

Arithmetic Unitarity IVb

Counting Ghosts

Eisenstein Anomalies, Hyperbolic Geometry, the Ample-Class Obstruction, and
Mordell–Weil Lattices

Thermodynamics of Arithmetic Action, Lemmas 66–85

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Grok Bot is an AI assistant; see the Statement of Intellectual Honesty.

Volume IV, second of two parts (companion: Volume IVa, Lemmas 48–65)

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Abstract

This is Volume IVb, the second of the two parts of Volume IV of the programme *Thermodynamics of Arithmetic Action*. It collects the standalone notes TAA Lemmas 66–85; the companion Volume IVa contains Lemmas 48–65. The two parts form one volume and are split only because of the archive’s 5 MB file limit. Volumes I–III (ai.viXra:2610.0045v2, 2610.0054, 2610.0052) recast the Riemann hypothesis (RH) as the absence of negative-norm (“ghost”) directions for the Weil form, measured that obstruction, and asked which structure could supply the positivity. Here we continue the tests through modular forms and Eisenstein anomalies (Chs. 66–71), hyperbolic geometry, 3D gravity, Farey fractions and codes (Chs. 72–77), and a final cluster that isolates the obstruction (Chs. 78–85). The ghost count κ (the negative index of a Weil-type form) recurs throughout: for a Weil polynomial it equals the number of negative directions of the Rosati trace form, a classical linear-algebra fact that we restate and check exhaustively for 13 135 genus-2 polynomials (Ch. 84). We prove a no-go theorem for positivity built from scaling-commuting multipliers (Ch. 80). We show numerically that the Davenport–Heilbronn ghost already appears at finite Connes cutoff, with primes $n \leq 41$ (Ch. 82). Closed geodesics on Bianchi orbifolds see only Euler-product channels, not the off-line zeros of Epstein zeta functions (Ch. 83). In Mordell–Weil lattices over function fields, the BSD count is one Frobenius eigenvalue, and Weil’s proof ties it to RH through algebraicity of Frobenius (Ch. 85). Certified computations show that extremal weight enumerators of self-dual codes satisfy Duursma’s RH in stated ranges (Ch. 77). What we claim across both parts, and where, is stated precisely at the start. **RH is not proved.** No chapter supplies the missing positivity for ζ ; most identify why a candidate cannot.

Statement of Intellectual Honesty

The Riemann Hypothesis is NOT proved in this volume (Volume IVa or Volume IVb).

Nothing here implies it, and no chapter reduces it to an easier statement.

Neither GRH nor the Birch–Swinnerton-Dyer conjecture is proved here.

1. No claim of proof. No statement in this volume is, or should be quoted as, a proof of RH, or as progress towards one beyond the modest items listed in “What this volume does and does not advance” below. Every RH-equivalent condition in the notes is labelled [Restatement]. Results about the Davenport–Heilbronn function f_{DH} , about function fields, or about codes are statements about *those* objects. They do not transfer to ζ unless a chapter says how, and the transfer is then labelled.

2. Authorship and AI assistance. The questions, hunches and direction of every chapter are J. W. McGreevy’s, and each note records the request it answers (tags such as [t352u]). The written notes are a different matter. Their wording, proofs, literature checks and all computations (scripts in Python/mpmath, PARI/GP, FLINT/Arb) were produced by an AI assistant (Grok Bot) at his direction. That is why every note carries the byline “J. W. McGreevy; formalization by Grok Bot”. The material has not been refereed or checked by an independent human expert. Read it as a research record, not as vetted mathematics.

3. Pasted texts were critiqued, not trusted. Many chapters respond to text pasted into the working conversation: excerpts from other AI systems, drafts, or the assistant’s own earlier chat replies. These were treated as *claims to check*, never as sources. Each note ends with a “Corrections” section that records which claims were right, which were wrong, and how they were fixed (consolidated in Appendix A). Where a pasted text could not be verified, the note says so.

4. Numerics. Computations are [Numerical] unless they are explicitly labelled computer-assisted (Arb ball arithmetic, or exact arithmetic with certified root isolation, [EXACT-COMP]). The certified items are listed in §A below. Floating-point and multiprecision results are reported with their stated tolerances. They are not proofs, and checks over finite ranges say nothing beyond those ranges.

5. References. The notes distinguish [CITED] (checked at the source), [CITED-sec] (checked through a named secondary source) and [BIB] (bibliographic record verified, content not re-read). Each note’s references are recorded in the project’s per-lemma reference logs (`lit/lemmaNN/verified_refs.md`). A source that could not be verified was dropped or explicitly marked.

6. Facsimile chapters. Chapters 66–85 (this part) and Chapters 48–65 (Volume IVa) are the compiled notes themselves, reproduced page by page (see “Status of this volume”). A statement “Thm. $s.n$ of Ch. k ” is Theorem $s.n$ of TAA Lemma k . Each page carries the note’s own page number and the volume page number in its footer.

