54°$ on the segment. So the change of argument across the segment equals the principal argument of $f(q)/f(p)$, which is less than $108°$ in absolute value. Otherwise the segment is bisected. 72661 segments were needed (32637 bisections), and the total winding is 0. *The square* Q_0 contains $s = 1$, where Arb's Hurwitz-zeta route has a removable pole. With $c_0 = 1.025 + 0.025i$: $|f(c_0)| \geq 0.9263$, $|f'(c_0)| = 0.1376$, and $M = \max|f| \leq 1.995$ on $|s - c_0| = \frac{1}{2}$ (balls). For $s \in Q_0$, with $d = |s - c_0| \leq 0.0354$, Cauchy's estimate gives $|f(s) - f(c_0)| \leq |f'(c_0)|d + M(2d)^2/(1 - 2d) < 0.0156$. So $f \neq 0$ on Q_0. □

Numerically (`num/lemma56v3_beta1.py`, $\varepsilon = 5 \cdot 10^{-4}$ engine, chunks of length 50 in z on $\mathrm{Im}\, z \in (-1.85, -1)$, i.e. $1 < \sigma < 1.425$): there is no zero up to height 10^4 (total winding 0, largest step

0.30 rad) [NUM]. *So no ghost with $\beta > 1$ exists below height* 10^4, and none could be followed. J.W.'s question about their t_c has no instance in this range.

Λ_{DH}*, certified.*

Theorem 7.6 (Lower bound for Λ_{DH}; computer-assisted)**.** *H_t^{DH} has a non-real zero for $t = 0.230607$. Hence $0.230607 \le \Lambda_{\mathrm{DH}} \le 1.6025$. [THM] (the upper bound as in Prop. 7.4, [BIB])*

Proof. $H = H_t^{\mathrm{DH}}$ is real entire of order 1 (Def. 7.3, (iii)). If all its zeros were real, then by Hadamard's factorization $H = cz^m e^{bz} \prod (1 - z/z_k)e^{z/z_k}$, so on $\mathbb{R}$

$$(H'/H)' = -\frac{m}{z^2} - \sum_k \frac{1}{(z - z_k)^2} < 0, \qquad \text{i.e.} \qquad L := H'^2 - HH'' > 0$$

away from zeros; at a zero, $L = H'^2 \ge 0$. This is the Laguerre inequality. So $L(x_0) < 0$ at a single real x_0 proves a non-real zero. At $t = 0.230607$ and $x_0 = 480.69024$ we computed, with Arb's rigorous integration (`acb.integral`, 520 bits; `num/lemma56v3_lagcert.py`, `num/l56v3/lagcert.jsonl`),

$$I_k = \int_0^2 (iu)^k e^{tu^2} \Phi_{\mathrm{DH}}(u)\, \{\cos,\ i^{-1}\sin,\ \cos\}(x_0 u)\, du \quad (k = 0, 1, 2)$$

(up to the positive factor $8\sqrt{\pi/5}$). The ingredients:

- The theta tail is enclosed by $(N+1)q^{N+1}/(1-q)^2$ with $q = e^{-\pi(N+1)\rho/5}$ and $\rho \le \operatorname{Re} e^{4u}$ on each ball.
- For $u \ge 2$ the integrand is at most $e^{-\pi e^{4u}/10}$, so the tail of the u-integral is below e^{-900}.

The result is $I_0 = 2.0566\ldots \cdot 10^{-88}$, $I_1 = 4.9235\ldots \cdot 10^{-87}$, $I_2 = 5.68226\ldots \cdot 10^{-82}$, and $L \propto I_1^2 - I_0 I_2 = [-1.16838 \cdot 10^{-169} \pm 1.9 \cdot 10^{-184}] < 0$. To conclude: by de Bruijn's forward preservation (real-rootedness persists for larger t; Laguerre–Pólya theory, as in Prop. 7.4), every $t \le 0.230607$ lies below Λ_{DH}. □

The same test, at $t = 0.23$ and 0.2306, also certifies (`lagcert.jsonl`). The ghost responsible is $0.8695 + 240.4047i$, and its numerical $t_c = 0.2306074$ lies $4 \cdot 10^{-7}$ above the certified value.

Large-δ ghosts between 1000 *and* 10^4 [NUM].

- *Scans.* Argument principle on the strips $\operatorname{Im} z < -0.74, -0.70, -0.678$ (i.e. $\beta > 0.87, 0.85, 0.839$) for $z \le 2 \cdot 10^4$ (`num/lemma56v3_strip.py`): 4, 18 and 23 zeros above height 1000.
- *Location.* By $|f_{\mathrm{DH}}|$-grid and Newton, residual $\le 2 \cdot 10^{-12}$ (`lemma56v3_locate*.py`).
- *Landing times.* By the $\varepsilon = 5 \cdot 10^{-4}$ engine, two independent ways: bisection on a real-zero count (`lemma56v3_tchigh*.py`), and Newton continuation of the ghost itself (`lemma56v3_track.py`). They agree to $\le 4 \cdot 10^{-4}$ wherever both exist (22 ghosts). The tracked value is always the smaller, by 0.9–$3.6 \cdot 10^{-4}$; the linear extrapolation of y^2 and the finite grid of the count are the likely causes. The narrow-window count ($2\gamma \pm 1.2$) failed for 8 ghosts because a neighbouring real zero drifted across the window edge. The wide window (± 10, `lemma56v3_tchigh4.py`) fixed 6 of them. For 4935.62 the bracket had to be lowered ($[0.05, 0.5] \cdot 2\delta^2$). For 8470.59 the wide window contains another, unlisted ghost ($\beta < 0.839$) that lands first, so only the tracked value is used.
- *Why $\beta > 0.839$ suffices.* A ghost can beat 0.2306 only if $2\delta^2 > 0.2306$, i.e. $\beta > 0.8396$, *provided* $t_c \le 2\delta^2$. That inequality holds in every case computed here, but it is not a theorem for individual ghosts; de Bruijn's strip theorem is the global version.

Surprises.

(a) No ghost with $\beta \ge 1$ below 10^4, although infinitely many exist.

(b) The ratio $t_c/2\delta^2$ falls with height, from 0.80–1.00 below 1000 to 0.59–0.78 between 1000 and 10^4, with one outlier at 0.44 ($\gamma = 4935.62$, $t_c = 0.1115$, confirmed by tracking). The neighbouring real zeros are denser there, so the term $-4y^2 \sum (d_k^2 + y^2)^{-1}$ of v2 (iv) is larger.

TABLE 6. DH ghosts with $\beta > 0.839$ between heights 1000 and 10^4 [NUM] (strip scans; the largest phase step in some chunks was 3.1 rad, so a zero very close to a strip edge could be missed). t_c by bisection with the $\varepsilon = 5 \cdot 10^{-4}$ engine (bracket $[\delta^2, 2\delta^2]$, 12 bisections, error $\le \delta^2/4096$, or 34 bisections for the first three found). Real zeros are counted in a window $2\gamma \pm 1.2$, or, marked ^{w}, in a window of half-width about 10 with both ends in gaps (`lemma56v3_tchigh4.py`), which is needed when a neighbouring real zero drifts across the edge of the narrow window. The column t_c^{tr} is an independent check: the ghost itself is followed by Newton continuation of $H_t(z) = 0$ from $z = 2\gamma + 2i\delta$ (direct quadrature, step 0.004; `lemma56v3_track.py`), and t_c^{tr} is the linear extrapolation of $(\mathrm{Im}\, z)^2$ to 0 (accuracy about 10^{-3}). The ratio uses the window-count value where available, otherwise t_c^{tr}. A dash means not completed.

γ	β	$2\delta^2$	t_c	t_c^{tr}	$t_c/2\delta^2$
1092.454	0.8546	0.2515	0.19572	0.1956	0.778
2442.530	0.8978	0.3164	0.22319	0.2231	0.705
2887.939	0.8604	0.2598	0.19102	0.1909	0.735
3059.519	0.8679	0.2706	0.18552	0.1853	0.685
3214.144	0.8613	0.2610	0.17704^w	0.1769	0.678
3294.609	0.8604	0.2597	0.17706^w	0.1769	0.682
3757.398	0.8427	0.2349	0.17102^w	0.1709	0.728
4237.969	0.8577	0.2558	0.17108^w	0.1709	0.669
4318.603	0.8389	0.2298	0.16796^w	0.1678	0.731
4392.376	0.8535	0.2500	0.17592	0.1758	0.704
4935.618	0.8551	0.2522	0.11148^w	0.1111	0.442
5090.318	0.8664	0.2685	0.17724^w	0.1770	0.660
5833.282	0.8741	0.2799	0.19017	0.1901	0.679
5913.706	0.8594	0.2584	0.18337	0.1832	0.710
6113.929	0.8582	0.2566	0.16564^w	0.1654	0.645
7011.512	0.8663	0.2683	0.18138	0.1812	0.676
7909.445	0.8546	0.2514	0.16021^w	0.1600	0.637
8235.593	0.8753	0.2818	0.18623	0.1861	0.661
8470.592	0.8460	0.2394	—	0.1636	0.683
8561.594	0.8836	0.2944	0.17364^w	0.1733	0.590
9340.123	0.8414	0.2331	0.16080^w	0.1606	0.690
9694.579	0.8516	0.2473	0.16900	0.1688	0.683
9911.484	0.8484	0.2428	0.16052^w	0.1603	0.661

As a result the ghost with the largest δ below 10^4 ($\delta = 0.3978$, $\gamma = 2442.53$) lands at $0.2232 < 0.2306$.

(c) The lower bound is set by a *low* ghost ($\gamma = 240.40$).

(d) One ghost ($\gamma = 4935.62$, $\delta = 0.355$) lands at less than half of $2\delta^2$. We have not diagnosed why; a local excess of real zeros, or an unlisted nearby ghost with $\beta < 0.839$, are candidates.

(e) The whole forward flow up to Λ (in this window) is a pure sequence of zero collisions: no saddle ever leaves the line.

Updated bound: $0.230607 \le \Lambda_{\mathrm{DH}} \le 1.6025$, with the lower bound certified (Thm. 7.6) and unchanged numerically by the ghosts up to height 10^4 that were checked. Ghosts with $\beta > 1$, which lie higher, would give t_c up to about $2(0.895)^2 \cdot$ratio. We make no claim about them.

8. Ghost diagnostics side by side (v3: with Lemma 54 v2's invariants)

The columns, with κ (official) first:

- $\kappa = \tau + S^+$ ($\kappa_{\mathrm{FF}} = \tau + \frac{1}{2} Z_{\mathrm{off}}$).

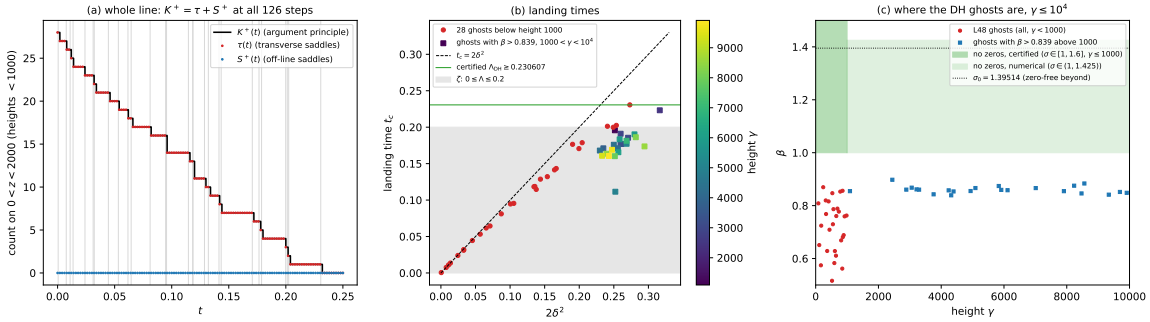

FIGURE 5. Whole-line DH flow [NUM]. (a) $K^+(t)$ from the argument principle (staircase), with $\tau(t)$ and $S^+(t)$ on $0 < z < 2000$; grey lines mark the 28 landing times. (b) t_c against $2\delta^2$ for the 28 ghosts below 1000 (red) and the large-δ ghosts between 1000 and 10^4 (squares, coloured by height), with the certified bound 0.230607 and ζ's band $[0, 0.2]$. (c) Positions (γ, β) of these ghosts, with the zero-free regions for $\sigma \geq 1$: certified up to 1000, numerical up to 10^4.

- k = number of ghosts: L48's count of roots outside the circle over $\mathbb{F}_q$; K^+ on windows; the couples J on full symmetric windows.
- τ and S^+.
- ν = Lemma 54 v2's *frequency-form index* $n_-(h_F)$, $h_F(x, y) = \omega(x, (F - Ve)y)$ (flow version $\omega(x, Ay)$). Over $\mathbb{F}_q$ it is computed in L50's trace model.
- r = Lemma 54 v2's *drift rank*, the number of eigenvalues of g_{n} outside the unit circle (flow: $\#\{\mathrm{Re}\,\mu > 0\}$), i.e. the number of linearly growing Siegel rapidities.
- ℓ = L54 drift rate $2(\sum_{|\lambda|>1} \sigma^2)^{1/2}$.
- The L53 charge.

For ξ-type rows, ν, r and the charge are computed on the full symmetric truncation (L54 §7), where they count ρ and $\bar{\rho}$. So they equal $\kappa_{\mathrm{full}} = 2\kappa + R$, with R the number of real pairs. "—" means not defined or not computed; "$= k$" means given by Lemma 54 v2's trace-model theorem and not run separately. All entries are [NUM], from `num/l56/*.json`, `num/l56v3/*`, `num/l54v2/{ffrows,proxies,invariants_trace}.json`.

Reading.

(a) κ equals k in every row [THM][NUM].

(b) ν and r now equal k wherever they are defined, *real-pair ghosts included.* v1's n_- was undefined there (Lemma 54 v2).

(c) ℓ is a size, not a count. It is 0 iff $k = 0$.

(d) τ undercounts exactly when saddles have left the line.

(e) The DH forward flow removes ghosts only at zero collisions, with $S^+ \equiv 0$, on the whole window $0 < z < 2000$ (§7.6).

9. CORRECTIONS

(C1) *J.W.'s answer 3 ("ghost counter = transverse-saddle count").* This is exact only when there are no off-line saddles. The correct counter is $\kappa = \tau + S^+$ (Thm. 4.3, Thm. 6.2). Counterexamples: Ex. 6.4 (FF); the L55 toy with $\delta \geq 0.6$; every ghost of H_t after its saddle collision (Table 3); $\tau \neq k$ in 1510/3000 random FF tests and $\tau \neq J$ in 2622/4000 random toys [NUM]. We adopt $\mathfrak{o}$ (answer 1) as the invariant and κ as the counter, with τ reported as its on-line part. If J.W. prefers to keep τ as the official counter, every statement "$\tau =$ ghosts" must carry the hypothesis $S^+ = 0$.

TABLE 7. Ghost diagnostics, κ first [NUM]. In every row $\kappa = k$. Where defined, $\nu = r = k$ (FF) or $= 2k + R$ (full ξ-type truncations), *including* real-pair ghosts, where v1's index n_- was undefined. $\tau = k$ fails exactly in the rows with $S^+ > 0$.

family	case	κ	k	τ	S^+	ν	r	ℓ	charge
FF curves	10 cases[a]	0	0	0	0	0	0	0	0
FF fakes	$g = 1$: $a = 8, q = 13$	1	1	1	0	1	1	0.927	-1
	$g = 1$: $a = 7, q = 11$	1	1	1	0	1	1	0.662	-1
	$g = 1$: $a = 13, q = 13$	1	1	1	0	1	1	2.390	-1
	$g = 2$ real pair, $q = 11$	2	2	2	0	2	2	0.730	-2
	$g = 2$ quadruple, $q = 11$	2	2	2	0	2	2	0.960	-2
	$g = 2$ mixed, $q = 11$	1	1	1	0	1	1	1.091	-1
	$g = 2$, $(c_1, c_0) = (-5, 9)$	2	2	1	1	2	2	2.440	-2
	$g = 2$, $(c_1, c_0) = (-4.5, 8)$	2	2	1	1	2	2	2.270	-2
	$g = 3$, $1 + 10u^2 + 10u^4 + u^6$	2	2	0	2	2	2	3.090	-2
	1480 random[b], $g \le 5$	k	k			k	k^b		
DH	10 windows of length 100[c]	k	k	k	0	—	—	—	—
	total $(0, 1000]$	28	28	28	0	56[d]	56[d]	3.635[d]	-56
DH flow[e]	$t = 0.05$	20	20	20	0	$= 40$	$= 40$	—	
	$t = 0.1$	14	14	14	0	$= 28$	$= 28$	—	
	$t = 0.2$	3	3	3	0	$= 6$	$= 6$	—	
	$t = 0.232$	0	0	0	0	0	0	0	0
ζ	$(0, 1000]$ (649 zeros)	0	0	0	0	0[d]	0[d]	0	0
ζ flow[f]	$t = -5$	6	6	4	2	—	—	—	
	$t = -10$	6	6	0	6	—	—	—	
	$t = -15.8$	8	8	0	8	—	—	—	
proxies[d]	ζ_{50} + real pair $\beta = 0.7$	—	1	—	—	1	1	—	-1
	DH_{80} + real pair $\beta = 0.8$	—	57	—	—	57	57	—	-57
	toy: quadruple + real pair	—	3	—	—	3	3	—	-3
	toy: real pairs $\beta = 0.55, 0.9$	—	2	—	—	2	2^g	—	-2
toys[f]	L55 pair, $\delta \le 0.4$	2	2	2	0	—	—	—	
	L55 pair, $\delta \ge 0.6$	2	2	0	2	—	—	—	

[a] 11a at $p = 5, 7, 13, 17, 19, 23$; genus-2 curves $x^5 - x + 1$, $x^5 + x^2 + 3$ at $p = 11, 13$. [b] Lemma 54 v2, random test: 1500 random trace models, 20 degenerate ($\pm\sqrt{q}$ an eigenvalue); $\nu = k$ in 1480/1480 (in 779/779 with real pairs); orbit estimate $r_{\mathrm{orbit}} = k$ in 295/295 tested. [c] $k = 1, 3, 1, 3, 2, 5, 4, 2, 5, 2$ (Table 1). [d] Full symmetric truncations of L54 §7, conjugated by a random symplectic matrix: DH 216-dim (28 ghosts, 52 on-line) and 1808-dim (all 848 on-line): $\nu = 56$; ζ 200-dim and 2000-dim (100 and all 649 zeros): $\nu = 0$. r is spectral. The finite-time Lyapunov estimate gives 54 for DH because the two smallest exponents ($\delta = 0.016$) lie below the 300-step threshold. [e] κ, k, τ, S^+ from §7.6; ν, r "=" are $2k$ by Lemma 54 v2, with no separate truncation. [f] Full symmetric windows: $k = J$, $S^+ = J'$. [g] Spectral $r = 2$; the 300-step orbit estimate sees only 1 ($\sigma = 0.05$ is below threshold).

(C2) *Lemma 55 abstract (2), "for polynomials, #transverse = #ghost pairs".* Lemma 55 Thm. 4.2(e) has the needed hypothesis (all critical points real), but the abstract dropped it. The general statement is $\tau = J - J'$. Likewise Lemma 55 §9's "$\#T = k$ on the circle" holds for the 16 cases, but not in general (Ex. 6.4).

(C3) *Lemma 55 §4 and Q2 ("counting step neither proved nor found").* The window count is Kim 1996, Thm. 1 [14]. Jensen localization is Jensen/Pólya. Lemma 55 cited Kim96 only for a different statement (local reality of high derivatives, via Far22). Lemma 55's suggested route ("$\#\{\Xi' = 0\} \ge N(T) - C$ along a sequence") is now Thm. 4.2, an exact equality at admissible heights. As predicted, it does not by itself give the converse: the content moves to (A_∞) (Prop. 4.5).

(C4) *Lemma 55 §9 (Cohn).* Lemma 55 quoted Cohn only through Lalín–Smyth (unimodularity criterion). It left $\#\{L' = 0, |u| > 1\} = \#\text{ghosts}$ as an observation in 16 cases. Cohn's Fall IV (p. 113, read here) gives it exactly, for all self-inversive P (Thm. 6.1).

(C5) *The task brief's "Polymath 15 ($\Lambda \le 0.2$)".* Polymath proved $\Lambda \le 0.22$ [21]. $\Lambda \le 0.2$ is Platt–Trudgian 2021, Cor. 2 [19] (Polymath's method plus RH to $3 \cdot 10^{12}$).

(C6) *"Orientation flips $\iff$ zero collisions"* (the framing of task 5). Only half true. Flips also occur at saddle collisions (Thm. 7.1(ii)), and there the flip removes a transverse saddle *without* removing the ghost.

(C7) *Lemma 55 "no off-line saddle was found" for DH.* Confirmed with a sharper method: only Jensen discs need to be searched (Thm. 3.1), and all 28 discs contain exactly one zero of Λ', the transverse one [NUM].

(C8) *(v2) v1 Cor. 4.4(d), Prop. 4.5, Rem. 4.6, the v1 abstract item (2) ("this is where the argument breaks") and v1 Q2.* The converse (H)⇒RH was called [OPEN] and reduced to an unproved explicit density bound. That was wrong: the unconditional global count (Thm. 4.8) excludes covering under (H), so (H)⇒RH is a theorem (Thm. 4.10). The resulting equivalence RH $\iff$ (H) is a restatement of RH.

(C9) *(v2) v1 Rem. 4.6, "$T \lesssim 10^{232}$".* v1 marked this as heuristic, and it is the right asymptotic value for the two-sided mean square on $[T, 2T]$. With the verified explicit inputs, which give only an *upper* bound for $\int_0^X |\zeta|^2$, the rigorous range is $[305, 10^{116.7}]$ (Thm. 4.13). An explicit two-sided mean square on $[T, 2T]$ would push it towards 10^{232}.

(C10) *(v2) v1 Prior art listed Prop. 4.5 as the "covering reduction" of an open converse.* It is now a lemma inside the proof route of Prop. 4.14, and it is not needed for Thm. 4.10.

(C11) *(v2) v1 Q4 expected the DH forward flow "eventually [to] make κ vanish".* Confirmed for the 28 ghosts below 1000, which are gone by $t = 0.231$. The full line was not followed, and DH has infinitely many ghosts, so no global statement is made (§7.5).

(v3) Corrections to earlier claims by Grok Bot, in v2 and in the chat summaries of v2. None of these is J.W.'s claim.

(C12) *v2 Q4 and v2 §7.5 ("push to DH ghosts with $\beta > 1$ above height 1000").* This suggested that such ghosts are within easy reach. There is *none* below height 10^4: certified to 1000 (Thm. 7.5), numerical to 10^4.

(C13) *v2 §7.5: "DH ghosts above 1000 with larger δ would raise the lower bound".* This is false for every such ghost below 10^4. The largest δ there (0.3978 at $\gamma = 2442.53$) gives $t_c = 0.2232 < 0.2306$, because $t_c/2\delta^2$ falls with height (Table 6).

(C14) *v2 U5 and abstract ("$t_c \approx 2\delta^2$, ratio 0.80–1.00").* This holds below height 1000 only. Between 1000 and 10^4 the ratio is 0.44–0.78.

(C15) *v2 abstract (iv) and U5 stated $\Lambda_{\mathrm{DH}} \ge 0.2306$ under the mixed tag [NUM][THM].* It was numerical. It is now a computer-assisted theorem with the sharper 0.230607 (Thm. 7.6).

(C16) *v2 Table 4, sixth digit of t_c.* The independent FFT values are lower by up to $4.2 \cdot 10^{-5}$ (e.g. 0.010991 against 0.011034 at $\gamma = 166.48$). v2's t_c should be read to $\pm 5 \cdot 10^{-5}$; Table 5 supersedes them.

(C17) *v2 caveat "only windows around the 28 ghosts were followed".* This is resolved (§7.6): there are no events off the ghost windows.

(C18) *v2 Q5 ("should the L54 invariants (n_-, ℓ) be redefined so that they see real-pair ghosts").* The premise was half wrong. ℓ already sees real pairs ($\ell > 0$ iff $k > 0$, Table 7); only the index n_- was blind, being undefined. The fix (Lemma 54 v2) does not need κ_{FF}: the form $\omega(x, (F - Ve)y)$ has index k in the trace model by Hermite's theorem.

(C19) *v2 Table 7, DH charge "(-28)".* This is inconsistent with Lemma 54's -56 for the same truncation. L53's charge counts ρ and $\bar{\rho}$, so it is -56 on the full symmetric truncation. v3 uses that convention.

10. Prior art; what is new and what is restatement

- *Classical and cited (no novelty):* Jensen discs (Jensen 1913; Pólya 1943 Thm. I); the window count $J - J'$ (Kim 1996, Thm. 1); Cohn's Fall IV count (1922); the de Bruijn–Newman facts (Λ, Hermite splitting, zero dynamics) [22, 21]; preservation of real-rootedness by e^{-tD^2}, $t \geq 0$ (Laguerre–Pólya theory). The Fourier–Pólya and Pólya–Wiman circle of results [4, 5, 13] concerns all derivatives $f^{(n)}$, not the single ξ' used here.
- *Restatements:* Cor. 4.4(b),(c) and Thm. 4.10 (RH $\iff$ (H)) [RESTATEMENT]. Related equivalences: Lagarias 1999, Sondow–Dumitrescu 2010, Spira 1973 (records; Rem. 4.11).
- *New here, to our knowledge (searches not exhaustive):* the official orientation and counter notation; Thm. 4.2 as an exact formula for $N_{\xi'}(T)$ at admissible heights; the zero-naive admissibility condition; Prop. 4.5 and Remark 4.6; (v2) Thm. 4.8 in this form, Thm. 4.10 (likely known; not found stated), Thm. 4.13 (numbers), the DH heat flow H_t^{DH} and its landing times, and Table 7; Cor. 3.2; the circle identity Thm. 6.2 with the genus-2 criterion (an elementary analogue of Kim's theorem); the normal-form residue $1/(t_c - t)$ and the two-event picture for H_t; all numerics. Each of these is elementary given the cited results. We claim presentation and computation, not depth.

11. Questions for J.W.

v2's questions and J.W.'s answers ([t218u]): Q1 yes (Def. 5.1); Q2 explanation only (Rem. 5.2); Q3 yes (Vol. 3 addendum, test build); Q4 yes (§7.6); Q5 yes (Lemma 54 v2).

Question 11.1 (Q1)**.** DH has no zero with $\beta \geq 1$ below 10^4, but it has infinitely many higher up. Should we search for the first one, beyond 10^4, and follow its landing? Its t_c could exceed 0.5 and would sharpen Λ_{DH} considerably. Or is the certified $0.230607 \leq \Lambda_{\mathrm{DH}} \leq 1.6025$ enough for TAA?

Question 11.2 (Q2)**.** The ratio $t_c/2\delta^2$ falls with height (Table 6). Should we model it, e.g. against the local zero density $\frac{1}{2\pi} \log \frac{5\gamma}{2\pi}$, as a numerical TAA law ([CONJ]), or leave it as an observation?

Question 11.3 (Q3)**.** Certification: should the per-ghost t_c brackets also be certified? That would need a Laguerre test below t_c and an argument-principle test above it, for each of the 28 ghosts. Should the count "exactly 28 ghosts below 1000" be certified with balls as well? Currently both are [NUM].

Question 11.4 (Q4)**.** Ledger: record the FF twin of the saddle form, "$\mathrm{RH}_{\mathbb{F}_q} \iff$ every symmetric saddle lies on the circle and is longitudinal" (Thms. 6.1, 6.2), as an official FF restatement next to (H)?

Question 11.5 (Q5)**.** Volume 3: move the test-built addendum (now FF and ξ-side, with figures) into the replacement queue as the definitive chapter text, to be applied at the next volume revision? J.W.'s approval is needed before anything is applied or submitted.

Status

- RH: **not proved**; not advanced. All RH-equivalences are tagged [RESTATEMENT].
- Proved or cited: Def. 2.1, Lemma 2.2, Thm. 3.1, Cor. 3.2, Thm. 4.2, Thm. 4.3, Cor. 4.4, Prop. 4.5, Thm. 6.1, Thm. 6.2, Prop. 6.3, Thm. 7.1, Cor. 7.2 (conditional on the strip hypothesis for $t < 0$).
- v2 proved: Thm. 4.8, Cor. 4.9, Thm. 4.10 (RH $\iff$ (H), [RESTATEMENT]), Thm. 4.13, Prop. 4.14, Prop. 7.4.
- v3: Def. 5.1 (official, [RESTATEMENT]); Thm. 7.5 and Thm. 7.6 (computer-assisted, Arb balls); the whole-line flow and the large-δ ghosts above 1000 ([NUM]).

- Open: RH (of course); an explicit $c < \frac{1}{2}$ for all large T; the strip property of H_t for $t < 0$ (no reference found); the DH flow above height 1000 in full, and for ghosts with $\beta > 1$, the first of which lies above 10^4.
- Numerics: ζ and DH to 1000 (v2: the global count at all heights; the DH forward flow for 28 ghosts; the barrier in multiprecision); 19 FF cases plus 3000 random; toys and 4000 random; H_t for $z < 128$, $-16 \le t \le 0.3$; polynomial flows.
- Read at the source in v2: Conrey 1989 (BAMS, Thm. 3), Simonič 2020 (arXiv v2: Thm. 1, Cor. 5, §4.2). Records only in v2: Lagarias 1999, Sondow–Dumitrescu 2010, Spira 1973, BCHB 1985, Kadiri–Lumley–Ng 2018.
- v3 sources: Arb [11] (software; record verified); Davenport–Heilbronn's zeros in $\sigma > 1$ via [18] (introduction read for Lemma 53). Bombieri–Ghosh (2011) was *not* read (download failed), and no value from it is used.
- Not read (records only): Pólya 1943, Jensen 1913, Craven–Csordas–Smith 1987, Csordas–Smith–Varga 1994, Csordas–Varga 1990, de Bruijn 1950, Newman 1976. Kim 1996 and Ki–Kim 2000 were read only in excerpts.

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TAA LEMMA 57: THE WEIL POSITIVITY LADDER. WHERE DOES THE POLARIZATION COME FROM?

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. **Statement of intellectual honesty.** RH is not proved here, and nothing below brings it closer. Weil's criterion (RH $\iff$ the Weil functional is positive) is in the literature (Weil 1952, Yoshida 1992, Bombieri 2000) and is tagged [RESTATEMENT]. Tags: [THM] proved here (or classical with proof given); [THM, computer-assisted] a finite rigorous computation with ball arithmetic (Arb/`python-flint`) whose correctness is conditional on the correctness of the implementation, which was validated independently against zero sums; [CITED] checked at the source (`lit/lemma57/verified_refs.md`); [BIB] bibliographic; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical, with [NUM, certified] (rigorous enclosure) or [NUM, estimated] (floating point, not certified); [CROSS-CHECK, uses zeros] an after-the-fact check that uses zeros; [CONJ] conjecture; [OPEN] open.

Question (J.W., [t208u]–[t209u]). Do steps 1–3 (a certified prime-side Weil ladder; the Davenport–Heilbronn (DH) comparison; attribution of positivity to primes, ground states and spectral flow), *but only if* this can elucidate the polarization and is not merely equivalent to zero counting. Lemma 58 ranked this as target #1.

Answer in brief. (1) *Ladder.* Two-sided certified enclosures of the lowest Weil eigenvalue $\lambda^*(e^A)$ for ζ on windows $A = \log\lambda^2$, computed from primes and the archimedean term only. New here are the rigorous upper bounds (Thm. 3.1): $\lambda^*(e^{1.8}) \in [3.5600, 4.2973] \cdot 10^{-23}$, $\lambda^*(e^2) \in [4.9946, 6.2683] \cdot 10^{-30}$, $\lambda^*(e^{2.2}) \le 2.4731 \cdot 10^{-38}$, $\lambda^*(e^{2.4}) \le 6.9484 \cdot 10^{-48}$, $\lambda^*(e^{3.5}) \le 4.4461 \cdot 10^{-95}$ [THM, computer-assisted]. The lower bounds are Lemma 24's, with a new run at $A = 2.1$ still in progress. A prime-only lower bound at window A needs a tail height $T^\star(A) \approx 2\pi e^{\mathcal{M}(A)}$, which grows doubly exponentially (Prop. 2.1). This is the same wall as zero verification. (2) *DH.* The DH Weil form is indefinite for every $A \ge 3.43$, i.e. $\lambda^2 \ge 30.88$ (Thm. 4.1) [THM, computer-assisted]. No negative direction was found for $A \le 3.42$ ($\lambda^2 \le 30.57$) [NUM, estimated]. The first ghost $0.8085 + 85.6993i$ drives the crossing [CROSS-CHECK, uses zeros]. The chat value "$\lambda^2 \approx 60$" is corrected to ≈ 31. (3) *Attribution.* Every local piece of the ground-state energy is $O(1)$. Their sum is 10^{-23}–10^{-48}. Positivity has essentially no slack: perturbing the weight of the single term $n = 2$ by a relative amount $1.2 \cdot 10^{-46}$ destroys it at $A = 2.4$ (Thm. 5.1) [THM, computer-assisted]. The "dominant archimedean term + bounded prime corrections" mechanism already fails at $A = 0.8$, the first window containing a prime (Prop. 5.3) [NUM, estimated]. As primes are switched on, the negative index of the partial forms first *rises* (to 13 at $A = 3$), then falls to 0 only when the last prime below e^A is added. Positivity is collective, not cumulative [NUM, estimated]. The ground state is a near-prolate bump, and its Fourier zeros reproduce $\gamma_1, \ldots, \gamma_{18}$ to 10^{-3} at $A = 2$ [CROSS-CHECK, uses zeros]. (4) *Verdict.* J.W.'s condition is *not met* by the ladder. By the explicit formula, W_A *is* the zero measure compressed to the Paley–Wiener space $PW_{A/2}$. Positivity at window A is therefore a resolution-limited statement about zeros. It is weaker than RH to any height, and DH is positive despite its ghost until $\lambda^2 \approx 31$. Its prime-side certificate costs as much as resolving the zeros up to $T^\star(A)$. The zero-independent structure we found is real but negative: a no-slack theorem and a failed-dominance proposition. These rule out a whole class of "local positivity" mechanisms. They do not produce one. Positivity at fixed λ already follows (in principle) from Platt–Trudgian for $\lambda^2 \le 67$ with a finite hybrid computation (Remark 2.2).

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1. Setting, normalizations, and what TAA already has

1.1. **The window form.** Let $A > 0$ and $\lambda = e^{A/2}$. In the log variable u, a test function $F \in L^2(-\frac{A}{2}, \frac{A}{2})$ (real-valued; the complex case decomposes) corresponds to a function on $[\lambda^{-1}, \lambda]$. Put

$$\widehat{F}(z) = \int F(u) e^{izu}\, du, \qquad g(y) = \int F(u+y)F(u)\, du \quad \text{(even, supported in } [-A, A]).$$

For ζ the Weil form on the window is

$$W_A(F) = \underbrace{4\int_0^A g(y)\cosh\tfrac{y}{2}\, dy}_{\text{POLE} = 2\widehat{F}(\frac{i}{2})\widehat{F}(-\frac{i}{2})} + \underbrace{\frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{F}(t)|^2 \Big(\operatorname{Re}\psi(\tfrac{1}{4}+\tfrac{it}{2}) - \log\pi\Big) dt}_{\text{ARCH} = W_\infty} - \sum_{n<e^A} \underbrace{\frac{2\Lambda(n)}{\sqrt{n}}\, g(\log n)}_{W_n}. \tag{1}$$

In the code, ARCH is evaluated in the equivalent real-variable form $g(0)(\psi(\frac{1}{4}) - \log\pi) + \int_0^A 2e^{-y/2}\frac{g(0)-g(y)}{1-e^{-2y}}dy + g(0)\sum_{k\geq 0}\frac{2e^{-(2k+1/2)A}}{2k+1/2}$. The last series (the part of the integral beyond $y = A$) is summed rigorously. This is Lemma 24's `PrimeWeight` normalization. For a prime p write $W_p = \sum_k W_{p^k}$.

For DH, $f_{\mathrm{DH}} = \frac{1}{2}(1 - i\kappa)L(s,\chi) + \frac{1}{2}(1 + i\kappa)L(s,\bar{\chi})$ (Lemma 48 §5) has no pole, gamma factor $\Gamma(\frac{s+1}{2})$ and conductor 5. Equation (1) holds with POLE = 0, with $\psi(\frac{3}{4}+\frac{it}{2}) + \log\frac{5}{\pi}$ in ARCH, and with $\Lambda(n)$ replaced by the coefficients $c(n)$ of $-f'_{\mathrm{DH}}/f_{\mathrm{DH}} = \sum c(n)n^{-s}$. These are computed exactly from the Dirichlet recursion with $a(n) = [0, 1, \kappa, -\kappa, -1]_{n \bmod 5}$. They do not vanish at composite n (there is no Euler product).

Proposition 1.1 (explicit formula; [CITED] Weil 1952 / Bombieri 2000, normalization [NUM, certified]-validated)**.** *For F as above,*

$$W_A(F) = \sum_\rho \widehat{F}(\gamma_\rho)\widehat{F}(-\gamma_\rho), \qquad \rho = \tfrac{1}{2} + i\gamma_\rho,$$

summed symmetrically over the nontrivial zeros. If ρ is on the line, its term is $|\widehat{F}(\gamma_\rho)|^2$. An off-line quadruple contributes $4\,\mathrm{Re}[\widehat{F}(\gamma-i\delta)\overline{\widehat{F}(\gamma+i\delta)}]$, which can be negative. The same holds for DH with its zeros.

Our implementation of the right-hand side of (1) was checked against zero sums for the smooth test functions $(1-x^2)^m$. The agreement is $\sim 10^{-20}$ for ζ and $\sim 10^{-15}$ for DH, limited by the zero lists (`num/l57/validate.json`) [NUM].

Definition 1.2. $\lambda^*(e^A) = \inf\{W_A(F)/\|F\|^2\}$. This is Volume 2's $\lambda^*(d)$ with $d = \lambda^2 = e^A$.

Lemma 1.3 (monotonicity; [THM])**.** *$A \mapsto \lambda^*(e^A)$ is non-increasing.*

Proof. g is unchanged when F is translated. So a function supported in a sub-interval of length A of $[-\frac{A'}{2}, \frac{A'}{2}]$ has $W_{A'}(F) = W_A(F_{\text{centered}})$. Taking infima gives the claim. □

Corollary 1.4 ([RESTATEMENT])**.** *RH $\iff$ $\lambda^*(e^A) \geq 0$ for all A (Weil's criterion restricted to L^2 windows; Yoshida, Bombieri [CITED]). Under RH, $W_A \geq 0$ for each A is immediate from Prop. 1.1. For a* fixed *A, positivity is strictly weaker than RH: see DH, §4.*

1.2. **Engines.**

- *Galerkin/Ritz* in the orthonormal Legendre basis $\varphi_j(u) = \sqrt{(2j+1)/A}\,P_j(2u/A)$. The matrix entries are exact polynomial integrals for POLE and W_n. ARCH uses rigorous quadrature in Arb. Eigenvalues are computed in mpmath at 256–520 bits (`num/lemma57_leg.py`, `lemma57_struct.py`). A Ritz value is an upper bound on λ^* up to floating-point error [NUM, estimated].
- *Rigorous Rayleigh quotient* (`lemma57_rq.py`). The Ritz vector is rounded to rationals (40–60 digits). Then g is an exact rational polynomial in y on $[0, A]$. Every term of (1) is enclosed in Arb: exact polynomial moments, `acb.integral` for the archimedean kernel $z/(1-e^{-z})$, $\log n$ and $\sqrt{n}$ as balls, and the tail series summed with a rigorous geometric remainder. The output is a ball for $W_A(F)/\|F\|^2$, hence a *certified upper bound* on $\lambda^*(e^A)$, or certified negativity [THM, computer-assisted].
- *Negative-direction search* (`lemma57_dhhp.py`). The matrix is built in Arb at 240–560 bits. An LDL^T factorization is done in mpmath. Each negative pivot d_k gives $x = L^{-\mathsf{T}}e_k$ with $x^\mathsf{T}Mx = d_k$, which is then certified by the rigorous RQ.
- *Certified lower bounds* are those of Lemma 24 (spline basis, rigorous high-frequency tail β, Sylvester inertia), run prime-only. The new driver `num/lemma57_runpri.py` is the prime-only branch of Lemma 24's `runhyb2` with the β-cache read removed, because the cache stores a 30-digit string that can round up.

1.3. **Overlap with earlier lemmas.** Lemma 22 (Connes–Consani ζ-cycles), Lemma 24 (certified prime-only and hybrid lower bounds, Volume 2), Lemma 52 §5 (mp Weil windows: ζ positive to $\lambda^2 = 13$; DH positive at 20, indefinite at 60; Fourier zeros of the lowest vector $\approx \gamma_k$, the CCM-ZST observation), Lemma 54 Prop. 7.1 (negative index $= 2 \cdot$ #quadruples in a symplectic model). Nothing there is redone. The DH bracket below answers Lemma 52 Q9.4.

2. What is and is not new: the cost wall

Proposition 2.1 (prime-only cost wall; [THM] for the criterion, [NUM] for the table)**.** *Let $\mathcal{M}(A) = \sum_{n<e^A} 2\Lambda(n)n^{-1/2}$. Since $|g(y)| \leq \|F\|^2$, the prime part of (1) is bounded by $\mathcal{M}(A)\|F\|^2$. In Lemma 24's method the frequencies $|t| > T_c$ are controlled by the archimedean weight alone. This requires*

$$\beta(T_c) = \mathrm{Re}\,\psi\big(\tfrac{1}{4} + \tfrac{iT_c}{2}\big) - \log\pi - \mathcal{M}(A) > 0, \qquad \textit{i.e. } T_c > T^\star(A) \approx 2\pi e^{\mathcal{M}(A)}.$$

The finite-dimensional part then has dimension $\asymp A\,T_c$.

A	2.0	2.1	2.2	2.4	2.6	3.0	3.4	3.8	4.2	4.3
$T^\star$	2187	3571	7427	$3.15{\cdot}10^4$	$1.3{\cdot}10^5$	$2.8{\cdot}10^6$	$1.06{\cdot}10^8$	$1.5{\cdot}10^{10}$	$2.41{\cdot}10^{12}$	$5.05{\cdot}10^{13}$

Here $\mathcal{M}(A) \sim 4e^{A/2}$ by the prime number theorem, so $T^\star$ grows doubly exponentially in A [THM].

Remark 2.2 (Platt–Trudgian; [CITED] for the height, [CONJ] for the execution)**.** RH is verified up to $H = 3.0001753328 \cdot 10^{12}$ [8]. Since $\mathcal{M}$ is constant for $\log 64 < A \le \log 67$ and $T^\star(4.2) = 2.41 \cdot 10^{12} < H$, while adding $n = 67$ multiplies $T^\star$ by $e^{2\log 67/\sqrt{67}} \approx 2.8$ (to $\approx 6.7 \cdot 10^{12} > H$), Lemma 24's *hybrid* method could in principle certify $W_A \ge 0$ for every $A \le \log 67$ ($\lambda^2 \le 67$). It would use verified zeros for $|t| \le H$ and the archimedean tail above. We did not execute this, and the matrix size ($\asymp 10^{12}$) makes it impractical as stated. The point is logical: positivity at a fixed moderate λ is *implied* by published zero verification. A certified window is therefore a genuine theorem, but it is not new information about RH. For very small supports it is Yoshida's theorem [2, 3]. *The value of the lemma must come from structure.*

Remark 2.3 (resolution, not height; [THM] + [ANALOGY])**.** By Prop. 1.1, $W_A \ge 0$ says that the zero measure is a positive functional on $\{|\widehat{F}|^2 : \widehat{F} \in PW_{A/2}\}$. A ghost at height γ with offset δ is detected only when some F supported in a window of length A makes $\mathrm{Re}[\widehat{F}(\gamma - i\delta)\overline{\widehat{F}(\gamma + i\delta)}]$ beat the on-line mass near γ, at resolution $\sim 2\pi/A$. So window-A positivity is a statement about zeros at *all heights* but at *finite resolution*. It is neither "RH up to height $T(A)$" nor implied by it [ANALOGY]. DH (§4) shows the gap concretely.

3. Step 1: the certified ladder for ζ

Theorem 3.1 (certified upper bounds; [THM, computer-assisted])**.** *For the explicit rational test functions stored in* `num/l57/rq.jsonl` *and* `dhhp.jsonl`*, the Rayleigh quotients enclose as follows:*

A	K	$W_A(F)/\|F\|^2$ *(Arb ball)*	*hence* $\lambda^*(e^A) \le$
9/5	*90*	$[4.2972356581585233 \cdot 10^{-23} \pm 4.7 \cdot 10^{-43}]$	$4.2972357 \cdot 10^{-23}$
2	*100*	$[6.2682798261702295 \cdot 10^{-30} \pm 2.2 \cdot 10^{-51}]$	$6.2682799 \cdot 10^{-30}$
11/5	*115*	$[2.4730976104004096 \cdot 10^{-38} \pm 2.9 \cdot 10^{-58}]$	$2.4730977 \cdot 10^{-38}$
12/5	*130*	$[6.9483957778136649 \cdot 10^{-48} \pm 3.7 \cdot 10^{-68}]$	$6.9483958 \cdot 10^{-48}$
7/2	*220*	$[4.4460070362599392 \cdot 10^{-95} \pm 3.7 \cdot 10^{-115}]$	$4.4460071 \cdot 10^{-95}$

Since these are values of the form, they are upper bounds and do not show positivity. At $A = 7/2$ the test vector came from an LDL factorization at 320 bits whose four "negative pivots" were precision artifacts. The rigorous RQ of the corresponding vector is positive*, so it gives an upper bound only.*

Remark 3.2. At $A = 1.8$ and 2.0 the enclosures are tight to a factor 1.21 and 1.26, so λ^* is known to within about 25%. The Ritz values at $K \le 130$ are not converged for $A \ge 2.4$. Connes–Consani report $2.389 \cdot 10^{-48}$ at $\lambda^2 = 11$ (via Lemma 22; comparison only, the normalizations are not re-verified). This is consistent with our weaker upper bound $6.95 \cdot 10^{-48}$ at $\lambda^2 = e^{2.4}$, which satisfies $\lambda^*(e^{2.4}) \le \lambda^*(11)$ by Lemma 1.3. Figure 1 summarizes the ladder.

4. Step 2: Davenport–Heilbronn, the first negative window

Theorem 4.1 (DH indefiniteness; [THM, computer-assisted])**.** *The DH Weil form W_A^{DH} is indefinite for every $A \ge \frac{343}{100}$, i.e. for every window $\lambda^2 \ge e^{3.43} = 30.877$. Explicit rational test functions (*`num/l57/dhhp.jsonl`*,* `rq.jsonl`*) give the rigorous Rayleigh quotients*

TABLE 1. The ζ ladder. "cert. lower" is from Lemma 24 (prime-only, β-tail, inertia; `num/lemma24_progress.md`). "cert. upper" is from Thm. 3.1. Ritz values are [NUM, estimated] (K Legendre modes, mp eigensolver).

A	λ^2	primes	cert. lower	cert. upper	Ritz (K)	status
0.6	1.82	–	–	–	$7.572 \cdot 10^{-3}$ (40)	[NUM, estimated]
1.0	2.72	2	–	–	$9.360 \cdot 10^{-7}$ (50)	[NUM, estimated]
1.4	4.06	2,3	–	–	$4.268 \cdot 10^{-13}$ (60)	[NUM, estimated]
1.5	4.48	2,3	$3.2380 \cdot 10^{-15}$	–		L24 [NUM, certified]
1.55	4.71	2,3	$2.2870 \cdot 10^{-16}$	–		L24 [NUM, certified]
1.6	4.95	2,3	$1.4646 \cdot 10^{-17}$	–	$1.679 \cdot 10^{-17}$ (80)	L24 [NUM, certified]
1.8	6.05	2,3,5	$3.5600 \cdot 10^{-23}$	$4.2973 \cdot 10^{-23}$		two-sided [NUM, certified]
2.0	7.39	2–7	$4.9947 \cdot 10^{-30}$	$6.2683 \cdot 10^{-30}$		two-sided [NUM, certified]
2.1	8.17	2–7	*running*	–		pri21, §10
2.2	9.03	2–7	*queued*	$2.4731 \cdot 10^{-38}$		pri22
2.4	11.0	2–11	(L24 hybrid hp running)	$6.9484 \cdot 10^{-48}$		prime-only infeasible
3.5	33.1	2–31	–	$4.4461 \cdot 10^{-95}$		upper only

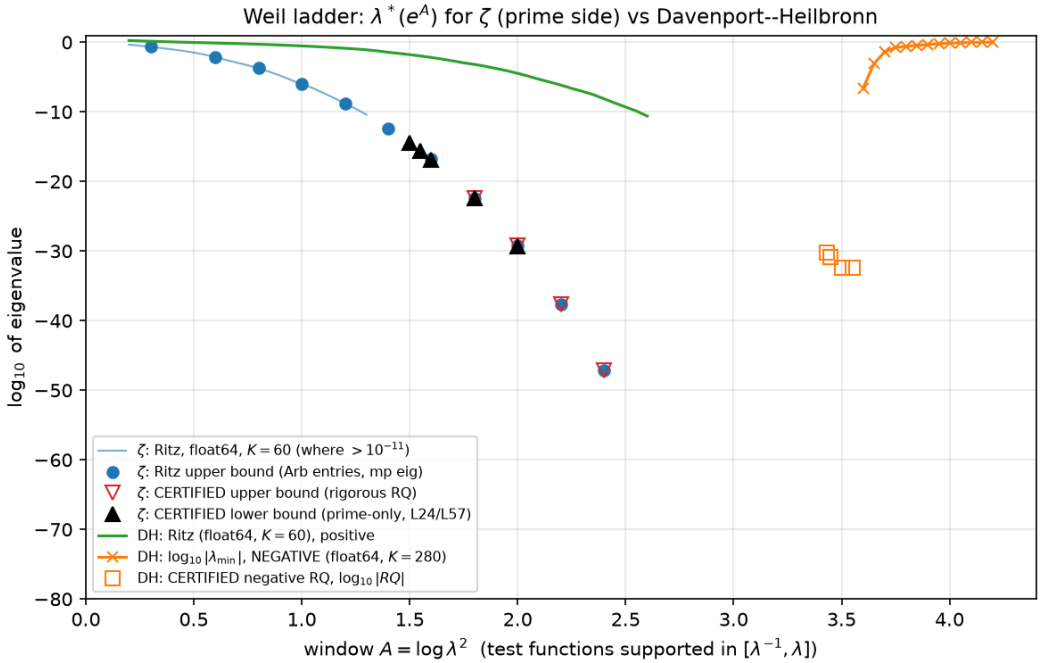

FIGURE 1. $\log_{10}$ of the lowest Weil eigenvalue: ζ (certified lower and upper bounds, Ritz) versus DH (positive Ritz at $K = 60$; negative values at $K = 280$; certified negative RQs). DH's positive branch is *not* converged at $K = 60$ (the ghost at $t = 85.7$ needs $K \gtrsim 43A$), which is why it looks positive beyond 3.43.

A	λ^2	K, *bits*	*neg. pivots*	*certified RQ*
343/100	*30.88*	*220, 280*	*1*	$[-6.4159671850 \cdot 10^{-31} \pm 3.5 \cdot 10^{-51}]$
69/20	*31.50*	*220, 240*	*2*	$[-1.3035539328 \cdot 10^{-31} \pm 2.4 \cdot 10^{-52}]$
7/2	*33.12*	*220, 240*	*2*	$[-4.3253242740 \cdot 10^{-33} \pm 2.0 \cdot 10^{-53}]$
71/20	*34.81*	*220, 240*	*2*	$[-3.5673578521 \cdot 10^{-33} \pm 1.8 \cdot 10^{-53}]$
37/10	*40.45*	*200 (Ritz)*	*2*	$[-0.035838843755 \pm 3.6 \cdot 10^{-23}]$

By Lemma 1.3, $\lambda^*_{\mathrm{DH}}(e^A) \le -6.41 \cdot 10^{-31}$ *for all* $A \ge 3.43$*. The smaller magnitudes at* $A = 3.5$ *and* 3.55 *only mean that the LDL vectors there are far from the eigenvectors.*

Proposition 4.2 (lower side of the bracket; [NUM, estimated])**.** *The LDL factorization finds no negative pivot at* $A = 2.7, 2.8, \ldots, 3.4$ *(*$K = 220$*, 240–320 bits), at* $A = 3.41, 3.42$ *(*$K = 220$*, 280 bits), or at* $A = 3.40$ *with* $K = 300$*. So the first negative window is numerically* $A_c^{\mathrm{DH}} \in (3.42, 3.43]$*, i.e.* $\lambda_c^2 \in (30.57, 30.88]$*. The upper end is certified. The lower end is* not*: a positive Galerkin matrix does not prove positivity, and the* ζ *control at* $A = 3.5$ *shows that 320-bit pivots can be wrong at the* 10^{-95} *level. Repeating at 560 bits (*$K = 220$*) again finds no negative pivot at* $A = 3.2$*,* 3.3 *and* 3.4*. The same 560-bit run for* ζ *at* $A = 3.5$ *finds no negative pivot, confirming that the four 320-bit pivots were artifacts.*

Proposition 4.3 (the crossing is the first ghost; [CROSS-CHECK, uses zeros])**.** *At* $A = 3.7$ *the certified negative function* F *was split by Prop. 1.1 using the DH zeros to height* 1000 *from Lemma 48. On-line zeros contribute* $+0.5967$*. The ghost quadruple at* $0.80852 + 85.69935i$ *(agreeing with Spira* [11]*) contributes* -0.6385*. All other ghosts together contribute* $< 2 \cdot 10^{-4}$ *in absolute value. The zero-side total* -0.0386 *matches the prime-side value* -0.0358 *up to the truncation of the zero list. Two negative directions appear from* $A \approx 3.45$ *(one at* 3.43*). This is consistent with Bombieri's count of* $\frac{1}{2}\#\{$*off-line zeros*$\}$ *negative eigenvalues,* $= 2$ *for one quadruple* [3]*, and with Lemma 54 Prop. 7.1. The analogy is only partial [ANALOGY]: DH has infinitely many ghosts, and Bombieri's statement is for finitely many.*

Remark 4.4 (comparison with ζ at the same windows)**.** For ζ, the same machinery gives no negative pivot at $A = 3.0$ ($K = 220$, 320 bits) and a certified *positive* upper bound $4.45 \cdot 10^{-95}$ at $A = 3.5$. So near $A = 3.5$ the certified scales are $\lambda^*_{\mathrm{DH}} \le -6.4 \cdot 10^{-31}$ versus $0 < \lambda^*_\zeta \le 4.4 \cdot 10^{-95}$ (positivity of λ^*_ζ there is not certified; it follows from RH). The DH low spectrum lives on a vastly larger scale than ζ's at the same window [NUM, estimated]. The two forms are not small perturbations of each other: DH has a different gamma factor ($\psi(\frac{3}{4} + \cdot)$, no pole) and conductor 5. This matters for the attribution question (Correction 2).

5. Step 3: attribution

5.1. **Energy budget of the ground state.** Let F_0 be the normalized Ritz ground state and $\langle W_v \rangle = W_v(F_0)$ (`num/l57/struct_zeta_*.json`, cross-checked in Arb by `rq.jsonl`) [NUM, estimated]/[NUM, certified]:

A	ARCH	POLE	W_2	W_3	W_5	W_7	W_{11}	λ^* (Ritz)
0.6	-1.0235	1.0311	–					$7.6 \cdot 10^{-3}$
1.0	-1.2558	1.2683	-0.01251					$9.4 \cdot 10^{-7}$
1.4	-1.3496	1.3812	-0.03154	$-2.5 \cdot 10^{-5}$				$4.3 \cdot 10^{-13}$
1.8	-1.4014	1.4470	-0.04546	$-1.7 \cdot 10^{-4}$	$-1.6 \cdot 10^{-12}$			$4.3 \cdot 10^{-23}$
2.0	-1.4188	1.4697	-0.05067	$-2.7 \cdot 10^{-4}$	$-5.5 \cdot 10^{-11}$	$-2.6 \cdot 10^{-23}$		$6.3 \cdot 10^{-30}$
2.4	-1.4443	1.5035	-0.05874	$-4.9 \cdot 10^{-4}$	$-1.3 \cdot 10^{-9}$	$-9.7 \cdot 10^{-17}$	$-8.3 \cdot 10^{-48}$	$6.9 \cdot 10^{-48}$

The pole and the archimedean term are $O(1)$ and opposite. $p = 2$ and $p = 3$ absorb the difference. The cancellation ratio $\lambda^* / \sum_v |\langle W_v \rangle|$ is $3.7{\cdot}10^{-3}$, $3.7{\cdot}10^{-7}$, $1.5{\cdot}10^{-13}$, $1.5{\cdot}10^{-23}$, $2.1{\cdot}10^{-30}$, $2.3{\cdot}10^{-48}$ at $A = 0.6, 1.0, 1.4, 1.8, 2.0, 2.4$ [NUM, estimated]. At $A = 2.4$ the edge prime 11 contributes about as much as λ^* itself.

Theorem 5.1 (no slack; [THM, computer-assisted])**.** *Let* $W_A^{(\varepsilon)}$ *be* (1) *with the single term* W_2 *(i.e.* $n = 2$*) multiplied by* $1 + \varepsilon$*. For the certified test functions* F *of Thm. 3.1,* $W_A^{(\varepsilon)}(F) < 0$ *as soon as* $\varepsilon > \varepsilon^*(A)$*, with*

$$\varepsilon^*(1.8) = 9.45 \cdot 10^{-22}, \quad \varepsilon^*(2.0) = 1.24 \cdot 10^{-28}, \quad \varepsilon^*(2.2) = 4.50 \cdot 10^{-37}, \quad \varepsilon^*(2.4) = 1.18 \cdot 10^{-46}.$$

The same holds for multiplying ARCH by $1+\varepsilon$ with $\varepsilon > 4.8 \cdot 10^{-48}$, for multiplying POLE by $1-\varepsilon$ with $\varepsilon > 4.6 \cdot 10^{-48}$, and for W_3 with $\varepsilon > 1.4 \cdot 10^{-44}$ (all at $A = 2.4$).

Proof. $W_A^{(\varepsilon)}(F) = W_A(F) + \varepsilon W_2(F)$, with $W_2(F)/\|F\|^2 \in [-0.0587394373534 \pm 3 \cdot 10^{-14}]$ at $A = 2.4$ (Arb, `rq.jsonl`). Then $\varepsilon^* = \mathrm{RQ}_{\mathrm{upper}}/|W_2|_{\mathrm{lower}}$. The same argument applies to the other pieces and windows. □

Remark 5.2 (what this does and does not say)**.** Algebraically this is trivial: any positive form with a tiny minimum loses positivity under a perturbation of that size. The content is in the numbers. The local data enter at $O(1)$, and the margin is super-exponentially small in A. Positivity at moderate A is an *identity-level cancellation*, not an inequality with room. A proof of $W_A \geq 0$ must use $\Lambda(2) = \log 2$ to ~ 47 significant digits at $A = 2.4$. Any argument insensitive to perturbations of the local weights at the 10^{-46} level cannot prove it.

5.2. **Failure of archimedean dominance.** Let $V_0 = \{F : \widehat{F}(\pm\frac{i}{2}) = 0\}$ be the primitive subspace. This is where POLE vanishes, the analogue of primitive H^1 in Weil's proof for curves (Lemma 58's "polarization" language [ANALOGY]).

Proposition 5.3 ([NUM, estimated])**.** *On V_0 (QR complement of the two constraint vectors, Legendre basis):*

A	$\lambda^*\|_{V_0}$	$\min \mathrm{ARCH}\|_{V_0}$	*#neg ARCH*$\|_{V_0}$	*spec(primes*$\|_{V_0}$*)*	*dominance?*
0.6	*0.692*	*0.692*	*0*	–	*(no primes)*
0.8	*0.225*	*0.403*	*0*	$[-0.490, 0.490]$	*fails* ($0.403 < 0.490$)
1.0	*0.0119*	*0.178*	*0*	$[-0.490, 0.490]$	*fails*
1.2	$6.7 \cdot 10^{-5}$	-0.0071	*1*	$[-0.895, 0.894]$	*ARCH indefinite*
1.6	$5.4 \cdot 10^{-12}$	-0.302	*1*	$[-1.21, 1.02]$	*ARCH indefinite*
2.0	$1.5 \cdot 10^{-23}$	-0.535	*2*	$[-1.65, 1.65]$	*ARCH indefinite*
2.4	$6.4 \cdot 10^{-41}$	-0.732	*3*	$[-2.19, 1.84]$	*ARCH indefinite*

So the bound "$W_\infty \geq c\|F\|^2$ on V_0 and $\|\sum_p W_p\| < c$" fails at $A = 0.8$, the first window containing a prime ($A > \log 2$), and fails at every larger window tested. From $A \approx 1.2$ ($\lambda^2 \approx 3.3$) on, ARCH itself is indefinite on V_0. In the ground state on V_0, $\langle W_2 \rangle$ changes sign (negative at $A \leq 1.8$, positive at 2.0–2.4): primes create positivity in some directions.

Remark 5.4 (Connes–Consani archimedean positivity)**.** CC21 [4] prove an operator-theoretic strengthening of archimedean positivity, $W_\infty(g * g^*) \geq \mathrm{Tr}(\vartheta(g) S \vartheta(g)^*)$, for g supported in $[2^{-1/2}, 2^{1/2}]$ (log-length $\log 2$) with $\hat{g}$ vanishing at $\frac{i}{2}$ and 0 (their Thm. 1; Thm. 6.11 with $-c|\hat{g}(0)|^2$, $c = 4\gamma/\log 2$). They attribute the underlying inequality on $(\frac{1}{2}, 2)$ to Yoshida [CITED]. Our finding that ARCH is positive on V_0 up to $A \approx 1$ and indefinite from $A \approx 1.2$ is consistent with the scope "small windows only" [NUM, estimated]. Their constraint set (also $\hat{g}(0) = 0$) differs from our V_0, so the thresholds are not directly comparable.

5.3. **Switching primes on: the positive cone.** Let $W_A^{(\leq P)}$ be ARCH + POLE + $\sum_{p \leq P} W_p$. Its negative index (eigenvalues $< -10^{-9}$, $K = 60$, float64) is [NUM, estimated]:

A	ARCH alone	ARCH+POLE, then adding $p = 2, 3, 5, \ldots$ in order
1.0	2	$2 \to 0$
1.4	3	$2 \to 4 \to 0$
2.0	4	$3 \to 4 \to 7 \to 3 \to 0$
2.6	5	$4 \to 6 \to 8 \to 11 \to 7 \to 2 \to 0$
3.0	6	$5 \to 7 \to 10 \to 12 \to 13 \to 10 \to 5 \to 3 \to 0$
3.4	7	$6 \to 9 \to 12 \to 12 \to 15 \to 14 \to 11 \to 11 \to 7 \to 2 \to 0$

The negative eigenvalues of the partial forms are $O(0.1\text{–}3)$. For example, at $A = 3.0$, $W^{(\leq 17)}$ has $\lambda_{\min} = -0.226$: omitting $p = 19$ ($\log 19 = 2.944$, close to the edge) leaves three robust negative directions. They are edge-localized, whereas the ground state sits in the middle of the window (Fig. 2).

Proposition 5.5 ([NUM, estimated])**.** *The positive cone does not grow as primes switch on. The small primes* enlarge *the negative index, and it falls to* 0 *only when (essentially) all primes below e^A are present. A prime within $\sim 10^{-3}$ of the edge is invisible at float resolution, e.g.* 11 *at $A = 2.4$. DH behaves the same way, with all n (composite ones included) playing the role of primes. Positivity is a collective property of the complete local data, not a cumulative one.*

Each single term is indefinite: $F \mapsto g(y)$ is the compression of $\frac{1}{2}(T_y + T_{-y})$, whose spectrum is symmetric about 0 to leading order. So $\|W_n\| \leq 2\Lambda(n)n^{-1/2}$ and no W_n is semidefinite [THM].

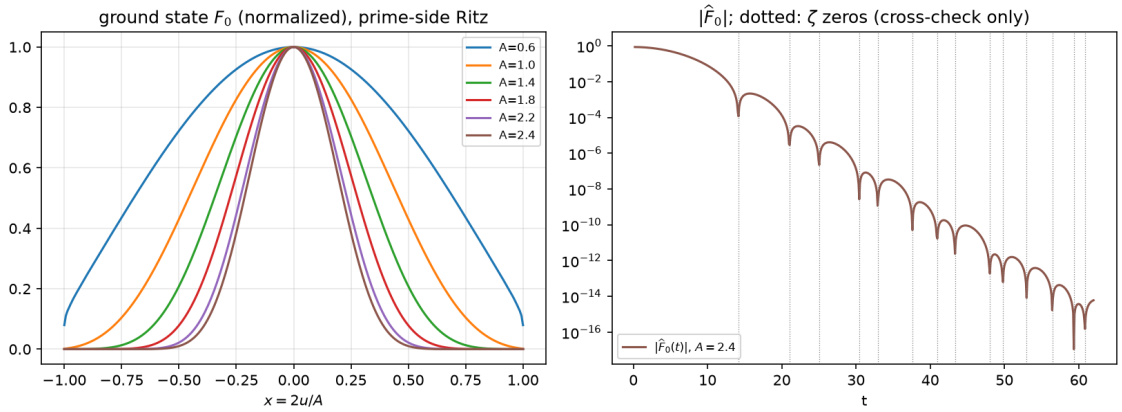

FIGURE 2. Left: prime-side ground states F_0, all even, as bumps narrowing with A. Right: $|\widehat{F}_0|$ at $A = 2.4$, with ζ zeros dotted (cross-check only).

5.4. **Ground-state shape, prolate comparison, and the zeros.**

Proposition 5.6 ([NUM, estimated], [CROSS-CHECK, uses zeros] for the zero comparison)**.** *(a) F_0 is even at every A tested. Its L^2 mass in $|x| > 0.8$ ($x = 2u/A$) is $2.5 \cdot 10^{-2}$, $5.3 \cdot 10^{-4}$, $2.3 \cdot 10^{-6}$, $1.1 \cdot 10^{-9}$, $8.0 \cdot 10^{-12}$ at $A = 0.6, 1.0, 1.4, 1.8, 2.0$. The ground state does not use the edges of the window. (b) Its overlap with the best prolate spheroidal function $\psi_0(c, \cdot)$ [9] is* 0.9997–0.99992, *with best $c = 3.5, 7.25, 12.5, 19.0, 22.75$ ($c/A \approx$ 6–11). The overlap saturates at $1 - 8 \cdot 10^{-5}$ and does* not *tend to* 1. *F_0 is prolate-like but not prolate, and the difference carries the arithmetic. (c) The real zeros of $\widehat{F}_0$ match $\gamma_1, \gamma_2, \ldots$ (to 10^{-3}) for the first $0, 1, 2, 4, 7, 12, 18, 26$ zeros at $A = 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 2.2$. At $A = 1.8$ the first three match to $2 \cdot 10^{-14}$. This is the CCM-ZST phenomenon* [6]*, reproduced in Lemma 52 for $\lambda^2 \leq 13$.*

Remark 5.7 (this is the zero counting)**.** (c) explains the no-slack theorem. $W_A(F_0) = \sum_\rho |\widehat{F}_0(\gamma_\rho)|^2$ is tiny because the variational problem drives $\widehat{F}_0$ to vanish at the zeros it can resolve. The ground state is a prime-side *zero detector*. Prolate-type stabilization (Connes–Moscovici [7], [BIB]) is the right language for the bulk shape (b), but the arithmetic content lives in (c), which is zeros.

5.5. **Spectral flow.** The low eigenvalues decrease monotonically in A (Lemma 1.3 for λ^*; numerically for the next ones). They form a geometric-looking ladder, e.g. $6.3 \cdot 10^{-30}, 1.7 \cdot 10^{-26}, 2.4 \cdot 10^{-23}, 2.3 \cdot 10^{-20}$ at $A = 2$. The number of eigenvalues below any fixed η grows with A, and no

eigenvalue crosses zero for ζ in the tested range [NUM, estimated]. For DH one eigenvalue crosses at $A_c \in (3.42, 3.43]$ and a second at ≈ 3.45 (Prop. 4.2). The sign changes are spectral flow in the sense of Lemma 54 §7 (a pair of directions for one quadruple) [ANALOGY].

6. Relation to Lemmas 50–54 and to G5

- *Lemma 54 (J_F and Krein signature).* For DH, the negative index of W_A^{DH} is 2 for $A \gtrsim 3.45$, matching Lemma 54 Prop. 7.1's $n_- = 2 \cdot \#\text{quadruples}$ for the one quadruple visible at this resolution [NUM]/[ANALOGY]. For ζ the ladder supplies no flow and no symplectic form on the window space, so Lemma 54's canonical J_F has nothing to act on. What the ladder supplies is a positive form *without* a complex structure. The "polarization" in Weil's curve proof is the pairing of this form with a correspondence action, and the window form alone does not carry one [OPEN].
- *Lemmas 50–52 (Siegel space, vacua).* Lemma 52 showed that scaling acts as a boost on the CCM vacuum. Here, scaling changes the window and the form only through Lemma 1.3. The ground states for different A are different functions (Fig. 2) and do not converge to an invariant vacuum [NUM, estimated]. This is consistent with Lemma 52's verdict.
- *G5.* The ladder certifies instances of the G5-type positivity at fixed windows. Each instance is implied by finite zero verification (Remark 2.2). Theorem 5.1 and Prop. 5.3 show that no "local positive part plus small correction" decomposition of W_A can prove G5 on growing windows. The polarization, if it exists, is not visible as a positive *piece* of W_A [THM]/[NUM, estimated].

7. Verdict on J.W.'s condition

(1) *Prime side only: satisfied.* All certificates (Thm. 3.1, 4.1, 5.1, Lemma 24's lower bounds) use primes, prime powers, the gamma factor and the pole. Zeros enter only in [CROSS-CHECK, uses zeros] checks.

(2) *Not equivalent to zero counting: not satisfied in substance.* Prop. 1.1 makes W_A the zero measure tested on $|PW_{A/2}|^2$. Prop. 2.1 makes the prime-side cost equal to resolving the spectrum to $T^\star(A)$. Prop. 5.6(c) shows that the extremal function literally locates the zeros. Positivity at window A is zero counting at resolution $2\pi/A$. It is weaker than height-limited RH, as DH shows (positive with a ghost at 85.7 until $\lambda^2 \approx 31$).

(3) *Fixed λ from Platt–Trudgian.* Yes, in principle, for $\lambda^2 \le 67$ (Remark 2.2). The bare ladder adds certified numbers, not new mathematics about RH.

(4) *Structure: real, but negative.* Zero-independent structural facts: (i) no slack (Thm. 5.1); (ii) failure of archimedean dominance from the first prime on (Prop. 5.3); (iii) non-monotone, collective positive cone; (iv) pole/archimedean balance in the ground state. These rule out a "positive archimedean part + bounded prime correction" mechanism, and any mechanism that is stable under small changes of local weights. They do not identify a source of positivity.

(5) *Recommendation.* Stop extending the bare ladder beyond $A \approx 2.2$ except to supply Volume numbers. If a polarization exists, it must come from a structure where positivity has *slack* (CC21's trace-class lower bound is the only verified candidate with slack, and it is archimedean and small-window), or from an extra structure on the window space (a correspondence or flow) that the form alone does not have [OPEN].

8. Corrections to the chat claims

(1) *"Pushing certified Weil positivity to larger windows would be genuine theorems, not restatements."* Each certified window is a genuine (computer-assisted) theorem and not a restatement of RH. But each is implied by RH, and for $\lambda^2 \le 67$ in principle by Platt–Trudgian plus a finite hybrid computation (Remark 2.2). Small windows are Yoshida's theorem. They are theorems that carry no new information about RH. The new information is structural (§5).

(2) *"DH goes negative by $\lambda^2 \approx 60$, so below that the Euler product must start doing work."* DH is certified indefinite already at $\lambda^2 = 30.88$ (Thm. 4.1), and numerically positive at $\lambda^2 \leq 30.57$. Lemma 52 only tested 20 and 60. The inference is a non sequitur. DH and ζ differ in gamma factor, pole, conductor and coefficients, so there is no "same form minus Euler product" to compare. The DH crossing is caused by its first ghost (Prop. 4.3). ζ's positivity at the same windows is the absence of ζ-ghosts at that resolution, not a localized prime effect (§5).

(3) *"Connes–Consani proved archimedean positivity for small windows."* CC21 attribute the archimedean inequality on $(\frac{1}{2}, 2)$ to Yoshida [their ref. 34] and prove an operator-theoretic strengthening with vanishing at $\frac{i}{2}$ and 0 (their Thms. 1, 6.11) [CITED]. Correct statement: Yoshida proved it, and CC sharpened it.

(4) *"Each fixed λ is a finite checkable statement."* Only in principle. The prime side is a finite sum, but the form lives on an infinite-dimensional space. A certificate needs a rigorous high-frequency tail, which needs $T_c > T^\star(A) \approx 2\pi e^{\mathcal{M}(A)}$, doubly exponential in A (Prop. 2.1). With the present code, prime-only runs are feasible to $A \approx 2.2$; Lemma 24's zero-assisted hybrid at $A = 2.4$ failed once and is being retried.

(5) *"RH $\iff$ positivity for all λ."* Correct (Weil; Yoshida; Bombieri) [RESTATEMENT]. Fine print: for a *fixed* λ, positivity is strictly weaker (DH: positive up to $\lambda^2 \approx 30.6$ although a ghost exists).

9. Prior art; what is new

Prior art [CITED]/[BIB]: Weil's explicit formula and criterion [1]; small-support positivity [2]; the attained minimum, Yoshida's result reproved, and the negative-index count [3]; archimedean positivity with trace-class bound [4]; the zero-detecting lowest eigenvector [6], and the $\lambda^2 = 11$ value [5]; prolate functions [9] and the prolate/zeros picture [7]; DH [10], Spira's first off-line zero [11]; RH to $3 \cdot 10^{12}$ [8]. New here (to our knowledge, not searched exhaustively): the certified *upper* bounds and two-sided enclosures (Thm. 3.1); the certified DH first-negative window ≤ 30.88 with numerical bracket (Thm. 4.1); the no-slack constants $\varepsilon^*(A)$; the dominance-failure table; the partial-sum negative-index profiles. None of these is a step towards RH.

10. Questions for J.W.

Q1. Do you accept the verdict that the bare ladder fails your condition, and should it be frozen at $A \leq 2.2$ (finishing pri21/pri22 for Volume 2 only)?

Q2. Should the no-slack theorem be developed into a general statement, $\varepsilon^*(A) \lesssim \lambda^*(e^A)$ for every local weight, as an obstruction lemma for "local positivity" proofs of G5?

Q3. Should we test CC21's trace-class bound semilocally (does $\mathrm{Tr}(\vartheta S \vartheta^*)$-type slack survive adding $p = 2$ on V_0?), as the one verified source of slack?

Q4. Should the DH crossing $\lambda_c^2 \in (30.57, 30.88]$ be used to calibrate Lemma 54's index detector (negative index $1 \to 2$ at $A = 3.43 \to 3.45$)?

Q5. May the two-sided enclosures and the DH certificate go into a Volume 2 revision? Nothing is submitted anywhere without your approval.

Status

Certified so far: Thm. 3.1 (5 windows), Thm. 4.1 (5 windows), Thm. 5.1, and Lemma 24's lower bounds. Still running when this version was compiled: the prime-only lower bound at $A = 2.1$ (`num/l57/ladder.status`, `run_pri21.log`; $A = 2.2$ queued after it), (the DH and ζ high-precision checks in `num/l57/dhhp.status` have finished). Their results go to `num/l57/ladder.jsonl` and `dhhp.jsonl`, and will be recorded in `lemma57_progress.md` and STATE §66. Partial-sum and pieces figures: `build57/fig_partial.png`, `fig_pieces.png`. RH is not proved.

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TAA LEMMA 58: INVENTORY AND SYNTHESIS. WHAT THE RESTATEMENTS ARE CIRCLING: POLARIZATION, COMPACT REAL FORMS, THE THERMAL ARROW, AND WHAT THE MISSING OBJECT CANNOT BE

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Date: October 22, 2026.

ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. This lemma is a synthesis: it proves only elementary statements, and every RH-equivalent statement in it is a [RESTATEMENT] of results in the literature or in earlier TAA lemmas. Its value, if any, is organizational: it says precisely which structure the earlier restatements share, which known mathematics already carries that structure, and which realizations of the missing object are excluded. Tags: [THM] proved here (elementary) or classical with proof given; [CITED] checked at the source (`lit/lemma58/verified_refs.md`); [CITED-sec] checked only via a named secondary source; [BIB] bibliographic record verified, content standard; [TAA-L] a result of an earlier TAA lemma, cited by number and not redone; [RESTATEMENT] restatement (equivalent to RH or to a known statement); [ANALOGY] analogy only; [METAPHOR] metaphor, not mathematics; [HEUR] heuristic; [CONJ] conjecture; [OPEN] open; [NUM] numerical (none new here).

Question (J.W., [t210u]). "All of these concepts we have been talking about—positivity, rotations, boosts—must be circling around a few very precise concepts that we have yet to identify... akin to a Grothendieck motive, infinite topos and homotopy type theory... each of our restatements is iteratively refining what these concepts could be and what they could not be. Take an inventory, identify the common themes or motives, and put these ideas into words or formalized math." *Additions:* [t211u] the polarization as a flow from hot to cold (the thermal arrow); [t212u] Frobenius/scaling as holomorphic versus heat flow as antiholomorphic or Kramers–Kronig causal.

Answer in brief. (1) *Inventory* (§2). 57 completed items (Lemmas 1–4 and 6–55, Lemma 24's supplements folded into its row, Definition 5, the contact and Schmid notes) plus Lemmas 56–57 (in progress) are tabulated by statement, label, ingredients and what they rule out. Of the RH-related results, *every* RH-equivalence is a restatement, and every genuinely new theorem is either a no-go, a finite certificate, or a structural identity. (2) *The common core* (§3). Everything reduces to four precise objects: a *reflection datum* $(V, \omega, \Phi, \sigma)$ (free: functional equation + flow), the *Krein form* it induces (free, indefinite), a *polarization* J (the missing positive datum), and the *real form* in which the normalized flow lives. The thesis "RH $\iff$ the normalized flow lies in a *compact* real form (rotation), not a split one (boost); ghosts = split directions" is a [THM] in finite dimensions (Prop. 3.4) and a [RESTATEMENT] in the infinite-dimensional Weil setting (Prop. 3.6). It is *not* a new route to RH. (3) *Known mathematics already carries this shape, exactly* (§4). Serre's Kähler analogue (1960) and the standard conjectures (Grothendieck 1969, Kleiman 1968) are literally "polarization $\Rightarrow$ normalized Frobenius is a rotation" [CITED]. In Deligne–Milne's theory of polarized Tannakian categories, a polarization is a *Hodge element* C whose inner twist is compact, and an element preserves the C-polarization iff it lies in the maximal compact subgroup $\mathrm{Cent}(C)$ [CITED]/[THM]: J.W.'s "rotation not boost" is this, word for word. Deninger's and Connes–Consani's programs supply the flow, the reflection and the trace formula for $\mathrm{Spec}\,\mathbb{Z}$, and *explicitly* lack the positivity [CITED]. Topoi, ∞-topoi, condensed mathematics and HoTT supply *spaces and coefficients*, never positivity; using them for RH is at present [ANALOGY]/[METAPHOR]. (4) *Via negativa* (§5): 17 realizations of the missing object are excluded by our lemmas, and 11 necessary properties are extracted. (5) *The thermal arrow* (§7). The de Bruijn–Newman parameter is a genuine arrow (Pólya, de Bruijn, Newman; $0 \le \Lambda \le 0.2$ by Rodgers–Tao and Platt–Trudgian), and RH $\iff \Lambda = 0$ [CITED][RESTATEMENT]: the second law guarantees eventual, not present, equilibrium; the gap *is* RH. But this flow is not the Frobenius/scaling flow: it is backward heat in the height, *forward* heat transverse to the line, and the imaginary-time continuation of a *chirp*, not of scaling (Prop. 7.6, [THM]). The imaginary-time continuation of scaling is Cramér's function (Lemma 32). Real time and imaginary time are restrictions of one holomorphic semigroup e^{-zH}, which is contractive iff $H \ge 0$ (Prop. 7.5, [THM]): positive energy = polarization = Euclidean continuation. Causality (Titchmarsh) is *weaker* and free for ζ; RH is minimum phase. "RH $\iff$ scaling is the modular flow of a positive state on the missing space" is a [RESTATEMENT] whose positivity is circular. (6) *Dictionary* (§9), *diagram* (§8), five *ranked targets* (§10), *corrections* (§11, 13 items), and questions Q1–Q5.

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1. Sources and conventions

What was read. `STATE.md` (§§0–64), and for every lemma file `lemma1.tex`–`lemma55.tex` (plus `definition5`, `contact_note`, `schmid_note`, `tropical_note` = Lemma 23, and the two Lemma 24 supplements) the title, abstract and "Answer in brief". Volumes 1–3 are the compiled forms of these lemmas and were not re-read. *Lemmas 56 and 57 are being written in parallel*: Lemma 56 (saddle orientation as invariant; the ξ converse; transverse saddle count; Cohn 1922; heat-flow births) has only literature downloaded (`lit/lemma56/`); Lemma 57 (prime-side Weil ladder, lowest eigenvector, prime-by-prime positive cone, Davenport–Heilbronn comparison) has nothing on disk. Both enter only as "in progress".

Precursor papers. Only abstracts are local (`archive/abstracts.txt`, `ARCHIVE_REVIEW.md`): GRQFT (ai.viXra 2509.0047, 2510.0007, 2511.0031), ARE (2602.0059, 2602.0095), RFTP (2604.0024). They are used lightly in Remark 2.2.

Conventions. $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$; a *ghost* is a zero ρ with $\operatorname{Re}\rho > \frac{1}{2}$, $\delta = \operatorname{Re}\rho - \frac{1}{2}$; N_+ is the number of distinct ghosts. $\mathcal{Q}$ is the Weil form on $C_c^\infty(\mathbb{R})$ (log-scale u), T_t the scaling flow (Lemma 1). For a curve $X/\mathbb{F}_q$ of genus g, F is Frobenius on H^1, $V = qF^{-1}$ the Verschiebung, $g_F = F/\sqrt{q}$ the normalized Frobenius. "Lemma n" always means TAA Lemma n.

2. Inventory

Ingredient codes. R reflection/functional equation; E Euler product/primes; F a flow (Fr Frobenius, Sc scaling, Ht de Bruijn–Newman heat, Md modular/KMS); P positivity/polarization J; S spin/metaplectic double cover; K Krein/indefinite structure; N counting (indices, windings, ghost numbers). **Label** is the dominant label of the lemma's main result (each lemma tags its own statements individually; Thm = proved, Thm^c = classical/cited, R = restatement, A = analogy, Num*

= interval-certified computation, Num = floating point, O = open). The last column is the *via negativa*: what the item excludes.