Claim labels

Every claim in the notes carries a label. The local tag variants map to seven volume classes:

Volume class	Meaning	Tags used in the chapters
Theorem	Proved in the note, or a published theorem used as stated.	[THEOREM]/[THM]; [CITED], [CITED-sec], [BIB] for cited results; [EXACT-COMP] and “[THM, computer-assisted]” for certified finite computations; [TAA-L] for a result of an earlier TAA lemma
Restatement	Equivalent to RH, GRH or a known statement; no new content.	[RESTATEMENT]
Analogy	A structural parallel; proves nothing.	[ANALOGY], [METAPHOR]
Conjecture	A stated expectation, with its evidence.	[CONJECTURE]/[CONJ]
Open	A question or gap explicitly left open.	[OPEN]; “Questions for J.W.”
Numerical	Floating-point or multiprecision computation; not a proof.	[NUMERICAL]/[NUM]; [CROSS-CHECK]
Heuristic	A plausibility argument or estimate, weaker than a conjecture.	[HEURISTIC]

What this volume does and does not advance

Scope of this statement. This statement covers the whole of Volume IV, both parts: Volume IVa (Chapters 48–65) and Volume IVb (Chapters 66–85). It is printed identically in both parts. Locations in this part (IVb) give the note page and the volume page of this part; locations in the companion part are marked “in Volume IVa”.

Summary. This volume does not advance RH for ζ . Its contributions are of four modest kinds. (A) A few certified computations, mostly about the Davenport–Heilbronn function, a fake with a functional equation but no Euler product. (B) Elementary theorems that we did not find stated in the literature we checked. The searches were not exhaustive, so each is “new to our knowledge” at most, and none bears directly on RH. (C) Re-packagings of known results, credited to their authors. (D) Negative results: proofs or computations showing that a proposed source of positivity cannot work. Items are labelled with the classes above. Locations give the chapter, the statement number inside the note, the note page and, for this part, the volume page.

A. Certified computer-assisted results [Theorem, computer-assisted]

- A1. *No zeros of f_{DH} with $1 \leq \sigma \leq 1.6$, $0 \leq t \leq 1000$* (Arb ball arithmetic, winding number 0). Ch. 56, Thm. 7.5 (note p. 22, in Volume IVa). Davenport–Heilbronn proved that f_{DH} has infinitely many zeros with $\sigma > 1$; the certificate only says none lie in this box.
- A2. *Lower bound $\Lambda_{\text{DH}} \geq 0.230607$ for the de Bruijn–Newman-type constant of f_{DH} : H_t^{DH} has a non-real zero at $t = 0.230607$* (certified Laguerre sign). Ch. 56, Thm. 7.6 (note p. 23, in Volume IVa); the upper bound 1.6025 (de Bruijn’s strip theorem, [BIB]) is in Prop. 7.4. The analogous statement for ζ is the Rodgers–Tao theorem $\Lambda \geq 0$, which we only cite.
- A3. *Certified upper bounds for the smallest eigenvalue $\lambda^*(e^A)$ of the truncated Weil form at $A = 1.8, 2, 2.2, 2.4, 3.5$* , giving two-sided enclosures when combined with the Volume II lower bounds. Also certified: *the f_{DH} Weil form is indefinite for all $A \geq 3.43$* , and a *no-slack theorem* for a single local term. Ch. 57, Thms. 3.1, 4.1 (note p. 4, in Volume IVa); Thm. 5.1 on p. 6 (in Volume IVa). These are finite-window facts. They neither prove nor test RH.
- A4. *Extremal weight enumerators of self-dual codes satisfy Duursma’s RH*, with $\kappa = 0$ certified, for Types I, III, IV with $n \leq 200$ and Type II with $n \leq 1000$ [EXACT-COMP]. Also: exact RH sets on the Type II lines $n = 24, 32, 40$. Ch. 77, Answer (2) (note p. 1, vol. p. 166); Thm. 6.3 on p. 7 (vol. p. 172). Duursma’s RH for codes is not RH for ζ .

B. Elementary theorems, new to our knowledge [Theorem]

Each is a short argument from standard facts. We claim the formulation, not depth, and we may have missed prior statements.

- B1. *Multiplier no-go.* On a single copy of L^2 (scaling of multiplicity one), every positivity built from scaling-commuting operators is one of four things. It is blind (positive for fakes too), a restatement of Weil’s criterion, useless, or dependent on knowing the zeros. Ch. 80, Thm. 3.2 (note p. 2, vol. p. 208). The commutant statement it rests on (Prop. 3.1: such operators are Fourier multipliers) is classical. The consequence that a candidate polarization needs a space

on which scaling has multiplicity is drawn in Ch. 78, §7(viii) (note p. 16, vol. p. 194) and Ch. 80, §§3–4.

- B2. *Visibility criterion and strip visibility law.* A zero is visible to the scaling action on a weighted space if and only if an explicit weight integral converges. On Meyer’s spaces S_I , the spectrum consists exactly of the zeros with real part in I , each one a Jordan block of size equal to its order, given Meyer’s theorem. Ch. 61, Prop. 2.1 (note p. 3, in Volume IVa); Thm. 3.3 on p. 4 (in Volume IVa). The RH reading (“no ghosts $\iff$ nothing lost between the analytic and smooth classes”) is [Restatement].
- B3. *Zero sums in modular and gravitational partition functions.* Three expansions: the reflected half of the Maloney–Witten–Keller spin-0 density at high temperature equals an anomaly term plus a sum over ζ zeros (Ch. 72, Thm. 3.3 (note p. 6, vol. p. 97)); its Bianchi analogue (Ch. 73, Thm. 6.1 (note p. 10, vol. p. 115)); and a tropical limit and smoothed height count for $E(\tau, s)$ (Ch. 74, Thms. 5.2, 6.1 (note p. 9, vol. p. 130)). Each says where the zeros enter. The negativity of the density is shown to be RH-blind; RH enters only as a decay rate ([Restatement]).
- B4. *Ghost charge and Krein-vacuum statements* (presentation-level, labelled as such in the notes). The ghost charge of the Hardy-space vacuum ($\text{ch} = -\#\text{ghosts}$) is in Ch. 53, Thm. 4.3 (note p. 5, in Volume IVa). The emergent complex structure J_F and the drift detector are in Ch. 54, Thm. 6.1 (note p. 11, in Volume IVa), with their prior art (Segal, Weinless, Kay; the matrix sign function) stated in the note. The exact count of ξ' zeros at admissible heights and the circle identity $\kappa_{\text{FF}} = \#\text{ghosts}$ over $\mathbb{F}_q$ are in Ch. 56, Thm. 4.2 (note p. 7, in Volume IVa) and Thm. 6.2 on p. 15 (in Volume IVa).
- B5. *Forced zero at the Fricke fixed point.* For a newform f of level N with root number $\varepsilon = -1$, $f(i/\sqrt{N}) = 0$. Ch. 85, Thm. 10.2 (note p. 13, vol. p. 255). This is an immediate consequence of $f|W_N = wf$, $\varepsilon = -w$, and is almost certainly folklore.

C. Known results re-proved or re-packaged [Theorem / Restatement; exposition]

- C1. *κ is the negative index of the Rosati/Weil trace form.* For a squarefree Weil polynomial, $\text{Tr}(xx^\dagger)$ on $\mathbb{Q}[F]$, with $F^\dagger = qF^{-1}$, is positive definite iff all roots have absolute value $\sqrt{q}$. Its number of negative directions equals κ , the number of off-circle reciprocal pairs. Ch. 84, Thm. 3.2 (note p. 3, vol. p. 236). The finite Toeplitz form of the zero power sums has the same Krein inertia: Ch. 77, Thm. 4.1 (note p. 4, vol. p. 169), which Ch. 77 itself labels a restatement “of the kind Li’s criterion is”. These are Sylvester’s law of inertia and Hermite’s signature count for trace forms. The link to RH for curves is Weil’s proof via Rosati positivity (Mumford, §21, [CITED-sec]). *Exposition, no novelty.* The accompanying computation is [Numerical] and exact in integer arithmetic. It is an exhaustive scan over $q \in \{2, 3, 5, 7, 11\}$ of all 13 135 squarefree genus-2 Weil-shaped polynomials, of which 785 satisfy RH, with 0 mismatches between “positive definite” and “RH” and $\#\text{neg} = \kappa$ in every case (Ch. 84, p. 4, vol. p. 237). It confirms a proved theorem; it is a consistency check, not new evidence. The same κ shows up on the zero side. In a 25-Gaussian Weil Gram matrix there is one negative eigenvalue (-1.078) for f_{DH} , and none for ζ or $L(s, \chi)$ (Ch. 81, §5 (note p. 4, vol. p. 215); [Numerical]).
- C2. *Every ghost is visible at finite Connes cutoff.* If a self-dual entire Λ has a zero off the line, the finite-support Weil form Q_L fails to be positive semidefinite for all $L > L^*$, with $L^* < \infty$. Ch. 82, Thm. 5.1 (note p. 3, vol. p. 224). The proof is Weil’s criterion plus Paley–Wiener, so the statement is a *finite-support restatement of Weil’s criterion*, in the spirit of Connes’s cutoff (Connes 1999, [CITED]). What is new is numerical. For f_{DH} , Q_L becomes indefinite at $L^* \approx 3.73$ (cutoff $\Lambda^* \approx 6.4$, primes $n \leq 41$) on the frequency band $[70, 102]$, with $\lambda_{\min} = -1.340$ at $L = 4$. For ζ , $\lambda_{\min} \geq -9.7 \cdot 10^{-10}$ (the numerical floor) on the same band (Fig. 1, p. 3; [Numerical], not certified).