#	main statement	label	ingred.	rules out
1	Weil form $\mathcal{Q}$ Hermitian; T_t always $\mathcal{Q}$-isometric; RH $\iff$ $\mathcal{Q} \geq 0$ $\iff$ unitary Hilbert realization $\iff$ bounded matrix coefficients; $\kappa(\mathcal{Q}) = N_+$	R	R E Sc P K N	— (sets the frame: RH = unitarity)
2	dBN forms $\mathcal{Q}_t$: $\kappa = N_+(t) < \infty$; second law ($Z_{\mathbb{C}}(H_t)$ non-increasing); $\mathcal{Q}_t \geq 0 \iff t \geq \Lambda$; cooling law	Thm/R	Ht K N	a second law for the *distinct* ghost count (it can rise)
3	Curves/$\mathbb{F}_q$: flow at $\log q = q^{-1/2}$Frob, unitary; positivity from Hodge index; $\kappa \in \{0, \infty\}$	Thm^c	R Fr P	— (the template where RH = unitarity is a theorem)
4	Iwasawa pillar: Mazur–Wiles as p-adic spectral realization; 691	Thm^c	Fr N	p-adic spectral realization as a source of positivity
D5	Action $S = \mathcal{Q}$; primes are closed orbits ($\log p$), not critical points of S; OS positivity fails at prime orbits	Thm	E Sc	"prime $\iff$ $\delta S = 0$"; OS reflection positivity as the right notion
C	Contact vs foliation: sign-correct models are foliation-like and singular/non-compact	Thm	Sc N	compact smooth foliated models; contact/Reeb models
Sch	Schmid weight cone: SL_2-orbit theorem is a boost of rapidity $\frac{1}{2}\log y$	Thm^c/A	Fr K	identification of distinct "cones"
6	Window = semilocal; positive trace τ_S with symbol ≥ 0; finite index per window	Thm/Num	E P K N	direct transfer of the CC archimedean mechanism to $\lvert S\rvert \geq 2$
7	Lax–Phillips/Pavlov–Faddeev; spectral bound $= -\frac{1}{4} + \frac{1}{2}(\Theta - \frac{1}{2})$	Thm	Sc R	any contraction/stability property of $Z(t)$ alone as RH criterion
8	Zero-free intertwiner LP core $\to$ Weil; pulled-back form Krein-diagonal	Thm	Sc K	"Weil = LP norm + positive"
9	Abscissa σ: $\sigma = 1$ full product rank-one defect	Thm/Num	E P	free positivity at $\sigma = 1$ transferring to $\frac{1}{2}$
10	Certified Weil positivity on windows $\leq \log 3$	Num*	E P	—
11	Margin law $\lambda^*(d)$; ghost identity $\lvert a+b\rvert^2 - \lvert a-b\rvert^2$	Thm/Num	K N	early detection of small-δ ghosts
12	Explicit detection $A_0 \approx c/\delta$; converse vacuous today	Thm	K N	current window certificates excluding any zero
13	Squaring map at i, CM Sato–Tate	Thm^c/A	S	"$\lvert\alpha\rvert = \sqrt{p}$ is a dynamical Levi-Civita coordinate"
14	Sommerfeld$\to$Hodge index chain; Frob graph = diagonal boosted by $\frac{1}{2}\log q$	Thm^c/A	Fr P K	— (records absence of NS(Spec $\mathbb{Z}^2$))
15	Lorentzian index of the Weil form ≤ 1 on all finite subspaces $\iff$ RH; certified inertia	R/Num*	R E Sc K N	finite certificates proving RH
16	Certified windows $\log 4$, 1.6	Num*	E P	—
17	Arithmetic action as Lagrangian; RH $\iff$ no conjugate points (Jacobi–Morse)	R	Sc N	—
18	σ-window index; $\Theta = \min\{\sigma : \kappa_{\sigma,A} = 0\ \forall A\}$	Thm	K N	monotonicity of the count in σ
19	One trace bound $\lvert a\rvert \leq 2\sqrt{q}$ is "halfway"; coin polynomial	Thm	Fr P	any finite family of trace bounds $\Rightarrow$ RH ($g \geq 2$)
20	T_p as two boosts in NS($X_0(N)^2$)	Thm	Fr K	—
21	Zero-side certification no-go at $A \approx 2.11$	Thm	N	pure zero-side certificates beyond 2.11
22	Prolate degeneration: prolate objects see ghosts no earlier	Thm/Num	Sc K	prolate/CCM shortcut to earlier detection
23	Tropicalization keeps $\lvert\cdot\rvert$, loses positivity	Thm/Num	P	tropical/$\mathbb{F}_1$/Newton-polygon Hodge index

#	main statement	label	ingred.	rules out
24	Frame bounds on Platt zeros (+2 supplements); hybrid not run	Num*	N	—
25	Telegrapher: FE $\Rightarrow$ mean loss $\frac{1}{2}$; residual δ^2 is FE-even; Weil = Krein form $\langle J_{\mathrm{fe}}x, y\rangle$	Thm	R K	reflection forcing $\delta = 0$
26	Viro/hyperfield $\hbar$-deformation: Weil index lives only at $\hbar = 1$	Thm	P	dequantization as positivity source
27	Off-line pair = hyperbolic plane; boost rapidity $\delta \log x$; two-stage ghost birth	Thm	Sc K	"ghost birth = one Krein collision"
28	No real telegrapher line for nontrivial zeros; GRSH by k; $\Theta < 1$	Thm/O	E R	spatial-mode realization
29	Index vs negativity mass (false as posed); pole threshold	Thm	K N	symbol-mass bounds on the index
30	Positive involution: (i),(ii),(iv) free; (iii) trace positivity = RH	Thm/R	R P	purely algebraic positive involution as target
31	Thermal/KMS/Tomita: shape yes, source no; modular realization $\Rightarrow$ RH restates G5	Thm/R	Md P	thermodynamics as positivity source
32	Weil is a degenerate OS form; Cramér S_+ completely monotone $\iff$ RH	Thm/R	R Sc P	half-space OS positivity
33	Causality free; passivity = RH	Thm	P	causality/Kramers–Kronig as missing link
34	de Branges/Suzuki line of ξ from prime side; indefinite case	Thm/Num	E K	—
35	Post–Widder check blind below order $4.7 \cdot 10^{12}$	Thm	E P	finite-order sign checks
36	Positivity ledger; G5 = domination of the line part by the prime part	Thm/R	all	positivity on line/windows/line-blind structures
37	Passive line official target; finite Euler cascade	R/Num	E P	—
38	Rapidity and Galois: commuting factors of $C_{\mathbb{Q}}$	Thm	E Sc	"Iwasawa = p-adic rapidity"; p-adic positivity
39	Prime clock; BC scaling is the dual action, not modular	Thm/Num	Md E	entropy–margin link; BC modular = scaling
40	Hodge split = Krein space; Pick kernel of ξ'/ξ = zero Krein form	Thm	P K	— (unification)
41	Two stages coincide canonically; no constraint on ζ	Thm	K	—
42	HSMI lead impossible; orthogonality pincer; Klein four-group	Thm	Md R	positive-energy HSMI with zeros as energies
43	Reversible holomorphic flows; "RH $\iff$ all centers" false	Thm	R	center-classification as criterion
44	Cavities: Hasse $\iff$ stable cavity $\iff$ invariant positive form	Thm/A	Fr P	—
45	Bose/Fermi primon gas; zeros never spectral data	Thm	E S	UV condensation picture
46	Algebra gives Krein, geometry gives Hilbert; $n_+ = 1 + k$	Thm	Fr R K	$FV = q$ forcing positivity; statistics forcing purity
47	Reflection $\mathbb{N} \to \mathbb{Z}$ only prime by prime; Toeplitz vs Hankel	Thm	E R	global reflection producing Weil's form
48	ζ minimum phase (outer) $\iff$ RH	R	R K N	—
49	Betti–de Rham for elliptic curves; E_2^* as polarization term	Thm^c	P	crystalline (unit-root) splitting as positivity
50	Missing polarization = missing invariant positive fermionic vacuum; spinors don't create positivity	Thm/R	S P K	spin structure as source; half Tate twist over $\mathbb{Q}$
51	At ghost birth the positive vacuum escapes to the Siegel boundary	Thm	S P K	"positive vacuum becomes a ghost"
52	CCM: scaling is a boost; no invariant vacuum; DH windows negative by $\lambda^2 \approx 60$	Thm/Num	Sc S P	scaling-invariant vacuum on L^2

#	main statement	label	ingred.	rules out
53	Bosonic twin; ghost charge = −#ghosts; RH $\iff$ outer	Thm/R	S P K N	boson–fermion correspondence as positivity
54	J_F emergent; exists iff no real eigenvalue; RH $\iff$ J_F positive	Thm	Fr S P K	"emergence of J" as proof
55	Cassinian/separatrix geometry; saddle type sees ghosts	Thm/R	R N	saddle *location* as criterion
56	(in progress) orientation invariant; ξ converse; transverse count	—	R N	—
57	(in progress) prime-side Weil ladder; lowest eigenvector; DH	—	E P	—

Remark 2.1 (Statistics of the inventory)**.** Of the 57 completed items, the ingredient P (positivity) appears in 25 and K (Krein) in 25; every item that touches RH for ζ directly is either R (an equivalence), a *no-go*, or a *finite certificate. No item supplies P for ζ from R, E and F alone.* Items carrying a non-trivial positivity *as a theorem* (3, 14, 19, 44, 46, 49–51, 54) are all over finite fields or about curves; the ζ-side positivities (6, 9, 10, 16, 31, 33, 36) are on windows, at $\sigma \geq 1$, or line-blind (Lemma 36's four places). This count is by inspection of the table, [THM] in the trivial sense.

Remark 2.2 (Precursor papers, used lightly)**.** RFTP (2604.0024, abstract) posits "a universal cohomological object that factors through all Weil cohomologies". That is, verbatim, the defining property of a Grothendieck motive $h(X)$ [ANALOGY]; for $\operatorname{Spec}\mathbb{Z}$ no such object is constructed there or anywhere [OPEN]. ARE (2602.0059) calls ζ "the maximally symmetric pre-geometric vacuum, with perfect functional-equation symmetry around $\operatorname{Re} s = \frac{1}{2}$": in the language below this is the *reflection datum*, which is free (§3); the vacuum's *positivity* is what is missing. GRQFT (2511.0031) speaks of "Monster moonshine polarization"; no polarization in the sense of Definition 3.2 is defined there [METAPHOR]. The audit of these papers is in `ARCHIVE_REVIEW.md` and is not redone.

3. The common core: four objects

Reading the table column by column, every RH-equivalent row is built from the same four objects. We define them, then test the candidate thesis.

Definition 3.1 (Reflection datum)**.** A *reflection datum* is $(V, \omega, \Phi, \sigma)$ where V is a real vector space (possibly infinite-dimensional, with a topology), ω a nondegenerate bilinear form on V (alternating: "symplectic presentation"; or the real part of a Hermitian form on $V_{\mathbb{C}}$: "Krein presentation"), $\Phi : \Gamma \to \mathrm{GL}(V)$ a representation of $\Gamma = \mathbb{Z}$ (Frobenius) or $\Gamma = \mathbb{R}$ (scaling) that scales ω by a character, $\omega(\Phi(\gamma)x, \Phi(\gamma)y) = c(\gamma)\,\omega(x, y)$ with $c(\gamma) > 0$, and σ an involutive similitude with $\sigma\Phi(\gamma)\sigma^{-1} = c(\gamma)\Phi(\gamma)^{-1}$ (the functional equation; for curves σ realizes $V = qF^{-1}$, Lemma 46). The *normalized flow* is $\tilde{\Phi}(\gamma) = c(\gamma)^{-1/2}\Phi(\gamma) \in \mathrm{Aut}(V, \omega)$.

The normalization needs $c^{1/2}$: for curves $c(\mathrm{Frob}) = q$ and $c^{1/2} = \sqrt{q}$ is a *choice* (the half Tate twist, a $\mathbb{Z}/2$-torsor; Lemma 50); for scaling $c(t) = e^t$ and $c^{1/2} = e^{t/2}$ is the half-density normalization (Lemma 1). In both cases the datum is *free*: it is supplied by the functional equation and the flow [TAA-L].

Definition 3.2 (Polarization)**.** A *polarization* of (V, ω) in the symplectic presentation is $J \in \mathrm{End}(V)$ with $J^2 = -1$, $\omega(Jx, Jy) = \omega(x, y)$ and $g_J(x, y) := \omega(x, Jy)$ symmetric positive definite. In the Krein presentation ($[x, y]$ Hermitian, indefinite) it is a *fundamental symmetry* J ($J^2 = 1$, $[Jx, Jy] = [x, y]$) with $[Jx, x] > 0$ for $x \neq 0$; the polarization is *trivial* ($J = 1$) iff $[\cdot, \cdot]$ is itself positive. In Deligne–Milne's language (§4) the polarizing operator is a *Hodge element* C and the positive form is $\psi_C(x, y) = \psi(x, Cy)$.

Definition 3.3 (Arithmetic polarization datum)**.** An *arithmetic polarization datum* (APD) is a reflection datum together with a polarization J that commutes with the normalized flow: $\tilde{\Phi}(\gamma)J =$

$J\tilde{\Phi}(\gamma)$ for all γ. It is *non-circular* if $V, \omega, \Phi, \sigma, J$ are constructed without reference to the zeros (Lemma 30's axioms; "G6" of Lemma 32), and *spectral* if it carries a trace formula: for test functions f, $\Phi(f) = \int f(\gamma)\tilde{\Phi}(\gamma)\, d\gamma$ is trace class on the relevant degree and $\operatorname{Tr}\Phi(f) = \sum_\rho m_\rho \hat{f}(\gamma_\rho)$, the zero side of the explicit formula.

3.1. **Finite dimensions: rotation versus boost is a theorem.**

Proposition 3.4 (Compact versus split, finite-dimensional)**.** *Let (V, ω) be a real symplectic space of dimension $2n$ and $g \in \mathrm{Sp}(V)$. The following are equivalent: (a) g commutes with some polarization J; (b) g lies in a compact subgroup of $\mathrm{Sp}(V)$; (c) $\{g^k : k \in \mathbb{Z}\}$ is bounded; (d) g is diagonalizable over $\mathbb{C}$ with all eigenvalues of modulus 1 ("elliptic"). If $g = g_e g_h g_u$ is the real Jordan decomposition (commuting elliptic, hyperbolic and unipotent parts), these hold iff $g_h = g_u = 1$. [THM]*

Proof. (a)⇒(b): g lies in $\mathrm{Sp}(V) \cap O(g_J)$, which is closed and bounded, hence compact. (b)⇒(c) is trivial. (c)⇒(d): bounded powers for $k \to \pm\infty$ force $|\lambda| \le 1$ and $|\lambda^{-1}| \le 1$ for every eigenvalue, and a nontrivial Jordan block would make $\|g^k\|$ grow like $|k|$. (d)⇒(c) is clear. (c)⇒(a): the closure K of $\{g^k\}$ is a compact abelian subgroup of $\mathrm{Sp}(V)$. Average any inner product over the Haar measure of K to get a K-invariant inner product h. Write $\omega(x, y) = h(Ax, y)$; then A is h-skew, invertible and commutes with K. Its polar decomposition $A = J|A|$, $|A| = (-A^2)^{1/2}$, gives $J = A(-A^2)^{-1/2}$ with $J^2 = -1$, J commuting with K, $\omega(Jx, Jy) = \omega(x, y)$ and $\omega(x, Jx) = h(|A|x, x) > 0$. The last sentence: g_h has eigenvalues of modulus $\neq 1$ unless trivial, g_u is a Jordan-block part, and (d) holds iff both are trivial. □

The formula $J = A(-A^2)^{-1/2}$ is the Segal–Weinless–Kay positive-energy complex structure of Lemma 54 [TAA-L]; Lemma 54's $J_F = (F - V)(4q - (F + V)^2)^{-1/2}$ is the case h replaced by the Frobenius-defined form, and is positive only when all Krein signs are + [TAA-L]. Prop. 3.4 is the precise content of "rotation, not boost": the boost is g_h, a parabolic (Jordan) obstruction is g_u, and Lemma 54's drift rate $\ell = 2(\sum_{|\lambda|>1} \log^2 |\lambda|)^{1/2}$ measures g_h alone [TAA-L].

Corollary 3.5 (Curves)**.** *For a curve $X/\mathbb{F}_q$ (or a fake L-polynomial) in the trace model of Lemmas 50/54, RH for X $\iff$ g_F is elliptic $\iff$ g_F fixes a point of Siegel space. Ghosts are exactly the hyperbolic part g_h: a real ghost pair is one boost of rapidity 2σ per step, a quadruple a loxodromic of $\mathrm{SL}_2(\mathbb{C}) = \mathrm{Spin}(3, 1)_0$ (Lemma 54). [RESTATEMENT] (of Weil's theorem; Lemma 50 Thm 4.4, Lemma 54)*

3.2. **Infinite dimensions: bounded orbits, and the sign of the rotations is free.** In the Weil setting "compact real form" has no literal meaning (the unitary group of an infinite-dimensional Hilbert space is not compact). The correct replacement is condition (c).

Proposition 3.6 (Weil setting)**.** *(i)* Zero-free form *(Lemma 1): T_t is $\mathcal{Q}$-isometric unconditionally, and RH $\iff$ every matrix coefficient $t \mapsto \mathcal{Q}(T_t F, G)$ is bounded $\iff$ T_t is unitary for the positive form $\mathcal{Q}$ [TAA-L][RESTATEMENT]. (ii)* Zero model *(Lemmas 25, 27, 40): on the span of the zero functionals the Krein Gram is $[e_\rho, e_{\rho'}] = m_\rho \delta_{\rho', \rho^*}$, $\rho^* = 1 - \bar{\rho}$. On-line zeros ($\rho = \rho^*$) are positive one-dimensional blocks on which $\tilde{T}_t$ acts by the rotation $e^{i\gamma t}$; each ghost pair spans a hyperbolic plane $\left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$ on which $\tilde{T}_t$ is a boost of rapidity δt times a rotation [TAA-L]. (iii) Consequently RH $\iff$ $\tilde{T}_t$ has no hyperbolic planes $\iff$ the Krein form is positive. [RESTATEMENT]*

Remark 3.7 (Why rotation suffices for ζ, and what is free)**.** In a general Krein space an elliptic unitary may have eigenvectors of *negative* type (Lemma 54's mixed Krein patterns), and then "rotation" does *not* imply "positive form". For ζ the Krein type of every rotation (on-line zero) is + because $m_\rho > 0$: the explicit formula's zero side is a positive combination of point evaluations. The same holds in Lemma 50's trace model for curves (pattern $(+, \dots, +)$). So, in the arithmetic setting, *the sign of the rotation sector is free, and RH is exactly the absence of boosts.* This is the precise form of J.W.'s "RH is a statement about unitarity" [RESTATEMENT], and it isolates the content: *exclude boosts.*

3.3. **The thesis, tested.**

Thesis 3.8 (Candidate thesis of TAA, as tested)**.** Every TAA restatement has the shape of a reflection datum $(V, \omega, \Phi, \sigma)$, and RH (or its analogue) is equivalent to the normalized flow preserving a polarization; equivalently, to $\tilde{\Phi}$ having bounded orbits ("compact real form", rotation) rather than hyperbolic planes ("split", boost); ghosts = split directions.

Test against the inventory.

- *Fits literally (finite-dimensional, Prop. 3.4):* 3, 14, 19, 20, 44, 46, 49–51, 54 (curves and fakes); 13 (CM Frobenius as a quarter turn).
- *Fits in the Krein form (Prop. 3.6):* 1, 8, 11, 15, 17, 18, 25, 27, 29, 40, 41 (RH = no hyperbolic planes / $\kappa = 0$ / Lorentzian index 1).
- *Fits after a change of presentation:* 33, 37, 48, 53 (the polarization is the Hardy-space splitting $H_+ \oplus H_-$, i.e. the sign of the frequency; RH = the ζ-subspace is all of H_+, "outer"); 31, 32, 42 (positivity of a state or of a Euclidean kernel); 55 (Herglotz positivity of ξ'/ξ, Lagarias).
- *Does not fit:* 2 (the heat parameter is a *deformation* of the datum, not an element of Γ; §7); 4, 38 (p-adic: no ordered field, no positivity); 23, 26 (deformations that destroy the real structure); the numerical items.

Verdict. The thesis is correct as an *organizing restatement*: it is a theorem in finite dimensions, a restatement for ζ, and it identifies the content as "exclude boosts". It is false that it gives a new route: the polarization J is exactly what no item constructs for ζ. [RESTATEMENT]

Theorem 3.9 (Spectral APD $\Rightarrow$ RH; conditional and elementary)**.** *Suppose a spectral* APD *exists for ζ with V a Hilbert space, $\tilde{\Phi}$ a strongly continuous unitary representation of $\mathbb{R}$, and the trace formula of Definition 3.3 for all $f \in C_c^\infty(\mathbb{R})$. Then RH holds.*

Proof. For $g \in C_c^\infty$ put $f = g * \tilde{g}$, $\tilde{g}(t) = \overline{g(-t)}$. Then $\Phi(f) = \Phi(g)\Phi(g)^* \geq 0$, so $\sum_\rho m_\rho \hat{f}(\gamma_\rho) = \operatorname{Tr} \Phi(g)\Phi(g)^* \geq 0$. The left side is Weil's form evaluated at g (Lemma 1), so Weil positivity holds and RH follows (Weil 1952; Lemma 1). [THM] □

Remark 3.10 (Where the circularity hides)**.** Under the hypotheses the generator is self-adjoint, so its spectrum $\{\gamma_\rho\}$ is real: the trace formula hypothesis *contains* RH. Any genuine construction must prove the trace formula without knowing that the γ_ρ are real. This is exactly the status of Connes' trace formula: its global validity is equivalent to RH (Connes 1999, abstract) [CITED]. Theorem 3.9 is therefore a [RESTATEMENT] with a clear division of labour: *construct V and J zero-free; the rest is formal.*

Conjecture 3.11 (TAA target, stated precisely)**.** *There exists a non-circular spectral arithmetic polarization datum for* $\operatorname{Spec} \mathbb{Z}$*: a real space V built from the idele class group (or a geometric substitute) without reference to zeros, carrying the idele-class scaling flow, the functional-equation reflection, and a flow-invariant polarization J, whose Lefschetz-type trace is Weil's explicit formula. [CONJ]*

By Theorem 3.9, Conjecture 3.11 implies RH; conversely, under RH Lemma 1's GNS completion gives a Hilbert space with unitary scaling, and the zero model of Prop. 3.6(ii) (completed, with on-line zeros only) carries the trace formula on suitable test functions; neither is zero-free. So the conjecture is RH plus a non-circularity requirement; it is not weaker than RH. [RESTATEMENT]

4. What known mathematics already carries this shape

4.1. **(a) Weil, Castelnuovo–Severi, Hodge index, standard conjectures.**

Proposition 4.1 (Positive involution $\Rightarrow$ compact unitary group)**.** *Let A be a finite-dimensional semisimple $\mathbb{R}$-algebra with an involution $\alpha \mapsto \alpha'$ that is positive, $\operatorname{Tr}(\alpha\alpha') > 0$ for $\alpha \neq 0$. Then $U = \{u \in A : uu' = 1\}$ is compact. If $\alpha\alpha' = q$ then every eigenvalue of α (in any representation) has absolute value $\sqrt{q}$. [THM] (the second statement is Milne,* pRH*, Prop. 3.10 [CITED])*

Proof. U is closed, and $\mathrm{Tr}(uu') = \mathrm{Tr}(1)$ for $u \in U$; since $(\alpha, \beta) \mapsto \mathrm{Tr}(\alpha\beta')$ is a positive definite form, U lies in a sphere and is compact. If $\alpha\alpha' = q$ then $u = q^{-1/2}\alpha \in U$, and the closure of $\{u^k\}$ is compact, so the eigenvalues of u are of modulus 1. □

This is the whole of Weil's argument in modern dress, and it is the finite-dimensional thesis: *a positive (Rosati) involution makes the normalized Frobenius lie in a compact group.*

What Weil actually used. Weil (1948, ASI 1041 and 1064) proved positivity of $\sigma(\xi \circ \xi')$ on correspondences, the Castelnuovo–Severi inequality, directly, by an argument in the geometry of the Jacobian and Rosati positivity; Mattuck–Tate (1958) gave a proof of the Castelnuovo–Severi inequality from Riemann–Roch on the surface $X \times X$, and Grothendieck (1958) observed that it is a consequence of the Hodge index theorem on that surface (Milne, *pRH*, §§2.4–2.7, and the original papers' titles and abstracts) [CITED]. The index theorem itself goes back to Hodge, Segre (1937) and Bronowski (1938) [CITED-sec] (via Milne).

Serre's Kähler analogue (1960). For X smooth projective over $\mathbb{C}$ and f an endomorphism with $f^*L \cong L^{\otimes q}$, the positive Hermitian form $A(x, Cy)$ on primitive cohomology is preserved up to q^r by f, hence $q^{-r/2}f$ is unitary and its eigenvalues have modulus $q^{r/2}$ (Serre 1960, Thm. 1; Milne *pRH* §3.1) [CITED]. This is Definition 3.3 with J the Weil operator C: the template is *literally* TAA's thesis.

Standard conjectures. Grothendieck's standard conjectures of Lefschetz type (B) and Hodge type (I) together imply that the form $\langle x, y\rangle = (-1)^r\, x \cdot L^{d-2r} * y$ on algebraic classes is a Weil form (a positive involution in the sense of Prop. 4.1), whence the Riemann hypothesis for $X/\mathbb{F}_q$ (Kleiman 1968, Thm. 3.11 [CITED-sec]; Milne *pRH* Thm. 3.15, 3.17 [CITED]). Hodge type I is known in characteristic 0 (Hodge theory) and, in characteristic p, for surfaces, where it *is* the Hodge index theorem [CITED]. Deligne proved RH over finite fields (Weil I, 1974) without the standard conjectures [BIB].

4.2. **(b) Polarized Tannakian categories.** We used the 2022 online version of Deligne–Milne ("DM"), checking numbering there [CITED].

- DM Def. 4.10: a *polarization* of a Tannakian category $\mathcal{C}$ over $\mathbb{R}$ is a compatible family of equivalence classes of Weil forms ("positive" sesquilinear forms) of parity ε; Prop. 4.11: a polarizable category is semisimple.
- DM Prop. 4.22–4.23: a *Hodge element* C of $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ (with C^2 central) defines a polarization π_C; $\mathrm{Rep}_{\mathbb{R}}(G)$ admits one iff G has a Hodge element iff G is an inner form of a compact group. DM Lemma 4.24: C is a Hodge element iff the real form of G obtained by twisting with $\mathrm{ad}\, C$ is compact. DM Rem. 4.25(a): the centralizer of a Hodge element is a maximal compact subgroup (Saavedra V.2.7.3.5).
- DM Thm. 4.17(a): an algebraic Tannakian category over $\mathbb{R}$ is polarizable iff its band is defined by a *compact* real algebraic group. Thm. 4.27: a symmetric polarization gives a real fibre functor and $\mathcal{C} \simeq \mathrm{Rep}_{\mathbb{R}}(K)$ with K compact.
- DM §5: graded polarizations of parity $(-1)^n$ (Def. 5.12); Ex. 5.21: real Hodge structures are polarized by $C = h(i)$, the Weil operator.

Proposition 4.2 (Element-level form of DM 4.23–4.25)**.** *In the situation of DM 4.23 with $C \in G(\mathbb{R})$, an element $\gamma \in G(\mathbb{R})$ preserves the positive form $\psi_C(x, y) = \psi(x, Cy)$ for all ψ in the polarization iff γ commutes with C iff $\gamma \in K_C := \mathrm{Cent}_G(C)(\mathbb{R})$, a maximal compact subgroup. [THM] (given DM)*

Proof. γ preserves each ψ (it is a tensor automorphism), so $\psi(\gamma x, C\gamma y) = \psi(x, \gamma^{-1}C\gamma y)$, which equals $\psi(x, Cy)$ for all x, y iff $\gamma^{-1}C\gamma = C$. The rest is DM Rem. 4.25(a). □

So J.W.'s sentence "RH $\iff$ Φ preserves a positive polarization $\iff$ Φ lies in a compact real form; ghosts are split directions" is, for a Tannakian category, Prop. 4.2 *word for word*, with two precisions: (1) it is the *inner twist* of G by C that is compact, not G itself, and the element statement needs $\gamma \in \mathrm{Cent}(C)$; (2) Tannakian objects are finite-dimensional, while the "H^1" of ζ is

infinite-dimensional with continuous (Lebesgue-type) spectrum at every finite stage (Lemmas 6, 52), so no Tannakian category of finite-dimensional objects can carry ζ's datum literally [THM] (finite-dimensionality is part of the definition; see Target 3). The candidate homes are the category of motives with the conjectural Hodge-type polarization (Milne *pRH* §4.3–4.4: $\mathcal{M}_{\mathrm{num}}(\mathbb{F}_q)$ is a graded Tannakian category whose canonical polarization is equivalent to the standard conjectures [CITED]) and its unknown $\operatorname{Spec}\mathbb{Z}$ analogue [OPEN].

4.3. **(c) Deninger, Connes, Connes–Consani. Deninger** (ICM 1998) conjectures a foliated dynamical system whose leafwise cohomology carries a flow with Θ having spectrum the zeros, a Hodge $*$ and a Kähler-type positivity which would give RH as in Serre 1960 [BIB] (bibliographic record verified; content as summarized in TAA Lemma 5 and the contact note). TAA's contact note and Lemma 5 show sign-correct models must be singular/non-compact [TAA-L]. **Connes** (Selecta 1999) realizes the zeros as an absorption spectrum of the idele-class scaling action; RH is reduced to the validity of a global trace formula, equivalently a positivity (abstract) [CITED]. TAA Lemmas 6, 22, 52 analyse finite-S and prolate versions [TAA-L]. **Connes–Consani** build the arithmetic site (CRAS 2014; Adv. Math. 2016) and scaling site (CRAS 2016), toposes over $\mathbb{Z}_{\max}/\mathbb{R}_{\max}$ whose points recover the adele-class space and whose structure sheaf is tropical; they propose a Riemann–Roch strategy (2019) and prove Weil positivity for test functions supported in a small interval at the archimedean place (Selecta 2021) [CITED]. Lemma 23/26: tropical structures keep $|\cdot|$ and lose positivity, so the positivity must come from elsewhere [TAA-L]. **Summary.** These programs supply V (or its points), Φ, σ and the trace formula; *each explicitly lacks* J. In the language of §3: they are reflection data with a trace formula, awaiting a polarization.

4.4. **(d) Topoi, ∞-topoi, condensed mathematics, HoTT (stated honestly).** A (Grothendieck) topos is a category of sheaves; an ∞-topos (Lurie, HTT) its homotopical refinement; condensed sets (Clausen–Scholze, 2019) give a good category of topological algebra; HoTT (Univalent Foundations, 2013) is a type theory whose models include ∞-topoi [BIB]. Their role for RH is at most to provide the *space* V and its *coefficients* (Weil-étale cohomology of Lichtenbaum 2005/2009 is the closest existing arithmetic example [BIB]; Connes–Consani's sites are toposes [CITED]). *None of them carries a positivity*: positivity is an ordered-field phenomenon, absent from the internal logic of a topos and from type theory. Statements "HoTT/univalence $\leftrightarrow$ RH" or "$\leftrightarrow$ SU(3)" are [METAPHOR]. The only formal bridge between "motive-like category" and "rotation not boost" known to us is DM 4.23–4.25 (Prop. 4.2).

5. Via negativa: what the missing object must and cannot be

5.1. **Necessary properties.** Each property is forced by an earlier result. "Forced" means: a candidate lacking it is excluded by the cited lemma.

- **P1** *Real, infinite-dimensional, with continuous spectrum at every finite stage*: finite windows/semilocal pieces have Lebesgue spectrum (Lemmas 6, 52); only the full limit can have pure point spectrum $\{\gamma_\rho\}$.
- **P2** *The flow is idele-class scaling, with Galois/p-adic directions as a compact commuting factor* (Lemma 38): rapidity lives only at the archimedean place.
- **P3** *Singular or non-compact base* if foliated (contact note, Lemma 5).
- **P4** *Toeplitz, not Hankel*: the reflection acts prime by prime and globally only as a Toeplitz-type symbol (Lemma 47).
- **P5** *A geometric (ample/Kähler-type) positivity source*, not algebra: $FV = q$ and other algebraic identities give only a Krein form (Lemmas 30, 46).
- **P6** *Integrality must fail at the archimedean place*: the positivity cannot be a lattice/integrality statement (Lemmas 4, 38, 49).

P7 *The pole is a Lorentzian exceptional direction*: index exactly 1 on the Lorentzian cone (Lemmas 14, 15): the missing NS has signature $(1, \infty)$ like Hodge index.

P8 *It must see the whole critical strip interior*: line-blind positivities are insufficient (Lemma 36's ledger; Lemma 21's 2.11 no-go).

P9 *Zero-free construction* (Remark 3.10; Lemmas 30, 32).

P10 *It must distinguish ζ from Davenport–Heilbronn*: same functional equation, no Euler product, zeros off the line; so E is essential (Lemmas 47, 52; Lemma 57 in progress).

P11 *Compatible with the Connes–Consani archimedean positivity* (Selecta 2021): it must restrict to that theorem on small intervals (Lemma 6).

5.2. **Impossible realizations (each excluded by a theorem).**

N1 (F, V) or $FV = q$ alone (Lemma 46: algebra gives Krein, geometry gives Hilbert).

N2 The functional equation alone (Lemma 25: reflection forces mean loss $\frac{1}{2}$, the residual δ^2 is FE-even).

N3 Spinors/metaplectic structure alone (Lemma 50: the double cover exists in all cases; Lemma 53 ghost charge).

N4 A scaling-invariant positive vacuum on L^2 (Lemma 52: scaling is a boost, no invariant vacuum).

N5 A Hilbert–Pólya operator with positive energy and zeros as energies (Lemma 42).

N6 The thermal arrow (dBN, KMS) as a source of positivity (Lemmas 2, 31; §7).

N7 Causality/Kramers–Kronig (Lemma 33: free for ζ; §7.4).

N8 Half-space OS positivity (Lemma 32: Weil's form is degenerate OS; D5).

N9 p-adic/Iwasawa spectral realizations (Lemmas 4, 38).

N10 Tropical, $\mathbb{F}_1$, hyperfield, Viro dequantization as positivity (Lemmas 23, 26).

N11 Finite certificates proving RH (Lemmas 15, 19, 21, 35).

N12 Compact smooth foliations and contact/Reeb dynamics (contact note).

N13 The emergent J_F alone (Lemma 54: it exists iff no real eigenvalue, positivity is RH).

N14 Equidistribution/Sato–Tate (Lemma 13: statistics do not force purity; Lemma 46).

N15 Saddle *location* alone (Lemma 55: symmetry free, saddle type not).

N16 A purely algebraic positive involution (Lemma 30: (iii) = RH).

N17 The prolate spectrum as an earlier detector (Lemma 22).

Remark 5.1. N1–N17 say: *nothing free produces positivity.* The positive datum must come from P5 (geometry), and it must be invisible on every finite window (P1, P8), distinguish ζ from DH (P10), and be zero-free (P9). This is the same list that Grothendieck's standard conjectures fill over $\mathbb{F}_q$ (Hodge type I for P5/P7), which is why the motive analogy is apt [ANALOGY], and why it is not a proof [RESTATEMENT].

6. Six themes

T1 Rotation versus boost. Prop. 3.4, 3.6, 4.2: a polarization is the same thing as a fixed point in the symmetric space (Siegel space, or G/K_C), and the flow preserves it iff it is elliptic. Ghosts are the hyperbolic Jordan parts. Over $\mathbb{F}_q$ the rotation is a theorem; for ζ, scaling on L^2 is a boost (Lemma 52), and RH says the *Weil-renormalized* flow is a rotation [RESTATEMENT].

T2 $\sqrt{q}$ and the half Tate twist. Normalizing requires $c^{1/2}$ (Def. 3.1). Over $\mathbb{F}_q$, $\sqrt{q}$ is a choice of square root of q, a $\mathbb{Z}/2$-torsor, and the metaplectic/spin double cover is where this choice lives (Lemmas 13, 50, 53). Over $\mathbb{Q}$ there is no half Tate twist motive (Lemma 50 [TAA-L]), but the half-density $e^{t/2}$ always exists. The double cover is *free* and therefore cannot produce positivity (N3) [THM].

T3 Reflection is free, positivity is hard. The reflection datum is supplied by the functional equation in every case (Lemmas 25, 46, 47, 55); every attempt to extract positivity from it fails (N1,

N2, N16). In Tannakian terms: the *band* and the *fibre functor* are cheap, the *Hodge element* is the theorem.

T4 The Krein ghost sector. The Weil form is a Krein form whose negative index equals the number of ghosts (Lemma 1); each ghost pair is a hyperbolic plane on which the flow boosts (Lemma 27); ghost birth is two-stage (Lemmas 27, 41); the Hodge split is a Krein space (Lemma 40); the bosonic twin's ghost charge is $-\#\text{ghosts}$ (Lemma 53). The ghost sector *is* the split part of the real form.

T5 Level sets and saddles. Lemma 55: Cassinian geometry, separatrices, the saddle structure of $|\xi|$ off the line. Symmetry of the configuration is free; its *type* sees ghosts. This is the topological shadow of T1: an elliptic fixed point versus a hyperbolic one. Lemma 56 (in progress) studies whether saddle orientation is an invariant.

T6 Finite fields versus $\operatorname{Spec}\mathbb{Z}$**.** Every non-trivial positivity in the inventory is over $\mathbb{F}_q$ or on a window (Remark 2.1). The difference is P1, P5, P7: no finite-dimensional H^1, no ample class, no $\mathrm{NS}(\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z})$. In the APD language, over $\mathbb{F}_q$ the polarization comes from geometry (Hodge index on $X\times X$) and the flow is in a compact real form by Prop. 4.1; over $\mathbb{Z}$ the flow is given, the reflection is given, and J is missing.

7. The thermal arrow

7.1. **Reversible group flows versus an irreversible deformation.** The Frobenius flow F^k ($k\in\mathbb{Z}$) and the scaling flow T_s ($s\in\mathbb{R}$) are *groups* acting on a fixed datum: they act on test functions or cohomology, and ζ is unchanged. The de Bruijn–Newman flow is of a different kind: it is a one-parameter *deformation of the datum itself*,

$$H_t(z)=\int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du,\qquad H_0(z)=\tfrac{1}{8}\,\xi\big(\tfrac{1}{2}+\tfrac{iz}{2}\big),$$

defined for *all* $t\in\mathbb{R}$ (RT 2020, eq. (4) and the line before it) [CITED]. As a family of functions it is a group too (formally $H_t=e^{-t\partial_z^2}H_0$, with H_t entire for every real t); *the irreversibility is in the property "all zeros real"*, not in the family.

Proposition 7.1 (Heat sign)**.** *(i)* $\partial_t H_t=-\partial_z^2 H_t$*: backward heat in the real height variable* $z=x$ *(RT 2020, p. 2, citing Csordas–Smith–Varga, their ref. [11, p. 114]) [CITED]. (ii) Writing* $z=x+iy$*,* $\partial_t H_t=-\partial_x^2 H_t=+\partial_y^2 H_t$*:* forward *heat in the direction transverse to the line. [THM]*

Proof. (i) Differentiate under the integral: ∂_t brings down u^2, and $\partial_z^2\cos(zu)=-u^2\cos(zu)$. (ii) H_t is holomorphic, so $\partial_x^2 H_t+\partial_y^2 H_t=0$. □

So the dBN flow smooths in the imaginary direction, which is the direction that measures distance from the line. That is the precise sense in which it "cools toward the line".

7.2. **The arrow and its gap.**

- *Pólya*: if H_t has only real zeros then so has $H_{t'}$ for every $t'>t$ (RT 2020, p. 2) [CITED]. Lemma 2: the number $Z_{\mathbb{C}}(H_t)$ of non-real zeros is non-increasing [TAA-L]; the number of *distinct* ghosts can rise (Lemma 2) [TAA-L].
- *de Bruijn 1950*: all zeros real for $t\ge\frac{1}{2}$; if at time t_0 all zeros satisfy $|\operatorname{Im} z|\le y_0$, then $\Lambda\le t_0+\frac{1}{2}y_0^2$ (Polymath 2019, Fig. 1 and §1) [CITED]: the strip containing the zeros narrows like $(y_0^2-2(t-t_0))_+^{1/2}$.
- *Newman 1976*: there is Λ with $-\infty<\Lambda\le\frac{1}{2}$ such that H_t has only real zeros iff $t\ge\Lambda$; RH $\iff\Lambda\le 0$. Newman conjectured $\Lambda\ge 0$ and remarked that this says "if the Riemann hypothesis is true, it is only 'barely so'" (RT 2020, p. 2) [CITED].
- *Ki–Kim–Lee 2009*: $\Lambda<\frac{1}{2}$ [CITED-sec] (via Polymath 2019, §1). *Polymath 2019*: $\Lambda\le 0.22$ [CITED]. *Platt–Trudgian 2021*: $\Lambda\le 0.2$ (Cor. 2) [CITED]. *Rodgers–Tao 2020*: $\Lambda\ge 0$ [CITED].

Proposition 7.2 (The gap is RH)**.** *Given Rodgers–Tao, RH* $\iff \Lambda = 0$*.* *[RESTATEMENT]* Thermal reading *[ANALOGY]: the heat flow guarantees equilibrium (all zeros real) at* $t = \frac{1}{2}$*, in fact at* $t = 0.2$*; it does not guarantee equilibrium at the present time* $t = 0$*; and since* $\Lambda \geq 0$*, the present is at best exactly at the critical point. "The second law guarantees eventual equilibrium, not equilibrium now; the gap is RH" is correct as a restatement and contains no information beyond* $\Lambda \leq 0.2$*.*

7.3. **Equilibrium states: Bost–Connes, Tomita–Takesaki, thermal time. Bost–Connes 1995.** A C^*-dynamical system with Hamiltonian $H\varepsilon_n = (\log n)\varepsilon_n$ on $\ell^2(\mathbb{N}^*)$, partition function $\operatorname{Tr} e^{-\beta H} = \zeta(\beta)$ for real $\beta > 1$, a unique KMS_β state for $0 < \beta \leq 1$, and spontaneous symmetry breaking for $\beta > 1$ with the extremal KMS states permuted by $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ [CITED]. The transition occurs at the *pole* $\beta = 1$; the zeros never enter (Lemmas 31, 39, 45) [TAA-L].

Tomita–Takesaki and KMS. A faithful normal state φ on a von Neumann algebra M determines the modular group $\sigma_t^{\varphi}(x) = \Delta^{it} x \Delta^{-it}$, and φ is a $(\sigma^{\varphi}, \beta)$-KMS state with $\beta = -1$ in Takesaki's convention, equivalently $\beta = 1$ for the flow $\alpha_t = \sigma^{\varphi}_{-t}$ (Takesaki 1970; Connes–Rovelli 1994, §2: "Any faithful state is a KMS state (with inverse temperature $\beta = 1$) with respect to the modular automorphism it itself generates", where they set $\alpha_t = \Delta^{-it} \cdot \Delta^{it}$) [CITED]. Uniqueness: σ^{φ} is the unique flow for which φ is KMS at that β (Takesaki's theorem) [BIB].

Thermal time. Connes–Rovelli propose that physical time is the modular flow of the state (the *thermal time hypothesis*) [CITED]; this is a physical hypothesis [ANALOGY].

Positive energy = polarization. For a one-parameter unitary group with generator $H \geq 0$ (no zero modes), the positive-frequency subspace gives a complex structure $J = H|H|^{-1}$ on the real symplectic space of solutions (Segal; the formula of Prop. 3.4's proof, Lemmas 53, 54) [TAA-L]. In this sense "positive energy" and "polarization" are the same datum.

Proposition 7.3 (Modular reading of RH)**.** *(i) If there is a von Neumann algebra* M*, a faithful normal state* φ *and a spectral* APD *whose scaling flow is* σ^{φ}*, then RH holds (Lemma 31; Theorem 3.9). [RESTATEMENT] (ii) Under RH, Lemma 1's Hilbert space carries a unitary scaling flow; that it is the modular flow of a state on an algebra* constructed zero-free *is [OPEN]. (iii)* Circularity. *The positivity in (i) is the positivity of* φ*, i.e. of the GNS inner product, i.e. Weil positivity: the modular formulation relocates* J *into* φ *and does not produce it (Lemma 31) [RESTATEMENT]. (iv) Bost–Connes' scaling is the* dual *action, not the modular flow of a state on the semilocal algebra in the sense needed (Lemma 39), and the positive-energy Hilbert–Pólya reading is impossible (Lemma 42) [TAA-L].*

Conjecture 7.4 (Thermal formulation; Conjecture/Analogy)**.** *"RH* $\iff$ *the arithmetic flow is the modular flow of a positive KMS state on the missing space." As an equivalence this is Prop. 7.3, i.e. [RESTATEMENT] with circular positivity. As a* statement about physics *(that the laws of thermodynamics force it) it is [ANALOGY]: the laws of thermodynamics are physical principles, not theorems about* ζ*, and no lemma derives a property of* ζ *from them.*

7.4. **Real time, imaginary time, and causality. (1) Wick rotation, not conjugation.** e^{itH} (real time) and $e^{-\tau H}$ (imaginary time) are related by $t = -i\tau$, a rotation of the time variable by $90°$ in $\mathbb{C}$; they are not a holomorphic/antiholomorphic pair (complex conjugation of the time variable sends e^{itH} to e^{-itH}, its inverse). [THM]

Proposition 7.5 (One holomorphic semigroup)**.** *Let* H *be self-adjoint and define* e^{-zH} *by the spectral theorem on its natural domain. (i)* $\|e^{-zH}\| = \sup_{\lambda \in \mathrm{spec} H} e^{-\operatorname{Re} z\, \lambda}$*. Hence* e^{-zH} *is bounded for one (equivalently all)* z *with* $\operatorname{Re} z > 0$ *iff* H *is bounded below, and it is a contraction for all* $\operatorname{Re} z \geq 0$ *iff* $H \geq 0$*. (ii) If* $H \geq 0$*,* $z \mapsto e^{-zH}$ *is holomorphic on* $\operatorname{Re} z > 0$*, strongly continuous on* $\operatorname{Re} z \geq 0$*, and its restrictions to* $z = -it$ *and* $z = \tau > 0$ *are the unitary group* e^{itH} *and the contraction semigroup* $e^{-\tau H}$*. [THM] (iii) Conversely (Osterwalder–Schrader 1973/1975): Euclidean correlation functions satisfying reflection positivity (and the other OS axioms) reconstruct a Hilbert space and a Hamiltonian* $H \geq 0$*, i.e. the real-time theory is the boundary value of the Euclidean one [CITED].*

Proof. (i) Functional calculus: the norm of a multiplication operator is the sup of the multiplier on the spectrum, and $|e^{-z\lambda}| = e^{-\operatorname{Re} z\,\lambda}$. (ii) Dominated convergence in the spectral integral. □

So the steering claim "both are boundary values of one holomorphic e^{-zH}, which exists iff $H \geq 0$" is correct with one precision: the half-plane extension exists (boundedly) iff H is bounded below, and is contractive iff $H \geq 0$; the two are equivalent up to the shift $H \mapsto H + c$.

(3) Causality. Titchmarsh's theorem (Titchmarsh 1948, Thm. 95 [CITED-sec], via Lemma 33): for $f \in L^2(\mathbb{R})$ the following are equivalent: $\hat{f}$ vanishes on a half-line (causality); f is the boundary value of a function in the Hardy space H^2 of a half-plane; the real and imaginary parts of f are Hilbert transforms of each other (Kramers–Kronig). By Paley–Wiener (via Mellin; Burnol, as in Lemma 53), $L^2(0,1)$ corresponds to $H^2(\operatorname{Re} s > \frac{1}{2})$, which is *exactly* the space H_+ of Lemma 53 [TAA-L]. Hence:

$$\begin{aligned}\text{polarization } (H_+ \subset L^2) \;&=\; \text{positive energy}\\ &=\; \text{analytic continuation to a half-plane} \;=\; \text{the space of causal responses.}\end{aligned}$$

This chain is [THM] given Titchmarsh and Paley–Wiener. **But causality of ζ is free.** The relevant ζ-response is causal unconditionally (Lemma 33) and lies in H_+; RH is the strictly stronger statement that it is *outer* (minimum phase; Lemmas 48, 53) [TAA-L][RESTATEMENT]. So "polarization = positive energy = causality = analytic continuation" is correct for the *space*; for ζ the content is not causality but outerness. (N7.)

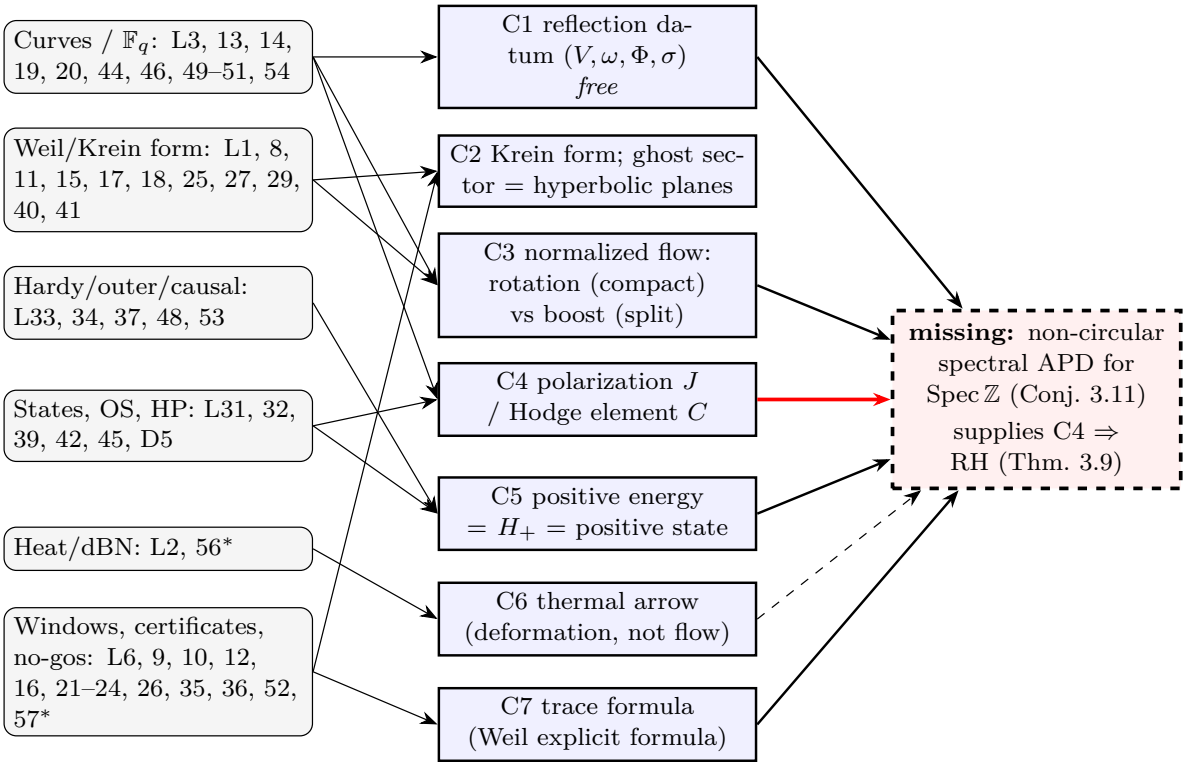
(4) dBN is not the Wick rotation of scaling.

Proposition 7.6 (Chirp versus translation)**.** *Work in the log-scale variable u, Fourier-dual to the height z. (i) Scaling acts on test functions by translation, $(T_s F)(u) = F(u - s)$ (multiplication by e^{isz} on the height side), with generator $-i\partial_u$. Its imaginary-time continuation $s = -i\tau$ on the zero side is $\sum_\gamma e^{-\tau\gamma}$, Cramér's function $S_+(\tau)$ (Lemma 32: RH $\iff$ S_+ completely monotone $\iff$ OS positivity of the associated kernel) [TAA-L]. (ii) The dBN deformation is multiplication of the* kernel $\Phi(u)$ *by e^{tu^2}, which is the continuation $a = -it$ of the unitary* chirp e^{iau^2} *on $L^2(\mathbb{R}, du)$ (generator u^2; on the height side the free Schrödinger group $e^{ia\partial_z^2}$). [THM] (iii) The two generators do not commute: $[\partial_u, u^2] = 2u$. [THM] (iv) The chirp does not preserve Weil's form: writing $\mathcal{Q}(F,G) = \iint W(u-v)F(u)\overline{G(v)}$ with W the Weil distribution, the chirp replaces $W(u-v)$ by $e^{ia(u^2-v^2)}W(u-v)$, which is a function of $u - v$ only if W is supported at 0; Weil's W has the prime-power atoms at $\pm\log n$ [THM].*

Proof. (ii) is the definition of H_t. (iii) direct. (iv) $u^2 - v^2 = (u-v)(u+v)$; on the support of $W(u-v)$ away from $u = v$ the factor depends on $u + v$. □

Do they relate? Both are imaginary-time objects built from the same Φ: the dBN flow is the Euclidean version of a chirp acting on the datum; Cramér's S_+ is the Euclidean version of scaling acting on the zero side. RH is "$\Lambda \leq 0$" for the first and "S_+ completely monotone" for the second. Whether there is a direct implication between the *rates* (e.g. a "cooling law" relating Λ-type margins to Cramér/OS margins) is [OPEN] (Target 5); at present the relation is [ANALOGY].

8. Diagram



Left: lemma groups (* in progress). Middle: the seven concepts. Right: the missing object. Over $\mathbb{F}_q$ every arrow into the right box exists (Weil, Serre, Grothendieck–Kleiman); for Spec $\mathbb{Z}$ the red arrow C4 is missing and C6 is only an analogy (dashed).

9. Dictionary

TAA word	precise mathematics	known analogue	label
ghost	zero with $\operatorname{Re}\rho > \frac{1}{2}$; hyperbolic plane of the Krein form	split torus direction	R
rotation / boost	elliptic / hyperbolic Jordan part of $\tilde{\Phi}$ (Prop. 3.4)	compact / split Cartan	T
polarization	J of Def. 3.2; Hodge element C	Weil operator $h(i)$; Rosati involution	T/C
vacuum	fixed point of $\tilde{\Phi}$ in Siegel space	K_C-fixed point of G/K_C	T
action	Weil form $\mathcal{Q}$ (D5)	intersection form on $X \times X$	A
light cone	Lorentzian cone, index 1 (L14, 15)	Hodge index cone of NS	R
$\sqrt{q}$	choice of $c^{1/2}$ (half Tate twist)	metaplectic/spin double cover	T
ghost charge	$-$#ghosts (L53)	index of a Krein space	T
passive line	ζ-response passive/outer (L37, 48)	minimum-phase filter	R
causality	H^2 boundary values (Titchmarsh)	Kramers–Kronig	C^s
positive energy	$H \geq 0$; $J = H\|H\|^{-1}$ (Segal)	polarization	T
Wick rotation	$t = -i\tau$ on e^{-zH} (Prop. 7.5)	OS reconstruction	T/C
Euclidean scaling	Cramér $S_+(\tau) = \sum e^{-\tau\gamma}$ (L32)	OS positivity	R
second law	$Z_{\mathbb{C}}(H_t)$ non-increasing; Pólya persistence	entropy increase	C/A
hot $\to$ cold	dBN: forward heat transverse to the line (Prop. 7.1)	cooling	T/A
temperature $\beta = 1$	BC pole; Tomita KMS normalization	phase transition	C

TAA word	precise mathematics	known analogue	label
thermal time	modular flow σ^{φ}	Connes–Rovelli hypothesis	A
Cassinian / saddle	level sets of $\lvert\xi\rvert$, saddle type (L55)	Morse theory	T
motive	universal cohomology, polarized (standard conj.)	$\mathcal{M}_{\mathrm{num}}(\mathbb{F}_q)$	A
topos / ∞-topos	sheaf (higher) categories: spaces, coefficients	arithmetic/scaling site	A
HoTT	type theory with models in ∞-topoi	—	M

10. Ranked open targets

Score = 0.6·significance+0.4·probability of a clean result in one or two lemmas (both 0–10, judgment calls [HEUR]).

(1) **Structured prime-side Weil ladder** (sig 6, prob 7, **6.4**). Isolate the prime-by-prime positive cone and its lowest eigenvector on growing windows, and measure the margin against DH (P10). Synergy with Lemma 57 in progress. [OPEN]

(2) **Partial Lorentzian index theorem on zero-free non-window families** (sig 7, prob 4, **5.8**). Prove Lemma 15's index ≤ 1 on an infinite-dimensional family not reducible to finite windows (P7, P8). A genuine Hodge-index-type statement for ζ in a restricted class. [OPEN]

(3) **Tannakian no-go at finite** S (sig 5, prob 7, **5.8**). Prove that no polarized Tannakian category with a Frobenius-type element realizes the semilocal Weil datum at finite S, via its Lebesgue spectrum (P1; Prop. 4.2). Turns the motive analogy into a theorem about what the missing category cannot be. [OPEN]

(4) **Modular-state no-go or construction on the semilocal algebra** (sig 6, prob 5, **5.6**). Decide whether the scaling flow on a semilocal crossed product is the modular flow of a faithful normal state, and compute where positivity fails (Prop. 7.3; Lemma 39). [OPEN]

(5) **dBN versus Cramér cooling law** (sig 5, prob 6, **5.4**). Relate the dBN zero dynamics (Euclidean chirp) to Cramér/OS margins (Euclidean scaling), Prop. 7.6; e.g. bound the OS defect of S_+ for H_t in terms of $t - \Lambda$ (thermal arrow, §7). [OPEN]

Runner-up: the ξ saddle converse (Lemma 56, sig 4, prob 7, 5.2).

11. Corrections

C1 *"Polarized Tannakian categories: polarization $\iff$ Cartan involution making the real form compact."* **Essentially correct, with precisions.** DM 4.23/4.24: a Hodge element C exists iff G is an inner form of a compact group, and C is Hodge iff the *inner twist* of G by $\mathrm{ad}\, C$ is compact (so $\mathrm{ad}\, C$ is a Cartan involution); G itself need not be compact. In the non-neutral case the statement is about the band (Thm. 4.17). In the graded case one needs $C^2 = w(-1)$ (§5). It is a category-level statement; for an *element* the statement is Prop. 4.2 ($\gamma \in \mathrm{Cent}(C)$). Finite-dimensionality prevents literal transfer to ζ.

C2 *"Hodge standard conjecture $\Rightarrow$ Weil RH over finite fields."* **Incomplete.** One needs Hodge type I *and* Lefschetz type B (Kleiman 1968, 3.11; Milne *pRH* 3.15). For surfaces I is the Hodge index theorem; RH was proved by Deligne without the standard conjectures.

C3 *"Weil's proof for curves uses Castelnuovo–Severi/Hodge index."* **Half correct.** Weil used the Castelnuovo positivity of correspondences (proved directly) and, in the second proof, Rosati positivity on the Jacobian. Deriving Castelnuovo–Severi from the Hodge index theorem on $X \times X$ is due to Mattuck–Tate (via Riemann–Roch) and Grothendieck (1958).

C4 *"Forward heat flow only pushes zeros onto the line."* **Imprecise.** Correct statements: real zeros stay real (Pólya); $Z_{\mathbb{C}}(H_t)$ is non-increasing (Lemma 2); the zero strip narrows (de Bruijn); the flow is forward heat transverse to the line (Prop. 7.1). Not claimed: monotone motion of each zero, and the number of distinct ghosts can rise (Lemma 2).

C5 *"RH $\iff \Lambda = 0$ given Rodgers–Tao."* **Correct** (Newman: RH $\iff \Lambda \le 0$; RT: $\Lambda \ge 0$).

C6 *"Polymath: $\Lambda \le 0.2$."* **Corrected:** Polymath 2019 gives $\Lambda \le 0.22$; $\Lambda \le 0.2$ is Platt–Trudgian 2021 (Cor. 2).

C7 *"BC phase transition at $\beta = 1$ with ζ as partition function."* **Correct**, for real $\beta > 1$; the transition is at the pole, and the zeros play no role.

C8 *"A faithful state determines its modular flow and satisfies KMS at $\beta = 1$."* **Correct up to convention**: faithful *normal* on a von Neumann algebra (or via GNS for C^*); $\beta = -1$ for $\sigma_t = \Delta^{it} \cdot \Delta^{-it}$, $\beta = 1$ for the reversed flow (Connes–Rovelli's convention).

C9 *Steering (1): "e^{itH} and $e^{-\tau H}$ are Wick rotations of each other, not holomorphic vs antiholomorphic."* **Correct** (§7.4).

C10 *Steering (2): "both are boundary values of the holomorphic e^{-zH}, which exists iff $H \ge 0$."* **Correct with precision**: bounded extension iff H bounded below; contractive iff $H \ge 0$ (Prop. 7.5).

C11 *Steering (3): "causal $\iff$ analytic in a half-plane $\iff$ H_+; so polarization = positive energy = causality = analytic continuation."* **Correct for the space, wrong as an RH criterion**: causality of the ζ-response is free (Lemma 33); RH is outerness (Lemma 48).

C12 *Steering (4): "dBN is backward heat $\partial_t H_t = -\partial_z^2 H_t$ in the height, not the Wick rotation of scaling."* **Correct**, plus: it is forward heat in $\operatorname{Im} z$, and the Wick continuation of a chirp; the Wick continuation of scaling is Cramér's S_+ (Prop. 7.6); their relation is open.

C13 *[t212u] "Frobenius/scaling is holomorphic, heat flow antiholomorphic."* **Not correct as stated.** All three flows act by holomorphic multipliers in the height (e^{isz}, and H_t is entire in z for every t). The true contrast is group acting on a fixed datum versus deformation of the datum, and unitary (real time) versus Euclidean (imaginary time).

12. Questions for J.W.

Q1 Adopt the arithmetic polarization datum (Def. 3.3) and Conjecture 3.11 as TAA's official statement of the missing object, replacing the informal "G5"?

Q2 Adopt the via-negativa sheet (P1–P11, N1–N17) as a standing checklist: every future proposal must say which P it supplies and that it avoids N1–N17?

Q3 Which target next: the prime-side ladder (synergy with Lemma 57), the partial index theorem (higher risk), or the Tannakian no-go (cleanest)?

Q4 Adopt the thermal phrasing "the second law guarantees equilibrium at $t = 0.2$, not at $t = 0$; the gap is RH" (Prop. 7.2), with dBN recorded as a *deformation* (Euclidean chirp) and not as the Wick rotation of scaling?

Q5 Keep motives, topoi and HoTT as [ANALOGY]/[METAPHOR], with DM 4.23–4.25 (Prop. 4.2) as the only formal bridge, and drop HoTT claims from future drafts?

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TAA LEMMA 59: THE DERIVATIVE TOWER OF ξ, LAGUERRE–PÓLYA POSITIVITY, AND GHOSTS

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

Date: October 22, 2026.

ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Every RH-equivalent statement below is a [RESTATEMENT] of known results. The unconditional theorems quoted (Turán, higher-order Turán, Jensen polynomials of small degree or large shift, Conrey, Kim, Ki) are the work of their authors, and none of them implies RH. What this lemma adds is organizational: an elementary generating identity that shows exactly what the Laguerre–Pólya positivities are, a tower form of Lemma 56's ghost counter, and numerics on ζ and on Davenport–Heilbronn (DH). Tags: [THM] proved here (elementary) or classical with proof given; [CITED] checked at the source (`lit/lemma59/verified_refs.md`); [CITED-sec] checked only via a named secondary source; [BIB] bibliographic record only; [TAA-L] earlier TAA lemma; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEUR] heuristic; [CONJ] conjecture; [OPEN] open; [NUM] numerical (`num/lemma59_*.py`, data in `num/l59/`).

Answer in brief. (1) *The class.* $\Xi(t) = \xi(\frac{1}{2} + it)$ is real entire of order 1. RH $\iff$ $\Xi \in \mathcal{L}\text{-}\mathcal{P}$, the Laguerre–Pólya class: locally uniform limits of real-rooted real polynomials. $\mathcal{L}\text{-}\mathcal{P}$ is closed under $D = d/dt$ and under $e^{-\lambda D^2}$ ($\lambda \geq 0$, the de Bruijn–Newman direction) [CITED][THM]. So RH $\Rightarrow$ every $\Xi^{(n)}$ has only real zeros (Conrey) [CITED].

(2) *What the positivity inequalities are* (§3). With $L_n(f) = \frac{1}{(2n)!}\sum_k (-1)^{k+n}\binom{2n}{k} f^{(k)} f^{(2n-k)}$, every real entire f satisfies the identity

$$f(x+iy)\, f(x-iy) = \sum_{n\geq 0} L_n(f)(x)\, y^{2n} \qquad (x, y \in \mathbb{R})$$

(Thm. 3.1, [THM], checked numerically to 10^{-13}). Hence $f \in \mathcal{L}\text{-}\mathcal{P} \Rightarrow L_n \geq 0$, and conversely a non-real zero $x_0 \pm iy_0$ forces $L_n(f)(x_0) < 0$ for some $n \geq 1$ *at its own abscissa* (Thm. 3.2). So RH $\iff$ $L_n(\Xi)(x) \geq 0$ for all n and x [RESTATEMENT]. The *Laguerre inequalities are the Taylor coefficients, in y^2, of $|\Xi|^2$ off the line.* The Turán inequalities are their moment shadow at $t = 0$.

(3) *Verdict on J.W.'s standing condition* (§3.4). The Laguerre–Pólya positivities are *zero location in disguise*. They are not zero *counting*: they are pointwise and local, since each x tests only the zeros with real part x. But they carry no new positivity. Each L_n is an *indefinite* form of signature $(n+1, n)$ on jets, evaluated on the jet curve of Ξ (Prop. 3.5). That is the Krein shape of Lemma 58, not a polarization: there is no positive form and no flow-invariant J. The rungs that are proven unconditionally (Turán, higher Turán, $d \leq 8$, large shift, Ki's cosine limit) are blind to RH: they hold for DH too, which has ghosts (numerically, and by Farmer for Ki's theorem). The one genuine zero-free positivity in sight is $\Phi > 0$ (Lemma 31). It gives the *opposite* inequality $\beta_m^2 \leq \beta_{m-1}\beta_{m+1}$, while LP demands $\beta_m^2 \geq \frac{2m-1}{2m+1}\beta_{m-1}\beta_{m+1}$. The window between these two inequalities is where RH lives at order 2 (Thm. 3.6).

(4) *Tower and ghosts* (§4). Let G_n be the number of ghost couples of $F_n = \Xi^{(n)}$ in a window and τ_n the number of transverse critical points of F_n. Applying Kim's window theorem at every level gives $\kappa_n = \tau_n + G_{n+1} = G_n$, so $G_{n+1} = G_n - \tau_n \leq G_n$: a discrete second law. With Kim's local reality theorem this gives $G_0 = \sum_{n\geq 0} \tau_n$ (Thm. 4.4; Fourier's real-axis count, Ki–Kim). Hence RH $\iff$ no level of the tower has a transverse critical point [RESTATEMENT]. This purely real-axis form of Lemma 56's (H) replaces the off-line saddles S^+ by $\tau_1, \tau_2, \ldots$.

(5) *Numerics* [NUM]. For ζ, $t \in [-5, 200]$: $G_n = \tau_n = \kappa_n = 0$ for $n \leq 10$, $L_1, L_2, L_3 \geq 0$ on $[0, 1000]$, Turán margins positive for $m \leq 149$, $J^{d,0}$ hyperbolic for $d \leq 150$. For DH: every one of the 28 ghosts below height 1000 has $G_0 = \tau_0 = 1$ and $G_n = 0$ for $n \geq 1$, so *one derivative removes every ghost*, whatever its $\delta \in [0.016, 0.370]$, while the heat flow needs $\lambda_c \approx \delta^2/2$. Every ghost has $L_1(\gamma) < 0$; on $[0, 1000]$ the set $\{L_1(\Xi_{\mathrm{DH}}) < 0\}$ consists of exactly 28 intervals, one around each ghost, of length $\approx 2\delta$, while $L_2, L_3 \geq 0$ throughout. Yet DH satisfies all Turán inequalities checked ($m \leq 149$), and its kernel satisfies the Csordas–Norfolk–Varga log-concavity hypothesis. Differentiation and heat flow are both LP-preserving and ghost-non-increasing. Within that common class, the single-step operators $1 - \lambda D^2$ and $1 - cD$ remove an isolated couple at $\lambda = \delta^2/2$ and at $c = \delta$ respectively (§4.1).

(6) *The C^∞/analytic threshold* (§6). On the Fourier side, the threshold is the passage from subexponential to exponential decay of the kernel. Ξ (doubly exponential kernel) is entire of order 1 and lies in $\bigcap_{s>0} G^s$; the de Bruijn–Newman flow never moves it. Regularity invariants cannot see RH (Prop. 6.9). QED/QCD divergent series and the flat Λ_{QCD} are an [ANALOGY] only, with the tagline corrected.

(7) *Atlas* (§7): analyticity is stronger than C^∞, ξ is entire so one chart suffices, and analytic continuation is the power-series atlas [ANALOGY]. *Prior work* (§8), *corrections* to Grok Bot's chat claims (§9), and Q1–Q6.

Question (J.W., [t220u], [t221u]). "Is there a second and third or perhaps even up to infinite derivative of Ξ? ... to keep a smooth transition between charts there must be an infinite differentiable transition function. Is this applicable to derivatives of Ξ?" Then: "Write it up as a short lemma. This topic seems high yield. Can you give me more information on the Laguerre–Pólya membership as a family of positivity inequalities? Has any of our previous lemmas touched on this work or added to it, refined it?"

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1. Conventions

$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and $\Xi(t) = \xi(\frac{1}{2} + it)$: even, real on $\mathbb{R}$, entire of order 1 [BIB]. A *ghost* is a zero $\rho = \frac{1}{2} + \delta + i\gamma$ with $\delta > 0$. In the t-plane it is the conjugate couple $t = \gamma \mp i\delta$; together with $\bar{\rho}$ it gives four zeros of Ξ. We write $F_n := \Xi^{(n)} = d^n\Xi/dt^n$; then $F_n(t) = i^n\xi^{(n)}(\frac{1}{2} + it)$, so the zeros of F_n are those of $\xi^{(n)}$. DH: $\Xi_{\mathrm{DH}}(t) = \Lambda_{\mathrm{DH}}(\frac{1}{2} + it)$ with $\Lambda_{\mathrm{DH}}(s) = (5/\pi)^{s/2}\Gamma(\frac{s+1}{2})f_{\mathrm{DH}}(s)$, as in Lemmas 55–56. It is real on $\mathbb{R}$ (the constant phase was checked to be 1 to 10^{-17}), even, and of order 1. De Bruijn–Newman: $H_t(z) = \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du$ with $H_0(z) = \frac{1}{8}\Xi(z/2)$ [16] [CITED]. In the t-variable, $H_\tau(2t) = e^{-\lambda D^2}H_0(2t)$ with $\lambda = \tau/4$. So Lemma 56's landing times t_c correspond to $\lambda_c = t_c/4$, and an isolated couple at height δ lands at $\lambda = \delta^2/2$ ($t_c = 2\delta^2$).

2. The Laguerre–Pólya class

Definition 2.1 ([4] (1.4), [17] (1.2))**.** $\mathcal{L}$-$\mathcal{P}$ is the set of real entire functions

$$f(x) = Cx^m e^{-\alpha x^2+\beta x}\prod_j (1-x/x_j)e^{x/x_j}, \qquad \alpha \geq 0,\ \beta, C \in \mathbb{R},\ x_j \in \mathbb{R}\setminus\{0\},\ \textstyle\sum_j x_j^{-2} < \infty.$$

Proposition 2.2. *(a) $\mathcal{L}$-$\mathcal{P}$ is exactly the set of locally uniform limits on $\mathbb{C}$ of real polynomials with only real zeros* [2] *[CITED-sec] (classical: Laguerre, Pólya).*

(b) $f \in \mathcal{L}$-$\mathcal{P} \Rightarrow f' \in \mathcal{L}$-$\mathcal{P}$. More generally, a real entire function of order < 2 with only real zeros keeps this property under differentiation [9, §2] *[CITED].*

(c) For $\lambda \geq 0$ and $c \in \mathbb{R}$, the operators $e^{-\lambda D^2}$, $1-\lambda D^2$ and $1-cD$ map $\mathcal{L}$-$\mathcal{P}$ into $\mathcal{L}$-$\mathcal{P}$, where they converge (for $e^{-\lambda D^2}$, on functions of order < 2 such as Ξ) [THM]. For $e^{-\lambda D^2}$ applied to H_0 this is Pólya's theorem: if H_t has only real zeros then so does $H_{t'}$ for every $t' > t$ [16] *[CITED].*

(d) A real entire f of order < 2 with only real zeros lies in $\mathcal{L}$-$\mathcal{P}$ (Hadamard; the factor $e^{-\alpha x^2}$ is absent) [THM]. Hence RH $\iff \Xi \in \mathcal{L}$-$\mathcal{P}$ [4, §1] *[RESTATEMENT].*

Proof of (c) and (b). For a real-rooted real polynomial p and $c \neq 0$, $p - cp' = -c\,e^{x/c}\,(e^{-x/c}p)'$. By Rolle, $e^{-x/c}p$ has a critical point between consecutive zeros and one more beyond the extreme zero on the side where $e^{-x/c} \to 0$. Counting with multiplicity, $p-cp'$ has $\deg p$ real zeros. So $1-cD$ preserves real-rootedness, and so does $1-\lambda D^2 = (1-\sqrt{\lambda}D)(1+\sqrt{\lambda}D)$. Further, $e^{-\lambda D^2} = \lim_N (1-\lambda D^2/N)^N$ on polynomials, and on $\mathcal{L}$-$\mathcal{P}$ by (a) and Hurwitz's theorem. Lemma 2 proves the stronger statement that the number of non-real zeros never increases [TAA-L]. The derivative is $p' = \lim_{c\to\infty}(p-cp')/(-c)$, or directly Rolle. □

Remark 2.3 (Fourier side)**.** Write $\Xi(t) = \int_{\mathbb{R}} \widetilde{\Phi}(u)e^{itu}\,du$ with $\widetilde{\Phi}(u) = 2\Phi(u/2) > 0$ even (RT20 normalization; Lemma 31) [THM]. Then $D \leftrightarrow iu$, $e^{-\lambda D^2} \leftrightarrow e^{\lambda u^2}$, $1-\lambda D^2 \leftrightarrow 1+\lambda u^2$, and $D^{2n} \leftrightarrow (-1)^n u^{2n}$. The even-order ones are positive multipliers (up to the sign $(-1)^n$), and all of them *shift weight to large $|u|$*, i.e. towards the cusp of the theta function in Φ. The translation group $e^{-cD} \leftrightarrow e^{-icu}$ is unitary and moves no zero off or onto the line [THM].

3. Laguerre–Pólya membership as positivity inequalities

3.1. **The Laguerre inequalities and a generating identity.** For a real entire f and $n \geq 0$ put

$$L_n(f)(x) := \frac{1}{(2n)!}\sum_{k=0}^{2n}(-1)^{k+n}\binom{2n}{k}f^{(k)}(x)f^{(2n-k)}(x).$$

So $L_0 = f^2$, $L_1 = f'^2 - ff''$, and $L_2 = \frac{1}{24}(6f''^2 - 8f'f''' + 2ff'''')$. Wang–Yang's normalization [17] is $\frac{(2n)!}{2}L_n$, a positive multiple, so the signs agree. The inequalities $L_n(f) \geq 0$ are Jensen's higher-order Laguerre inequalities (1913; $n = 1$ is Laguerre's) [17] [CITED-sec].

Theorem 3.1 (generating identity)**.** *For every real entire f and all real x, y,*

$$f(x+iy)\,f(x-iy) = |f(x+iy)|^2 = \sum_{n=0}^{\infty} L_n(f)(x)\,y^{2n},$$

with absolute convergence. [THM]

Proof. By Taylor's theorem, $f(x+iy)f(x-iy) = \sum_{j,k} f^{(j)}(x)f^{(k)}(x)\,i^j(-i)^k y^{j+k}/(j!k!)$, absolutely convergent since f is entire. The coefficient of y^m is $i^m\sum_{j+k=m}(-1)^k f^{(j)}f^{(k)}/(j!k!)$. For odd m the terms (j,k) and (k,j) cancel. For $m = 2n$ it is $(-1)^n\sum_k(-1)^k\binom{2n}{k}f^{(2n-k)}f^{(k)}/(2n)! = L_n(f)(x)$. Since f is real, $f(x-iy) = \overline{f(x+iy)}$. □

Check [NUM] (`lemma59_laguerre.py ident`). Truncating at $n = 20$ reproduces $|\Xi(x+iy)|^2$ with relative error $1.8 \cdot 10^{-16}$ at $(30, 0.2)$ and 0 at $(100, 0.4)$, and $|\Xi_{\mathrm{DH}}|^2$ with $5 \cdot 10^{-16}$ at $(85.7, 0.1)$ and $9.5 \cdot 10^{-14}$ at $(85.7, 0.3)$.

Theorem 3.2 (Laguerre inequalities characterize real zeros, pointwise)**.** *Let $f \not\equiv 0$ be real entire.*

(a) If $f \in \mathcal{L}\text{-}\mathcal{P}$, then $L_n(f)(x) \geq 0$ for all $n \geq 0$ and all real x. More precisely, for each x the function $E_x(w) := f(x + i\sqrt{w})f(x - i\sqrt{w}) = \sum_n L_n(f)(x)w^n$ is entire in w, lies in $\mathcal{L}\text{-}\mathcal{P}$, and has only zeros $w \leq 0$.

(b) (Localization.) *If $f(x_0 \pm iy_0) = 0$ with $y_0 \neq 0$, then $\sum_n L_n(f)(x_0)y_0^{2n} = 0$. Hence $L_n(f)(x_0) < 0$ for some $n \geq 1$, unless f and all its derivatives vanish at x_0, which is impossible. In terms of E_x: the positive zeros of E_x are exactly the numbers y_0^2 for the non-real zeros $x \pm iy_0$ with real part x.*

(c) Consequently: f has only real zeros $\iff$ $L_n(f)(x) \geq 0$ for all n, x $\iff$ no E_x has a positive zero. For f of order < 2 these are equivalent to $f \in \mathcal{L}\text{-}\mathcal{P}$. In particular

$$RH \iff L_n(\Xi)(x) \geq 0 \quad \forall n \geq 1, \ \forall x \in \mathbb{R},$$

and a ghost $\gamma \mp i\delta$ violates some $L_n(\Xi)(\gamma) \geq 0$ exactly at its own height γ. [THM][RESTATEMENT]

Proof. (a) For a factor $(x - a)$, $a \in \mathbb{R}$: $|x + iy - a|^2 = (x - a)^2 + w$ with $w = y^2$. For $e^{x/a}$: $|e^{(x+iy)/a}|^2 = e^{2x/a}$, independent of w. For $e^{\beta x}$, the same. For $e^{-\alpha x^2}$: $|e^{-\alpha(x+iy)^2}|^2 = e^{-2\alpha x^2}e^{2\alpha w}$. Each factor is in $\mathcal{L}\text{-}\mathcal{P}$ as a function of w, with non-negative Taylor coefficients and zeros only at $w = -(x - a)^2 \leq 0$. The canonical product converges locally uniformly in w, so E_x is a locally uniform limit of products of such factors. Hence it has non-negative coefficients and only non-positive zeros (Hurwitz). (b) Put $w = y_0^2$ in Thm. 3.1. If all $L_n(x_0) \geq 0$, every term vanishes; $L_0 = f(x_0)^2$ gives $f(x_0) = 0$. Inductively, if $f^{(j)}(x_0) = 0$ for $j < n$, then $L_n(x_0) = f^{(n)}(x_0)^2/(n!)^2$, so $f^{(n)}(x_0) = 0$, and $f \equiv 0$. For the zeros of E_x: $x + i\sqrt{w} = x_0 + iy_0$ gives $w = (y_0 - i(x_0 - x))^2$, which is real and positive iff $x = x_0$; real zeros a give $w = -(x - a)^2$. (c) follows from (a), (b) and Prop. 2.2(d). □

Remark 3.3 (attribution)**.** (a) is classical (Laguerre for $n = 1$; Jensen 1913 for higher n, as reported in [17]) [CITED-sec]. "RH $\iff$ all Laguerre inequalities" is attributed to Csordas–Varga [4], and "$L_n(f) \geq 0$ for all $n, x \Rightarrow f \in \mathcal{L}\text{-}\mathcal{P}$" to Csordas–Vishnyakova [5], both as reported in [17, §1] [CITED-sec]. The proof above is self-contained and short, and the identity of Thm. 3.1 is surely classical. *We claim no novelty.* What we use is the *form* of the statement: the inequalities are the w-Taylor coefficients of $|f|^2$ at distance $\sqrt{w}$ from the line.

Corollary 3.4 (relations to earlier TAA positivities)**.** *(a) If all $L_n(f)(x) \geq 0$, then $y \mapsto |f(x + iy)|$ is non-decreasing in $|y|$. For Ξ this is the statement that $|\xi(\sigma + it)|$ increases with $|\sigma - \frac{1}{2}|$: Lagarias/Sondow–Dumitrescu, i.e. Lemma 37's $G5_{\mathrm{line}}$ and Lemma 55's "valley along the line" [THM][TAA-L]. So the Laguerre family is a* coefficientwise *strengthening of the line-monotonicity positivity. Both are equivalent to RH, so logically nothing is gained [RESTATEMENT].*

(b) At a critical point $f'(x) = 0$ we have $L_1 = -ff''$. So Lemma 55's orientation (longitudinal $\iff f''/f < 0$) is the sign of L_1 at critical points, and a transverse saddle is a point where $L_1 < 0$ [THM][TAA-L].

3.2. **The quadratic-form view: indefinite, not positive.** Let $j_{2n}(x) = (f(x), f'(x), \ldots, f^{(2n)}(x)) \in \mathbb{R}^{2n+1}$ be the jet of f at x, and let K_n be the anti-diagonal symmetric matrix with $(K_n)_{k,2n-k} = (-1)^{k+n}\binom{2n}{k}/(2n)!$. Then $L_n(f)(x) = j_{2n}(x)^\top K_n j_{2n}(x)$.

Proposition 3.5. *K_n has signature $(n+1, n)$. The quadratic form $j \mapsto j^\top K_n j$ is therefore indefinite for $n \geq 1$, and the Laguerre inequality of order n asks only that the jet curve $x \mapsto j_{2n}(x)$ stay in its*

positive cone. The translations $x \mapsto x + a$ act on jets by the unipotent matrix e^{aN} (N the shift) and do not preserve K_n. Example: for $n = 1$ on quadratic jets, L_1 changes by a $f'f'' + \frac{a^2}{2} f''^2$. [THM]

Proof. For $k < n$ the pair of indices $\{k, 2n-k\}$ spans a block $\left(\begin{smallmatrix} 0 & a_k \\ a_k & 0 \end{smallmatrix}\right)$, with eigenvalues $\pm a_k$ and $a_k \neq 0$. The middle entry is $\binom{2n}{n}/(2n)! > 0$. The translation formula is a direct expansion. □

3.3. **Turán inequalities: the moment shadow at $t = 0$.** Let $\beta_m := (-1)^m \Xi^{(2m)}(0) = \int_{\mathbb{R}} u^{2m} \widetilde{\Phi}(u)\, du > 0$. Up to the normalization $\beta_m \propto 4^m b_m$, these are the moments $b_m = \int_0^\infty t^{2m} \Phi(t)\, dt$ of [3], and Turán ratios are invariant under $b_m \mapsto c r^m b_m$. Put $\rho_m = \beta_m^2/(\beta_{m-1}\beta_{m+1})$.

Theorem 3.6 (the moment sandwich)**.** *For every $m \geq 1$,*

$$\frac{2m-1}{2m+1} \;\leq\; \rho_m \;\leq\; 1.$$

The right inequality, and more generally positive-definiteness of the Hankel form $p \mapsto \int |p(u)|^2 \widetilde{\Phi}(u)\, du$, follows from $\Phi > 0$ by Cauchy–Schwarz. It is free, zero-independent, and is the positive-definiteness of Ξ in Lemma 31 [THM][TAA-L]. CNV state it as their (1.10) [3] *[CITED]. The left inequality is the family of* Turán inequalities*. They are necessary for $\Xi \in \mathcal{L}$-$\mathcal{P}$ (Pólya–Schur; apply the Turán condition $\gamma_m^2 \geq \gamma_{m-1}\gamma_{m+1}$ to the order-$\frac{1}{2}$ function $\Xi(\sqrt{z}) = \sum \gamma_m z^m/m!$, $\gamma_m = (-1)^m \beta_m m!/(2m)!$). Csordas–Norfolk–Varga proved all of them unconditionally, settling Pólya's 1927 problem* [3] *[CITED].*

Thus the free positivity bounds ρ_m from above and the LP positivity bounds it from below. The Turán window $[\frac{2m-1}{2m+1}, 1]$ closes like $1/m$.

Theorem 3.7 (what is proven beyond Turán)**.**

(a) Higher-order Turán inequalities *for ξ (equivalently, the degree-3 Jensen polynomials are hyperbolic): Dimitrov–Lucas* [8] *[CITED].*

(b) Jensen polynomials. $J_\gamma^{d,n}(X) = \sum_{j \leq d} \binom{d}{j} \gamma(n+j) X^j$. *Pólya 1927: RH $\iff$ all $J_\gamma^{d,n}$ are hyperbolic, and shift $n = 0$ suffices* [10] *[CITED-sec][RESTATEMENT]. Griffin–Ono–Rolen–Zagier: for each $d \geq 1$, $J_\gamma^{d,n}$ is hyperbolic for all sufficiently large n (Thm. 1), and for $d \leq 8$ for all n (Thm. 2), with Hermite-polynomial limits* [10] *[CITED]. Chasse: $J_\gamma^{d,0}$ is hyperbolic for $d \leq 2 \cdot 10^{17}$* [10] *(footnote 1) [CITED-sec]. As Farmer reports, RH up to height T gives $d < T^2$* [9, §4.2] *[CITED].*

(c) Laguerre inequalities for the Maclaurin coefficients *of order r hold for shifts $n > cr^3$* [17] *[CITED].*

(d) Mechanism behind Turán *(as described in* [8]*): CNV prove that if a kernel $K > 0$ is even, sufficiently smooth, fast decaying, and $\log K(\sqrt{t})$ is concave, then the Turán inequalities hold, and they prove that $\log \Phi(\sqrt{t})$ is concave [CITED-sec].*

Each item covers a finite *part of the LP condition (fixed degree, or large shift). The full family is a [RESTATEMENT].*

Remark 3.8 (Farmer's critique)**.** Farmer [9] argues that Jensen polynomials are not a plausible route to RH. (i) Large-shift hyperbolicity follows from the general fact that repeated differentiation tends to the cosine (Ki's theorem), which also holds in the extended Selberg class, i.e. without an Euler product. (ii) Jensen polynomials "disperse the information": in his example X_{10} (cosine with two zero pairs moved to $\pm 10 \pm i$), every zero of X_{10}' is real, and $J^{d,0}$ is real-rooted up to $d = 118$. (iii) In his taxonomy they are a *repackaging* (type (B)) [CITED]. Our numerics (§5) confirm (i)–(ii) for DH.

3.4. **Verdict: is Laguerre–Pólya positivity a polarization?** J.W.'s condition ([t209u]) is to pursue only directions that could shed light on the polarization and are not zero counting in disguise. Measured against it:

(V1) *Not counting, but location.* By Thm. 3.2, the full Laguerre family at x is the statement "E_x has no positive zero", i.e. "no non-real zero has real part x". It is *local and pointwise*, and it detects a ghost at its own abscissa. It does not count zeros up to a height, as Lemma 56's κ does or as Lemma 57's Weil ladder does implicitly. It is still *zero location in disguise*: the coefficients of $|\Xi|^2$ off the line [THM][RESTATEMENT].

(V2) *Krein, not polarization.* Each L_n is an indefinite form of signature $(n+1, n)$ evaluated on the jet orbit (Prop. 3.5). This is Lemma 58's Krein shape: an indefinite form, with positivity demanded only on the vectors that occur. It is not an arithmetic polarization datum. There is no positive form, and the natural flow (translation along the line) does not preserve K_n, so Lemma 58's reflection-datum axioms fail for this datum [THM][ANALOGY].

(V3) *Proven rungs are RH-blind.* Turán, higher Turán, $d \le 8$, large shift, and Ki's cosine limit hold far beyond ζ. Numerically, DH, which has ghosts, satisfies every Turán inequality checked, its kernel satisfies the CNV log-concavity hypothesis, and its $J^{d,0}$ are hyperbolic for all tested $d \le 150$ (Table 2) [NUM]. The rungs that can see a ghost of height γ start only at degree $\gtrsim \gamma^2$ (Chasse, via Farmer). This is dispersal, the opposite of what a polarization would do.

(V4) *The only zero-free positivity in sight is the wrong one.* $\Phi > 0$ gives a genuine positive-definite form, the moment/Hankel form, which is Lemma 31's positive-definiteness. But it yields $\rho_m \le 1$, the *upper* side of the sandwich. LP requires the lower side, a reverse Cauchy–Schwarz. In Lemma 58's terms, $\Phi > 0$ is the free Krein sign, and the reverse inequalities are "exclude boosts". CNV's sufficient condition, log-concavity of $\Phi(\sqrt{t})$, is a zero-free kernel property that *does* yield the first reverse rung. It is the only place in this lemma where a zero-free positivity reaches across. It is RH-blind (DH satisfies it), so it cannot be the polarization [THM][CITED-sec][NUM].

Label. The Laguerre–Pólya positivities are a [RESTATEMENT]: localized zero location, of Krein (indefinite) type. They are not a polarization, and their finite rungs are RH-blind. As a route to J they *fail* J.W.'s condition. As a ghost detector they pass: they are local and pointwise, with every ghost seen at its own height (§5). *Open, and the one honest lead*: is there a hierarchy of zero-free kernel conditions (log-concavity of $\Phi(\sqrt{t})$ at order $1, \dots$) implying the order-k Laguerre inequalities, and if so, at which order does it first separate ζ from DH? [OPEN]

4. The derivative tower and ghosts

Definition 4.1 (tower counters)**.** Fix a window $a < b$ and a height Y with all zeros of all F_n in $|\operatorname{Im} t| < Y$ (Prop. 4.2). For $F_n = \Xi^{(n)}$ let $G_n(a,b)$ be the number of conjugate couples of non-real zeros of F_n with real part in (a,b) (the *level-n ghosts*), and $\tau_n(a,b)$ the number of transverse critical points of F_n in (a,b), i.e. real zeros of F_{n+1} with $F_{n+2}/F_n > 0$, counted with Kim's multiplicity convention as in Lemma 56. Lemma 56's counter for F_n is $\kappa_n := \tau_n + S_n^+$ with $S_n^+ = G_{n+1}$. So $\kappa_0 = \kappa$ is the official TAA ghost counter.