- C3. *Epstein zeta functions have off-line zeros* (Davenport–Heilbronn 1936; Potter–Titchmarsh 1935; [BIB] in Ch. 83). New in Ch. 83 are a count for $x^2 + 5y^2$ and a channel theorem. The count: $N(200) = 253$ zeros, of which 183 are on the line and 70 off it, the first at $0.93297 + 15.66825i$ (Ch. 83, §5 (note p. 4, vol. p. 230); [Numerical]). The theorem: the closed-geodesic side of the Selberg trace formula for $\mathrm{PSL}_2(\mathcal{O}_K)$ sees only the Euler-product channels $\det \Phi = \pm \prod_{\chi} \Lambda(s, \chi) / \Lambda(1 + s, \chi)$, so Epstein ghosts contribute nothing to it (Ch. 83, Thm. 5.1 (note p. 4, vol. p. 230); “channel blindness”, an elementary consequence of the cited trace formula). The *transport gap* says that the elliptic terms at $\tau = i$ are zero-independent and cannot carry Rosati positivity at the elliptic point to the zeros (Ch. 83, Thm. 6.4 (note p. 6, vol. p. 232)). The Bianchi link remains [Open].
- C4. *Mordell–Weil lattices, Shioda–Tate, Tate’s conjecture and BSD over function fields* are cited (Shioda; Schütt–Shioda; Tate; Artin–Swinnerton-Dyer), as are modularity (Wiles; Taylor–Wiles; Breuil–Conrad–Diamond–Taylor) and Sato–Tate (Barnet-Lamb–Geraghty–Harris–Taylor and coauthors); see Ch. 85, §§2, 9, 10. Exposition, with explicit examples computed: a rational elliptic surface over $\mathbb{F}_5$ with Mordell–Weil lattice $A_3 \oplus A_1$ and a K3 family over $\mathbb{F}_7$ (§3); EDSAC-type products $\prod_{p \leq x} N_p/p$ for ranks 0–3 (§4). The *split-invariance detector*, i.e. independence of the Fricke-split Mellin integral from the split point (the classical device of Riemann’s second proof), detects fakes with the wrong root number or altered coefficients. But it is *RH-blind*: f_{DH} passes it to 10^{-41} despite its off-line zero $0.808517 \dots + 85.699348 \dots i$, while the Ch. 82 cutoff detector rejects f_{DH} . See Ch. 85, §10 (note p. 14, vol. p. 256); [Theorem]/[Numerical].
- C5. *Riemann’s bilinear relations for the $(\chi, \bar{\chi})$ sector*. The first relation holds unconditionally (character orthogonality); the second is $\mathrm{GRH}(\chi)$, a restatement of Weil’s criterion. Ch. 81, Thm. 3.1 (note p. 2, vol. p. 213).
- C6. *Earlier chapters* state their own novelty assessments, and we adopt them. Ch. 48: every RH-equivalence is a restatement (Lagarias, Burnol, BSY), the remaining theorems are presentation-level and probably folklore, and the new item is numerical (an argument-principle count of all 904 f_{DH} zeros to height 1000: 28 ghosts with $\beta > \frac{1}{2}$, i.e. 56 off-line zeros). Chs. 49 and 50: classical results re-derived, with “no new mathematics”. Ch. 55: the separatrix picture is correct and classical. Ch. 59: Laguerre–Pólya positivity is zero location in disguise. Ch. 63: Michałowski’s PF_5 witness is reproduced (credited), and the cone-metric dictionary is standard. See each note’s “What is new” or “Prior art” section.

D. Negative results (what cannot supply the positivity)

These are the volume’s main content: precise reasons why a candidate source of positivity fails. They are labelled [Theorem] where proved, otherwise [Numerical].