Proposition 4.2 (the strip is inherited)**.** *If every zero of Ξ (resp. Ξ_{DH}) lies in $|\operatorname{Im} t| < A$, then so does every zero of every F_n [THM].*

Proof. F_n is real entire of order 1, even or odd. By Hadamard, $F_n(t) = ct^\epsilon \prod_k (1 - t^2/w_k^2)$ with $\sum |w_k|^{-2} < \infty$; the factor $e^{\beta t}$ is absent by parity. Hence $F_n'/F_n = \epsilon/t + \sum_k \big[(t-w_k)^{-1} + (t+w_k)^{-1}\big]$. For $\operatorname{Im} t > A$ and $|\operatorname{Im} w| < A$, $\operatorname{Im}(t-w)^{-1} < 0$, so $\operatorname{Im} F_n'/F_n < 0$ and $F_{n+1}(t) \neq 0$; similarly for $\operatorname{Im} t < -A$. Induct on n. □

For ζ, $A = \frac{1}{2}$. For DH, de Bruijn's strip bound of Lemma 56 gives $A = 0.8952$ in the t-variable (1.7903 in $z = 2t$) [TAA-L].

Theorem 4.3 (tower window identity; discrete second law)**.** *Let $a < b$ lie outside the closed Jensen intervals of $F_0, \dots, F_N$; for each level these are finitely many intervals in any bounded window, so*

such a, b are dense. Then for $0 \le n < N$:

$$\kappa_n(a,b) = G_n(a,b), \qquad i.e. \qquad G_{n+1}(a,b) = G_n(a,b) - \tau_n(a,b) \le G_n(a,b).$$

Hence $G_0(a,b) = \sum_{n<N} \tau_n(a,b) + G_N(a,b)$ [THM].

Proof. F_n is real entire of genus ≤ 1 (order 1). Kim's Theorem 1 [14] (wording recorded in `lit/lemma56/verified_refs.md`) says: if f has $2J$ non-real zeros and f' has $2J'$ non-real zeros in $a \le \operatorname{Re} z \le b$, then f has exactly $J - J'$ critical zeros in $[a,b]$. Apply it with $f = F_n$: here $J = G_n$, $J' = G_{n+1}$, and the critical zeros are the transverse critical points τ_n. This is Lemma 56 Thm. 4.2/4.3 at level n [CITED-sec][TAA-L]. Then sum over n. □

Theorem 4.4 (Fourier's real-axis count for ξ)**.** *For every bounded window with endpoints admissible at all levels,*

$$G_0(a,b) = \sum_{n \ge 0} \tau_n(a,b) \qquad (a \text{ finite sum}).$$

Every ghost of ξ with height in (a,b) is spent as exactly one transverse critical point at some level of the tower. Consequently

$$RH \iff \tau_n(a,b) = 0 \ \text{ for all } n \ge 0 \text{ and all windows} \iff \kappa_n \equiv 0 \ \forall n.$$

[THM][RESTATEMENT]

Proof. Kim's local theorem [9, §2] [CITED]: if f is real entire of order < 2 with all zeros in a horizontal strip, then for every R, $f^{(n)}$ has only real zeros in $|z| < R$ for n large. Prop. 4.2 puts every level-n ghost of the window inside a fixed disc, so $G_N(a,b) = 0$ for $N \ge N(a,b)$. Insert this in Thm. 4.3. For "⇐": all $\tau_n = 0$ gives $G_0 = 0$ in every window. For "⇒": under RH, $F_n \in \mathcal{L}\text{-}\mathcal{P}$ for all n (Prop. 2.2(b)), and $L_1(F_n) \ge 0$ gives $-F_n F_{n+2} \ge 0$ at the critical points of F_n, so there are no transverse ones. □

Remark 4.5. (a) This is the ξ-instance of Fourier's 1831 programme, completed by Ki–Kim: count the non-real zeros of a real entire function by watching its derivatives on the real axis alone [13] [CITED-sec]. For genus 0 Ki–Kim prove Pólya's global form ("as many critical points as couples of non-real zeros"). For ξ (genus 1) we only use the window form above. *No novelty is claimed.* (b) Compared with Lemma 56's saddle form (H): "$\tau_0 = 0$ and $S^+ = 0$" $\iff$ RH. The tower form trades the off-line saddles $S^+ = G_1$ for the real-axis counts $\tau_1, \tau_2, \ldots$. It is *purely real-axis data*, but it is still a restatement [RESTATEMENT]. (c) The ghost count is non-increasing up the tower, exactly as $Z_{\mathbb{C}}(H_t)$ is non-increasing along the heat flow (Lemma 2) [TAA-L].

Proposition 4.6 (what is known about high derivatives)**.** *(a) RH ⇒ all zeros of every $\xi^{(m)}$ lie on $\sigma = \frac{1}{2}$. Unconditionally, the proportion α_m of zeros of $\xi^{(m)}$ on the line satisfies $\alpha_m = 1 + O(m^{-2})$ (Conrey [6], as stated in [7, §1]) [CITED].*

(b) Kim: local reality of high derivatives (above) [CITED]. Ki: $(-1)^n A_n \Xi^{(2n)}(C_n z) \to \cos z$ locally uniformly [12] via [9, Thm. 2.1] [CITED-sec]. The analogue holds in the extended Selberg class [9] [CITED].

(c) Pólya–Wiman, proved by Craven–Csordas–Smith: a real entire function of order < 2 with only finitely many *non-real zeros has $f^{(n)}$ with only real zeros for all large n [1] (zbMATH review) [CITED-sec]. For ξ this applies only if there are finitely many ghosts. It is not an unconditional global statement for ξ, and* we found none*: Kim's theorem is local, and Conrey's is a proportion.*

4.1. **Two arrows: differentiation and heat.**

Proposition 4.7 (isolated couple)**.** *For $q(t) = (t-x)^2 + \delta^2$: $e^{-\lambda D^2} q = (1 - \lambda D^2) q = (t-x)^2 + \delta^2 - 2\lambda$, which becomes real-rooted at $\lambda = \delta^2/2$. Also $(1 - cD)q = (t - x - c)^2 + \delta^2 - c^2$, real-rooted at $|c| = \delta$, and q' is real-rooted [THM].*

So for an isolated ghost couple the heat flow and its Euler step $1 - \lambda D^2$ agree to leading order and cost $\lambda \approx \delta^2/2$, quadratic in δ. The first-order step $1 - cD$ costs $c \approx \delta$, linear in δ. A full derivative (the limit $c \to \infty$) removes the couple outright. The semigroup generated by D is translation, which removes nothing (Remark 2.3). *So differentiation is not a flow.* It is a single, coarse, LP-preserving step, and $\{D^n\}$ is a discrete monoid. Heat is a continuous LP-preserving semigroup. What both share, as theorems, is *monotonicity*: the ghost count is non-increasing (Thm. 4.3; Lemma 2) [THM][TAA-L]. Calling both "smoothing arrows" in the sense of Lemma 58's thermal arrow is [ANALOGY]. On the Fourier side both push weight to large $|u|$ (Remark 2.3). In Lemma 58's language, translation e^{-cD} is the real-time (unitary) group, and $e^{-\lambda D^2}$ is the Euclidean deformation [ANALOGY].

5. Numerics

Methods. Jets of Ξ and Ξ_{DH} were computed with python-flint ball arithmetic: `acb_series` of the completed Λ at $s = \frac{1}{2} + it$, 160-bit working precision, series cap 64, explicit rescaling of t against underflow (`lemma59_core.py`). Non-real zeros were counted by the argument principle on rectangles $|\operatorname{Im} t| \le 1$; the strip is inherited, Prop. 4.2. Real zeros were counted by sign changes on a grid of step 0.002, and zeros were then refined by Newton's method. Window endpoints were chosen away from small $|F_n|$. The Turán/Jensen computation used 12000 bits with all constants created at that precision, kernels used `mpmath` (60 digits), and the Gevrey data are in §6. Scripts are in `num/lemma59_*.py` and data in `num/159/`. All statements below are [NUM].

5.1. **The generating identity.** The identity $|f(x+iy)|^2 = \sum_n L_n(f)(x) y^{2n}$ (Thm. 3.1) was checked against direct evaluation: ζ at $(x, y) = (30, 0.2), (100, 0.4)$ and DH at $(85.7, 0.1), (85.7, 0.3)$, a DH ghost abscissa. The relative errors were 2×10^{-16}, 0, 5×10^{-16} and 1×10^{-13} (`lag_ident.json`).

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
ζ G_n	0	0	0	0	0	0	0	0	0	0	0
ζ τ_n	0	0	0	0	0	0	0	0	0	0	0
ζ κ_n	0	0	0	0	0	0	0	0	0	0	–
DH G_n	4	0	0	0	0	0	0	0	0	0	0
DH τ_n	4	0	0	0	0	0	0	0	0	0	0
DH κ_n	4	0	0	0	0	0	0	0	0	0	–

Table 1. Tower counters on $t \in [-5, 199.4]$ (ζ, 21 windows) and $[-5, 199.5]$ (DH, 21 windows), $|\operatorname{Im} t| \le 1$; sums over windows [NUM]. Windings were integral to 2e-14, maximal phase step 0.12 rad. DH windows with $G_0 > 0$: $[79.75, 89.65]$, $[109.5, 119.55]$, $[159.6, 169.7]$, $[169.7, 179.6]$. $\kappa_n = G_n$ in every window (Thm. 4.3).

5.2. **The tower. Global scan** (Table 1). ζ, $t \in [-5, 199.4]$, $n \le 10$: $G_n = \tau_n = \kappa_n = 0$ at every level. DH, same range: four ghost couples ($\gamma = 85.70, 114.16, 166.48, 176.70$), with $\tau_0 = 4$ and $G_n = 0$ for $1 \le n \le 10$. The window identity $G_{n+1} = G_n - \tau_n$ of Thm. 4.3 holds in all $2 \times 21 \times 10$ window–level pairs, with no violations. Windings were integral to 2×10^{-14}, and the maximal phase step was 0.12 rad.

All 28 DH ghosts below height 1000 (Table 3; ghost list from Lemma 56 v2). In the window $[\gamma - 2, \gamma + 2]$, for $n \le 8$, every ghost has $G_0 = \tau_0 = 1$ and $G_n = \tau_n = 0$ for $1 \le n \le 8$. Windings were integral to 8×10^{-15}, and the maximal phase step was 0.22 rad. So $n^* = 1$ for every ghost, across the whole range $\delta \in [0.0159, 0.3695]$: *one derivative removes each ghost, independently of its depth.* The ghost is spent as a transverse critical point of F_0 itself, exactly as Thm. 4.3 requires. Fig. 1 shows the zeros of $F_0, \dots, F_3$ near the first DH ghost; for ζ, all of them are real.

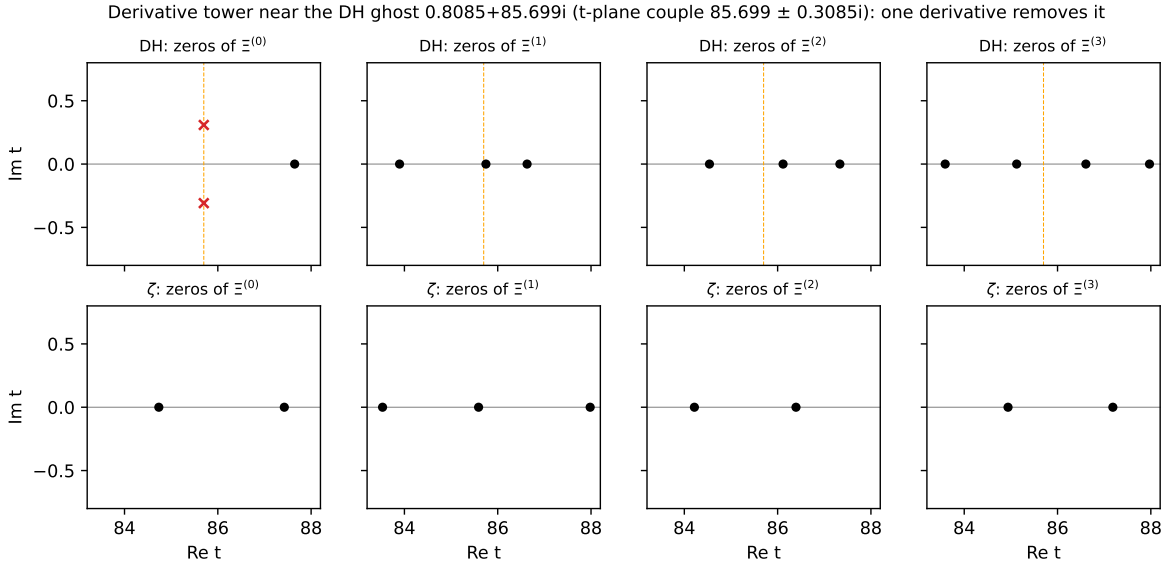

FIGURE 1. Zeros of $\Xi^{(n)}$, $n = 0, \dots, 3$, in $[83.2, 88.2] \times [-0.8, 0.8]$. Top: DH, whose couple $85.699 \pm 0.3085i$ is present at $n = 0$ and gone at $n = 1$; the critical points interlace afterwards. Bottom: ζ, all zeros real (`zeroplot.json`).

5.3. **Laguerre expressions on the line and at ghosts.** With $\widetilde{L}_n = L_n/M_n \in [-1, 1]$ (M_n the sum of absolute values of the terms):

- ζ, $t \in [0, 1000]$, step 0.01 (10^5 points): $\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_3$ are never negative. The minima are 0.173 ($t = 10.79$), 0.0818 ($t = 12.65$) and 0.0556 ($t = 14.55$).
- DH, same grid: $\widetilde{L}_1 < 0$ on exactly 28 intervals, one around each ghost with $\gamma \le 1000$, and nowhere else (1089 grid points in all). The minimum is -0.999999, at $t = 240.44$. Each interval is centred on γ to within 0.013, and its length is $(0.77\text{–}1.01) \times 2\delta$. This matches the isolated couple $q = (t - \gamma)^2 + \delta^2$, for which $L_1(q) = q'^2 - qq'' = 2((t - \gamma)^2 - \delta^2)$ is negative exactly on the Jensen interval [THM]. $\widetilde{L}_2$ and $\widetilde{L}_3$ are never negative on the DH grid (minima 0.267 at $t = 737.59$ and 0.329 at $t = 10.23$). So for DH's ghosts the first Laguerre inequality alone is the detector (`lag_grid_dh.json`; computed in six chunks by `lemma59_lagchunk.py`).
- At each of the 28 DH ghost abscissae γ: $\widetilde{L}_1(\gamma) \in [-0.99993, -0.818]$, so L_1 is negative at every ghost, and the first negative index is $n = 1$ in every case (Thm. 3.2 guarantees some $n \ge 1$).

	$\beta_m > 0$, $m \le 150$	$\min_m$ Turán margin	$\min_m(1 - \rho_m)$	$J^{d,0}$ hyperbolic
ζ	yes	0.0037 ($m = 149$)	0.0030	all tested $d \le 150$
DH	yes	0.0043 ($m = 149$)	0.0024	all tested $d \le 150$

TABLE 2. Taylor coefficients of Ξ and Ξ_{DH} at $t = 0$ (12000-bit ball arithmetic, all radii negligible) [NUM]. Turán margin $= \frac{2m+1}{2m-1}\rho_m - 1$, $1 \le m \le 149$; both margins decrease slowly in m. Jensen polynomials (even, shift 0) tested for $d = 2, \dots, 20, 30, 40, 60, 80, 100, 120, 150$ by certified complex root isolation (python-flint) of the rounded rational polynomial.

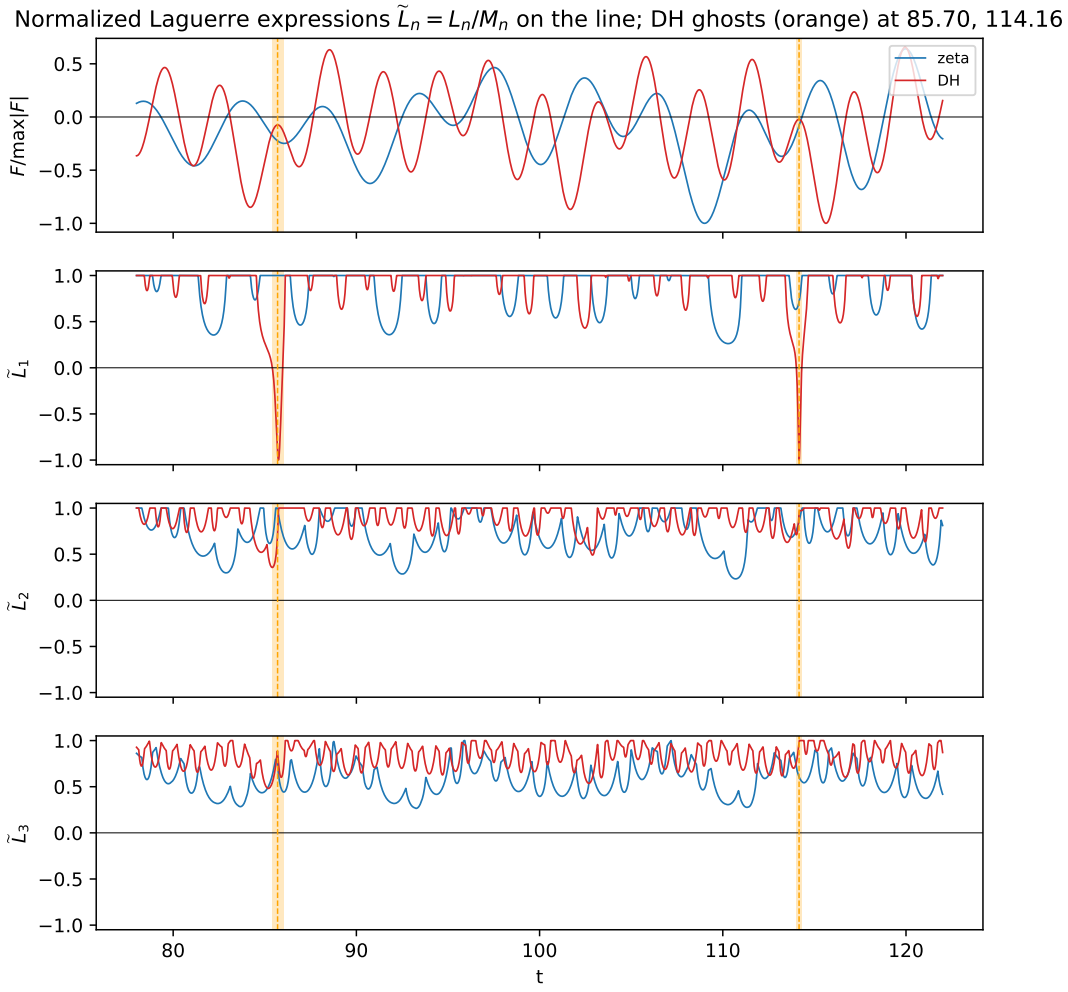

FIGURE 2. $F/\max|F|$ and $\widetilde{L}_1, \widetilde{L}_2, \widetilde{L}_3$ on $t \in [78, 122]$ for ζ (blue) and DH (red). Orange: Jensen intervals $[\gamma - \delta, \gamma + \delta]$ of the DH ghosts at 85.70 and 114.16. The Laguerre expressions dip below 0 exactly over the ghosts, a local and pointwise detector.

5.4. **Turán inequalities and Jensen polynomials at $t = 0$.** Table 2 and Fig. 3: for both ζ and DH, $\beta_m > 0$ for $m \le 150$. Both sides of the moment sandwich hold for $1 \le m \le 149$: the Turán margin (min 0.00369 for ζ, 0.00435 for DH, both at $m = 149$) and the Cauchy–Schwarz margin $1 - \rho_m$ (min 0.00303, 0.00237). The even Jensen polynomials $J^{d,0}$ are hyperbolic for every tested $d \le 150$. *DH, which has ghosts, passes every one of these tests*: they are RH-blind at this range (§3.4). On $50 \le m \le 149$ the log–log slopes are -0.84 (Turán) and -1.16 (Cauchy–Schwarz) for ζ, and -0.91 and -1.15 for DH. Both margins shrink roughly like $1/m$, so the window in which RH lives at order 2 (Thm. 3.6) narrows.

5.5. **Kernels.** Φ_ζ and Φ_{DH} are positive on the sampled range $[0, 1]$ (step 0.01). The minimal ratio $\Phi(u)/\Phi(0)$ is 1.1×10^{-69} and 4.4×10^{-14} respectively; these numbers are small because of the doubly exponential decay, not because of sign changes. Both kernels are even to 10^{-43} (checked at three points). $\psi(v) = \log \Phi(\sqrt{v})$ is strictly concave on $(0, 4]$, with $\max \psi'' = -174$ (ζ) and -40.7 (DH). So

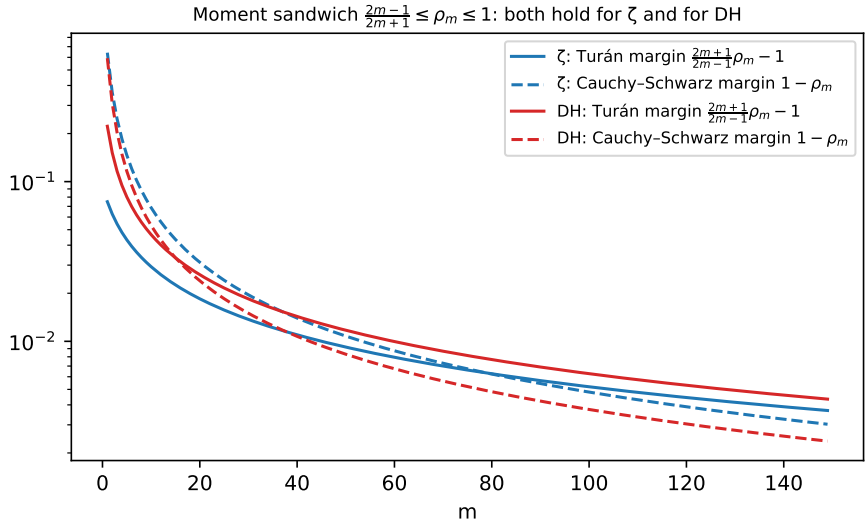

FIGURE 3. Turán margin $\frac{2m+1}{2m-1}\rho_m - 1$ and Cauchy–Schwarz margin $1 - \rho_m$, $\rho_m = \beta_m^2/(\beta_{m-1}\beta_{m+1})$, for ζ and DH (`turan.json`).

the Csordas–Norfolk–Varga hypothesis [3] holds for DH as well, consistent with DH satisfying the Turán inequalities.

5.6. **Landing costs: heat, $1 - \lambda D^2$, $1 - cD$.**

Table 3: The 28 DH ghosts with $\gamma \le 1000$ [NUM]. n^*: first derivative order with no ghost couple in the window $[\gamma - 2, \gamma + 2]$ (tower counts G_0, τ_0, G_1 shown). $\lambda_{\text{heat}} = t_c/4$ (Lemma 56 v2), λ^* for $1 - \lambda D^2$ (grid $5 \cdot 10^{-4}$), c^* for $1 - cD$ (grid 10^{-3}). $\widetilde{L}_1(\gamma)$: normalized first Laguerre expression at the ghost height. Isolated-couple values: $\lambda = \delta^2/2$, $c = \delta$. †: landing value not certified (maximal phase step along the contour > 1.5 rad, or the count did not stay at 0 to the end of the parameter grid). λ^* for $\gamma = 520.94$ is at the grid floor $5 \cdot 10^{-4} > \delta^2/2$.

γ	δ	$\delta^2/2$	G_0, τ_0, G_1	n^*	λ_{heat}	λ^*	c^*	$\widetilde{L}_1(\gamma)$	first n: $L_n < 0$
85.699	0.3085	0.04759	1,1,0	1	0.04413	0.0450	0.491†	-0.919	1
114.163	0.1508	0.01137	1,1,0	1	0.01113	0.0115†	0.169	-0.993	1
166.479	0.0744	0.00276	1,1,0	1	0.00276	0.0030	0.081†	-0.995	1
176.702	0.2243	0.02515	1,1,0	1	0.02367	0.0245	0.321†	-0.946	1
240.405	0.3695	0.06828	1,1,0	1	0.05765	0.0740	0.567†	-0.969	1
320.876	0.3195	0.05106	1,1,0	1	0.04468	0.0490†	0.617	-0.893	1
331.050	0.2682	0.03597	1,1,0	1	0.03218	0.0345	0.480	-0.906	1
366.641	0.1285	0.00826	1,1,0	1	0.00803	0.0085	0.145	-0.993	1
411.797	0.3159	0.04989	1,1,0	1	0.04266	0.0515†	0.820	-0.834	1
440.485	0.2089	0.02182	1,1,0	1	0.02027	0.0230	0.239†	-0.997	1
520.944	0.0159	0.00013	1,1,0	1	0.00013	0.0005	0.017	-1.000	1
531.280	0.3470	0.06019	1,1,0	1	0.05032	0.0660	0.530†	-0.973	1
548.907	0.2295	0.02634	1,1,0	1	0.02387	0.0280	0.287	-0.990	1
566.510	0.2866	0.04106	1,1,0	1	0.03536	0.0435	0.435	-0.966	1
595.023	0.0829	0.00343	1,1,0	1	0.00339	0.0035†	0.087	-0.999	1
611.775	0.1283	0.00822	1,1,0	1	0.00779	0.0080	0.189	-0.937	1
646.987	0.1108	0.00613	1,1,0	1	0.00600	0.0065	0.128	-0.988	1
657.108	0.2606	0.03396	1,1,0	1	0.02968	0.0325	0.531	-0.884	1

γ	δ	$\delta^2/2$	G_0, τ_0, G_1	n^*	λ_{heat}	λ^*	c^*	$\widetilde{L}_1(\gamma)$	first n
692.892	0.2887	0.04167	1,1,0	1	0.03584	0.0445	0.435	-0.967	1
737.767	0.2774	0.03846	1,1,0	1	0.03303	0.0345†	0.772†	-0.818	1
783.653	0.3530	0.06231	1,1,0	1	0.04997	0.0650	0.816	-0.896	1
811.766	0.1686	0.01420	1,1,0	1	0.01332	0.0150	0.203†	-0.990	1
847.466	0.0619	0.00192	1,1,0	1	0.00190	0.0020	0.064†	-1.000	1
857.296	0.3561	0.06341	1,1,0	1	0.05062	0.0655	0.913†	-0.880	1
864.118	0.1809	0.01636	1,1,0	1	0.01542	0.0165	0.240	-0.967	1
892.149	0.1884	0.01775	1,1,0	1	0.01612	0.0160	0.371†	-0.874	1
921.173	0.2594	0.03363	1,1,0	1	0.02959	0.0365†	0.304	-1.000	1
983.752	0.2625	0.03445	1,1,0	1	0.02866	0.0390	0.302†	-0.995	1

For each ghost we found the smallest parameter at which the window couple becomes real (Table 3, Fig. 4). The isolated-couple predictions are $\lambda = \delta^2/2$ and $c = \delta$ (Prop. 4.7).

- *Heat flow* (Lemma 56 v2 collision times, $\lambda_c = t_c/4$): $\lambda_c/(\delta^2/2) \in [0.798, 1.000]$, median 0.907, all 28 ghosts. Neighbouring zeros *help* the flow.
- *Euler step* $1 - \lambda D^2$: $\lambda^*/(\delta^2/2) \in [0.901, 1.132]$, median 1.043. This uses the 21 certified values, excluding the grid-floor case $\gamma = 520.94$ ($\delta = 0.0159$).
- *First-order step* $1 - cD$: $c^*/\delta \in [1.05, 2.60]$, median 1.47, from 15 certified values. It costs more than an isolated couple.

In 6 ($1 - \lambda D^2$) and 12 ($1 - cD$) of the 28 runs, the landing value is not certified († in Table 3): the maximal phase step reached ≈ 3 rad, or the count re-opened later in the parameter grid. These runs are reported but excluded from the ranges above. The full derivative D needs no parameter: $n^* = 1$ for all 28.

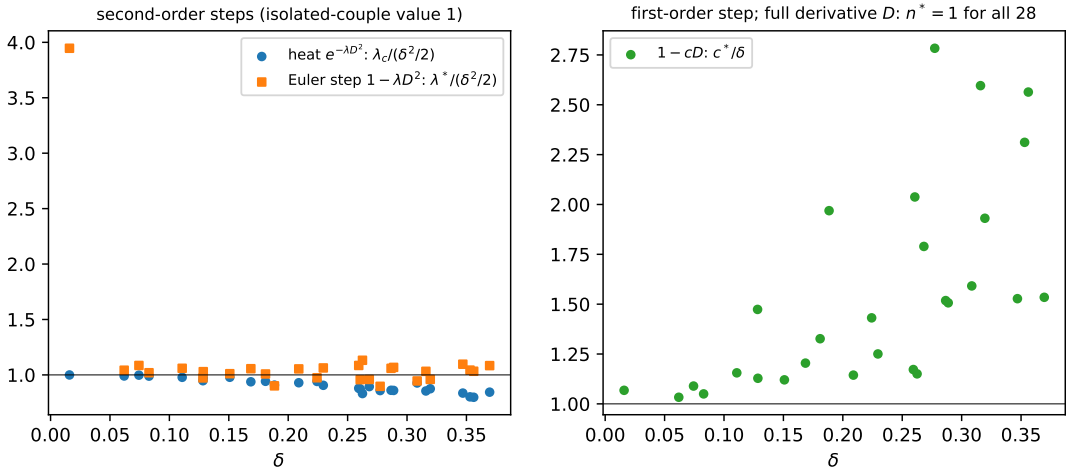

FIGURE 4. Landing costs relative to the isolated-couple values against the depth δ, for the 28 DH ghosts. Left: heat flow and $1 - \lambda D^2$ (value 1 = isolated couple). Right: $1 - cD$. Uncertified points († in Table 3) are included in the plot.

6. THE THRESHOLD BETWEEN C^∞ AND ANALYTIC (J.W. [T225U])

J.W. asked where Ξ sits relative to the line between "infinitely differentiable" and "analytic", and whether that line has anything to do with Laguerre–Pólya membership and with the polarization. In short: the line is real and sharp, and it has a clean Fourier description. Ξ sits as far on the analytic

side as a function of order 1 can. The de Bruijn–Newman flow never moves it, and *no growth or regularity invariant can detect RH* (Prop. 6.9). The physics of divergent perturbation series is a genuine instance of the threshold, but for the polarization it is an analogy only (§6.4).

6.1. **Cauchy estimates, Gevrey classes, the flat function.**

Proposition 6.1 (Cauchy estimates)**.** *Let $f \in C^\infty(I)$, $I \subset \mathbb{R}$ open. Then f is real-analytic on I if and only if for every compact $K \subset I$ there are $C, R > 0$ with $\sup_K |f^{(n)}| \le C\,R^{-n}\,n!$ for all n. Further, f extends to an entire function if and only if this holds on $K = [-r, r]$ for* every *R (with $C = C_{R,r}$). [THM]*

Proof. ($\Rightarrow$) f extends holomorphically to a complex neighbourhood containing the closed R-discs about the points of K; apply Cauchy's formula on those circles. ($\Leftarrow$) The Taylor remainder satisfies $|R_N(x_0, h)| \le \sup |f^{(N+1)}|\,|h|^{N+1}/(N+1)! \le C(|h|/R)^{N+1} \to 0$ for $|h| < R$. The entire case is the same with R arbitrary. □

Definition 6.2 (Gevrey classes [24])**.** For $s > 0$, $G^s(I)$ is the set of $f \in C^\infty(I)$ such that for every compact K there is C with $\sup_K |f^{(n)}| \le C^{n+1}(n!)^s$. Thus $G^1 = C^\omega$ (Prop. 6.1). For $s > 1$, G^s contains nonzero compactly supported functions; for $s \le 1$ it does not [29] [CITED-sec]. For $s < 1$, the same bound on all of $\mathbb{R}$ forces f to be entire of order $\le 1/(1-s)$ [THM] (Taylor coefficients $\le C^{n+1}(n!)^{s-1}$).

Indexing warning. Physicists call a formal series $\sum a_n g^n$ "Gevrey-1" when $|a_n| \le CA^n n!$. Its coefficients are $f^{(n)}(0)/n!$, so the derivatives grow like $(n!)^2$, which in the function-class indexing above is G^2. The two indexings differ by one. J.W.'s "$n!$ growth (Gevrey-1)" is correct in the series indexing, and we use that indexing in §6.4 [THM].

Example 6.3 (The flat function)**.** Let $\varphi(x) = e^{-1/x^2}$ for $x \neq 0$ and $\varphi(0) = 0$. Induction gives $\varphi^{(n)}(x) = P_n(1/x)\,e^{-1/x^2}$ with P_n a polynomial, so every $\varphi^{(n)}(0) = 0$. The Taylor series at 0 converges everywhere, to the wrong function 0. So φ is C^∞ but not analytic at 0. In the complex plane, $e^{-1/z^2} = e^{+1/y^2}$ on $z = iy$: 0 is an essential singularity, which is why no Cauchy estimate holds at 0 [THM]. Locally $\varphi \in G^{3/2}$, by the standard computation for e^{-1/x^ω}, which gives $G^{1+1/\omega}$ [29] [CITED-sec]. The usual bump $e^{-1/(1-x^2)}\mathbf{1}_{|x|<1}$ is built the same way. Bump functions give partitions of unity, which is why smooth atlases (J.W.'s [t220u], §7) are built in C^∞ and not in C^ω: by the identity theorem, an analytic function vanishing on an open set vanishes identically [THM].

Theorem 6.4 (Denjoy–Carleman [20, 22]; statement as in [30], proof in [21])**.** *For $M = (M_k)$ positive, let $C^M([a,b])$ be the set of f with $|f^{(k)}| \le A^{k+1}k!\,M_k$. Call C^M* quasianalytic *if $f \in C^M$ and $f^{(k)}(x_0) = 0$ for all k at one point force $f \equiv 0$. Then C^M is quasianalytic if and only if $\sum_j 1/L_j = \infty$, where $L_j = \inf_{k \ge j} k\,M_k^{1/k}$. [CITED-sec]*

Corollary 6.5. *G^s corresponds to $M_k = (k!)^{s-1}$, so $L_j \asymp j^s$ and $\sum_j j^{-s}$ diverges if and only if $s \le 1$. Hence G^s is quasianalytic if and only if $s \le 1$ [THM] (from Thm. 6.4). Denjoy's sequences $M_n = (\log n)^n$, $(\log n)^n(\log\log n)^n, \ldots$ give quasianalytic classes that are strictly larger than C^ω* [30] *[CITED-sec].*

So "the threshold" is really two thresholds. One is *convergence of the Taylor series to f* (analyticity, G^1). The other is *the jet determining f* (quasianalyticity, the Carleman condition). They are strictly nested: $C^\omega \subsetneq$ quasianalytic $\subsetneq C^\infty$, and e^{-1/x^2} lies outside both.

6.2. **The Fourier side: the decay ladder.**

Proposition 6.6 (Decay of the kernel versus regularity)**.** *Let $\phi \in L^1(\mathbb{R})$ and $f(t) = \int_{\mathbb{R}} \phi(u)e^{itu}\,du$. Write $M_n = \int |u|^n|\phi(u)|\,du$.*

(a) *If every $M_n < \infty$, then $f \in C^\infty$ with $\sup_{\mathbb{R}} |f^{(n)}| \le M_n$. If moreover $\phi \ge 0$, then $|f^{(2m)}(0)| = M_{2m} = \sup_{\mathbb{R}} |f^{(2m)}|$ [THM].*

(b) *If $|\phi(u)| \le Ce^{-|u|^{1/s}}$ with $s > 1$, then $M_n \le 2Cs\,\Gamma(s(n+1))$, so $f \in G^s(\mathbb{R})$ [THM].*

(c) *If $|\phi(u)| \le Ce^{-a|u|}$, then f extends holomorphically to the strip $|\operatorname{Im} t| < a$ [THM]. Conversely, L^2-type holomorphy in a strip forces exponential decay of ϕ (Paley–Wiener* [28]*) [BIB].*

(d) *If $|\phi(u)| \le Ce^{-c|u|^p}$ with $p > 1$, then f is entire of order $\le q = p/(p-1)$. Gaussian decay ($p = 2$) gives order ≤ 2 [THM].*

(e) *If $|\phi(u)| \le C\exp(-c\,e^{\kappa|u|})$, then f is entire of order ≤ 1 [THM]. If ϕ has compact support in $[-\sigma, \sigma]$, then f is entire of exponential type $\le \sigma$ (Paley–Wiener* [28]*) [BIB].*

Proof. (a) Differentiate under the integral. For $\phi \ge 0$ and $n = 2m$, $|f^{(2m)}(t)| = |\int u^{2m}\phi e^{itu}| \le \int u^{2m}\phi = |f^{(2m)}(0)|$. (b) $\int |u|^n e^{-|u|^{1/s}} du = 2s\Gamma(s(n+1))$, and $\Gamma(s(n+1)) \le C_1^{n+1}(n!)^s$ by Stirling. (c) $|\phi(u)e^{i(x+iy)u}| \le Ce^{-(a-|y|)|u|}$, then dominated convergence and Morera. (d) $|f(x+iy)| \le C\int e^{-c|u|^p + |y||u|} du \le C'e^{C''|y|^q}$ by Young's inequality $|y||u| \le \frac{c}{2}|u|^p + C_3|y|^q$. (e) $\int \exp(-ce^{\kappa u} + |y|u)\,du \le \exp\left((1+o(1))\frac{|y|}{\kappa}\log|y|\right)$ by Laplace's method at the maximiser $u \approx \kappa^{-1}\log(|y|/c\kappa)$. □

Reading the ladder from the bottom: polynomial decay of ϕ gives finitely many derivatives, superpolynomial decay gives C^∞, decay like $e^{-|u|^{1/s}}$ with $s > 1$ gives G^s, exponential decay gives a strip, Gaussian decay gives order 2, and doubly exponential decay gives order 1. *The C^∞/analytic threshold is exactly the passage from subexponential to exponential decay of the kernel*, and the strip width is the exponential rate [THM].

6.3. Where Ξ sits.

Proposition 6.7. *Let $\widetilde{\Phi}(u) = 2\Phi(u/2) > 0$ (Remark 2.3).*

(a) *$\widetilde{\Phi}(u) \sim 4\pi^2 e^{9|u|/2}\exp(-\pi e^{2|u|})$ as $|u| \to \infty$ (the $n = 1$ term of Φ) [THM]. Numerically, $\log(-\log\widetilde{\Phi}(u)) - 2u = 1.144670$ at $u = 6$, against $\log\pi = 1.144730$ [NUM].*

(b) *Hence Ξ is entire of order 1 and of maximal (infinite) type (Prop. 6.6(e);* [27]*: "from the super-exponential decay of $e^{tu^2}\Phi(u)$ ... the H_t are entire of order 1") [CITED][THM].*

(c) *$\sup_{\mathbb{R}} |\Xi^{(2m)}| = \beta_m = |\Xi^{(2m)}(0)|$ (Prop. 6.6(a)) [THM]. The moments $\beta_m = \int u^{2m}\widetilde{\Phi}$, computed by quadrature (`lemma59_gevrey.py`), agree with the β_m from the functional side (`lemma59_turan.py`, flint) to 3×10^{-15} relative error for $m \le 5$ [NUM].*

(d) *$\beta_m^{1/2m}$ grows only like $u_m^* \sim \frac{1}{2}\log m$, where $2m/u + \frac{9}{2} = 2\pi e^{2u}$ is the Laplace saddle. At $m = 150$, $\beta_m^{1/2m} = 1.284$ and $u_m^* = 1.685$; the ratio 0.762 is close to the predicted first correction $e^{-1/2u^*} = 0.743$ [NUM][HEUR]. Consequently, for every $s > 0$ there is C_s with $\sup_{\mathbb{R}} |\Xi^{(n)}| \le C_s^{n+1}(n!)^s$: $\Xi|_{\mathbb{R}} \in \bigcap_{s>0} G^s(\mathbb{R})$ [THM] (since $(A\log n)^n \le C_\epsilon(n!)^\epsilon$).*

(e) *The root-test radius $|a_n|^{-1/n}$, with $a_n = \Xi^{(n)}(0)/n!$, grows without bound: 9.3, 16.3, 35.6, 65.4, 87.1 at $n = 2, 20, 80, 200, 300$. For contrast, $\operatorname{sech} t$, whose kernel decays exponentially, has radius $\pi/2$ (Fig. 5(b)) [NUM]. The naive order estimator $n\log n/\log(1/|a_n|)$ is 1.28 at $n = 300$. It tends to 1 only like $1 + O(\log\log n/\log n)$, so the finite-n numbers do not "see" order 1, and we do not claim they do.*

For DH the same holds with π replaced by $\pi/5$. Numerically, $\log(-\log\widetilde{\Phi}_{\mathrm{DH}}(u)) - 2u = -0.46481$ at $u = 6$, against $\log(\pi/5) = -0.46471$. Ξ_{DH} is entire of order 1 and maximal type, its root-test radius is unbounded (59.0 at $n = 300$), and $\Xi_{\mathrm{DH}}|_{\mathbb{R}} \in \bigcap_{s>0} G^s$ [NUM][THM]. (The kernel Φ_{DH} of Lemma 56 v2 Def. 6.3 is normalized so that $\Xi_{\mathrm{DH}}(t) = \int \frac{1}{2}\Phi_{\mathrm{DH}}(u/2)e^{itu}du$. The quadrature reproduces the flint coefficients up to this constant factor 4, with 10^{-15} relative error. This is only a convention.)

Proposition 6.8 (de Bruijn–Newman never touches the threshold)**.** *For every $\lambda \in \mathbb{R}$, $e^{\lambda u^2}\widetilde{\Phi}(u)$ still decays doubly exponentially, so H_λ is entire of order 1 for every λ, on both sides of Λ* [27] *[CITED].*

Numerically, at $\lambda = 0.5$ and $\lambda = -1$ (kernel multiplier $e^{\lambda u^2}$), the root-test radii at $n = 300$ differ from those at $\lambda = 0$ by 0.5% and 1.0% (ζ) and by 0.9% and 1.8% (DH). For $20 \le n \le 300$ the deviation is at most 8.4%, and it is largest at small n, where the Gaussian multiplier matters most (Fig. 5(b)) [NUM]. The deviation shrinks as n grows, as it must: the multiplier changes neither the order nor the type. A kernel that would *cross a threshold is a Gaussian e^{-au^2}: the multiplier $e^{\lambda u^2}$ destroys integrability at $\lambda = a$, which is the order-2 ceiling of $\mathcal{L}$-$\mathcal{P}$. $\widetilde{\Phi}$ is nowhere near it.*

Proposition 6.9 (Regularity is blind to RH)**.** *(a) $\cos t \in \mathcal{L}$-$\mathcal{P}$ and $\cosh t \notin \mathcal{L}$-$\mathcal{P}$ (zeros $\pm i\pi/2$), yet $|\cos^{(n)}(0)| = |\cosh^{(n)}(0)|$ for every n. So no condition on the moduli $|f^{(n)}(x_0)|$, and in particular no Cauchy, Gevrey, Denjoy–Carleman or order/type condition, can decide membership in $\mathcal{L}$-$\mathcal{P}$ [THM].*

(b) Every $f \in \mathcal{L}$-$\mathcal{P}$ is entire of order ≤ 2 (Def. 2.1: genus-≤ 1 product times $e^{-\alpha x^2+\beta x}$, $\alpha \ge 0$), so $\mathcal{L}$-$\mathcal{P}$ lies entirely on the analytic side, strictly inside "entire". Among functions of order < 2, membership is a pure zero condition (Prop. 2.2(d)). Only at order exactly 2 does growth enter, through the sign of α: $e^{-t^2} \in \mathcal{L}$-$\mathcal{P}$, $e^{t^2} \notin \mathcal{L}$-$\mathcal{P}$ [THM].

(c) Ξ and Ξ_{DH} occupy the same place in every class considered here (order 1, maximal type, $\bigcap_{s>0} G^s$, doubly exponential kernel). So do H_λ for $\lambda < \Lambda$ and $\lambda \ge \Lambda$. DH has ghosts [NUM][TAA-L], and H_λ has non-real zeros for $\lambda < \Lambda$ [27] [CITED]. Hence the C^∞/analytic threshold is not where RH lives*. RH lives in the signs and second-order ratios of the jet (Turán/Laguerre, §3), not in its growth [THM].*

Remark 6.10 (Ghosts are not flat)**.** Since Ξ is entire, its jet at 0 determines it, so a ghost changes *every* Taylor coefficient. A ghost is not a "non-perturbative" e^{-1/x^2}-type term invisible to the jet. What is true is weaker and numerical: DH's ghosts are invisible to every low-order, shift-0 positivity test we ran (Turán for $m \le 149$, $J^{d,0}$ for $d \le 150$, §5). Yet Thm. 3.2 shows they are visible to L_1 at their own abscissa [NUM][THM]. "Invisible to every test we ran, but present in the jet" is the right description, and it is different from flatness.

6.4. **Physics remark: perturbative series in QED and QCD. Label: [ANALOGY] / standard background.** Nothing in this subsection is a TAA result. Each physics statement carries its source and verification level.

(i) Both series diverge. Dyson's 1952 argument [23] (abstract read [CITED]) leads "tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge". The mechanism, as usually summarized: convergence for small $e^2 > 0$ would force convergence for small $e^2 < 0$, where like charges attract and the vacuum is unstable to pair creation [CITED-sec]. Generically, an observable $R = \sum r_n \alpha^n$ in QED or QCD has $r_n \sim K a^n n!\, n^b$: Gevrey-1 in the series indexing [19, (1.2),(2.2)] [CITED]. Optimal truncation is at $N_* \sim 1/(|a|\alpha)$, with error $\sim e^{-1/(|a|\alpha)}$ [19, (2.3),(2.4)] [CITED].

(ii) Borel summation and its obstructions. The Borel transform $B[R](t) = \sum r_n t^n/n!$ and the Laplace integral $\int_0^\infty e^{-t/\alpha} B[R](t)\, dt$ resum a Gevrey-1 series [19, (2.5),(2.6)] [CITED]. This is a Laplace/Fourier integral of a kernel, structurally the same move as $\Xi = \int \widetilde{\Phi} e^{itu}$ [ANALOGY]. Non-alternating divergence ($a > 0$) puts singularities on the positive axis. The integral is then ambiguous by $\sim e^{-1/(a\alpha)}$ [19, (2.10)] [CITED]. *Renormalons* are such singularities, coming from large or small loop momenta. In QCD the IR renormalons lie on the positive axis and the ambiguity scales like $(\Lambda/Q)^4$, matched by condensates (power corrections). In QED the corresponding problem sits in the UV [19, §2.2–2.4, (2.28)] [CITED]. *Instantons* also produce factorial divergence (Lipatov, as reviewed in [19, §2.4] [CITED-sec]). Instanton–anti-instanton pairs give Borel singularities at $t = 4\pi n$ in the α_s-normalization [19, §2.4] [CITED]. This corresponds to action $2\pi/\alpha_s = 8\pi^2/g^2$ per (anti-)instanton, the BPST action [18] [BIB], so the effects are $\sim e^{-8\pi^2/g^2}$ [THM] (arithmetic, $\alpha_s = g^2/4\pi$). Beneke states the key logical point exactly [19, §2.1] [CITED]: "while a divergent series implies that R is

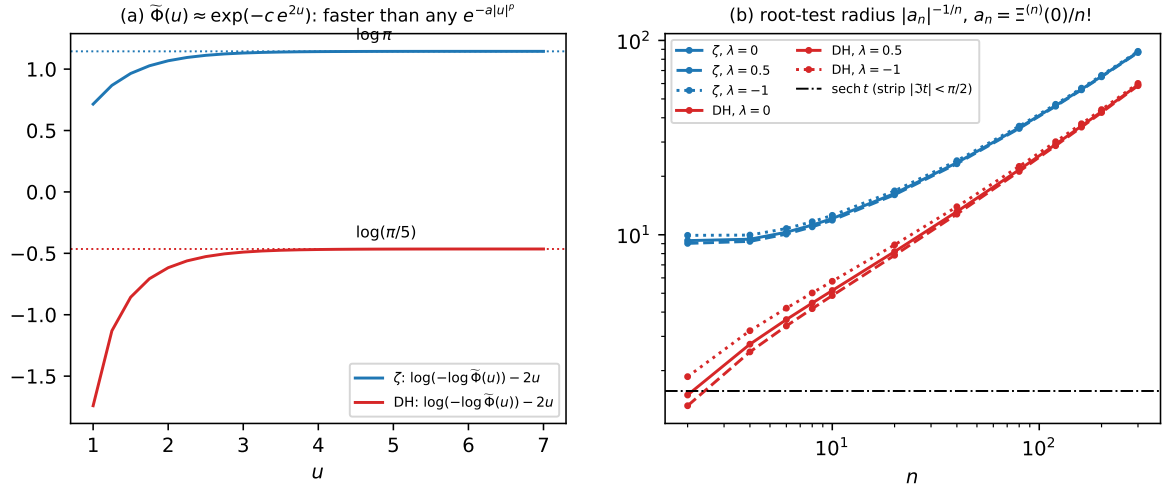

FIGURE 5. (a) Doubly exponential decay of the kernels: $\log(-\log\widetilde{\Phi}(u)) - 2u \to \log c$, with $c = \pi$ (ζ) and $c = \pi/5$ (DH). (b) Root-test radius $|a_n|^{-1/n}$ of the Taylor series at 0. It is unbounded (entire) for ζ and DH and for the de Bruijn–Newman deformations $\lambda = 0.5, -1$, which barely move it. For contrast, $\operatorname{sech} t$ (exponential kernel, strip of width $\pi/2$) has radius $\pi/2$. Script `num/lemma59_figgevrey.py`, data `num/l59/gevrey.json`.

non-analytic at $\alpha = 0$, non-analyticity does not imply divergence." He also notes that there is "no rigorous non-perturbative definition of R in theories such as QED and QCD" [CITED].

(iii) Λ_{QCD} *is a flat function of* g. Write $\mu\frac{dg}{d\mu} = -\beta_0 g^3 + O(g^5)$ with $\beta_0 = \frac{1}{16\pi^2}(11 - \frac{2}{3}n_f) > 0$ (asymptotic freedom [25, 26] [BIB]). This coefficient is converted from [19, (2.17),(2.18)], where $\mu^2\frac{d\alpha_s}{d\mu^2} = \beta_0^{\mathrm{B}}\alpha_s^2$ with $\beta_0^{\mathrm{B}} = -\frac{1}{4\pi}(11 - \frac{2}{3}n_f)$ [CITED][THM]. At one loop, $g^{-2}(\mu) = 2\beta_0\log(\mu/\Lambda)$, i.e.

$$\Lambda = \mu\,\exp\Big(-\frac{1}{2\beta_0 g^2}\Big) \qquad \text{(cf. [19, (2.27)])}\ \text{[THM]}.$$

As a function of real g (extended by 0 at $g = 0$), this is $\mu\,\varphi(g\sqrt{2\beta_0})$ with φ from Ex. 6.3: C^∞, every derivative zero at $g = 0$, not analytic [THM]. The same holds for every power correction $(\Lambda/Q)^p = \exp(-p/(2\beta_0 g^2(Q)))$. Perturbation theory is the jet of an observable at $g = 0$, and the flat terms are exactly the kernel of the jet map. So *even the complete perturbative series cannot determine* Λ*-terms* [THM]. In complex g^2, $\exp(-1/(2\beta_0 g^2))$ has an essential singularity. It is flat only along directions where $\mathrm{Re}(1/g^2) > 0$, which is the sectorial picture behind Borel summation [THM].

(iv) Correcting the tagline "QED is analytic, QCD isn't." As stated it is false. Neither perturbative series converges: both are believed to be divergent, Gevrey-1 asymptotic series of radius 0 (QED: [23], "tentatively"; both: [19]). Neither is known to define an observable analytic at coupling 0, since divergence already implies non-analyticity there (Beneke, quoted above), and neither 4D theory is rigorously constructed. The true distinctions are these. (1) In QCD the flat, non-perturbative pieces (Λ_{QCD}, condensates, $(\Lambda/Q)^p$) are large at accessible scales, and the IR renormalons on the positive Borel axis signal them. (2) In QED at accessible energies ($\alpha \approx 1/137$) the analogous ambiguities $\sim e^{-1/(|a|\alpha)}$ are numerically negligible, and QED's own obstruction is in the ultraviolet [19] [CITED][CITED-sec]. A precise one-line version: *both expansions are divergent asymptotic series; QCD's flat (invisible-to-all-orders) terms are physically essential, while QED's are numerically negligible at accessible energies.*

(v) Does this shed light on the polarization? Honestly: **no, it is an analogy**, for three reasons. (1) The object of RH, Ξ, has a convergent Taylor series everywhere and sits deep on the analytic side (Prop. 6.7). There is no divergent series, no flat term, and no Borel ambiguity in Ξ itself. (2) The one structural parallel is "a resummation integral with a kernel" ($B[R]$ versus $\widetilde{\Phi}$), together with the positivity that kernel can carry. The positivity available there is $\widetilde{\Phi} > 0$, and Lemma 31 and §3.4 already show that it is insufficient: it holds for DH, which has ghosts, and it gives the inequality opposite to Turán's. (3) Prop. 6.9: every regularity invariant on either side of the threshold takes the same value for ζ, DH, and H_λ on both sides of Λ. A polarization must separate exactly these cases, so it cannot be a regularity statement [THM][HEUR]. What the physics does supply is *vocabulary* [ANALOGY]: "visible to all orders but only through fine combinations" (ghosts, Remark 6.10) versus "invisible to all orders" (flat terms). In that vocabulary, RH-type information is perturbative and fine-grained, not non-perturbative.

7. Atlas remark (J.W. [t220u])

J.W. asked whether the infinite differentiability required of chart transitions in a smooth atlas is related to the derivatives of Ξ.

(i) *Analytic is stronger than C^∞.* A C^∞ atlas gives a smooth manifold, and a smooth function is not determined by its jet at a point (e^{-1/x^2}). Ξ is entire, so it equals its Taylor series everywhere. The whole tower $\{\Xi^{(n)}(t_0)\}_n$ at one point determines Ξ, hence every zero and every ghost [THM].

(ii) *One chart suffices.* Since ξ is entire, the atlas of $\mathbb{C}$ needs one chart, and there is nothing to glue. The power-series atlas is the classical construction of analytic continuation (Weierstrass), and the Riemann surfaces of multivalued functions are where the gluing matters. For ξ it does not.

(iii) *What does carry over* is rigidity. An analytic object is determined by local data, which is why a positivity statement about the tower at one point (Turán/Jensen at $t = 0$) can be equivalent to RH at all. It is also why such statements disperse information (Remark 3.8): a ghost at height γ can show up in the tower at 0 only at degree $\gtrsim \gamma^2$ (Chasse, via Farmer).

Label. The link between chart transitions and the derivative tower is [ANALOGY]. What the analogy points to (analytic rigidity, one chart) is [THM]. It gives no new handle on RH.

8. Prior TAA work on this topic, and what Lemma 59 adds

L2 *Second law for the heat flow.* The number $Z_{\mathbb{C}}(H_t)$ of non-real zeros is non-increasing (Laguerre-type step via $1 \pm cD$ and Hurwitz; the operators are LP-preserving). The number of distinct ghosts can rise (example $(z^2+1)^2$) [TAA-L]. *L59 adds* the discrete tower analogue $G_{n+1} = G_n - \tau_n$ (Thm. 4.3).

L31 Ξ *is positive-definite unconditionally* ($\Phi > 0$; Bochner/KMS reading), whereas RH is the Laguerre–Pólya property, which $\Phi > 0$ does not imply [TAA-L]. *L59 makes this quantitative*: $\Phi > 0$ gives $\rho_m \le 1$, LP needs $\rho_m \ge \frac{2m-1}{2m+1}$ (Thm. 3.6), and $\Phi > 0$ is a positive Hankel form, while LP is Krein-type (Prop. 3.5).

L55 *Gauss–Lucas, LP closure, Rolle.* RH $\Rightarrow$ every saddle of $|\xi|$ lies on the line, and the zeros of every $\xi^{(m)}$ are on the line (Conrey). Theorem 4.2 ("Laguerre criterion"): an on-line saddle is longitudinal iff $L_1 = \Xi'^2 - \Xi\Xi'' > 0$ there; RH $\Rightarrow$ all longitudinal; transverse $\Rightarrow$ ghost nearby. It cites Farmer 2022 and treats Kim/Ki as RH-blind [TAA-L]. *L59 adds* all orders L_n, the generating identity explaining them, the localization at the ghost's abscissa, and the observation that $L_1 < 0$ at every DH ghost height.

L56 $\kappa = \tau + S^+$ (Kim 1996 window count), the saddle form (H) of RH, Jensen discs, DH heat-flow landings $t_c \approx 2\delta^2$. It notes that "longitudinal" is the first Laguerre inequality at critical points

and records Csordas–Varga [TAA-L]. *L59 adds* the tower counters $\kappa_n = G_n$, the real-axis form RH $\iff \tau_n \equiv 0$, and the comparison of one derivative ($n^* = 1$ for all 28 ghosts) with heat ($\lambda_c \approx \delta^2/2$).

L58 *Thermal arrow*: Pólya persistence, RH $\iff \Lambda = 0$, dBN as a Euclidean deformation (not the Wick rotation of scaling), and the Krein-versus-polarization dichotomy [TAA-L]. *L59 adds* the classification of LP positivity as Krein-type (indefinite form on the jet orbit), and of differentiation as a discrete LP-preserving step rather than a flow.

L17/33/37 *Line monotonicity* (Lagarias, Sondow–Dumitrescu, $G5_{\text{line}}$) [TAA-L]. *L59*: the Laguerre family is its coefficientwise strengthening (Cor. 3.4).

Net novelty (modest, and none claimed for the classical statements): the TAA-internal synthesis, i.e. the verdict (V1)–(V4), the moment sandwich read as "free positivity on one side, LP on the other", and the tower form of κ. Plus new numerics: the DH tower census, Laguerre negativity at all 28 ghosts, DH Turán/Jensen/kernel checks, and the one-step-versus-heat landing table.

9. Corrections to Grok Bot's chat claims

These were Grok Bot's statements in chat ([t220u] reply), not J.W.'s.

C1 *"$\Xi \in \mathcal{L}$-$\mathcal{P}$; $\mathcal{L}$-$\mathcal{P}$ is closed under differentiation, so RH $\Rightarrow$ every $\Xi^{(n)}$ is real-rooted."* **Correct**, with the hypothesis made explicit: closure holds for order < 2 (Prop. 2.2(b); fails for e^{z^2}-type growth). Conrey states the conclusion for $\xi^{(m)}$ [7].

C2 *"Pólya's criterion: RH $\iff$ all Jensen polynomials (built from the derivatives of Ξ at a point) are hyperbolic."* **Correct** (all degrees d and shifts n; shift 0 suffices) [10]. Precision: the point is the centre of symmetry, and the sequence is the even Taylor coefficients.

C3 *"Griffin–Ono–Rolen–Zagier (2019) proved this for each fixed degree once the shift is large enough. That's partial evidence, not RH."* **Correct, and incomplete.** They also proved $d \leq 8$ for all shifts. "Partial evidence" overstates it: by Farmer's analysis the large-shift statement follows from a universal cosine limit that also holds without an Euler product. So it is RH-blind, not evidence (Remark 3.8; DH numerics).

C4 *"Conrey (1983) showed that the share of zeros of $\xi^{(m)}$ on the critical line tends to 100%."* **Correct**, with the rate: $1 + O(m^{-2})$ [6, 7]. The paper is J. Number Theory 16 (1983); the sequel in vol. 17 adds simplicity.

C5 *"Differentiation pushes zeros toward the line, just as the heat flow does; ghosts disappear as you climb the tower."* **Partly correct.** True as theorems: the ghost count is non-increasing up the tower (Thm. 4.3), high derivatives are locally real-rooted (Kim), and the proportion on the line tends to 1 (Conrey). Numerically every DH ghost is gone after one derivative. *But* differentiation is not a flow: it is a discrete, coarse step, and the semigroup generated by D is translation, which moves nothing towards the line. "Twin of the heat flow" is [ANALOGY]. And the effect is universal (RH-blind), which is Farmer's point.

C6 *"Laguerre–Pólya membership is literally a family of positivity inequalities (the Turán inequalities), so this route touches positivity directly, not just zero counting."* **Needs correction.** (i) The Turán inequalities are only the first, necessary rung, and they are proven unconditionally (CNV). The characterization uses the Laguerre inequalities of all orders, or Jensen hyperbolicity in all degrees. (ii) These are positivity of *values of indefinite forms* on the jet curve, not positivity of a form (Prop. 3.5). (iii) By the generating identity they are *zero location in disguise* (Thm. 3.2). It is true that they are not zero *counting*: they are pointwise and local. They do not "touch positivity directly" in the polarization sense (§3.4).

10. Summary of labels

THM (elementary, proved here): Thm. 3.1; Thm. 3.2; Cor. 3.4; Prop. 3.5; the free half of Thm. 3.6; Prop. 4.2; Prop. 4.7; Prop. 2.2(c),(d). **THM given cited theorems:** Thm. 4.3 (Kim96 Thm. 1), Thm. 4.4 (plus Kim's local theorem). **CITED:** CNV86, DL11, GORZ19, WY24, Con83, Far22 (and through it Kim96/Ki06), CCS87, RT20. **RESTATEMENT:** RH $\iff$ $\Xi \in \mathcal{L}\text{-}\mathcal{P}$ $\iff$ all $L_n \geq 0$ $\iff$ all Jensen polynomials hyperbolic $\iff$ $\tau_n \equiv 0$. **ANALOGY:** "both are smoothing arrows"; Krein-shape comparison with Lemma 58; the atlas. **§6:** Prop. 6.1, Cor. 6.5, Prop. 6.6, Prop. 6.9 [THM]; Def. 6.2, Thm. 6.4 [CITED-sec]; Prop. 6.7, Prop. 6.8 [THM][CITED][NUM]; §6.4 [ANALOGY] (standard background, cited). **OPEN:** zero-free kernel hierarchy for higher Laguerre orders. **NUM:** §5. *RH is not proven.*

11. Questions for J.W.

Q1 Adopt the verdict (V1)–(V4)? That is: LP positivity is localized zero location of Krein type, not a polarization. It is kept as a ghost *detector*, at its own height, but not pursued as a source of J.

Q2 Adopt the tower counters $\kappa_n = G_n$ and the real-axis form "RH $\iff$ $\tau_n \equiv 0$" (Thm. 4.4) alongside Lemma 56's (H) in the diagnostics table, with $\kappa = \kappa_0$ still the official counter?

Q3 Pursue the one open lead (§3.4, end): find zero-free conditions on the kernel Φ (log-concavity of $\Phi(\sqrt{t})$ and higher analogues) that imply the order-k Laguerre/Turán inequalities, and test numerically at which order they first separate ζ from DH? This is the only item here that is zero-free and positive. Expected outcome: RH-blind at every finite order (Farmer), but the failure order would itself be informative.

Q4 Extend the DH numerics beyond Turán: higher-order Turán and Jensen degrees up to $d \sim 10^3$–10^4 at shift 0, to locate the degree at which a DH ghost of height γ first becomes visible (Chasse predicts $\gtrsim \gamma^2$)? This would quantify Farmer's "dispersal" for a function that really has ghosts.

Q5 ([t225u]) Adopt the conclusion of §6? It has three parts. (a) Regularity and growth invariants are blind to RH (Prop. 6.9). (b) Ghosts are "visible in the jet but only through fine combinations", not flat (Remark 6.10). (c) The QED/QCD material is [ANALOGY] only, with the corrected tagline. If you want a non-analogical test, the natural one is Q4 rephrased: at which Jensen degree or Turán order does a DH ghost become visible? That is the "order of perturbation theory" at which the ghost appears.

Q6 Include this lemma (with figures) in Volume 4, and record the atlas remark as [ANALOGY] only?

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TAA LEMMA 60: GHOSTS, CLASS NUMBERS, AND PRINCIPALIZATION. DEURING–HEILBRONN, EPSTEIN ZETA FUNCTIONS AS POLARIZED LATTICES, IWASAWA TOWERS, CAPITULATION, AND AN AUDIT OF "PRINCIPALIZATION AS HAMILTONIZATION"

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. J.W.'s hunch is that the *ghosts* of TAA (off-line zero couples, counted by $\kappa = \tau + S^+$, Lemma 56) touch Iwasawa theory, class-number growth and the principalization of ideals, and that his earlier slogan "principalization as Hamiltonization" and the Wick rotation are part of the picture. We (i) audit the ten papers in which J.W. uses these words, (ii) collect the rigorous links between zeros and class numbers (Hecke, Deuring, Mordell, Heilbronn, Siegel, Goldfeld–Gross–Zagier, Davenport–Heilbronn for Epstein zeta functions), (iii) prove a small structural result: binary quadratic forms of discriminant $d < 0$ *are* integral positive polarizations J_Q of $\mathbb{Z}^2$, so $h(d)$ counts polarizations and an Epstein zeta function is the zeta function of a polarized lattice, and (iv) test which of these structures see ghosts. The answer to J.W.'s standing condition ("does it shed light on the polarization J?") is a qualified yes in one place only: *positivity of a polarization is free and does not prevent ghosts; what prevents them, where anything provably does, is Hecke-equivariance (the Euler product).* For fundamental $d < 0$: $h(d) = 1$ if and only if every polarized-lattice zeta function of discriminant d is zero-free in $\sigma > 1$ (Thm. 3.3). Iwasawa-type (p-adic) growth is provably ghost-blind in the function-field model, while archimedean class-number growth in the constant tower detects ghosts exactly (Thm. 5.1), whose number-field analogue is von Koch's criterion (a restatement). "Principalization as Hamiltonization" and "Wick rotation = Hilbert class field" are, as written, analogies with several false statements; three precise salvageable versions are given.

Statement of intellectual honesty. *RH is not proved here*, and nothing below brings it closer. Every equivalence with RH is tagged [RESTATEMENT]. Tags: [THM] proved here, or classical with proof or exact source; [CITED] checked in a source we read (`lit/lemma60/verified_refs.md`, level SRC/SEC); [BIB] bibliographic record verified (Crossref), content standard; [RESTATEMENT] restatement of RH or of a known result; [ANALOGY] analogy only; [NUM] numerical (`num/lemma60_*.py`, data in `num/l60/`); [CONJ] conjecture; [OPEN] open; [WRONG] wrong as stated; [SALVAGEABLE] salvageable with repair. Nothing here has been or will be submitted anywhere without J.W.'s approval.

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1. Task A: what J.W.'s papers say

All 48 of J.W.'s viXra papers were already in `archive/`; the ten in which "principalization" is used substantively were copied to `lit/lemma60/jw/`. Quotations below are verbatim from the extracted text (`archive/txt/`); page numbers are PDF pages.

viXra	Title (abridged)	Role of "principalization" / Wick
2604.0068v1	*The Relativistic Field Theory of Primes: Hamiltonization as Principalization* (Apr. 17, 2026; with Gemini AI, Grok AI)	Founding paper. The 691 defect is a "singular Legendre point"; "conductor-9 triality clutching" is "the arithmetic process of principalization". No Wick rotation.
2604.0106v1	*RFTP: From the 691 Topological Defect to Unified Number Theory ... and an approach to the Riemann Hypothesis*	"principalization as Hamiltonization: a singular Legendre point (non-invertible Hessian) is resolved into a regular Hamiltonian structure".
2606.0062v1	*Color Confinement and the Yang–Mills Mass Gap from Arithmetic Principalization*	Confinement as failure of an arithmetic Legendre transform (principalization).
2606.0066v1	*The Standard Model and Einstein–Cartan Gravity from a Single Arithmetic Object*	CM points i, ρ; Kummer regularity.
2606.0071v1	*Arithmetic Principalization as a Legendre Transform*	Dirac reduction; a complex structure J with $J^2 = -\mathrm{Id}$.
2608.0068v2	*Thermodynamics of Arithmetic Action: Higher Gauge Moduli, ...*	Wick rotation "as the absolute physical realization of Arithmetic Principalization"; Hilbert class field.
2608.0072v1	*TAA: The Adelic Field Quantization (Monograph II)*	"Wick-Principalization Transform".
2608.0081v1	*Hydrodynamics of Arithmetic Action (Monograph III)*	"$u_4 = 0$" as "the literal, first-principles execution of the Wick Rotation".
2608.0085v1	*The Optical Mechanics of Adelic Action: Conformal Regularization of Dual Potentials and the Spacetime Maxwell Relation*	"Scheme-theoretic principalization": Rees algebra, blow-up, Kustaanheimo–Stiefel.

2609.0007v1	*TAA: A Non-Hermitian Operator Framework on Derived Adélic Modular Stacks*	"cosmic Wick rotation to open up the Hilbert Class Field".

1.1. **The claims, quoted and classified.**

(J1) *Status of the dictionary.* 2604.0068, p. 2: "These mappings are not analogies; they are precise identifications". [WRONG] in the sense that no map between the mechanical and arithmetic objects is defined in the paper. Everything below that survives does so as an analogy or after a repair.

(J2) *The ideal condition.* 2604.0068, §3: "At prime 691 this condition $[a \cdot s' \in S]$ fails locally in the χ_{11}-eigenspace". [WRONG]. A non-principal ideal is still an ideal, so the ideal condition $as' \in S$ never fails. What fails is *being principal*. The correct 691 statement is the Herbrand–Ribet theorem: $691 \mid B_{12}$ gives a nonzero ω^{-11}-eigenspace of the 691-part of $\mathrm{Cl}(\mathbb{Q}(\zeta_{691}))$ [BIB] [23, 29].

(J3) *The congruence.* 2604.0068, p. 6: "$E_{12}(\tau) = 1 - \frac{65520}{691}\sum \sigma_{11}(n)q^n + O(q^2)$" and "At the cusp ... both E_{12} and Δ vanish simultaneously modulo 691: $E_{12} \equiv \Delta \pmod{691}$". [WRONG] in three places. The sign is +: $E_{12} = 1 + \frac{65520}{691}\sum \sigma_{11}(n)q^n$. The "$+O(q^2)$" is spurious, since the series is exact. E_{12} has constant term 1 and is not 691-integral, so it neither vanishes at the cusp nor is congruent to Δ. The correct statement is Ramanujan's $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$, i.e. $691(E_{12} - 1) = 65520\sum \sigma_{11}(n)q^n$ and $\Delta \equiv \sum \sigma_{11}(n)q^n \pmod{691}$. Correct in the same passage: weight 12 is the least weight with a cusp form, and the mod-691 representation of Δ is reducible while the 691-adic one is irreducible [BIB].

(J4) *The leakage residue.* 2604.0068, p. 6: $\Omega_{\mu\nu} = \mathrm{Res}_{\tau\to i\infty} R(\tau)\, d\tau \wedge d\gamma \approx \frac{\lambda_{691}}{2}\, d\tau \wedge d\gamma$ with $R = \Delta/E_{12}$. [WRONG]. $R = q + O(q^2)$ is holomorphic at the cusp and $d\tau = dq/(2\pi i q)$, so $\mathrm{Res}_{q=0} R\, dq/(2\pi i q)$ is the constant term of R, which is 0.

(J5) *Principalization.* 2604.0068, p. 7: "This clutching is the arithmetic process of principalization: non-principal ideals (the failure $a \cdot s' \notin S$) are absorbed and mapped to principal ideals in an effective sense". Later, the triality tensor is said to be "reducing the effective class number", and p. 8: "The residual class number after clutching measures the small 'waste heat' ". [ANALOGY]. No field extension, ideal or class group is specified, so no class number can be computed. Principalization in the literature is the capitulation map $\mathrm{Cl}(k) \to \mathrm{Cl}(K)$ for an extension K/k (§6). That is the only precise reading, and it requires naming K.

(J6) *Hamiltonization.* 2604.0106, p. 1: "principalization as Hamiltonization: a singular Legendre point (non-invertible Hessian) is resolved into a regular Hamiltonian structure". 2606.0071, p. 5: Dirac reduction by a first-class constraint $C = \sum_p n_p I'_p\, \omega(p) - 12k$. [ANALOGY]. Dirac reduction of a degenerate Lagrangian is standard mechanics. No ideal or class group appears in the constraint, so nothing about principal ideals is being reduced. Salvaged versions are in §7.

(J7) *The complex structure.* 2606.0071, p. 6: "This operator J is uniquely determined (up to sign) as the real structure that rotates the local shear direction". [SALVAGEABLE]. $J^2 = -1$ alone is not a polarization: one needs positivity $\omega(x, Jx) > 0$ (Lemma 54). The "up to sign" ambiguity is *exactly* the choice of polarization (J vs. $-J$, i.e. τ vs. $\bar{\tau}$). Prop. 4.1 gives a precise integral version in which the class number counts such J.

(J8) *CM points.* 2606.0066, p. 3: $\tau = \rho$ and $\tau = i$ are "the only CM points of class number one". [WRONG]. There are thirteen imaginary quadratic orders of class number one (nine maximal, discriminants $-3, -4, -7, -8, -11, -19, -43, -67, -163$), hence thirteen CM points on $X(1)$ with rational j. What singles out i and ρ is that they are the *elliptic* points (nontrivial stabilizers $\mathbb{Z}/4, \mathbb{Z}/6$ in $SL_2(\mathbb{Z})$). Correct in the same paper: p is irregular iff $p \mid h(\mathbb{Q}(\zeta_p))$ (Kummer) [BIB].

(J9) *RH claim.* 2604.0068, §9.1, p. 15: "Thus the Riemann Hypothesis holds in RFTP because the hamiltonization process necessarily produces a self-adjoint operator whose spectrum lies on $\mathrm{Re}(s) = 1/2$". [WRONG] as a proof: no operator is constructed whose spectrum is shown to be the zeros. App. C, p. 17, gives a formula $t_n \approx 2\pi n + \frac{\lambda_{691}}{2}\frac{h_{\text{eff}}-1}{3}\ln\frac{n}{2\pi}$. This is numerology: the true zeros satisfy $t_n \sim 2\pi n/\log n$.

(J10) *Wick = Hilbert class field.* 2608.0068v2, p. 4: "nature invokes Wick Rotation ($t \to -i\tau$) as the absolute physical realization of Arithmetic Principalization. ... In algebraic number theory, this is the exact equivalent of constructing the Hilbert Class Field extension. By injecting complex algebraic extension roots into the field structure, the non-principal ideal loops are allowed to completely unwind and become principal ideals". Similar wording appears in 2608.0072, p. 9 ("Wick-Principalization Transform"), 2608.0081, p. 10 ("This parameter freeze is the literal, first-principles execution of the Wick Rotation") and 2609.0007, p. 27 ("a cosmic Wick rotation to open up the Hilbert Class Field"). [WRONG] as an equivalence, [ANALOGY] at best. See §7: the Hilbert class field of $\mathbb{Q}(\sqrt{10})$ is totally real (Ex. 6.2), so no "complex roots" are injected.

(J11) *Scheme-theoretic principalization.* 2608.0085, pp. 8–9: the Rees algebra $R(I) = \bigoplus I^n$, the blow-up $\tilde{X} = \operatorname{Proj} R(I)$, and $\pi^{-1}I \cdot O_{\tilde{X}} = O_{\tilde{X}}(-E)$. On p. 6, "the non-principal ideal obstructions are completely unrolled and principalized into an invertible line bundle". The algebra is *rigorous and standard*: blowing up makes an ideal sheaf invertible (universal property of the blow-up). This is a different notion from number-theoretic principalization, and conflating the two is [WRONG]. An ideal of a Dedekind ring O_K is already *locally* principal (invertible), so blowing up $\operatorname{Spec} O_K$ along it is an isomorphism and changes no ideal class. Non-principality of an ideal class is a *global* obstruction. Also [WRONG]: "the Exceptional Divisor E (the 1D Hopf circle fiber)". Blowing up a point of a smooth complex surface gives $E \cong \mathbb{P}^1 \cong S^2$, which is the *base* of the Hopf fibration $S^3 \to S^2$, not its circle fiber.

Earlier TAA verdicts that are consistent with this audit: Lemma 32 §9 (Wick $\leftrightarrow$ capitulation is a weak analogy; $\zeta_H = \prod L$ *adds* positivity conditions; "$u_4 = 0$" is a gauge choice) and Lemma 58 (Wick is $t = -i\tau$ with Osterwalder–Schrader reconstruction; de Bruijn–Newman is not the Wick rotation of scaling).

2. Ghosts and class numbers: what is actually proved

Throughout, $d < 0$ is a fundamental discriminant, χ_d the Kronecker character, $h(d)$ the class number, w the number of roots of unity, and

$$h(d) = \frac{w\sqrt{|d|}}{2\pi}\, L(1, \chi_d) \qquad \text{(Dirichlet) [BIB].} \tag{1}$$

A *ζ-ghost* is a zero $\rho_0 = \beta_0 + i\gamma_0$ of ζ with $\beta_0 > \frac{1}{2}$. An *$L(\cdot, \chi_d)$-ghost* is the same for $L(s, \chi_d)$. A *Siegel ghost* is a real $L(\cdot, \chi_d)$-ghost $\beta \in (\frac{1}{2}, 1)$.

Theorem 2.1 (Hecke, Deuring, Mordell, Heilbronn; as stated in Goldfeld 1985, §3)**.** *(a) (Hecke) If $L(s, \chi_d)$ has no real zero in $(1 - c/\log|d|, 1)$, then $h(d) > c'\sqrt{|d|}/\log|d|$.*
(b) (Deuring 1933) If RH is false, then $h(d) \geq 2$ for all sufficiently large $|d|$.
(c) (Mordell 1934) If RH is false, then $h(d) \to \infty$.
(d) (Heilbronn 1934) If GRH is false for some $L(s, \chi)$, then $h(d) \to \infty$.
Together these give $h(d) \to \infty$ unconditionally (Gauss's conjecture), with an ineffective constant. [CITED] [11, 7, 21, 14]

Theorem 2.2 (Siegel 1935; Tatuzawa 1951; Goldfeld–Gross–Zagier)**.** *For every $\varepsilon > 0$, $h(d) > c(\varepsilon)|d|^{1/2-\varepsilon}$ with ineffective $c(\varepsilon)$* [24]*. Tatuzawa: this is effective with at most one exceptional d* [28]*. Goldfeld's theorem, with Gross–Zagier's elliptic curve of rank 3, gives the effective bound*

$h(d) > c\,(\log|d|)^{1-\varepsilon}$*; Oesterlé's constant is* $1/7000$ *(Goldfeld 1985, using the conductor-5077 curve)* [11, 13]. *[CITED]*

Proposition 2.3 (A Siegel ghost makes h *small*)**.** *If* $L(\beta,\chi_d)=0$ *with* $1-1/\log|d|\le\beta<1$*, then* $h(d)\le C(1-\beta)\sqrt{|d|}\log^2|d|$ *with an absolute* C*. [THM]*

Proof. By the mean value theorem, $L(1,\chi)=(1-\beta)L'(\sigma^*,\chi)$ with $\sigma^*\in(\beta,1)$. Write $D=|d|$ and $-L'(\sigma,\chi)=\sum\chi(n)\log n\,n^{-\sigma}$. For $\sigma\ge 1-1/\log D$ and $n\le D$ we have $n^{1-\sigma}\le e$, so the terms with $n\le D$ contribute $\le e\sum_{n\le D}\log n/n\ll\log^2 D$. For the tail, partial summation with $|\sum_{n\le x}\chi(n)|\le D$ gives $\ll D\int_D^\infty(\log x)x^{-\sigma-1}dx\ll\log D$. Insert this into (1). □

Remark 2.4 (Which ghost does what)**.** The signs are opposite. A ghost of $L(s,\chi_d)$ near $s=1$ (a Siegel ghost) *lowers* $h(d)$ (Prop. 2.3). A ζ-ghost *raises* class numbers, but only through Deuring–Mordell. Heilbronn uses an $L(s,\chi)$-ghost for some *other* character to force growth of $h(d)$. None of (a)–(d) distinguishes RH from not-RH at the level of conclusions: whichever case holds, $h\to\infty$. The phenomenon is *RH-neutral* [THM] (logic of Thm. 2.1). This agrees with STATE.md §16a.

2.1. **The Deuring mechanism: ghost inheritance.** Let Q_0 be the principal form of discriminant $d<-4$ ($w=2$), so that $Q_0=x^2+xy+\frac{1-d}{4}y^2$ or $x^2+\frac{|d|}{4}y^2$, and let $y_d=\sqrt{|d|}/2$. Let $Z_Q(s)=\sum'_{(m,n)}Q(m,n)^{-s}$ be the Epstein zeta function. The Chowla–Selberg/Deuring expansion [4] reads

$$Z_{Q_0}(s)=\underbrace{2\zeta(2s)+\frac{2\sqrt{\pi}\,\Gamma(s-\frac12)\zeta(2s-1)}{\Gamma(s)}\,y_d^{1-2s}}_{CT(s)}+R_d(s), \tag{2}$$

where R_d is a sum of K-Bessel terms. On compact subsets of $\mathbb{C}\setminus\{\frac12,1\}$, $R_d(s)=O_K(e^{-2\pi y_d(1-\epsilon)})$ [BIB].

Theorem 2.5 (Ghost inheritance; Deuring 1933)**.** *(a) If* $h(d)=1$ *then* $Z_{Q_0}=2\zeta(s)L(s,\chi_d)$*. So every* ζ*-ghost is a ghost of* Z_{Q_0} *for* every *class-number-one* d*. [THM]*

(b) If ρ_0 *is a* ζ*-ghost, then* $Z_{Q_0}(\rho_0)\to 2\zeta(2\rho_0)\ne 0$ *as* $|d|\to\infty$*, because* $\mathrm{Re}(2\rho_0)>1$ *and* $y_d^{1-2\beta_0}\to 0$*. Hence only finitely many* d *have* $h(d)=1$ *(Deuring). [THM][CITED]*

Reading. A ghost is a zero of ζ, and $\zeta(s)\mid Z_{Q_0}$ forces it onto every class-number-one lattice. But the lattice zeta function tends to $2\zeta(2s)$, which has no zero in $\sigma>\frac12$ (it lives on the doubled variable). The contradiction lives in the map $s\mapsto 2s$, which sends the critical line to $\mathrm{Re}=1$. TAA Lemma 13 *(squaring)* uses the same doubling. [ANALOGY] for the Lemma 13 link.

d	$s=0.75+10i$	$s=0.75+30i$	$s=0.9+5i$	$e^{-2\pi y_d}$
-7	1.44	1.54	0.241	$2.5\cdot10^{-4}$
-43	$2.2\cdot10^{-4}$	0.51	$3.9\cdot10^{-7}$	$1.1\cdot10^{-9}$
-67	$2.5\cdot10^{-6}$	0.96	$3.5\cdot10^{-9}$	$6.8\cdot10^{-12}$
-163	$1.7\cdot10^{-12}$	$3.0\cdot10^{-3}$	$1.1\cdot10^{-15}$	$3.8\cdot10^{-18}$

TABLE 2. Relative error $|Z_{Q_0}-CT|/|CT|$ for the class-number-one discriminants (`num/lemma60_deuring.py`, `l60/deuring.json`) [NUM]. The remainder is small on compacta but *not uniformly in* t: at $t=30$ even $d=-163$ is only accurate to $3\cdot10^{-3}$. Deuring's argument uses a fixed ρ_0, so this is harmless for the theorem. It does mean that no uniform-in-height ghost statement follows from (2).

2.2. **Numerics on $L(1,\chi_d)$.** For the 30 392 fundamental $d<0$ with $|d|\le 10^5$ (`num/lemma60_classno.py`), $h(d)=1$ exactly for $d=-3,-4,-7,-8,-11,-19,-43,-67,-163$, and $\max h = 533$ [NUM]. Spot checks: $h(-15)=h(-20)=2$, $h(-23)=3$, $h(-47)=5$, $h(-71)=7$, $h(-167)=11$, $h(-191)=13$, $h(-239)=15$, $h(-3299)=27$, $h(-5923)=7$. The smallest values of $L(1,\chi_d)$ are 0.23636 ($d=-85507$, $h=22$), 0.23981 (-90787, $h=23$) and 0.24607 (-163, $h=1$). For the twelve smallest, $L(\sigma,\chi_d)$ was evaluated on $\sigma\in\{0.5,0.525,\dots,1\}$ by a validated theta/incomplete-gamma formula (agreement with `mpmath.dirichlet` to all printed digits). It is positive at every sample. This is a sampled check, not a proof of no real zero [NUM]. The minima are attained at $\sigma=\frac12$: e.g. $L(\frac12,\chi_{-163})=0.0685$, and 0.0220 for $d=-34483$.

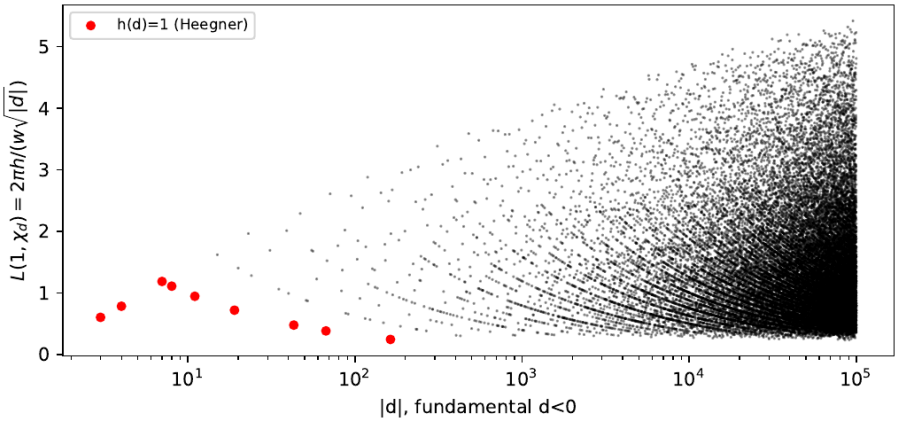

FIGURE 1. $L(1,\chi_d)$ for fundamental $d<0$, $|d|\le 10^5$. The class-number-one discriminants are marked. $d=-163$ is the third smallest value in the whole range: class number one forces $L(1,\chi)$ small, which is the content of (1).

3. Products, sums and κ

Let $K=\mathbb{Q}(\sqrt{d})$. Then $\zeta_K=\zeta\cdot L(\cdot,\chi_d)$, and for a class character ψ of $\mathrm{Cl}(K)$, $L_K(s,\psi)=\sum_C \psi(C)\zeta_K(s,C)$, where $\zeta_K(s,C)=\sum_{\mathfrak{a}\in C} N\mathfrak{a}^{-s}$. If Q corresponds to the class C, then $Z_Q = w\,\zeta_K(s,C)$, hence

$$Z_Q(s)=\frac{w}{h}\sum_{\psi\in\widehat{\mathrm{Cl}}}\overline{\psi(C)}\,L_K(s,\psi),\qquad \zeta_H(s)=\prod_{\psi}L_K(s,\psi)\quad (H=\text{Hilbert class field})\ \text{[BIB]}. \tag{3}$$

For $d=-20$ ($h=2$, $w=2$):

$$Z_{x^2+5y^2}=\zeta L(\cdot,\chi_{-20})+L(\cdot,\chi_{-4})L(\cdot,\chi_5),\qquad Z_{2x^2+2xy+3y^2}=\zeta L(\cdot,\chi_{-20})-L(\cdot,\chi_{-4})L(\cdot,\chi_5). \tag{4}$$

These were checked against direct lattice sums at $s=2$ [NUM]. All four functions share the completion $\Lambda(s)=(\sqrt{20}/2\pi)^s\Gamma(s)Z(s)$, with $\Lambda(s)=\Lambda(1-s)$.