- D1. *Single-copy operators are multipliers, so multiplicity is needed* (B1 above; Ch. 78, §7(viii) (note p. 16, vol. p. 194); Ch. 80, Thm. 3.2 (note p. 2, vol. p. 208)).
- D2. *Local positivity at every prime does not give global positivity*. One positive-definite binary form per good prime exists for every $E/\mathbb{Q}$ (Ch. 64, Prop. 6.5 (note p. 12, in Volume IVa)). In Ch. 85, §9, the normalized $a_{p^k}/p^{k/2}$ are spin- $k/2$ characters $U_k(\cos \theta_p)$ (Hecke relations; 1088 exact checks), and Sato–Tate is cited. “Local unitarity plus Sato–Tate” controls equidistribution, a density statement with no rate, whereas RH for $L(E, s)$ is a rate, $\sum_{p \leq x} 2 \cos \theta_p \log p = O(x^{1/2+\varepsilon})$. So there is no route from local to global positivity this way (Ch. 85, §9 (note p. 12, vol. p. 254); [Theorem] for the identities, [Heuristic] for the moral).
- D3. *A positivity counter and an algebraicity counter catch different ghosts*. In a rational elliptic surface over $\mathbb{F}_5$, a fake with an on-circle Frobenius eigenvalue that is not q times a root of unity has $\kappa = 0$ and a positive trace form. It is caught only by the algebraic count $\bar{\rho} = b_2$,

which holds for rational surfaces (Ch. 85, §3.3, fake G3 (note p. 5, vol. p. 247); [Numerical], reading [Theorem]). For ζ there is no Néron–Severi group, so only the first kind of counter is conceivable.

- D4. *Modularity sees the functional equation, Weil positivity sees the zeros* (C4 above; Ch. 85, §10 (note p. 14, vol. p. 256)).
- D5. *Other no-go results*, each with a proof or computation in its note. Laguerre–Pólya positivity is of Krein type, not a polarization (Ch. 59, verdict). Small-support and archimedean positivity are blind (Ch. 82, §5(i)–(ii) (note p. 3, vol. p. 224)). Real- s positivity of the Mayer transfer operator cannot detect off-line zeros (Ch. 75, Prop. 7.2 (note p. 11, vol. p. 147)). No positivity in Ch. 76 sees off-line zeros (Ch. 76, Prop. 8.1 (note p. 12, vol. p. 163)). The MWK negativity is RH-blind (Ch. 72). Passivity and Kramers–Kronig causality of the modular scattering matrix are RH-blind (Ch. 78).

What remains open

The obstruction isolated in Chapters 84–85 has three parts [Open]:

- (1) *No rational lattice*. There is no finite-dimensional $\mathbb{Q}$ -algebra $\mathbb{Q}[F]$ for ζ that contains a Frobenius (Ch. 84, §6, [Theorem]). An infinite-dimensional substitute would need a positive state, and that positivity is RH.
- (2) *No ample class*. There is no known $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$ carrying an ample class whose Hodge index theorem would give the sign. (Arithmetic Hodge index theorems, Faltings–Hriljac, give positivity of heights, not RH; Connes–Consani’s scaling site is a candidate setting, not a construction.)
- (3) *Zero global margin despite full local margin*. Every local positivity holds (Hasse at each prime, Sato–Tate), but no global margin is available.

Refining this, compare where the three kinds of data have a global algebraic framework over $\mathbb{Q}$:

- The Fourier coefficients of a newform do: the Hecke algebra acts on $X_0(N)$, a curve over $\mathbb{Q}$.
- The local zeros, i.e. the roots of the local factor at each prime p , have one at each prime: Frobenius acting on $X_0(N) \bmod p$.
- The global zeros of $L(s)$ have none.

What is missing is a *global operator whose spectrum is the set of zeros, together with a positive form coming from a global ample class*, as Frobenius and the Hodge index theorem on $C \times C$ provide in Weil’s proof. Nothing in this volume supplies it. The “Questions for J.W.” sections of this part’s chapters are collected in Appendix B; those of the other chapters are in the corresponding appendix of Volume IVa.

Status of this volume

Two parts. Volume IV is published as two files: Volume IVa (Chapters 48–65) and Volume IVb (Chapters 66–85). The split is made only because the archive accepts files of at most 5 MB; IVa and IVb together are one Volume IV. The Statement of Intellectual Honesty, the claim labels, “What this volume does and does not advance”, “What remains open” and this status section cover the whole volume and appear in both parts. The introduction and the appendices cover only the chapters of the part they are in. Volume pages are counted separately in each part.

Assembly. This volume was assembled on 24 October 2026 from the latest compiled versions of TAA Lemmas 48–85. These are Lemma 54 v2, Lemma 56 v3, Lemma 81 with its two addenda (Möbius strip and Verschiebung; Cassini ovals and the de Bruijn–Newman flow), and Lemma 85 with Sections G–I. Chapter k is Lemma k .

Facsimile inclusion. Unlike Volumes II and III, the chapters are not re-typeset. Each note’s compiled pages are reproduced at 93% scale, with its own title, numbering, figures and reference list. The reason is technical. Lemmas 60–85 are multi-file builds with conflicting local macro sets, and a faithful source conversion would have required editing them. Consequences:

- Statement numbers restart in each chapter.
- Each chapter keeps its own bibliography; there is no consolidated bibliography.
- Cross-references between chapters are by lemma number.

A re-typeset edition can be produced later without changing any content.

Raster figures. To keep the file size within the archive’s limit, some heavy vector figures (dense scatter, contour and heat-map plots) are reproduced as raster images at 200 dpi, at the same size and position. All text and equations, and all other figures, remain vector. The vector originals are unchanged in the standalone notes.

Lemma 50. Volume III included only an eight-item condensation of Lemma 50 (an editorial section of its Introduction). The full note was not in Volume III: proofs, the sections on Riemann’s θ and the metaplectic cover, Shale–Stinespring and the Krein–Fock construction, the half Tate twist and Sato–Tate, what $\text{Spec } \mathbb{Z}$ would need, corrections and questions. It is therefore included in full as Chapter 50 (in Volume IVa). The overlap with Volume III is that condensation only.

Lemma 63. Lemma 63 (“Ghosts as dents in a flat cone metric, the right-angle cross, zero-by-zero Weil positivity, the kernel-condition hierarchy, and DH Jensen polynomials to degree $2 \cdot 10^4$ ”) was completed on 22 October 2026. Its compiled PDF remained in the build directory and was never copied to the project’s top level, which is why it was briefly thought missing. It is included unchanged as Chapter 63 (in Volume IVa).

Not included. The project’s working notes (`STATE.md`, `NOTATION.md`), scripts and data are not part of the volume. Each note names its scripts and output files.

Place in the series

Volume	Title	Contents and status
I	Arithmetic Unitarity I: Where the Riemann Hypothesis Lives in the Scaling Flow	Lemmas 1–13; ai.viXra:2610.0045v2
II	Arithmetic Unitarity II: The Light Cone of the Weil Form	Lemmas 14–29; ai.viXra:2610.0054
III	Arithmetic Unitarity III: In Search of Positivity — G5, Polarization, and Hilbert versus Krein	Lemmas 30–47 (and a condensation of Lemma 50); ai.viXra:2610.0052
IVa	Arithmetic Unitarity IVa: Counting Ghosts — Spinor Vacua, Saddles, the Weil Positivity Ladder, and the Prime Side	Lemmas 48–65; companion part of Volume IV
IVb	Arithmetic Unitarity IVb: Counting Ghosts — Eisenstein Anomalies, Hyperbolic Geometry, the Ample-Class Obstruction, and Mordell–Weil Lattices	Lemmas 66–85; this part

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Introduction

The thread

Volume III ended with a slogan proved in several forms: *a reflection gives the functional equation; positivity needs a polarization*. Algebra gives Krein unitarity, an indefinite form with κ negative directions, one per pair of ghosts (off-line zeros). Hilbert unitarity ($\kappa = 0$) needs a positive polarization, which for curves over $\mathbb{F}_q$ comes from the Hodge index theorem on $C \times C$ through Rosati positivity. For ζ no such structure is known.

Volume IV pursues this along six lines, in roughly the order the notes were written. The first three are in Volume IVa (Chapters 48–65), the last three in Volume IVb (Chapters 66–85):

- polarizations as vacua and complex structures;
- saddles and zero counting;
- the prime side and $\mathbb{F}_1$;
- modular forms and Eisenstein anomalies;
- hyperbolic geometry, gravity, Farey fractions and codes;
- the final cluster on causality, the multiplier no-go, the Hodge–Riemann form, Connes’s cutoff, Albert’s classification and Mordell–Weil lattices.

Two diagnostics run throughout. One is the ghost counter κ ; the other is the Davenport–Heilbronn function f_{DH} , which has a functional equation of Riemann type but no Euler product, and has off-line zeros (the first at $0.808517\dots + 85.699348\dots i$). A structure that f_{DH} also satisfies cannot prove RH. Most chapters apply this test explicitly. Volume IVa (Chapters 48–65) treated the first three lines; its main items are listed in “What this volume does and does not advance” above.

Standing conventions

$s = \sigma + it$; $\rho = \beta + i\gamma$ is a nontrivial zero, a *ghost* if $\beta \neq \frac{1}{2}$; $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. The Weil form is Q (or W , W_A , Q_L for truncations, as defined per chapter). Over $\mathbb{F}_q$, F is Frobenius, $V = qF^{-1}$ the Verschiebung, and κ the number of off-circle reciprocal pairs of Frobenius eigenvalues. Objects such as h^1 (a motive, a categorical object), H^1 (a vector space, its realization), Hodge–Riemann relations (inequalities on $H^{1,0}$) and the Hodge index theorem (signature of the intersection form on Néron–Severi of a surface) are restated in each chapter that uses them. Appendix C lists the conventions that differ between chapters.

Chapter by chapter (this part)

Modular forms and Eisenstein anomalies (Chs. 66–71).