Proposition 3.1 (Ghost counts under products and sums)**.** *Let $F=F_1F_2$ with F_i real entire of genus ≤ 1 and real on the line. On a window admissible for F, the ghost count is additive: $K_F^+=K_{F_1}^+ + K_{F_2}^+$. By Kim's identity (Lemma 56, Thm. 4.3), $\kappa_F=\kappa_{F_1}+\kappa_{F_2}$. The two summands of κ are* not *separately additive: a product of two ghost-free functions has no ghosts, so $\tau_F=S_F^+=0$ there, but products of functions with ghosts can trade τ for S^+. Sums are not controlled at all (Thm. 3.2). [THM]*

Theorem 3.2 (Davenport–Heilbronn; Bateman–Grosswald; Stark)**.** *As stated in Stark 1967:*

(a) *If $h(d) > 1$, then for every form Q of discriminant d, $Z_Q(s)$ has infinitely many zeros in $\sigma > 1$, and zeros arbitrarily close to $\sigma = 1$ (Davenport–Heilbronn I for h even, II for h odd > 1)* **[CITED]** [5, 6], [25].
(b) *$Z_{x^2+ky^2}$ has a real zero in $(\frac{1}{2}, 1)$ if $k > 7.0556$ (Bateman–Grosswald)* [2, 25]. **[CITED]**
(c) *For large k, the zeros of $Z_{x^2+ky^2}$ with $|t| \le 2k$ lie on the line, apart from two real zeros (Stark 1967, Thm. 1)* [25]. **[CITED]**
(d) *Potter–Titchmarsh: Z_Q has infinitely many zeros on the line, and an off-line zero was exhibited for $x^2 + 5y^2$* [22]. **[BIB]** *(statement via McPhedran's survey, secondary).*

Theorem 3.3 (Class number one $\iff$ no ghosts beyond $\sigma = 1$)**.** *For a fundamental discriminant $d < 0$ the following are equivalent: (i) $h(d) = 1$; (ii) for every positive definite primitive form Q of discriminant d, Z_Q has no zero in $\sigma > 1$.* **[THM]** *(from Thm. 3.2(a))*

Proof. (i)⇒(ii): $Z_{Q_0} = w\zeta L(\cdot, \chi_d)$ is an Euler product, absolutely convergent and nonzero in $\sigma > 1$. (ii)⇒(i): if $h > 1$, Thm. 3.2(a) gives zeros in $\sigma > 1$. □

This is the precise form of J.W.'s older hunch "cusp forms vs Eisenstein series ↔ principal vs non-principal ideals". The *class sum* Z_Q for a non-principal class carries ghosts. The *character sums* $L_K(s, \psi)$, i.e. the Hecke eigenfunctions obtained by Fourier transform on Cl, are Euler products. For ψ of order > 2 they are L-functions of weight-one cusp forms (Hecke) **[BIB]**.

function	sign changes	real critical pts	transverse τ	ghosts K^+ / off-line saddles S^+
$\zeta_K = \zeta L(\chi_{-20})$	54	53	0	(0 / 0 expected)
$L(\chi_{-4})L(\chi_5)$	52	53	0	(0 / 0 expected)
$Z_{x^2+5y^2}$	39	51	6	7 / 1
$Z_{2x^2+2xy+3y^2}$	28	32	2	6 / 4

TABLE 3. Census on $0 < t \le 60$ **[NUM]**. Max relative imaginary part on the line: $3.5 \cdot 10^{-18}$. Consistency check (proof of Lemma 56, Thm. 4.3, with $t = 0$ a longitudinal maximum): $C = 2\tau + (n-1) + 1$ gives $51 = 12 + 38 + 1$ and $32 = 4 + 27 + 1$. The Euler-product pieces have $\tau = 0$.

Kim's identity on $d = -20$ **[NUM]**. The search `num/lemma60_ghosts2.py` used a grid $0.55 \le \sigma \le 2.95$, $0 < t \le 61$ (steps 0.1, 0.25), analytic F' via Hurwitz zeta functions, and secant refinement. It found the following. $Z_{x^2+5y^2}$: ghosts at $0.933 + 15.668i$, $0.938 + 29.983i$, $0.697 + 36.374i$, $0.823 + 44.000i$, $0.635 + 46.758i$, $0.597 + 52.743i$, $0.811 + 58.508i$; one off-line saddle at $0.649 + 44.174i$. Hence $K^+ = 7 = 6 + 1 = \tau + S^+$. $Z_{2x^2+2xy+3y^2}$: ghosts at $2.079 + 7.629i$, $1.994 + 23.020i$, $1.964 + 38.349i$, $0.862 + 41.767i$, $1.739 + 53.593i$, $0.837 + 56.183i$; off-line saddles at $1.723 + 8.225i$, $1.687 + 23.326i$, $1.675 + 38.543i$, $1.478 + 53.770i$. Hence $K^+ = 6 = 2 + 4$. The identity #ghosts $= \kappa$ (Lemma 56, Thm. 4.3) holds exactly on both windows. An independent check is the total zero count. Riemann–von Mangoldt for a Dirichlet series whose first term is n_0^{-s} gives $N(60) \approx \frac{60}{\pi} \log \frac{60\sqrt{20}}{2\pi e\, n_0}$, i.e. 52.6 for Q_1 ($n_0 = 1$) and 39.4 for Q_2 ($n_0 = 2$, since $\min Q_2 = 2$). The search gives $39 + 2 \cdot 7 = 53$ and $28 + 2 \cdot 6 = 40$. So no ghost was missed at this resolution. *Pattern* (Fig. 3). Every *near* ghost ($\beta < 1$) sits within $\Delta t \le 0.2$ of a transverse saddle, except the one at 44.00, which is paired with the off-line saddle at 44.17. The *far* ghosts of Q_2 ($\beta \approx 1.7$–2.1, beyond $\sigma = 1$ as Davenport–Heilbronn predicts) each carry an off-line saddle at the same height. This is the Epstein version of the Lemma 56 picture: there, DH had 28 ghosts, 28 transverse saddles and 0 off-line saddles. The far-ghost/off-line-saddle pairing is new here **[NUM]**.

Ghost births. In the Epstein family a real ghost is born at the centre $s = \frac{1}{2}$, where a double zero on the line splits. McPhedran reports the same mechanism for complex zeros as the lattice ratio varies: an off-line zero lands on the line at a second-order zero [19]. This is the same codimension-one

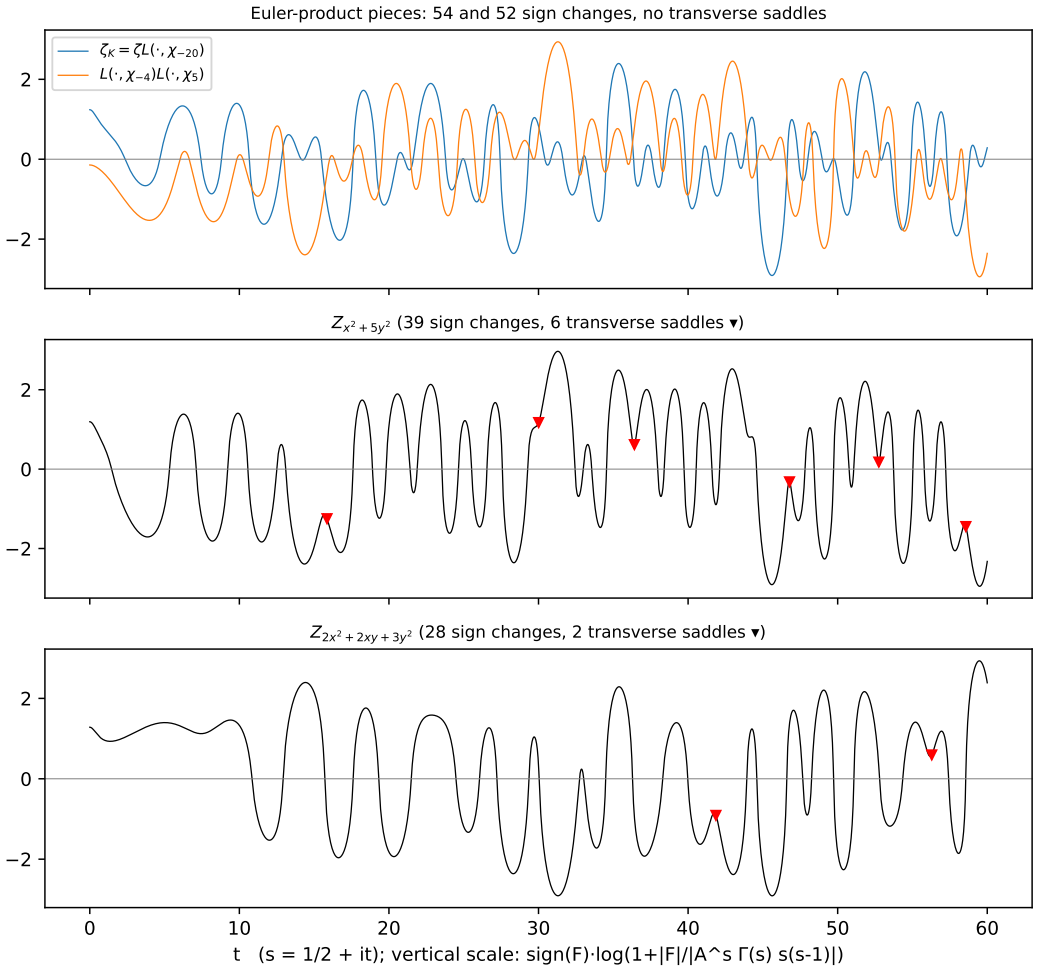

FIGURE 2. $d = -20$ on the critical line, $0 \le t \le 60$ (step 0.01). Top: the two Euler-product pieces of (4). Below: the two class zeta functions. Red triangles mark transverse saddles. `num/lemma60_epstein.py`, `l60/epstein_line.json` [NUM].

event as in the de Bruijn–Newman flow (collision on the line, then exit) [ANALOGY]. Unlike Λ for ξ, here the parameter is a *polarization* (the lattice), not a heat time.

4. Forms are polarizations

Proposition 4.1 (Forms $\leftrightarrow$ integral positive polarizations)**.** *Let* $\omega(x, y) = x_1 y_2 - x_2 y_1$ *on* $\mathbb{R}^2 \supset \mathbb{Z}^2$. *For* $Q = [a, b, c]$ *positive definite of discriminant* $d = b^2 - 4ac < 0$, *put*

$$J_Q = \frac{1}{\sqrt{|d|}} \begin{pmatrix} -b & -2c \\ 2a & b \end{pmatrix}.$$

Then:

(a) $J_Q^2 = -1$ *and* $\det J_Q = 1$.

(b) $\omega(x, J_Q x) = 2Q(x)/\sqrt{|d|}$, *so* $g_Q(x, y) = \omega(x, J_Q y)$ *is a positive definite compatible metric.* J_Q *is a* positive *polarization, and* $-J_Q$ *is the negative one (J7's "up to sign").*

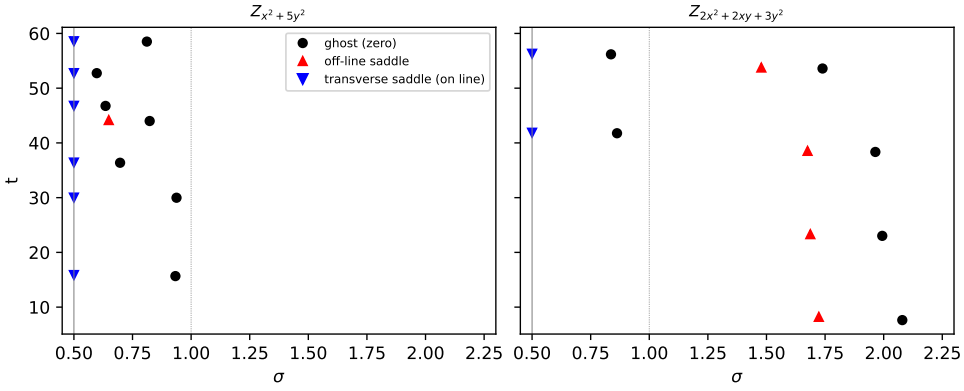

FIGURE 3. Ghost map for $d = -20$, $0 < t < 60$: ghosts (black), off-line saddles (red), and transverse saddles on the line (blue). Near ghosts leave transverse saddles; far ghosts carry off-line saddles. `160/ghosts2.json` [NUM].

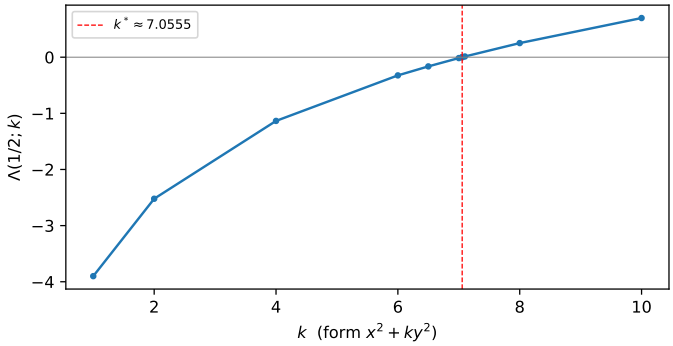

FIGURE 4. Bateman–Grosswald, reproduced: $\Lambda(\frac{1}{2}; k)$ for $Q = x^2 + ky^2$. The sign change is at $k^* = 7.05550795544818\ldots$, matching BG's 7.0556. For $k > k^*$ a real ghost is born *at* $s = \frac{1}{2}$ and moves right: 0.5864 ($k = 7.2$), 0.7055 ($k = 8$), 0.8134 ($k = 10$). `num/lemma60_bg.py` [NUM]. The sanity check at $k = 1$ against $4\zeta L(\chi_{-4})$ agrees to 20 digits at $s = 2$ and $s = 0.7$.

(c) *The fixed point of J_Q acting by Möbius transformations on $\mathbb{H}$ is the CM point $z_Q = (-b + \sqrt{d})/2a$.*
(d) *For $\gamma \in SL_2(\mathbb{Z})$, $Q \circ \gamma \leftrightarrow \gamma^{-1} J_Q \gamma$.*
(e) *Conversely, every J with $J^2 = -1$, $\sqrt{|d|}\, J \in M_2(\mathbb{Z})$ with entries of the shape above, and $\omega(x, Jx) > 0$ comes from a unique positive form of discriminant d. Primitivity of Q corresponds to primitivity of $\sqrt{|d|}J$.*

Hence $h(d)$ is the number of $SL_2(\mathbb{Z})$-conjugacy classes of primitive integral positive polarizations of $\mathbb{Z}^2$ of discriminant d. Moreover

$$Z_Q(s) = \left(\frac{\sqrt{|d|}}{2}\right)^{-s} \sum_{x \in \mathbb{Z}^2 \setminus 0} g_Q(x, x)^{-s},$$

the zeta function of the polarized lattice $(\mathbb{Z}^2, \omega, J_Q)$. [THM]

Proof. (a) $J_Q^2 = \frac{1}{|d|}(b^2-4ac)I = -I$, and $\det J_Q = (4ac-b^2)/|d| = 1$. (b) $J_Q x = \frac{1}{\sqrt{|d|}}(-bx_1 - 2cx_2,\ 2ax_1 + bx_2)$, so $\omega(x, J_Q x) = \frac{1}{\sqrt{|d|}}(2ax_1^2 + 2bx_1x_2 + 2cx_2^2)$. (c) $z = (-bz-2c)/(2az+b)$ gives $az^2+bz+c=0$. (d) ω is SL_2-invariant, so $\omega(x, \gamma^{-1}J\gamma x) = \omega(\gamma x, J\gamma x) = 2Q(\gamma x)/\sqrt{|d|}$. (e) Read (b) backwards: a, b, c are recovered from the entries. □

Corollary 4.2 (Refinement of Lemma 58 P10)**.** *Positivity of the polarization is free. Every J_Q is positive, yet Z_Q has ghosts for every Q once $h > 1$ (Thm. 3.2), already in $\sigma > 1$. What excludes ghosts in $\sigma > 1$ is Hecke-equivariance: the class is fixed by all Hecke correspondences (true iff $h = 1$), or one passes to Hecke eigen-combinations* (3)*. So P10 ("must distinguish ζ from DH") sharpens to* ***P10′****:* a candidate polarization J must be Hecke-equivariant; positivity alone cannot be the mechanism. *[THM] (for $\sigma > 1$); for the strip $\frac{1}{2} < \sigma \le 1$ it is [OPEN].*

Remark 4.3 (Cross-references: Lemma 54 v2 and Lemma 56 v3)**.** Lemma 54 v2 constructs invariants $\nu = r$ (a Krein-type negative index of a form built from the zeros) which equal the number of real-pair ghosts k in every trace model tested. Positivity there *does* detect ghosts. This is not in conflict with Cor. 4.2. The polarization J_Q of Prop. 4.1 lives on the lattice $\mathbb{Z}^2$ and is positive for every Q, while the Lemma 54 form lives on the spectral side. P10′ says that *lattice* positivity is free. Whether the Lemma 54 index applied to $\Lambda_{Q_1}, \Lambda_{Q_2}$ returns 7 and 6 is Q3 [OPEN]. Lemma 56 v3 tracks the whole-line DH heat flow, including $\beta > 1$ zeros. The far ghosts of Q_2 at $\beta \approx 2$ are a second family of $\beta > 1$ ghosts on which that flow could be run [OPEN].

This is the one place where the class-number circle of ideas *sheds light on the polarization J* in J.W.'s sense. It is not a zero count: it is a statement about which structure (J vs. Hecke) carries the burden.

4.1. **Arakelov class groups.** For a number field F, van der Geer–Schoof [10] have the exact sequence

$$0 \to V/\varphi(O_F^*) \to \mathrm{Pic}^{(0)}(F) \to \mathrm{Cl}(O_F) \to 0, \qquad \mathrm{vol}\,\mathrm{Pic}^{(0)}(F) = hR,$$

where $V/\varphi(O_F^*)$ is a torus of dimension $r_1 + r_2 - 1$. Hence $\mathrm{Cl}(O_F) = \pi_0(\mathrm{Pic}^{(0)}(F))$ [THM][CITED]. Their $h^0(D) = \log \sum_{x \in D} e^{-\pi \|x\|^2}$ satisfies a Riemann–Roch theorem. But (quoted): "we do not have a definition of $h^1(D) := h^0(\kappa - D)$ for an Arakelov divisor without recourse to duality" [CITED]. For $F = \mathbb{Q}$, $\mathrm{Pic}^{(0)}$ is a point. The Arakelov theta function is the natural candidate for a "theta divisor = polarization" with $h = \#\pi_0$ [ANALOGY]. Without an intrinsic h^1 there is no cohomological positivity statement to transfer (direction D2).

5. Towers: archimedean vs. p-adic class-number growth

5.1. **Function fields: archimedean growth sees ghosts.** Let $P(T) = \prod_{i=1}^{2g}(1 - \alpha_i T)$ be the numerator of the zeta function of a curve over $\mathbb{F}_q$, or more generally any real polynomial with the functional-equation symmetry $\alpha \mapsto q/\alpha$ (so $\prod \alpha_i = q^g$) and $1 < |\alpha_i| < q$. Put $h_n = \prod_i (1 - \alpha_i^n)$, the class number over $\mathbb{F}_{q^n}$.

Theorem 5.1 (Archimedean tower criterion)**.** $\limsup_{n\to\infty} |h_n q^{-gn} - 1|^{1/n} = \dfrac{1}{\min_i |\alpha_i|} = q^{-1/2+\delta_{\max}}$, *where* $\min |\alpha_i| = q^{1/2-\delta_{\max}}$*. Hence "RH" ($|\alpha_i| = \sqrt{q}$ for all i) holds iff the limsup equals $q^{-1/2}$. For $g = 1$, a ghost ($a^2 > 4q$) violates the Hasse bound $|h_n - q^n - 1| \le 2q^{n/2}$ for all large n. [THM]*

Proof. $1 - \alpha^n = -\alpha^n(1 - \alpha^{-n})$ and $\prod \alpha_i^n = q^{gn}$, so $h_n q^{-gn} = \prod_i (1 - \beta_i^n)$ with $\beta_i = \alpha_i^{-1}$, $|\beta_i| < 1$. Expanding, $h_n q^{-gn} - 1 = \sum_{\emptyset \ne I} (-1)^{|I|} \beta_I^n$ with $\beta_I = \prod_{i \in I} \beta_i$. The generating function $\sum_n (h_n q^{-gn} - 1) T^n$ is rational with simple poles at $1/\beta_I$. A product β_I with $|I| \ge 2$ has modulus $< \max |\beta_i|$. So the poles of maximal modulus $|\beta| = \max |\beta_i|$ come from singletons, with coefficient $-$(multiplicity) $\ne 0$,

and no cancellation occurs. Cauchy–Hadamard gives the limsup. For $g = 1$ with real $\alpha > \sqrt{q}$: $|\alpha^n + \bar{\alpha}^n| \geq \alpha^n - (q/\alpha)^n > 2q^{n/2}$ eventually. □

Weil's theorem says the hypothesis "RH" is true for curves, so Thm. 5.1 is a *framework* statement: in the model, archimedean class-number growth detects ghosts exactly. The number-field analogue (there is no constant-field tower over $\mathbb{Q}$) is the growth of the error term in the prime number theorem, i.e. von Koch's $\pi(x) - \mathrm{li}(x) = O(x^{1/2+\varepsilon}) \iff$ RH [RESTATEMENT]. Direction D1 is therefore a restatement.

5.2. ℓ-adic growth is ghost-blind.

Proposition 5.2 (Iwasawa invariants of the constant $\mathbb{Z}_\ell$-tower)**.** *Let $\ell \nmid q$ be an odd prime and $\mathfrak{m}$ the maximal ideal of $\overline{\mathbb{Z}}_\ell$. Then for $k \geq k_0$,*

$$v_\ell(h_{\ell^k}) = \lambda k + \nu, \qquad \lambda = \#\{i : \alpha_i \equiv 1 \bmod \mathfrak{m}\}, \quad \nu = \sum_{\alpha_i \equiv 1} v_\ell(\alpha_i - 1) \ \ (\textit{suitably stabilized}),$$

and $\mu = 0$. λ depends only on $P \bmod \ell$, and ν only on $P \bmod \ell^N$ for $N > \nu$. Archimedean RH is not determined by $P \bmod \ell^N$ for any N. Hence the ℓ-adic class-number growth cannot distinguish a true P from a congruent ghost-carrying fake. [THM] (elementary sketch)

Sketch. If $\bar{\alpha} \neq 1$ in $\overline{\mathbb{F}}_\ell$ then $\bar{\alpha}^{\ell^k} \neq 1$ (Frobenius is bijective), so $v(1 - \alpha^{\ell^k}) = 0$. If $\alpha \equiv 1$, then $v(\alpha^\ell - 1) = v(\alpha - 1) + 1$ once $v(\alpha - 1) > 1/(\ell - 1)$. Roots depend continuously on coefficients ℓ-adically. Finally, a and $a + \ell^N m$ cannot both satisfy $a^2 \leq 4q$ for large m. □

Numerics ($q = 5$, $\ell = 3$, $g = 1$, $P = 1 - aT + 5T^2$; `num/lemma60_fftower.py`) [NUM]. True $a = 3$: $v_3(h_{3^k}) = 1, 2, 3, 4, 5$ ($k = 0..4$). True $a = -3$: $2, 3, 4, 5, 6$. The fakes $a = 30, 12, 84$ ($\equiv 3$) and $a = -84$ ($\equiv -3 \bmod 27$) give *identical* sequences. The fake $a = -9$, congruent only mod 3, has the same $\lambda = 1$ but a different ν. Every fake violates the Hasse bound at $n = 1$; the true ones never do for $n \leq 40$ (exact integer check). See Fig. 5.

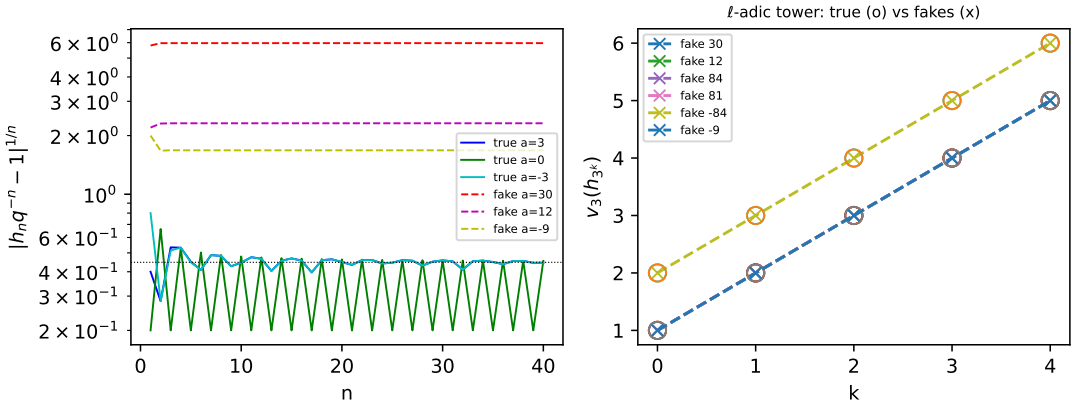

FIGURE 5. Left: $|h_n q^{-n} - 1|^{1/n}$ (exact integer arithmetic). True curves oscillate below and up to $q^{-1/2}$ (dotted). Fakes sit above, and above 1 when $\max|\alpha| > q$; these small-q fakes lie outside the hypothesis $1 < |\alpha_i| < q$ of Thm. 5.1, which only strengthens the separation. Right: $v_3(h_{3^k})$; the true values (circles) and the congruent fakes (crosses) coincide.

5.3. **Iwasawa theory for number fields.** For a $\mathbb{Z}_p$-extension K_∞/K with layers K_n, $p^{e_n} \| h(K_n)$ and $e_n = \mu p^n + \lambda n + \nu$ for $n \gg 0$ (Iwasawa [15, 16]) [BIB]. For abelian $K/\mathbb{Q}$ and the cyclotomic $\mathbb{Z}_p$-extension, $\mu = 0$ (Ferrero–Washington [8]) [BIB]. The main conjecture (Mazur–Wiles [20]; Wiles [30] for totally real fields) identifies the characteristic power series of the minus-part Iwasawa modules with p-adic L-functions, normalized as in TAA Lemma 4 [BIB]. *p-adic ghosts.* The zeros of a p-adic L-function in the open unit disc are counted by λ (Weierstrass preparation). There is no functional equation with a symmetry axis, hence no notion of "off-line" and no p-adic κ (Lemma 38(e)) [THM] (structural remark). Together with Prop. 5.2: λ, μ, ν are ghost-blind invariants [ANALOGY] for number fields, [THM] in the model.

Proposition 5.3 (Greenberg's conjecture is total capitulation; Greenberg 1976, as stated by Sumida, Prop. 1)**.** *Let K_∞/k be a $\mathbb{Z}_p$-extension in which every ramified prime is totally ramified. Let A_n be the p-part of $\mathrm{Cl}(k_n)$, $A_\infty = \varinjlim A_n$, and $H_n = \ker(A_n \to A_\infty)$. Then: $\#A_n$ is bounded $\iff$ $A_n = H_n$ for all n $\iff$ $A_\infty = 0$. [CITED]* [12, 26]

For totally real k, Greenberg's conjecture ($\lambda = \mu = 0$ for the cyclotomic $\mathbb{Z}_p$-extension) [CONJ] is therefore the statement that *every p-class eventually principalizes in the tower.* This is the literal, rigorous "principalization" in the vicinity of J.W.'s idea. It is open, p-adic, and ghost-blind (direction D5).

6. Principalization and capitulation

Theorem 6.1 (Classical capitulation theorems)**.**
(a) (Hilbert's Satz 94) If K/k is cyclic unramified of prime degree, then some non-principal ideal class of k becomes principal in K. [BIB]
(b) (Principal ideal theorem; Artin 1929, Furtwängler 1929) Every ideal of k becomes principal in the Hilbert class field H. Artin reduced this to group theory: capitulation is the transfer $V : G/G' \to G'/G''$ for $G = \mathrm{Gal}(H^{(2)}/k)$, and Furtwängler proved $V = 0$ [1, 9]*. [BIB]*
(c) (Suzuki 1991) For unramified abelian K/k, the order of the capitulation kernel is divisible by $[K : k]$ [27]*. [BIB] (statement via secondary sources)*
(d) For $p \neq 2$ there is no capitulation on the minus part in the cyclotomic $\mathbb{Z}_p$-extension of a CM field containing μ_p ("well known", quoted in [3]*). [CITED]*

Example 6.2 ($\mathbb{Q}(\sqrt{10})$: principalization without complex roots)**.** $k = \mathbb{Q}(\sqrt{10})$ has $h = 2$, narrow class number 2 and signature $(2, 0)$ (LMFDB 2.2.40.1) [BIB]. $\mathfrak{p} = (2, \sqrt{10})$ is not principal, since $x^2 - 10y^2 = \pm 2$ has no solution mod 5. The genus field $H = \mathbb{Q}(\sqrt{2}, \sqrt{5})$ is unramified over k and of degree $2 = h$, so it is the Hilbert class field. It is *totally real.* In H: $\mathfrak{p}O_H = \sqrt{2}\,(\sqrt{2}, \sqrt{5})$ and $(\sqrt{2}, \sqrt{5}) \ni \sqrt{5} \cdot \sqrt{5} - 2\sqrt{2} \cdot \sqrt{2} = 1$, so $\mathfrak{p}O_H = (\sqrt{2})$ [THM]. Principalization needed no complex number at all.

Remark 6.3 (The 691 class never principalizes in the natural tower)**.** The 691 class of 2604.0068 lives in the ω^{-11}-eigenspace of $\mathrm{Cl}(\mathbb{Q}(\zeta_{691}))[691]$ (Herbrand–Ribet). Since $1 - 12 = -11$ is odd, this is the *minus part.* By (d) it does *not* capitulate anywhere in the cyclotomic $\mathbb{Z}_{691}$-tower [THM] (from (d)). It does capitulate in the Hilbert class field, but by (b) so does every class, so that statement carries no information about 691. Any "principalization of the 691 defect" must therefore leave the cyclotomic tower. This is a hard constraint on J.W.'s narrative.

7. Verdict on "principalization as Hamiltonization" and on Wick

As written (J5, J6, J10): analogy [ANALOGY], with false supporting statements (J2–J4, J8, J9). **Salvageable precise versions** [SALVAGEABLE]:

(i) *Forms are polarizations* (Prop. 4.1): a class is a positive polarization of $\mathbb{Z}^2$ up to $SL_2(\mathbb{Z})$, and h counts polarizations [THM]. "Principalization" then means passing from a polarization

class to the Hecke-diagonal combinations (3), the Fourier transform on Cl. That is what separates ghost-carrying class sums from ghost-free Euler products [THM] ($\sigma > 1$) / [NUM] (strip, $d = -20$).

(ii) $\mathrm{Cl} = \pi_0(\mathrm{Pic}^{(0)})$ (§4.1): principalization kills π_0 and leaves the connected torus $V/\varphi(O^*)$ of "Hamiltonian" (translation) flows [THM]; the Hamiltonian reading is [ANALOGY].

(iii) *Dirac reduction*: a degenerate ("singular Legendre") Lagrangian reduced by first-class constraints. There is no arithmetic content until the constraint surface is identified with something like the torus in (ii) [ANALOGY].

Wick $\leftrightarrow$ Hilbert class field: [WRONG] as an equivalence. (1) Wick rotation is a continuous analytic continuation $t \mapsto -i\tau$; H/k is a finite extension with no parameter. (2) H can be totally real (Ex. 6.2), so no imaginary unit is "injected". (3) Passing to H *multiplies* the RH content: $\zeta_H = \prod_\psi L_K(\psi)$ requires GRH for every class character, so it does not "stabilize" anything (Lemma 32). (4) The only rigorous i in sight is the CM point $z_Q = (-b + \sqrt{d})/2a$ of Prop. 4.1(c): choosing $\sqrt{d}$ in the upper half-plane is choosing the positive polarization. "Wick rotation = choosing the sign of J" is a defensible *analogy* [ANALOGY], consistent with OS-positivity in Lemma 58. It gives no mechanism for RH.

8. Ranked directions

Score = 0.6·significance+0.4·probability (both judgmental, out of 10).

	Direction	sig	prob	score
D3	Epstein/class-group test bed: J_Q, DH ghosts, κ census, BG/McPhedran births vs dBN; P10′	6	8	**6.8**
D2	Arakelov theta function / h^0 as polarization source (needs intrinsic h^1)	9	2	6.2
D4	Deuring ghost inheritance ($\zeta \mid Z_{Q_0}$, $s \mapsto 2s$, Lemma 13 link); RH-neutral	5	7	5.8
D5	Greenberg = total capitulation; Iwasawa invariants (ghost-blind, N9)	4	6	4.8
D1	Archimedean tower class numbers (Thm. 5.1); over $\mathbb{Q}$ = von Koch [RESTATEMENT]	2	8	4.4
D6	Literal "principalization as Hamiltonization" via Dirac reduction	5	3	4.2

Only D3 and D2 pass J.W.'s standing condition (light on J, not merely zero counting). D3 already yields P10′. D2 is the high-significance, low-probability bet.

9. Corrections

Correction 9.1 (to Grok Bot's chat claim)**.** The chat said: "the Deuring–Heilbronn phenomenon, where off-line zeros force class numbers to grow". This is inaccurate in four ways. (1) Deuring alone gives only $h \geq 2$ for large $|d|$; growth $h \to \infty$ under not-RH is Mordell's, and under not-GRH Heilbronn's. (2) It is ineffective and RH-neutral: Hecke's theorem covers the complementary case, so $h \to \infty$ holds either way (Rem. 2.4). (3) The sign depends on whose ghost it is. A Siegel ghost of $L(s, \chi_d)$ itself makes $h(d)$ *small* (Prop. 2.3). (4) In modern usage, "Deuring–Heilbronn phenomenon" usually means *zero repulsion*: an exceptional zero repels the other zeros of L-functions near $s = 1$ (Linnik; Jutila's statistical version [17]). It is not the class-number statement.

Correction 9.2 (to J.W.'s papers; for J.W.'s errata list, nothing to be posted without approval)**.** J2 (ideal condition), J3 (sign of E_{12}, spurious $O(q^2)$, the false congruence $E_{12} \equiv \Delta$), J4 (the residue is 0), J8 (thirteen class-number-one CM points, not two), J9 (no proof of RH), J10 (the Hilbert class field can be totally real), J11 (local vs. global principality; E is the Hopf base, not the fiber).

10. Status

Statement	Status
HDMH, Siegel, GGZ (Thms. 2.1, 2.2)	[CITED]; RH-neutral
Siegel ghost $\Rightarrow$ h small (Prop. 2.3)	[THM]
Ghost inheritance $\zeta \mid Z_{Q_0}$; Deuring via $s \mapsto 2s$ (Thm. 2.5)	[THM][CITED]
$h(d) = 1 \iff$ all Z_Q zero-free in $\sigma > 1$ (Thm. 3.3)	[THM]
Forms = positive polarizations; h counts them (Prop. 4.1)	[THM]
P10′: Hecke-equivariance, not positivity (Cor. 4.2)	[THM] ($\sigma > 1$), [OPEN] (strip)
Archimedean tower criterion (Thm. 5.1); over $\mathbb{Q}$	[THM]; [RESTATEMENT]
ℓ-adic growth ghost-blind (Prop. 5.2)	[THM] (sketch), [NUM]
Greenberg = total capitulation (Prop. 5.3)	[CITED]; conjecture [CONJ]
691 class never capitulates in the $\mathbb{Z}_{691}$-tower	[THM] (from [3])
Principalization as Hamiltonization	[ANALOGY]; three [SALVAGEABLE] versions
Wick = Hilbert class field	[WRONG]; sign-of-J reading [ANALOGY]
$d = -20$ census; BG birth; Deuring table; $L(1, \chi_d)$	[NUM]
RH	**not proved**

11. Questions for J.W.

Q1. In your principalization papers, which field extension K/k is meant: the Hilbert class field of an imaginary quadratic field, the cyclotomic $\mathbb{Z}_{691}$-tower, or something on $X(1)$? The 691 class is in the minus part and never capitulates in the cyclotomic tower (§6). Would you accept Prop. 4.1 (class = positive polarization J_Q) as the precise meaning of "principalization as Hamiltonization"?

Q2. P10′ says the polarization must be Hecke-equivariant and that positivity is free. Does this match your intuition for the TAA polarization J, or do you expect positivity to carry weight in the strip $\frac{1}{2} < \sigma \leq 1$ where Thm. 3.3 says nothing?

Q3. Should D3 (the $d = -20$ and $x^2 + ky^2$ test bed: κ census, now with Kim's identity confirmed, $7 = 6 + 1$ and $6 = 2 + 4$; ghost births as the lattice varies; the Lemma 56 v3 heat flow on the far ghosts; the Lemma 54 v2 index ν on $\Lambda_{Q_1}, \Lambda_{Q_2}$) become a standing numerical companion to Lemmas 54–59?

Q4. Are you willing to drop "Wick = Hilbert class field" in favour of "Wick $\approx$ choice of sign of J (CM point in $\mathbb{H}$)", labelled as an analogy?

Q5. Which corrections J2–J11 do you want in an errata note for the RFTP/TAA papers? Nothing will be posted or submitted without your approval.

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TAA LEMMA 61: THE SMOOTH/ANALYTIC THRESHOLD. DENINGER'S EMISSION VERSUS CONNES' ABSORPTION

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Every RH-equivalent statement below is labelled [RESTATEMENT]. Labels: [THEOREM] proved here (elementary, proof given); [THEOREM, cited] a theorem of the literature checked at the source (`lit/lemma61/verified_refs.md`); [cited-sec] checked only through a named secondary source; [BIB] bibliographic record verified, content standard; [TAA-L] an earlier TAA lemma; [RESTATEMENT] restatement; [ANALOGY] analogy only; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical; [WRONG] a translation that is false as stated.

Question (J.W., [t227u]/[t228u]). The threshold between smooth and analytic (derivatives growing like $n!$) is "an absolute core element of our work". Does it tease out a property difference between Deninger's spectral flow (zeros as an *emission* spectrum of a generator Θ) and Connes / Connes–Consani (zeros as an *absorption* spectrum, a cokernel; tropical sites in characteristic one)? Additions: [t229u] a spectroscopy dictionary (Zeeman, fine structure, zero-point energy, g-factor); [t230u] the Heisenberg/Ritz–Rydberg combination principle.

Answer in brief. (1) *The visibility law* (§3, [THEOREM]). In the Connes–Meyer absorption picture the class of the test-function space decides exactly which zeros are seen. On the log-scale line, let the space carry weights of growth type a (equivalently: Mellin transforms analytic in the strip $|\operatorname{Re} s - \frac{1}{2}| < a$). Then the spectrum is *exactly* the set of zeros with $|\beta - \frac{1}{2}| < a$ (Theorem 3.3, proved here from Meyer's description of the range). Every sub-exponential class has $a = 0$ and sees only critical zeros: Sobolev (Connes' L^2_δ), C^∞/Schwartz, every Gevrey class $s > 1$, and even the quasi-analytic but non-analytic classes. The elementary criterion (Prop. 2.1) reproduces both of Connes' unexplained constraints, $\delta > 1$ and multiplicity $< \frac{1+\delta}{2}$. So the $n!$ threshold, read on the spectral side, is literally the line between "the ghosts are invisible" and "the ghosts are visible". *Ghosts are precisely the spectral content lost on passing from the analytic to the smooth class*, and RH $\iff$ nothing is lost [RESTATEMENT]. (2) *The growth ladder* (§4). The same threshold reappears as the growth rate of the scaling flow: bounded (unitarizable by Sz.-Nagy, hence a polarization) $\subset$ polynomial (Connes) $\subset$ sub-exponential $\subset$ exponential type a. Deninger's Hodge-$*$ argument lives on the *bounded* rung, strictly below the threshold. *New elementary result* (Thm. 4.3, [THEOREM]): in the absorption realization a zero of order m is a single Jordan block of size m. Hence a polarization on a zero-complete absorption space forces RH *and simplicity of all zeros*, while an emission realization with a polarization must be semisimple. This is a property difference between the two pictures, and it concerns the polarization, not zero counting. (3) *Deninger's side is intrinsically C^∞* (§5, [THEOREM, cited]). It uses leafwise Hodge theory (Álvarez López–Kordyukov, Riemannian foliations), distributional trace formulas, and the 2024/2026 Álvarez López–Kordyukov–Leichtnam formula with *two* cohomologies (conormal and dual-conormal). His positivity argument needs no analyticity of the foliation. It needs a Hausdorff Hodge theory, a $*$ commuting with Θ, and positivity. The function-space dependence exists on his side too. (4) *Tropical/Maslov* (§6). Dequantization $h \log(e^{u/h} + e^{v/h}) \to \max$ has a *flat* $e^{-|u-v|/h}$ remainder, exactly an entropy term (Connes–Consani) [THEOREM]/[THEOREM, cited]. Its link to the zeros is [ANALOGY]. (5) *Emission vs absorption and J* (§7). Eigen-distributions $e^{\lambda u}$ are tempered iff $\operatorname{Re}\lambda = 0$ [THEOREM]: emission eigenvectors of ghosts exist only among non-tempered (analytic) functionals. **Verdict**: the threshold is a precise handle on *where ghosts are stored*, not on J. J sits on the bounded rung, and the threshold controls location, not form. The one genuine J-statement is the simplicity corollary. (6) Spectroscopy dictionary (§8), Ritz groupoid (§9), numerics with 4 figures (§10; for Davenport–Heilbronn, an argument-principle count of the slabs reproduces all 26 ghosts with $10 < t < 900$ class by class, and ζ shows none, [NUMERICAL]), ranked directions (§11), corrections (§12), Q1–Q5.

Date: October 22, 2026.

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1. Sources, scope and conventions

Read for this lemma. `STATE.md` (§0, §§64–67), `NOTATION.md`; Lemma 7 (Lax–Phillips; Remark "absorption sign and temperedness"), Lemma 28 (§Candidate (4): Deninger foliations), Lemma 33 (boundary absorption spectrum, off-line zeros invisible on the line; causality free, passivity not), Lemma 42 (no positive-energy Hilbert–Pólya reading), Lemma 47 (Connes cokernel; finite-field degeneration), Lemma 48 (Herglotz/Pick, Bode), Lemma 53 (H_+, Paley–Wiener), Lemma 58 (§"(c) Deninger, Connes, Connes–Consani"; reflection datum, polarization J, APD; Prop. "elliptic $\iff$ bounded $\iff$ commutes with a polarization"). Primary sources read: Connes, Selecta 1999 (arXiv v1); Meyer's note arXiv:math/0412277; Deninger, ICM 1998 (Documenta PDF) and the 2005 survey arXiv:math/0505354; Connes–Consani, *Characteristic 1, entropy and the absolute point* and *The scaling site*; Litvinov's survey; Connes' book *Noncommutative Geometry*, Ch. I §1; Sz.-Nagy 1947; Titchmarsh Thm. 9.3. Records and what was checked: `lit/lemma61/verified_refs.md`.

Division of labour. Lemma 59 (in preparation in parallel) treats the derivative tower of ξ, Laguerre–Pólya positivity, and the Gevrey/Denjoy–Carleman/QED–QCD material. We use from it only the definitions: a smooth f is *Gevrey of order* s if $|f^{(n)}| \le CR^{-n}(n!)^s$ locally; $s = 1$ is real-analytic (Cauchy estimates), and $s > 1$ admits compactly supported functions [Gev18]. Lemma 60 (ghosts and class numbers) is not used.

Conventions. $\xi_R(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ is Riemann's ξ (entire). Meyer writes $\xi_M(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, so $\xi_R = \frac{1}{2}s(s-1)\xi_M$. A *ghost* is a zero $\rho = \beta + i\gamma$ with $\beta \neq \frac{1}{2}$, with $\delta_\rho = |\beta - \frac{1}{2}|$. On $\mathbb{R}_+^\times$ we use the log-scale $u = \log x$, and the Mellin transform $\hat{f}(s) = \int_0^\infty f(x)x^s\, d^\times x$ (Meyer's normalization; the critical line is $\operatorname{Re} s = \frac{1}{2}$). DH is the Davenport–Heilbronn function of Lemmas 55–59.

2. The threshold, read on the spectral side

Three classical facts are used. (i) *Cauchy/Gevrey.* Real-analytic means derivatives $\lesssim n!R^{-n}$, and Gevrey $s > 1$ is strictly larger and contains bump functions (Lemma 59). (ii) *Paley–Wiener for strips* [BIB] [PW34]. $e^{a|u|}F \in L^2(\mathbb{R})$ iff $\hat{F}$ extends holomorphically to $|\operatorname{Im} t| < a$ with $\sup_{|y|<a} \|\hat{F}(\cdot + iy)\|_2 < \infty$. Faster-than-every-exponential decay gives an entire $\hat{F}$. (iii) *Polynomial weights.* $(1+u^2)^{k/2}F \in L^2$ iff $\hat{F} \in H^k$ (Sobolev; finite smoothness) [BIB].

So on the log-scale line, *decay class of F in u ↔ regularity class of its spectral transform in t.* The smooth/analytic threshold on the spectral side is the threshold between sub-exponential and exponential weights in u.

Proposition 2.1 (visibility criterion; [THEOREM]). *Let $W:\mathbb{R}\to(0,\infty)$ be measurable and locally bounded above and below, and let $L^2(W)=\{F:\int|F|^2W\,du<\infty\}$. For $\lambda=\mu+i\gamma\in\mathbb{C}$ and $k\in\mathbb{N}$, the functional $\ell_{\lambda,k}(F)=\int F(u)\,u^k e^{\lambda u}\,du$ (defined on $C_c(\mathbb{R})$) extends to a continuous functional on $L^2(W)$ iff*

$$\int_{\mathbb{R}} u^{2k}e^{2\mu u}\,W(u)^{-1}\,du<\infty.$$

Proof. $L^2(W)'\cong L^2(W^{-1})$ under $\langle F,G\rangle=\int F\bar G$ (Riesz, using the unitary $F\mapsto W^{1/2}F$). The functional is represented by $G=u^ke^{\bar\lambda u}$, and $|G|^2W^{-1}=u^{2k}e^{2\mu u}W^{-1}$. Cauchy–Schwarz gives sufficiency, and testing on $F=\mathbf{1}_{[-L,L]}\bar GW^{-1}$, $L\to\infty$, gives necessity. □

Corollary 2.2 (the four classes; [THEOREM]). *Let $\mu=\beta-\frac12$ be the distance of a zero from the line, measured after the half-density normalization. (a)* Polynomial $W=(1+u^2)^{\delta/2}$ *(Connes' L^2_δ): $\ell_{\lambda,k}$ is continuous iff $\mu=0$ and $k<\frac{\delta-1}{2}$. In particular no character is continuous unless $\delta>1$, and Jordan chains $\{\ell_{\lambda,0},\dots,\ell_{\lambda,n-1}\}$ have length $n<\frac{1+\delta}{2}$. (b)* Gevrey-type $W=e^{2c|u|^{1/s}}$, $s>1$, *and more generally every W with $\log W(u)=o(|u|)$ and $W\ge(1+u^2)^{1/2+\epsilon}$: continuous iff $\mu=0$. (c)* Exponential type a, $W=e^{2a|u|}$*: continuous iff $|\mu|<a$. (d)* All exponential types *(Meyer's $\bigcap_s$): every λ.*

Proof. Integrability of $u^{2k}e^{2\mu u}W^{-1}$ at $\pm\infty$. In (b), $e^{2|\mu||u|}/W\to\infty$ unless $\mu=0$. □

Remark 2.3 (consistency with Connes' Theorem 1; [THEOREM] + [THEOREM, cited]). Connes' norm on $L^2_\delta(C_{\mathbb{Q}})$ is $\int|\xi(g)|^2(1+\log^2|g|)^{\delta/2}d^*g$ (Con99, eq. (11) of App. II and (7) of §III), i.e. $W=(1+u^2)^{\delta/2}$ in $u=\log|g|$. His Theorem 1 [THEOREM, cited] requires $\delta>1$ and gives each critical zero with multiplicity "the largest integer $n<\frac{1+\delta}{2}$, $n\le$ order". Corollary 2.2(a) predicts *both* constraints from the function space alone: $\delta>1$ is exactly when the critical characters become continuous, and $n<\frac{1+\delta}{2}$ is exactly the length of continuous Jordan chains $u^ke^{i\gamma u}$. Connes calls δ "an unnatural parameter … which plays the role of a Sobolev exponent and allows to see the absorption spectrum as a point spectrum" and says the space "artificially eliminates the non-critical zeros" (Con99, Intro and §VIII) [THEOREM, cited]. Prop. 2.1 is the mechanism: the polynomial (finitely smooth) class has no exponential room.

Remark 2.4 (quasianalyticity is not the threshold; [THEOREM]/[BIB]). Denjoy–Carleman quasianalyticity sits *below* analyticity. On the weight side, the Beurling–Domar condition $\int\log W(u)\,\frac{du}{1+u^2}<\infty$ is the standard non-quasianalyticity condition for weighted algebras (it is what makes compactly supported spectral bumps available) [Dom56, Koo88] [BIB]. The weight $W=e^{2|u|/\log(e+|u|)}$ violates it, so it is on the quasianalytic side. It is still $e^{o(|u|)}$, so by Cor. 2.2(b) it sees no ghost. The exact visibility threshold is therefore *analyticity in a strip* (Gevrey $s=1$ with positive radius), not quasianalyticity: *the $n!$ line, not the Denjoy–Carleman line, separates the two pictures.*

3. The visibility law: Connes' δ and Meyer's spaces as the two ends

Theorem 3.1 (Connes 1999, Thm. 1; [THEOREM, cited]). *For $\delta>1$ and a character χ of the norm-one idele classes, the generator D_χ of the scaling action on the cokernel $H_\chi\subset L^2_\delta(C_k)/\overline{\mathrm{Im}E}$ has discrete spectrum contained in $i\mathbb{R}$, equal to the set of $\rho\in i\mathbb{R}$ with $L(\tilde\chi,\frac12+\rho)=0$.*

Theorem 3.2 (Meyer, arXiv:math/0412277, Thm. 4.1, Cor. 4.2; [THEOREM, cited]). *Let $\mathcal{S}_s=\{f:\mathbb{R}_+^\times\to\mathbb{C}:\ x\mapsto f(x)x^s\in\mathcal{S}(\mathbb{R}_+^\times)\}$, $\mathcal{S}_I=\bigcap_{s\in I}\mathcal{S}_s$, $H_-=\mathcal{S}_{(-\infty,\infty)}$ (nuclear Fréchet). The Mellin transform identifies H_- with the entire functions that are Schwartz on every vertical line, and identifies $\mathcal{S}_I$, for open I, with the functions holomorphic on $\Sigma_I=I+i\mathbb{R}$ that are Schwartz on every*

line $\operatorname{Re} s = \sigma \in I$. *The subspace* $ZH_\cap \subset H_-$ *corresponds to the entire* h *with* $h/\zeta(s)$ *Schwartz on* $\operatorname{Re} s \geq \frac{1}{2}$ *and* $h/\zeta(1-s)$ *Schwartz on* $\operatorname{Re} s \leq \frac{1}{2}$. *The eigenvalues of the transpose of the generator* D *(acting as* $h \mapsto sh$*) on* $(H_-/ZH_\cap)'$ *are exactly the zeros of* ξ*, with algebraic multiplicity equal to the order. No RH is assumed.*

Theorem 3.3 (strip visibility law; [THEOREM], given Thm. 3.2)**.** *Let* $I \subset \mathbb{R}$ *be an open interval,* R_I *the closure of* $ZH_\cap$ *in* $\mathcal{S}_I$*, and* $\mathcal{H}_I = \mathcal{S}_I/R_I$*, with the scaling generator* D*. For* $\lambda \in \mathbb{C}$*:* λ *is an eigenvalue of the transpose* ${}^tD$ *on* $\mathcal{H}_I'$ *iff* $\xi_R(\lambda) = 0$ *and* $\operatorname{Re}\lambda \in I$*. Each such eigenvalue has a one-dimensional eigenspace, and its generalized eigenspace has dimension equal to the order* m_λ *of the zero, so it is a single Jordan block of size* m_λ*. In particular, for* $I = (\frac{1}{2} - a, \frac{1}{2} + a)$*:*

$$\operatorname{Sp}({}^tD \,|\, \mathcal{H}_I') = \{\rho : \ \xi_R(\rho) = 0, \ \delta_\rho < a\}.$$

Proof. Work on the Mellin side, where D is multiplication by s.

Continuity of jets. Fix $\lambda \in \Sigma_I$ and a closed sub-strip $[\sigma_1, \sigma_2] + i\mathbb{R} \subset \Sigma_I$ containing a disc around λ. For $h = \hat{f}$ and $\sigma \in [\sigma_1, \sigma_2]$, $|h(\sigma+it)| \leq \int |f| x^\sigma d^\times x \leq \int |f|(x^{\sigma_1} + x^{\sigma_2}) d^\times x$, a continuous seminorm on $\mathcal{S}_I$. Cauchy's estimate then makes $h \mapsto h^{(j)}(\lambda)$ continuous for every j.

Division. Let $e_\lambda(s) = e^{(s-\lambda)^2}$; on every vertical line it is a Gaussian in t, so $e_\lambda \in \mathcal{S}_I$, and $e_\lambda(\lambda) = 1$. If $\lambda \notin \Sigma_I$, then $1/(s-\lambda)$ is holomorphic on Σ_I with all t-derivatives bounded on each line $\operatorname{Re} s = \sigma \in I$ (the distance $|\sigma - \operatorname{Re}\lambda| > 0$), so multiplication by $s - \lambda$ is onto $\mathcal{S}_I$. If $\lambda \in \Sigma_I$, then $h - h(\lambda)e_\lambda = (s-\lambda)g$ with $g \in \mathcal{S}_I$ (removable singularity; smooth and Schwartz on each line).

Eigenvalues. Let $l \neq 0$ be continuous with $l|_{R_I} = 0$ and $l((s-\lambda)g) = 0$ for all g. If $\lambda \notin \Sigma_I$, then $l = 0$ by surjectivity, a contradiction. If $\lambda \in \Sigma_I$, then $l = l(e_\lambda)\,\mathrm{ev}_\lambda$, so the eigenspace is at most one-dimensional. The function $h_0(s) = 2\xi_R(s)e^{(s-\frac{1}{2})^2} = s(s-1)\xi_M(s)e^{(s-\frac{1}{2})^2}$ lies in $ZH_\cap$ by Thm. 3.2: $h_0/\zeta(s) = s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})e^{(s-\frac{1}{2})^2}$ is Schwartz on lines with $\operatorname{Re} s \geq \frac{1}{2}$, and by $\xi_M(s) = \xi_M(1-s)$, $h_0/\zeta(1-s) = s(s-1)\pi^{-(1-s)/2}\Gamma(\frac{1-s}{2})e^{(s-\frac{1}{2})^2}$ is Schwartz on lines with $\operatorname{Re} s \leq \frac{1}{2}$. So $0 = l(h_0) \propto \xi_R(\lambda)$. Conversely, if $\xi_R(\lambda) = 0$ and $\operatorname{Re}\lambda \in I$, then every $h \in ZH_\cap$ vanishes at λ to order $\geq m_\lambda$: if $\operatorname{Re}\lambda \geq \frac{1}{2}$, h/ζ has no pole at λ; if $\operatorname{Re}\lambda < \frac{1}{2}$, use $h/\zeta(1-s)$ and $\zeta(1-\lambda) = 0$. The functionals $h \mapsto h^{(j)}(\lambda)$, $j < m_\lambda$, are continuous, vanish on $ZH_\cap$ and hence on R_I, and are nonzero on $\mathcal{H}_I$ (test on $(s-\lambda)^j e_\lambda$). So ev_λ is an eigenvector of ${}^tD$.

Jordan structure. Iterating the division step, $({}^tD - \lambda)^k l = 0$ iff $l = \sum_{j<k} a_j\, h \mapsto h^{(j)}(\lambda)$ (Meyer's argument, unchanged). Vanishing on R_I forces $a_j = 0$ for $j \geq m_\lambda$: test on $h_0 \cdot p$ with p a polynomial, which is still in $ZH_\cap$ and has arbitrary jets of order $\geq m_\lambda$ at λ. So the generalized eigenspace has dimension m_λ and contains the one-dimensional eigenspace, giving a single Jordan block. □

Corollary 3.4 (ghosts = the content lost from analytic to smooth; [THEOREM], and [RESTATEMENT] for the last clause)**.** *For* $0 < a < b$, $\operatorname{Sp}(\mathcal{H}'_{(\frac{1}{2}-b,\frac{1}{2}+b)}) \setminus \operatorname{Sp}(\mathcal{H}'_{(\frac{1}{2}-a,\frac{1}{2}+a)}) = \{\rho : \ a \leq \delta_\rho < b\}$. *As* $a \downarrow 0$ *the spectrum decreases to the critical zeros, which is the content of Connes' Theorem 3.1 (polynomial class).* RH $\iff$ the spectrum is independent of the class $\iff$ the passage from the analytic class to the smooth (Sobolev, C^∞, Gevrey $s > 1$) class loses nothing *[RESTATEMENT].*

Remark 3.5 (what this is and is not)**.** Theorem 3.3 answers J.W.'s Q1 in the strongest form we could hope for. There *is* a theorem "the absorption spectrum in the space X consists of exactly the zeros with $|\beta - \frac{1}{2}| < f(X)$", with $f(X) =$ the exponential type of the weight, i.e. the half-width of the analyticity strip of the spectral transforms. It is also sobering. Visibility is *chosen* by the analyst, not imposed by ζ. Connes' δ-space is RH-neutral (it sees critical zeros unconditionally and is silent about ghosts). Meyer's space is RH-neutral (it sees everything and is silent about where). Neither carries a positive form. The visibility law is linear-topological; positivity is not.

4. The growth ladder, and a property difference: Jordan blocks versus semisimplicity

The same threshold reappears as the growth rate of the flow itself.

Proposition 4.1 (growth ⇒ spectral strip; [BIB] (standard semigroup theory, EN00)). *Let $T(t) = e^{tA}$ be a C_0-group on a Banach space with $\|T(t)\| \le Me^{\omega|t|}$. Then* $\mathrm{Sp}(A) \subset \{|\operatorname{Re}\lambda| \le \omega\}$. *If $\|T(t)\| = O(e^{\epsilon|t|})$ for every $\epsilon > 0$ (sub-exponential growth), then* $\mathrm{Sp}(A) \subset i\mathbb{R}$.

This is exactly Connes' remark after (22) in Con99 ($\|W(g)\| = O(\log|g|)^{\delta/2}$, so the generator has purely imaginary spectrum) [THEOREM, cited]. It is also Deninger's Thm. 5.4 in Den05 (spectrum of Θ in $-\omega \le \operatorname{Re} s \le \omega$; if ϕ^{t*} is orthogonal, $-i\Theta$ is self-adjoint) [THEOREM, cited].

Theorem 4.2 (Sz.-Nagy 1947, Thms. I–II; [THEOREM, cited]). *A uniformly bounded group of operators T^n ($n \in \mathbb{Z}$), or a strongly continuous uniformly bounded one-parameter group T_s ($s \in \mathbb{R}$), on a Hilbert space is similar to a unitary group: QT_sQ^{-1} is unitary for a bounded positive invertible Q. The new inner product is a Banach-limit average $\langle f, g\rangle = L(T_sf, T_sg)$.*

The ladder. For the normalized scaling flow on a space realizing the zeros, consider four rungs:

rung	growth of the flow	spectral consequence	where it occurs
R0	bounded	on the line, *semisimple*; unitarizable (Thm. 4.2)	Deninger's Hodge-$*$ ($\Theta = \frac{1}{2} + A$)
R1	polynomial $O(\lvert t\rvert^{\delta/2})$	on the line, Jordan chains $< \frac{1+\delta}{2}$	Connes' L^2_δ (Thm. 3.1)
R2	sub-exponential	on the line	Gevrey/C^∞ classes (Cor. 2.2(b))
R3	exponential type a	strip $\delta_\rho \le a$	strip spaces $\mathcal{S}_I$ (Thm. 3.3); Meyer ($a = \infty$)

The smooth/analytic threshold is the line between R2 and R3. The polarization J of Lemma 58 lives on R0, strictly below the threshold.

Theorem 4.3 (absorption is non-semisimple at multiple zeros; [THEOREM]). *(a) In the absorption realizations of Theorems 3.2 and 3.3, a zero of order m carries a single Jordan block of size m. In Connes' L^2_δ the corresponding object is a Jordan chain of length* $\min(m, \lceil\frac{1+\delta}{2}\rceil - 1)$. *(b) Hence there is no continuous inner product on a zero-complete absorption space for which the scaling flow is uniformly bounded, unless* every *zero lies on the line* and is simple. *More precisely, a uniformly bounded flow restricts to a bounded flow on each finite-dimensional generalized eigenspace, and $e^{t(\lambda+N)}$ with $N \neq 0$ nilpotent is unbounded. (c) In an emission realization with a polarization (Deninger: $\Theta = \frac{1}{2} + A$ with A skew on a Hilbert space), Θ is semisimple. A zero of order m must appear as an m-dimensional* eigenspace*, i.e. a degenerate level, not a Jordan block.*

Proof. (a) is Theorem 3.3 (and Remark 2.3 for L^2_δ). (b): $\|e^{t(\lambda+N)}\| \ge c|t|^k e^{t\operatorname{Re}\lambda}$ with $N^k \neq 0$; boundedness for $t \in \mathbb{R}$ forces $\operatorname{Re}\lambda = 0$ (after normalization) and $N = 0$. (c): skew-adjoint operators are diagonalizable on their finite-dimensional generalized eigenspaces. □

Remark 4.4 (what Theorem 4.3 adds). This is the one place where the threshold analysis says something about J rather than about zero location. *Any polarization on the absorption (cokernel) side would prove RH and the simplicity of all zeros.* Simplicity is a separate open conjecture, so this is a strictly stronger target. A polarization on the emission side proves RH and requires degenerate levels for multiple zeros. The two pictures are inequivalent as *modules*: the cokernel is cyclic (one generator, $\mathbb{C}[s]/$(local factor $(s-\rho)^m$) at each zero), while a polarized emission space is semisimple. The finite-field analogue is consistent: for a curve over $\mathbb{F}_q$, Frobenius acts semisimply on H^1 (on the Tate module of the Jacobian, by Weil's positivity of the Rosati involution) [BIB], and its H^1 is a Deninger-type (emission) space, not a cyclic cokernel. Whether an absorption model of a curve's H^1 can be made semisimple is [OPEN] and is direction D1 (§11).

5. Deninger's side is intrinsically C^∞

The formalism (Den98 §3; Den05 §3) [THEOREM, cited]. Write $\hat\zeta(s) = \zeta(s)\zeta_\infty(s)$, $\zeta_\infty(s) = 2^{-1/2}\pi^{-s/2}\Gamma(\frac{s}{2})$. Deninger proves that $\zeta_p(s) = \det_\infty\left(\frac{1}{2\pi}(s-\Theta)\,|\,\mathcal{R}_p\right)^{-1}$ for every place, with $\mathcal{R}_\infty =$

$\mathbb{R}[\exp(-2y)]$, $\Theta = d/dy$ (Prop. 3.1, via Lerch's formula). He conjectures cohomologies $H^i(\overline{\operatorname{Spec}\mathbb{Z}}, \mathbb{R})$ with $\hat{\zeta}(s) = \prod_{i=0}^{2} \det_\infty(\frac{1}{2\pi}(s-\Theta)|H^i)^{(-1)^{i+1}}$. This forces $H^0 = \mathbb{R}$ ($\Theta = 0$), H^1 infinite-dimensional with spectrum the nontrivial zeros, and $H^2 = \mathbb{R}$ ($\Theta = \mathrm{id}$). The consequence $\frac{s}{2\pi}\frac{s-1}{2\pi}\hat{\zeta}(s) = \prod_\rho \frac{1}{2\pi}(s-\rho)$ (zeta-regularized) "turned out to be true" [Den92, SchS94]. If a Hodge $*$ on H^1 makes $\langle f, f'\rangle = \operatorname{tr}(f \cup *f')$ positive definite and commutes with Θ, then Θ being a derivation for $\cup$ gives $\Theta = \frac{1}{2} + A$ with A skew, "hence the Riemann conjecture would follow" (Den98 p. 169) [THEOREM, cited]. In our language, that is R0.

The toy theorem (Den05 Thm. 4.1) [THEOREM, cited]. Let X be a compact 3-manifold with a Riemannian foliation by surfaces with a dense leaf, and ϕ^t a foliation-compatible flow that is conformal on TF with factor $e^{\alpha t}$. Then on $\bar{H}^1$, $\Theta = \frac{\alpha}{2} + S$ with S skew. This is a *proved* "Riemann hypothesis" in a C^∞ model, by exactly the Hodge-$*$ mechanism. Deninger adds that he does "not think that such a [conformal] metric will exist for dynamical systems relevant to number fields"; one would need Kähler identities instead (Den05 §4).

Where the analysis lives. (i) The leafwise Hodge isomorphism $\ker \Delta_F \cong \bar{H}^n$ is a theorem of Álvarez López–Kordyukov for *Riemannian* foliations [ALK01] [cited-sec]. It "is quite deep since Δ_F is only elliptic along the leaves", and it fails for general foliations [DS01] [cited-sec]. (A corrigendum to ALK01 appeared in Compos. Math. 161 (2025); we have not read it, [OPEN] for us.) (ii) The trace formulas are identities of *distributions* on $\mathbb{R}$ tested by $C_c^\infty(\mathbb{R})$: Conjecture 5.1 and Theorem 5.3 (ALK02, flows transversal to the leaves) in Den05 [cited-sec]. Their Fourier–Laplace transforms are entire of exponential type, so on the spectral side the explicit formula is an R3/Meyer-class object and sees all zeros, ghosts included. (iii) The recent trace formula for foliated flows *with* preserved leaves (Álvarez López–Kordyukov–Leichtnam, arXiv:2402.06671; LNM 2387, 2026) uses *two* reduced leafwise cohomologies, of currents conormal and dual-conormal to the preserved leaves. Its definition involves renormalized (b-)traces, and it solves a conjecture of Deninger; the publisher's summary notes that "further generalization is needed for arithmetic applications" [THEOREM, cited] (abstract and publisher's summary only). So the *function-space dependence also exists on the emission side*: a conormal space and its dual-conormal partner play the roles of $\mathcal{S}_I$ and its dual. (iv) Deninger asks that the arithmetic cohomologies be the dense subspaces of *smooth vectors*, i.e. finite sums of generalized eigenvectors of the flow, inside reduced leafwise cohomology (Den98 p. 181) [THEOREM, cited]. These are entire (Θ-finite) vectors.

Proposition 5.1 (what Deninger's positivity needs; [RESTATEMENT]/[THEOREM, cited])**.** *In the formal argument for $\Theta = \frac{1}{2} + A$, the needed inputs are: (1) a Hausdorff (reduced) H^1 with a Hodge isomorphism, a* metric rigidity *(Riemannian foliation), not analyticity; (2) $*$ commuting with Θ (conformality or Kähler identities); (3) positivity of $\operatorname{tr}(f \cup *f)$, the polarization. Real-analyticity of the foliation or the flow enters nowhere. The smooth/analytic distinction enters only through the choice of test functions and cohomology classes in the trace formula.*

Verdict for §5. The threshold does *not* separate Deninger from Connes as programs: each has an analytic-class side (trace formula with C_c^∞ test functions; Meyer) and a smooth/bounded side (Hodge-$*$ Hilbert space; Connes' δ). What separates them is *where the spectrum is housed*: as eigenvectors in the space (Deninger; R0 if polarized) or as eigen*functionals* on a quotient (Connes/Meyer). Theorem 4.3 shows the two are inequivalent at multiple zeros.

6. Connes–Consani in characteristic one: Maslov dequantization

The facts [THEOREM, cited]. Connes–Consani recall that $h\ln(e^{w_1/h} + e^{w_2/h}) \to \max\{w_1, w_2\}$ as $h \to 0$. Hence $D_h(u) = u^h$ deforms $(\mathbb{R}_+, +, \cdot)$ to $\mathbb{R}_+^{\max}$. In the limit there is a one-parameter group $\vartheta_\lambda(u) = u^\lambda$, which "corresponds to the arithmetic Frobenius" (CC-char1, Intro). They reconstruct addition from the degenerate limit by the entropy formula $\log(e^a + e^b) = \sup_{x\in[0,1]} S(x) + xa + (1-x)b$, $S(x) = -x\log x - (1-x)\log(1-x)$ (*ibid.*, (4) and §2.4, a characteristic-one Witt construction). The scaling site $[0,\infty) \rtimes \mathbb{N}^\times$ is the extension of scalars of the arithmetic site to $\mathbb{R}_+^{\max}$. Its points

are the adele classes modulo $\hat{\mathbb{Z}}^*$, and its periodic orbits give, for each prime p, a curve C_p with Jacobian $\mathbb{Z}/(p-1)\mathbb{Z}$ and a Riemann–Roch formula with real dimensions (CC-scal, abstract). Litvinov: idempotent mathematics is the dequantization "as the Planck constant tends to zero taking imaginary values" (Lit07, §1; see also Viro [Vir01]) [THEOREM, cited].

Proposition 6.1 (the dequantization remainder is flat and is an entropy; [THEOREM]). *For $u \neq v$, $r(h) := h\log(e^{u/h} + e^{v/h}) - \max(u,v) = h\log(1 + e^{-|u-v|/h})$ satisfies $0 < r(h) \leq he^{-|u-v|/h}$, so r extends to a C^∞ function on $[0,\infty)$ with* all *derivatives zero at $h = 0$: smooth but not analytic, of exactly the e^{-1/g^2} type. For $u = v$, $r(h) = h\log 2$ is analytic. Moreover $h\log(e^{a/h} + e^{b/h}) = \sup_x[hS(x) + xa + (1-x)b]$, with the supremum attained at the Gibbs weight $x^* = e^{a/h}/(e^{a/h} + e^{b/h})$. So the flat remainder is $hS(x^*)$ minus the loss $(1-x^*)(a-b)$ (for $a > b$): the non-analytic sector is precisely an entropy.*

Proof. $\log(1+y) \leq y$; the bound $he^{-d/h}$ and all its derivatives tend to 0 as $h \downarrow 0$. The variational formula is the Legendre duality of $\log(e^a + e^b)$ with $-S$ (CC-char1 (4)), rescaled by h; the first-order condition gives x^*. □

Assessment. *Genuine* ([THEOREM]): the tropical limit is a flat, non-analytic limit, and it discards exactly the exponentially small sector, as perturbation theory discards $e^{-8\pi^2/g^2}$. Connes–Consani's entropy reconstruction is a mechanism that *recovers* the flat sector exactly, which perturbation theory cannot do. Also genuine: in CC-char1 the deformation parameter h and the Frobenius exponent λ are the same family $u \mapsto u^{(\cdot)}$, and Litvinov's "imaginary Planck constant" makes h a Wick-rotated $\hbar$. *Only analogy* ([ANALOGY]): that this flatness has anything to do with the visibility of ghosts. The scaling site sees the primes (periodic orbits of length $\log p$), the emission/orbit side. No statement in CC-scal or CC-char1 concerns zeros off the line. TAA Lemmas 23 and 26 found that tropical structures keep $|\cdot|$ and lose positivity [TAA-L]. The Riemann–Roch strategy of Connes–Consani (2019) remains a programme [OPEN].

7. Emission versus absorption, boundary values, and the polarization

Proposition 7.1 (temperedness of eigen-distributions; [THEOREM]). *The distributions F on $\mathbb{R}$ with $F' = \lambda F$ are $ce^{\lambda u}$. Such a distribution is tempered iff $c = 0$ or $\operatorname{Re}\lambda = 0$. More generally, $u^k e^{\lambda u} \in \mathcal{S}'(\mathbb{R})$ iff $\operatorname{Re}\lambda = 0$.*

Proof. Uniqueness for the ODE in $\mathcal{D}'$; $e^{\mu u}$ with $\mu \neq 0$ grows exponentially, so it pairs unboundedly with Schwartz functions supported far out. □

So an emission eigenvector for a ghost exists only among *non-tempered* functionals: duals of analytic (exponential-type) classes. The same split runs through earlier lemmas:

object	class	ghosts seen?	positivity
L33 boundary absorption spectrum $\pi\sum m\delta_t$	boundary values on the line (tempered)	no	positive measure, free [TAA-L]
L48 Herglotz/Pick, Bode	boundary values; ghosts in interior factor	only via interior	passivity free, outer $\iff$ RH [TAA-L]
L7 Lax–Phillips	Hardy space (analytic in half-plane)	yes, as resonances	contraction free, RH-neutral [TAA-L]
L53 H_+	Hardy/Paley–Wiener	via Blaschke factors	$\omega(f, Jf) > 0$ for loops, not for ζ [TAA-L]
Connes L^2_δ	polynomial (Sobolev)	no	Hilbert, but not the Weil form
Meyer H_-, $\mathcal{S}_I$	entire / strip	yes / $\delta < a$	none
Deninger H^1 (conj.)	smooth vectors in a Hilbert completion	n/a (RH if R0)	Hodge-$*$: the polarization

Theorem 7.2 (the threshold versus J; [THEOREM] for (a)–(b), [RESTATEMENT] for (c))**.** *(a) A polarization in the sense of Lemma 58 (a complex structure J commuting with the normalized flow, with $\omega(\cdot, J\cdot) > 0$) makes the normalized flow unitary on the completion. This is rung R0, so the spectrum is on the line and semisimple. (b) Conversely, rung R0 for a flow on a Hilbert space yields an invariant inner product (Thm. 4.2), but by a Banach-limit average, i.e. non-constructively, and with no reality or Hodge-type compatibility. R1–R2 give location without any invariant form. (c) The smooth/analytic threshold (R2/R3) therefore controls* where the ghosts are stored *(Cor. 3.4: in the analytic, non-tempered part). It does not produce J. RH $\iff$ "no loss from R3 to R2 in the zero-complete space"; RH + simple zeros $\Leftarrow$ "R0 on a zero-complete absorption space" (Thm. 4.3).*

Verdict on J.W.'s standing condition. Corollary 3.4 is, at bottom, a statement about which zeros a space can see. It is close to zero counting, and we do not recommend pursuing it further as such. Theorem 4.3 and Theorem 7.2(a)–(b) *do* bear on the polarization. They say J must live strictly below the $n!$ threshold (bounded flow). They say that on the absorption side J would force simplicity. And they say that the only general existence theorem for an invariant inner product (Sz.-Nagy) is an averaging with no Hodge content. *The threshold is a precise structural fact, not a new handle on J; the honest label for "the threshold is the core of the polarization" is [ANALOGY].*

8. A spectroscopy dictionary (J.W. [T229U])

Each row tests one translation. Labels: [THEOREM]/[THEOREM, cited] = theorem-backed, [ANALOGY] = analogy, [WRONG] = wrong as stated.

physics	arithmetic counterpart	verdict
(a) Zeeman: sectors by m	$H = \bigoplus_{\chi\in\hat{K}} H_\chi$ by characters of the norm-one idele classes (Con99 (23)); for abelian $K/\mathbb{Q}$, $\zeta_K = \prod_\chi L(s,\chi)$	sector decomposition [THEOREM, cited]/[BIB]; "a field splits a degenerate level" [ANALOGY]; "the zeros of ζ split into those of ζ_K" [WRONG]
(b) $\sigma_\pm$ circular polarization	$J^2 = -1$ on real V: $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ ($\pm i$ eigenspaces); for a curve, $H^1 = H^{1,0} \oplus H^{0,1}$	linear algebra [THEOREM]; pairing of $\frac{1}{2} \pm i\gamma$ [THEOREM] in polarized models; positivity of $\omega(\cdot, J\cdot)$ for ζ [OPEN]
(c) zero-point energy	ζ-regularized determinants: $\zeta_\infty(s) = \det_\infty(\frac{1}{2\pi}(s - \Theta)\lvert\mathbb{R}[e^{-2y}])^{-1}$; $\prod_\rho \frac{s-\rho}{2\pi} = \frac{s}{2\pi}\frac{s-1}{2\pi}\hat{\zeta}(s)$	[THEOREM, cited] (Den98 Prop. 3.1; Den92, SchS94); checked [NUMERICAL]; "= ZPE" needs the refinement below
(d) fine structure	lower-order terms in $N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(\frac{1}{T})$ (Tit86 Thm. 9.3)	the formula [THEOREM, cited]; "relativistic correction" [ANALOGY]; Hodge grading as fine structure [ANALOGY]
(e) $g = 2$, $g - 2$	Dirac $g = 2$ [Dir28]; $a_e = \frac{\alpha}{2\pi} + \ldots$ [Sch48], 5-loop coefficients known [AKN19]; series believed asymptotic (Dyson)	physics [BIB]; link to ζ or to Lemma 50 spinors [ANALOGY]; "$g = 2$ because of the double cover" [WRONG]

(a) Zeeman. *Theorem-backed part.* Connes' cokernel splits canonically into $\hat{K}$-sectors, $H = \bigoplus_\chi H_\chi$, and each sector carries the zeros of $L(\tilde{\chi}, \cdot)$ (Con99 (23), Thm. 1) [THEOREM, cited]. For $\mathbb{Q}$, $K = \hat{\mathbb{Z}}^*$ and the sectors are Dirichlet characters, the "spatial modes" of Lemma 28. For abelian $K/\mathbb{Q}$, $\zeta_K(s) = \prod_\chi L(s,\chi)$ [BIB], so the zero set of ζ_K is the union, with multiplicities, of the character sectors. The labels are characters of a compact abelian group, as m labels characters of $\mathrm{U}(1) \subset \mathrm{SO}(3)$. *Analogy part.* In Zeeman a *continuous* field B interpolates between a degenerate level ($B = 0$) and its split components. No such parameter connects ζ to the $L(s,\chi)$: they are distinct functions with no common degenerate limit. *Wrong as stated.* ζ is one factor of ζ_K ($\chi = 1$); its zeros are not split, only joined by other sectors. Splitting/inert/ramified primes and Chebotarev label *orbits* (Frobenius

classes), i.e. the prime side, not levels. With Theorem 4.3(c), the honest Zeeman-type question is: if a zero were multiple, would the emission picture show a *degenerate level* that a symmetry-breaking deformation splits, while the absorption picture shows an *exceptional point* (Jordan block)? That is [ANALOGY], but it isolates a testable difference.

(b) Polarization and helicity. For a rotation generator on $\mathbb{R}^2$, $(1, \mp i)$ are the $\pm i$ eigenvectors. These are the two circular polarizations, and in a polarized real flow the eigenvalues $\frac{1}{2} \pm i\gamma$ sit on $V^{1,0}$ and $V^{0,1}$ [THEOREM] (Lemma 58, finite-dimensional dichotomy). This is genuine linear algebra and goes exactly as far as the existence of *some* J commuting with the flow. The TAA polarization asks for more: $\omega(\cdot, J\cdot) > 0$. Zeeman's $\sigma_\pm$ say nothing about that positivity. Beyond the decomposition, it is [ANALOGY].

(c) Zero-point energy. Theorem-backed for the Γ-factor and, formally, for the zeros. With $Z(z) = \sum_{\nu \geq 0} \left(\frac{s+2\nu}{2\pi}\right)^{-z} = \pi^z \zeta_H(z, \frac{s}{2})$ and Lerch's $\partial_z \zeta_H(0, a) = \log \frac{\Gamma(a)}{\sqrt{2\pi}}$, one gets $\exp(Z'(0)) = 2^{-1/2}\pi^{-s/2}\Gamma(\frac{s}{2})$, i.e. Deninger's Prop. 3.1 at $p = \infty$. We checked this to relative error $\leq 6 \times 10^{-31}$ at four points (§10) [NUMERICAL]. *Refinement.* $\det_\infty = \exp(-\zeta'_\Theta(0))$ is the one-loop functional determinant (cf. Hawking 1977). Its logarithm is the vacuum free energy. The zero-point energy $\frac{1}{2}\sum \omega$ is instead $\frac{1}{2}\zeta_\omega(-1)$, a different value of the same spectral zeta function. "Deninger's Γ-factor = a zero-point energy" is correct only in the free-energy sense. The textbook instance in Den98: $\det_\infty(\Theta)$ for spectrum $1, 2, 3, \ldots$ is $\sqrt{2\pi}$.

(d) Fine structure. The constant $\frac{7}{8}$ and the $-\frac{T}{2\pi}$ term come from Γ (Stirling) and from $s(s-1)$; this is Theorem 9.3 of Titchmarsh [THEOREM, cited]. In Deninger's picture the archimedean factor is the contribution of the stationary point x_∞ (Den98 p. 181) [THEOREM, cited]. Calling it a "relativistic correction" is [ANALOGY]: there is no small coupling of which it is the next order.

(e) Gyromagnetic ratio. Dirac's equation gives $g = 2$ [BIB]. The anomaly $a_e = (g-2)/2 = \frac{\alpha}{2\pi} + O(\alpha^2)$ (Schwinger) is a perturbation series whose divergence is expected from Dyson's 1952 argument [BIB] (heuristic). Its $n!$ growth sits exactly at the Gevrey-1 threshold of Lemma 59 and §2. That is a true statement about QED and an [ANALOGY] for ζ, which has no coupling constant. "$g = 2$ comes from the double cover $\mathrm{Spin}(3) \to \mathrm{SO}(3)$" is [WRONG]: spin $\frac{1}{2}$ alone does not fix g, and $g = 2$ follows from the minimally coupled Dirac (or squared Pauli) operator. Lemma 50's spinor material (half Tate twist, Sato–Tate SU(2)) is unaffected.

9. The Ritz–Rydberg combination principle and the zeros (J.W. [T230U])

(1) The statement [THEOREM, cited] (Con94, Ch. I §1). Rydberg found that spectral lines satisfy $1/\lambda = R/m^2 - R/n^2$. The spectrum is the set of differences $\nu_{ij} = \nu_i - \nu_j$ of a set I of frequencies, and $\nu_{ik} = \nu_{ij} + \nu_{jk}$ is the Ritz–Rydberg combination principle. Classically, emitted frequencies form a *group* $\Gamma \subset \mathbb{R}$ and observables form the commutative convolution algebra of Γ. Experiment instead dictates the *groupoid* $\Delta = \{(i,j)\}$, $(i,j)\cdot(j,k) = (i,k)$, whose convolution algebra is the matrix algebra, $(ab)_{ik} = \sum_j a_{ij} b_{jk}$. Time evolution is $q_{ij}(t) = q_{ij} e^{2\pi i \nu_{ij} t}$. This is Heisenberg's 1925 matrix mechanics (Hei25), with matrix algebra identified by Born and Jordan (Connes, *loc. cit.*).

(2) The Ritz groupoid of the zeros. Set $\Delta_Z = Z \times Z$ for the multiset Z of nontrivial zeros, with frequency map $(\rho, \rho') \mapsto \gamma_\rho - \gamma_{\rho'}$ (complex if ghosts exist) and the Heisenberg dynamics $\sigma_t(q)_{\rho\rho'} = e^{it(\gamma_\rho - \gamma_{\rho'})} q_{\rho\rho'}$ on finitely supported matrices.

Proposition 9.1 ([THEOREM] / [RESTATEMENT])**.** *σ_t is a $*$-automorphism group ($\sigma_t(q^*) = \sigma_t(q)^*$) iff $\mathrm{Im}(\gamma_\rho - \gamma_{\rho'}) = 0$ for all pairs, iff all zeros lie on the line. So "the Ritz groupoid of the zeros carries a Heisenberg dynamics" is a restatement of RH.*

Proof. $\sigma_t(q^*)_{\rho\rho'} = e^{it(\gamma_\rho - \gamma_{\rho'})} \bar{q}_{\rho'\rho}$, while $(\sigma_t q)^*_{\rho\rho'} = e^{-it\overline{(\gamma_{\rho'} - \gamma_\rho)}} \bar{q}_{\rho'\rho}$. These agree iff $\gamma_\rho - \gamma_{\rho'}$ is real. A ghost pair $\rho, 1-\bar{\rho}$ has $\mathrm{Im}\,\gamma = \mp\delta_\rho$, so it violates this. □

The *statistics* of the Ritz frequencies is Montgomery's pair correlation: the normalized differences have density $1-(\frac{\sin \pi u}{\pi u})^2$ (Mon73; under RH, for restricted test functions) [BIB]. The single frequencies γ are Fourier-dual to the prime powers by the explicit formula: $\sum_\gamma \cos(\gamma u)$ has negative spikes at $u = \log p^k$ (Lemma 1, Weil) [TAA-L]. Both are reproduced in Fig. 4 [NUMERICAL]. Both are zero statistics in the sense of J.W.'s standing condition.

Which algebra carries transition amplitudes between zero levels in Connes' picture? None, canonically. The idele class group C_k is *abelian*. The operators $W(h)$, $h \in \mathcal{S}(C_k)$, are trace class with $\operatorname{Tr} W(h) = \sum \hat{h}(\tilde{\chi}, \rho)$ (Con99 Cor. 2) [THEOREM, cited], and they are functions of the generators, diagonal on the spectral decomposition. In Connes' own Chapter I terms, the action on the zeros is *classical*: a group of frequencies, not a groupoid of transitions. The noncommutativity of Connes' picture lives in the *space* $X = \mathbb{A}_k/k^*$ (the action of k^* on adeles), i.e. on the geometric and prime side.

Is multiplicativity the arithmetic Ritz rule? In the Bost–Connes system, $H\varepsilon_n = (\log n)\varepsilon_n$ and $\sigma_t(\mu_m) = m^{it}\mu_m$ with $\mu_m\varepsilon_n = \varepsilon_{mn}$ (Lemmas 38, 39, 44, 58) [TAA-L]. The transition $n \to mn$ has Bohr frequency $\log(mn) - \log n = \log m$, a difference of levels: *yes, $\mu_m\mu_{m'} = \mu_{mm'}$ is a Ritz rule* [THEOREM]. It is a very special one, because the level differences $\log(n/n')$ fill the *group* $\log \mathbb{Q}^\times_{>0}$. In Connes' Chapter I sense, every sum of frequencies is a frequency: classical-looking frequencies with a noncommutative (isometry) algebra. And it is on the emission side, with levels $\log n$, the spectrum of H. $\zeta(\beta) = \operatorname{Tr} e^{-\beta H}$ sees only the pole; BC time is dual to rapidity (Lemmas 38, 39, 44, 58) [TAA-L].

(3) The obstruction [RESTATEMENT]/[TAA-L]. A Ritz groupoid of the zeros realized as an operator algebra on a Hilbert space with real frequencies is a Hilbert–Pólya operator $H = \operatorname{diag}(\gamma)$. That presupposes RH (Prop. 9.1). Lemma 42 shows the Cramér reconstruction $\ell^2(\{\gamma > 0\})$, $H = \operatorname{diag}(\gamma)$, cannot be the positive-energy part of any half-sided modular inclusion. The prime levels $\log n$ have a canonical Ritz groupoid (BC); the zero levels, which are dual to them through the explicit formula, do not.