- 66:** The 691 bridge: Eisenstein congruences, the ω^{-11} class, where the polarization degenerates.
- 67:** The archimedean lift of the 691 congruence: the quasi-modular anomaly and the Ramanujan–Serre derivative.
- 68:** Rotation versus boost for Δ and E_{12} : spinors, multiplicativity, duality, parity.
- 69:** Curvature, the Eisenstein pole and ghosts: how far the positivity of the anomaly reaches.
- 70:** The weight-2 Eisenstein core and the cusp: can 691 and Mazur’s congruences be read off the low-temperature expansion?

71: The Kustaanheimo–Stiefel lift of the local triple: Frobenius as a null vector, the Weil light cone on $E \times E$.

Hyperbolic geometry, gravity, Farey fractions and codes (Chs. 72–77).

72: Pure 3D gravity: where ζ zeros meet the negative densities of the Maloney–Witten–Keller partition function (negativity is RH-blind).

73: Bianchi groups and the arithmetic (3,1): integral light rays, the class group, the Bianchi scattering matrix.

74: Horocycles, heights and the hyperbolic Legendre transform: the tropical limit of $E(\tau, s)$.

75: Mayer–Hensley: localizing the zeros in the cusp tail of the Gauss transfer operator; a no-go for real- s positivity.

76: Rationals in $[0, 1)$: Farey discrepancy, Franel–Landau (restatements of Littlewood’s criterion), the factorial motive.

77: Codes as a finite laboratory for J : Duursma zeta functions, Toeplitz–Weil positivity, certified results for extremal enumerators.

The final cluster (Chs. 78–85).

78: Causality versus linewidth: passivity is RH-blind; RH is a uniform linewidth $\frac{1}{4}$; any scaling-commuting quarter turn is a multiplier.

79: The holomorphic half and its conjugate: Riemann–Siegel, the Γ -phase as gluing, conjugacy and f_{DH} .

80: A no-go for multiplier polarizations.

81: The $(\chi, \bar{\chi})$ Hodge–Riemann form (first relation free, second = GRH); addenda on the Möbius strip, branched covers and Verschiebung, and on Cassini ovals and the de Bruijn–Newman flow.

82: Connes’s cutoff: the f_{DH} ghost is visible at finite cutoff, before any unitarity step.

83: From Laplace–Runge–Lenz to $\mathbb{H}^3$: Coulomb phases are the archimedean place; closed orbits see only Euler-product channels; the transport gap at $\tau = i$.

84: Albert’s classification, the $\mathbb{Q}$ -structure and the ample class: Rosati positivity is Weil’s proof; one negative direction per ghost pair.

85: Mordell–Weil lattices, the ample class and BSD, with the chord–tangent law on the Möbius circle (§8), spinor characters and Sato–Tate (§9), and modularity and the divergent region (§10).

Editorial remarks

- Chapters were not edited. Where a later chapter corrects an earlier one, the correction is recorded in the later chapter and listed in Appendix A.
- The order of chapters is numerical. Chapter 66 opens this part; Chapters 48–65 are in Volume IVa.

TAA LEMMA 66: THE 691 BRIDGE.
EISENSTEIN CONGRUENCES, THE ω^{-11} CLASS, THE DEGENERATION OF
THE POLARIZATION,
AND WHERE GHOSTS ENTER

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA; nor is GRH for $L(\Delta, s)$. Nothing below is a step toward a proof that we can certify. Tags: [THEOREM] proved here (elementary) or classical with proof indicated; [CITED] checked at the source (`lit/lemma66/verified_refs.md`); [CITED-sec] via a named secondary source (mostly Mazur’s 2011 Bulletin survey [13], read in full for this lemma); [BIB] bibliographic record verified, content standard but not opened; [TAA-L] earlier TAA lemma; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (exact integer arithmetic in PARI/GP 2.17.2 unless said).*

Question (J.W., t252u). “If principalization kills the discrete part, then the χ^{11} minus class group eigenvalue ought to be a bridge between these multiplicativity and duality worlds.” Steering (t253u): “the 691 bridge is one piece of the positivity, and ghost killing is collective ... we need to find either a generalization or the elements around the 691 bridge where ghosts come into play”; and: how does mod 691 connect to ℓ -adics?