(4) Would a positive-definite kernel on the groupoid be J? A positive-definite kernel on Δ_Z is a positive semidefinite matrix $K(\rho, \rho')$. The natural one is Weil's form in the zero basis: it pairs ρ with $1-\bar{\rho}$. Under RH it is diagonal and positive ($\sum m_\rho |\hat{f}(\gamma_\rho)|^2$). A ghost pair contributes the off-diagonal block $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, a hyperbolic plane with one negative square (Lemma 1: exactly N_+ negative squares) [TAA-L]. So "the transition kernel is positive on the Ritz groupoid" $\iff$ RH, and in the zero basis J is the trivial diagonal structure. *It is J only in the circular sense* ([RESTATEMENT]): it needs the zeros as input. A non-circular J must be constructed on the prime/geometric side (Lemma 58's APD).

10. Numerics

All runs stayed far below 1 GB (resident size $\approx$ 26 MB per strip job) at low priority. Scripts: `num/lemma61_figs.py`, `num/lemma61_strip.py`, `num/lemma61_strip2.py`; outputs in `num/l61/`.

N1 (visibility norms; illustration of Cor. 2.2). $\|\ell_{\lambda,0}\|$ on $L^2(W)$ truncated to $[-L, L]$, for $\delta_\rho \in \{0, 0.0159, 0.1508, 0.3085\}$ (the critical line and three DH ghosts) and four weights (Fig. 1). Bounded in L iff visible. This is an exact computation by quadrature, an illustration, not evidence.

N2 (the strip visibility law, counted). Thm. 3.3 says the absorption spectrum in the class $\mathcal{S}_{(\frac{1}{2}-a, \frac{1}{2}+a)}$ consists of the zeros with $|\beta - \frac{1}{2}| < a$. Counting it therefore means counting zeros in the slabs $\frac{1}{2} + \varepsilon < \Re s < \frac{1}{2} + a$. We count them by the argument principle for $\Lambda(s)$ (completed ζ, and the completed Davenport–Heilbronn function of Lemmas 56/59), around rectangles $[\frac{1}{2} + \varepsilon, \frac{1}{2} + a] \times [10.3, 900.3]$ with $\varepsilon = 0.01$. Vertical sides use step 0.002 (on $\Re s = 0.51$) or 0.01, with adaptive midpoint refinement whenever a phase increment exceeds $\pi/4$. Arithmetic is in `python-flint`. An earlier coarse run (step 0.25) produced spurious windings from phase aliasing near on-line zeros. It

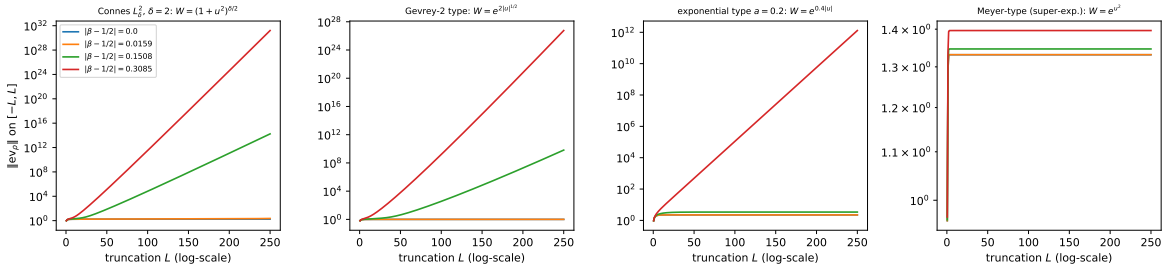

FIGURE 1. [NUMERICAL] Norm of the eigen-functional of a zero at distance δ_ρ from the line, on weighted spaces truncated to $|u| \le L$. Left to right: Connes' L^2_δ ($\delta = 2$), a Gevrey-2-type weight, exponential type $a = 0.2$, and Meyer-type (super-exponential). Theory (Prop. 2.1): only the critical character is bounded in the first two; in the third, exactly the ghosts with $\delta_\rho < 0.2$; in the last, all. For $\delta_\rho = 0.0159$ under the Gevrey-2 weight, the growth starts only near $L \approx \delta_\rho^{-2} \approx 4 \cdot 10^3$, beyond the plotted range. A truncated computation cannot certify (un)boundedness.

was discarded, and the aliasing was confirmed by box tests. By the functional equation, each zero in the right slab has a mirror on the left, so the number of *visible ghosts* is twice the count.

a	DH: winding count in $\frac{1}{2}+0.01 < \Re s < \frac{1}{2}+a$	ghosts with $0.01 < \delta_\rho < a$ in the L56/59 list
0.02	1.0000	1
0.05	1.0000	1
0.10	4.0000	4
0.15	7.0000	7
0.20	11.0000	11
0.25	14.0000	14
0.30	19.0000	19
0.35	23.0000	23
0.40	26.0000	26
0.45	26.0000	26
0.49	26.0000	26

Here $\varepsilon = 0.01$ for DH. For ζ (baseline $\Re s = 0.51$, the same a-grid) every slab count is 0 to within $9e-12$ [NUMERICAL]. The two DH columns agree exactly. The argument-principle count and the ghost list of Lemmas 56/59 are independent computations, so the agreement is a consistency check on both. Numerically, the absorption spectrum of DH grows with the class exactly as Thm. 3.3 predicts. This is an illustration, not evidence about ζ.

N3 (Deninger's Γ-factor as a determinant). We checked the archimedean case of Deninger's formula, $\zeta_\infty(s) = 2^{-1/2}\pi^{-s/2}\Gamma(\frac{s}{2}) = \det_\infty(\frac{1}{2\pi}(s-\Theta)\,|\,\mathbb{R}[\exp(-2y)])^{-1}$, where $\Theta = d/dy$ has eigenvalues -2ν, $\nu \ge 0$. The determinant was computed as $\exp(-Z'(0))$ with $Z(z) = \pi^z\zeta_H(z, \frac{s}{2})$ (§8(c)) in `mpmath` at 30 digits, at $s = 3$, $\frac{1}{2} + 14.134725i$, $2.3 - 7.1i$ and $-0.7 + 0.4i$. The maximal relative error was $5.7 \cdot 10^{-31}$ [NUMERICAL]. This checks our transcription and is not new mathematics.

N4 (Maslov dequantization). Fig. 3 shows $u \oplus_h v - \max(u,v) = h\log(1 + e^{-|u-v|/h})$ for $|u - v| = 1$. It is $1.03 \cdot 10^{-10}$ at $h = 0.05$ and flat at $h = 0$, with every Taylor coefficient zero (Prop. 6.1). For $u = v$ it is analytic ($h \log 2$) [NUMERICAL].

N5 (Ritz frequencies of the zeros). We took the first $N = 20000$ zeros ($\gamma \le 18046.46$), used the unfolded differences $w_i - w_j \le 3$ (50063 pairs), and compared their histogram with $1 - (\sin \pi u/\pi u)^2$. The rms deviation was 0.041 for $u > 0.25$. For $\sum_{\gamma<2000}\cos(\gamma u)$ the values were

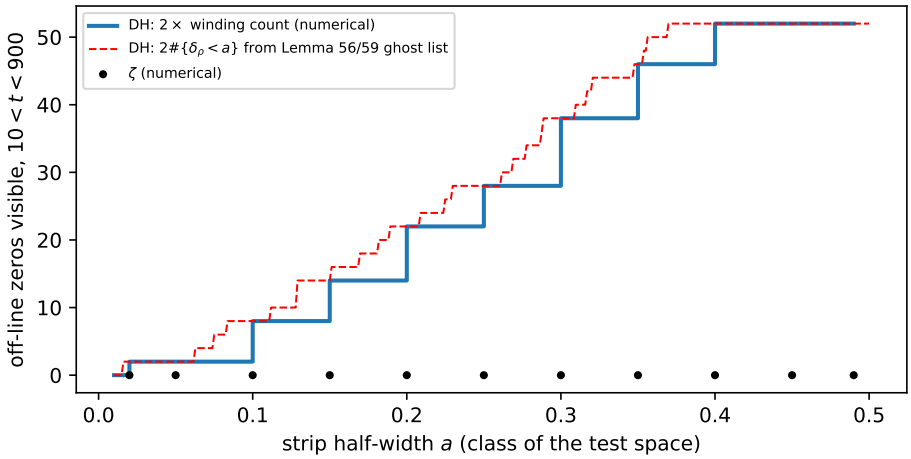

FIGURE 2. [NUMERICAL] Off-line zeros with $10.3 < t < 900.3$ that become visible as the test class widens from the strip of half-width ε to half-width a (Thm. 3.3). Blue: argument-principle count for the Davenport–Heilbronn function. Red: the prediction from the independently computed ghost list of Lemmas 56/59. Black: ζ (identically 0).

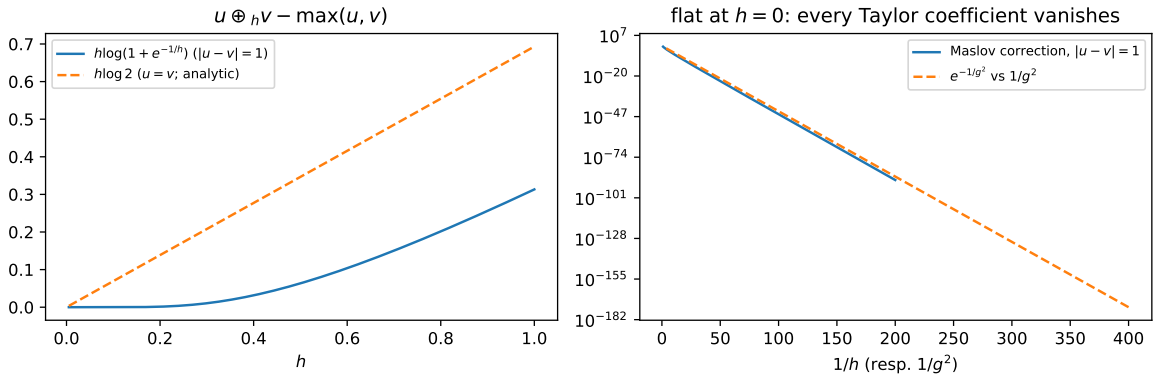

FIGURE 3. [NUMERICAL] Left: the Maslov correction. Right: on a log scale against $1/h$ it decays like $e^{-1/h}$, the same flatness as e^{-1/g^2} (plotted against $1/g^2$) behind non-analytic Λ_{QCD} (correction C2).

$-156.3, -201.7, -108.5, -227.3, -232.3$ at $u = \log 2, \log 3, \log 4, \log 5, \log 7$, against -1.08 at the reference point $u = 1.2$ [NUMERICAL]. Both are classical (Mon73; explicit formula). We include them only to make the "Ritz groupoid" of §9 concrete. Neither bears on J.

11. RANKED FOLLOW-UP DIRECTIONS

Score = 0.6×significance +0.4×probability of a clean result (both in $[0, 1]$, our estimates).

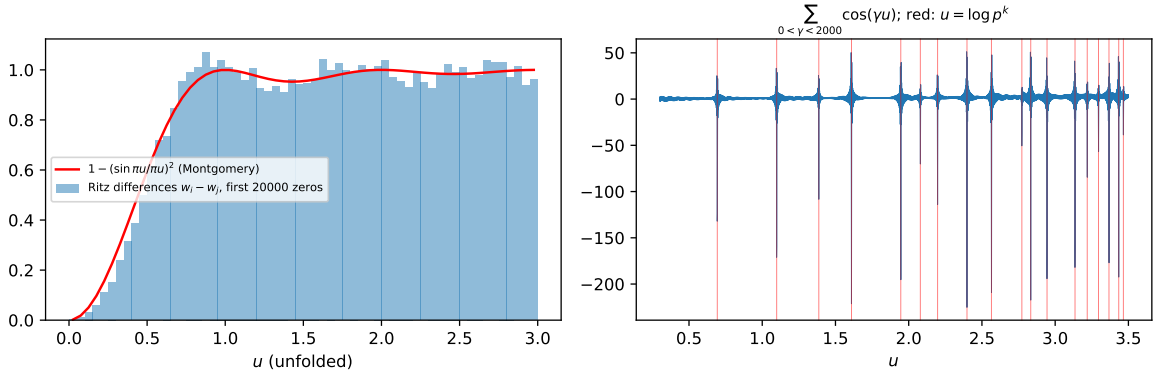

FIGURE 4. [NUMERICAL] Left: pair correlation of the first 20000 zeros (the statistics of the Ritz frequencies $\gamma_i - \gamma_j$). Right: the single frequencies are Fourier-dual to $\log p^k$ (red lines).

#	direction	sig.	prob.	score
D1	*Polarization ⇒ semisimplicity on the absorption side.* Formalize Thm. 4.3 for the finite-field cokernel of Lemma 47's "next step". Realize H^1 of a curve over $\mathbb{F}_q$ as a Connes-type cokernel, and test whether the Rosati/theta polarization survives there (semisimple Frobenius) or the cokernel forces Jordan structure. This is about J, not counting.	.70	.80	.74
D2	*Class-dependent Weil index.* Compute the number of negative squares of the Weil form restricted to test functions of class $\mathcal{S}_I$ (DH and ζ). Expected: $\#\{\delta_\rho < a\}$, the Lemma 1 count filtered by class. Cheap and diagnostic, but close to counting.	.55	.80	.65
D3	*Conormal versus dual-conormal (ALKL) as $\mathcal{S}_I$ versus its dual.* Map the two cohomologies of the Álvarez López–Kordyukov–Leichtnam trace formula onto Meyer's $\rho_+ \ominus \rho_-$, and ask where a Hodge $*$ could act.	.70	.35	.56
D4	*Growth exponent of truncated scaling flows.* On the Lemma 22/52 prolate and cutoff truncations, measure the growth rate of the compressed flow (R0/R1/R3) as the cutoff grows.	.60	.50	.56
D5	*Ritz/Montgomery as line intensities.* Excluded by the standing condition (zero statistics). Listed for completeness.	.20	.90	.48
D6	*Degenerate level versus exceptional point.* Deform a function with a forced double zero (e.g. a product $L(s)^2$) along the de Bruijn–Newman flow, and compare the splitting in an emission model (semisimple) with the Jordan block of the cokernel.	.30	.70	.46
D7	*Entropy as the flat sector.* Ask whether Connes–Consani's archimedean Weil positivity (Selecta 2021) has a dequantization (entropy) form.	.50	.25	.40

12. Corrections to Grok Bot's chat claims

C1. *"QED and QCD perturbation series both diverge with $n!$ growth (Dyson 1952)."* **Overstated.** Dyson's 1952 note [Dys52] is a heuristic argument for QED (instability at $\alpha < 0$), not a proof.

Factorial growth of large-order coefficients is the standard *expectation* from instanton and renormalon analysis (e.g. Beneke's review [Ben99]) in both theories. Divergence is not a theorem for either full theory. "Both grow like $n!$" should read "both are expected to grow like $n!$".

C2. *"$\Lambda_{\mathrm{QCD}} \propto \exp(-1/(2\beta_0 g^2))$ is non-analytic in g."* **Correct up to convention.** At one loop with $\mu\, dg/d\mu = -\beta_0 g^3$, $1/g^2(\mu) = 2\beta_0 \log(\mu/\Lambda)$, so $\Lambda = \mu e^{-1/(2\beta_0 g^2)}$. With the usual $\beta(g) = -\beta_0 g^3/(16\pi^2)$ and $\beta_0 = 11 - \frac{2}{3}n_f$ it reads $\Lambda = \mu e^{-8\pi^2/(\beta_0 g^2)}$. Either way it is flat at $g = 0$: all derivatives vanish as $g \downarrow 0$, the same type as Prop. 6.1.

C3. *"Analyticity-strip width $\leftrightarrow$ kernel decay (Paley–Wiener)."* **Correct** for exponential decay $e^{-a|u|} \leftrightarrow$ strip $|\operatorname{Im}| < a$ (in L^2 form). Double-exponential decay of Riemann's Φ makes Ξ entire. Lemma 53's H_+ is the one-sided (half-plane) version.

C4. *"dBN multiplies Φ by $e^{\lambda u^2}$."* **Correct**: $H_\lambda(z) = \int_0^\infty e^{\lambda u^2}\Phi(u)\cos(zu)\,du$, real zeros iff $\lambda \geq \Lambda$ [New76, dB50, RT20].

C5. *"The ghosts sit on a threshold of the same kind (kernel decay vs zero location) — an analogy."* **Half upgraded, half corrected.** For *test-function classes* it is now a theorem: Thm. 3.3 and Cor. 3.4 (ghosts are exactly what is lost from the analytic to the smooth class). For the *de Bruijn–Newman flow* it is not the same threshold. The multiplier $e^{\lambda u^2}$ never changes the analyticity class of Ξ_λ (it stays entire for every λ), so Λ is not a smooth/analytic threshold. That part stays [ANALOGY] at best.

C6. *"Deninger's ICM paper is Doc. Math. ICM I 163–186."* **Correct** (verified PDF). EuDML and Den05's bibliography give 23–46 (alternative pagination); Lemma 58 cited its content only bibliographically, and now it is [THEOREM, cited].

C7. *"Connes: RH $\iff$ positivity in the global trace formula."* **Correct with wording**: the validity of the global trace formula is equivalent to RH for all L-functions with Grössencharakter; positivity of the Weil distribution enters the proof (Con99, Intro) [THEOREM, cited].

13. Questions for J.W.

Q1. Should D1 become a separate lemma (Lemma 62 is now taken)? It is the one result here that bears on J: on the absorption side a polarization forces simple zeros, and the finite-field cokernel test would show whether that is a real obstruction or an artifact of the cokernel.

Q2. Do you accept the verdict that the smooth/analytic threshold governs *where* ghosts are stored (visibility), not the positive form J? If yes, we stop treating it as a route to J and keep Cor. 3.4 as a structural fact.

Q3. For Volume 4: should the spectroscopy dictionary carry only its theorem-backed rows (sector decomposition, helicity, ζ-regularized determinants, $N(T)$), with the analogies relegated to a remark?

Q4. Run D2 (the Weil form's negative index on strip-class test functions for DH)? It is cheap and diagnostic, but close to zero counting.

Q5. Ritz groupoid: its only positivity is the circular Weil-in-zero-basis kernel (Prop. 9.1 and (4)). Keep it as a via-negativa entry, or drop it?

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TAA LEMMA 62: HELICITY. THE POLARIZATION AS CIRCULAR POLARIZATION; THE WICK LOOP; SIGNS, SUPERTRACES AND THE MÖBIUS FUNCTION

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

Date: October 22, 2026.

ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Every RH-equivalent statement below is a [RESTATEMENT] of known results or of earlier TAA lemmas. Tags: [THM] proved here (elementary) or classical with proof indicated; [CITED] checked at the source (`lit/lemma62/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [CONJ] conjecture; [OPEN] open; [NUM] numerical (floating point, not certified).

Questions (J.W.). [t231u] "Doesn't this suggest that the precise kind of polarization we ought to be looking for would be a sort of circular / helical polarization? I would imagine this ought to also feed into our positivity condition." [t232u] Does the cos / cosh fact mean there are "loops or orbits between real and non-real zeros that close at every order"; trivial or worth exploring? [t233u, t234u] Connect the coefficient signs to fermionic Fock space, the alternating Betti sum, and the Möbius *function* $\mu(n)$. [t235u] Does Littlewood's $M(x) = O(x^{1/2+\varepsilon})$ connect to smooth-versus-analytic and to positivity (Ingham–Landau)? [t237u] Are the Cassini separatrices of Lemmas 55–56 the phase-paired object that route needs?

Answer in brief. (1) *Linear algebra* (§2). A compatible complex structure J on (V, ω) splits $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ into its $\pm i$ eigenspaces. In the 2-dimensional real rotation representation these are literally the two circular polarizations (Jones vectors $(1, \mp i)$). Positivity $\omega(\cdot, J\cdot) > 0$ selects which eigenspace satisfies $i\omega(\xi, \bar{\xi}) > 0$; with our conventions it is $V^{0,1}$, the antiholomorphic one, matching the "positive Kähler polarization" of geometric quantization [THM][CITED]. In optics, *helicity* needs *two* commuting complex structures: positive frequency (time) and the transverse quarter turn. TAA's polarization is the first one, not the second (Prop. 2.4). (2) *Rotation versus boost* (§3). cos is a matrix coefficient of a rotation group, cosh of a boost group. In Weil's explicit formula an on-line zero contributes a pure phase $e^{i\gamma u}$; an off-line pair contributes $2\cosh(\delta u)\, e^{i\gamma u}$, a rotation times a boost [THM]. On an elliptic plane the Krein sign equals the *handedness* of the rotation relative to the orientation of ω [THM]. Ghost planes are Krein-neutral: they have *no* helicity, not the wrong one. Lemma 54's $\nu = k + 2m_-$ reads "ν = ghosts + 2× wrong-helicity planes" [TAA-L]. (3) *The helix* (§4). $\zeta(\frac{1}{2} + it) = e^{-i\vartheta(t)} Z(t)$ with Z real (Titchmarsh §4.17) [CITED]. For $t > 6.2898$ the phase turns in one fixed sense ($\vartheta' > 0$): a definite chirality. It is *unconditional*, holds for Davenport–Heilbronn (DH) too, and comes from the Γ-factor. Ghosts appear as amplitude minima without sign change (Fig. 1) [NUM]. Reading ghosts off the helix is zero counting. (4) *The Wick loop* (§5). $f \mapsto f(e^{i\varphi} t)$ multiplies a_n by $e^{in\varphi}$, so all $|a_n|$ are invariant, and $\varphi = \pi/2$ is the Wick rotation, $\cosh t = \cos(it)$. This is *trivial* and explains why every modulus/growth test is blind. The orbit meets $\mathcal{L}$-$\mathcal{P}$ only at $\varphi \in \{0, \pi\}$. The non-trivial generalization is Pólya–Schur/Borcea–Brändén theory. Via Borcea–Brändén's symbol we verify that e^{-tD^2} preserves real-rootedness iff $t \geq 0$. The coefficient signs of Ξ are *unconditional* (from $\Phi > 0$), so RH is not in them. Score 4.2. (5) *Signs and* μ (§6). Zeros live in *odd* cohomology, and odd degree is exactly what makes the cup product *alternating*, i.e. symplectic [THM]. That is the one structural link between the "fermionic" grading and the polarization: the grading produces ω, positivity then picks J, and the sign of the degree map fixes the handedness [THM] for curves. Spector's $\mu = (-1)^F$ is [CITED]. RH $\iff$ $M(x) = O(x^{1/2+\varepsilon})$ (square-root cancellation of fermion parity) is a [RESTATEMENT]. The growth exponent of M is a *convergence* abscissa, and RH says it equals the holomorphy abscissa $\frac{1}{2}$. Smooth, Gevrey or analytic weights damp zeros by height γ but never change the exponent x^β: *regularity controls* γ*, RH lives in* β [THM]. A one-sided bound $M \geq -K\sqrt{x}$ would give RH, simple zeros (classical, via Landau) and infinitely many linear relations among the γ (Ingham 1942) [CITED]. Boundedness of $M/\sqrt{x}$ is open, with $\limsup \geq 1.6383$. Positivity + Landau is a genuine route that lacks a sign-definite object [OPEN]. (6) *Phase pairing* (§7). There are two pairings: helicity $\rho \leftrightarrow 1 - \rho$ ($e^{\pm i\gamma u}$) and mirror $\rho \leftrightarrow 1 - \bar{\rho}$ (Cassini, (H), Weil). RH $\iff$ they coincide on every zero. Weil's form pairs by the mirror, and a ghost pair is a signature-$(1,1)$ plane. A concrete helicity-projected Weil functional (tanh weight) equals *exactly half* the Weil functional for real test functions, because Ξ is even. So for ζ it reduces to Weil, hence to Lemma 57's zero counting [THM]. (7) *Verdict* (§8). "Positive helicity" (positive energy) is in general *stronger* than Lemma 58's APD: it also forbids wrong-helicity planes, $m_- = 0$. For ζ every Krein sign is already + (Lemma 58 Rem. 3.7), so the helicity condition is *equivalent* to the APD and to RH: a [RESTATEMENT]. Helicity adds *one* clarification. The handedness of J is canonical: it is fixed by the direction of the flow together with the orientation of ω, and its global sign is a convention. It adds no new constraint for ζ and no non-circular construction of J.

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1. Sources and conventions

Read for this lemma: STATE.md §§59, 62, 63, 65, 67, 69; NOTATION.md; Lemma 58 (Def. 3.3 APD; Prop. 3.4 compact $\iff$ elliptic; Prop. 3.6 Weil setting; Rem. 3.7 "sign of rotations is free"); Lemma 54 v2 (Thm. `thm:nu` $\nu = k + 2m_-$; Prop. 5.1 positive energy; Prop. 5.2 $J_A = -i$ on H_+); Lemma 53 (Siegel positivity; ghost charge); Lemma 50 (Thm. 4.2 Hodge $*$, $+i$-eigenspace $H^{0,1}$; Thm. 4.4; Prop. 6.5 spinor structure ghost-blind); Lemma 59 §6 (Prop. `prop:blind`). Lemmas 59, 60, 61 are being written concurrently and were not edited.

Conventions. (V, ω) real symplectic; J *compatible* if $J^2 = -1$ and $\omega(Jx, Jy) = \omega(x, y)$; *positive* if $g_J(x, y) := \omega(x, Jy)$ is positive definite (Lemma 58 Def. 3.2). $V^{1,0} = \ker(J - i)$, $V^{0,1} = \ker(J + i) = \overline{V^{1,0}}$. On $\mathbb{R}^2$: e_1, e_2, $\omega(e_1, e_2) = 1$, $J_0 e_1 = e_2$, $J_0 e_2 = -e_1$ (counterclockwise quarter turn). The Krein form is $h_K(x, y) = i\,\omega(\bar{x}, y)$ (Lemma 50). $\Xi(t) = \xi(\frac{1}{2} + it)$; DH denotes the Davenport–Heilbronn function $f_{\mathrm{DH}} = c_A L(s, \chi) + c_B L(s, \bar{\chi})$ mod 5 with $\chi(2) = i$, as in `num/lemma59_core.py`. Numerics: `num/lemma62_{helix,theta,figs,mertens,phasor}.py` → `num/l62/`.

2. Linear algebra: compatible complex structures are pairs of helicities

Theorem 2.1 (Helicity splitting; [THM])**.** *Let (V, ω) be a real symplectic space of dimension $2n$ and J a compatible complex structure.*

(a) $V_{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$, each summand has complex dimension n, complex conjugation swaps them, and both are Lagrangian for the $\mathbb{C}$-bilinear extension of ω.

(b) For $v \in V^{1,0}$ and $x = v + \bar{v} \in V$: $g_J(x, x) = -2i\,\omega(v, \bar{v})$. Hence J is positive iff $i\,\omega(\xi, \bar{\xi}) > 0$ for all $0 \neq \xi \in V^{0,1}$, iff $i\,\omega(\xi, \bar{\xi}) < 0$ on $V^{1,0} \setminus 0$. Equivalently, $h_K(\xi, \xi) = i\omega(\bar{\xi}, \xi)$ is > 0 on $V^{1,0}$ and < 0 on $V^{0,1}$.

(c) Replacing J by $-J$ swaps $V^{1,0}$ and $V^{0,1}$ and changes the sign of g_J. Exactly one of $\pm J$ can be positive, so positivity is a choice of one *of the two helicity subspaces.*

Proof. (a) J is real and $J^2 = -1$, so $V_{\mathbb{C}}$ splits into the $\pm i$ eigenspaces, and conjugation swaps them. For $v, w \in V^{1,0}$, $\omega(v,w) = \omega(Jv, Jw) = i^2\omega(v,w)$, so $\omega(v,w) = 0$. (b) $Jx = iv - i\bar{v}$, so $\omega(x, Jx) = \omega(v+\bar{v}, iv - i\bar{v}) = -i\omega(v,\bar{v}) + i\omega(\bar{v}, v) = -2i\,\omega(v,\bar{v})$, using (a). Put $\xi = \bar{v} \in V^{0,1}$: $-2i\omega(v,\bar{v}) = 2i\omega(\xi,\bar{\xi})$ (as ω is alternating). (c) is immediate from (b). □

Check on $\mathbb{R}^2$. For J_0, $v = e_1 - ie_2 \in V^{1,0}$ ($J_0 v = e_2 + ie_1 = iv$) and $\omega(v,\bar{v}) = 2i$. For $x = v + \bar{v} = 2e_1$, $g(x,x) = \omega(2e_1, 2e_2) = 4 = -2i \cdot 2i$ ✓. With Lemma 50's convention $J = *$ on H^1 of a curve, $V_J^{1,0} = H^{0,1}$ (Lemma 50 Thm. 4.2(ii)), and $h_K(d\bar{z}, d\bar{z}) = i\int dz \wedge d\bar{z} = 2\,\mathrm{area} > 0$, consistent with (b) [THM].

Proposition 2.2 (Real versus positive (Kähler) polarizations in geometric quantization; [CITED] [12]; [BIB] [19, 7, 14])**.** *In geometric quantization a* polarization *of* (M,ω) *is a complex involutive Lagrangian distribution* $F \subset T_{\mathbb{C}}M$ *with* $\dim(F \cap \bar{F})$ *constant. It is* real *if* $F = \bar{F}$ *and* positive *if* $i\omega(\xi,\bar{\xi}) \geq 0$ *for all* $\xi \in F$*. For admissible positive polarizations the leaf spaces carry a Kähler structure in which* F *projects onto the* antiholomorphic *vectors. The vertical real polarization of* T^*Q *gives the Schrödinger representation, and the horizontal one gives the momentum representation; the two are related by Fourier transform* [12]*, §4 [CITED]. On a vector space, a positive polarization with* $F \cap \bar{F} = 0$ *is exactly* $F = V^{0,1}$ *for a positive compatible* J *(Thm. 2.1(b)) [THM].*

The sign conventions for ω differ between references. What is invariant is that positivity selects one of the two conjugate Lagrangians, and Thm. 2.1 shows which one in our convention. "Real polarization" (a real Lagrangian $L = \bar{L}$, e.g. position space) is the *linear* polarization of optics below; "Kähler polarization" is the *circular* one.

2.1. **The optics dictionary, and where it is literal.**

Proposition 2.3 (The 2-dimensional rotation representation; [THM]; conventions [CITED-sec] [5])**.** *Let* $R_\phi = e^{\phi J_0}$ *act on* $\mathbb{R}^2$ *(the transverse plane of a light wave).*

- *(a) The eigenvectors of* R_ϕ *in* $\mathbb{C}^2$ *are* $e_1 \mp ie_2$*, i.e. the Jones vectors* $(1, \mp i)/\sqrt{2}$*, with eigenvalues* $e^{\pm i\phi}$*. These are the two* circular polarizations*. The labels "right" and "left" depend on the phase convention (*$kz - \omega t$ *versus* $\omega t - kz$*) and on the point of view (source or receiver)* [5]*, so only the* relative *handedness is invariant.*
- *(b) A real vector* $x \in \mathbb{R}^2$ *is a* linear *polarization; it is* $v + \bar{v}$*, an equal-weight superposition of the two helicities. A real line* $\mathbb{R}x$ *is a real Lagrangian, i.e. a real polarization in the sense of Prop. 2.2.*
- *(c) The quarter turn* $R_{\pi/2} = J_0$ *is a compatible positive complex structure for* $\omega = dx \wedge dy$*, and its* $\pm i$*-eigenspaces are the two circular polarizations.*

Proposition 2.4 (Helicity needs two commuting complex structures; [THM])**.** *For a monochromatic transverse wave the real field space is* $W = \mathbb{R}^2_{\text{transverse}} \otimes \mathbb{R}^2_{\text{phase}}$*. Time evolution by a quarter period is a complex structure* $J_t = 1 \otimes J_0$ *(the positive-frequency structure; for the generator* $A = \omega_0 J_t$ *with* $\omega_0 > 0$ *it is* $J_A = A(-A^2)^{-1/2}$ *of Lemma 54 Prop. 5.1). The transverse quarter turn* $J_r = J_0 \otimes 1$ *is a second complex structure, and* $[J_t, J_r] = 0$*.* Helicity *is the eigenvalue* $\pm i$ *of* J_r *on the positive-frequency space* $W^{1,0}_{J_t} \cong \mathbb{C}^2$*; it is well defined because* J_r *commutes with* J_t*.*

Proof. Linear algebra on a tensor product; J_r preserves the $\pm i$ eigenspaces of J_t, and on $W^{1,0}_{J_t} \cong \mathbb{C}^2$ it acts as J_0 with eigenvectors $(1, \mp i)$. □

Where the analogy is literal, and where it stops. *Literal* [THM]: (i) in a suitable symplectic basis a compatible J *is* the quarter turn in each 2-plane, and its eigenspaces *are* circular polarizations (Prop. 2.3); (ii) the TAA polarization J, which commutes with the flow and is positive, is the analogue of J_t: the *positive-frequency* structure of the flow (Lemma 54 Prop. 5.1; Lemma 58 Prop. 3.4). *Stops* [ANALOGY]: physical helicity needs the *second* structure J_r, a transverse rotation commuting with

the flow. For ζ no such structure is known. Arithmetic candidates are a CM action commuting with Frobenius (Lemma 13), or the Galois/character sectors of the "Zeeman" dictionary planned for Lemma 61 (in preparation; not read), which split but do not rotate. So J.W.'s "circular polarization" is literally right for the *positive-frequency* meaning of J. "Helicity" in the photon sense is an extra structure that TAA does not have.

3. Rotation versus boost: what separates cos from cosh

Lemma 59 §6, Prop. "Regularity is blind to RH(a): $|\cos^{(n)}(0)| = |\cosh^{(n)}(0)|$ for all n, yet $\cos \in \mathcal{L}\text{-}\mathcal{P}$ and $\cosh \notin \mathcal{L}\text{-}\mathcal{P}$. What separates them is the *conjugacy class of a one-parameter group.*

Proposition 3.1 (Matrix coefficients; [THM]). *On $\mathbb{R}^2$ let $R_t = e^{tJ_0}$ (elliptic: eigenvalues $e^{\pm it}$, preserves $g = dx^2 + dy^2$) and $B_t = e^{tK}$, $K = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (hyperbolic: eigenvalues $e^{\pm t}$, preserves the Lorentzian form $dx^2 - dy^2$). Then $\cos t = \langle e_1, R_t e_1 \rangle$ and $\cosh t = \langle e_1, B_t e_1 \rangle$. Both groups lie in $\mathrm{SL}_2(\mathbb{R}) = \mathrm{Sp}(2, \mathbb{R})$; R_t commutes with the positive J_0, and B_t commutes with no compatible complex structure. In helicity language: $\cos t = \frac{1}{2}(e^{it} + e^{-it})$ is the sum of the two* unit-modulus *circular modes, while $\cosh t = \frac{1}{2}(e^t + e^{-t})$ is the sum of a growing and a decaying* real *mode, which is helicity-neutral (a real eigenline cannot carry J, since $J^2 = -1$).*

Proof. Direct computation. A J commuting with B_t preserves its real eigenlines, which is impossible for $J^2 = -1$ (Lemma 50 Thm. 4.4, proof). □

So the cos / cosh pair is separated exactly by elliptic versus hyperbolic, i.e. by Lemma 58 Prop. 3.4: cos is a matrix coefficient of a group that commutes with a polarization, and cosh is not. Regularity does not enter: the Taylor moduli of a matrix coefficient $\langle e_1, e^{tA} e_1 \rangle$ depend only on |eigenvalues|, and $\pm i$, ± 1 have the same modulus.

Proposition 3.2 (On-line zeros are phases, off-line pairs are phase × cosh; [THM]). *In Weil's explicit formula, write the zero side as $\sum_\rho x^{\rho - 1/2}$ (formally, against a test function in $u = \log x$). An on-line zero $\rho = \frac{1}{2} + i\gamma$ contributes $e^{i\gamma u}$, a unit-modulus rotation. An off-line pair $\rho = \frac{1}{2} + \delta + i\gamma$, $\rho^* = 1 - \bar{\rho} = \frac{1}{2} - \delta + i\gamma$ ($\delta > 0$) contributes*

$$e^{(\delta + i\gamma)u} + e^{(-\delta + i\gamma)u} = 2\cosh(\delta u)\, e^{i\gamma u} = 2\cos(i\delta u)\, e^{i\gamma u},$$

a rotation times a boost of rapidity δ. This is the zero model of Lemma 58 Prop. 3.6(ii): on-line zeros are positive 1-dimensional blocks with $\tilde{T}_u = e^{i\gamma u}$, and each pair is a hyperbolic plane with Gram $\left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$ [TAA-L].

Proof. $\rho - \frac{1}{2} = \pm\delta + i\gamma$; add. □

For a real entire function in $\mathcal{L}\text{-}\mathcal{P}$ with real zeros $\pm\gamma_k$, each Hadamard factor $1 - t^2/\gamma_k^2$ is a "rotation factor": its zeros are the frequencies $\pm\gamma_k$ of a unit-modulus mode. A ghost contributes the real quartic factor $(1 - t^2/(\gamma + i\delta)^2)(1 - t^2/(\gamma - i\delta)^2)$, whose zeros $\pm\gamma \pm i\delta$ are the complex frequencies of the rotation×boost modes of Prop. 3.2 [THM]. In optics language, $e^{(\pm\delta + i\gamma)u}$ is a circular wave in a medium with gain or loss: a conical spiral, not an elliptical polarization [ANALOGY].

Theorem 3.3 (Krein sign = handedness; ghost planes have none; [THM]). *Let $A \in \mathfrak{sp}(V, \omega)$ be semisimple and let $P \subset V$ be an A-invariant symplectic 2-plane on which A has eigenvalues $\pm i\nu$, $\nu > 0$. Orient P by $\omega|_P$. Then:*

- *(a) $e^{uA}|_P$ is a rotation, and it is counterclockwise for the ω-orientation iff $h_A(x, x) = \omega(x, Ax) > 0$ on P, iff the Krein sign κ of the eigenline $E_{i\nu} \subset P_{\mathbb{C}}$ is $+1$ (for $h_K(x, y) = i\omega(\bar{x}, y)$; Lemma 54 v2).*
- *(b) The unique positive compatible J on P commuting with A is the counterclockwise quarter turn J_P. So $A|_P = \kappa\nu J_P$: the plane has* positive helicity *with respect to the positive polarization iff $\kappa = +1$.*

(c) On a hyperbolic plane or a quadruple block (eigenvalues off $i\mathbb{R}$), h_K and h_A are neutral *(signature $(1,1)$, resp. $(2,2)$; Lemma 54 v2 Thm. `thm:nu`). Such blocks carry no helicity at all.*

Hence Lemma 54 v2's index reads $\nu(A) = k + 2m_- = \#\{$ghost eigenvalues with $\operatorname{Re}\mu > 0\} + 2\#\{$wrong-helicity planes$\}$ [TAA-L].

Proof. (a) Choose a Darboux basis of P with $\omega(e_1, e_2) = 1$. $A|_P \in \mathfrak{sl}_2(\mathbb{R})$ is elliptic, hence conjugate in $\mathrm{SL}_2(\mathbb{R})$ (which preserves ω and the orientation) to cJ_0 with $c = \pm\nu$. Then $h_A(x,x) = \omega(x, cJ_0x) = c|x|^2$, which has the sign of c, and $c > 0$ is counterclockwise. For the eigenvector $v = e_1 \mp ie_2$ of cJ_0 with eigenvalue $i\nu$ ($c = \pm\nu$), $h_K(v,v) = i\omega(\bar{v}, v) = i\omega(e_1 \pm ie_2, e_1 \mp ie_2) = \pm 2$, so $\kappa = \operatorname{sgn} c$. (b) A compatible J commuting with cJ_0 is $\pm J_0$, and positivity forces J_0 (Prop. 2.3(c)). (c) is Lemma 54 v2 Thm. `thm:nu` (ghost blocks are neutral). □

Connection to Lemma 58 Prop. 3.4. Prop. 3.4 says the flow commutes with *some* positive J iff it is elliptic (no ghosts). Thm. 3.3 adds that *such a J exists whatever the helicities*: on a wrong-helicity plane the positive J is still J_P, and the flow just turns clockwise there. So "preserves a polarization" allows mixed helicity; "positive energy" ($h_A > 0$, Lemma 54 Prop. 5.1) forbids it. We return to this in §8.

4. The helix on the critical line

Theorem 4.1 (Riemann–Siegel phase; [CITED] [17, §4.17])**.** *Let $\chi(s) = \zeta(s)/\zeta(1-s)$, $\vartheta(t) = -\frac{1}{2}\arg\chi(\frac{1}{2}+it)$ (continuous, $\vartheta(0) = 0$), so $\chi(\frac{1}{2}+it) = e^{-2i\vartheta(t)}$, and $Z(t) = e^{i\vartheta(t)}\zeta(\frac{1}{2}+it)$. Then Z is real for real t and $|Z(t)| = |\zeta(\frac{1}{2}+it)|$. Hence*

$$\zeta(\tfrac{1}{2}+it) = e^{-i\vartheta(t)}\,Z(t)\,:$$

a point turning with phase $-\vartheta(t)$, with real signed *amplitude $Z(t)$. On-line zeros are the zeros of Z; at a simple zero the amplitude changes sign, i.e. the curve passes through the axis and its phase jumps by π.*

[NUM] $|\zeta(\frac{1}{2}+it) - e^{-i\vartheta}Z| \le 1.2\times10^{-30}$ at six test heights (mpmath, 30 digits).

Proposition 4.2 (Chirality is fixed and free; [THM] + [NUM])**.** *(a) $\vartheta'(t) = \frac{1}{2}\operatorname{Re}\psi(\frac{1}{4}+\frac{it}{2}) - \frac{1}{2}\log\pi$, so $\vartheta'(t) = \frac{1}{2}\log\frac{t}{2\pi} + O(t^{-2})$, and ϑ has its minimum at $t_0 = 6.28984$ ($\vartheta(t_0) = -3.53097$) [NUM]. On a grid of step 0.05 on $[6.3, 1006]$, $\vartheta' > 8\times10^{-4}$ [NUM]. So for $t > t_0$ the phase $-\vartheta$* decreases*: the helix $t \mapsto (t, \operatorname{Re}\zeta, \operatorname{Im}\zeta)$ winds clockwise in the* $(\mathrm{Re},\mathrm{Im})$*-plane as t increases. Its handedness is set by the Γ-factor alone. (b) The same holds for DH. With $\Lambda_{\mathrm{DH}}(s) = (5/\pi)^{s/2}\Gamma(\frac{s+1}{2})f_{\mathrm{DH}}(s)$, $\Lambda_{\mathrm{DH}}(\frac{1}{2}+it)$ is real (relative imaginary part $\le 6\times10^{-30}$ at four test heights [NUM], as the functional equation predicts). So $f_{\mathrm{DH}}(\frac{1}{2}+it) = e^{-i\vartheta_{\mathrm{DH}}(t)}Z_{\mathrm{DH}}(t)$, with $\vartheta_{\mathrm{DH}} = \operatorname{Im}\log\big((5/\pi)^{s/2}\Gamma(\frac{s+1}{2})\big)$, Z_{DH} real ($|\operatorname{Im}| \le 10^{-28}$ on $[80,92]$), and $\vartheta'_{\mathrm{DH}} > 0$ for $t > 1.2116$ (sampled to $t \approx 1000$) [NUM]. (c) Hence the helix's chirality is* unconditional *and is* shared by a function with ghosts*. It carries no RH information [THM].*

Proposition 4.3 (Ghosts are missing half-turns of the amplitude; [RESTATEMENT] (zero counting))**.** *Let $N(T)$ count all nontrivial zeros with $0 < \gamma \le T$, and $N_0(T)$ the sign changes of Z on $(0,T]$. Then $N(T) = \vartheta(T)/\pi + 1 + S(T)$ (equivalent to the $\frac{7}{8}$-form of* [17, Thm. 9.3] *via $\vartheta(T)/\pi = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} - \frac{1}{8} + O(1/T)$) [TAA-L], and $N_0(T) \le N(T)$. If all zeros up to height T are simple, equality holds iff they all lie on the line. A ghost pair at height $\gamma \le T$ adds 2 to N and 0 to N_0. [NUM]: for ζ on $[10,40]$, $N_0 = 6 = N(40) - N(10)$. For DH on $[80,92]$, $N_0 = 6$ (on-line zeros* 80.328, 81.925, 83.109, 87.647, 89.439, 90.725*) and $(\vartheta_{\mathrm{DH}}(92) - \vartheta_{\mathrm{DH}}(80))/\pi = 8.069 \approx 6 + 2$. The gap* 83.11–87.65 *is where the two missing half-turns sit.*

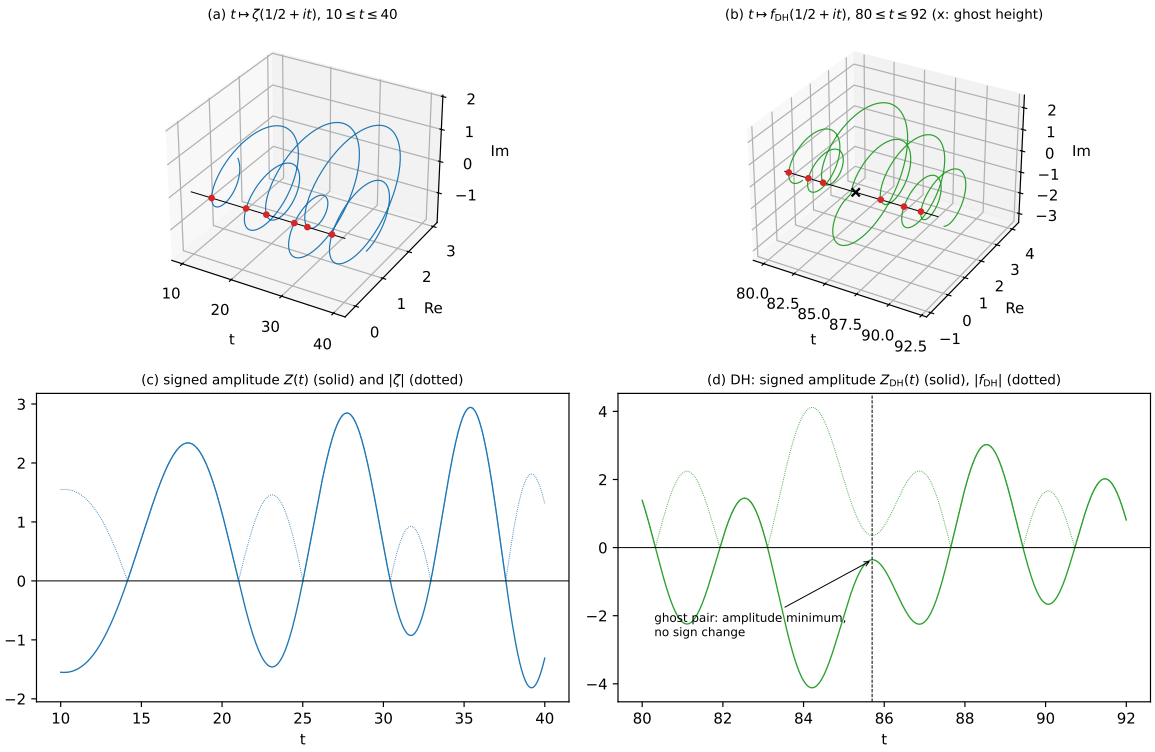

FIGURE 1. (a) $t \mapsto \zeta(\frac{1}{2}+it)$ as a space curve $(t, \mathrm{Re}, \mathrm{Im})$, $10 \le t \le 40$, with zeros (red) on the axis. (b) The same for f_{DH} on $80 \le t \le 92$; $\times$ marks the height 85.6993 of the ghost pair $\beta + i\gamma$, $1-\beta+i\gamma$ with $\beta = 0.808517$, $\gamma = 85.699348$ (Newton; agrees with Spira [15]). (c), (d) The signed amplitudes Z, Z_{DH} (solid) and moduli (dotted). Near the ghost, Z_{DH} has a local extremum -0.357 at $t \approx 85.71$ *without a sign change* (grid step 0.01). The transverse saddle of $|\xi_{\mathrm{DH}}|$ found in Lemma 55 is at 85.7472, a slightly different function ($|\Lambda_{\mathrm{DH}}|$ versus $|f_{\mathrm{DH}}|$). Script `num/lemma62_figs.py`, data `num/l62/helix.json`. *What the picture shows*: a curve that passes through the axis at on-line zeros and fails to do so at a ghost. *What it does not show*: anything about why ghosts cannot occur. The curve is not a constant-radius helix; "helix" refers only to the monotone phase.

This is the idea behind Turing's method for verifying RH numerically. *It is zero counting*, so under J.W.'s standing condition it sheds no light on the polarization. We record it only to say what the helix picture is.

Remark 4.4 (H_+ is one sense of rotation; [TAA-L])**.** In Lemma 53 the Hardy space H_+ (positive frequencies, boundary values from $\mathrm{Re}\, s > \frac{1}{2}$) is the "one sense of rotation". Its positive-energy structure is $J_A = -i$ on H_+ for the rotation generator $A = -d/d\theta$ (Lemma 54 Prop. 5.2). The ζ-subspace W_ζ always lies in H_+ (causality is free, Lemma 33), and RH $\iff$ $W_\zeta = H_+$ $\iff$ charge 0, where the charge is $-\#$ghosts (Lemma 53 Thm. 4.3) [RESTATEMENT]. In helicity words: a ghost is a positive-frequency mode *missing* from W_ζ, the Hardy-space form of the missing half-turns of Prop. 4.3. It is again a count.

5. The Wick loop (J.W. [T232U])

J.W. asked whether the cos / cosh fact means there are "loops or orbits between real and non-real zeros that close at every order". Yes, and the loop is a $U(1)$-orbit. The basic part is *trivial*, and we say so. The non-trivial generalization is classical (Pólya–Schur, Borcea–Brändén).

5.1. The loop, and why it is trivial.

Proposition 5.1 (The circle action on the spectral variable; [THM] (elementary))**.** *For an entire $f(t) = \sum a_n t^n$ put $(R_\varphi f)(t) = f(e^{i\varphi}t)$. Then: (a) $R_\varphi f = \sum e^{in\varphi} a_n t^n$, so every $|a_n|$ is invariant; $R_\varphi R_\psi = R_{\varphi+\psi}$, $R_{2\pi} = 1$, and for even f already $R_\pi f = f$ (period π). (b) The zero set rotates rigidly: $Z(R_\varphi f) = e^{-i\varphi} Z(f)$. (c) $\varphi = \pi/2$ is the Wick rotation $t \mapsto it$; in particular* $\cosh t = \cos(it) = (R_{\pi/2}\cos)(t)$*. (d) Consequently any test depending only on $(|a_n|)_n$ (Cauchy, Gevrey, Denjoy–Carleman, order, type, root-test radius) is constant on each orbit, while real-rootedness is not. This is the reason behind Lemma 59 §6's "regularity is blind to RH".*

All of this is immediate; the "closing at every order" is just $e^{in\cdot 2\pi} = 1$ for all n at once. The loop is the action of the rotation group of the *spectral* plane $\mathbb{C} \ni t$, which is a different plane from the transverse plane of §2.

Proposition 5.2 (Where the orbit meets $\mathcal{L}$-$\mathcal{P}$; [THM])**.** *Let $f \in \mathcal{L}$-$\mathcal{P}$ have a zero $\gamma \neq 0$. Then $R_\varphi f$ is a constant multiple of an element of $\mathcal{L}$-$\mathcal{P}$ iff $\varphi \in \pi\mathbb{Z}$. If moreover f is even with $f(0) \neq 0$ and real coefficients, then $R_\varphi f$ has real Taylor coefficients iff $\varphi \in \frac{\pi}{2}\mathbb{Z}$. At $\varphi = \pm\pi/2$ one gets the "*cosh*-side" $f(it)$, which has only purely imaginary zeros $\pm i\gamma_k$ and is not in $\mathcal{L}$-$\mathcal{P}$.*

Proof. $e^{-i\varphi}\gamma \in \mathbb{R}$ forces $e^{i\varphi} = \pm 1$; conversely $f(-t) \in \mathcal{L}$-$\mathcal{P}$. For even f, $R_\varphi f = \sum a_{2m} e^{2im\varphi} t^{2m}$; if $a_2 \neq 0$ (true for Ξ, since $b_1 > 0$, Rem. 5.3) reality of the t^2-coefficient forces $e^{2i\varphi} = \pm 1$, and then every coefficient is real. At $\varphi = \pi/2$ the zeros are $e^{-i\pi/2}(\pm\gamma_k) = \mp i\gamma_k$. □

Remark 5.3 (The signs of Ξ's coefficients are free; correction to the steering; [THM] + [TAA-L])**.** $\Xi(t) = \int \widetilde{\Phi}(u)\cos(tu)\,du$ with $\widetilde{\Phi} > 0$ (Lemma 59 Prop. "Where Ξ sits", Lemma 31). Hence $\Xi(t) = \sum_m (-1)^m b_m t^{2m}$ with $b_m = \frac{1}{(2m)!}\int u^{2m}\widetilde{\Phi} > 0$, *unconditionally.* The same holds for DH, whose kernel is also positive (Lemma 59 §6(v)), and Lemma 59's `turan.json` confirms $b_m > 0$ for $m \leq 150$ for both ζ and DH [NUM][TAA-L]. The alternating sign is the Wick-loop phase i^{2m}: $\Xi = R_{\pi/2}^{-1} F$ with $F(t) = \sum b_m t^{2m}$ having positive coefficients. So the steering sentence "real-rootedness is a statement about the phases of the coefficients" needs a precision: *the phases are necessary for $\mathcal{L}$-$\mathcal{P}$ and are free.* Given them, RH is a condition on the *moduli* b_m, but not a growth condition. It is the infinite family of Turán/Laguerre/Jensen inequalities (Lemma 59 §§3–5). This also refines Lemma 59's summary "RH lives in the signs": it lives in sign-*patterns of quadratic expressions* in the b_m, not in the signs of the b_m themselves.

(We do not restate the exact Turán normalization here; see Lemma 59 and `num/lemma59_turan.py`, where the Turán ratios are $\rho_m = \beta_m^2/(\beta_{m-1}\beta_{m+1})$ with $\beta_m = (-1)^m \Xi^{(2m)}(0) = (2m)!\, b_m$, tested against $(2m-1)/(2m+1)$ in the Csordas–Norfolk–Varga form.)

5.2. Helicity: Wick is J on the spectral plane, and Wick^2 is time reversal.

Proposition 5.4 ([THM])**.** *Identify the spectral plane $\mathbb{C} \ni t$ with $\mathbb{R}^2$, oriented by $\omega = d(\mathrm{Re}\, t) \wedge d(\mathrm{Im}\, t)$. Then R_φ is the rotation $e^{\varphi J_0}$ and the Wick rotation is $R_{\pi/2} = J_0$, the counterclockwise quarter turn. "Anti-Wick" $t \mapsto -it$ is $-J_0$. For even f, $R_{\pi/2} f = R_{-\pi/2} f$:* the sign of J on the spectral plane is invisible to Ξ. *Further, $R_{\pi/2}^2 = R_\pi$ is $t \mapsto -t$. On a mode $e^{i\gamma u}$ (§3) this is $\gamma \mapsto -\gamma$, i.e. time reversal, which flips every helicity in Thm. 3.3 at once. Finally, $R_{\pi/2}$ sends the elliptic frequency $\gamma \in \mathbb{R}$ to the hyperbolic $i\gamma$:* Wick exchanges rotation and boost *($e^{i\gamma u} \leftrightarrow e^{-\gamma u}$), and a ghost at angle* $\arg t_\rho = \pm\arctan(\delta/\gamma)$ *is an intermediate loxodromic.*

Remark 5.5 (Lemma 60's "Wick ≈ sign of J", made precise; [THM] for the facts, [ANALOGY] for the identification)**.** Lemma 60 recorded "Wick ≈ sign of J" as an analogy (STATE §69, Q4). Prop. 5.4 separates two things that the analogy merges. (i) The *sign* of J (global helicity flip, $J \leftrightarrow -J$, $\omega \leftrightarrow -\omega$, or time reversal) corresponds to $R_\pi = \mathrm{Wick}^2$, which *preserves* $\mathcal{L}$-$\mathcal{P}$. (ii) Wick itself, $R_{\pi/2}$, corresponds to *rotation* ↔ *boost*, the elliptic/hyperbolic exchange, which *destroys* $\mathcal{L}$-$\mathcal{P}$. So the precise version is "Wick^2 = sign of J; Wick = the exchange of compact and split". Both statements concern the *spectral* plane. Neither identifies the Wick quarter turn with the polarization J on V, so the link to the TAA polarization remains an analogy.

5.3. **The non-trivial generalization: independent multipliers.**

Theorem 5.6 (Pólya–Schur [11]; statement as in [1, Thm. 1] [CITED])**.** *A real sequence λ is a* multiplier sequence *($\sum a_k z^k$ hyperbolic $\Rightarrow \sum \lambda_k a_k z^k$ hyperbolic or 0) iff $\Phi(z) = \sum \lambda_k z^k/k!$ is an entire function that is a locally uniform limit of real polynomials whose zeros are all real and of the same sign; equivalently, $\Phi(z)$ or $\Phi(-z)$ equals $Cz^n e^{az}\prod(1+\alpha_k z)$ with $a, \alpha_k \geq 0$, $\sum \alpha_k < \infty$.*

Corollary 5.7 (The loop in Pólya–Schur terms; [THM])**.** *The diagonal operator $z^k \mapsto e^{ik\varphi}z^k$ maps every real-rooted polynomial to a complex multiple of a real-rooted one iff $\varphi \in \pi\mathbb{Z}$. In that case $\lambda_k = (\pm 1)^k$ is a multiplier sequence with $\Phi(z) = e^{\pm z}$.*

Proof. $(z-1)(z-2) \mapsto (e^{i\varphi}z)^2 - 3e^{i\varphi}z + 2$ has zeros $e^{-i\varphi}, 2e^{-i\varphi}$. Conversely $e^{\pm z}$ is of the form in Thm. 5.6. □

Proposition 5.8 (Heat flow and differentiation as hyperbolicity preservers; [THM] from [1, Thm. 5] [CITED])**.** *By* [1, Thm. 5]*, a linear $T : \mathbb{R}[z] \to \mathbb{R}[z]$ of rank > 2 preserves hyperbolicity iff its symbol $G_T(z,w) = \sum_n (-1)^n T(z^n) w^n/n! = T_z\, e^{-zw}$, or $G_T(z,-w)$, lies in $\overline{\mathcal{H}}_2(\mathbb{R})$, the closure of real stable polynomials in two variables. (a) $T = e^{-tD^2}$, $D = d/dz$: $G_T = e^{-tw^2}e^{-zw}$. For $t \geq 0$ this is a limit of $(1 - tw^2/N)^N(1 - zw/N)^N$, and both factors are real stable ($1 - zw \neq 0$ when $\operatorname{Im} z, \operatorname{Im} w > 0$, since then $\arg(zw) \in (0, 2\pi)$), so T preserves hyperbolicity. For $t < 0$, setting $z = 0$ in $G_T(z, \pm w)$ gives $e^{|t|w^2} \notin \mathcal{L}$-$\mathcal{P}$, so T does not preserve hyperbolicity. (b) $T = D$: $G_D = -we^{-zw} \in \overline{\mathcal{H}}_2(\mathbb{R})$, so D preserves hyperbolicity (Rolle).*

Proof. For (a), use $T_z e^{-zw} = e^{-zw}e^{-tw^2}$, because $D_z^2 e^{-zw} = w^2 e^{-zw}$. The restriction $w \mapsto F(0,w)$ of an element of $\overline{\mathcal{H}}_2(\mathbb{R})$ lies in $\overline{\mathcal{H}}_1(\mathbb{R}) = \mathcal{L}$-$\mathcal{P}$ by Hurwitz, and $e^{|t|w^2} \notin \mathcal{L}$-$\mathcal{P}$ (Lemma 59 Prop. "blind"(b)). The rank of e^{-tD^2} is infinite. □

dBN in this language [RESTATEMENT] (Lemma 58 §7; [13]). $H_t = e^{-tD^2}H_0$ with $H_0 = \Xi/8$ in the normalization of [13] ($\partial_t H = -\partial_{zz} H$). By Prop. 5.8(a) (the polynomial version; for H_t itself, Pólya and de Bruijn via [13]), real-rootedness persists forward ($t \uparrow$). Newman's Λ is the first t from which H_t is real-rooted. RH $\iff$ $\Lambda \leq 0$: *RH asks whether Ξ is reached from the real-rooted H_t ($t > \Lambda$) by the backward heat operator e^{+tD^2}, which is not a preserver.* Rodgers–Tao: $\Lambda \geq 0$. Nothing here is new.

5.4. **The numerical picture, and an honest assessment. Assessment.** The loop is real, and it is the precise reason that regularity tests are blind (Prop. 5.1(d)). But it is a *rigid* rotation of the zero set, so it cannot move a ghost relative to the on-line zeros (Fig. 2(c)). It carries no information beyond the zero set. The non-trivial theory of zero-preserving operators (Pólya–Schur, Borcea–Brändén, Craven–Csordas, the GORZ Jensen-polynomial results of Lemma 59) is mature and already contains the dBN reformulation. We see no new route here. The one useful output is conceptual: Wick^2 is the sign of J, and Wick is the compact/split exchange (Rem. 5.5). **Score** (significance 3, probability of a useful result 6): $0.6 \cdot 3 + 0.4 \cdot 6 = \mathbf{4.2}$. The one cheap, well-defined next step is the "Wick-twisted heat flow" $e^{-te^{2i\varphi}D^2}$ (complex heat time: for which φ do DH's ghosts still land?), listed as F5 in §10. It is classical in spirit, and we did not search the literature for it.

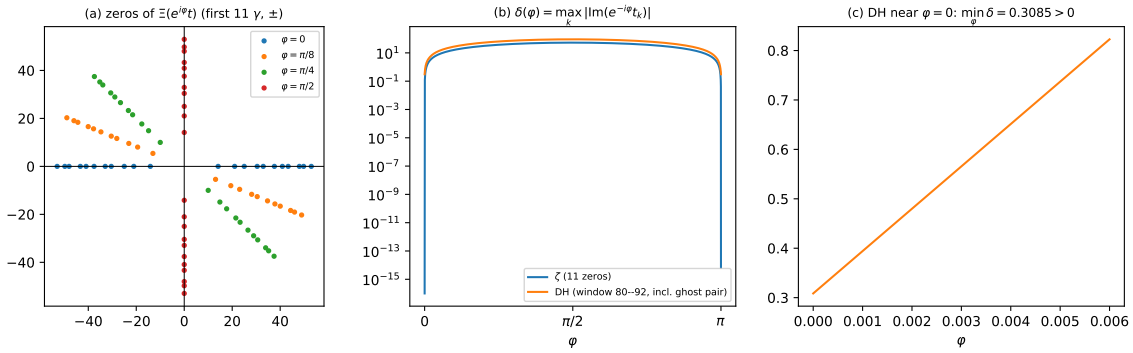

FIGURE 2. The Wick loop. (a) Zeros of $\Xi(e^{i\varphi}t)$ (first 11 ordinates, $\pm$) for $\varphi = 0, \pi/8, \pi/4, \pi/2$: a rigid rotation. At $\varphi = \pi/2$ the zeros are purely imaginary (the cosh-side). (b) $\delta(\varphi) = \max_k |\operatorname{Im}(e^{-i\varphi}t_k)|$ for ζ (11 zeros) and for DH (the window $80 \le t \le 92$: 6 real zeros and the ghost pair $85.6993 \mp 0.3085i$, with negatives). $\delta = 0$ only at $\varphi \in \{0, \pi\}$ for ζ, and never for DH. (c) For DH, $\min_\varphi \delta = 0.3085 = \beta - \frac{1}{2}$, attained at $\varphi = 0$: because the zero set is symmetric under $t \mapsto -t$, no rotation can reduce the ghost's distance from the real axis. Script `num/lemma62_figs.py` [NUM].

6. Signs: supertraces, Euler characteristic, and the Möbius function

J.W.'s questions [t233u, t234u, t235u] concern three different sources of signs. We keep them apart. They are (i) the cohomological grading $(-1)^i$, (ii) the fermion parity $\mu(n)$, and (iii) the coefficient signs of Ξ. Only (i) has a theorem linking it to the polarization.

6.1. Curves: the zeta function is a superdeterminant.

Theorem 6.1 (Grothendieck–Lefschetz for a curve; [THM] classical, [BIB]). *Let $C/\mathbb{F}_q$ be a smooth projective geometrically connected curve of genus g, F the Frobenius, and $H^i = H^i_{\text{et}}(C_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ ($\dim H^0 = \dim H^2 = 1$, $\dim H^1 = 2g$). Then*

$$Z(C,T) = \prod_{i=0}^{2} \det(1 - TF \,|\, H^i)^{(-1)^{i+1}} = \frac{P_1(T)}{(1-T)(1-qT)}, \qquad \#C(\mathbb{F}_{q^n}) = \sum_i (-1)^i \operatorname{Tr}(F^n | H^i).$$

So Z is the superdeterminant of $1 - TF$ on the $\mathbb{Z}/2$-graded space $H^{\text{ev}} \oplus H^{\text{odd}}$. The zeros come from the odd ("fermionic") part, the poles from the even part. At $F = \mathrm{id}$ the Lefschetz number is $\sum (-1)^i b_i = \chi = 2 - 2g$.

Grothendieck's trace formula; for curves, Weil 1948. TAA has this as Lemma 44 Thm. 2.1–2.2 (super-Fock form) [TAA-L].

Proposition 6.2 (Odd degree is what makes the form symplectic; [THM]). *The cup product $H^i \times H^j \to H^{i+j}$ is graded-commutative, $a \cup b = (-1)^{ij} b \cup a$. Hence $\langle a, b\rangle := \operatorname{tr}(a \cup b)$ on H^1, with $\operatorname{tr} : H^2 \xrightarrow{\sim} \mathbb{Q}_\ell(-1)$, is* alternating, *and by Poincaré duality it is nondegenerate: $(H^1, \langle,\rangle)$ is symplectic. The same grading sign makes the even-degree pairing $H^0 \times H^2 \to H^2$ symmetric. F is a symplectic similitude, $\langle Fa, Fb\rangle = q\langle a, b\rangle$. Over $\mathbb{C}$ (complex curve) $H^1(C, \mathbb{R})$ with the intersection form is a real symplectic space. The Hodge star $*$ is a compatible J, positive by the Riemann bilinear relations, with $+i$-eigenspace $H^{0,1}$ (Lemma 50 Thm. 4.2) [TAA-L]. Over $\mathbb{F}_q$, Weil's positivity (Rosati) gives the analogous positive form on the ℓ-adic side via the Jacobian (Lemma 50 Thm. 4.4; Lemma 58 §2) [TAA-L].*

Proof. Graded commutativity with $i = j = 1$ gives $a \cup b = -b \cup a$. Nondegeneracy is Poincaré duality. Equivariance holds because F^* acts on H^2 by q. □

What this gives for handedness [THM] (curves). The symplectic form on the zero-carrying space is produced by the odd grading, and its *sign* is fixed by the orientation (trace map), i.e. by the positive degree of a point. Positivity then picks J *including its handedness*: of $\pm *$, only $*$ is positive for this orientation. So for curves the handedness of J is canonical, given by the fundamental class. This is the precise sense in which "fermionic" (odd) degree and the polarization are linked. The archimedean analogue is Deninger's conjectural formula

$$\xi(s) \stackrel{?}{=} \prod_{i=0}^{2} \det{}_{\infty}\Big(\tfrac{1}{2\pi}(s-\Theta)\,\Big|\,H^i\Big)^{(-1)^{i+1}}$$

[3, formula (3)]. Deninger has *proved* the right-hand factorization $\xi(s) = \prod_\rho (s-\rho)/2\pi$ in the regularized sense, but the cohomology H^1 with its pairing is conjectural [CITED] (carried from Lemma 61). No orientation or positive J is known on it. That is Lemma 58's missing APD.

6.2. **The Möbius function as fermion parity.**

Theorem 6.3 (Spector; [CITED] abstract, [TAA-L] Lemma 44 Thm. 3.1)**.** *In the primon gas (Julia* [6]*), take one fermionic mode b_p per prime of energy $\log p$, the Fock vacuum, and $n = \prod p^{a_p}$. Then square-free n label the fermionic states, $(-1)^F|n\rangle = \mu(n)|n\rangle$, and $\operatorname{Tr}\left((-1)^F e^{-sH}\right) = \sum \mu(n) n^{-s} = 1/\zeta(s)$* [16]*. The supersymmetric gas has $\operatorname{Tr}((-1)^F e^{-sH}) = \zeta(s)\cdot\zeta(s)^{-1} = 1$, i.e. Witten index 1, and this identity is Möbius inversion.*

Proposition 6.4 (Möbius inversion is a superdimension; [THM])**.** *For $n > 1$ with $\omega(n) = r$ distinct prime factors, $\sum_{d\mid n}\mu(d) = \sum_{k=0}^{r}(-1)^k\binom{r}{k} = \operatorname{sdim}\Lambda^\bullet(\mathbb{C}^r) = (1-1)^r = 0$. Inclusion–exclusion is the vanishing of the supertrace of the identity on an exterior algebra (equivalently χ of a simplex boundary plus the empty face).*

Restatement [RESTATEMENT] (Littlewood [9]; Titchmarsh §14.25(C); Lemma 44 Thm. 3.2 [TAA-L]). RH $\iff$ $M(x) = \sum_{n\le x}\mu(n) = O(x^{1/2+\varepsilon})$ for every $\varepsilon > 0$. So RH says that the net fermion parity of $n \le x$ shows square-root cancellation, "like a random walk". This is a restatement. Nothing in the fermion picture supplies the cancellation. The Mertens conjecture $|M(x)| < \sqrt{x}$ is false [10] (§6.3).

Remark 6.5 (The two other sign sources; [ANALOGY])**.** (a) *Coefficient signs of Ξ.* $\Xi(t) = \sum(-1)^n b_n t^{2n}$ with $b_n > 0$ follows from $\Phi > 0$ (Lemma 31) for every kernel of this shape. These are the Wick-loop phases i^{2n} (Rem. 5.3). They are unconditional and are *not* the RH information. RH is in the Turán/Laguerre inequalities among the b_n. Reading them as fermionic signs is an analogy, since no supertrace producing them is known. (b) *Möbius strip.* The nontrivial real line bundle on S^1 and the spinor sign -1 under a 2π rotation (the double cover Spin $\to$ SO) are $\mathbb{Z}/2$ twists. Lemma 50 Prop. 6.5 showed that the spinor structure is ghost-blind [TAA-L]. The resemblance to $(-1)^F$ is an analogy only.

6.3. **Mertens, regularity, and the rotation/boost split.** Let $\Theta := \sup\{\operatorname{Re}\rho\}$ ($\frac12 \le \Theta \le 1$) and $\alpha_M := \inf\{\theta : M(x) = O(x^\theta)\}$.

Theorem 6.6 (Mellin; [THM])**.** *(a) For $\sigma > 1$, $1/\zeta(s) = s\int_1^\infty M(x)x^{-s-1}dx$. (b) If $M(x) = O(x^\theta)$, the integral converges absolutely for $\sigma > \theta$, so $1/\zeta$ is holomorphic there. Hence $\Theta \le \alpha_M$, and $\alpha_M \ge \frac12$. (c) With Littlewood: RH $\iff$ $\alpha_M = \frac12$ $\iff$ $\alpha_M = \Theta = \frac12$.*

Proof. (a) Abel summation: $\sum\mu(n)n^{-s} = \int_{1^-}^\infty x^{-s}dM(x) = s\int_1^\infty M(x)x^{-s-1}dx$, because $M(x)x^{-s} \to 0$. (b) The integrand is $O(x^{\theta-\sigma-1})$. Zeros ρ with $\operatorname{Re}\rho = \frac12$ exist, so $\Theta \ge \frac12$. (c) $\Leftarrow$ by (b); $\Rightarrow$ is Littlewood's theorem. □

Link to Lemma 59 (precise). For a power series, coefficient growth *equals* the radius of analyticity (Cauchy–Hadamard plus a singularity on the circle). For a Dirichlet series this fails. α_M is the

abscissa of *convergence* of $\sum \mu(n)n^{-s}$ (classical for abscissae ≥ 0; Hardy–Riesz [BIB]). The abscissa of *holomorphy* of $1/\zeta$ is Θ, and in general the two can differ (e.g. $\sum(-1)^n n^{-s}$ converges exactly for $\sigma > 0$ but is entire). The Mertens form of RH is therefore the statement that *the convergence abscissa of* $\sum \mu(n)n^{-s}$ *equals its holomorphy abscissa,* $\frac{1}{2}$. Whether $\alpha_M = \Theta$ also when $\Theta > \frac{1}{2}$ we have not checked in the literature, so we do not claim it. Landau's theorem (Thm. 6.9) is exactly the statement that *positivity* restores the Cauchy–Hadamard coincidence for Dirichlet integrals. That is why positivity enters below.

Proposition 6.7 (Smoothed sums: regularity controls γ, RH lives in β; [THM])**.** *For φ on $(0,\infty)$ let $S_\varphi(x) = \sum_n \mu(n)\varphi(n/x)$ and $\tilde{\varphi}(s) = \int_0^\infty \varphi(y)y^{s-1}dy$.*

- *(a) $\int_0^\infty S_\varphi(x)x^{-s-1}dx = \tilde{\varphi}(s)/\zeta(s)$ wherever both sides converge absolutely. Hence if $\varphi \in C_c^\infty(0,\infty)$ and $S_\varphi(x) = O(x^\theta)$, then every zero ρ with $\tilde{\varphi}(\rho) \neq 0$ has $\operatorname{Re}\rho \leq \theta$. $\tilde{\varphi}$ is entire and $\not\equiv 0$, so for any given ρ some φ has $\tilde{\varphi}(\rho) \neq 0$.*
- *(b)* Regularity sets the decay in γ. *With $u = \log y$, $\tilde{\varphi}(\sigma + i\gamma) = \int \varphi(e^u)e^{\sigma u}e^{i\gamma u}du$ is a Fourier transform. If $\varphi \in C_c^k$, then $|\tilde{\varphi}(\sigma+i\gamma)| \leq C_{k,\sigma}|\gamma|^{-k}$. If $\varphi \in C_c^\infty$, the decay is faster than any power. For Gevrey-s ($s > 1$) compactly supported φ, the decay is* $\exp(-c|\gamma|^{1/s})$ *(standard, [BIB]). An analytic φ cannot have compact support in $(0,\infty)$ (identity theorem; Lemma 59 §6). For the analytic choice $\varphi(y) = e^{-y}$: $\tilde{\varphi} = \Gamma$, $|\Gamma(\beta+i\gamma)| \sim \sqrt{2\pi}|\gamma|^{\beta-1/2}e^{-\pi|\gamma|/2}$ (Stirling), and since Γ has no zeros, $S_e(x) = \sum \mu(n)e^{-n/x} = O(x^\theta) \Rightarrow \Theta \leq \theta$.*
- *(c) Formally, each zero contributes $\tilde{\varphi}(\rho)x^\rho/\zeta'(\rho)$: a* weight $|\tilde{\varphi}(\rho)|$, *controlled by regularity and $|\gamma|$, times a* size x^β *that no choice of φ changes.*

Proof. (a) Fubini: $\int_0^\infty \varphi(n/x)x^{-s-1}dx = n^{-s}\tilde{\varphi}(s)$ (substitute $y = n/x$). For φ supported in $[a,b]$, $S_\varphi(x) = 0$ for $x < 1/b$, so $O(x^\theta)$ makes the left side holomorphic in $\sigma > \theta$. At a zero ρ with $\operatorname{Re}\rho > \theta$ and $\tilde{\varphi}(\rho) \neq 0$, the right side has a pole, a contradiction. Given ρ, any $\varphi \geq 0$, $\varphi \not\equiv 0$, concentrated near y_0 has $\tilde{\varphi}(\rho) \approx y_0^{\rho-1}\int\varphi \neq 0$. (b) Integrate by parts k times. Stirling for Γ. Absolute convergence for S_e: $e^{-n/x}$ and $\int_0^\infty S_e(x)x^{-\sigma-1}dx$ converge for $\sigma > \max(1,\theta)$; continuation then as in (a), since $\Gamma(s)/\zeta(s)$ would be holomorphic in $\sigma > \theta$ and $\Gamma(\rho) \neq 0$. (c) Residue at $s = \rho$ of $\tilde{\varphi}(s)x^s/\zeta(s)$ (simple zero). □

This is the same split as Lemma 59 §6, Prop. "blind". There, Taylor *moduli* (regularity) could not see whether zeros are real. Here the *regularity class* of the weight only reshapes the γ-profile $|\tilde{\varphi}(\rho)|$ of the zero contributions, while the exponent of x is β. Numerically (Fig. 3), the analytic weight $e^{-n/x}$ damps the first zero's term to $2|\Gamma(\rho_1)/\zeta'(\rho_1)| = 1.44 \cdot 10^{-9}$ (times $\sqrt{x}$). The curve $S_e(x)/\sqrt{x}$ is then visibly the pole term $1/\zeta(0) = -2$ divided by $\sqrt{x}$ [NUM]. Smoothing makes the zeros invisible at finite x without moving their exponent.

Theorem 6.8 (Explicit formula for M; [CITED] Titchmarsh Thm. 14.27)**.** *Assume all zeros of ζ are simple. There is a sequence $T_\nu \in [\nu, \nu+1]$ such that, for $x > 0$ not an integer,*

$$M(x) = \lim_{\nu\to\infty} \sum_{|\gamma|<T_\nu} \frac{x^\rho}{\rho\,\zeta'(\rho)} - 2 + \sum_{n\geq 1} \frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!\,n\,\zeta(2n+1)}.$$

Rotations and spirals [RESTATEMENT] (hypotheses: simple zeros, the limit along T_ν). Put $u = \log x$. Under RH, $M(x)/\sqrt{x} = \lim \sum c_\rho e^{i\gamma u} + O(x^{-1/2})$ with $c_\rho = 1/(\rho\zeta'(\rho))$: a superposition of *pure rotations*, each a matrix coefficient of an elliptic one-parameter group (Prop. 3.1). An off-line simple zero $\rho = \frac{1}{2} + \delta + i\gamma$ contributes $c_\rho e^{\delta u}e^{i\gamma u}$, a growing spiral. Together with its partner $1 - \bar{\rho}$ it contributes $(c_\rho e^{\delta u} + c_{1-\bar{\rho}}e^{-\delta u})e^{i\gamma u}$, the rotation×boost mode of Prop. 3.2. So RH for M is "no boosts", as in Lemma 58 Prop. 3.4 (compact $\iff$ elliptic). *Caveat* [THM]: Lemma 58 Prop. 3.4 is finite-dimensional. An infinite superposition of bounded rotations can be unbounded, and under linear independence of the γ it *is* unbounded (Thm. 6.10(c)). "All modes elliptic" gives $O(x^{1/2+\varepsilon})$, not $O(\sqrt{x})$. Numerically (Fig. 3(a)), truncating at 10 and 200 zeros, the RMS error against $M(x)/\sqrt{x}$ on $[10^3, 10^7]$ is 0.061 and 0.022 (RMS of $M/\sqrt{x}$ is 0.172), with $|2c_{\rho_1}| = 0.178$ [NUM].

Theorem 6.9 (Landau)**.** *Let $A(x) \geq 0$ for $x \geq x_0$, and let $F(s) = \int_1^\infty A(x)x^{-s-1}dx$ have abscissa of convergence σ_c. Then F has a singularity at the real point $s = \sigma_c$* [8] *[BIB] (content standard).*

Theorem 6.10 (One-sided bounds; [THM] classical, [CITED] [4])**.** *Suppose $M(x) \geq -K\sqrt{x}$ for all sufficiently large x (or $M(x) \leq K\sqrt{x}$).*

(a) RH holds.
(b) All zeros are simple.
(c) (Ingham 1942, Thm. A, contrapositive) The ordinates $\gamma_n > 0$ satisfy infinitely many *relations $\sum_{n\leq N} c_n\gamma_n = 0$, $c_n \in \mathbb{Z}$ not all 0. Ingham proved: if there are no such relations, or only finitely many, then $\limsup x^{-1/2}M(x) = +\infty$ and $\liminf x^{-1/2}M(x) = -\infty$. The same holds for Liouville's $L(x)$.*

Proof of (a), (b). For $\sigma > 1$, $\frac{1}{s\zeta(s)} + \frac{K}{s-\frac{1}{2}} = \int_1^\infty (M(x) + K\sqrt{x})\, x^{-s-1}dx$, and the integrand is eventually ≥ 0. The left side is holomorphic on the real segment $(\frac{1}{2}, \infty)$, because ζ has no zeros on $(0, 1)$ and $1/\zeta$ is 0 at $s = 1$. By Thm. 6.9 the integral converges in $\sigma > \frac{1}{2}$, so $1/\zeta$ is holomorphic there: (a). For $\sigma \downarrow \frac{1}{2}$ and $s = \sigma + i\gamma$,

$$\Big|\frac{1}{s\zeta(s)} + \frac{K}{s-\frac{1}{2}}\Big| \leq \int_1^\infty (M + K\sqrt{x})x^{-\sigma-1}dx + O(1) = \frac{1}{\sigma\zeta(\sigma)} + \frac{K}{\sigma-\frac{1}{2}} + O(1) = O\big((\sigma-\tfrac{1}{2})^{-1}\big).$$

A zero of multiplicity ≥ 2 at $\frac{1}{2} + i\gamma$ would force growth $(\sigma - \frac{1}{2})^{-2}$: (b). (c) is [4, Thm. A]. Ingham's §1 calls (a), (b) "well known" and reproduces the proof in §3 [CITED]. □

Status [CITED]. Odlyzko–te Riele [10]: $\limsup M(x)x^{-1/2} > 1.06$, $\liminf < -1.009$. So $K = 1$ fails on both sides, though no explicit counterexample x is known. Best–Trudgian [2]: ± 1.6383. *Unboundedness of $M(x)/\sqrt{x}$ is not known.* Odlyzko–te Riele state that $\limsup |M(x)|x^{-1/2} = \infty$ "remains unproved". It would follow from Ingham's theorem under linear independence, and Best–Trudgian (2015) record that no linear relation among the γ is known, but none is excluded either. We found no later unconditional proof. By Thm. 6.10, unboundedness *does* follow if RH fails or some zero is multiple.

Positivity + Landau ⇒ analyticity ⇒ RH: an honest assessment [OPEN]/[ANALOGY]. The mechanism is genuine. Thm. 6.10 *is* this route, and Landau's theorem is the Dirichlet-series substitute for Cauchy–Hadamard. The obstruction is the object. $M + K\sqrt{x} \geq 0$ is false for $K \leq 1.6383$, and under linear independence it fails for every K. A useful object $A(x)$ needs (i) a Mellin transform equal to known factors times $1/\zeta$ (or ζ'/ζ) and (ii) *provable* eventual one-sidedness. By Thm. 6.8, one-sidedness of a sum $\sum c_\rho x^{i\gamma}$ requires the phases of the c_ρ to conspire. For M, $c_\rho = 1/(\rho\zeta'(\rho))$ carries the uncontrolled phase of $\zeta'(\rho)$. A "phase-paired" or "helicity-projected" object would have $c_\rho \geq 0$ (paired with $\bar{c}_{\bar{\rho}}$), and then the natural positivity is *positive-definiteness* (Bochner), not pointwise positivity (Landau). That is Weil's criterion, which brings us to §7. No new sign-definite arithmetic object is proposed here.

7. Phase pairing: Cassini separatrices, Ingham–Landau, and Weil

J.W. [t237u] asks whether the longitudinal/transverse separatrices of Lemmas 55–56, i.e. the Cassini level sets of $|(s-\rho)(s-(1-\bar{\rho}))|$ near a mirror pair and the saddle form (H), are the "better phase-paired object" that §6.3 asks for. Short answer: *they use the same pairing, but on the wrong side.* (H) is a zero-side restatement, while Landau needs sign-definiteness on the arithmetic side. Weil's formula is the bridge, and a helicity projection of it adds nothing for ζ (Prop. 7.5).

7.1. Two pairings, not one.

Proposition 7.1 (The Klein four-group of ξ; [THM])**.** *ξ is invariant under $s \mapsto 1 - s$ (functional equation) and $s \mapsto \bar{s}$ up to conjugation, $\xi(\bar{s}) = \overline{\xi(s)}$. Hence the zero set is invariant under $V_4 =$*

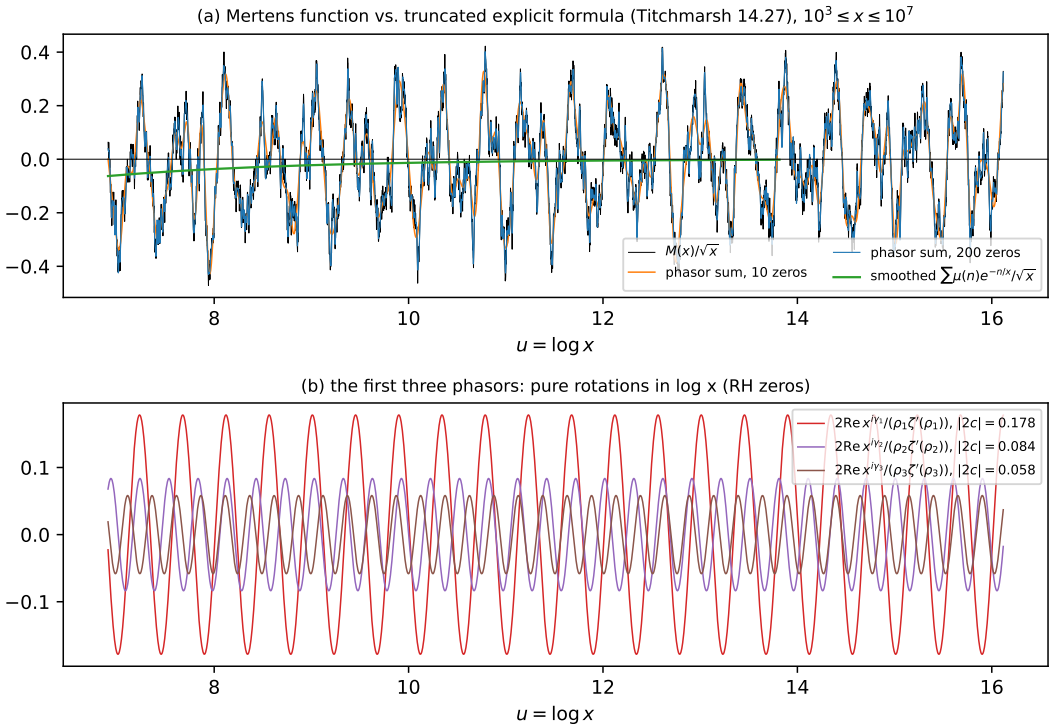

FIGURE 3. (a) $M(x)/\sqrt{x}$ for $10^3 \le x \le 10^7$ (black), against the truncated Titchmarsh 14.27 sum $2\,\mathrm{Re}\sum_{k\le K} x^{i\gamma_k}/(\rho_k\zeta'(\rho_k)) - 2/\sqrt{x}$ for $K = 10$ (orange) and $K = 200$ (blue). Green: the analytically smoothed $\sum \mu(n)e^{-n/x}/\sqrt{x}$, which is $\approx -2/\sqrt{x}$ because $\Gamma(\rho)$ damps every zero by $e^{-\pi\gamma/2}$. (b) The first three phasors, pure rotations in $u = \log x$ with amplitudes $0.178, 0.084, 0.058$. Script `num/lemma62_phasor.py` [NUM]; truncated sums are heuristic, not a convergence proof.

$\{\mathrm{id},\ \kappa : s \mapsto \bar{s},\ \iota : s \mapsto 1 - s,\ R = \kappa\iota : s \mapsto 1 - \bar{s}\}$. *In the variable* $\gamma = (\rho - \frac{1}{2})/i$ *these act by* $\gamma \mapsto \bar{\gamma}$ *(R),* $\gamma \mapsto -\gamma$ *(ι), and* $\gamma \mapsto -\bar{\gamma}$ *(κ).*

(a) The helicity pairing $\{\rho, \iota\rho\} = \{\rho, 1 - \rho\}$ *is* $\gamma \leftrightarrow -\gamma$. *Its two terms are* $e^{\pm i\gamma u}$, *and for real* γ *they sum to* $2\cos\gamma u$: *linear = sum of the two circular modes (Prop. 3.1).*

(b) The mirror (Cassini) pairing $\{\rho, R\rho\} = \{\rho, 1 - \bar{\rho}\}$ *is* $\gamma \leftrightarrow \bar{\gamma}$. *R fixes the critical line pointwise and* $\xi(Rs) = \overline{\xi(s)}$, *so* ξ *is real on the line. The transverse lemniscate of (H) points at such a pair (Lemma 56 §sec:saddlemeaning) [TAA-L].*

(c) RH $\iff$ *R fixes every zero* $\iff$ *the two pairings coincide on every zero* $(1 - \rho = \bar{\rho})$. *An on-line zero has orbit* $\{\rho, \bar{\rho}\}$; *a ghost has a quartet. This is Lemma 47 Prop. 5.2 ("half-turn = conjugation on every zero") [TAA-L][RESTATEMENT].*

So the chat sentence "both pair ρ with $1 - \bar{\rho}$ … $e^{i\varphi}$ paired with $e^{-i\varphi}$" merges two different pairings. The $e^{\pm i\varphi}$ pairing is ι, the helicity pairing. The Cassini pairing is R. They agree exactly on on-line zeros, and their disagreement on a zero *is* a ghost.

Proposition 7.2 (Level sets of $|\xi|$ versus phase; [THM], cf. Lemma 55 Lemma (a) [TAA-L])**.** *Away from zeros and critical points of* ξ, *Cauchy–Riemann for* $\log\xi = \log|\xi| + i\arg\xi$ *gives* $\nabla\arg\xi = J_0\nabla\log|\xi|$, *with* $|\nabla\arg\xi| = |\nabla\log|\xi||$. *So: (a) the level curves of* $|\xi|$ *(Cassini curves) are* orthogonal

to the level curves of $\arg\xi$*; they are not level sets of phase; (b) the* gradient lines *of* $|\xi|$ *(steepest-descent paths)* are *the level curves of* $\arg\xi$*; (c) at a simple saddle* s_0*,* $\log\xi \approx c_0 + a(s-s_0)^2$*; the two level curves of* $|\xi|$ *through* s_0 *(the lemniscate) cross at right angles, as do the two phase curves, and the two crosses are rotated by* $\pi/4$ *against each other; (d) the critical line is a phase level set (*$\arg\xi \in \{0,\pi\}$*). So at an on-line saddle the phase cross is {critical line, horizontal}, and the lemniscate crosses at* $\pm\pi/4$ *to the line, for longitudinal and transverse saddles alike. The orientation (Lemma 56 Def. 2.1) is which pair of opposite sectors is low, not the crossing angle.*

Proof. (a), (b): $\nabla\arg\xi \perp \nabla\log|\xi|$. (c): $\mathrm{Re}(az^2)=0$ and $\mathrm{Im}(az^2)=0$ are pairs of orthogonal lines, rotated by $\pi/4$. (d): Ξ is real on the line, so $\arg\xi$ is constant there between zeros. The line is therefore one phase curve through s_0, and the other is orthogonal to it. □

J.W.'s phrasing "separatrices are level sets of phase" is thus right for the separatrices of the *gradient flow* of $|\xi|$, and these are phase curves by (b). It is wrong for the *Cassini* separatrices (level sets through saddles), which cross phase curves at right angles.

7.2. **Weil pairs termwise; helicity projection adds nothing for** ζ**.** Let $f \in C_c^\infty(\mathbb{R})$ (in $u = \log x$), $g = f * \tilde f$ with $\tilde f(u) = \overline{f(-u)}$, so $\hat g(\gamma) = \hat f(\gamma)\overline{\hat f(\bar\gamma)}$ (entire). Weil's explicit formula $W(g) = \sum_\rho \hat g(\gamma_\rho) = (\text{pole}) + (\text{arch}) - \sum_n \Lambda(n) n^{-1/2}(g+g^-)(\log n)$, in the normalization of Lemma 57 [TAA-L] [18].

Proposition 7.3 (Weil uses the mirror pairing; [THM])**.** *An on-line zero contributes* $|\hat f(\gamma)|^2 \ge 0$*. A ghost R-pair* $\{\gamma,\bar\gamma\}$ *contributes* $\hat f(\gamma)\overline{\hat f(\bar\gamma)} + \hat f(\bar\gamma)\overline{\hat f(\gamma)} = 2\,\mathrm{Re}\,(\hat f(\gamma)\overline{\hat f(\bar\gamma)})$*. This is the Hermitian form* $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ *in* $(\hat f(\gamma), \hat f(\bar\gamma))$*, of signature* $(1,1)$*: the hyperbolic plane of Prop. 3.2 and Lemma 58 Prop. 3.6 [TAA-L]. So Weil's quadratic form pairs zeros by R, exactly the Cassini pairing of (H). By Bochner–Schwartz, positivity of W on all* $f * \tilde f$ *is positive-*definiteness *of the Weil distribution, i.e. a positive zero measure on real* γ*. That is the "Landau on the zero side" version of positivity, not pointwise positivity of an arithmetic function [THM][BIB].*

Definition 7.4 (Helicity-projected Weil functional)**.** Fix $a > 1/\pi$ and $h_a(\gamma) = \frac12(1+\tanh(\gamma/a))$. Then h_a is holomorphic and bounded in $|\mathrm{Im}\,\gamma| \le c < \pi a/2$ (so on a strip wider than $|\mathrm{Im}\,\gamma| \le \frac12$), $h_a > 0$ on $\mathbb{R}$, $h_a(\gamma) + h_a(-\gamma) \equiv 1$, and $h_a \to \mathbf{1}_{\gamma>0}$ as $a \to 0$. This is a smoothed projector onto positive frequency, i.e. positive helicity, H_+ (Lemma 53). Set

$$W_h(f) := \sum_\rho h_a(\gamma_\rho)\,\hat g(\gamma_\rho).$$

The transform of $h_a\hat g$ is the inverse Fourier transform of $\frac12\hat g$ plus that of $\frac12\tanh(\gamma/a)\hat g$. The latter is g convolved with a p.v. kernel $\propto \mathrm{csch}(\pi a u/2)$, so it is *not* compactly supported and decays like $e^{-\pi a|u|/2}$. Its prime side therefore runs over *all* n, with weights $\ll n^{-1/2-\pi a/2}$, and it converges exactly in the range $a > 1/\pi$ that the strip condition requires [THM] (standard Fourier pair $\tanh \leftrightarrow \mathrm{csch}$).

Proposition 7.5 (Helicity projection reduces to Weil; [THM])**.** *(a) The zero multiset of* ζ *is* ι*-invariant (*$\gamma \mapsto -\gamma$*), so* $W_h(f) = \sum_\rho \hat G_h(\gamma_\rho)$ *with the* even *function* $\hat G_h(\gamma) = \frac12[h_a(\gamma)\hat g(\gamma) + h_a(-\gamma)\hat g(-\gamma)]$*, which is* ≥ 0 *on* $\mathbb{R}$*. (b) If f is real-valued,* $\hat g$ *is even and* $W_h(f) = \frac12 W(f)$ identically*: ghost terms included, for every a. (c) RH* $\Rightarrow W_h(f) \ge 0$ *for all f. Since W on real f already detects every ghost (Lemma 57: RH* $\iff$ *the ladder is positive for all A) [TAA-L], (b) gives RH* $\iff$ $W_h \ge 0$ *on real f. This is a [RESTATEMENT]. (d) Consequently* W_h *is the zero measure integrated against an even test function that is nonnegative on* $\mathbb{R}$*. That is the object Lemma 57 identified as resolution-limited zero counting, and the projection makes it worse: compact support is lost (Paley–Wiener: a compactly supported* $f \ne 0$ *cannot have* $\hat f$ *vanishing on a half-line, so exact helicity projection is impossible), and with it the finite prime window.*

Proof. (a) Reindex the sum by $\gamma \mapsto -\gamma$ and average. On $\mathbb{R}$, $\hat{g}(\pm\gamma) = |\hat{f}(\pm\gamma)|^2$ and $h_a > 0$. (b) For real f, $\hat{f}(-\gamma) = \overline{\hat{f}(\bar{\gamma})}$, so $\hat{g}(\gamma) = \hat{f}(\gamma)\hat{f}(-\gamma)$ is even, and then $\hat{G}_h = \frac{1}{2}(h_a(\gamma) + h_a(-\gamma))\hat{g} = \frac{1}{2}\hat{g}$. (c) Under RH all γ are real and $\hat{G}_h \geq 0$ there. The converse is (b) with Lemma 57. (d) Paley–Wiener: $\hat{f}$ is entire. □

Verdict. For ζ the two helicities carry the *same* zeros (Ξ is even), so projecting onto one helicity is invisible. This is the Weil-side form of the abstract's item (6): every Krein sign is +, and helicity adds no constraint. In optics, by contrast, the two circular modes are independent. A helicity projection could only matter for an object *without* the ι-symmetry: for instance a single $L(s, \chi)$ with χ complex, whose zeros are not symmetric under $\gamma \mapsto -\gamma$, paired with $L(s, \bar{\chi})$. We record this as F6 (§10) and did not pursue it. The Cassini/(H) side and the Weil side therefore pair the same way (R). Neither supplies Landau-type pointwise positivity on the arithmetic side, and the open question in §6.3 stays open [OPEN].

8. Positivity as helicity: restatement or new constraint?

Definition 8.1 (Helicity condition)**.** Let (V, ω) carry a flow e^{uA}, $A \in \mathfrak{sp}(V, \omega)$. The flow has *positive helicity* if it preserves a positive compatible J and $h_A(x, x) = \omega(x, Ax) > 0$ for $x \neq 0$. Equivalently, on every invariant elliptic plane it is a counterclockwise rotation for the ω-orientation, $A|_P = \nu J_P$ with $\nu > 0$. That is, $-iA > 0$ on the positive eigenspace, the positive-energy condition of Lemma 54 Prop. 5.1 [TAA-L].

Proposition 8.2 ([THM] finite-dimensional, [RESTATEMENT] for ζ)**.** *(a) Positive helicity $\iff$ the flow is elliptic* and *every Krein sign is +1 ($m_- = 0$). By Thm. 3.3, Lemma 58's* APD *("preserves some positive J") is elliptic only. So helicity is* strictly stronger *in general, and by Lemma 54 v2 the index becomes $\nu = k$. (b) In the Weil zero model of ζ (Lemma 58 Prop. 3.6), every on-line zero gives a positive block, and the Krein sign of rotations is free (Lemma 58 Rem. 3.7) [TAA-L]. Hence for ζ: positive helicity $\iff$ APD $\iff$ no hyperbolic planes $\iff$ RH. This is a [RESTATEMENT].* (c) Handedness. *Given the flow, the positive J commuting with it is unique on each elliptic plane (Thm. 3.3(b)). Its handedness is fixed by the direction of the flow (the sign of u, i.e. $\log p > 0$: Frobenius acts by p^{-s}, not p^{+s}) together with the orientation of ω. Reversing both is the global convention $J \leftrightarrow -J$ (Wick², Prop. 5.4). For curves the orientation is canonical (fundamental class, Prop. 6.2). For ζ the candidates are H_+ (Lemma 53/54: $J_A = -i$ on H_+, canonical from causality but ghost-blind, Rem. 4.4), the spinor structure (ghost-blind, Lemma 50 Prop. 6.5), and Deninger's H^1 (conjectural). None constructs J non-circularly.*

Answer to [t231u]. Yes: the polarization TAA looks for *is* a circular polarization (Prop. 2.3), and "positive helicity" is the right name for the positivity condition. It has two parts. Positive *frequency* is TAA's J. Transverse helicity (Prop. 2.4) has no counterpart for ζ: Ξ is even, so both helicities carry the same zeros (Prop. 7.5). As a condition it adds *no* constraint for ζ beyond the APD, and it does not construct J. What it does add is the clarification (c): the handedness is canonical up to one global convention, and ghosts are helicity-*neutral*, not wrong-handed.

9. Corrections to chat claims

C1. *[t231u] "circular/helical polarization".* Correct in the sense of Prop. 2.3 (the $\pm i$ eigenspaces of J are the circular modes). Optical helicity needs a second, transverse complex structure, which TAA lacks (Prop. 2.4).

C2. *[t232u]* $\cosh t = \cos(it)$*; the circle action multiplies a_n by $e^{in\varphi}$; Wick is $\varphi = \pi/2$.* All correct (Prop. 5.1). For even f the loop has period π, and $\varphi = -\pi/2$ gives the same function.

C3. *[t232u] "real-rootedness is about the phases of the coefficients".* The phases are necessary and *free* (unconditional from $\Phi > 0$). RH is in quadratic inequalities among the moduli (Rem. 5.3).

C4. *[t232u] Pólya–Schur multiplier sequences; Lemma 2 cites Pólya–Schur.* Correct. Lemma 2 mentions "Pólya–Schur theory; Craven–Csordas" in a remark (line 146) without a bibliography entry. Pólya–Schur classify *diagonal* operators (multiplier sequences); arbitrary linear preservers are Borcea–Brändén [1] (Thm. 5.6).

C5. *[t233u] $Z(T) = \prod \det(1-TF|H^i)^{(-1)^{i+1}}$; superdeterminant; Lefschetz at $F = \mathrm{id}$ gives χ.* Correct (Thm. 6.1). Deninger's archimedean formula is conjectural as a cohomological statement; the product side is proved (Den98) [CITED].

C6. *[t233u] Spector: $\mu = (-1)^F$, $1/\zeta$ partition function; "the Witten index connects to χ".* The first two are correct [16]. In the supersymmetric primon gas the Witten index is $\zeta \cdot \zeta^{-1} = 1$ (Möbius inversion). A link to an Euler characteristic of a space is an analogy, since there is no manifold (Prop. 6.4 gives the exact combinatorial version).

C7. *[t233u] "Ξ's alternating Taylor signs are fermionic".* Analogy only (Rem. 6.5(a)).

C8. *[t234u] RH $\iff M(x) = O(x^{1/2+\varepsilon})$ (Littlewood); Mertens disproved (Odlyzko–te Riele 1985).* Correct. Littlewood's note is C.R. 154 (1912) 263–266. Odlyzko–te Riele gave no explicit counterexample.

C9. *[t235u] "Ingham showed $M \geq -K\sqrt{x}$ implies RH + simple zeros + linear relations among γ."* Essentially correct, with three precisions. (i) RH and simplicity were already "well known" (a Landau-type argument); Ingham reproduces them in §3, and his new result is the linear relations. (ii) They are *infinitely many* integer relations. (iii) The bound need only hold for all sufficiently large x, and the same holds for $M \leq K\sqrt{x}$ and for Liouville's $L(x)$ (Thm. 6.10).

C10. *[t235u] "growth exponent of M equals the abscissa of analyticity of $1/\zeta$".* Proved here only as $\Theta \leq \alpha_M$, with equality iff RH (Thm. 6.6). In general the growth exponent is the abscissa of *convergence*. Equality with the holomorphy abscissa when $\Theta > \frac{1}{2}$ was not checked.

C11. *[t235u] "unboundedness of $M/\sqrt{x}$ is not unconditionally known".* Correct. It is open (stated as unproved in [10]; no later proof found). It follows if RH fails, if some zero is multiple, or under linear independence.

C12. *[t237u, Grok Bot chat] "Both pair ρ with $1-\bar{\rho}$ … $e^{i\varphi}$ paired with $e^{-i\varphi}$, like linear = two circular."* This merges two pairings. $e^{\pm i\gamma u}$ is $\iota : \rho \mapsto 1-\rho$ (helicity), and the Cassini/Weil pairing is $R : \rho \mapsto 1-\bar{\rho}$. They coincide exactly on on-line zeros, and RH $\iff$ they coincide everywhere (Prop. 7.1).

C13. *[t237u, chat] "Weil's explicit formula … gives positivity of a prime sum."* Imprecise. Weil's criterion makes RH *equivalent* to positivity of the prime-side functional on $\{f * \tilde{f}\}$; it supplies no positivity. The positivity is positive-definiteness (Bochner), not pointwise (Landau) (Prop. 7.3).

C14. *[t237u, chat] "L57 showed the window ladder is zero counting."* Correct as Lemma 57's verdict. Precisely: W_A is the zero measure tested on $|\hat{F}|^2$, and certifying it from primes costs resolving zeros to $T^*(A)$.

C15. *[t237u, chat] "By Cauchy–Riemann, level sets of $|\xi|$ are orthogonal to the constant-phase lines."* Correct away from zeros and critical points. At a saddle both families cross, at $\pi/4$ to each other. J.W.'s "separatrices are level sets of phase" is right for gradient-flow separatrices and wrong for Cassini separatrices (Prop. 7.2).

10. Ranked follow-ups

Score = 0.6·significance+0.4·probability (each 0–10).

	Follow-up	sig	prob	score
F1	Function-field helicity table: genus 2 over $\mathbb{F}_{11}$, Weil curves vs Lemma 45/48 fakes. Krein signs and handedness from the fundamental class (Prop. 6.2); do fakes fail by neutral planes only, or also by wrong helicity?	5	7	5.8
F2	A sign-definite arithmetic object for the Landau route (§6.3): a weighted Möbius/Liouville sum with provable eventual one-sidedness and Mellin transform known up to $1/\zeta$.	9	1	5.8
F3	Orientation on Deninger's H^1: does the conjectural pairing come with a fundamental-class orientation as in Prop. 6.2?	8	1	5.2
F4	Literature: is $\alpha_M = \Theta$ known when $\Theta > \frac{1}{2}$ (Thm. 6.6)?	3	8	5.0
F6	Helicity projection without ι-symmetry: $L(s,\chi)$, χ complex, and the DH pair $L(s,\chi), L(s,\bar{\chi})$ (Prop. 7.5).	5	5	5.0
F5	Wick-twisted heat flow $e^{-te^{2i\varphi}D^2}$: for which φ do DH's ghosts reach the real axis? (§5)	3	6	4.2

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TAA LEMMA 63: GHOSTS AS DENTS IN A FLAT CONE METRIC, THE RIGHT-ANGLE CROSS, ZERO-BY-ZERO WEIL POSITIVITY, THE KERNEL-CONDITION HIERARCHY, AND DH JENSEN POLYNOMIALS TO DEGREE $2 \cdot 10^4$

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

Date: October 22, 2026.

ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Every RH-equivalent statement below is a [RESTATEMENT] of known results. What this lemma adds is a precise translation of three pictures (curvature, saddle crosses, Weil pairing) and new numerics on Davenport–Heilbronn (DH), the standard function with "ghosts" (off-line zeros) but no Euler product. Tags: [THM] proved here (elementary) or classical with proof given; [CITED] checked at the source (`lit/lemma63/verified_refs.md`); [CITED-sec] checked only via a named secondary source; [BIB] bibliographic record only; [TAA-L] earlier TAA lemma; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEUR] heuristic; [CONJ] conjecture; [OPEN] open; [NUM] numerical (`num/lemma63_*.py`, data in `num/l63/`).

Answer in brief. (A) *Ghosts as dents* (§2). The metric $g = |\xi(s)|^2|ds|^2$ is flat away from the zeros. A zero of order m is a cone point of angle $2\pi(m+1)$, so it carries curvature $-2\pi m$: *negative* (an excess-angle saddle point of the surface), not a positive dent [THM]. Gauss–Bonnet on a rectangle is literally Green's theorem for $\log|\xi|$ and is equivalent to the argument principle. "Total curvature $= -2\pi N$" [RESTATEMENT]. Poincaré–Hopf for $\nabla|\xi|^2$ (zeros $+1$, critical points -1) gives $N - N'$ as a boundary winding. The *bound* $|N' - N| \le C\log T$ is not topological; it is Lemma 56's analytic estimate. Kim's identity $K^+ = \tau + S^+$ is the 2D count minus the 1D Morse count on the line [RESTATEMENT]. The genuine "dent" lives on the line. The vertical convexity $\kappa(x) = \partial_y^2 \log|\Xi(x+iy)|_{y=0} = \sum_\rho \mathrm{Re}(x-t_\rho)^{-2}$ gets from each ghost pair a mean-zero profile $2(a^2-\delta^2)/(a^2+\delta^2)^2$, a well of depth $2/\delta^2$ and width 2δ [THM]. All 28 DH ghosts below height 1000 dig a visible well (margin $\delta^2 B/2 \le 0.141$); $\min\kappa_\zeta = 0.046 > 0$ on $[0,1000]$ [NUM]. A wide ghost can hide (Ex. 2.10).

(A′) *The right-angle cross* (§3; J.W. [t238u]). At a simple critical point with $\xi \ne 0$, the level set of $|\xi|$ is an exact 90° cross of real-analytic arcs, with $c = \xi''/(2\xi)$. On the line it sits at $\pm 45°$, and the sign of c is Lemma 56's orientation [THM]. Degenerate saddles ($m+1$ arcs at angle $\pi/(m+1)$) occur at *saddle* collisions, not at zero collisions. At a zero collision $c \sim \frac{1}{2}/(t_c - t)$, and the cross shrinks to the double zero and flips (DH: $c(t_c - t) = 0.497$–0.503) [NUM]. The 90° is automatic, not an equilibrium.

(W) *Zero-by-zero Weil positivity* (§4; J.W. [t239u]). Write Weil's form as $W(f) = \sum_\rho \hat f(\rho)\overline{\hat f(1-\bar\rho)}$. An on-line zero contributes $|\hat f(\rho)|^2$. An off-line pair contributes $2\,\mathrm{Re}(a\bar b)$, a Hermitian form of signature $(1,1)$ [THM]. The sum converges absolutely for $f \in C_c^\infty$. Weil: RH $\iff$ $W \ge 0$ [RESTATEMENT]. DH: $W < 0$ with the ghost block alone (-0.529, designed; -1.35, scanned) [NUM]. *Verdict: zero location in disguise.* The only non-circular content is the prime side (Lemma 57).

(B) *Kernel hierarchy* (§5). From $\Phi > 0$ through CNV log-concavity, PF_k, higher Turán/Jensen, Pólya/de Bruijn/Newman, up to Gröchenig's PF_∞ criterion: *every rung that DH satisfies is RH-blind, and every RH-equivalent rung is zero location in disguise.* DH has a real, even, positive kernel. J.W.'s polarization condition (a zero-free positive form that sees ghosts) is *not* met by any kernel rung. Bochner's form $\sum c_j\bar c_k \Xi(t_j - t_k) \ge 0$ is the one genuine positive form, and it is RH-blind [THM].

(C) *DH Jensen polynomials to degree* $2\cdot 10^4$ (§6). Moments are computed to double-precision agreement with Lemma 59. DH and ζ: $J^{d,0}$ is *globally* hyperbolic for $d \le 2000$ (certified by sign changes) [NUM]. Near the first DH ghost ($85.699 \pm 0.309i$), the even Jensen polynomials carry a ghost-induced real pair whose gap shrinks like $\sqrt{d^* - d}$. That gives $d^*_{\mathrm{ev}} = 20786$ (the real pair survives at $d = 20785$; bisection, see `l63/dstar.jsonl`). Classical $J^{d,0}$ is predicted at $\approx 4d^*_{\mathrm{ev}} \approx 8.3\cdot 10^4$. So across J.W.'s range 10^3–10^4 *DH is indistinguishable from ζ by Jensen hyperbolicity.* The ghost becomes visible only at $d^*_{\mathrm{ev}} = 20786 \approx 2.83\,T^2$ (even), while L_1 sees it at order 2. This quantifies Farmer's "dispersal" for a function that really has ghosts.

Prior work (§7), corrections (§8), labels, Q1–Q5.

Decisions and questions (J.W., [t236u], [t238u], [t239u]; via the parent thread). [t236u]: adopt the Lemma 59 verdict, the tower counters, and the smooth/analytic conclusions; pursue the kernel hierarchy; extend the DH Jensen/Turán tests to degree 10^3–10^4; put L59 in Volume 4. Then the question behind (A): "ghosts as dents" in a curvature picture. [t238u]: "the right-angle cross at saddles." [t239u]: "zero-by-zero Weil positivity: $1 - \bar\rho = \rho$ on the line, the helicity partner is $1 - \rho = \bar\rho$."

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1. Conventions

$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and $\Xi(t) = \xi(\frac{1}{2} + it)$: even, real on $\mathbb{R}$, entire of order 1. The zeros of Ξ are the numbers t_ρ with $\rho = \frac{1}{2} + it_\rho$. A *ghost* is a zero $\rho = \frac{1}{2} + \delta + i\gamma$ with $\delta > 0$. In the t-plane it is the conjugate couple $t = \gamma \mp i\delta$; with $-t$ it gives four zeros. DH: $\Xi_{\mathrm{DH}}(t) = \Lambda_{\mathrm{DH}}(\frac{1}{2} + it)$, real and even, as in Lemmas 55–59 [TAA-L]. Its 28 ghosts below height 1000 have $\delta \in [0.016, 0.370]$ (L59 `lag_ghosts.json`). The first one is $85.6993484854 \pm 0.3085171825\,i$. Kernels: $\Phi_\zeta(u) = \sum_{n\geq 1}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})e^{-\pi n^2 e^{4u}}$ (even in u), with $\int_0^\infty \Phi_\zeta(u)\cos(zu)\,du = \frac{1}{8}\Xi(z/2)$ [15] [CITED]. $\Phi_{\mathrm{DH}}(u) = 4\sqrt{\pi/5}\,e^{3u}\sum_{n\geq 1} n\,a_n e^{-\pi n^2 e^{4u}/5}$, where a_n is $0, 1, \varkappa, -\varkappa, -1$ for $n \equiv 0, \ldots, 4 \pmod 5$ and $\varkappa = (\sqrt{10 - 2\sqrt{5}} - 2)/(\sqrt{5} - 1)$. It is even in u (checked to 12 digits), and its cosine transform is proportional to $\Xi_{\mathrm{DH}}(z/2)$ [TAA-L] (L56 v2 Def. 6.3, L59). De Bruijn–Newman: $H_\tau(z) = \int_0^\infty e^{\tau u^2}\Phi(u)\cos(zu)\,du$, $z = 2t$ [15] [CITED]. Lemma 56 counters: ℓ, τ are the longitudinal and transverse on-line critical points; S^+ are off-line critical points with $\sigma > \frac{1}{2}$; K^+ are ghosts. The orientation is $\mathfrak{o} = -\operatorname{sgn}(\Xi''/\Xi)$ at an on-line critical point ($+1$ longitudinal, -1 transverse). $\kappa = \tau + S^+$ is the official counter.

2. (A) Ghosts as dents: the flat cone metric $|\xi|^2|ds|^2$

2.1. **The Riesz measure and the cone metric.** Let f be entire, $f \not\equiv 0$, and $\varphi = \log|f|$. Then $\Delta\varphi = 2\pi\sum_\rho m_\rho\,\delta_\rho$ in the sense of distributions, where m_ρ is the multiplicity (Poincaré–Lelong in one variable) [THM] (classical: $\log|z - a|$ is 2π times the fundamental solution).

Consider the conformal metric $g = e^{2\varphi}|ds|^2 = |f(s)|^2|ds|^2$. Its Gaussian curvature is $K = -e^{-2\varphi}\Delta\varphi$, so the curvature *measure* is

$$K\,dA_g = -\Delta\varphi\,dx\,dy = -2\pi\sum_\rho m_\rho\,\delta_\rho. \tag{1}$$

Proposition 2.1 (cone points)**.** *[THM]g is flat off the zero set. Near a zero of order m, the developing map $F(s) = \int_{s_0}^{s} f$ satisfies $F \sim \frac{f^{(m)}(s_0)}{(m+1)!}(s-s_0)^{m+1}$, and $g = |dF|^2$ is the pull-back of the Euclidean metric. Hence s_0 is a cone point of total angle $2\pi(m+1)$, and its curvature atom is $2\pi - 2\pi(m+1) = -2\pi m$, in agreement with* (1)*.*

Proof. $g = |f|^2|ds|^2 = |dF|^2$, and F is a local isometry wherever $f \neq 0$. Near s_0, $F = w^{m+1}$ in a holomorphic coordinate w, so a small loop around s_0 winds $m+1$ times around $F(s_0)$. The cone angle is therefore $2\pi(m+1)$. This is the standard dictionary "zero of degree d of the holomorphic 1-form $\omega = f\,ds$ $\leftrightarrow$ cone angle $2\pi(d+1)$" of translation surfaces [19] [CITED] (§3.3). □

Remark 2.2 (sign)**.** Zeros are cone points of *excess* angle, i.e. concentrated *negative* curvature. Intrinsically they are "saddle points" of the flat surface, not sphere-like dents. Positive cone curvature (angle $< 2\pi$) would require the poles of a meromorphic f of order ≥ 2, or $|f|^{2\alpha}$ with $-1 < \alpha < 0$.

2.2. **Gauss–Bonnet = Green = argument principle.** Let R be a rectangle, positively oriented, with no zeros on ∂R, and let N_R be the number of zeros inside, with multiplicity. For the conformal metric, the geodesic curvature of ∂R is $k_g\,ds_g = (k_0 + \partial_\nu\varphi)\,ds_0$, where ν is the outward Euclidean normal and $k_0 = 0$ on the sides. The exterior angles are conformally invariant, so they still sum to 2π.

Theorem 2.3 (Gauss–Bonnet for $|f|^2|ds|^2$)**.** *[THM] [RESTATEMENT]*

$$\iint_R K\,dA_g + \oint_{\partial R} k_g\,ds_g + \sum \textit{(exterior angles)} = 2\pi\chi(R) = 2\pi$$

reads $-2\pi N_R + \oint_{\partial R}\partial_\nu \log|f|\,ds_0 + 2\pi = 2\pi$. By Cauchy–Riemann, $\partial_\nu \log|f| = \partial_\tau \arg f$, so this is $\frac{1}{2\pi}\Delta_{\partial R}\arg f = N_R$: the argument principle. *Both sides are Green's theorem $\iint_R \Delta\varphi = \oint \partial_\nu\varphi$. In particular, with $R = [-1,2]\times[0,T]$ and $f = \xi$,* the total curvature in R is $-2\pi N(T)$.

So "the argument principle is a Gauss–Bonnet theorem" is correct, and it adds nothing: Gauss–Bonnet for a conformal flat metric with cone points *is* Green's theorem for the weight. The cone-metric convention (atom $= 2\pi -$ angle) is that of Troyanov [17] [BIB].

Remark 2.4 (the logarithmic-derivative metric)**.** A second natural metric is $g' = |\xi'/\xi|^2|ds|^2$, the pull-back of the flat cylinder by $\log\xi$. A simple zero of ξ becomes a complete cusp-like end ($|ds|^2/|s-s_0|^2$, a half-infinite cylinder) carrying $+2\pi$. A simple zero of ξ' with $\xi \neq 0$ is a cone point of angle 4π carrying -2π. Then Gauss–Bonnet gives $2\pi(N - N') =$ boundary turning of ξ'/ξ [THM]. Here zeros *are* positive dents, but only because the cylinder metric puts them at infinity.

2.3. **Poincaré–Hopf; $N - N'$; Kim's identity.** Identify $V = \nabla|\xi|^2$ with the complex number $2\,\xi\,\overline{\xi'}$ (since $\nabla u \leftrightarrow 2\partial_{\bar s}u$).

Proposition 2.5. *[THM]At a zero of ξ of order m, $\operatorname{ind} V = m - (m-1) = +1$. At a zero of ξ' of order k with $\xi \neq 0$, $\operatorname{ind} V = -k$ (a simple saddle has -1). Poincaré–Hopf on R (no zeros of $\xi\xi'$ on ∂R) gives*

$$\sum \operatorname{ind} V = N_R - N'_R = \frac{1}{2\pi}(\Delta_{\partial R}\arg\xi - \Delta_{\partial R}\arg\xi') = -\frac{1}{2\pi}\Delta_{\partial R}\arg(\xi'/\xi).$$

This is the claimed "Poincaré–Hopf with a boundary term" [RESTATEMENT]. But the *inequality* $|N'(T) - N(T)| \leq C\log T$ is not topological. It is the analytic estimate of $\Delta\arg(\xi'/\xi)$ on the boundary (horizontal sides by the sign of $\operatorname{Re}\xi'/\xi$, vertical sides by Jensen's formula), proved in Lemma 56 v2 Thm. 4.8 [TAA-L]. Poincaré–Hopf supplies the identity, not the bound.

Proposition 2.6 (Kim's identity as 2D minus 1D)**.** *[RESTATEMENT]On a window $[a, b]$ with admissible ends (L56), the rectangle count of L56 reads $(\ell + \tau + 2S^+) - (Z + 2K^+) = \beta$, where $\beta = \frac{1}{2}(\varepsilon(a) - \varepsilon(b))$ is its boundary term. The 1D Morse count on the line says that the extrema of $|\Xi|$ on $[a, b]$ alternate. The maxima are the ℓ longitudinal points; the minima are the Z real zeros and the τ transverse points (where $\Xi''/\Xi > 0$). This gives $\ell - (\tau + Z) = \beta$. Subtracting the two counts gives $2\tau + 2S^+ - 2K^+ = 0$, i.e. $K^+ = \tau + S^+$ (Kim [8] Thm. 1, L56 Thm. 4.3) [CITED-sec][TAA-L].*

In the language of the flat surface: S^+ and τ are the cone points of g', K^+ the cusps off the line. The identity says that the off-line curvature of g' balances. No new information.

2.4. **The line convexity: where the dent really is.**

Definition 2.7. For real x with $\Xi(x) \neq 0$ put $\kappa(x) := \partial_y^2 \log |\Xi(x + iy)|\big|_{y=0}$.

Theorem 2.8. *[THM]*$\kappa(x) = -(\log \Xi)''(x) = \dfrac{\Xi'(x)^2 - \Xi(x)\Xi''(x)}{\Xi(x)^2} = \dfrac{L_1(\Xi)(x)}{\Xi(x)^2} = \sum_\rho \operatorname{Re}(x - t_\rho)^{-2}$, *where the sum runs over all zeros t_ρ of Ξ (absolutely convergent).*

(a) A real zero contributes $(x - t)^{-2} > 0$.

(b) A ghost couple $\gamma \pm i\delta$ contributes, with $a = x - \gamma$,

$$k_{\gamma,\delta}(x) = \frac{2(a^2 - \delta^2)}{(a^2 + \delta^2)^2}. \tag{2}$$

This is negative exactly for $|a| < \delta$, has minimum $-2/\delta^2$ at $a = 0$, maximum $1/(4\delta^2)$ at $|a| = \sqrt{3}\,\delta$, and satisfies $\int_{\mathbb{R}} k_{\gamma,\delta} = 0$.

(c) Hence RH $\Rightarrow \kappa > 0$ on $\mathbb{R} \setminus \{\text{zeros}\}$.

Proof. $\log|\Xi|$ is harmonic off the zeros, so $\partial_y^2 = -\partial_x^2$ on the line, and $\partial_x^2 \log|\Xi| = (\log \Xi)''$ where Ξ is real. L_1 is L59's first Laguerre expression. Hadamard: Ξ is even of order 1, so $\Xi(t) = \Xi(0)\prod_\rho (1 - t/t_\rho)$, with the zeros paired $\pm t_\rho$ so the product converges. Then $-(\log \Xi)'' = \sum (t - t_\rho)^{-2}$, which converges absolutely because $\sum |t_\rho|^{-2} < \infty$. For (b), $(a - i\delta)^{-2} + (a + i\delta)^{-2} = 2\operatorname{Re}(a + i\delta)^{-2}$, and $\int_{\mathbb{R}} (a + i\delta)^{-2} da = 0$. □

So the precise version of "a ghost is a dent" is this. *A ghost couple redistributes, but does not create, vertical convexity along the line.* Its contribution has total mass zero: a well of depth $2/\delta^2$ and width 2δ, flanked by shoulders. A real zero instead contributes a non-integrable positive spike. The 2D picture (Thm. 2.3) cannot see this, because there every zero, on or off the line, is the same cone point -2π.

Proposition 2.9 (first-order visibility)**.** *[THM] [HEUR]Write $\kappa = k_{\gamma,\delta} + B$ near γ, where B is the contribution of all other zeros. If B is constant on $[\gamma - \delta, \gamma + \delta]$, then $\kappa(\gamma) < 0 \iff \delta^2 B/2 < 1$. The well $\{\kappa < 0\} = \{L_1 < 0\}$ is then the interval $|a| < a_*$ with*

$$a_*^2 = \delta^2 \,\frac{\sqrt{1 + 8\beta} - 1 - 2\beta}{2\beta}, \qquad \beta := \delta^2 B/2,$$

the positive root of $Ba^4 + 2(B\delta^2 + 1)a^2 + B\delta^4 - 2\delta^2 = 0$; it exists iff $\beta < 1$. For $\beta \to 0$, $a_* \to \delta$.

Example 2.10 (a ghost that hides)**.** [NUM]A Farmer-type toy: $\cos z$ with the zeros $\frac{5}{2}\pi, \frac{7}{2}\pi$ replaced by $10 \pm i\delta$ (and mirrored). Then $\min_{0<x<30} \kappa = -1.81, -0.306, -0.029, +0.110$ for $\delta = 1, 2, 3, 5$. For $\delta = 5$, $\kappa > 0$ everywhere although there is a ghost. So $\kappa \geq 0$, i.e. $L_1 \geq 0$, is necessary but not sufficient. By Lemma 59 (its Thm. "lag"), some $L_n(x_0) < 0$, $n \geq 1$, always detects a ghost at its own abscissa.

[NUM] Numerics. (`num/lemma63_curv.py`, `163/curv.json`). (i) Hadamard check: κ computed from the L59 jet grids agrees with $\sum \mathrm{Re}(x-t_\rho)^{-2}$ over the 649 (ζ) and 848 (DH) real zeros $|t| \le 1000$ plus ghosts plus a density tail, to relative error $\le 3.6 \cdot 10^{-7}$ at 7 test points. Example: DH at $x = 85.7$, $\kappa = -20.159$. (ii) $\min \kappa_\zeta = 0.0462 > 0$ on the grid $[0, 1000]$ (step 0.01). (iii) For all 28 DH ghosts, the margin $\delta^2 B/2 \in [0.00027, 0.141]$, so every ghost is visible at first order. The exact length of $\{L_1 < 0\}$ divided by 2δ lies in $[0.778, 0.9995]$, and the constant-background prediction of Prop. 2.9 is within 0.019 of it. Example: ghost 85.699 has $B = 0.853$, ratio 0.9251 (predicted 0.9272). This replaces L59's "$(0.77\text{–}1.01) \cdot 2\delta$": the value 1.01 was a grid-step artifact (C8).

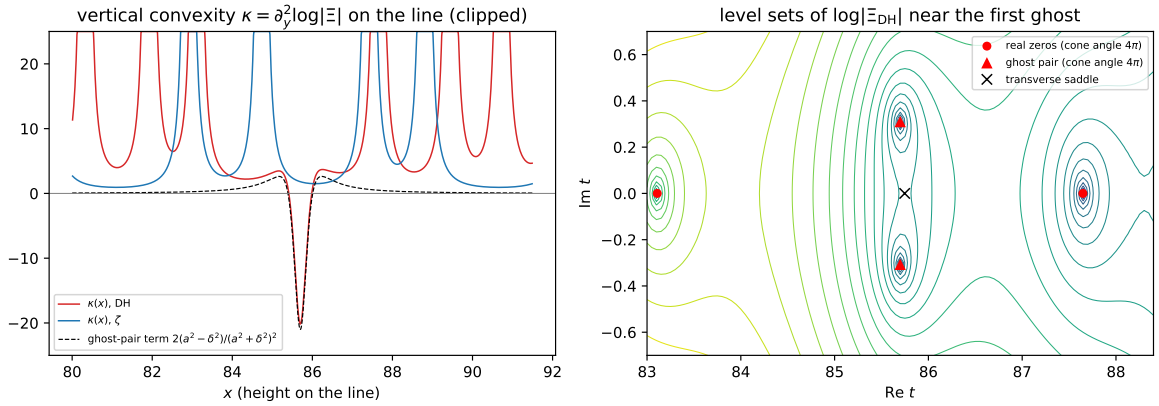

FIGURE 1. Left: $\kappa = \partial_y^2 \log|\Xi|$ on the line near the first DH ghost (red) and for ζ (blue); dashed: the ghost-couple term (2), which accounts for the well. Right: level sets of $\log|\Xi_{\mathrm{DH}}|$ in the t-plane. Real zeros and the ghost couple are all cone points of angle 4π (curvature -2π) of $|\Xi|^2|dt|^2$; the transverse saddle sits between the ghosts. (`num/lemma63_figs.py`)

Remark 2.11 ("ghosts are not flat", made precise)**.** [THM]L59 showed $\Xi(x+iy)\Xi(x-iy) = \sum_n L_n(\Xi)(x) y^{2n}$. A ghost at $x_0 + iy_0$ forces this real-analytic function of y to vanish at y_0. So *the finite jet of Ξ at x_0 detects the ghost*: some $L_n(x_0) < 0$. For all 28 DH ghosts, $n = 1$ suffices. A local curvature functional that sees ghosts is the dent $D(x) = \max(0, -\kappa(x))$: RH $\Rightarrow D \equiv 0$. It is pointwise zero location [RESTATEMENT]. Integrated against a weight, $\int w\,\kappa = \sum_\rho \int w(x)\,\mathrm{Re}(x-t_\rho)^{-2}dx$ is a sum over zeros of a transform of w, i.e. an explicit-formula functional. Its positivity for all $w \ge 0$ is again zero location.

3. (A′) The right-angle cross at saddles (J.W. [T238U])

Theorem 3.1 (conformal Morse lemma)**.** *[THM]Let f be holomorphic near s_0 with $f(s_0) \neq 0$, $f'(s_0) = 0$, $f''(s_0) \neq 0$, and put*

$$c := \frac{f''(s_0)}{2f(s_0)}, \qquad \log f(s) = \log f(s_0) + c\,(s-s_0)^2(1 + h(s-s_0)),\ h(0) = 0.$$

(The coefficient is c because $(\log f)'' = f''/f - (f'/f)^2 = f''/f$ at s_0.) Then $\zeta := \sqrt{c}\,(s-s_0)\sqrt{1+h}$ is a holomorphic coordinate with $\log|f| - \log|f(s_0)| = \mathrm{Re}\,\zeta^2$. Consequently:

- *(a) the level set $\{|f| = |f(s_0)|\}$ near s_0 is the union of two real-analytic arcs crossing at* exactly $90°$. *Their tangents are* $\arg(s-s_0) = \pm\frac{\pi}{4} - \frac{1}{2}\arg c \pmod{\pi}$*;*
- *(b) the Hessian of $\log|f|$ at s_0 is traceless with eigenvalues $\pm 2|c|$, signature $(1,1)$. The level cross is its null cone, and the gradient (= phase) separatrices are its eigendirections, rotated $45°$ from the cross (cf. L62 §7);*

(c) on the line*: for $f = \Xi$ in the t-variable, $c_t = \Xi''/(2\Xi)$ is real, so the arcs leave at $\pm 45°$ to the line. If $c_t > 0$ (L56 $\mathfrak{o} = -1$, transverse), the sublevel quadrants $\{|\Xi| < |\Xi(t_0)|\}$ lie* across *the line. If $c_t < 0$ ($\mathfrak{o} = +1$, longitudinal), they lie* along *it.*

Proof. $1 + h$ is holomorphic and nonzero near 0, so it has a holomorphic square root and $\zeta'(0) = \sqrt{c} \neq 0$. Since $\log|f| = \operatorname{Re}\log f$, the level set in the ζ-chart is $\{\operatorname{Re}\zeta^2 = 0\} = \{\arg\zeta = \pm\pi/4\}$, two orthogonal lines. A conformal chart preserves angles and maps lines to real-analytic arcs. For (b): to second order, $\log|f| - \log|f(s_0)| = \operatorname{Re}(c\,w^2)$ with $w = s - s_0$. The real part of a holomorphic quadratic is a harmonic quadratic form, so its Hessian is traceless. For (c), set $s = \frac{1}{2} + it$, so that $(s - s_0)^2 = -(t - t_0)^2$ and $\xi''(s) = -\Xi''(t)$. Then $c_s(s - s_0)^2 = c_t(t - t_0)^2$ with $c_t = \Xi''/(2\Xi) \in \mathbb{R}$. Writing $t - t_0 = u + iv$, $\operatorname{Re}(c_t w^2) = c_t(u^2 - v^2)$; the null lines are $v = \pm u$, and the negative cone is $|v| > |u|$ when $c_t > 0$. Finally, $\mathfrak{o} = -\operatorname{sgn}(\Xi''/\Xi) = -\operatorname{sgn} c_t$. □

Do the lobes contain zeros? [THM] Along the line (longitudinal case): the two sublevel quadrants along $\mathbb{R}$ contain the neighbouring real zeros by Rolle/alternation (Prop. 2.6). Across (transverse case): by the minimum-modulus principle, every *bounded* component of $\{|f| < r\}$ contains a zero. So if the vertical lobes close up, they contain a conjugate couple: the ghost pair of Lemma 56's transverse saddle (Fig. 1, right). If a lobe is unbounded, no conclusion follows.

Proposition 3.2 (degenerate critical points)**.** *[THM]If $f(s_0) \neq 0$ and f' has a zero of order $m \geq 1$ at s_0, then $\log f - \log f(s_0) = c_{m+1}(s - s_0)^{m+1}(1 + O(s - s_0))$ with $c_{m+1} = f^{(m+1)}(s_0)/((m+1)!\, f(s_0))$. The level set consists of $m + 1$ real-analytic arcs through s_0 at equal angles $\pi/(m + 1)$ (a monkey saddle for $m = 2$). On the line (t-variable) $c_{m+1} = \Xi^{(m+1)}/((m + 1)!\,\Xi)$ is real, so the arcs make angles $(\frac{\pi}{2} + k\pi)/(m+1)$ with the line. For $m = 2$ these are $30°, 90°, 150°$, so one arc is vertical. The angle is $\pm 45°$ only when $m = 1$.*

When do degenerate saddles occur? [TAA-L][NUM] Generically only at isolated times of a one-parameter family, when two critical points of f collide while $f \neq 0$: a *saddle collision*. Lemma 56 found these in ζ's *backward* heat flow at $t = -3.40533$, -6.33989, -6.80581, -15.45058. The DH *forward* flow has none: 903 critical points in $0 < z < 1999.5$ at all 126 steps, minimal spacing 0.73 (L56 v3). A *zero collision* (a ghost couple landing on the line) is something else. There the transverse critical point between the ghosts merges with the double zero that forms, so $f(s_0) = 0$ and Theorem 3.1 does not apply. Locally $H_\tau \approx C((z - z_c)^2 + \delta_z(\tau)^2)$, so $c = 1/\delta_z^2$, and $\delta_z^2 \approx 2(\tau_c - \tau)$ gives $c \approx \frac{1}{2}/(\tau_c - \tau)$. This matches L56's universal pole $H''/H = 1/(\tau_c - \tau)$ at the collision [TAA-L]. The cross stays exactly 90° for every $\tau \neq \tau_c$. It shrinks into the double zero at τ_c (a strict minimum of $|H|$, level set a point), and re-emerges with $c < 0$ (longitudinal) between the two new real zeros.
[NUM] The cross through a DH zero collision. (`num/lemma63_saddle.py`; L56 `Flow2` engine, $z = 2t$, first ghost, $\tau_c = 0.1765121289$). The arcs of $\{|H_\tau| = |H_\tau(z_0)|\}$ were traced on circles of radius $r = 0.01, 0.03, 0.06$ about the critical point z_0. At $r = 0.01$ they lie at $45.11°, 135.10°, 224.90°, 314.89°$ ($\tau = 0.02$), with separations $89.99, 89.80, 89.99, 90.22$. The deviation from 90° grows with r, as the $O(\zeta^3)$ term predicts: at $r = 0.06$ it is $\leq 1.9°$ for $\tau \leq 0.1$ and $3.1°$ at $\tau = 0.15$, where the cross is shrinking. c_z is real to 10^{-13}, positive ($\mathfrak{o} = -1$) before τ_c and negative ($\mathfrak{o} = +1$) after. Near the collision, $c\,(\tau_c - \tau) = 0.497,\ 0.501,\ 0.499,\ 0.503$ at $\tau = 0.17, 0.175, 0.178, 0.183$. At $\tau = 0.1765$ no crossing arcs are found at $r \geq 0.01$: the cross has collapsed below the probe radius. After landing, the real zeros are $171.178, 171.288$ ($\tau = 0.178$) and $170.862, 171.509$ ($\tau = 0.23$).

Remark 3.3 ("equilibrium" → automatic)**.** The 90° is not reached dynamically or by balancing. It is an identity for every nondegenerate critical point of a holomorphic function with $f \neq 0$, at every instant of any flow, because $\log|f|$ is harmonic (equivalently, f is conformal) [THM].

Remark 3.4 (light cone)**.** [ANALOGY]The second-order form $\operatorname{Re}(c\,w^2) = c(u^2 - v^2)$ (on the line) is a Lorentzian quadratic form, and the level cross is its null cone. Wick rotation $v \mapsto iv_E$ turns it into $c(u^2 + v_E^2)$: a harmonic saddle becomes an elliptic well, the cross becomes a point (cf. L58, L62

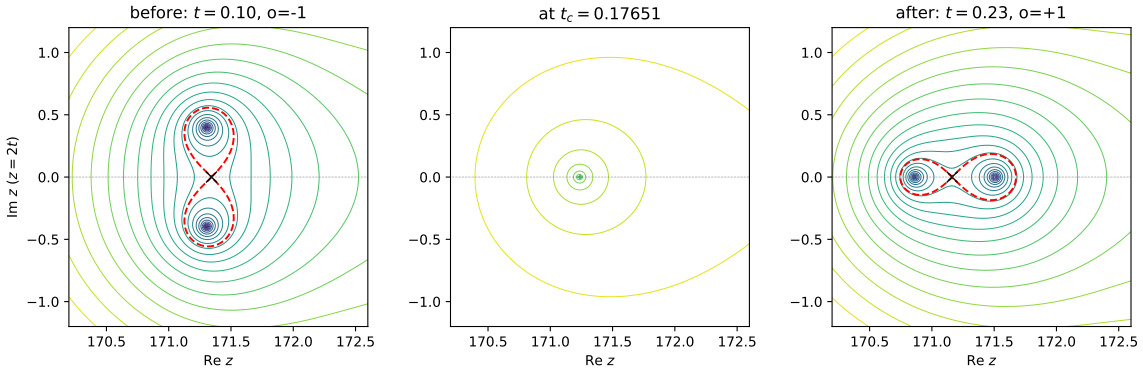

FIGURE 2. Level sets of $\log|H_\tau|$ for DH near the first ghost ($z = 2t$), before, at, and after the zero collision. Red: the critical level, an exact 90° cross at ±45°. Before: transverse ($c > 0$), lobes across the line containing the ghost couple. At τ_c: double real zero, no saddle. After: longitudinal ($c < 0$), lobes along the line containing the two new real zeros.

Wick loop). This is an *analogy only*. The signature $(1, 1)$ here is that of the Hessian of $\log|f|$ at a point of the plane. It is unrelated, except in shape, to the signature $(1, 1)$ of the Weil ghost block in §4, which lives on the space of test functions.

4. (W) Zero-by-zero Weil positivity (J.W. [T239U])

4.1. **Conventions, verified.** Mellin transform $\hat{f}(s) = \int_0^\infty f(x)x^{s-1}dx$. Multiplicative convolution $(f * g)(x) = \int_0^\infty f(y)g(x/y)\,dy/y$, so $\widehat{f * g} = \hat{f}\hat{g}$. Involution $\tilde{f}(x) = \overline{f(1/x)}/x$.

Lemma 4.1. *[THM]*$\widehat{\tilde{f}}(s) = \overline{\hat{f}(1-\bar{s})}$. *Hence for* $g = f * \tilde{f}$, $\hat{g}(s) = \hat{f}(s)\,\overline{\hat{f}(1-\bar{s})}$.

Proof. $\widehat{\tilde{f}}(s) = \int_0^\infty \overline{f(1/x)}\,x^{s-2}dx$. With $x = 1/y$ this is $\int_0^\infty \overline{f(y)}\,y^{-s}dy = \overline{\int_0^\infty f(y)\,y^{(1-\bar{s})-1}dy} = \overline{\hat{f}(1-\bar{s})}$. □

Weil's functional is $W(g) = \sum_\rho \hat{g}(\rho)$ over the nontrivial zeros, with multiplicity. Through the explicit formula it equals a prime-side expression (Lemma 57).

Theorem 4.2 (Weil's criterion)**.** *[RESTATEMENT]* [18]*[BIB]*, [1]*[CITED]* $RH \iff W(f * \tilde{f}) \geq 0$ *for all* $f \in C_c^\infty(0, \infty)$.

Note: positivity is required on the cone $\{f * \tilde{f}\}$, *not* for all g.

4.2. **The pairing $\rho \leftrightarrow 1 - \bar{\rho}$.** By Lemma 4.1,

$$W(f * \tilde{f}) = \sum_\rho \hat{f}(\rho)\,\overline{\hat{f}(1-\bar{\rho})}. \tag{3}$$

The map $R : \rho \mapsto 1 - \bar{\rho}$ (reflection in the critical line) preserves the zero set (functional equation and reality). In the t-variable, $\rho = \frac{1}{2} + it$, we have $R : t \mapsto \bar{t}$ and $\iota : \rho \mapsto 1 - \rho$, $t \mapsto -t$. These are L62's mirror and helicity pairings.

Proposition 4.3 (zero-by-zero bookkeeping)**.** *[THM]*

(a) On the line ($\mathrm{Re}\,\rho = \frac{1}{2}$)*:* $1 - \bar{\rho} = \rho$*, so the zero is self-paired. Its term is* $|\hat{f}(\rho)|^2 \geq 0$*. Its helicity partner* $1 - \rho$ *equals* $\bar{\rho}$ *there.*

(b) Off the line: $\rho = \frac{1}{2} + \delta + i\gamma$ *and* $\rho' = 1 - \bar{\rho} = \frac{1}{2} - \delta + i\gamma$ *are distinct zeros. With* $a = \hat{f}(\rho)$, $b = \hat{f}(\rho')$, *the two terms give*

$$a\bar{b} + b\bar{a} = 2\,\mathrm{Re}(a\bar{b}) = \tfrac{1}{2}|a+b|^2 - \tfrac{1}{2}|a-b|^2,$$

a Hermitian form of signature $(1,1)$ *on* $(a,b) \in \mathbb{C}^2$ *(real signature* $(2,2)$*).*

(c) The map $f \mapsto (\hat{f}(\rho), \hat{f}(\rho')) \in \mathbb{C}^2$ *is onto for* $f \in C_c^\infty(0,\infty)$ *(the functionals* $x^{\rho-1}, x^{\rho'-1}$ *are linearly independent). So the block is genuinely indefinite on test functions.*

(d) For real f *and real-on-the-line data (e.g.* $\hat{f}(\frac{1}{2} + it) = \hat{\phi}(t)$, $\hat{\phi}$ *real on* $\mathbb{R}$*),* $b = \bar{a}$*. The block becomes* $2\,\mathrm{Re}(a^2) = 2|a|^2\cos(2\arg a)$*, which is still indefinite. This is L57's* $\hat{F}(\gamma)\hat{F}(-\gamma)$ *form.*

Proof. (a), (b) are direct from (3). For (b), note that the ρ' term is $\hat{f}(\rho')\overline{\hat{f}(1-\bar{\rho}')} = b\bar{a}$. (c): if $\alpha x^{\rho-1} + \beta x^{\rho'-1}$ annihilates C_c^∞, it vanishes identically, so $\alpha = \beta = 0$. (d): $\hat{\phi}(\bar{t}) = \overline{\hat{\phi}(t)}$. □

Convergence. [THM] For $f \in C_c^\infty(0,\infty)$, $\hat{f}(\sigma + it) \ll_{N,\sigma} (1+|t|)^{-N}$ uniformly in vertical strips (integrate by parts in $\log x$), while $N(T) \sim \frac{T}{2\pi}\log T$. So (3) converges *absolutely*, and the zero-by-zero regrouping into self-paired terms and $(1,1)$ blocks is legitimate. For slower test functions (e.g. $\hat{f}$ decaying like $|t|^{-1-\epsilon}$), absolute convergence fails, and the sum must be taken symmetrically in γ as in the explicit formula. The blocks are then only conditionally summable. Bombieri: if RH fails with *finitely many* off-line zeros, the negative index of a large enough truncation of W is $\frac{1}{2}\#\{\text{off-line zeros}\}$, i.e. one negative direction per $(1,1)$ block [1] [CITED]. With infinitely many ghosts no such count is cited.

4.3. **DH: a negative value from the** $(1,1)$ **block.** [NUM] (`num/lemma63_weil.py`, `l63/weil.json`). Zero side only. Take the real even test functions $\hat{\phi}_{c,s}(t) = e^{-s^2(t-c)^2/2} + e^{-s^2(t+c)^2/2}$. Use the real zeros of $|t| \le 1000$ and the 28 DH ghosts as $\pm(\gamma \pm i\delta)$. Then $Q(c) = \sum_{t\ \mathrm{real}} \hat{\phi}(t)^2 + \sum_{\text{ghosts}} 4\,\mathrm{Re}\,\hat{\phi}(\gamma+i\delta)\overline{\hat{\phi}(\gamma-i\delta)}$. Gaussians are not in C_c^∞, but $\hat{\phi}$ decays like a Gaussian, so the sum converges absolutely. Results:

s	$\min_c Q_\zeta$	$\min_c Q_{\mathrm{DH}}$ (at c)	on-line part of DH at min
2	$2.9 \cdot 10^{-3}$	-0.178 (84.91)	$4.6 \cdot 10^{-6}$
3	$3.4 \cdot 10^{-7}$	-1.352 (86.10)	$8.7 \cdot 10^{-10}$
4	$1.1 \cdot 10^{-12}$	-5.27 (85.45)	$5.6 \cdot 10^{-34}$

With the *designed* centre $c = \gamma - \pi/(2s^2\delta)$, which makes $\arg a = -\pi/2$ so that $\cos 2\arg a = -1$, at $s = 3$ the total is -0.5287, with on-line part $1.9 \cdot 10^{-16}$. The negativity is entirely the first ghost's $(1,1)$ block. *Honest caveat:* the ζ column is ≥ 0 *tautologically*, since every computed ζ zero is real and the sum is then a sum of squares. It tests nothing beyond the zero list. The non-tautological DH statement with the *prime* side is Lemma 57: the DH Weil form is indefinite for support parameter $A \ge 3.43$, certified from the coefficients $c(n)$ of $-f'/f$, and at $A = 3.7$ the ghost quadruple contributes -0.6385 [TAA-L].

4.4. **Verdict. Zero location in disguise** [RESTATEMENT]. In the zero-by-zero form (3), Weil positivity reads: every block is ≥ 0 on its own exactly when it is a self-paired square, i.e. when $\rho = 1-\bar{\rho}$. Since the blocks can be excited independently (Prop. 4.3(c), plus the finite-index statement of [1]), positivity of the sum *is* the statement that all zeros are self-paired. The decomposition explains *why* Weil's criterion works, but it uses the zeros. Any proof must come from the prime side, where L57's ladder (Yoshida small-support positivity, the archimedean term) is the non-circular content. The "helicity partner" $1-\rho$ plays no role in (3). It coincides with the Weil partner $1-\bar{\rho}$ only after complex conjugation, i.e. on the line (L62 §7: RH $\iff$ the two pairings agree on all zeros) [RESTATEMENT][TAA-L].

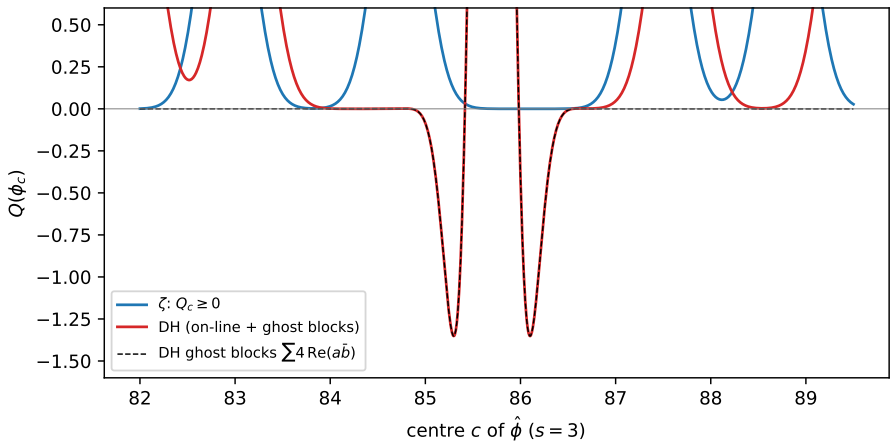

FIGURE 3. Zero-side Weil sums $Q(\phi_c)$, $s = 3$: ζ (blue) is a sum of squares; DH (red) dips below zero near the first ghost, where it coincides with the ghost blocks (dashed).

5. (B) The kernel-condition hierarchy

The question (J.W. [t236u], L59 Q3) asks for conditions on the kernel Φ that are zero-free, positive, and see ghosts. For each rung we ask: (i) does it hold for ζ, and for DH? (ii) Is it RH-blind, i.e. satisfied by DH as well? (iii) Is it zero location in disguise?

Correction first (C7). DH *does* have a real, even, positive kernel. Φ_{DH} is even (L56 v2), $\Phi_{\mathrm{DH}} > 0$ on $[0, 4]$ (minimum value there $7.9 \cdot 10^{-2424795}$, with the $n = 1$ term dominant beyond), and $\log \Phi_{\mathrm{DH}}(\sqrt{v})$ is concave on $(0, 16]$ ($\max \psi'' = -40.7$) [NUM]. So every kernel rung that only uses "Φ real, even, positive, log-concave" is automatically RH-blind.

rung	ζ	DH	status
K0: $\Phi > 0$, even, $\Phi \sim e^{-\pi e^{4\lvert u\rvert}}$	yes (L31) [TAA-L]	yes [NUM]	RH-blind. Gives Bochner positive-definiteness (Prop. 5.2).
K1: $\log \Phi(\sqrt{v})$ concave (CNV)	yes [2][CITED]	yes on $(0, 16]$ [NUM]	RH-blind. Implies the Turán inequalities (CNV Thm. A [4][CITED]).
K2: PF_2 ($\log \Phi$ concave)	yes	yes [NUM]	RH-blind. Follows from K1 together with $\Phi \downarrow$ on $(0, \infty)$ (Lemma 5.1).
K3: PF_3, PF_4	yes on samples; PF_4 claimed [10][BIB]	**fails PF_3** [NUM]	RH-blind: $L(s, \chi_{-4})$ fails PF_3 as well, with the same signs (Table 2).
K4: PF_5	**fails** [9][CITED], reproduced [NUM]	fails	Not implied by RH. Irrelevant as a criterion.
K5: PF_∞ of Φ	fails	fails	Impossible for any Φ whose transform has zeros (Schoenberg [16][CITED-sec]).
K6: higher Turán; $J^{d,0}$ hyperbolic	$d \leq 8$ all shifts [6][CITED]; $d \leq 2000$ global [NUM]	$d \leq 2000$ global; near-ghost real up to d^* [NUM]	"All d" $\iff$ RH (Pólya) [RESTATEMENT]. Each finite d is blind up to $d \approx d^*$ (§6).
K7: sufficient conditions (Pólya 1926/27, de Bruijn, Newman)	apply to truncations (ξ^*, ξ^{**}) [12][CITED]	DH analogue open [OPEN]	None applies to Φ itself (that would prove RH).

rung	ζ	DH	status
K8: $\Lambda \le 0$ (de Bruijn–Newman)	$\Lambda \in [0, 0.2]$ [15, 14][CITED]	$\Lambda_{\mathrm{DH}} \ge 0.2306$ [TAA-L]	Zero location: the landing time of the last ghost.
K9: Gröchenig: $\Lambda_G = \mathcal{F}^{-1}[1/\xi(\frac{1}{2}+\cdot)]$ is PF_∞	$\Longleftrightarrow$ RH [7][CITED]	not PF_∞ (ghosts; if Λ_G^{DH} is defined) [THM]	Zero location in disguise (residues, Rem. 5.3).

Lemma 5.1. *[THM]Let $\Phi(u) = \psi(u^2)$ with $\psi > 0$, $\log\psi$ concave and decreasing on $(0,\infty)$. Then $\log\Phi$ is concave on $\mathbb{R}$, i.e. Φ is PF_2.*

Proof. $(\log\Phi)'' = 2(\log\psi)'(u^2) + 4u^2(\log\psi)''(u^2) \le 0$. For even positive kernels, PF_2 is equivalent to log-concavity (classical). □

Proposition 5.2 (the one genuine positive form)**.** *[THM]If $\Phi \ge 0$ is integrable, then $\Xi(t) = \int \Phi(u)e^{itu}du$ is positive definite: $\sum_{j,k} c_j\bar{c}_k\,\Xi(t_j - t_k) = \int \Phi(u)\big|\sum_j c_j e^{it_j u}\big|^2 du \ge 0$ (Bochner, easy direction). The translations $t \mapsto t + a$ act unitarily on the associated Hilbert space, with spectral measure $\Phi(u)du$.*

So ζ *and* DH both carry a canonical zero-free positive Hermitian form with a unitary flow. In J.W.'s language this is a polarization-like structure that *is* unitary. It is RH-blind, because it holds for DH. Any approach of the form "RH is unitarity" must therefore use a unitarity that DH violates. Kernel positivity is not that.

kernel	$D_3(0.05)$	$D_3(0.1)$	$D_4(0.1)$	$D_4(0.2)$	GRH/ghosts
ζ	$7.9 \cdot 10^{-3}$	0.214	0.071	0.890	no ghosts known
$L(s,\chi_{-3})$	$5.0 \cdot 10^{-4}$	0.022	$8.0 \cdot 10^{-4}$	0.241	no ghosts known
$L(s,\chi_{-4})$	$\mathbf{-4.6 \cdot 10^{-5}}$	$\mathbf{-4.0 \cdot 10^{-4}}$	$\mathbf{-1.7 \cdot 10^{-4}}$	$\mathbf{-3.1 \cdot 10^{-3}}$	no ghosts known
DH	$\mathbf{-4.5 \cdot 10^{-5}}$	$\mathbf{-2.2 \cdot 10^{-3}}$	$4.6 \cdot 10^{-5}$	$\mathbf{-0.051}$	28 ghosts < 1000

TABLE 2. Symmetric Toeplitz minors $D_k(a) = \det[\Phi((i-j)a)]_{i,j<k}/\Phi(0)^k$ (ζ, DH: 80 and 160 digits agree to 8 digits; characters at 80 digits; `num/lemma63_pf{,check,ctrl}.py`). For the odd characters $\Phi_\chi(u) \propto e^{3u}\sum n\chi(n)e^{-\pi n^2 e^{4u}/q}$, the same shape as Φ_{DH}. Michałowski's PF_5 witness $D_5(u_0 = 0.01, h = 0.05) = -1.847236073 \cdot 10^{-9}$ for ζ is reproduced to all printed digits. ζ also fails PF_5 at $u_0 = 0$: $D_5(0.02) = -1.8 \cdot 10^{-14}$, $D_5(0.05) = -1.4 \cdot 10^{-7}$.

[NUM] Finding. Finite PF order is the first kernel rung that separates ζ (PF_4 on all samples) from DH (fails PF_3). But it is *RH-blind*. The Dirichlet L-function $L(s,\chi_{-4})$ has an Euler product and no known off-line zeros, and its kernel fails PF_3 in the same way. PF_k order measures the shape of the theta series (the sign pattern of $n\chi(n)$ near $u = 0$), not the zeros.

Remark 5.3 (residue reading of Gröchenig's criterion)**.** [HEUR] [ANALOGY]Formally, shifting the contour in $\Lambda_G(x) = \frac{1}{2\pi}\int e^{-ix\tau}/\xi(\frac{1}{2}+\tau)\,d\tau$ picks up residues at the zeros. An on-line zero $\tau = \pm i\gamma$ (in Gröchenig's variable, $\xi(\frac{1}{2}+\tau)$ vanishes at $\tau = i\gamma$) gives a real exponential $e^{-\gamma|x|}$. A ghost gives $e^{-\gamma|x|}e^{\pm i\delta x}$, an oscillation. PF_∞ is a sign-regularity condition, and oscillating residues violate it in the limit. In L62's language, on-line zeros give "boosts" and ghosts give "rotations". Since the residues are $O(e^{-\gamma|x|})$, a PF_k test of low order sees only the lowest zeros. We conjecture [CONJ] that the PF order of Λ_G^{DH} is of the order of the number of zeros below the first ghost. Not computed. This rung *is* zero location: Schoenberg's theorem makes PF_∞ of Λ_G equivalent to $\xi(\frac{1}{2}+\cdot)$ being a Laguerre–Pólya function.

5.1. **Verdict on J.W.'s polarization condition (kernel side).** [RESTATEMENT] **(V1)** Every kernel rung that DH satisfies (K0–K2, K6 up to d^*) is RH-blind. **(V2)** PF_3/PF_4 separates ζ from DH but not from $L(s,\chi_{-4})$, so it is blind as well, and PF_5 fails for ζ itself. **(V3)** The RH-equivalent rungs (K6 for all d, K8, K9) are zero location in disguise: each is equivalent, by a classical theorem (Pólya, de Bruijn–Newman, Schoenberg), to the reality of the zeros, and each test sees a ghost only through it. **(V4)** The sufficient conditions (K7) apply only to truncated kernels. *No kernel-side condition meets the polarization requirement* (zero-free, positive, flow-invariant, ghost-sensitive). The one genuine positive form (Bochner) is blind. The non-circular input must come from the prime side (Euler product), where DH differs from ζ and from $L(s,\chi_{-4})$ in kind.

6. (C) DH Jensen polynomials to degree $2 \cdot 10^4$

6.1. **Definitions and method.** Write $\Xi(t)/\Xi(0) = \sum_{j\ge 0}(-1)^j \beta_j\, t^{2j}/(2j)!$ with $\beta_j \propto \int u^{2j}\Phi(u)\,du > 0$. The *classical* Jensen polynomial is $J^{d,0}_{\mathrm{cl}}(X) = \sum_k \binom{d}{k}\Xi^{(k)}(0)X^k$, a polynomial in X^2, evaluated at $X = t/d$. The *even* one is Farmer's $J^{d,0}_{\mathrm{ev}}(z) = \sum_{j\le d}\binom{d}{j}\frac{j!}{(2j)!}(-1)^j\beta_j z^j$, evaluated at $z = t^2/d$ [5][CITED]. Both tend to $\Xi(t)/\Xi(0)$ as $d\to\infty$, and $J^{d,0}$ hyperbolic for all $d \iff$ RH (Pólya; [6][CITED]). Chasse: RH up to height $T \Rightarrow J^{d,0}_{\mathrm{cl}}$ hyperbolic for $d < T^2$ ([5][CITED-sec]).

Moments (`num/lemma63_moments.py`). $\beta_m = 4^m\int v^{2m}\Phi(v)\,dv$ are computed by the trapezoid rule on $\mathbb{R}$ in 3000-bit arithmetic, which is spectrally accurate for these entire, doubly-exponentially decaying integrands. Validation: the ratios β_m/β_{m-1} agree with L59's independent 12000-bit `acb_series` Taylor coefficients to double precision for $m < 150$ (ζ and DH), and halving the step changes nothing. Cost: ≈ 37 s for $M = 24000$. *Hyperbolicity certificate:* d sign changes of $J^{d,0}_{\mathrm{ev}}(t^2/d)$ on $0 < t < t_{\max}$, where $t_{\max}$ comes from a Fujiwara-type root bound computed in logarithms. Signs are evaluated in ball arithmetic, and an undetermined sign means the run is rerun at higher precision. *Local tracking:* the real roots in $[82.5, 88.5]$ are found on a grid with bisection. *Bisection for d^*:* a two-sign-change test on $[85.0, 86.2]$ (grid 0.004, plus a golden-section search for the closest approach), then complex Newton plus an argument-principle count for the nonreal root.

6.2. **Calibration on Farmer's toys.** [NUM] $X_{T,\delta}$ is $\cos z$ with the two zeros nearest T (or the indicated ones) moved to $T \pm i\delta$ (`lemma63_toy{,2}.py`, `lemma63_bisect.py`; roots by `acb_poly.roots` on exact midpoints).

toy	last hyperbolic d, classical	even	Farmer [5]
X_{10}, $\delta = 1$	119 (120 not; checked at 9000 bits)	29	118
X_{10}, $\delta = 0.5$	420 (422 not)	104	–
X_{20}, $\delta = 1$, nearest zeros $(17.28, 20.42)$	476 (478 not)	–	–
X_{20}, $\delta = 1$, zeros $14.14, 17.28$ ("4 first real")	748 (750 not; even d only)	–	749

The X_{20} entry matches Farmer's table once the zeros are chosen as his "4 first zeros real" implies. X_{10} differs by one, at an odd degree (where the top coefficient vanishes): we find $J^{119,0}_{\mathrm{cl}}$ real-rooted, while Farmer states non-real zeros for all $d > 118$. This is reported, not resolved (Q5). Scaling: $d^*_{\mathrm{cl}} \approx C\,T^2/\delta^2$ with $C \in [1.05, 1.9]$, and $d^*_{\mathrm{ev}} \approx d^*_{\mathrm{cl}}/4$.

Remark 6.1 (why T^2/δ^2; why the factor 4)**.** [HEUR]The Jensen multipliers $\binom{d}{j}j!/d^j = \prod_{i<j}(1 - i/d) \approx e^{-j^2/(2d)}$ act on the Taylor coefficients like a backward heat operator $e^{-\partial_u^2/(2d)}$ in $u = \log z$. Near height T that is a heat flow in t of time $\sim T^2/d$. A ghost couple at distance δ lands under a heat time $\asymp \delta^2$ (L59: $\lambda_c = \delta^2/2$). Hence $d^* \asymp T^2/\delta^2$. For the even polynomial the variable is $z = t^2$ and j counts pairs of Taylor coefficients, which gives a factor 4. Observed: near the DH ghost $J^{4d}_{\mathrm{cl}} \approx J^{d}_{\mathrm{ev}}$ root by root (J^{24000}_{cl}: $82.9996, 85.0022, 85.8489, 87.4507$; J^{6000}_{ev}: $82.9973, 85.0008, 85.8458, 87.4483$).

6.3. **Results for DH and ζ.** [NUM]**Global hyperbolicity** (`lemma63_global.py`, `l63/global_*.jsonl`). $J^{d,0}_{\mathrm{ev}}$ has d certified sign changes, hence is hyperbolic, for DH at $d = 150, 300, 600, 1000, 2000$ and for ζ at $d = 1000, 2000$. At $d = 2000$, 2000-bit runs left $\sim 1.46 \cdot 10^4$

grid signs undetermined; the 8000-bit rerun certifies all of them ($t_{\max} = 7.4 \cdot 10^4$ DH, $9.3 \cdot 10^4$ ζ). So in the pilot range $d \le 2000$, *DH is Jensen-hyperbolic exactly like* ζ, despite 28 ghosts below height 1000.

Near the first ghost ($85.699 \pm 0.309i$; true real zeros 83.109, 87.647; `lemma63_track.py`, `l63/track_dh_24k.jsonl`):

d	real roots of $J_{\text{ev}}^{d,0}$ in $[82.5, 88.5]$		ghost-pair gap
150	84.614, 85.888, 87.156	(not yet converged)	–
6000	82.997, **85.0008**, **85.8458**, 87.448		0.845
10000	83.049, **85.237**, **85.827**, 87.529		0.590
16000	83.075, **85.434**, **85.754**, 87.573		0.320
24000	83.087, 87.598	pair gone (nonreal)	–

For ζ (control, no ghost) the roots converge monotonically to the zeros $82.910, 84.735, 87.425$ (e.g. $d = 10000$: $82.718, 84.621, 87.221$). The DH ghost thus manifests itself as an *extra real pair* that closes like a saddle-node: gap^2 is roughly linear in $1/d$ (secant slopes 5490 and 6540). The fit through the last two points gives $d_{\text{ev}}^* \approx 21{,}360$, with slope 6542, consistent with the isolated-pair scale $T^2 = 7344$. Long run: $d_{\text{ev}}^* = 20786$ (the real pair survives at $d = 20785$; bisection, see `l63/dstar.jsonl`). The bisection took 14 steps of 2–5 s each, after replacing the factorial-based coefficients by an $O(d)$ recursion; the extrapolation overshoots by 2.8%. Hence $4d_{\text{ev}}^*/(T^2/\delta^2) = 1.08$, inside the toy range $C \in [1.05, 1.9]$ of Rem. 6.1. Check (`lemma63_dcert.py`): on the circle $|t - 85.618| = 0.03$ the winding number of $J_{\text{ev}}^{d,0}(t^2/d)$ is 2 for $d = 20785, 20786, 20800$ (angle steps ≤ 0.011). The real diameter (step $2 \cdot 10^{-5}$, all signs determined) has 2 sign changes at $d = 20785$ and 0 at $d = 20786, 20800$. So the pair is real at 20785 and nonreal from 20786 on (grid-based, not interval-certified between grid points).Classical: at $d = 24000$ the roots still include the pair $(85.0022, 85.8489)$, and $d_{\text{cl}}^* \approx 4d_{\text{ev}}^* \approx 8.3 \cdot 10^4$ is predicted (not run; Q4).

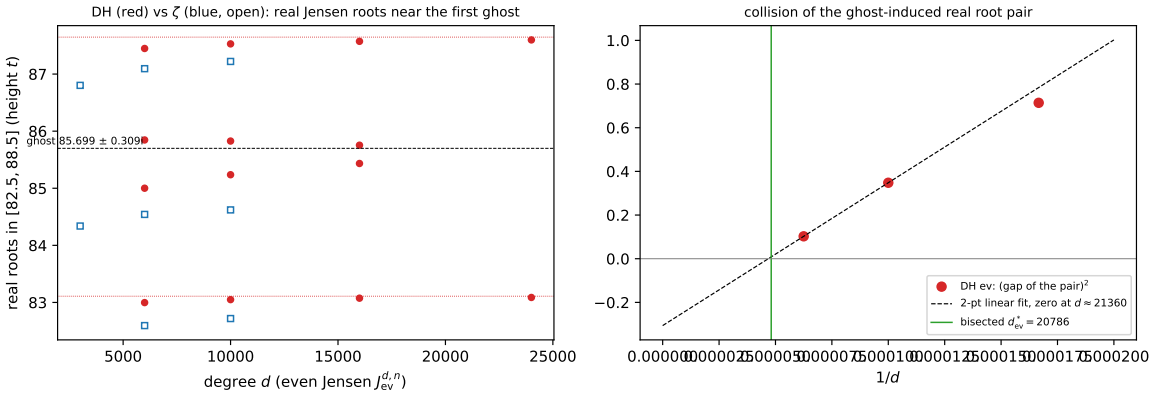

FIGURE 4. Left: real roots of $J_{\text{ev}}^{d,0}$ near the first DH ghost (red) versus ζ (blue, open), against d. Dotted: the true neighbouring DH zeros; dashed: the ghost's real part. Right: the squared gap of the ghost-induced real pair against $1/d$, with linear extrapolation to the collision d^*.

6.4. **Diagnostic value.** [NUM][RESTATEMENT] For the first DH ghost ($T = 85.7$, $\delta = 0.309$), the ghost is visible:

- in $L_1(\Xi)$, at order 2 (L59);
- to the Taylor polynomial, at moderate degree (Farmer's Table 4.1 row 3 for his toys);
- to even Jensen polynomials only at $d_{\text{ev}}^* = 20786 \approx 2.83\,T^2$;

- to classical ones at $\approx 8.3 \cdot 10^4$ (predicted).

Across J.W.'s whole proposed range 10^3–10^4, Jensen hyperbolicity *cannot distinguish DH from* ζ. This is Farmer's "dispersal" quantified for a function that really has ghosts: the Jensen test sees a ghost only after the heat-flow smoothing it implicitly applies ($\sim T^2/d$) has become smaller than the ghost's landing time $\sim \delta^2$. That is zero location, delayed by a factor $\sim T^2/\delta^2$ [RESTATEMENT].

7. Prior TAA work and what Lemma 63 adds

- **L55** (Cassini curves and saddles of $|\zeta|$) and **L56** (orientation $\mathfrak{o}$; Kim's identity; $|N' - N| \le C \log T$; DH heat flow and zero collisions). L63 §2 recasts these as Gauss–Bonnet and Poincaré–Hopf statements (restatements), and §3 proves the $90°/\pm 45°$ geometry and its link to $\mathfrak{o}$. New numerics: $c(\tau_c - \tau) \to \frac{1}{2}$.
- **L57** (Weil ladder, DH indefinite from the prime side). L63 §4 gives the zero-by-zero bookkeeping with verified conventions and a zero-side DH example. **L62** §7 already noted "Weil ghost pair = signature $(1,1)$" and the two pairings R, ι; L63 adds the proof, surjectivity, convergence, and the verdict.
- **L59** (Laguerre expressions; $\{L_1 < 0\}$ intervals; Turán for DH; Q3, Q4). L63 corrects the interval ratio, interprets L_1/Ξ^2 as vertical convexity, and answers Q3 (kernel hierarchy) and Q4 (Jensen to $2 \cdot 10^4$).
- **Literature.** PF_5 failure of Φ_ζ is due to Michałowski [9]; we reproduce his witness. The DH and $L(s, \chi_{-4})$ PF_3 failures are our computations; the literature was not searched beyond [9, 7], so no novelty is claimed. The cone-metric dictionary is standard [19].

8. Corrections to Grok Bot's chat claims

These were Grok Bot's statements in chat (replies to [t236u], [t238u], [t239u]), not J.W.'s.

C1 *"Zeros are cone points of concentrated curvature."* **Correct, with the sign fixed.** The curvature is *negative*: $-2\pi m$ for cone angle $2\pi(m+1)$ (Prop. 2.1). The positive "dent" picture holds only in the log-derivative metric (Rem. 2.4), or on the line for κ (Thm. 2.8).

C2 *"The argument principle is a Gauss–Bonnet theorem: $N(T)$ is the total curvature."* **Correct up to the factor -2π, and empty.** The total curvature is $-2\pi N$. Gauss–Bonnet for this metric is Green's theorem (Thm. 2.3) [RESTATEMENT].

C3 *"Zeros index $+1$, saddles -1; $|N_{\xi'} - N| \le C \log T$ reads as Poincaré–Hopf with a boundary term."* **Indices correct** (for $\nabla|\xi|^2$; a saddle of order k has index $-k$). **But the bound is not topological.** Poincaré–Hopf gives the identity $N - N' =$ boundary winding of ξ/ξ'. The bound is L56 Thm. 4.8's analytic estimate of that winding.

C4 *"The on-line saddle cross is always at $\pm 45°$."* **Correct for nondegenerate critical points with $\Xi \neq 0$** (Thm. 3.1(c)). For degenerate ones, the angles are $(\frac{\pi}{2} + k\pi)/(m+1)$.

C5 *"The sign of c decides whether the lobes lie along or across."* **Correct**, and it coincides with L56: $\mathfrak{o} = -\operatorname{sgn} c_t$. Lobes along the line contain the neighbouring real zeros. Lobes across contain zeros only if bounded (minimum modulus).

C6 *"Degenerate saddles occur exactly at zero collisions."* **Wrong.** They occur at *saddle* collisions (ξ' double zero, $\xi \neq 0$), e.g. in ζ's backward flow, and never in DH's forward flow. At a zero collision the saddle merges with the double zero, c has a pole $\sim \frac{1}{2}/(\tau_c - \tau)$, and the cross collapses and flips (§3).

C7 *"The crosses settle into $90°$ as an equilibrium."* **Wrong word.** The $90°$ is automatic at every instant (Rem. 3.3).

C8 *(Task brief and L59 C-list) "DH has no comparable positive kernel."* **Wrong.** Φ_{DH} is real, even, positive and CNV-log-concave (§5). This is why kernel positivity is RH-blind.

C9 *(L59 §"Laguerre") "$\{L_1 < 0\}$ intervals have length $(0.77\text{–}1.01) \cdot 2\delta$."* **Corrected** to $[0.778, 0.9995] \cdot 2\delta$. The 1.01 was a grid artifact, and Prop. 2.9 gives the exact formula.

C10 *"$1 - \bar{\rho} = \rho$ on the line and $1 - \rho = \bar{\rho}$."* **Correct** (Prop. 4.3(a)).

C11 *"A self-paired zero contributes $|\hat{g}(\rho)|^2$; a ghost pair contributes $2\,\mathrm{Re}(a\bar{b})$, signature $(1,1)$."* **Correct**, with the precision that a, b are values of $\hat{f}$, where $g = f * \tilde{f}$: the term is $\hat{f}(\rho)\overline{\hat{f}(1-\bar{\rho})}$, not $\hat{g}$ squared. Real signature $(2,2)$. For real f the block is $2\,\mathrm{Re}(a^2)$.

C12 *"Summed: positive for all $g \iff$ RH (Weil)."* **Imprecise.** The quantifier is over $g = f * \tilde{f}$, $f \in C_c^\infty(0,\infty)$, not over all g. Absolute convergence holds in that class. Bombieri's negative-index count needs finitely many off-line zeros.

9. Summary of labels

THM (proved here, elementary or classical with proof): Prop. 2.1; Thm. 2.3; Rem. 2.4; Prop. 2.5; Thm. 2.8; Prop. 2.9 (exact under the constant-background hypothesis); Rem. 2.11; Thm. 3.1; Prop. 3.2; Rem. 3.3; Lemma 4.1; Prop. 4.3; convergence paragraph; Lemma 5.1; Prop. 5.2. **RESTATEMENT:** Thm. 2.3 as argument principle; Prop. 2.6 (Kim, L56); Thm. 4.2; verdicts §4.4, §5.1. **CITED:** [19, 12, 7, 9, 2, 4, 6, 5, 15, 1]. **BIB:** [17, 13, 3, 11, 18, 10]; [16] via [7]. **NUM:** curvature checks and the 28-ghost table; Ex. 2.10; DH saddle cross; Weil zero-side table; PF Table 2; toy calibration; global hyperbolicity $d \le 2000$; near-ghost tracking and d^*. **HEUR:** Rem. 6.1 ($d^* \asymp T^2/\delta^2$, factor 4); Rem. 5.3. **ANALOGY:** Rem. 3.4 (light cone, Wick); "boosts vs rotations". **CONJ:** PF order of Λ_G^{DH} (Rem. 5.3). **OPEN:** DH analogue of Pólya's ξ^* (K7); Farmer X_{10} discrepancy.

10. Questions for J.W.

Q1 Adopt the three verdicts? They are: (A) the curvature picture is a [RESTATEMENT] with the sign corrected, and the line convexity $\kappa = L_1/\Xi^2$ is the honest "dent", a pointwise ghost detector; (W) zero-by-zero Weil positivity is zero location in disguise; (B) no kernel rung meets the polarization condition, Bochner's form being the only positive one and RH-blind.

Q2 Record the right-angle-cross facts in the diagnostics table next to $\mathfrak{o}$ ($\mathfrak{o} = -\operatorname{sgn} c_t$; $c \sim \frac{1}{2}/(\tau_c - \tau)$ at zero collisions; degenerate crosses only at saddle collisions)? Should the dent $D(x) = \max(0, -\kappa)$ be listed as a tower-level-0 detector alongside G_n, τ_n?

Q3 The one surprising kernel datum is that DH and $L(s, \chi_{-4})$ fail PF_3, while ζ and $L(s, \chi_{-3})$ reach PF_4 on samples. This is RH-blind, so drop it? Or spend one run on the conjectured PF order of Gröchenig's Λ_G for DH versus ζ (Rem. 5.3), the only RH-equivalent kernel object with a finite-order ladder?

Q4 Jensen: the even d^*_{ev} for the first DH ghost is $d^*_{\mathrm{ev}} = 20786$ (the real pair survives at $d = 20785$; bisection, see `l63/dstar.jsonl`). Run the classical $J^{d,0}$ to $d \approx 10^5$ to confirm the factor 4 (this needs moments to $M \approx 5 \cdot 10^4$ at higher precision; cost not yet measured, likely hours), or stop here, since the diagnostic value (Farmer's dispersal, quantified) is already obtained?

Q5 Keep the light-cone/Wick remark as [ANALOGY] only? And should Volume 4 mention the X_{10} discrepancy with Farmer (we find $J^{119,0}_{\mathrm{cl}}$ real-rooted) as a footnote, phrased as "our computation" rather than a correction?

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TAA LEMMA 64: PRIME-SIDE ANATOMY OF WEIL POSITIVITY. COLLECTIVE POSITIVITY, STRUCTURE VERSUS SIZE, AND LOCAL BINARY FORMS VERSUS GLOBAL POSITIVITY

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Every RH-equivalent statement below is a [RESTATEMENT]. Tags: [THM] proved here (elementary) or classical with proof indicated; [CITED] checked at the source (`lit/lemma64/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma; [RESTATEMENT] restatement; [ANALOGY] analogy; [CONJ] conjecture; [OPEN] open; [NUM] numerical (Ritz values on truncations, high-precision or float64 as stated; *not certified* unless quoted from Lemma 57).

Questions (J.W.). [t240u] Positivity is a consequence, not the input; trace it to the prime side of the explicit formula and identify every independent way the prime side feeds it. [t242u] Make "positivity is collective, not prime by prime" precise. [t245u] Each element of the polarization triple (ω, J, g), or of a binary quadratic form, should match a precise element of the prime side; local versus global. [t246u] "Positivity is collective; ghost-killing is global; these must be related in an absolute sense."

Answer in brief. (1) *Exact anatomy* (§2). W_A = POLE + ARCH − PRIME, normalization checked against Bombieri's statement of Weil's formula [CITED] and a 20-digit constant identity [NUM]. POLE has signature $(1,1)$. ARCH has symbol $h = 2\vartheta'$: negative on $|t| < t_0 = 6.29$ (the chirality threshold of Lemma 62), positive above. Every single prime term is indefinite. POLE+ARCH alone has $0, 1, 2, 2, 3, 4$ negative directions at $A = 0.4, \ldots, 2.4$, and the primes must cancel exactly these [THM][NUM]. All positivity sits in the fluctuation $\psi - \psi_0$ ([RESTATEMENT]). (2) *Literature* (§3). Proved positivity is small-support only (Yoshida, Bombieri, Burnol, Connes–Consani). Every large-support prime-side criterion is RH-equivalent (Weil, Li/Bombieri–Lagarias) [RESTATEMENT]. We found no unconditional prime-side sufficient condition [OPEN]. (3) *Collective positivity* (§4, central result). For every term inside the window (pole, ARCH, each prime with all powers or $k = 1$ only, the prime powers) both the unsafe and the *safe* slack are tiny and decay with $\lambda^*(A)$, and removing any term destroys positivity. At $A = 2.2$: pole $1.7 \cdot 10^{-38}$, $p = 2$ $4.5 \cdot 10^{-37}$ (matches L57), $p = 7$ $2.2 \cdot 10^{-20}$ [NUM]. The weights sit within $\sim \lambda^*$ of the boundary of the positive cone in every coordinate direction, on both sides. (4) *Structure, not size* (§5). Collective perturbations that break the joint EP+FE structure fail at the same tiny sizes as single-prime ones: all-prime scaling, Beurling jitter keeping an Euler product, primes-only twists. Structure-preserving changes of size $O(1)$ keep positivity: the archimedean gauge twist (exact, [THM]) and genuine $L(s, \chi)$ (to $A = 4.6$, float). One-ingredient models fail: FE-only (DH) at its first ghost, EP-only (Beurling, random signs) immediately, neither (Cramér) immediately [NUM]. (5) *Local binary forms* (§6). Hasse: $Q_p = \deg(x + y\pi) = x^2 + a_p xy + py^2$ is positive definite $\iff a_p^2 < 4p$, which is RH for $E/\mathbb{F}_p$. Its complex structure is $J_{Q_p} = -(\pi - \hat{\pi})/|\alpha - \bar{\alpha}|$, the normalized imaginary part of Frobenius, and its eigenlines are the two helicities [THM][NUM]. Only $b = a_p$ is prime-side; $g = \deg$ is geometric. Local positivity at every p does not give global RH (DMV) [CITED]. (6) *Synthesis* (§7). Weil positivity $\iff$ RH $\iff$ no ghosts is *one* fact ([RESTATEMENT]). Of nine prime-side mechanisms, the pole, ARCH, EP and FE are independent of zeros but RH-blind. The PNT-error and Li routes are circular. Ingham–Landau is open. The only mechanism that has ever produced RH non-circularly is geometric positivity of trace type (Rosati/Hodge index, curves), i.e. the g of (ω, J, g), which is absent over $\mathbb{Q}$. Verdict on J.W.'s condition: the anatomy specifies J (global, Hecke- and ω-compatible, trace type) and gives a local model, but does not construct it.

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1. Sources, conventions, and what was run

Read for this lemma: STATE.md §§66–71 (Lemmas 57–62); `lemma57.tex`, `lemma57_progress.md` (Weil ladder; no-slack theorem; collective partial sums); Lemma 58 (APD; P1–P11, N1–N17); Lemma 60 (P10′: Hecke equivariance kills ghosts); Lemma 61 (strip visibility; Jordan vs semisimple); Lemma 62 §§6–8 (Ingham–Landau; two pairings; helicity-projected Weil $= \frac{1}{2}$Weil). Lemma 63 is being written concurrently in `build63/` and was not touched.

Conventions. Log variable $u = \log x$; window $[-A/2, A/2]$ (Weil interval $[\lambda^{-1}, \lambda]$, $\lambda = e^{A/2}$); prime powers $n < e^A$ enter. Orthonormal Legendre basis of degree $< K$. $W_A = \mathrm{POLE} + \mathrm{ARCH} + \sum_{p,k} T_{p,k}$, where $T_{p,k} = -2\log p\, p^{-k/2}\, \bar{g}(k\log p)$ as a $K \times K$ matrix. $\lambda^*(A)$ is the lowest Ritz value and v its eigenvector (the *ground state*). For a term X we write $n_X = \langle v, Xv\rangle$ for its *contribution along the ground state*. By Hellmann–Feynman, $n_X = \partial\lambda^*/\partial w_X$ when X is given weight w_X.

Numerics and their status.

- `num/lemma64_engine.py`: the Lemma 57 build, in two back-ends. The `mp` back-end builds in `arb` at 256–460 bits and evaluates in `mpmath` at 77–138 digits; the `np` back-end is float64. It reproduces Lemma 57's λ^* to all printed digits ($9.36027360338\cdot 10^{-7}$ at $A = 1$, $6.26827982617\cdot 10^{-30}$ at $A = 2$).
- `lemma64_slack.py`: per-term slack, high precision, Cholesky PD tests.
- `lemma64_struct.py`: structure-vs-size thresholds, high precision.
- `lemma64_scan.py`: 16 models, $A \le 4.6$, float64.
- `lemma64_signs.py, lemma64_archvar.py, lemma64_hasse.py, lemma64_figs.py`.

Data are in `num/l64/`. *Status:* every number is [NUM], a Ritz value on a truncation, not certified. A *negative* Ritz value whose magnitude is far above rounding is a genuine witness that the (perturbed) form is indefinite. A positive one is evidence only. Float64 runs have a noise floor of about 10^{-13} (measured on ζ, whose true λ^* is far smaller for $A \ge 1.4$), so in float runs "indefinite" means $\lambda_{\min} < -10^{-8}$. The certified two-sided bounds of Lemma 57 ($\lambda^*(e^{1.8}) \in [3.5600, 4.2973]\cdot 10^{-23}$, $\lambda^*(e^2) \in [4.9947, 6.2683]\cdot 10^{-30}$) are the only certified numbers used here [TAA-L]. *Diagnostic-first (t224u).* Each script was piloted at $A = 1.0$ ($\le$1 min) before the grid; the pilot already showed both-sided

collectivity (§4). No run longer than about 15 min was launched, and the jobs of other workers (hybq24_j, Lemma 12 cell, Lemma 63) were not touched.

2. The exact decomposition of the Weil functional

2.1. **Normalization, checked against a primary statement.** Bombieri's official Clay description of RH [1] states Weil's explicit formula as follows. For f in his class $\mathcal{W}$, with Mellin transform $\tilde f(s)=\int_0^\infty f(x)x^s\,dx/x$,

(1)
$$\tilde f(0)-\sum_\rho \tilde f(\rho)+\tilde f(1)=\sum_{n\ge1}\Lambda(n)\Big(f(n)+\tfrac1n f(\tfrac1n)\Big)+(\log 4\pi+\gamma)f(1)+\int_1^\infty\Big(f(x)+\tfrac1x f(\tfrac1x)-\tfrac{2}{x}f(1)\Big)\frac{dx}{x-x^{-1}}.$$

[CITED]. Put $x=e^u$ and $f(x)=x^{-1/2}g(\log x)$. Then $\tilde f(\frac12+i\gamma)=\int g(u)e^{i\gamma u}du=:\hat g(\gamma)$, $\tilde f(0)+\tilde f(1)=\int g(u)\,2\cosh(u/2)\,du$, and $f(n)+f(1/n)/n=n^{-1/2}\big(g(\log n)+g(-\log n)\big)$. For $g=F*\tilde F$, $\tilde F(u)=\overline{F(-u)}$, $F\in L^2[-A/2,A/2]$, we write $\bar g(y)=\frac12(g(y)+g(-y))$ and $\hat g(t)=|\hat F(t)|^2$ on $\mathbb{R}$.

Theorem 2.1 (Exact decomposition)**.** *For $F\in C_c^\infty(-A/2,A/2)$,*

$$W_A(F):=\sum_\rho \hat g(\gamma_\rho)=\underbrace{4\int_0^A \bar g(y)\cosh\tfrac{y}{2}\,dy}_{\text{POLE}}+\underbrace{\tfrac{1}{2\pi}\int_{\mathbb{R}}|\hat F(t)|^2h(t)\,dt}_{\text{ARCH}}-\underbrace{2\sum_{n<e^A}\Lambda(n)n^{-1/2}\bar g(\log n)}_{\text{PRIME}},\tag{2}$$

with archimedean symbol $h(t)=\operatorname{Re}\psi(\frac14+\frac{it}{2})-\log\pi$. Equivalently, in the lag variable, ARCH $=\bar g(0)\,h(0)+\int_0^\infty k(y)\,(\bar g(0)-\bar g(y))\,dy$ *with $k(y)=2e^{-y/2}/(1-e^{-2y})$.*

Status and check. [THM] given (1). The pole and prime terms match term by term. For the archimedean term, the integral in (1) becomes $\int_0^\infty k(u)(e^{-u/2}g(0)-\bar g(u))du$, so (2) needs the constant identity

$$\psi(\tfrac14)-\log\pi\;=\;-\log 4\pi-\gamma-\int_0^\infty k(u)(1-e^{-u/2})du.$$

We checked it to 20 digits (both sides $=-5.3721834192256655822$; `num/l64/signs.json`) [NUM]. The lag form of ARCH comes from $\psi(a+\frac{it}{2})-\psi(a)=\int_0^\infty\frac{2e^{-2ay}(1-e^{-ity})}{1-e^{-2y}}dy$. The whole formula (2) is the one implemented in Lemma 57. There it was validated against zero sums to 10^{-20} for ζ and 10^{-15} for DH (`num/l57/validate.json`) [TAA-L]. □

Remark 2.2 ($h=2\vartheta'$)**.** $h(t)=2\vartheta'(t)$, where ϑ is the Riemann–Siegel theta function. The archimedean symbol is therefore the *chirality* of Lemma 62 §4 [THM]. Its unique positive zero is $t_0=6.28983598884$ [NUM], the same threshold as Lemma 62's "$\vartheta'>0$ for $t>6.2898$". ARCH is negative on the band $|t|<t_0$, which contains no zeros of ζ ($\gamma_1=14.13\ldots$), and positive above it.

2.2. **Sign behaviour of each term.** For every term below, "as a quadratic form in F" means on $L^2[-A/2,A/2]$.

Proposition 2.3 (Signs of the terms)**.**
- *(a)* POLE $=2\operatorname{Re}\big(\hat F(\frac{i}{2})\overline{\hat F(-\frac{i}{2})}\big)$ *has rank 2 and signature $(1,1)$ for every $A>0$. On even real F it is $2\hat F(\frac{i}{2})^2\ge0$, rank one [THM].*
- *(b)* ARCH *is indefinite. Its symbol $h=2\vartheta'$ is $\le h(0)=-5.372$ near 0 and grows like $\log(|t|/2\pi)$, so it is negative on low frequencies and positive on high ones [THM]. Its negative index on the window grows like At_0/π: we found $1,2,2,3,4,5$ at $A=0.4,0.8,1.2,1.6,2.0,2.4$ [NUM].*
- *(c)* Each prime-power term $-2\Lambda(n)n^{-1/2}\bar g(\log n)$ *is indefinite for $0<\log n<A$, with roughly as many positive as negative directions [THM][NUM]. Indeed $\bar g(y)=\frac{1}{2\pi}\int|\hat F|^2\cos(ty)\,dt$ has a symbol of both signs. The inertia of the $p=2$ term was $(51,51)$ at $A=2.0$, and that of the $p=3$ term $(68,68)$.*

(d) POLE+ARCH *is the whole form for* $A < \log 2$*. Its positivity there is an instance of Yoshida's small-support theorem [CITED]. Numerically POLE+ARCH stays positive definite slightly* past $\log 2$*, losing positivity between* $A = 0.74$ *(*$\lambda_{\min} = 1.06\cdot 10^{-3}$*) and* $A = 0.76$ *(*$-2.36\cdot 10^{-2}$*) [NUM]. Its negative index is* $0, 1, 2, 2, 3, 4$ *at the same grid. These are exactly the negative directions that the prime terms must cancel.*

Proof. (a) $\hat{g}(z) = \hat{F}(z)\overline{\hat{F}(\bar{z})}$, so $\hat{g}(\frac{i}{2}) + \hat{g}(-\frac{i}{2}) = 2\,\mathrm{Re}(ab^*)$ with $a = \hat{F}(\frac{i}{2})$ and $b = \hat{F}(-\frac{i}{2})$. These are linearly independent functionals for $A > 0$, and $2\,\mathrm{Re}(ab^*) = \frac{1}{2}(|a+b|^2 - |a-b|^2)$. (b) Stirling. (c) For $0 < y < A$, take $F = \phi \pm \phi(\cdot + y)$ with ϕ supported in an interval of length less than $\min(y, A - y)$. The inertia counts are from `num/lemma64_signs.py` with tolerance 10^{-10}. □

Corollary 2.4 (Truncated Toeplitz form)**.** *$W_A = P_A\,\mathcal{Z}\,P_A$ is the compression to $L^2[-A/2, A/2]$ of convolution by the distribution*

$$\mathcal{Z}(u) = 2\cosh(u/2) + \mathcal{H}(u) - \sum_{n\geq 2} \Lambda(n) n^{-1/2}(\delta_{\log n} + \delta_{-\log n}), \qquad \widehat{\mathcal{H}} = h.$$

By the explicit formula, $\widehat{\mathcal{Z}} = \sum_\rho \delta_{\gamma_\rho}$ formally. "$W_A \geq 0$ for all A" says that $\mathcal{Z}$ is a positive-definite distribution. By the Bochner–Schwartz theorem this holds if and only if $\widehat{\mathcal{Z}}$ is a positive measure on $\mathbb{R}$, i.e. all γ_ρ are real: RH [RESTATEMENT] (this is Weil's criterion in Bochner form).

Remark 2.5 (Where the positivity sits on the prime side)**.** The smooth part of the prime distribution is $d\psi_0(x) = (1 - \frac{1}{x(x^2-1)})dx$, i.e. ψ without its zero terms: the pole term plus the trivial zeros. It reproduces POLE and the kernel part of ARCH identically, because $2e^{-u/2}(1 + \frac{1}{e^{2u}-1}) = k(u)$ [THM]. Hence, as distributions on $u \neq 0$,

$$\mathcal{Z}(u) = -e^{-|u|/2}\,\frac{d}{du}(\psi - \psi_0)(e^{|u|}),$$

and all of the positivity is carried by the *fluctuation* $\psi - \psi_0 = -\sum_\rho x^\rho/\rho$ of the primes about their mean, together with a local term at $u = 0$. "Positivity is a property of the prime fluctuations" is a [RESTATEMENT]: it is the explicit formula read from the right.

3. Literature map: what is actually proved

Each entry was checked at the source or at a named secondary source (`lit/lemma64/verified_refs.md`). Only the statements listed are used.

source	what it proves (as used here)	bearing on the prime side
Weil 1952 [2][BIB]	explicit formula; RH $\iff$ positivity of the quadratic functional for all admissible g	the equivalence itself ([RESTATEMENT])
Bombieri, Clay [1][CITED]	formula (1); RH $\iff$ "negativity of the right-hand side" for $f = g * g^*$ with $\int g\,dx/x = \int g\,dx = 0$; in the function-field case this negativity is "a rather easy consequence of the algebraic index theorem for surfaces"	normalization; function-field positivity is *geometric*
Yoshida 1992 [3][CITED]	Weil's form is positive definite on functions supported in a sufficiently small interval	small support: POLE+ARCH(+few primes) regime
Bombieri 2000 [4][CITED]	minimum on the unit ball of $L^2[-t,t]$ attained; Yoshida reproved; if RH fails with finitely many off-line zeros, the number of negative eigenvalues of a large truncation is half the number of off-line zeros	ghosts $\leftrightarrow$ negative directions
Burnol 2000 [5][CITED]	the local terms of the explicit formula are given by the dilation-invariant conductor operator $\log\lvert x\rvert_\nu + \log\lvert y\rvert_\nu$ (spectral multiplier $\Gamma'/\Gamma(s,\chi_\nu)$), uniformly for finite and infinite places; "we also check Weil's positivity criterion under a support condition"	ARCH and the p-terms are *local* operators of one type
Connes–Consani 2021 [6][CITED]	archimedean place: $W_\infty(g*g^*) \geq \mathrm{Tr}(\vartheta(g)S\vartheta(g)^*)$ for g supported in $[2^{-1/2}, 2^{1/2}]$ with $\hat{g}(\frac{i}{2}) = \hat{g}(0) = 0$ (Thm 1); the $(1/2, 2)$ positivity attributed to Yoshida	ARCH alone, small support, with vanishing conditions
Li 1997 [7][CITED]; Bombieri–Lagarias 1999 [8][CITED]	RH $\iff$ $\lambda_n \geq 0$ for all n; BL: valid for any multiset with the 4-fold symmetry, *not specific to* ζ; an arithmetic formula for λ_n via the Guinand–Weil explicit formula, related to Weil's criterion	another sequence of RH-equivalent prime-side sums ([RESTATEMENT])
Beurling 1937 [9][BIB]; Diamond–Montgomery–Vorhauer 2006 [10][CITED]	a Beurling system with $N_B(x) = \kappa x + O(x^\theta)$, $\frac{1}{2} < \theta < 1$, whose zeta function $\zeta_B = \prod(1-\lambda_j^{-s})^{-1}$ has infinitely many zeros on $\sigma = 1 - a/\log t$ and $\pi_B(x) = \mathrm{li}(x) + \Omega(xe^{-c\sqrt{\log x}})$	an *Euler product without functional equation* can fail RH maximally
Selberg 1992 [11][BIB]; Kaczorowski–Perelli 1999 [12][BIB]	the Selberg class (Dirichlet series with Euler product, functional equation, Ramanujan-type bound); GRH for it is a conjecture	EP + FE $\Rightarrow$ RH is [CONJ]
Platt–Trudgian 2021 [13][CITED]	RH holds up to height $3.0001753328 \cdot 10^{12}$	all windows $\lambda^2 \leq 67$ positive (via L57)
Ingham 1942 [14][CITED](L62)	a one-sided bound $M(x) > -Kx^{1/2}$ would imply RH and simple zeros (classical); Thm A on linear relations	the only known *sufficient* prime-side sign condition; not known to hold

Remark 3.1 (No unconditional prime-side sufficient condition for ζ)**.** Every prime-side criterion above that is *equivalent* to RH is a [RESTATEMENT]: Weil, Li–Bombieri–Lagarias, $\psi(x) - x = O(x^{1/2+\varepsilon})$, $M(x) = O(x^{1/2+\varepsilon})$. The only *sufficient* one-sided condition (Ingham–Landau) is not known to hold, and Lemma 62 explains why it is expected to fail. The proved positivity results (Yoshida, Bombieri, Burnol, Connes–Consani) are all small-support results, where the prime sum has few or no terms. We found no published unconditional prime-side condition that implies positivity at large support [OPEN].

4. Positivity is collective: per-term slack

J.W. [t242u] flagged "positivity is collective, not prime by prime" as important. Here it is made precise.

Definition 4.1 (Per-term slack)**.** Let X be one of: POLE, ARCH, a prime block $T_p = \sum_k T_{p,k}$ (all powers), the $k = 1$ term $T_{p,1}$ alone, or PP $= \sum_{k\geq 2} T_{p,k}$ (all higher powers). Replace X by $(1 + s)X$ in W_A. The *unsafe-side slack* $\varepsilon_X(A)$ is the least $|s|$ in the direction that lowers λ^* for which the form stops being positive definite. The *safe-side slack* $\delta_X(A)$ is the same quantity in the opposite direction. "Removing X" is $s = -1$.

Theorem 4.2 (Structure of the slack)**.** *(a) $s \mapsto \lambda_{\min}(W_A + sX)$ is concave, so $\{s : W_A + sX > 0\}$ is an open interval $(-\delta, \varepsilon)$ or its mirror [THM].*

(b) If λ^* is simple, then $\varepsilon_X = \lambda^*/|n_X|\,(1+O(\varepsilon_X|n_X|/\mathrm{gap}))$, and the unsafe direction is $\mathrm{sgn}(-n_X)$ (Hellmann–Feynman) [THM]. In particular $\varepsilon_X|n_X| \approx \lambda^*$: the margin, normalized by the term's own contribution along the ground state, is the same for every term.

(c) For the constant part of ARCH ($\psi(\frac{1}{4}) - \log\pi$, i.e. the conductor) the slack is exactly λ^*, because $X = I$ [THM].

Part (b) shows that a tiny unsafe-side ε_X alone only says "λ^* is tiny" (a *tiny margin*). Positivity is *genuinely collective* if, in addition, (i) the safe-side slack δ_X is also small, so no term can be dialled down freely, and (ii) removing any term destroys positivity. Both are tested below.

A	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
K	40	46	50	56	60	80	90	100	115
λ^*	$7.57{\cdot}10^{-3}$	$1.81{\cdot}10^{-4}$	$9.36{\cdot}10^{-7}$	$1.62{\cdot}10^{-9}$	$4.27{\cdot}10^{-13}$	$1.68{\cdot}10^{-17}$	$4.30{\cdot}10^{-23}$	$6.27{\cdot}10^{-30}$	$2.47{\cdot}10^{-38}$
POLE: n_X	$1.03{\cdot}10^{0}$	$1.18{\cdot}10^{0}$	$1.27{\cdot}10^{0}$	$1.33{\cdot}10^{0}$	$1.38{\cdot}10^{0}$	$1.42{\cdot}10^{0}$	$1.45{\cdot}10^{0}$	$1.47{\cdot}10^{0}$	$1.49{\cdot}10^{0}$
ε_X (unsafe)	$7.34{\cdot}10^{-3-}$	$1.54{\cdot}10^{-4-}$	$7.38{\cdot}10^{-7-}$	$1.21{\cdot}10^{-9-}$	$3.09{\cdot}10^{-13-}$	$1.18{\cdot}10^{-17-}$	$2.97{\cdot}10^{-23-}$	$4.26{\cdot}10^{-30-}$	$1.66{\cdot}10^{-38-}$
δ_X (safe)	$2.82{\cdot}10^{1}$	$9.53{\cdot}10^{-1}$	$9.40{\cdot}10^{-3}$	$2.48{\cdot}10^{-5}$	$9.64{\cdot}10^{-9}$	$5.51{\cdot}10^{-13}$	$2.41{\cdot}10^{-18}$	$5.22{\cdot}10^{-25}$	$2.78{\cdot}10^{-33}$
ARCH: n_X	$-1.02{\cdot}10^{0}$	$-1.17{\cdot}10^{0}$	$-1.26{\cdot}10^{0}$	$-1.31{\cdot}10^{0}$	$-1.35{\cdot}10^{0}$	$-1.38{\cdot}10^{0}$	$-1.40{\cdot}10^{0}$	$-1.42{\cdot}10^{0}$	$-1.43{\cdot}10^{0}$
ε_X (unsafe)	$7.40{\cdot}10^{-3+}$	$1.55{\cdot}10^{-4+}$	$7.45{\cdot}10^{-7+}$	$1.23{\cdot}10^{-9+}$	$3.16{\cdot}10^{-13+}$	$1.22{\cdot}10^{-17+}$	$3.07{\cdot}10^{-23+}$	$4.42{\cdot}10^{-30+}$	$1.73{\cdot}10^{-38+}$
δ_X (safe)	$9.65{\cdot}10^{-1}$	$4.74{\cdot}10^{-1}$	$5.81{\cdot}10^{-2}$	$5.02{\cdot}10^{-4}$	$5.53{\cdot}10^{-7}$	$6.60{\cdot}10^{-11}$	$7.02{\cdot}10^{-16}$	$2.74{\cdot}10^{-22}$	$2.39{\cdot}10^{-30}$
p=2: n_X	–	$-2.91{\cdot}10^{-3}$	$-1.25{\cdot}10^{-2}$	$-2.27{\cdot}10^{-2}$	$-3.15{\cdot}10^{-2}$	$-3.92{\cdot}10^{-2}$	$-4.55{\cdot}10^{-2}$	$-5.07{\cdot}10^{-2}$	$-5.49{\cdot}10^{-2}$
ε_X (unsafe)	–	$6.23{\cdot}10^{-2+}$	$7.48{\cdot}10^{-5+}$	$7.11{\cdot}10^{-8+}$	$1.35{\cdot}10^{-11+}$	$4.29{\cdot}10^{-16+}$	$9.45{\cdot}10^{-22+}$	$1.24{\cdot}10^{-28+}$	$4.50{\cdot}10^{-37+}$
δ_X (safe)	–	$2.19{\cdot}10^{-1}$	$1.21{\cdot}10^{-3}$	$2.84{\cdot}10^{-6}$	$1.05{\cdot}10^{-9}$	$5.93{\cdot}10^{-14}$	$2.59{\cdot}10^{-19}$	$5.59{\cdot}10^{-26}$	$2.98{\cdot}10^{-34}$
p=3: n_X	–	–	–	$-1.57{\cdot}10^{-6}$	$-2.49{\cdot}10^{-5}$	$-8.42{\cdot}10^{-5}$	$-1.70{\cdot}10^{-4}$	$-2.72{\cdot}10^{-4}$	$-3.78{\cdot}10^{-4}$
ε_X (unsafe)	–	–	–	$1.03{\cdot}10^{-3+}$	$1.72{\cdot}10^{-8+}$	$1.99{\cdot}10^{-13+}$	$2.52{\cdot}10^{-19+}$	$2.30{\cdot}10^{-26+}$	$6.55{\cdot}10^{-35+}$
δ_X (safe)	–	–	–	$3.68{\cdot}10^{-3}$	$1.99{\cdot}10^{-7}$	$5.02{\cdot}10^{-12}$	$1.38{\cdot}10^{-17}$	$2.23{\cdot}10^{-24}$	$9.71{\cdot}10^{-33}$
p=5: n_X	–	–	–	–	–	–	$-1.58{\cdot}10^{-12}$	$-5.47{\cdot}10^{-11}$	$-3.61{\cdot}10^{-10}$
ε_X (unsafe)	–	–	–	–	–	–	$2.72{\cdot}10^{-11+}$	$1.15{\cdot}10^{-19+}$	$6.84{\cdot}10^{-29+}$
δ_X (safe)	–	–	–	–	–	–	$1.69{\cdot}10^{-10}$	$1.78{\cdot}10^{-18}$	$1.92{\cdot}10^{-27}$
p=7: n_X	–	–	–	–	–	–	–	$-2.56{\cdot}10^{-23}$	$-1.13{\cdot}10^{-18}$
ε_X (unsafe)	–	–	–	–	–	–	–	$2.45{\cdot}10^{-7+}$	$2.20{\cdot}10^{-20+}$
δ_X (safe)	–	–	–	–	–	–	–	$5.27{\cdot}10^{-7}$	$1.69{\cdot}10^{-19}$
PP ($k \geq 2$): n_X	–	–	–	–	$-4.51{\cdot}10^{-13}$	$-3.26{\cdot}10^{-9}$	$-3.71{\cdot}10^{-8}$	$-1.41{\cdot}10^{-7}$	$-3.29{\cdot}10^{-7}$
ε_X (unsafe)	–	–	–	–	$9.46{\cdot}10^{-1+}$	$5.15{\cdot}10^{-9+}$	$1.16{\cdot}10^{-15+}$	$4.45{\cdot}10^{-23+}$	$7.51{\cdot}10^{-32+}$
δ_X (safe)	–	–	–	–	$7.24{\cdot}10^{-1}$	$3.44{\cdot}10^{-8}$	$2.09{\cdot}10^{-14}$	$1.59{\cdot}10^{-21}$	$4.39{\cdot}10^{-30}$

TABLE 1. Per-term slack (`num/l64/slack_*.json`; mp back-end, 256–460 bits; Cholesky PD tests). $n_X = \langle v, Xv\rangle$. ε_X is first order and was checked by Cholesky (positive definite at $0.9\varepsilon_X$, not at $1.1\varepsilon_X$) in every case except PP at $A = 1.4$. There the single term $n = 4$ sits at the window edge ($\log 4 = 1.386$), $n_{\mathrm{PP}} = -4.5 \cdot 10^{-13}$, first order fails, and positivity is lost already at 0.85. Superscript $\pm$: unsafe direction. δ_X: bisection to 0.1%. For $p = 2, 3$ the $k = 1$ term alone gives the same ε, δ to the digits shown, because powers contribute $\leq 10^{-6}$ relatively along v.

Theorem 4.3 (Collective positivity; numerical)**.** *On the grid $A \in \{0.6, \ldots, 2.2\}$, for every term X lying inside the window (POLE, ARCH, T_p and $T_{p,1}$ for $p < e^A$, PP):*

(i) both slacks are small and ordered: $\varepsilon_X \leq \delta_X$ in every case except PP at $A = 1.4$ (see (iii)); for $A \geq 1.2$ every $\delta_X \leq 4 \cdot 10^{-3}$ except PP at $A = 1.4$; both decay with A in step with $\lambda^*(A)$. At $A = 2.2$: POLE $(\varepsilon, \delta) = (1.7 \cdot 10^{-38}, 2.8 \cdot 10^{-33})$, ARCH $(1.7 \cdot 10^{-38}, 2.4 \cdot 10^{-30})$,

removed	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2
POLE	-1.1 [1]	-1.5 [1]	-2 [1]	-2.4 [1]	-2.9 [1]	-3.3 [1]	-3.8 [1]	-4.3 [1]	-4.8 [1]
ARCH	-0.009 [19]	-0.49 [23]	-0.49 [25]	-0.9 [28]	-0.97 [29]	-1.2 [39]	-1.5 [42]	-1.6 [48]	-1.9 [56]
p=2	–	-0.078 [1]	-0.32 [2]	-0.47 [2]	-0.49 [2]	-0.51 [3]	-0.53 [4]	-0.54 [6]	-0.54 [7]
p=3	–	–	–	-0.15 [2]	-0.5 [4]	-0.61 [4]	-0.63 [4]	-0.63 [4]	-0.63 [5]
p=5	–	–	–	–	–	–	-0.47 [6]	-0.67 [8]	-0.72 [8]
p=7	–	–	–	–	–	–	–	-0.18 [5]	-0.6 [10]
PP ($k \geq 2$)	–	–	–	–	-2.3e-10 [1]	-0.26 [4]	-0.34 [6]	-0.35 [6]	-0.38 [6]

TABLE 2. Removal ($s = -1$): $\lambda_{\min}$ [negative index] of $W_A - X$ (mp). Every removal of a term that lies inside the window makes the form indefinite.

p=2 $(4.5 \cdot 10^{-37}, 3.0 \cdot 10^{-34})$, p=3 $(6.6 \cdot 10^{-35}, 9.7 \cdot 10^{-33})$, p=5 $(6.8 \cdot 10^{-29}, 1.9 \cdot 10^{-27})$, p=7 $(2.2 \cdot 10^{-20}, 1.7 \cdot 10^{-19})$, *PP* $(7.5 \cdot 10^{-32}, 4.4 \cdot 10^{-30})$*;*

(ii) removing X makes the form indefinite, with negative index between 1 *(pole) and* 56 *(ARCH, $A = 2.2$);*

(iii) a term just entering the window has slack $O(10^{-2}$–$1)$: $p = 2$ at $A = 0.8$ (depth $A - \log 2 = 0.11$) has $(\varepsilon, \delta) = (0.062, 0.22)$; PP at $A = 1.4$ (depth 0.014) is $O(1)$; $p = 7$ at $A = 2.0$ (depth 0.054) already has $(2.5 \cdot 10^{-7}, 5.3 \cdot 10^{-7})$. The slack collapses as the term moves deeper into the window, i.e. as $A - \log n$ grows. Without primes ($A = 0.6$) the safe slack of the pole is large ($\delta = 28$): collectivity sets in with the primes.

[NUM]. The $p = 2$ values agree with Lemma 57's independently computed no-slack thresholds for the $n = 2$ term: $9.45 \cdot 10^{-22}$ ($A = 1.8$), $1.24 \cdot 10^{-28}$ (2.0), $4.50 \cdot 10^{-37}$ (2.2) [TAA-L].

Remark 4.4 (Tiny margin versus collective)**.** By Thm. 4.2(b), $\varepsilon_X \approx \lambda^*/|n_X|$ identically, so the unsafe slacks span many orders of magnitude (at $A = 2.2$ from 10^{-38} for the pole to 10^{-20} for $p = 7$) only because the contributions $|n_X|$ do. That part is "tiny margin". The collective part is (i) the *safe* side, where $\delta_X \ll 1$ means that weakening a term also destroys positivity, through a different eigenvector, and (ii) the removal table. Both hold for every term inside the window. In the language of convex geometry, the weight vector $(1, \ldots, 1)$ lies within $\sim \lambda^*$ of the boundary of the spectrahedron of positive weightings *in every coordinate direction and on both sides* [NUM].

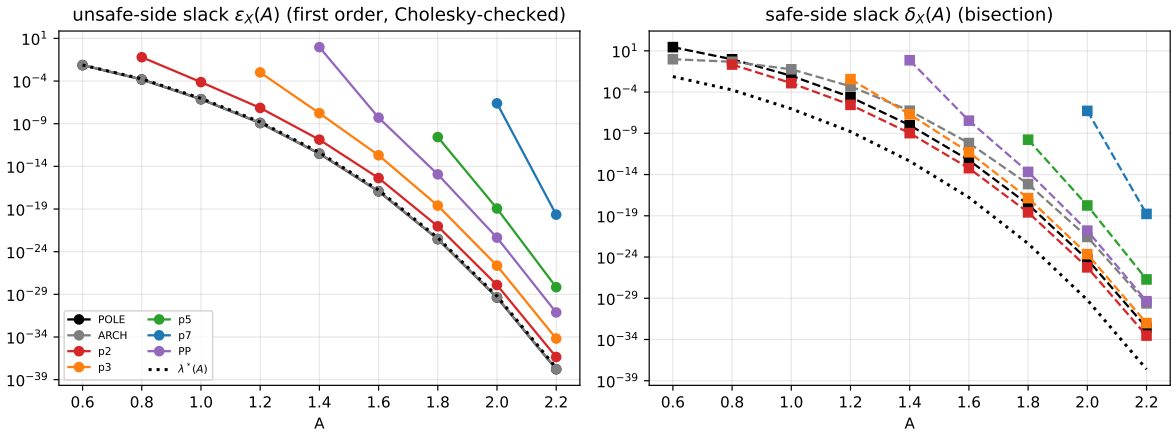

FIGURE 1. Unsafe (left) and safe (right) per-term slack versus A, with $\lambda^*(A)$ (dotted). [NUM]

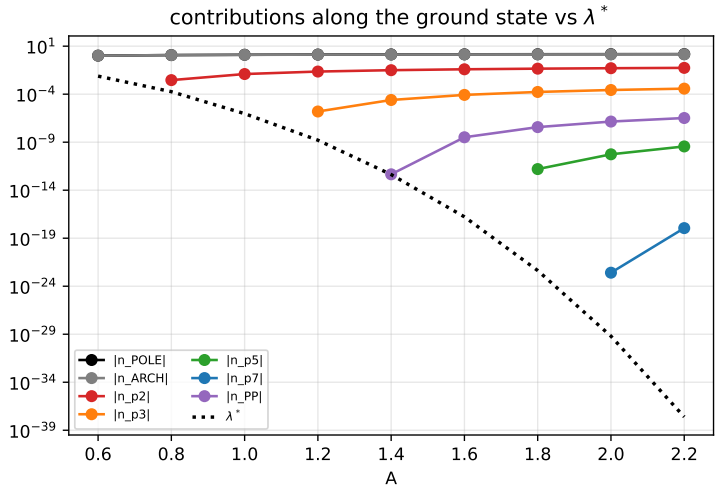

FIGURE 2. Contributions $|n_X|$ along the ground state versus λ^*. The gap between them is the cancellation. [NUM]

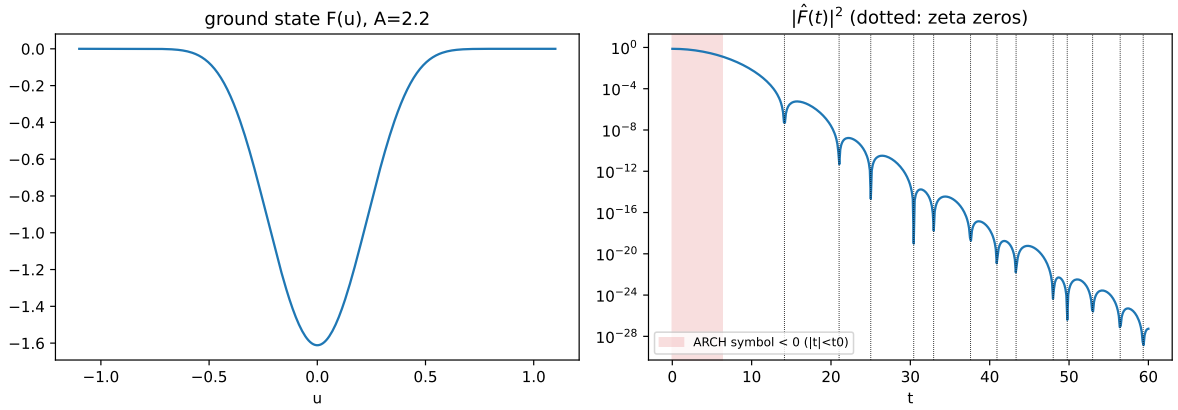

FIGURE 3. Ground state at the largest A of the grid and $|\hat{F}|^2$. The shaded band is where the archimedean symbol is negative ($|t| < t_0 = 6.29$). Dotted lines: zeros of ζ, where $\hat{F}$ nearly vanishes (CCM observation, reproduced in L52/L57) [NUM].

5. Structure, not size: ablation by ingredient class

Proposition 5.1 (Two exact invariances)**.** *(a)* Gauge. *Twisting* every *term (pole, ARCH and primes) by $n^{-i\tau}$, i.e. passing to $\zeta(s + i\tau)$, gives $W_A^{(\tau)}(F) = W_A(e^{-i\tau u}F)$. Multiplication by $e^{-i\tau u}$ is unitary on $L^2[-A/2, A/2]$, so the spectrum is unchanged for every τ [THM]. The perturbation has size $O(\tau)$ but leaves the slack infinite.*

(b) Convexity. *Positive definite forms make up a convex cone, so $(1-t)W_\zeta + tW_{L(\chi)}$ is positive whenever both ends are [THM]. We checked this at $A = 1$. Positivity is therefore not characteristic of L-functions.*

Proof. (a) $g \mapsto g(y)e^{-i\tau y}$ is the effect on $g = F * \tilde{F}$ of $F \mapsto e^{-i\tau u}F$, and every term of (2) is linear in g. □

perturbation \ A	1.0	1.4	1.8
λ^*	$9.36{\cdot}10^{-7}$	$4.27{\cdot}10^{-13}$	$4.30{\cdot}10^{-23}$
$p=2$ weight $1+x$ (single)	$7.44{\cdot}10^{-5}$	$1.35{\cdot}10^{-11}$	$9.41{\cdot}10^{-22}$
all prime weights $1+x$ (collective)	$7.44{\cdot}10^{-5}$	$1.35{\cdot}10^{-11}$	$9.41{\cdot}10^{-22}$
$q_2=2(1+x)$ (single Beurling move)	$1.39{\cdot}10^{-4}$	$1.98{\cdot}10^{-10}$	$6.21{\cdot}10^{-20}$
$q_p=p(1+x\eta_p)$ all p (collective Beurling, EP kept)	$5.57{\cdot}10^{-4}$	$7.38{\cdot}10^{-10}$	$2.01{\cdot}10^{-19}$
primes-only twist n^{-ix} (collective)	$3.96{\cdot}10^{-4}$	$1.55{\cdot}10^{-10}$	$1.98{\cdot}10^{-20}$
full twist (pole, ARCH, primes)	∞ (unitary gauge, Prop. 5.1)		
λ^* of $L(s,\chi_{-4})$	$1.40{\cdot}10^{-1}$	$6.53{\cdot}10^{-3}$	$2.62{\cdot}10^{-5}$
λ^* of $L(s,\chi_5)$	$9.01{\cdot}10^{-2}$	$4.77{\cdot}10^{-3}$	$4.30{\cdot}10^{-5}$

TABLE 3. Thresholds x_c (loss of positive definiteness; mp back-end; Cholesky bisection to 1%). Collective perturbations that do not preserve the joint structure break at the same scale as single-prime ones. The only collective deformation with infinite threshold is the gauge twist, which preserves EP+FE exactly.

model	first A with $\lambda_{\min}<-10^{-8}$	$\lambda_{\min}$ there
zeta	none (≤ 4.6)	$\min=-5.2e-13$
zeta_noPP	1.6	-0.26
L(χ-4)	none (≤ 4.6)	$\min=-2.3e-13$
L(χ-3)	none (≤ 4.6)	$\min=-2.3e-13$
L(χ5)	none (≤ 4.6)	$\min=-2.7e-13$
χ-4 primes, zeta arch+pole	0.8	-0.078
χ-4 primes+arch, zeta pole kept	1.6	-0.081
DH	3.6	-2.6e-07
Beurling liˆ-1 (Euler, zeta arch+pole)	1.2	-0.042
Cramer s0 (zeta arch+pole)	1.6	-0.25
random-sign Euler s0 (χ5 arch a=1/4,q=5)	1.0	-0.38
random-sign Euler s0 (χ-4 arch a=3/4,q=4, $\chi(2)$=0)	2.0	-0.059
Cramer s1 (zeta arch+pole)	1.6	-0.26
random-sign Euler s1 (χ5 arch a=1/4,q=5)	1.2	-0.035
random-sign Euler s1 (χ-4 arch a=3/4,q=4, $\chi(2)$=0)	1.8	-0.27
Cramer s2 (zeta arch+pole)	1.6	-0.25
random-sign Euler s2 (χ5 arch a=1/4,q=5)	1.8	-0.028
random-sign Euler s2 (χ-4 arch a=3/4,q=4, $\chi(2)$=0)	1.8	-0.27

TABLE 4. Gross positivity (float64, $K=\min(300,40+60A)$, $A=0.4,0.6,\dots$). "None" means no eigenvalue below -10^{-8}; the float noise floor is about 10^{-13}. Random-sign models: three seeds each; the signs of $p=2,3,5$ are recorded in `num/l64/struct_*.json`. A sign model coincides with χ_5 (resp. χ_{-4}) until the first prime where the signs differ.

Proposition 5.2 (Structure versus size; numerical)**.** *(a)* Collective non-structural perturbations break as easily as single-prime ones *(Table 3). At* $A=1.8$*, scaling only* $p=2$ *breaks positivity at relative size* $9.4\cdot10^{-22}$*, and scaling* all *primes together at the same size. Moving* $q_2=2$ *to* $2(1+x)$ *breaks at* $6.2\cdot10^{-20}$*. Collective Beurling jitter* $q_p=p(1+x\eta_p)$*, which keeps an Euler product, breaks at the same order, and so does the primes-only twist* n^{-ix}*.*

(b) Structure-preserving perturbations keep positivity at any size. *The full twist (Prop. 5.1) does, and so do the genuine L-functions* $L(s,\chi_{-4}),L(s,\chi_{-3}),L(s,\chi_5)$*. Each differs from* ζ *by* $O(1)$ *in coefficients, Gamma data and pole, and none shows a negative direction above float noise up to* $A=4.6$ *(Table 4).*

(c) One-ingredient models fail *(Table 4, Fig. 4). EP only: Beurling* li^{-1} *primes fail from* $A=1.2$*; random-sign Euler products fail once a sign differs from the character's. FE only: DH fails at* $A=3.43$ *(L57). Neither: Cramér models fail when the first wrong integer enters. Ablated* ζ*: removing prime powers fails at* $A=1.4$ *(mp) or* 1.6 *(float);* χ_{-4} *coefficients with* ζ*'s Gamma*

factor and pole fail at 0.8 *(this drops* $p = 2$*);* χ_{-4} *with its own Gamma factor but* ζ*'s pole kept fails at* 1.6.

(d) FE versus EP. *FE-only survives longest (DH to* $A = 3.42$*), because the FE gives the form a genuine zero-side representation. It then fails exactly when its first ghost is resolved. EP-only models with no FE fail as soon as their coefficients deviate. In the zero-side picture this is [ANALOGY]: FE makes* W *a zero sum, and EP together with FE decides (conjecturally) whether the zeros are ghost-free.*

Remark 5.3 (Archimedean variants)**.** (`num/l64/archvar.json`, float.) $\Gamma_{\mathbb{R}}(s+1)$ in place of $\Gamma_{\mathbb{R}}(s)$ ($a = \frac{3}{4}$), with ζ's pole and primes, stays positive up to $A = 2.0$. This is *monotonicity*, not structure: $h_{3/4} - h_{1/4} \geq 0$ (numerically, on $t \in [0, 200]$; it equals π at $t = 0$), so the variant is W_ζ plus a positive form. Scaling ARCH by 0.9 or 1.1, lowering its constant by 0.01, dropping its kernel, or dropping it altogether all destroy positivity once primes are present [NUM].

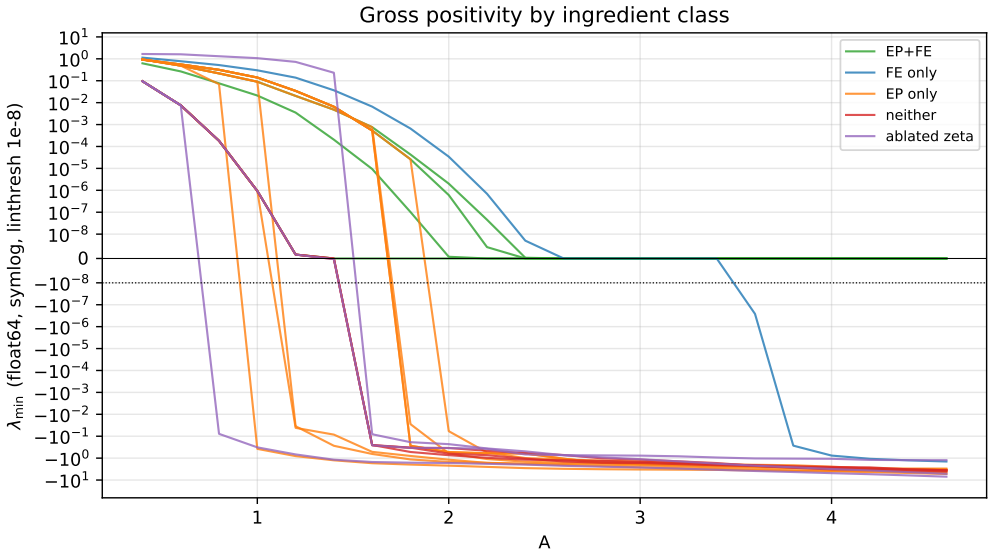

FIGURE 4. $\lambda_{\min}(A)$ by ingredient class (float64, symlog). Green: EP+FE; blue: FE only (DH); orange: EP only; red: neither; purple: ablated ζ. [NUM]

6. Local positivity as a binary quadratic form vs global collective positivity

J.W. [t245u]: each element of the polarization triple (ω, J, g), or of a binary quadratic form, should correspond to a precise element of the prime side.

6.1. Hasse's binary form.

Let $E/\mathbb{F}_p$ be an elliptic curve with Frobenius π and dual isogeny $\hat{\pi}$. Put $a_p = p + 1 - \#E(\mathbb{F}_p)$.

Theorem 6.1 (Hasse; Silverman, AEC, Thm. V.1.1)**.** *(a)* $\pi\hat{\pi} = p$ *and* $\pi + \hat{\pi} = a_p$*. Hence*

$$Q_p(x, y) := \deg(x + y\pi) = x^2 + a_p\, xy + p\, y^2, \qquad Q_p(1, -1) = \deg(1 - \pi) = \#E(\mathbb{F}_p).$$

With $\deg(x - y\pi)$ *one gets* $x^2 - a_p xy + py^2$*; the two conventions differ by* $y \mapsto -y$*.*

(b) deg *is a positive-definite quadratic form on* $\mathrm{End}(E)$*, so* $Q_p > 0$ *on* $\mathbb{Z}^2 \setminus 0$*, hence* $Q_p \geq 0$ *on* $\mathbb{R}^2$*. Therefore* $d_p := a_p^2 - 4p \leq 0$*, i.e.* $|a_p| \leq 2\sqrt{p}$*. Equality is impossible because* $4p$ *is not a square, so* $d_p < 0$ *and* Q_p *is positive definite.*

(c) $d_p < 0$ *is* exactly *RH for* $E/\mathbb{F}_p$. *Indeed* $Z(E/\mathbb{F}_p, T) = \frac{1-a_pT+pT^2}{(1-T)(1-pT)}$, *and the roots* $\alpha, \bar{\alpha}$ *of* $X^2 - a_pX + p$ *satisfy* $|\alpha| = \sqrt{p}$ *iff* $d_p \le 0$. *Writing* $\alpha, \bar{\alpha} = \sqrt{p}\, e^{\pm i\theta_p}$ *gives* $\cos\theta_p = a_p/2\sqrt{p}$.

Status. (b) is Theorem V.1.1 of [15], proved there from the positive definiteness of deg by a Cauchy–Schwarz lemma (V.1.2) [CITED-sec]. (a) and (c) are standard [THM]: $\deg(1-\pi) = (1-\pi)(1-\hat{\pi}) = 1 - (\pi+\hat{\pi}) + p$, and $\deg(x+y\pi) = (x+y\pi)(x+y\hat{\pi})$. □

Proposition 6.2 (J is the normalized imaginary part of Frobenius)**.** *For* $Q = ax^2 + bxy + cy^2$ *with* $d = b^2 - 4ac < 0$, *the matrix* $J_Q = \frac{1}{\sqrt{-d}}\left(\begin{smallmatrix} b & 2c \\ -2a & -b \end{smallmatrix}\right)$ *satisfies* $J_Q^2 = -1$ *and* $J_Q^{\mathsf{T}} Q J_Q = Q$. *For* $Q = Q_p$, *in the coordinates* (x, y) *of* $x + y\pi$,

$$J_{Q_p} = -\frac{2M_\pi - a_p}{\sqrt{4p - a_p^2}} = -\frac{\pi - \hat{\pi}}{|\alpha - \bar{\alpha}|},$$

where M_π *is the matrix of multiplication by* π. *Under* $x + y\pi \mapsto x + y\alpha \in \mathbb{C}$, *we have* $Q_p = |x + y\alpha|^2$ *and* J_{Q_p} *becomes multiplication by* $\mp i$. *The* $\pm i$ *eigenlines are the Frobenius eigenlines* $\alpha, \bar{\alpha}$, *i.e. the two helicities of Lemma 62 [THM].*

Proof. Direct computation: $\pi \cdot (x + y\pi) = -py + (x + a_p y)\pi$, and $2\pi - a_p = \pi - \hat{\pi}$. The identity $Q_p = |x + y\alpha|^2 = x^2 + xy(\alpha + \bar{\alpha}) + y^2|\alpha|^2$ uses $|\alpha|^2 = p$, which is RH for $E/\mathbb{F}_p$ itself. We checked all of this numerically for 24 primes (next example) [NUM]. □

Example 6.3 ($E : y^2 = x^3 + x + 1$, bad primes $2, 31$)**.** Brute-force point counts (`num/lemma64_hasse.py`, exact integers) give the following table [NUM].

p	$\#E$	a_p	Q_p	d_p	$\lvert a_p\rvert/2\sqrt{p}$	θ_p
3	4	0	$x^2 + 3y^2$	−12	0	90.0°
5	9	−3	$x^2 - 3xy + 5y^2$	−11	0.671	132.1°
7	5	3	$x^2 + 3xy + 7y^2$	−19	0.567	55.5°
11	14	−2	$x^2 - 2xy + 11y^2$	−40	0.302	107.5°
13	18	−4	$x^2 - 4xy + 13y^2$	−36	0.555	123.7°
37	48	−10	$x^2 - 10xy + 37y^2$	−48	0.822	145.3°
47	60	−12	$x^2 - 12xy + 47y^2$	−44	0.875	151.1°
97	97	1	$x^2 + xy + 97y^2$	−387	0.051	87.1°

For all 24 good $p \le 101$: $\deg(1-\pi) = Q_p(1,-1) = \#E(\mathbb{F}_p)$, $\|J^2 + 1\| \le 1.4 \cdot 10^{-15}$, $\|J^{\mathsf{T}} Q J - Q\| \le 1.5 \cdot 10^{-14}$, and $J_{Q_p} = -(2M_\pi - a_p)/\sqrt{-d_p}$ to rounding. The largest ratio is 0.875 ($p = 47$). Fig. 5 shows the helicity pairs $e^{\pm i\theta_p}$.

Dictionary (one prime).

form	arithmetic meaning	origin	prime side?
$a = 1$	deg(id)	geometry (normalization)	no
$b = a_p$	$\mathrm{Tr}\,\pi = p + 1 - \#E(\mathbb{F}_p)$	Lefschetz trace of Frobenius	**yes** (point count)
$c = p$	$\deg\pi = \pi\hat{\pi}$; Weil pairing $e(\pi x, \pi y) = e(x,y)^p$	geometry (symplectic scaling)	no
$d < 0$	RH for $E/\mathbb{F}_p$		
J_Q	$-(\pi - \hat{\pi})/\lvert\alpha - \bar{\alpha}\rvert$	Frobenius itself	built from b, c
$g = Q_p$	deg (Rosati trace form, $\mathrm{Tr}(\phi\phi^\dagger) = 2\deg\phi$)	positivity of degree	**no**

The only prime-side entry is b. Positivity is *not* inferred from the value of b; it constrains b, and it comes from $\deg \ge 0$, a geometric fact (a degree counts the points of a kernel). That is the template.

6.2. **Weil's function-field template.** Let $C/\mathbb{F}_q$ have genus g and Jacobian A.

- ω: the Weil pairing (equivalently Poincaré duality on H^1) is alternating and is scaled by q under Frobenius. This gives the functional equation $T \mapsto 1/(qT)$ of $Z(C,T)$. Status [THM] (standard; classically also from Riemann–Roch).

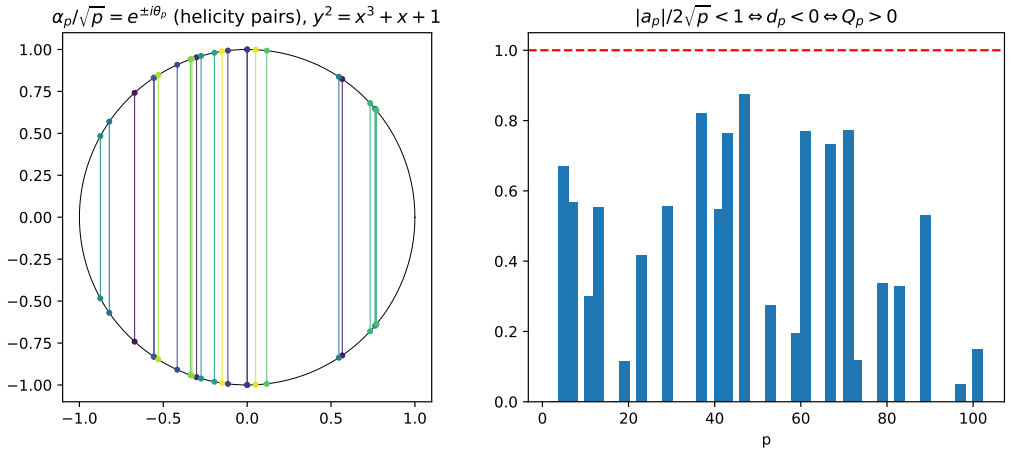

FIGURE 5. Left: normalized Frobenius eigenvalues $e^{\pm i\theta_p}$ for $y^2 = x^3 + x + 1$, $p \le 101$ good. Each pair is the two helicities ($\pm i$ eigenlines of J_{Q_p}). Right: $|a_p|/2\sqrt{p} < 1$, i.e. local positivity at every prime [NUM].

- The operator is Frobenius π. Its traces $\mathrm{Tr}(\pi^n|H^1) = q^n + 1 - \#C(\mathbb{F}_{q^n})$ are the prime side: point counts, i.e. counts of degree-n places (Lefschetz) [THM]. Bombieri [1] says the explicit formula "can be deduced from the Lefschetz fixed point formula" [CITED].
- g: positivity of the Rosati involution, $\mathrm{Tr}(\phi\phi^\dagger) > 0$ on $\mathrm{End}(A) \otimes \mathbb{Q}$ (Mumford §21 Thm. 1). Then $\pi^\dagger\pi = q$ forces every eigenvalue to have $|\alpha| = \sqrt{q}$ [CITED-sec] [16]. Equivalently, the Castelnuovo–Severi/Hodge index theorem on $C \times C$ ("the self-intersection form restricted to degree-0 divisors is negative semidefinite") [CITED] [1]. Kedlaya [16] describes the Rosati route as "Weil's second proof"; Bombieri describes the index-theorem route.

Proposition 6.4 (What kind of input the positivity is)**.** *In both proofs the positivity is (i) geometric, coming from ampleness or the index theorem rather than from any term of the explicit formula, and (ii) a* trace *over all of H^1 (dimension $2g$), so it is collective over the eigenvalues. It is applied to elements such as $m + n\pi^k$ and yields the prime-side bounds $|q^k + 1 - \#C(\mathbb{F}_{q^k})| \le 2g\, q^{k/2}$. Here the bound on the prime-side fluctuation is an* output *of geometric positivity [THM] (standard).*

This answers the correction "in Weil's proof positivity comes from Hodge index/Rosati, not the prime sum": correct (§8, K10).

6.3. The local/global split for $E/\mathbb{Q}$ and for ζ.

Proposition 6.5 (Local positivity everywhere does not give global positivity)**.**

(a) For $E/\mathbb{Q}$, Theorem 6.1 holds at every good prime: one positive-definite binary form per prime [THM]. It implies only that the Euler product of $L(E,s)$ converges absolutely and does not vanish for $\mathrm{Re}\, s > 3/2$ (arithmetic normalization; $\sigma > 1$ after centring at $\frac{1}{2}$), because every factor satisfies $|\alpha_p| = \sqrt{p}$ [THM]. The critical strip is reached only by analytic continuation (modularity) and the functional equation. GRH for $L(E,s)$ remains a [CONJ].

(b) For ζ the local factor at p is $(1-p^{-s})^{-1}$. Its "Frobenius eigenvalue" is 1*, so local positivity is* automatic *and carries no information.*

(c) Local positivity at every prime together with failure of RH occurs. The Diamond–Montgomery–Vorhauer Beurling zeta $\prod_j (1-\lambda_j^{-s})^{-1}$ [10] has every local factor of the same trivially "positive" type as ζ's, yet infinitely many zeros on $\sigma = 1 - a/\log t$ [CITED]. Numerically, Euler products with random signs ± 1 (each satisfying $|\pm 1| \le 1$ at every prime) and χ_5's Gamma factor lose Weil positivity at small support (§5) [NUM].

(d) In Weil's form for ζ, no prime has a positive piece of its own: each prime term is indefinite (Prop. 2.3(c)), and the per-prime slack is tiny on both sides (Thm. 4.3) [THM][NUM].

Remark 6.6 (Reading)**.** The binary-form dictionary works at one prime: $b = a_p$ is the prime side, and $g = \deg$ is geometric. Over $\mathbb{Q}$, RH for ζ is *not* a conjunction of local positivities. In function fields the global positivity is a trace over H^1, a single geometric statement that bounds all point counts at once. In Weil's form over $\mathbb{Q}$ (§4) positivity is likewise visible only after summing everything. "Positivity is collective" is therefore the expected shape, not an anomaly: the missing ingredient over $\mathbb{Q}$ is the global geometric g (Rosati/Hodge index on a would-be "$\overline{\operatorname{Spec}\mathbb{Z}} \times \overline{\operatorname{Spec}\mathbb{Z}}$"), consistent with Lemma 58's APD [ANALOGY].

7. Synthesis: one fact, several ingredients, one missing mechanism

7.1. One fact.

Proposition 7.1 (J.W. t246u, made precise)**.** *The following are equivalent:*

(i) $W_A \geq 0$ for every A (Weil positivity for all test functions);
(ii) the distribution $\mathcal{Z}$ of Cor. 2.4 is positive definite;
(iii) RH;
(iv) there are no ghosts, i.e. no zeros off the line. In the saddle form of Lemma 56 [TAA-L]: every saddle of $|\xi|$ is on the line and longitudinal.

[RESTATEMENT] (Weil; Bochner–Schwartz; Lemma 56). "Positivity is collective" (§4) and "ghost-killing is global" (Lemma 60 P10′) are therefore two descriptions of one *fact. The real question is not whether they are related; they are the same. It is whether there is* one mechanism *that produces both. At fixed A the relation is quantitative. A ghost $\rho = \frac{1}{2} + \delta + i\gamma$ contributes a signature-$(1,1)$ plane (Lemma 62 §7) and, once resolved ($A \gtrsim A_\gamma$, Lemma 57's $T^*(A)$), a negative direction (Bombieri 2000: index $= \frac{1}{2}$#off-line zeros) [CITED][TAA-L]. DH: certified indefinite at $A = 3.43$, first ghost at $\gamma = 85.70$ [TAA-L].*

7.2. Ingredient table (organizing hypothesis of the ablation).

class	example	what is known	our numerics (§5)
FE, no EP	DH	ghosts exist (Davenport–Heilbronn 1936; Spira 1994: $0.808517 + 85.699348i$) [CITED][TAA-L]	positive to $A = 3.42$, certified indefinite at 3.43 (L57) [TAA-L][NUM]
EP, no FE	Beurling systems	DMV 2006: infinitely many zeros on $\sigma = 1 - a/\log t$ [CITED]	Beurling li^{-1} primes: indefinite from $A = 1.2$; random-sign Euler products (χ_5 Gamma data): indefinite from $A = 1.0$ unless the signs agree with χ_5 [NUM]
neither	Cramér primes, shuffles	no structure	indefinite once the first non-prime/missing prime enters ($A \approx 1.4$–1.6) [NUM]
EP + FE	ζ, $L(s,\chi)$	GRH for the Selberg class: [CONJ] [11, 12]; RH verified to $3 \cdot 10^{12}$ [CITED]	no negative direction above float noise up to $A = 4.6$ for $\chi_{-4}, \chi_{-3}, \chi_5$ [NUM]
FE + Frobenius + geometric positivity	curves/$\mathbb{F}_q$	RH: Theorem (Weil; Rosati/Hodge index) [CITED][CITED-sec]	(§6) local binary forms checked [NUM]

Limitations. (1) No continuous deformation that keeps both EP and FE exactly is available, apart from the trivial ones: the archimedean twist $n^{-i\tau}$, a unitary gauge (Prop. 5.1), and dilations, which change A. EP+FE families are therefore sampled at isolated points (ζ, three Dirichlet L-functions). (2) In float64, "positive" means "no eigenvalue below -10^{-8}". (3) EP-only and neither-class models

need a Gamma factor and pole by fiat. We used ζ's data, or χ_5's for the sign models; the outcome depends on that choice only through the small-A regime.

7.3. **Mechanisms, with independent/circular verdicts.**

	mechanism	what it does (evidence)	independent of zeros?	sheds light on J?
M1	pole = PNT main term	rank 2, signature $(1,1)$; balances ARCH on the ground state ($+1.49$ vs -1.43 at $A = 2.2$); removing it gives one negative direction (Prop. 2.3, Table 1)	yes [THM]	no: RH-blind (absent for $L(s,\chi)$)
M2	archimedean term (Γ, Burnol's conductor operator)	symbol $h = 2\vartheta'$; deficit on $\lvert t\rvert < 6.29$, positive above; its constant shifts λ *exactly*, so $\lambda^*(A) =$ how far $\psi(\frac{1}{4}) - \log\pi$ could be lowered	yes [THM]	no: RH-blind; but its sign change at t_0 is the chirality of L62
M3	individual prime terms	each indefinite; two-sided slack $\le \varepsilon_X$, tiny; no prime removable	data yes, effect not separable [NUM]	negative result: J cannot be local in p
M4	Euler product (prime powers, multiplicativity, Hecke)	removing $k \ge 2$ powers breaks positivity at $A = 1.4$; but EP alone fails (DMV, sign models)	yes [THM]	constraint: J must commute with Hecke (L60 P10′)
M5	functional equation/Γ-matching (duality, ω)	gives W a zero-side representation; FE-only (DH) fails only at its first resolved ghost	yes [THM]	supplies ω, not g
M6	smallness of $\psi - \psi_0$ (PNT error)	positivity = positive-definiteness of the fluctuation (Rem. 2.5); $O(x^{1/2+\varepsilon}) \iff$ RH	**circular** [RESTATEMENT]	no (over $\mathbb{Q}$); over $\mathbb{F}_q$ it is an *output* of g
M7	Li/Bombieri–Lagarias λ_n	arithmetic formula via Guinand–Weil; $\forall n \iff$ RH	**circular** [RESTATEMENT]	no
M8	Ingham–Landau one-sided bound	sufficient; not known, expected false (L62)	would be independent [OPEN]	would give simplicity
M9	geometric positivity (Rosati/Hodge index; a trace over H^1)	the *only* mechanism that has ever produced RH non-circularly (curves)	yes [THM] (over $\mathbb{F}_q$)	**yes**: it *is* g in (ω, J, g); absent over $\mathbb{Q}$

Conjecture 7.2 (Organizing hypothesis; status)**.** *For Dirichlet series with (ζ-normalized) Weil forms, Weil positivity holds at all A iff the series has both an Euler product and a matching functional equation, i.e. lies in the Selberg class.*

Status. "If" is GRH for the Selberg class [CONJ]. "Only if" fails as stated: a positive combination need not be an L-function. Convexity (Prop. 5.1) makes every convex combination of positive forms positive, so positivity is not even characteristic of L-functions. The correct reading is weaker: *every model we tested that has only one of EP, FE fails, and every EP+FE model passes* [NUM]. "The mechanism lives in the combination of multiplicativity and duality" is thus at most a [CONJ] about where positivity holds, not a mechanism. In the one setting where the mechanism is known (curves), it needs a *third* input, geometric positivity (M9), which neither multiplicativity nor duality supplies (§6).

7.4. **Verdict on J.W.'s polarization condition.** Does the prime-side anatomy shed light on J, rather than being zero counting?

V1. *Negative, but sharp.* No term and no proper subset of terms of the prime side carries the positivity: every per-term slack is tiny on both sides (Thm. 4.3). So J (or g) cannot be assembled prime by prime or place by place. Any J must be *global*, a trace-type object as in M9 [NUM][ANALOGY].

V2. *Constraints, not construction.* J must be compatible with ω (FE, M5) and commute with the Hecke/multiplicative structure (EP, M4; L60 P10′). These are exactly the R+E ingredients of Lemma 58's APD. Lemma 58 already found that R+E+F alone gives no positivity [TAA-L]. The ablation confirms this numerically: each alone fails.

V3. *Local model of J found.* At one prime of an elliptic curve, $J_{Q_p} = -(\pi - \hat{\pi})/|\alpha - \bar{\alpha}|$ is the normalized imaginary part of Frobenius and $g = \deg$ (Prop. 6.2) [THM]. This is a precise instance of "Frobenius chooses its own J" (Lemma 54). Its global analogue over $\mathbb{Q}$ is missing, and that missing piece is the same as M9.

V4. *Circular routes identified.* M6 and M7 are zero location rewritten on the prime side. They do not meet J.W.'s condition.

Overall: the anatomy passes J.W.'s condition only in the negative-and-structural sense of V1–V3. It specifies what J must be (global, Hecke- and ω-compatible, of trace type) and exhibits a local model, but it does not construct J. **RH is not proven.**

8. Corrections to Grok Bot's chat claims

K1. *"The pole of ζ gives the one clearly positive term."* **Imprecise.** POLE has signature $(1,1)$: it is indefinite, and positive only on even test functions (Prop. 2.3(a)). ARCH also has a large positive part, at high frequency. Correct statement: on the (even) ground state the pole is the largest positive contribution ($n_{\mathrm{POLE}} = +1.488$ against $n_{\mathrm{ARCH}} = -1.432$ at $A = 2.2$), and removing it creates exactly one negative direction.

K2. *"Yoshida proved the archimedean term alone can carry positivity on small supports; Connes–Consani sharpened it."* **Partly wrong.** Yoshida proved that the *whole* Weil form is positive for sufficiently small support [CITED]. For support below $\log 2$ the whole form is POLE+ARCH, not ARCH alone. ARCH alone is indefinite on every window tested (negative index ≥ 1 already at $A = 0.4$). Connes–Consani attribute to Yoshida the positivity of the archimedean functional on $(1/2, 2)$ for positive-definite f with $\hat{f}(\pm\frac{i}{2}) = 0$. Those vanishing conditions remove the pole directions. CC's Theorem 1 strengthens this with a trace lower bound, under vanishing at $\frac{i}{2}$ and 0 [CITED]. "Archimedean term alone" is correct only with those conditions.

K3. *"The prime-power sum oscillates and must be held in check."* **Correct as a heuristic, too weak as a statement.** Each prime term is indefinite. The prime sum must cancel the negative directions of POLE+ARCH (index $0 \to 4$ as A goes $0.4 \to 2.4$) *without overshooting*. Increasing *or* decreasing any single prime's weight by more than its slack destroys positivity (two-sided slack, Table 1). "Tuned on both sides to $\sim \lambda^*$" replaces "held in check".

K4. *"DH keeps the functional equation and Gamma factor but has no Euler product."* **Partly wrong.** DH has the functional equation and Gamma factor of an odd Dirichlet character mod 5 ($\Gamma_{\mathbb{R}}(s+1)$, conductor 5, no pole), not those of ζ. The Euler-product inference from the comparison is ill-posed, as Lemma 57 Correction (2) already said [TAA-L].

K5. *"L57: $p = 2$ scaled by 10^{-46} breaks positivity at $A = 2.4$."* **Imprecise.** Lemma 57's theorem scales the $n = 2$ term (not all powers of 2) by $1 + \varepsilon$: positivity is lost for $\varepsilon > 1.18 \cdot 10^{-46}$ at $A = 2.4$ [TAA-L]. The present recomputation at $A = 2.2$ (all powers of 2) is in Table 1.

K6. *"No single prime has any slack; positivity is near-exact cancellation among pole, archimedean term and all primes."* **Correct, with two refinements.** (i) Every slack is $\ll 1$, but the slacks span many orders of magnitude, because $\varepsilon_X = \lambda^*/|n_X|$ to first order and the $|n_X|$ differ (at

$A = 2.2$, $p = 7$ has $\varepsilon = 2.2 \cdot 10^{-20}$). (ii) The slack is tiny on *both* sides, which is the genuinely collective part (Thm. 4.3).

K7. *"Slack per prime goes to zero as the support grows."* **Numerically correct** for $p = 2, 3$ on our grid. Each slack tracks $\lambda^*(A)$, which is non-increasing (Lemma 57, [THM]). That it tends to 0 is [NUM], not proved. Before a prime enters ($\log p \geq A$) its weight is irrelevant, so its slack is infinite.

K8. *"Unitarity is a property of the whole S-matrix."* **Correctly labelled [ANALOGY].** The precise mathematical counterpart is Cor. 2.4: positivity of a truncated Toeplitz form is a property of its whole symbol $\mathcal{Z}$, not of individual Fourier components.

K9. *"$Q_p = x^2 + a_p xy + py^2$."* **Correct** for $Q_p = \deg(x + y\pi)$ with $a_p = p + 1 - \#E$. With $\deg(x - y\pi)$ the middle sign flips.

K10. *"Positive definite $\iff$ $|a_p| < 2\sqrt{p}$."* **Correct.** The inequality is strict for prime p, since $4p$ is not a square; Silverman states $\leq$, for general q.

K11. *"In Weil's proof positivity comes from Hodge index/Rosati, not the prime sum."* **Correct.** The Castelnuovo–Severi/index-theorem route is the one Bombieri describes; the Rosati route is what Kedlaya calls Weil's second proof. Both inputs are geometric and of trace type, and the prime-side point-count bounds are *consequences* (Prop. 6.4).

K12. *"The functional equation comes from Poincaré duality."* **Correct for curves over $\mathbb{F}_q$** (also via Riemann–Roch). Over $\mathbb{Q}$ the functional equation comes from Poisson summation/theta (Riemann, Tate), and no Poincaré duality is known. There it is an [ANALOGY].

K13. *"Local Hasse bounds at all p do not imply GRH for $L(E, s)$."* **Correct.** They give only absolute convergence and non-vanishing for $\operatorname{Re} s > 3/2$ (Prop. 6.5).

K14. *"Beurling examples with Euler product where RH fails (Diamond–Montgomery–Vorhauer)."* **Correct, and stronger than stated.** DMV construct a Beurling system with $N_B(x) = \kappa x + O(x^\theta)$ whose zeta has *infinitely many* zeros on $\sigma = 1 - a/\log t$, i.e. as far from the line as the classical zero-free region allows [CITED]. It has no functional equation.

K15. *"Selberg class GRH conjectured."* **Correct** [11, 12] [BIB].

K16. *"The mechanism lives in the combination of multiplicativity and duality."* **Overstated.** As a statement about *where* positivity holds it is a [CONJ] (GRH for the Selberg class), consistent with our numerics. As a *mechanism* it is incomplete. In every proved case (curves) a third, geometric positivity input is needed (M9), and convexity shows positivity is not characteristic of L-functions (Conj. 7.2).

9. Ranked directions

Score = 0.6·significance+0.4·probability of a clear outcome, both in $[0, 1]$.

	direction	sig.	prob.	score
G1	*Function-field anatomy.* Repeat the slack/ablation study for the Weil form of a genus-2 curve over $\mathbb{F}_q$, where positivity is a theorem (Rosati). Is the slack also two-sided and tiny? Which endomorphism ($m + n\pi^k$) is the binding direction? This tests whether collectivity is the *signature of geometric positivity*	.80	.80	.80
G2	*Hasse → Weil for $L(E, s)$.* Build the Weil form of $L(E, s)$ for $y^2 = x^3 + x + 1$ and compare its ground state with the local forms Q_p. Does the eigenvector see the θ_p? Local versus global, quantitatively	.65	.60	.63
G3	*Tangent cone of positivity at ζ.* The normal vector (n_X) is the "prime-side shadow" of the ground state. Prove $n_p < 0$ for all $p < e^A$ (observed) and describe the safe directions	.45	.70	.55
G4	*Certified $L(s, \chi)$ windows at $A \geq 3.5$* (beyond DH's failure), to turn the EP+FE row into certified data	.45	.65	.53
G5	*Sign-definite object for Ingham–Landau* (carried from L62 F2)	.70	.25	.52
G6	*Beurling systems with a functional equation*: is EP+FE without being an L-function possible? (literature first)	.50	.30	.42

10. Questions for J.W.

Q1. Adopt Theorem 4.3 (two-sided slack $\ll 1$ for every term; removing any term breaks positivity) as the official form of "positivity is collective"?

Q2. Adopt the three-ingredient reading ($\omega \leftrightarrow$ FE, Frobenius/Hecke $\leftrightarrow$ EP and the prime side, $g \leftrightarrow$ geometric positivity, missing over $\mathbb{Q}$) in place of the two-ingredient "multiplicativity + duality"?

Q3. Run G1 next (function-field anatomy, score .80)?

Q4. Then G2 ($L(E, s)$ versus Hasse forms) or G4 (certified $L(s, \chi)$ windows)?

Q5. Include Lemma 64 with its five figures in Volume 4? Nothing will be submitted without your approval.

Statement of intellectual honesty (repeated)

The Riemann hypothesis is **not proven**. Nothing in this lemma advances it. The numerical results are Ritz values on finite truncations; only the numbers quoted from Lemma 57 are certified. Every RH-equivalent statement is labelled [RESTATEMENT].

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TAA LEMMA 65: THE FIELD WITH ONE ELEMENT AND THE SINGLE FROBENIUS

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ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Nothing below is a step toward a proof that we can certify; every RH-equivalent statement is a [RESTATEMENT]. Tags: [THEOREM] proved here (elementary) or classical with proof indicated; [CITED] checked at the source (`lit/lemma65/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (floating point or exact integer arithmetic, not certified unless said).

Questions (J.W.). [t247u] "If an elliptic curve E over $\mathbb{F}_p$ has prime-side data that is a single number, shouldn't this single number be the basis for the mythical field with one element?" Steerings: [t248u] two fermion signs (multiplicativity, duality, Lefschetz); [t249u] why the surface $C \times C$; [t250u] the Lefschetz motive $\mathbb{L}$.

Answer in brief. (1) *The model is right.* For $E/\mathbb{F}_p$ the single number a_p is the trace of the single Frobenius π on the 2-dimensional H^1, and $\#E(\mathbb{F}_{p^n}) = p^n + 1 - \alpha^n - \bar{\alpha}^n$ is a Lefschetz trace formula. Hasse's bound is positivity of the degree form $Q_p(x,y) = \deg(x + y\pi) = x^2 + a_p xy + py^2$ [CITED]. We prove that $2Q_p$ is *exactly* Weil's Castelnuovo–Severi defect form on $\mathrm{span}\{\Delta, \Gamma_\pi\} \subset \mathrm{NS}(E \times E)$ (Prop. 3.1) [THEOREM]: Hasse's proof is the genus-one shadow of the Hodge index theorem on the surface. (2) *What changes over* $\mathbb{F}_1$. In the Connes–Consani counting distribution for $\mathrm{Spec}\,\mathbb{Z}$, $N(u) = u - \frac{d}{du}\big(\sum_\rho \mathrm{ord}(\rho)\frac{u^{\rho+1}}{\rho+1}\big) + 1$ [CITED], the Frobenius power $n \in \mathbb{N}$ becomes a continuous parameter $u \geq 1$, and α_i^n becomes u^ρ. For curves this is literal: with $u = q^n$ and $\alpha_i = q^{\rho_i}$ one has $\alpha_i^n = u^{\rho_i}$ [THEOREM]. So the single number a_p (one trace, two eigenvalues) becomes the *whole* zero sum, one "eigenvalue" per zero, with infinite genus ($N(1) = -\infty$). The answer to J.W.'s question is therefore: *a_p is the right model of what $\mathbb{F}_1$ should supply, but over $\mathbb{F}_1$ the single trace is $\sum_\rho u^\rho$, not a single number.* For $E/\mathbb{Q}$, the collection $\{a_p\}_p$ is the L-function. Its local positivity (one binary form per prime, Hasse) is proved; its global positivity (RH for $L(E,s)$) is not. This is exactly Lemma 64's "collective, not prime by prime" split. (3) *Does $\mathbb{F}_1$ shed light on J?* Not yet. No $\mathbb{F}_1$ approach has a surface $\mathrm{Spec}\,\mathbb{Z} \times_{\mathbb{F}_1} \mathrm{Spec}\,\mathbb{Z}$ carrying a proved Hodge index. The Connes–Consani periodic orbits $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$ carry a genus-one Riemann–Roch formula [CITED], which is a striking per-prime "elliptic curve", but it is per prime, i.e. local. Faltings–Hriljac is a real global Hodge index, but for arithmetic surfaces (curves over $\mathrm{Spec}\,\mathcal{O}_K$), not for $\mathrm{Spec}\,\mathbb{Z} \times \mathrm{Spec}\,\mathbb{Z}$. Verdict: *zero counting is reproduced exactly (Connes–Consani); the polarization is not supplied.* The heuristic that positivity "lives on the surface" is a heuristic (§3). (4) *The Lefschetz motive* (§4). $\mathbb{L} = h^2(\mathbb{P}^1)$, Frobenius acts by q; $h(C) = 1 \oplus h^1 \oplus \mathbb{L}$; Poincaré duality $h^1 \otimes h^1 \to \mathbb{L}$ is the source of $F\,{}^tF = q$. The pole terms of the curve explicit formula are the h^0, h^2 traces [THEOREM]. For ζ, "POLE $= 1 \oplus \mathbb{L}$" is an [ANALOGY], with one checkable match: Lemma 64's POLE has signature $(1,1)$, exactly the hyperbolic plane $\langle C \times \mathrm{pt}, \mathrm{pt} \times C\rangle$ that carries the single positive direction of the Hodge index theorem. (4′) *The triple* (§5, central). For $E/\mathbb{F}_p$ the whole polarization triple is made from the single Frobenius. ω is the duality $\Lambda^2 h^1 \cong \mathbb{L}$, with $\omega(\pi x, \pi y) = p\,\omega(x,y)$. $J_p = (2\pi - a_p)/\sqrt{4p - a_p^2}$ satisfies $J_p^2 = -1$ and $\det J_p = 1$, its $\pm i$ eigenlines are the Frobenius eigenlines, and in the basis $(1, \pi)$ it equals Lemma 60's J_{Q_p} exactly. $g_p = \omega(\cdot, J_p\cdot) = 2\deg/\sqrt{|d|} > 0$ [THEOREM]. RH for $E/\mathbb{F}_p \iff$ Frobenius defines this polarization [RESTATEMENT], proved non-circularly by $\deg > 0$. The missing object for ζ is a "Frobenius" whose normalized imaginary part is J and whose g is positive for a reason other than zero location [OPEN]. (5) *Two fermion signs* (§9). The prime-side Fock space (Euler product = tensor factorization) and the cohomological super Fock space $\mathrm{Sym}(H^0) \otimes \Lambda(H^1) \otimes \mathrm{Sym}(H^2)$ carry *different* $\mathbb{Z}/2$ gradings. For curves, the Lefschetz formula on $\mathrm{Sym}^n C$ is literally the map between them, and it intertwines "bosons↔fermions" on the prime side ($\zeta \leftrightarrow 1/\zeta$, i.e. μ) with the parity shift Π of cohomology (Thm. 9.3) [THEOREM]. Multiplicativity + duality + trace formula give rationality and the functional equation; RH needs a *third* input, positivity, even for curves.

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1. Sources and conventions

Read for this lemma: STATE.md §§67–71 (Lemmas 58–62), the Lemma 64 drafts in `build64/` (read-only; Lemma 64 is in progress), Lemma 58 (APD, Prop. "compact $\iff$ elliptic"), Lemma 60 (forms = polarizations J_Q), Lemma 61 (emission vs absorption), Lemma 62 §6 (signs, Spector, odd degree $\Rightarrow$ symplectic), Lemma 42 Ex. 3.3 (Rosati Gram $\left(\begin{smallmatrix} 2 & a \\ a & 2p \end{smallmatrix}\right)$), Lemma 23 (tropical/$\mathbb{F}_1$ products). Primary sources downloaded and read for this lemma: Connes–Consani [2] (arXiv v3; equation and theorem numbers below are those of v3), Connes–Consani [5], [7], [8], López Peña–Lorscheid [25], Soulé [35], Borger [1], Moriwaki [32], and Milne's survey [31]. Tits [37] and Manin [29] were *not* available; they are cited only through [25, 35, 2].

Conventions. C is a smooth projective geometrically connected curve of genus g over $\mathbb{F}_q$, π its (q-power) Frobenius endomorphism, $N_n = \#C(\mathbb{F}_{q^n})$, $Z(C,T) = \exp\sum_n N_n T^n/n = P_1(T)/((1-T)(1-qT))$ with $P_1(T) = \prod_{i=1}^{2g}(1-\alpha_i T)$. On $C \times C$: $C_1 = C \times \mathrm{pt}$, $C_2 = \mathrm{pt} \times C$, $d_1(D) = D \cdot C_1$, $d_2(D) = D \cdot C_2$, Δ the diagonal, $\Gamma_\pi = \{(x, \pi x)\}$ the graph, so $d_1(\Gamma_\pi) = q$, $d_2(\Gamma_\pi) = 1$ (conventions of [31, §1]). $\zeta_{\mathbb{Q}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the complete zeta function. Scripts: `num/lemma65_{counts,figs}.py` $\to$ `num/l65/`.

2. The single Frobenius: $E/\mathbb{F}_p$ and curves over $\mathbb{F}_q$

Proposition 2.1 (Point counts as a trace formula)**.** *[CITED]* [31, §1]. *$Z(C,T) = \prod_P (1 - T^{\deg P})^{-1}$ over closed points; by Riemann–Roch $Z(C,T) = P_1(T)/((1-T)(1-qT))$ with $P_1 \in \mathbb{Z}[T]$ of degree $2g$, and*

$$N_n = q^n + 1 - \sum_{i=1}^{2g} \alpha_i^n. \tag{1}$$

If $|N_n - q^n - 1| \le 2g\, q^{n/2}$ for all n, then $|\alpha_i| = \sqrt{q}$ for all i ([31, §1, footnote 1]: the power series $\sum_i \alpha_i z/(1-\alpha_i z)$ converges for $|z| < q^{-1/2}$, and $\alpha \mapsto q/\alpha$ permutes the roots).

For an elliptic curve $g = 1$, so the "single number" is $a_p := \alpha + \bar{\alpha} = p + 1 - \#E(\mathbb{F}_p)$, and (1) reads $\#E(\mathbb{F}_{p^n}) = p^n + 1 - (\alpha^n + \bar{\alpha}^n)$. The whole sequence is determined by a_p alone.

Proposition 2.2 (Hasse via the positive degree form)**.** *[CITED]* [31, sketch "The proof of the Riemann hypothesis for elliptic curves"], [34, Cor. III.6.3, Thm. V.1.1]*[TAA-L]. For an endomorphism α of E with characteristic polynomial $T^2 + cT + d$ on $T_\ell E$, one has $m^2 + cmn + dn^2 = \deg(m - n\alpha) \geq 0$ for all $m, n \in \mathbb{Z}$, hence $c^2 \leq 4d$. For $\alpha = \pi$: $c = -a_p$, $d = \deg \pi = p$, so*

$$Q_p(x, y) := \deg(x + y\pi) = x^2 + a_p\, xy + p\, y^2 \ \geq 0 \quad \Longrightarrow \quad a_p^2 \leq 4p, \qquad Q_p(1, -1) = \#E(\mathbb{F}_p).$$

Strictness: $a_p^2 = 4p$ is impossible for p prime since $4p$ is not a square, so Q_p is positive definite and $|a_p| < 2\sqrt{p}$.

So the positivity used by Hasse is *positivity of the degree*: the degree of an isogeny is the number of geometric points in its (separable) kernel times an inseparable degree, a positive integer. Its matrix $\left(\begin{smallmatrix} 1 & a_p/2 \\ a_p/2 & p \end{smallmatrix}\right)$ is the J_Q-polarized binary form of Lemma 60 with discriminant $a_p^2 - 4p < 0$ [TAA-L].

Proposition 2.3 (Weil for curves: which positivity)**.** *[CITED]* [31, Thm. 1.2, Cor. 1.3, Thm. 1.5, Cor. 1.6, Ex. 1.7 and "Proof of the Riemann hypothesis for curves"], [20, V Thm. 1.9, Ex. 1.9, 1.10]*[TAA-L]. The positivity used is the* Hodge index theorem *on the surface $C \times C$: the intersection form on $\mathrm{NS}(C \times C) \otimes \mathbb{R}$ has exactly one positive eigenvalue. Its consequence used is the Castelnuovo–Severi inequality $D^2 \leq 2d_1 d_2$. Equivalently* [31, Thm. 1.27]*, Rosati positivity $\mathrm{Tr}(\varphi\varphi^\dagger) > 0$ on $\mathrm{End}^0(J_C)$. Historically, Hodge index in characteristic p is due to Grothendieck* [18] *(after Mattuck–Tate* [30]*); Weil's own 1948 proof* [38] *went through the Jacobian and Castelnuovo's equivalence defect (as reported in* [31]*).*

3. Why the surface? Multiplicativity, duality and Hodge index on $C \times C$

J.W. [t249u]: how do we know Hodge-index positivity would live on $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$, and where do multiplicativity and duality enter? We first write Weil's proof so that each ingredient sits in one line. Then we say precisely what is known, and what is only heuristic, for $\operatorname{Spec}\mathbb{Z}$.

3.1. **Weil's proof in five lines.** Define, following [31, §1], the *equivalence defect* $\mathrm{def}(D) = 2d_1(D)d_2(D) - D^2$ and Weil's trace $\sigma(D) = d_1(D) + d_2(D) - D \cdot \Delta$.

(W1) *Multiplicativity ⇒ point counts are intersection numbers.* The Euler product $Z = \prod_P (1 - T^{\deg P})^{-1}$ gives $N_n = \sum_{d|n} d \cdot \#\{P : \deg P = d\}$, and this equals the number of fixed points of π^n on $C(\bar{\mathbb{F}}_q)$. Since $d\pi = 0$, Γ_{π^n} meets Δ transversally, so

$$\Gamma_{\pi^n} \cdot \Delta = N_n \qquad \text{[31, "Proof of RH for curves"] [CITED].}$$

(W2) *Duality.* Let ${}^t\Gamma_\pi$ be the transpose (as a correspondence: the Verschiebung). Since π is a bijection on $\bar{\mathbb{F}}_q$-points, purely inseparable of degree q, ${}^t\Gamma_\pi \circ \Gamma_\pi = \Gamma_\pi \circ {}^t\Gamma_\pi = q\,\Delta$ [THEOREM] (elementary). On H^1 this is $\pi^\dagger \pi = q$ for the adjoint with respect to the Weil pairing, $\omega(\pi x, \pi y) = q\,\omega(x, y)$ ([31, Thm. 1.30]; Lemma 50 [TAA-L]). Hence $\alpha \mapsto q/\alpha$ permutes the eigenvalues: the functional equation.

(W3) *Positivity.* Hodge index ⇒ $\mathrm{def}(D) \geq 0$ for all D (Castelnuovo–Severi), and by polarization $|D \cdot D' - d_1 d_2' - d_2 d_1'| \leq (\mathrm{def}\, D\ \mathrm{def}\, D')^{1/2}$ [31, Thm. 1.5, Cor. 1.6][CITED].

(W4) *Computation.* $\mathrm{def}(\Gamma_f) = 2g \deg f$ [31, (7)], so $\mathrm{def}(\Delta) = 2g$ and $\mathrm{def}(\Gamma_\pi) = 2gq$. With $D = \Gamma_\pi$, $D' = \Delta$: $|N_1 - q - 1| \leq 2g\sqrt{q}$ [CITED].

(W5) *Powers.* Replacing π by π^n: $|N_n - q^n - 1| \leq 2gq^{n/2}$ for all n, hence $|\alpha_i| = \sqrt{q}$ by Prop. 2.1 [CITED].

Proposition 3.1 (The defect form on $\mathrm{span}\{\Delta, \Gamma_\pi\}$; Hasse = genus-one Weil)**.** *[THEOREM]. For $D = m\Delta + n\Gamma_\pi$,*

$$\mathrm{def}(m\Delta + n\Gamma_\pi) = 2g\, m^2 + 2\, a\, mn + 2g\, q\, n^2, \qquad a := \sigma(\Gamma_\pi) = q + 1 - N_1 = \mathrm{Tr}(\pi \mid H^1).$$

It is positive semidefinite iff $a^2 \le 4g^2q$, i.e. iff $|N_1 - q - 1| \le 2g\sqrt{q}$. For $g = 1$, $\mathrm{def}(m\Delta + n\Gamma_\pi) = 2\,Q_p(m,n)$ with Q_p the degree form of Prop. 2.2.

Proof. $\mathrm{def}(D) = \sigma(D \circ D')$ with D' the transpose [31, "Correspondences"]. Now $D \circ D' = m^2\Delta + mn(\Gamma_\pi + {}^t\Gamma_\pi) + n^2\,\Gamma_\pi \circ {}^t\Gamma_\pi = (m^2 + qn^2)\Delta + mn(\Gamma_\pi + {}^t\Gamma_\pi)$ by (W2). Further, $\sigma(\Delta) = 2 - \Delta^2 = 2 - (2 - 2g) = 2g$, and $\sigma(\Gamma_\pi) = \sigma({}^t\Gamma_\pi) = q + 1 - N_1$, since d_1, d_2 swap and $\Gamma_\pi \cdot \Delta = {}^t\Gamma_\pi \cdot \Delta$. Linearity of σ gives the formula. For $g = 1$ it is $2(m^2 + a\,mn + qn^2)$ with $a = a_p$. The Gram matrix $\left(\begin{smallmatrix} 2g & a \\ a & 2gq \end{smallmatrix}\right)$ is Lemma 42's Rosati Gram for $g = 1$ [TAA-L]. □

So J.W.'s binary form is not just "like" the surface proof: *it is the restriction of the Hodge-index form to the plane spanned by the diagonal and the Frobenius graph.* The plane is spanned by the identity and the single Frobenius; it is 2-dimensional because the prime-side datum is one number. Check: for the genus-2 curve $y^2 = x^5 + x + 1$ over $\mathbb{F}_{11}$ we find $N_1 = 8$, $a = 4$, and $a^2 = 16 \le 4 \cdot 4 \cdot 11$ (§10) [NUMERICAL].

Remark 3.2 (Where each ingredient enters)**.** Multiplicativity enters once, in (W1): the Euler product turns closed points into fixed points, and fixed points into the intersection number $\Gamma_\pi \cdot \Delta$. Duality enters in (W2) and fixes the shape $\alpha \leftrightarrow q/\alpha$. Positivity is a property of the *surface*, not of C and not of any single prime: the inequality compares Γ_π with Δ *inside* $\mathrm{NS}(C \times C)$. That is the precise sense in which "the polarization lives on the surface" for curves [THEOREM].

3.2. **What this says about** $\operatorname{Spec}\mathbb{Z}$**: heuristic, not knowledge.**

Proposition 3.3. *[THEOREM] (trivial).* $\operatorname{Spec}\mathbb{Z} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z} = \operatorname{Spec}(\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) = \operatorname{Spec}\mathbb{Z}$. *Since $\mathbb{Z}$ is initial among rings, any genuinely 2-dimensional "*$\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$*" must be formed over a base below $\mathbb{Z}$, outside rings and schemes; this is the stated motivation of $\mathbb{F}_1$* [1, 35]*[CITED]. In Durov's framework the fibre product over $\mathbb{F}_1$ is again* $\operatorname{Spec}\mathbb{Z}$ [25, §1.4]*[CITED-sec]. In Deitmar's and Borger's frameworks the analogous collapse is a* deduction *from the universal properties (Lemma 23 [TAA-L]), not a quotation.*

Remark 3.4 (Status labels)**.** "RH for ζ would follow from a Hodge index theorem on a surface $\operatorname{Spec}\mathbb{Z} \times_{\mathbb{F}_1} \operatorname{Spec}\mathbb{Z}$" is a [HEURISTIC] modelled on (W1)–(W5). It is Manin's question as reported by Soulé [35, §1.2][CITED-sec] and Connes's "what is missing is an intersection theory and a Riemann–Roch theorem on the square of the arithmetic site" [10][TAA-L]. It is *not* known that positivity must live on a surface: Weil's criterion (positivity of the explicit-formula functional) needs no surface [RESTATEMENT]; Connes's trace formula [9][TAA-L] and Connes–Consani's archimedean result [7, Thm. 1][CITED] ($W_\infty(g * g^*) \ge \mathrm{Tr}(\vartheta(g) S \vartheta(g)^*)$ for $\operatorname{supp} g \subset [2^{-1/2}, 2^{1/2}]$, $\hat{g}$ vanishing at $\frac{i}{2}$ and 0) are surface-free. Whether a surface is *needed*: [OPEN].

Remark 3.5 (Faltings–Hriljac is real, but it is a different surface)**.** An arithmetic surface is a regular 2-dimensional X, proper and flat over $\operatorname{Spec}\mathcal{O}_K$, whose generic fibre is a curve X_K. Arakelov intersection theory on X satisfies a Hodge index theorem: Faltings [17] and Hriljac [21], as stated by Moriwaki [32][CITED-sec] (his Thm. B is a higher-dimensional generalization, and his case $d = 1$ "can be derived from Faltings–Hriljac's Hodge index theorem"). For a divisor whose generic-fibre degree is zero, the arithmetic self-intersection is ≤ 0, with equality only for pullbacks from $\operatorname{Spec}\mathcal{O}_K$ [32, Thm. B, $d = 1$][CITED]. The Faltings–Hriljac identification of this form with (minus a positive multiple of) the Néron–Tate height pairing is [BIB]: Hriljac's title is "Heights and Arakelov's intersection theory", and we did not pin the constant. So "Faltings–Hriljac gives positivity of heights" is correct as a summary. But the surface is (geometric curve) × (arithmetic $\operatorname{Spec}\mathcal{O}_K$), with one arithmetic direction, not $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ with two. Its positivity is that of heights on $\mathrm{Jac}(X_K)(K)$, the Birch–Swinnerton-Dyer side, not that of the Weil functional of $L(X_K, s)$ [ANALOGY].

4. The Lefschetz motive: where duality lands and positivity is measured

J.W. [t250u]. All statements in (1)–(4) are about varieties over a field; (5) is about ζ.

(1) $\mathbb{L}$. In the axioms of a Weil cohomology, $\mathbb{Q}(-1) := H^2(\mathbb{P}^1)$ [31, "Poincaré duality" (b)][CITED]. In the category of motives $h(\mathbb{P}^1) = h^0(\mathbb{P}^1) \oplus h^2(\mathbb{P}^1)$, and $\mathbb{L} := h^2(\mathbb{P}^1)$ is formally inverted [31, §4][CITED]. The Frobenius endomorphism acts on $H^{2d}(V)$ by q^d [31, proof of Prop. 2.6][CITED], so it acts on $\mathbb{L}$ by q. *Convention caveat:* in ℓ-adic language $\mathbb{L} \leftrightarrow \mathbb{Q}_\ell(-1)$, on which the *geometric* Frobenius (the one induced by π) acts by q; the arithmetic Frobenius acts by q^{-1}.

(2) $h(C) = 1 \oplus h^1(C) \oplus \mathbb{L}$. For a curve with a rational point the motive splits with $h^2(C) \cong \mathbb{L}$ (Manin [28]; Murre's Chow–Künneth decomposition for surfaces [33] is the next case) [BIB]. Taking $Z(h, T) = \prod_i \det(1 - \pi T \mid h^i)^{(-1)^{i+1}}$ gives $Z(C, T) = P_1(T)/((1 - T)(1 - qT))$: $1 \mapsto (1 - T)^{-1}$, $\mathbb{L} \mapsto (1 - qT)^{-1}$ [THEOREM] (with Prop. 2.1).

(3) Duality lands in $\mathbb{L}$. Cup product $h^1 \otimes h^1 \to h^2 \cong \mathbb{L}$ is a perfect alternating pairing. It is alternating because cup product is graded-commutative and $1 \cdot 1$ is odd (Lemma 62 §6 [TAA-L]). For an elliptic curve the ℓ-adic version is the Weil pairing $T_\ell E \times T_\ell E \to \mathbb{Z}_\ell(1)$, i.e. $H^1 \times H^1 \to \mathbb{Q}_\ell(-1)$ after dualizing [BIB]. Because the pairing is Frobenius-equivariant and π acts on $\mathbb{L}$ by q, $\omega(\pi x, \pi y) = \pi \cdot \omega(x, y) = q\,\omega(x, y)$, i.e. $\pi^\dagger \pi = q$ ([31, Thm. 1.30][CITED]). *The q in $F\,{}^t\!F = q$ is the Frobenius eigenvalue of $\mathbb{L}$.*

(4) Positivity is measured in $\mathbb{L}$: the standard conjectures. A polarization of h^1 is an $\mathbb{L}$-valued alternating form whose associated symmetric form (via a complex structure, or via the Rosati involution on $\mathrm{End}(h^1)$) is positive. Over $\mathbb{C}$ this is Riemann's bilinear relations on $H^{1,0} \oplus H^{0,1}$ (Lemma 40 [TAA-L]). Over $\mathbb{F}_q$ there is no Hodge decomposition (Lemma 42 Rem. 3.6 [TAA-L]). Positivity is then available only on *algebraic* classes: $\mathrm{End}^0(h^1) \subset \mathrm{NS}(C \times C) \otimes \mathbb{Q}$, Rosati/Hodge index. Grothendieck's standard conjectures [19][BIB], as stated in [31, §3][CITED]:

- *Lefschetz type $B(V)$*: the inverse Lefschetz operator Λ is algebraic (equivalent to $A(V, L)$ for all V; implies $C(V)$, algebraicity of the Künneth projectors; [31, Prop. 3.11] = [24, Thm. 4-1]).
- *Hodge type $I(V, L)$*: $(-1)^r(x \cdot y \cdot L^{d-2r})$ is positive definite on primitive algebraic classes in $A^r(V)$.
- *Consequence:* assuming both, $x, y \mapsto (x \cdot *y)$ is a Weil form [31, Thm. 3.15] (proof: Kleiman [23], 3.11), and the eigenvalues of Frobenius on H^r have absolute value $q^{r/2}$ [31, Thm. 3.17][CITED].
- *Status* (as reported in [31]): B is known for abelian varieties and certain surfaces ([31, Ex. 3.13], Kleiman 1968). Hodge type is known in characteristic 0 via Hodge theory; "in nonzero characteristic, almost nothing is known except for surfaces", where it is the Hodge index theorem [CITED]. Deligne proved RH over finite fields by a different route [13][BIB] (content via [31]).

For a curve the relevant variety is the surface $C \times C$, where Hodge type is Hodge index, a theorem. So for curves "standard conjectures $\Rightarrow$ RH" is Weil's proof, and in that case the hypotheses are proved [THEOREM].

(5) For ζ. Deninger [14][TAA-L]: $\xi(s) := \frac{s}{2\pi}\frac{s-1}{2\pi}\zeta_{\mathbb{Q}}(s) = \prod_\rho \frac{s-\rho}{2\pi}$ (regularized product; proved, [14] citing Deninger and Schröter–Soulé), with conjectural $H^0 = \mathbb{R}$ ($\Theta = 0$) giving $s/2\pi$ and $H^2 = \mathbb{R}$ ($\Theta = \mathrm{id}$) giving $(s - 1)/2\pi$. So "the factor $s(s - 1)$ of ξ is $1 \oplus \mathbb{L}$" is an [ANALOGY] inside a [CONJECTURE] (the cohomology itself is conjectural); Manin's reading $h^0 = \mathrm{Spec}\,\mathbb{F}_1$, $h^2 =$ affine line over $\mathbb{F}_1$ is reported in [25][CITED-sec].

Proposition 4.1 (Pole terms $= 1 \oplus \mathbb{L}$: theorem for curves, analogy for ζ)**.** *(a) [THEOREM]. In* (1) *the terms* 1 *and* q^n *are* $\mathrm{Tr}(\pi^n \mid h^0)$ *and* $\mathrm{Tr}(\pi^n \mid h^2 = \mathbb{L})$*. In the explicit formula of* $\zeta_C(s) = Z(C, q^{-s})$ *they are the residues at the poles* $s = 0, 1$*. (b) [THEOREM]. In* $\mathrm{NS}(C \times C) \otimes \mathbb{R}$ *the classes* C_1, C_2 *(i.e.* $h^0 \otimes h^2$ *and* $h^2 \otimes h^0$*) span a plane with Gram matrix* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$*: hyperbolic, signature* $(1, 1)$*. By Hodge index, the orthogonal complement (which contains the classes* $D - d_2 C_1 - d_1 C_2$ *used in (W3), and the image of* $\mathrm{End}^0(h^1)$*) is negative definite. (c) [ANALOGY]. Lemma 64 finds that the POLE term of Weil's quadratic form for* ζ*,* $2\,\mathrm{Re}(\hat{F}(\frac{i}{2})\overline{\hat{F}(-\frac{i}{2})})$*, has rank 2 and signature* $(1, 1)$ *for every support [TAA-L]. It pairs the evaluations at* $s = 0$ *and* $s = 1$*, as Poincaré duality pairs* h^0 *with* h^2*. So the identification POLE* $\leftrightarrow 1 \oplus \mathbb{L}$ *matches in signature, not merely in name.*

Consequence for the chat claim "the pole gives the clearly positive term": in both settings the pole/fibre plane is *hyperbolic*, carrying one positive and one negative direction. The positivity that

RH needs is *negativity on its orthogonal complement.* For curves that is Hodge index; for ζ it is Weil positivity after removing the pole directions, which is what the vanishing conditions in [7] do (Lemma 64 [TAA-L]).

5. The polarization triple (ω, J, g) motivically

J.W. [t251u]: understand the Lefschetz motive through the triple (ω, J, g) and the binary forms of Lemma 60. This is the central section: for $E/\mathbb{F}_p$ the whole triple is *manufactured from the single Frobenius*, and each of its three pieces sits in one of h^0, h^1, $\mathbb{L}$.

Let $E/\mathbb{F}_p$ be an elliptic curve, $a = a_p$, $d = a^2 - 4p$. Write $V = \mathbb{Z}[\pi] \otimes \mathbb{R}$ with basis $(1, \pi)$. On this basis multiplication by π has matrix $M = \left(\begin{smallmatrix} 0 & -p \\ 1 & a \end{smallmatrix}\right)$, the companion matrix of $\chi_\pi(T) = T^2 - aT + p$, the characteristic polynomial of π on $T_\ell E$ [31, sketch for elliptic curves][CITED]. V is a rational/real model of H^1 with the same characteristic polynomial. (The ℓ-adic H^1 has no real structure; Lemma 42 Rem. 3.4 [TAA-L].) Let $\omega(x, y) = x_1 y_2 - x_2 y_1$ in this basis.

Theorem 5.1 (ω: duality lands in $\mathbb{L} = \Lambda^2 h^1$)**.** *[THEOREM]/[CITED]. (a) For E, cup product $\Lambda^2 H^1 \to H^2$ is an isomorphism onto the 1-dimensional H^2, which is $\mathbb{L}$ (§4(1)–(3)). ℓ-adically it is the Weil pairing $T_\ell E \times T_\ell E \to \mathbb{Z}_\ell(1) = \varprojlim \mu_{\ell^n}$ [BIB] (Silverman AEC III.8; theorem numbers not checked). (b) $\omega(\pi x, \pi y) = p\,\omega(x, y)$* [31, Thm. 1.30]*[CITED]. In dimension 2 this is automatic once* $\det(\pi \mid H^1) = \deg \pi = p$ [31, sketch]*[CITED], since any area form scales by the determinant. (c) Hence $\det(\pi \mid H^1) = p$ is the eigenvalue of π on $\mathbb{L}$:* the Frobenius eigenvalue of the Lefschetz motive is the determinant of the single Frobenius.

Theorem 5.2 ($(1, a_p, p)$ are traces on $h^0, h^1, \mathbb{L}$)**.** *[THEOREM].* $\mathrm{Tr}(\pi \mid \Lambda^k H^1) = 1,\ a_p,\ p$ *for $k = 0, 1, 2$, i.e. the traces on h^0, h^1, $h^2 = \mathbb{L}$. The degree form is*

$$Q_p(x, y) = \deg(x + y\pi) = \det(x + y\pi \mid H^1) = x^2 + a_p xy + py^2 = \mathrm{Nm}_{\mathbb{Q}(\pi)/\mathbb{Q}}(x + y\pi),$$

and $\det(x - y\pi) = x^2 - a_p xy + py^2 = Q_p(x, -y)$ *is the homogenized characteristic polynomial. So "Q_p is the characteristic polynomial written as a form" is* ***correct up to*** *$y \mapsto -y$. The sign convention $Q_p = \deg(x + y\pi)$ is the one used in Lemma 64. Lefschetz: $\#E(\mathbb{F}_p) = 1 - a_p + p = Q_p(1, -1) = \deg(1 - \pi)$.*

Theorem 5.3 (J: Frobenius chooses its own complex structure)**.** *[THEOREM] (elementary). Assume $a^2 < 4p$ (true by Prop. 2.2). Put*

$$J_p := \frac{2\pi - a}{\sqrt{4p - a^2}} = \frac{\pi/\sqrt{p} - \cos\theta_p}{\sin\theta_p}, \qquad a = 2\sqrt{p}\cos\theta_p,\ \theta_p \in (0, \pi).$$

(a) $J_p^2 = (4\pi^2 - 4a\pi + a^2)/(4p - a^2) = (a^2 - 4p)/(4p - a^2) = -I$, using $\pi^2 = a\pi - p$. (b) $\mathrm{Tr}\, J_p = (2a - 2a)/\sqrt{\cdot} = 0$ and $\det J_p = 1$, so $J_p \in SL_2(\mathbb{R}) = Sp(V, \omega)$ preserves ω; J_p commutes with π. (c) If $\pi v = \alpha v$ with $\alpha = \sqrt{p}e^{i\theta_p} = \frac{a + i\sqrt{4p - a^2}}{2}$, then $J_p v = \frac{2\alpha - a}{\sqrt{4p - a^2}} v = iv$. The $\pm i$ eigenspaces of J_p are the Frobenius eigenlines of $\alpha, \bar{\alpha}$: the two helicities of Lemma 62 [TAA-L]. (d) In the basis $(1, \pi)$,

$$J_p = \frac{1}{\sqrt{4p - a^2}} \begin{pmatrix} -a & -2p \\ 2 & a \end{pmatrix} = J_{Q_p},$$

exactly *Lemma 60's $J_Q = \frac{1}{\sqrt{|d|}}\left(\begin{smallmatrix} -b & -2c \\ 2a & b \end{smallmatrix}\right)$ for $Q = [1, a_p, p]$ [TAA-L]. Consequently (Lemma 60 Prop. (b)) $g_p(x, y) := \omega(x, J_p y)$ satisfies $g_p(x, x) = 2Q_p(x)/\sqrt{|d|} = 2\deg(x)/\sqrt{|d|} > 0$: the positive sign is $+J_p$, and $-J_p$ is the negative polarization. (e) For supersingular $a_p = 0$: $J_p = \pi/\sqrt{p}$ and $\theta_p = \pi/2$.*

Proof. (a)–(c) as displayed. (d) $2M - aI = \left(\begin{smallmatrix} -a & -2p \\ 2 & a \end{smallmatrix}\right)$. With ω as above, $g(e_1, e_1) = \omega(e_1, Je_1)$ is the lower-left entry $2/\sqrt{|d|} > 0$; the rest is Lemma 60 Prop. (b). □

Sign correction. The chat formula $J_Q = \frac{1}{\sqrt{|d|}}\left(\begin{smallmatrix} b & 2c \\ -2a & -b \end{smallmatrix}\right)$ is $-J_Q$ of the published Lemma 60, the *negative* polarization for $\omega = x_1y_2 - x_2y_1$. With it, J_{Q_p} would equal $-J_p$.

Proposition 5.4 (RH for $E/\mathbb{F}_p$ as a polarization statement)**.** *[RESTATEMENT], proved non-circularly. For $\pi \in GL(V)$, $\dim V = 2$, $\det \pi = p > 0$, the following are equivalent: (i) $a^2 < 4p$ (non-real eigenvalues; then $|\alpha| = \sqrt{p}$); (ii) $\pi/\sqrt{p} \in SL_2(\mathbb{R})$ is elliptic; (iii) there is a positive ω-compatible J commuting with π. Then $J = J_p$ is unique, up to the sign fixed by positivity. (Lemma 58 "compact $\iff$ elliptic $\iff$ commutes with a polarization" [TAA-L]; Lemma 54 "Frobenius chooses its own J" [TAA-L].) For $E/\mathbb{F}_p$, (i) is* proved *by degree positivity (Prop. 2.2), independently of any zero location. So the triple (ω, J_p, g_p) exists and is non-circular: ω from duality ($\mathbb{L}$), J_p from the single Frobenius, $g_p > 0$ from* deg > 0.

Over $\mathbb{C}$. For $E(\mathbb{C}) = \mathbb{C}/\Lambda$, g-positivity is the Riemann bilinear relation $i\int \eta \wedge \bar{\eta} > 0$ on $H^{1,0}$ (Lemma 40 [TAA-L]), a case of the Hodge–Riemann relations. In general the Hodge-type standard conjecture holds in characteristic 0 by Hodge theory [31, §3][CITED]. So "Hodge-type standard conjecture is a theorem over $\mathbb{C}$" is **correct**, and more generally true in characteristic 0. Over $\mathbb{F}_p$ the analogous positivity on $E \times E$ is Hodge index (§3), and for E alone it is deg > 0.

TABLE 1. The triple for $E : y^2 = x^3 + x + 1$ and two supersingular curves (brute-force counts). $J_p = \frac{1}{\sqrt{|d|}}\left(\begin{smallmatrix} -a_p & -2p \\ 2 & a_p \end{smallmatrix}\right)$. In every row $\|J_p^2 + I\|$, $|\det J_p - 1|$, $\|J_p - J_{Q_p}\|$, $\|g_p - 2Q_p/\sqrt{|d|}\|\|$, $\|J_p v_\alpha - i v_\alpha\| \le 1.4 \cdot 10^{-14}$, and $g_p > 0$ [NUMERICAL].

curve	p	$\#E(\mathbb{F}_p)$	a_p	$Q_p = [1, a_p, p]$	$d = a_p^2 - 4p$	θ_p/π
$x^3 + x + 1$	5	9	-3	$[1, -3, 5]$	-11	0.7341
	7	5	3	$[1, 3, 7]$	-19	0.3081
	11	14	-2	$[1, -2, 11]$	-40	0.5975
	13	18	-4	$[1, -4, 13]$	-36	0.6872
	17	18	0	$[1, 0, 17]$	-68	0.5000
	19	21	-1	$[1, -1, 19]$	-75	0.5366
	37	48	-10	$[1, -10, 37]$	-48	0.8071
	41	35	7	$[1, 7, 41]$	-115	0.3159
$x^3 + x$ (supersing. $p \equiv 3$ (4))	7, 11, 19, 23	$p + 1$	0	$[1, 0, p]$	$-4p$	0.5000
$x^3 + 1$ (supersing. $p \equiv 2$ (3))	5, 11, 17	$p + 1$	0	$[1, 0, p]$	$-4p$	0.5000

Example: $p = 5$, $J_5 = \frac{1}{\sqrt{11}}\left(\begin{smallmatrix} 3 & -10 \\ 2 & -3 \end{smallmatrix}\right)$, $g_5 = \frac{2}{\sqrt{11}}(x^2 - 3xy + 5y^2)$. Supersingular $p = 7$ on $y^2 = x^3 + x$: $J_7 = \pi/\sqrt{7} = \frac{1}{\sqrt{7}}\left(\begin{smallmatrix} 0 & -7 \\ 1 & 0 \end{smallmatrix}\right)$.

5.1. **TAA reading.** For $E/\mathbb{F}_p$ the polarization is *one* binary form, built from *one* Frobenius. ω lives in $\mathbb{L}$ ($\det \pi = p$), J is the normalized imaginary part $(\pi - \frac{a}{2})/\sqrt{p - a^2/4}$ of Frobenius, and $g > 0$ because degrees are positive. For a curve of genus g the same construction works eigenplane by eigenplane on $H^1 = \bigoplus_j V_j$, with $J = \bigoplus_j J_{p,j}$, and positivity comes from Rosati/Hodge index (Lemma 42 Prop. 3.2 [TAA-L]). **The missing object for** ζ is therefore sharply described. One needs (1) a "Frobenius" Φ, i.e. the scaling flow, with generator Θ [14]; (2) a duality into an $\mathbb{L}$ on which Φ acts by its "determinant"; (3) J as the normalized imaginary part, $J = (\Theta - \frac{1}{2})/|\Theta - \frac{1}{2}|$ on each eigenplane; and (4) $g = \omega(\cdot, J\cdot) > 0$ *for a reason other than zero location*, the analogue of deg > 0. Steps (1)–(3) are formal once the zeros are given. Deninger's Hodge-$*$ assumption is exactly (4) [TAA-L]; with the zeros as input, (3)–(4) are circular (Lemma 58's spectral APD [TAA-L]). The single-Frobenius case shows what a non-circular (4) looks like: positivity of a *degree*, i.e. of an intersection number on a surface (Prop. 3.1). For ζ, a non-circular (4) is [OPEN].

6. What $\mathbb{F}_1$ is meant to supply

The function-field template needs three things over a base field k: a curve C/k, the surface $C \times_k C$, and a Hodge index theorem on it. For $\operatorname{Spec}\mathbb{Z}$ the hoped-for base is $\mathbb{F}_1$, with $\operatorname{Spec}\mathbb{Z}$ as a "curve over $\mathbb{F}_1$". Table 2 records what each approach achieves; every row was checked in the source named, or in the survey [25].

Table 2. Approaches to $\mathbb{F}_1$: what is achieved, and what is not. "HI" = a proved Hodge index on a 2-dimensional $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$; it is achieved in no row.

approach	achieved (label)	HI on the square?
Tits 1956/57 [37] via [25, 35]	S_N as "$GL_N(\mathbb{F}_1)$"; q-analogue counts at $q=1$, e.g. $\#SL_N(\mathbb{F}_q)/P(\mathbb{F}_q) = \{N\}/\{N_1\}\cdots\{N_r\}$ with $[n] = q^{n-1}+\cdots+1$ [35, §1.1][CITED-sec]	not addressed
Kapranov–Smirnov (unpubl.) via [25]	$\mathbb{F}_{1^n} \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[\mathbb{Z}/n]$ [CITED-sec]	no
Manin 1995, Kurokawa, Deninger [29, 14]	$h^0 = \operatorname{Spec}\mathbb{F}_1$, $h^2 = \mathbb{A}^1_{\mathbb{F}_1}$; $\zeta_{h^0} = s/2\pi$, $\zeta_{h^2} = (s-1)/2\pi$; question: meaning of $\operatorname{Spec}\mathbb{Z} \times_{\mathbb{F}_1} \operatorname{Spec}\mathbb{Z}$ [CITED-sec]	question posed
Soulé 2004 [35]	first definition of varieties over $\mathbb{F}_1$; zeta via counting polynomials [CITED]	no
Deitmar 2005 [12]	$\mathbb{F}_1$-schemes from monoids; zeta functions; toric [TAA-L]	collapses (deduction, L23)
Toën–Vaquié, Haran, Durov via [25]	relative algebraic geometry; Haran and Durov build $\overline{\operatorname{Spec}\mathbb{Z}}$; Durov's square collapses to $\operatorname{Spec}\mathbb{Z}$ [CITED-sec]	no
Borger 2009 [1]	Λ-rings = commuting Frobenius lifts; $\operatorname{Spec}\mathbb{Z} \times_{\mathbb{F}_1} \operatorname{Spec}\mathbb{Z} := W^*(\operatorname{Spec}\mathbb{Z})$ (L46); "says nothing about the archimedean place" [CITED]	no
Lorscheid [26]	blueprints; Tits's Weyl-group idea realized [25, Thm. 2.6][CITED-sec]	no
Connes–Consani 2009–10 [2]	exact counting distribution $N(u)$ of "$\operatorname{Spec}\mathbb{Z}$ over $\mathbb{F}_1$", positive on $(1,\infty)$ (§7) [CITED]	no
Arithmetic / scaling site [4, 5, 6]	topos $[0,\infty) \rtimes \mathbb{N}^\times$ whose points are adèle classes; the square of the arithmetic site with Frobenius correspondences (L23); $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$ with RR $\operatorname{Dim} H^0(D) - \operatorname{Dim} H^0(-D) = \deg D$ [CITED]	square exists; its H^1 and intersection theory/RR "still open" [TAA-L]
Riemann–Roch for $\overline{\operatorname{Spec}\mathbb{Z}}$ [8]	integer-valued RR on the *curve* $\overline{\operatorname{Spec}\mathbb{Z}}$, over the sphere spectrum [CITED]	curve only

Proposition 6.1 (Status)**.** *[CITED]/[TAA-L]. No approach in Table 2 provides a 2-dimensional* $\operatorname{Spec}\mathbb{Z} \times_{\mathbb{F}_1} \operatorname{Spec}\mathbb{Z}$ *with a proved Hodge index theorem. Connes–Consani have a* square *(of the arithmetic site, with Frobenius correspondences), so the chat claim "the surface has never been built" is* ***imprecise****. The correct statement: candidate squares exist (Connes–Consani; Borger's* $W^*(\operatorname{Spec}\mathbb{Z})$ *by definition), but none has the sheaf cohomology* H^1*, intersection theory and Riemann–Roch needed to run (W3), and none has a Hodge index.*

7. The Connes–Consani counting distribution

Theorem 7.1 (Connes–Consani [2], Thm. 2.2 and eqs. (11)–(17), arXiv v3)**.** *[CITED]. There is a unique distribution* $N(u)$ *on* $[1,\infty)$ *with*

$$-\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = \int_1^\infty N(u)\, u^{-s}\, d^*u, \qquad d^*u = du/u, \quad \operatorname{Re} s > 1.$$

It is positive on $(1,\infty)$ *and is given on* $[1,\infty)$ *by*

$$N(u) = u - \frac{d}{du}\Big(\sum_{\rho\in Z} \operatorname{order}(\rho)\,\frac{u^{\rho+1}}{\rho+1}\Big) + 1, \tag{2}$$

with Z *the nontrivial zeros, the sum taken symmetrically, and the derivative in the sense of distributions. The boundary value is* $\omega(1) = \frac{1}{2} + \frac{\gamma}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)}$ *for* $\omega(u) = \sum \operatorname{order}(\rho) u^{\rho+1}/(\rho+1)$. *On the prime side,*

$$N(u) = \frac{d}{du}\varphi(u) + \kappa(u), \qquad \varphi(u) = \sum_{n<u} n\Lambda(n), \qquad \kappa(u) = \frac{u^2}{u^2-1} \text{ (principal value at } u=1), \tag{3}$$

and for $u > 1$ *not a prime power (eq. (16))*

$$\varphi(u) = \frac{u^2}{2} - \sum_{\rho} \operatorname{order}(\rho)\frac{u^{\rho+1}}{\rho+1} + a(u), \qquad a(u) = \operatorname{ArcTanh}(1/u) - \frac{\zeta'(-1)}{\zeta(-1)}. \tag{4}$$

Formally (dropping the principal value), $N(u) \sim u - \sum_\rho \operatorname{order}(\rho) u^\rho + 1$ *(eq. (17)). The value* $N(1)$*, "the Euler characteristic", is formally* $\sim -\frac{1}{2} E \log E$ *with* $E = 1/\epsilon$*, i.e.* $-\infty$*, reflecting the infinite genus (Rem. 2.3).*

TABLE 3. Term-by-term comparison. For curves $u = q^n$ and $\alpha_i = q^{\rho_i}$, ρ_i the zeros of $\zeta_C(s) = Z(C, q^{-s})$ (defined mod $2\pi i/\log q$).

	$C/\mathbb{F}_q$ ($\#C(\mathbb{F}_{q^n})$)	$\operatorname{Spec}\mathbb{Z}$ (Connes–Consani $N(u)$)
parameter	$n \in \mathbb{N}$ (powers of one Frobenius)	$u \in [1,\infty)$ (scaling, $\log u \in \mathbb{R}_+$)
$h^2 = \mathbb{L}$	q^n	u
h^0	1	1
h^1	$-\sum_{i=1}^{2g} \alpha_i^n = -\sum_i u^{\rho_i}$	$-\sum_\rho \operatorname{order}(\rho) u^\rho$ (as $-\frac{d}{du}\omega$)
genus / Euler char.	$2g$ eigenvalues; $N_0 = 2-2g$	infinitely many; $N(1) = -\infty$
prime side	$N_n = \sum_{d\mid n} d\,\#\{\deg P = d\}$	$\frac{d\varphi}{du} = \sum n\Lambda(n)\delta_n$, plus κ (archimedean)
zeta from counts	$\log Z = \sum N_n T^n/n$	$-\zeta_{\mathbb{Q}}'/\zeta_{\mathbb{Q}} = \int N u^{-s} d^*u$

Proposition 7.2 (The dictionary is literal for curves)**.** *[THEOREM]. (a) With* $u = q^n$ *and* $q^{\rho_i} = \alpha_i$*,* (1) *reads* $N_n = u - \sum_i u^{\rho_i} + 1$*, which is* (2) *in its formal form (17). The zeros* ρ_i *of* ζ_C *form finitely many classes modulo the period* $2\pi i/\log q$*, and the period tends to* ∞ *as* $q \to 1$*: "infinitely many zeros" is what a curve's zero set looks like with the period removed. (b) For a curve,* $-\zeta_C'/\zeta_C(s) = \log q \sum_n N_n q^{-ns}$*, which is the Riemann sum of* $\int N(u) u^{-s} d^*u$ *at the lattice* $u \in q^{\mathbb{N}}$ [2, Lemma 2.1]*[CITED]. Discrete Frobenius powers are the restriction of the continuous parameter to* $q^{\mathbb{N}}$*. (c)* $N \geq 0$ *on* $(1,\infty)$ *follows from* $n\Lambda(n) \geq 0$ *and* $\kappa > 0$ *alone* [2, proof of Thm. 2.2]*[CITED]. This mechanism is shared by every Euler product with nonnegative von Mangoldt weights, including Beurling systems that violate RH* [16]*. Hence* positivity of N cannot by itself force RH*: it is the analogue of* $N_n \geq 0$*, not of Hodge index [THEOREM].*

7.1. **Discrete Frobenius versus continuous scaling.** *Chat claim: "over* $\mathbb{F}_1$ *there is a continuous flow (scaling action of* $\mathbb{R}_+^*$ */ Deninger's flow) instead of one Frobenius."* **Correct, with labels.** (i) The scaling action of the idèle class group $C_{\mathbb{Q}} = \mathbb{R}_+^* \times \hat{\mathbb{Z}}^*$ on the adèle class space, and the expression of $N(u)$ as an intersection number for it, are theorems of Connes and Connes–Consani [9][TAA-L], [3][CITED]. (ii) Deninger's flow on a foliated space with closed orbits of length $\log p$ is conjectural for $\operatorname{Spec}\mathbb{Z}$ [14, 15][TAA-L] ([CONJECTURE]). (iii) The periodic orbit over p in the scaling site is the circle $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$ of length $\log p$. Connes–Consani call it "a variant of the classical Jacobi description $\mathbb{C}^*/q^{\mathbb{Z}}$ of an elliptic curve" and prove the genus-one Riemann–Roch

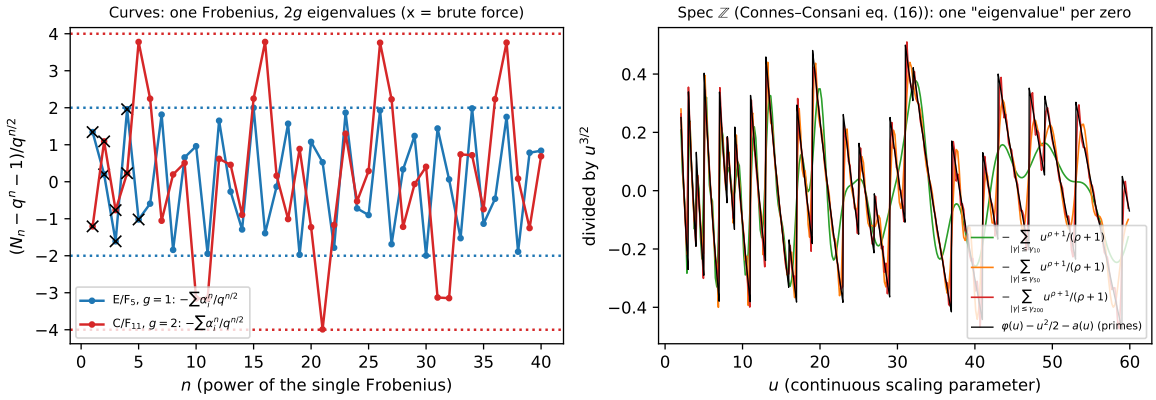

FIGURE 1. Left: curves. The normalized error $(N_n - q^n - 1)/q^{n/2} = -\sum_i \alpha_i^n/q^{n/2}$ for $E : y^2 = x^3 + x + 1$ over $\mathbb{F}_5$ ($a_5 = -3$; band ± 2) and $C : y^2 = x^5 + x + 1$ over $\mathbb{F}_{11}$ ($P_1 = 1 - 4T + 14T^2 - 44T^3 + 121T^4$; band ± 4). The crosses are brute-force counts over $\mathbb{F}_{q^n}$ ($n \le 5$, resp. ≤ 4). One Frobenius, $2g$ eigenvalues, discrete n. Right: $\operatorname{Spec}\mathbb{Z}$. The prime side of Connes–Consani's eq. (16), $\varphi(u) - u^2/2 - a(u)$ (black, exact step function), against the zero side $-\sum_{|\gamma|\le\gamma_K} u^{\rho+1}/(\rho+1)$ for $K = 10, 50, 200$ zero pairs, both divided by $u^{3/2}$. Continuous u, one "eigenvalue" per zero [NUMERICAL].

formula $\mathrm{Dim}_{\mathbb{R}} H^0(D) - \mathrm{Dim}_{\mathbb{R}} H^0(-D) = \deg D$, with real-valued dimensions [5, Thm. 5.17][CITED]. In the curve dictionary, however, the orbit of length $\log p$ corresponds to a *closed point* of degree $\log p/\log q$, not to the fibre $E/\mathbb{F}_p$ of an elliptic curve over $\mathbb{Q}$. That C_p is "elliptic" while $E/\mathbb{F}_p$ is elliptic is an [ANALOGY], not an identification.

8. Answer to J.W.'s question, and the polarization test

Theorem 8.1 (What the single number becomes)**.** *(a) [THEOREM] (Props. 2.1, 2.2). For $E/\mathbb{F}_p$ the prime-side datum is one integer $a_p = \mathrm{Tr}(\pi \mid H^1)$; it determines every $\#E(\mathbb{F}_{p^n})$; its positivity statement is the single binary form $Q_p > 0$. (b) [CITED] (Thm. 7.1) + [THEOREM] (Prop. 7.2). For "$\operatorname{Spec}\mathbb{Z}$ over $\mathbb{F}_1$" the role of $a_p^{(n)} = \alpha^n + \bar\alpha^n$ is played by $\sum_\rho \mathrm{order}(\rho) u^\rho$: one eigenvalue per zero, infinitely many, a distribution in a continuous parameter. The "single number" becomes the whole zero sum. (c) [THEOREM]/[RESTATEMENT]. For $E/\mathbb{Q}$ the collection $\{a_p\}_p$ (with the bad factors) is $L(E,s) = \prod_p (1 - a_p p^{-s} + p^{1-2s})^{-1}$ (good p). Local positivity, $Q_p > 0$ at every good p, is proved (Fig. 2, left). Global positivity, the Weil functional of $L(E,s)$, is RH for $L(E,s)$ [RESTATEMENT] and is open. Local does not imply global: this is the split "collective, not prime by prime" of Lemma 64 [TAA-L].*

So the precise answer is: *yes, a_p is the right model of what $\mathbb{F}_1$ is supposed to make visible, namely a single Frobenius whose trace is a point count. But over $\mathbb{F}_1$ the single trace is not a single number: it is $\sum_\rho u^\rho$.* The analogue of "a_p is one number" is "$\operatorname{Spec}\mathbb{Z}$ has one Frobenius-like flow", not "one eigenvalue".

8.1. J.W.'s standing condition: does this shed light on J?

(i) *Zero counting.* Connes–Consani's $N(u)$ reproduces the explicit formula exactly (Fig. 1). It is zero counting in the precise sense of Lemma 57: it holds for whatever zeros there are, and its positivity is prime-side positivity (Prop. 7.2(c)) [THEOREM].

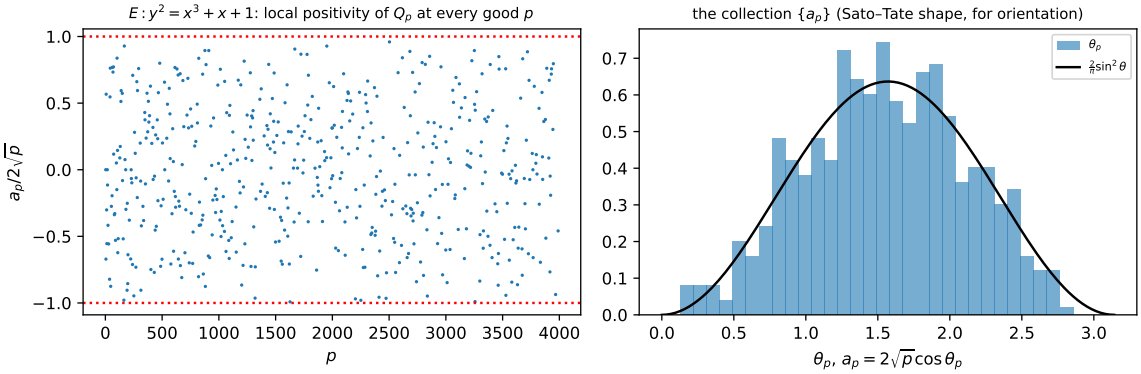

FIGURE 2. $E : y^2 = x^3 + x + 1$ over $\mathbb{Q}$ (bad primes $2, 31$). Left: $a_p/2\sqrt{p}$ for the 548 good primes $3 \le p < 4000$; every point is inside $(-1, 1)$ (max 0.9917), i.e. $Q_p > 0$ at every prime, as Hasse guarantees. Right: histogram of θ_p with $a_p = 2\sqrt{p}\cos\theta_p$ against $\frac{2}{\pi}\sin^2\theta$, for orientation only (no theorem is cited for this curve) [NUMERICAL]. The collective object is the L-function built from all of these at once.

(ii) *Per-prime "elliptic curves".* C_p carries a genus-one Riemann–Roch [CITED]. This is the closest $\mathbb{F}_1$ object to J.W.'s "single number", and it is per prime, i.e. local. It is the $\mathbb{F}_1$ counterpart of the local layer $Q_p > 0$, not of the global layer [ANALOGY].

(iii) *Global candidates.* (a) The Connes–Consani square: Riemann–Roch and positivity on it are "still open" [TAA-L]. (b) Faltings–Hriljac: a proved global Hodge index, on the wrong surface (Rem. 3.5). (c) Deninger's Hodge $*$: conjectural, and it would give $\Theta = \frac{1}{2} + A$ [TAA-L]. (d) Weil/Connes–Consani archimedean positivity: surface-free, partial [CITED].

Verdict. $\mathbb{F}_1$, in every current form, *reproduces the counting* (the 1, the u, the zero sum, with the right signs and the infinite genus) but *does not supply* J. The one structural lesson for J is Prop. 3.1 together with Prop. 4.1(b). J should be the negativity of the orthogonal complement of the hyperbolic $1 \oplus \mathbb{L}$ plane, on a space that contains the graphs of the identity and of "Frobenius". For $\operatorname{Spec}\mathbb{Z}$ no such space with a non-circular positivity is known [OPEN].

9. Two fermion signs: multiplicativity, duality, and the Lefschetz bridge

J.W. [t248u]. Lemma 64's synthesis [TAA-L]: Davenport–Heilbronn [11] has a functional equation (FE) but no Euler product (EP), and fails RH. Beurling systems have EP without FE and can fail RH maximally [16]. EP + FE is the Selberg class, where RH is a [CONJECTURE]. For curves, Frobenius + Poincaré duality + Hodge index give RH [THEOREM]. Lemma 62 §6 [TAA-L]: $\mu = (-1)^F$ (Spector [36][CITED], abstract), odd degree $\Rightarrow$ symplectic cup product. Here we make the two signs precise.

9.1. Prime side: Euler product = tensor factorization.

Let $\mathcal{H} = \ell^2(\mathbb{N})$ with basis $|n\rangle$ and $H|n\rangle = \log n\,|n\rangle$. For each prime p let $\mathcal{B}_p = \ell^2(\mathbb{N}_0)$ (boson occupation k, energy $k \log p$) and $\mathcal{F}_p = \mathbb{C}^2$ (fermion occupation $0, 1$).

Theorem 9.1 (Fock factorization)**.** *[THEOREM]. (a) $|n\rangle \mapsto \bigotimes_p |v_p(n)\rangle$ is a unitary isomorphism $\mathcal{H} \cong \bigotimes'_p \mathcal{B}_p$ (restricted tensor product with respect to the vacua $|0\rangle$), intertwining H with $\sum_p H_p$. This* is *unique factorization. (b) A diagonal operator $A|n\rangle = a_n|n\rangle$ factorizes as $\bigotimes'_p A_p$ iff a is multiplicative with $a_1 = 1$. Then $\operatorname{Tr}(Ae^{-sH}) = \prod_p \sum_k a_{p^k} p^{-ks}$, an Euler product, for $\operatorname{Re} s$ large. (c) Traces: bosons $\operatorname{Tr}_{\otimes \mathcal{B}_p} e^{-sH} = \zeta(s)$. Fermions $\operatorname{Tr}_{\otimes \mathcal{F}_p} e^{-sH} = \prod_p (1 + p^{-s}) = \zeta(s)/\zeta(2s) = \sum |\mu(n)| n^{-s}$. Fermions graded by $(-1)^F$, $F = \omega(n)$ on square-free n: $\operatorname{Str} = \prod_p (1 - p^{-s}) = 1/\zeta(s) = \sum \mu(n) n^{-s}$. Bosons graded by $(-1)^\Omega$: $\prod_p (1 + p^{-s})^{-1} = \zeta(2s)/\zeta(s) = \sum \lambda(n) n^{-s}$ (Liouville). Supersymmetric*

$\bigotimes_p(\mathcal{B}_p \otimes \mathcal{F}_p)$ *with* $(-1)^{F_{\mathrm{ferm}}}$*:* $\mathrm{Str} = \prod_p \frac{1-p^{-s}}{1-p^{-s}} = 1$*, i.e.* $\sum_{d|n} \mu(d) = \delta_{n,1}$ *(Möbius inversion as Witten index* 1*; Spector's interpretation* [36]*[CITED]).*

Proof. (a) Unique factorization; finitely many $v_p(n) \neq 0$. (b) Multiplicativity is factorization over coprime parts. (c) Geometric series per mode. □

Refinement of the parent's summary "Witten index $1/\zeta$*":* $1/\zeta$ is the graded trace of the *fermionic* sector alone. The full supersymmetric Witten index is 1 [THEOREM]. *Chat claim "Euler product = tensor factorization over primes":* **correct** (Thm. 9.1(b)). But a Beurling system, whose generalized integers form a free monoid, has the same factorization by construction and still fails RH [16]. Factorization alone is RH-blind [THEOREM].

9.2. **Zero side for curves: the super Fock space of cohomology.** Let $S^\bullet(V)$ denote the graded-symmetric algebra of a graded space V: Sym on even, Λ on odd parts. For an operator Φ write $\Gamma(\Phi)$ for its second quantization and N for the number operator.

Theorem 9.2 (Curves: zeta as a supertrace; duality)**.** *[THEOREM] (with* [31, §1, Prop. 2.6, Thm. 2.1]*[CITED]). (a)* $\det(1 - \pi T \mid H^1) = \sum_k (-1)^k \operatorname{Tr}(\Lambda^k \pi) T^k$ *and* $\det(1 - \pi T \mid H^{0,2})^{-1} = \sum_k \operatorname{Tr}(\operatorname{Sym}^k \pi) T^k$*, so*

$$Z(C,T) = \frac{P_1(T)}{(1-T)(1-qT)} = \operatorname{Str}_{S^\bullet(H^\bullet(C))}(\Gamma(\pi)\, T^N), \qquad S^\bullet(H^\bullet) = \operatorname{Sym}(H^0) \otimes \Lambda(H^1) \otimes \operatorname{Sym}(H^2),$$

with super-sign $(-1)^{\text{cohomological degree}}$ *(an element of* $\operatorname{Sym}^a H^0 \otimes \Lambda^b H^1 \otimes \operatorname{Sym}^c H^2$ *has degree* $b + 2c \equiv b$*). The zeros come from* H^1*, the odd fermionic part; the poles come from* H^0, H^2*, the even bosonic part* $1 \oplus \mathbb{L}$*. (b) Poincaré duality* $H^1 \times H^1 \to H^2 = \mathbb{L}$ *is alternating (odd degree, graded commutativity), so it is* symplectic*, and* $\omega(\pi x, \pi y) = q\, \omega(x,y)$*. Hence* $\alpha \mapsto q/\alpha$ *and* $Z(C, 1/qT) = q^{1-g} T^{2-2g} Z(C,T)$ *[CITED].*

$\Lambda^\bullet H^1$ is a genuine finite fermionic Fock space on the zero side: $2g$ fermion modes, one per eigenvalue α_i, and $P_1(T)$ is its graded partition function. We verified $(-1)^k \operatorname{Tr} \Lambda^k \pi = c_k$ for the three curves of §10 [NUMERICAL].

9.3. **The bridge: Lefschetz on** $\operatorname{Sym}^n C$**.**

Theorem 9.3 (Two Fock spaces, two gradings, one trace formula)**.** *[THEOREM] (curves), using* [31, Thm. 2.1, eq. (32)]*[CITED] and Macdonald* [27]*[BIB].*

(a) Prime-side Fock space. *The bosonic Fock space over closed points,* $\bigotimes'_P \mathcal{B}_P$ *(energy* $\deg P$*), has basis the effective* $\mathbb{F}_q$*-rational divisors. Its degree-n dimension is* $\#\operatorname{Sym}^n C(\mathbb{F}_q)$*, the coefficient of* T^n *in* $Z(C,T) = \prod_P (1 - T^{\deg P})^{-1}$*.*

(b) Cohomological Fock space. $H^\bullet(\operatorname{Sym}^n C) \cong S^\bullet(H^\bullet(C))_{(n)}$*, the degree-n part (Macdonald), with* π *acting by* $\Gamma(\pi)$*.*

(c) Bridge. *The Lefschetz trace formula on* $\operatorname{Sym}^n C$ *gives* $\#\operatorname{Sym}^n C(\mathbb{F}_q) = \sum_i (-1)^i \operatorname{Tr}(\pi \mid H^i(\operatorname{Sym}^n C)) = \operatorname{Str}_{(n)} \Gamma(\pi)$*. Summed over n, this is Thm. 9.2(a).*

(d) The two gradings. *The prime side is purely bosonic: no sign appears. The fermion signs on the zero side are the cohomological parity. Flipping the prime-side statistics to graded fermions gives* $\prod_P (1 - T^{\deg P}) = 1/Z(C,T) = \sum_{D \text{ reduced}} (-1)^{\#D} T^{\deg D}$*, the Möbius function* μ_C *of* $\mathbb{F}_q[C]$*. This equals* $\operatorname{Str}_{S^\bullet(\Pi H^\bullet)}(\Gamma(\pi) T^N)$*, with* Π *the parity shift:* $\Lambda(H^0) \otimes \operatorname{Sym}(H^1) \otimes \Lambda(H^2)$ *gives* $(1-T)(1-qT)/P_1(T)$*. So* the prime-side boson/fermion flip ($\zeta \leftrightarrow 1/\zeta$, i.e. μ) corresponds to the parity shift of cohomology.

Proof. (a) An $\mathbb{F}_q$-point of $\operatorname{Sym}^n C$ is a Galois-stable effective divisor of degree n, i.e. a multiset of closed points with total degree n. (c) Lefschetz [31, Thm. 2.1] for π on the smooth projective $\operatorname{Sym}^n C$. (d) Generating functions: $\sum_k (-1)^k \operatorname{Tr} \Lambda^k(\Phi) T^k = \det(1 - \Phi T)$ and $\sum_k \operatorname{Tr} \operatorname{Sym}^k(\Phi) T^k = \det(1 - \Phi T)^{-1}$. □

Chat claim "two different $\mathbb{Z}/2$ gradings": **correct**, and Thm. 9.3(d) gives the precise relation. They live on different spaces, and there is *no* state-by-state map: an effective divisor does not have a cohomological degree. Only graded traces match, degree by degree in n. A global flip on one side corresponds to a global flip on the other. *Chat claim "the Lefschetz trace formula is the bridge":* **correct for curves** (Thm. 9.3(c)).

Remark 9.4 ($\mathbb{Q}$: what survives)**.** Prime side: Thm. 9.1 holds verbatim; Julia's and Spector's primon gas [22, 36][TAA-L] is the bosonic Fock space, with $\operatorname{Tr} e^{-sH} = \zeta(s)$. The archimedean factor $\pi^{-s/2}\Gamma(s/2)$ is not a geometric series, so it has no Fock factor; in Connes–Consani it is the continuous term $\kappa(u)$ [THEOREM]. Zero side: Deninger's $\xi = \det_\infty(H^1)/(\det_\infty(H^0)\det_\infty(H^2))$ is a regularized superdeterminant. The regularized-product identity is proved; the cohomology is [CONJECTURE] [14][TAA-L]. "$\Lambda^\bullet H^1$ for ζ" is therefore an [ANALOGY]: an infinite fermionic Fock space whose regularized graded partition function would be $\prod_\rho (s-\rho)/2\pi$. Bridge: Weil's explicit formula is a theorem with the shape of a Lefschetz formula (Connes–Consani's $N(u)$ is its point-count form, Thm. 7.1). Its realization as a trace on a cohomology is [CONJECTURE] (Deninger), partially realized as an absorption spectrum (Connes; Lemma 61 [TAA-L]).

9.4. Multiplicativity + duality is not enough, even for curves.

Proposition 9.5. *(a) [CITED]* [31, §2, Props. 2.5, 2.6]. *From a Weil cohomology (Poincaré duality, Künneth, cycle map) and the Lefschetz trace formula one obtains rationality and the functional equation of $Z(V,T)$. RH did not follow; it required Deligne's Weil I* [13]*, or the standard conjectures (§4), or, for curves, the Hodge index theorem. (b) [THEOREM] (with Lemma 64 [TAA-L]). EP alone (free-monoid factorization) fails RH (Beurling,* [16]*). FE alone fails RH (DH). The proved case uses three inputs: multiplicativity (the bosonic prime Fock space, W1), duality (the symplectic H^1, W2) and positivity (Hodge index, W3). (c)* On "μ alone has multiplicativity but no duality"*: imprecise. $\sum \mu(n)n^{-s} = 1/\zeta(s)$ inherits the functional equation of ζ, since $1/\zeta_{\mathbb{Q}}(s) = 1/\zeta_{\mathbb{Q}}(1-s)$ [THEOREM]. The precise statement: sign changes $\mu(p) \mapsto \epsilon_p \mu(p)$ preserve multiplicativity and destroy the functional equation. Multiplicativity is a local property; duality is a global constraint on the whole collection [THEOREM]. Likewise "cohomology alone has duality but no primes" is a fair [ANALOGY] for DH (FE without EP).*

So the parent's sentence "RH needs both, linked by the trace formula" should read: *RH needs both, linked by the trace formula, plus positivity.* In the proved case the link supplies the shape (rationality, FE, $|\alpha| \in \{q^{w/2}\}$ to be shown) and Hodge index supplies the inequality. This is again Lemma 64's split: the shape is structural, and the inequality is collective [THEOREM].

10. Numerical checks

All numerics are light: integer brute force plus 200 zeros from `mpmath`, under a free-memory guard of 1600 MB. Scripts: `num/lemma65_counts.py`, `num/lemma65_figs.py`, `num/lemma65_triple.py`; data `num/l65/{counts,results,triple}.json`.

(1) *Lefschetz/exterior algebra* [NUMERICAL] (exact integers). Brute-force counts over $\mathbb{F}_{q^n}$ (fields built from an irreducible polynomial): $E : y^2 = x^3 + x + 1$ over $\mathbb{F}_5$: $N_n = 9, 27, 108, 675, 3069$ ($n \le 5$). Over $\mathbb{F}_7$: $5, 55, 380, 2475$. $C : y^2 = x^5 + x + 1$ over $\mathbb{F}_{11}$: $8, 134, 1304, 14670$. From N_1 (resp. N_1, N_2) and the functional equation: $P_1 = 1+3T+5T^2$, $1-3T+7T^2$, and $1-4T+14T^2-44T^3+121T^4$. These reproduce *all* remaining counts exactly. $|\alpha_i|/\sqrt{q} = 1$ to 10^{-15}, and $(-1)^k \operatorname{Tr}\Lambda^k \pi$ equals the coefficients c_k (Thm. 9.2).

(2) *Defect form* (Prop. 3.1): for $C/\mathbb{F}_{11}$, $a = 4$ and the Gram matrix $\left(\begin{smallmatrix} 4 & 4 \\ 4 & 44 \end{smallmatrix}\right)$ is positive definite. For $E/\mathbb{F}_5$, $\left(\begin{smallmatrix} 2 & -3 \\ -3 & 10 \end{smallmatrix}\right) = 2Q_5$ is positive definite, and $Q_5(1,-1) = 9 = \#E(\mathbb{F}_5)$.

(3) *Connes–Consani eq. (16)* (Fig. 1, right): $\zeta'(-1)/\zeta(-1) = 1.98505372\ldots$. On $2.05 \le u \le 59.95$ (300 points), the RMS difference between the prime side $\varphi(u) - u^2/2 - a(u)$ and the truncated zero side is 32.0, 21.1, 12.9 for $K = 10, 50, 200$ zero pairs, against a signal RMS of 50.0. Convergence is

slow because of the jumps (Gibbs) but visibly monotone. This checks the formula as transcribed, not a new statement [NUMERICAL].

(4) *Local Hasse* (Fig. 2): $\max_p |a_p|/2\sqrt{p} = 0.9917$ over 548 good primes < 4000.

(5) *Triple* (Table 1): all identities of Thm. 5.3 to $\leq 1.4 \cdot 10^{-14}$.

11. Corrections to Grok Bot's chat claims

C1. "Over $\mathbb{F}_1$ there is a continuous flow (scaling action of $\mathbb{R}_+^*$ / Deninger's flow) instead of one Frobenius." **Correct, with labels:** the scaling action on the adèle class space is a theorem; Deninger's flow on a foliated space is conjectural; the discrete Frobenius powers are the restriction $u \in q^{\mathbb{N}}$ (Prop. 7.2(b)).

C2. "Connes–Consani: $N(u) = u + 1 -$ (sum over zeros of u^ρ-type terms)." **Correct only as the formal version** (their eq. (17)). The exact statement is (2): the distributional derivative of $\sum \mathrm{order}(\rho) u^{\rho+1}/(\rho+1)$, summed symmetrically, on $[1, \infty)$, with a principal value at $u = 1$ and $N(1) = -\infty$.

C3. "The surface $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ has never been built; that's where Hodge-index positivity would live." **Imprecise in both halves.** Candidate squares exist (Connes–Consani's square of the arithmetic site; Borger's $W^*(\operatorname{Spec}\mathbb{Z})$), but none has H^1, intersection theory, Riemann–Roch and Hodge index (Prop. 6.1). "Would live" is a [HEURISTIC] (Rem. 3.4). The earlier chat sentence "the piece of Weil's proof never built over $\mathbb{F}_1$ is the surface ... where Hodge-index positivity would live" should have carried that label.

C4. "$\Gamma_F \cdot \Delta = \#C(\mathbb{F}_q)$." **Correct** (transversality since $d\pi = 0$). "$F \circ {}^tF = q$." **Correct**: $= q\Delta$ as correspondences, $\pi^\dagger \pi = q$ on H^1. "Hodge index $\Rightarrow |N - q - 1| \leq 2g\sqrt{q}$." **Correct**, via Castelnuovo–Severi and $\mathrm{def}(\Delta) = 2g$, $\mathrm{def}(\Gamma_\pi) = 2gq$. "$\operatorname{Spec}\mathbb{Z} \times_{\mathbb{Z}} \operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}$." **Correct** (trivial). "Faltings–Hriljac gives positivity of heights." **Correct as a summary.** The Hodge index on arithmetic surfaces is checked via Moriwaki; the height identity is [BIB] with the constant not pinned; and it is not $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ (Rem. 3.5).

C5. "$\mathbb{L} = h^2(\mathbb{P}^1) = \mathbb{Q}(-1)$, Frobenius acts by q." **Correct for the geometric Frobenius** (π); the arithmetic Frobenius acts on $\mathbb{Q}_\ell(-1)$ by q^{-1}. "Poincaré duality $h^1 \otimes h^1 \to \mathbb{L}$." **Correct** for curves; the pairing is alternating, i.e. it factors through $\Lambda^2 h^1$.

C6. "The polarization of h^1 is an $\mathbb{L}$-valued form positive in Hodge's sense." **Correct over** $\mathbb{C}$ (Riemann bilinear relations). Over $\mathbb{F}_q$ there is no Hodge decomposition, and positivity is available only on algebraic classes (Rosati on $\mathrm{End}^0(h^1)$, Hodge index on $C \times C$).

C7. "Grothendieck showed the standard conjectures would give Weil's RH." **Attribution refined:** Grothendieck formulated them [19]; the deduction (standard conjectures $\Rightarrow$ Weil form $\Rightarrow |\alpha| = q^{r/2}$) is Kleiman's [23], as reported in [31, Thms. 3.15, 3.17].

C8. "The pole term of Weil's explicit formula is the contribution of $1 \oplus \mathbb{L}$." **Theorem for curves, [ANALOGY] for ζ.** It has a checkable match: both are hyperbolic planes of signature $(1,1)$ (Prop. 4.1). The related phrase "the pole gives the clearly positive term" is wrong as stated: it is hyperbolic (Lemma 64 agrees).

C9. "Two different $\mathbb{Z}/2$ gradings." **Correct**; they are related only through graded traces (Thm. 9.3(d)). "Odd-degree pairing is symplectic." **Correct** (graded commutativity). "Euler product = tensor factorization over primes." **Correct** (Thm. 9.1), but RH-blind. "The Lefschetz trace formula is the bridge." **Correct for curves** (on $\mathrm{Sym}^n C$); for $\mathbb{Q}$ it is [CONJECTURE]. "RH needs both, linked by the trace formula." **Incomplete:** positivity is a third input, even for curves (Prop. 9.5). "μ alone has no $s \leftrightarrow 1 - s$ duality" is **imprecise** ($1/\zeta$ inherits the FE). "Witten index $1/\zeta$" should read: fermionic graded trace $= 1/\zeta$, supersymmetric Witten index $= 1$.

C10. "$J_p^2 = (a_p^2 - 4p)/(4p - a_p^2) = -I$" **correct**. "$\det J_p = 1$ so J_p preserves ω" **correct** ($SL_2 = Sp_2$). "Q_p is the charpoly written as a form" **correct up to** $y \mapsto -y$. "$\omega(\pi x, \pi y) = p\,\omega(x, y)$" **correct**. "Hodge-type standard conjecture is a theorem over $\mathbb{C}$" **correct** (more generally in

characteristic 0). The chat formula $J_Q = \frac{1}{\sqrt{|d|}}\left(\begin{smallmatrix} b & 2c \\ -2a & -b \end{smallmatrix}\right)$ has the **wrong sign** relative to Lemma 60 (it is the negative polarization).

C11. Earlier chat [t245u-reply]: "positive definite forces $a_p^2 - 4p < 0$." Positive *semi*definiteness gives $\leq$; strictness uses that $4p$ is not a square. "Euler product without functional equation: Diamond–Montgomery–Vorhauer (to be verified)" is now **verified** (Crossref; content per Lemma 64).

12. Ranked directions

Score = 0.6·significance +0.4·probability of a clear outcome.

	direction	sig.	prob.	score
G1	Genus-2 triple over $\mathbb{F}_{11}$: $J = \bigoplus J_{p,j}$ on Frobenius eigenplanes, Rosati Gram, defect form on span$\{\Delta, \Gamma_\pi, \Gamma_{\pi^2}\}$; merges with Lemma 62's F1	.50	.95	.68
G2	Defect form on the Connes–Consani square: is there a computable pairing on "span$\{\Delta, \Gamma_u\}$" (graphs of scaling) reproducing Weil's pairing with a hyperbolic pole plane?	.80	.30	.60
G3	$\mathrm{Sym}^n C$ check of Thm. 9.3: count effective divisors of $C/\mathbb{F}_{11}$ by closed points ($n \leq 6$) against $\mathrm{Str}_{(n)}$	.35	.95	.59
G4	Local layer over $\mathbb{F}_1$: does C_p (genus-one RR, Jacobian data $\mathbb{Z}/(p-1)$) carry an analogue of Q_p, a degree form?	.60	.50	.56
G5	Faltings–Hriljac vs. Weil functional of $L(E,s)$: document exactly why height positivity is not L-positivity	.50	.60	.54
G6	Real-valued (type II) dimensions on C_p vs. TAA's Krein/type-II traces	.50	.30	.42

13. Questions for J.W.

Q1. Do you accept the answer "a_p is the model; over $\mathbb{F}_1$ the single trace becomes $\sum_\rho u^\rho$", and should Prop. 3.1 ($2Q_p$ = Weil's defect form on span$\{\Delta, \Gamma_\pi\}$) become the official link between your binary forms and the surface?

Q2. Should the single-Frobenius triple (ω from $\mathbb{L}$, $J_p = (2\pi - a_p)/\sqrt{4p - a_p^2} = J_{Q_p}$, $g_p > 0$ from deg > 0) be adopted as the official model of J, with Lemma 60's sign convention?

Q3. Run G1 next (genus-2 triple, merged with Lemma 62's F1)?

Q4. Do you accept relabelling "positivity lives on $\mathrm{Spec}\,\mathbb{Z} \times \mathrm{Spec}\,\mathbb{Z}$" as a heuristic, and the three-input summary "multiplicativity + duality give the shape, positivity gives the inequality"?

Q5. Should the hyperbolic-plane match "POLE $\leftrightarrow 1 \oplus \mathbb{L}$, signature $(1,1)$" be added to Lemma 64? And G2 or G3 after G1?

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APPENDIX A

Consolidated corrections

Every note ends with a corrections section that records which claims made in the working conversation were wrong, and how they were fixed. Those claims came from pasted texts, from J.W.'s hunches stated as claims, or from the assistant's own chat replies. Chapter texts were not edited for this volume. The table points to each section and gives its gist; the full text is in the chapter. "Note p." is the page inside the note; "vol. p." is the volume page of this part. Only the chapters of this part (Chapters 48–65) are listed; the others are in the same appendix of Volume IVb.

Ch.	Section	note p.	vol. p.	Gist
48	5. Corrections to the pasted KK/causality text	6	6	§5: the pasted causality text, item by item. §10 (C1–C11): Lagarias's printed g is anti-Pick; Sondow–Dumitrescu venue corrected; "ghosts invisible in magnitude" false. Erratum: items on two earlier preprints of J.W., as recorded in the note.
	Erratum on J.W.'s ai.viXra:2606.0029 and 2609.0007	11	11	
	10. Corrections to the chat claims	12	12	
49	8. Corrections to the chat claims	10	24	C1–C9: Ramanujan's system is Gauss–Manin plus Legendre; the manuscript's (5.1)–(5.3) are off by $2\pi i$; E_2^* as polarization term correct.
50	8. Corrections to the chat claims	13	40	C1–C12: spinor bundle on S^2 is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$; spinor holonomy sign is the spin structure; Riemann's $\frac{1}{2}$ is bosonic.
51	7. Corrections to the chat claims and to the premise	13	56	Hasse's bound is $\leq$, with equality for square q; the positive vacuum escapes to the Siegel boundary rather than becoming a Krein vacuum.
52	7. Corrections	10	68	The premise is impossible for structural reasons (scaling is hyperbolic at every finite S); script labels and a heuristic corrected.
53	6. Corrections to the planned chat claims	9	80	The oscillator/spin twin holds with refinements (Shale, Shale–Stinespring); bosonic positivity is $\operatorname{Im}\Omega > 0$.
54	10. Corrections to the chat claims (made to J.W. on 2026-10-22)	17	99	v1 presented $n_-(g_{J_A})$ as the ghost index (undefined for real pairs); a Lemma 56 v2 premise was half wrong.
55	10. Corrections to the chat claims	15	117	Symmetry cannot exclude ghosts, but the type of on-line saddles detects them; saddles of $\lvert f\rvert$ vs. zeros of f' refined.

Ch.	Section	note p.	vol. p.	Gist
56	Corrections and updates in v2 and v3	4	125	$\beta > 1$ zeros not within reach below 10^4; a false claim about larger δ; certified bound replaces v2's mixed tag; sixth digit $0.230608 \to 0.230607$.
	9. Corrections	25	146	
57	8. Corrections to the chat claims	9	160	Certified windows are theorems but carry no new information on RH; DH is certified indefinite already at $\lambda^2 = 30.88$.
58	11. Corrections	16	178	C1–C13, notably: Polymath bound 0.22, not 0.2; "holomorphic vs. antiholomorphic" wrong.
59	9. Corrections to Grok Bot's chat claims	19	199	C1–C6; C6: "LP = Turán inequalities, touches positivity directly" corrected.
60	9. Corrections	13	214	Correction 9.1: the "Deuring–Heilbronn phenomenon" was misdescribed in four ways (Deuring/Mordell/Heilbronn; ineffective and RH-neutral; whose ghost; zero repulsion).
61	12. Corrections to Grok Bot's chat claims	13	229	C1–C7: Dyson 1952 is heuristic; Λ_{QCD} convention-dependent; ghost-threshold half upgraded to a theorem.
62	9. Corrections to chat claims	16	248	C1–C15, incl. the Ingham attribution and "Weil gives positivity" (imprecise).
63	8. Corrections to Grok Bot's chat claims	14	264	C1–C12: curvature sign; Gauss–Bonnet empty; Poincaré–Hopf bound not topological; $\pm 45°$ only for nondegenerate saddles; DH has a positive kernel.
64	8. Corrections to Grok Bot's chat claims	15	281	K1: the pole term has signature $(1, 1)$, not "clearly positive"; K2: Yoshida's small-support theorem concerns the whole Weil form.
65	11. Corrections to Grok Bot's chat claims	15	299	C1–C11: CC $N(u)$ only formal; Frobenius geometric vs. arithmetic; Hodge positivity over $\mathbb{C}$ vs. algebraic classes.

Corrections across chapters. Some later chapters correct or sharpen earlier ones; the later chapter states this in its "Prior TAA work" or "Corrections" section. Examples: Ch. 63 corrects Ch. 59's interval ratio and answers its Q3–Q4. Ch. 54 v2 corrects its own v1 and a premise of Ch. 56 v2. Ch. 81's second addendum (in Volume IVb) reconciles its Cassini saddle with Ch. 55, Prop. 2.2.

APPENDIX B

Open questions

Each note ends with "Questions for J.W." (usually Q1–Q5). They are reproduced in full in the chapters; this appendix lists their topics and locations for the chapters of this part (Chapters 48–65); the others are in the same appendix of Volume IVb. **Status.** Unless marked, no answer from J.W. is recorded in the notes, so the questions are open. Three kinds of exception: answers recorded in a later note (marked *Answered*); questions of the form "which direction next", which were in effect settled by the following lemma requests; and questions superseded by a later version of the same note (Lemma 54 v1, Lemma 56 v2). No volume-level computations are pending: unlike Volume III, this volume contains no placeholders.

Ch.	§	note p.	vol. p.	Topics
48	12	13	13	Which "2π" is meant (ghost, pole, or Lemma 36 deficit); Bode form of G5; θ as a stage; erratum placement; a strict-passivity lemma.
49	10	11	25	Polarization vs. dynamics dichotomy as official; Nash/Chern–Gauss–Bonnet to an appendix; manuscript fixes; Deninger target; CM/Damerell follow-up.
50	10	14	41	Thesis 5.4 as TAA language; presenting $\frac{1}{2}$ via the oscillator vacuum; Krein-collision follow-up; CCM vacuum numerics; genus-2 section; placement. *Answered* in [t196u] (see Chs. 51–52).
51	9	13	56	Ghost birth at the Siegel boundary as TAA language; spinor vs. bosonic measure; genus-2 presentation; $\mathrm{Sp}(1,1)$ exploratory lemma; rate-law normalization; $g \geq 3$.
52	9	11	69	Adopt "invariant positive vacuum for scaling excluded for structural reasons"; reword (S4); limit-circle proof; per-ghost split for f_{DH}; Volume IV placement.
53	8	10	81	"Ghost charge" as vocabulary; Krein form vs. de Branges–Pontryagin space [Open]; Krichever-type realization [Open]; 2-out-of-3 presentation; Burnol-distance numerics.
54	12	18	100	(v2) r vs. ν as primary count; record $\nu = r = -$charge; test on CCM truncations [Open]; Witt index; placement.
55	12	17	119	Lemniscate orientation as official invariant; the converse counting step [Restatement]; which saddle counter; reading Cohn 1922; heat-flow saddle births.
56	11	28	149	(v3) Locate the first $\beta > 1$ zero beyond 10^4; model the ratio vs. local density; certify per-ghost t_c; function-field twin of (H); Vol. 3 queue item. (v2 questions were answered in [t218u].)
57	10	10	161	Freeze the ladder; general no-slack obstruction; semilocal test of CC21 slack; calibrate L54's index with the DH crossing; Vol. II revision (approval needed).
58	12	17	179	Adopt the arithmetic polarization datum and Conj. 3.11; the via-negativa checklist; next target; thermal phrasing; motives/topoi as analogy only.

Ch.	§	note p.	vol. p.	Topics
59	11	20	200	Adopt verdict (V1)–(V4); tower counters; zero-free kernel conditions; DH Jensen/Turán to high degree; §6 conclusions. *Answered* in [t236u] (recorded in Ch. 63).
60	11	14	215	Which K/k is meant; P10′ in the strip; D3 as companion; Wick $\approx$ sign of J [Analogy]; which errata J2–J11.
61	13	14	230	D1 as its own lemma; accept "visibility, not J"; dictionary rows for Vol. 4; run D2; Ritz as via negativa.
62	–	17	249	No question section; Ranked follow-ups (§10) instead.
63	10	15	265	Adopt verdicts A/W/B; record cross facts and dent $D(x)$; PF order of Gröchenig's Λ_G; classical Jensen to 10^5 or stop; light-cone analogy footnote.
64	10	17	283	Adopt the collective theorem; three-ingredient reading; G1 next; then G2 or G4; Vol. 4 inclusion.
65	13	16	300	Accept the answer and Prop. (Hasse = Weil); adopt the triple with L60's sign; run G1; heuristic relabel; POLE $\leftrightarrow 1 \oplus \mathbb{L}$ in L64.

APPENDIX C

Consolidated notation

Each chapter defines its notation at the start of the note, and glosses objects such as h^1, H^1, the Hodge–Riemann relations and the Hodge index theorem where it uses them. This appendix lists the global conventions and the symbols that different chapters use differently. It covers the whole of Volume IV, so it also names chapters of Volume IVb; chapter numbers are the same in both parts.

Global conventions

- $s = \sigma + it$; nontrivial zeros $\rho = \beta + i\gamma$, $\delta_\rho = \beta - \frac{1}{2}$; mirror $\rho^* = 1 - \bar{\rho}$; $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$; $\Xi(t) = \xi(\frac{1}{2}+it)$. Some chapters use $s = \frac{1}{2} - iz$ (Lemma 33/34/37 coordinates; Ch. 48) or $\lambda = s(1-s)$ (Chs. 81, 85).
- *Ghost.* An off-line zero. Most chapters count both members of a pair $\{\rho, \rho^*\}$ together. Ch. 48 calls a zero with $\beta > \frac{1}{2}$ a ghost, so each pair contributes one ghost.
- *Davenport–Heilbronn.* f_{DH} is the Davenport–Heilbronn combination of the two L-functions mod 5, with completed form $\Lambda(s) = (5/\pi)^{s/2}\Gamma(\frac{s+1}{2})f_{\mathrm{DH}}(s)$, self-dual. It has a functional equation and no Euler product. Its first off-line zeros are $0.80851718 + 85.69934849i$ and $0.65083008 + 114.16334273i$ (Ch. 76; the same values recur in Chs. 78–85).
- *Over $\mathbb{F}_q$.* F is Frobenius on H^1 (an ℓ-adic vector space; h^1 is the motive, a categorical object). $V = qF^{-1}$ is the Verschiebung, written $\tilde{V}$ in Ch. 54. $F^\dagger = qF^{-1}$ is the Rosati-type involution on $\mathbb{Q}[F]$ (Ch. 84). On H^2 of a surface, Ch. 85 uses $F^\dagger = q^2F^{-1}$.
- *Rosati, Albert, Hodge index.* The Rosati involution is the adjoint for a polarization. Albert's classification lists the division algebras with a positive involution (types I–IV). The Hodge index theorem states that the intersection form on $\mathrm{NS}(S) \otimes \mathbb{R}$ of a surface has signature $(1, \rho - 1)$ (restated in Chs. 79, 81, 84, 85).

Symbols used differently in different chapters

Symbol	Meanings
κ	*Main meaning (Chs. 77, 81, 82, 84, 85):* the number of off-line (off-circle) reciprocal pairs, i.e. the negative index of the Weil-type form; a complex off-circle quartet counts as 2. Ch. 82 counts negative directions of Q_L on real functions, 2 per quartet. *Ch. 56 (and Chs. 59, 63):* the official saddle counter $\kappa = \tau + S^+$ (Kim's window identity), which equals the number of ghosts in the cases stated there; Ch. 59 adds tower counters $\kappa_n = G_n$, with $\kappa = \kappa_0$. *Chs. 50–51:* κ_λ is a Krein sign and $\hat{\kappa}$ a normalized Krein norm.
k	Winding number and ghost charge (ch $= -k$), Chs. 48, 53, 54; weight of a modular form, Chs. 66–70, 85; spin index of U_k, Ch. 85.
Q, W	The Weil form. Truncations: W_A (support $[-A/2, A/2]$; Chs. 57, 64), Q_L (finite support, Connes cutoff $\Lambda = e^{L/2}$; Ch. 82), Q_w (weighted, Ch. 80), Q_χ (sector form, Ch. 81), T_N (Toeplitz form of zero power sums, Ch. 77).
J	A compatible complex structure (the sought polarization) throughout. J_F is Frobenius's own (Ch. 54); J_ε (Ch. 50); J_p or $J_{\mathbb{Q}}$ local models (Chs. 64–66).

Symbol	Meanings
Λ	Completed L-function (most chapters); de Bruijn–Newman constant (Λ, Λ_{DH}; Chs. 56, 81); Connes's cutoff (Ch. 82).
E	An elliptic curve; the Eisenstein series $E(\tau, s)$ or $E(z, s)$ (Chs. 67–75); E_k holomorphic Eisenstein series; $E(\cdot, \cdot)$ an alternating form (Ch. 81).
ϕ, φ	Scattering coefficient of the Eisenstein series (normalizations stated per chapter: $E \sim y^s + \phi(s)y^{1-s}$ on $\mathbb{H}^2$; $E \sim y^{1+s} + \phi(s)y^{1-s}$ on $\mathbb{H}^3$, Ch. 73); Euler's totient in Farey counts (Ch. 76).
ε, w	Root number; for a newform, $f\|W_N = wf$ and $\varepsilon = -w$ (Ch. 85).
ρ, $\bar{\rho}$	Zero of an L-function; Picard number $\rho(S)$ and geometric Picard number $\bar{\rho}$ (Ch. 85).

Claim tags

The tags are mapped to volume classes in the table after the Statement of Intellectual Honesty. Local variants:

- Ch. 77 has [EXACT-COMP] (exact arithmetic plus certified root isolation).
- Ch. 57 has "[THM, computer-assisted]" and [CROSS-CHECK, uses zeros].
- Ch. 58 has [METAPHOR].
- Chs. 84–85 have [CITED-sec].
- Chs. 48–56 use the short forms [THM], [NUM], [CONJ].