ANSWER IN BRIEF

(1) *The bridge is real and precise (§§3–5).* Deligne’s 691-adic representation $\rho_{\Delta, 691}$ of the multiplicative object Δ is irreducible; its reduction has semisimplification $\mathbf{1} \oplus \omega^{11}$ (Ramanujan’s congruence) [CITED-sec]; a suitable lattice gives a non-split, everywhere unramified extension, i.e. a nonzero ω^{-11} -eigenspace in $\text{Cl}(\mathbb{Q}(\mu_{691}))$ [691] (Greenberg–Monsky for Δ ; Ribet in general via weight-2 forms) [CITED-sec]. Its eigencharacter is odd, which is forced by $-I \in SL_2(\mathbb{Z})$ (Prop. 3.2) [THEOREM]; the “minus” in J.W.’s phrase is the residue of complex conjugation, i.e. of the duality/real-place structure. (2) *What the bridge does to J (§4, central).* For Δ at p the TAA complex structure is $J_p = (2\pi - \tau(p))/\sqrt{4p^{11} - \tau(p)^2}$, $J_p^2 = -1$ [THEOREM] (Lemma 65 form). Mod 691, $(2\bar{\pi} - \bar{\tau})^2 = (1 - p^{11})^2$, so $\sigma_p = (2\bar{\pi} - \bar{\tau})/(1 - p^{11})$ is an *involution* ($\sigma_p^2 = +1$) whose eigenlines are $\mathbf{1}$ and $\mathbb{L}^{11}$; J_p does not reduce to $\mathbb{F}_{691}$ at all (since $691 \equiv 3 \pmod{4}$), and over $\mathbb{F}_{691^2}$ it reduces to $i\sigma_p$ [THEOREM]. The positive-definite local form of Δ is congruent to the *hyperbolic* local form $(x+y)(x+p^{11}y)$ of E_{12} [THEOREM]. In Lemma 64/65 language: mod 691, the h^1 -type piece degenerates into the *pole-type* piece $\mathbf{1} \oplus \mathbb{L}^{11}$, whose signature is $(1, 1)$ at every prime. (3) *Principalization.* No 691-class of $\mathbb{Q}(\mu_{691})$ capitulates anywhere in the cyclotomic $\mathbb{Z}_{691}$ -tower: the minus part by Kida’s theorem [CITED-sec], the plus part because it is 691-free [CITED-sec]. All classes die in the Hilbert class field (Furtwängler), which carries no 691-specific information. The precise surviving form of J.W.’s hunch is class field theory itself: the class is *converted* into the Galois group of the unramified Ribet extension [RESTATEMENT]. (4) *Where ghosts enter (§6).* $\Delta \equiv E_{12} \pmod{691}$ coefficientwise, both have Euler products and the functional equation $s \mapsto 12 - s$, yet *every* zero of $\Lambda(E_{12}, s)$ lies off the axis $\text{Re } s = 6$ (unconditionally; Thm. 6.1), while GRH predicts that every zero of $\Lambda(\Delta, s)$ lies on it [THEOREM]/[CONJECTURE]. So no statement about zero location is a function of the mod- ℓ data [THEOREM]. The bridge is a single closed point of $\text{Spec } \mathbb{T}_{12}$, where the Eisenstein and cuspidal components meet (Prop. 5.2) [THEOREM]/[NUMERICAL]. Ghosts and positivity live on the components

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and at the real place: (a) the sign of the local discriminant (proved for Δ by Deligne’s weights); (b) L^2 -ness/unitarity: $\langle \Delta, \Delta \rangle = 1.0353620568 \times 10^{-6} > 0$, while $E_{12} \notin L^2$; (c) the *archimedean size* of the very same class group, $h^- = 2p(p/4\pi^2)^{(p-1)/4} \prod_{\chi \text{ odd}} |L(1, \chi)|$, which is zero-sensitive, while its 691-part is not (Fig. 4); (d) GRH for $L(\Delta, s)$ (open). The generalization is the family of all Eisenstein congruences: for the six level-one weights with $\dim S_k = 1$ the Hecke algebra is $\{(a, b) \in \mathbb{Z}^2 : a \equiv b \pmod{N_k}\}$ with $N_k = |\text{num}(B_k/2k)|$ (691, 3617, 43867, $283 \cdot 617$, $131 \cdot 593$, 657931) [NUMERICAL], and the same verdict holds at each. (5) *Correction to Grok Bot’s “ghost-blind”*. The full ℓ -adic representation is *not* ghost-blind: it determines every $\tau(p)$, hence $L(\Delta, s)$ and its zeros, and Deligne’s weights prove local positivity. What is blind is the residual (mod ℓ) data, the ℓ -part of the class group and the Iwasawa invariants.

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1. SOURCES AND CONVENTIONS

Read for this lemma: STATE.md §§69, 73, 74 (Lemmas 60, 64, 65); Lemma 60 §§“Principalization and capitulation” and “Verdict” (source `build60/lemma60.tex`, including Remark “The 691 class never principalizes” and its citation of Ribet [16]); Lemma 64 (three-ingredient reading, ablation table); Lemma 65 (triple (ω, J_p, g_p) , $\mathbb{L}$, pole $\leftrightarrow \mathbf{1} \oplus \mathbb{L}$). Primary/secondary sources read in full for this lemma: Mazur’s Bulletin survey [13] (local copy `lit/mazur2011.txt`; all statements marked [CITED-sec] about Ribet, Deligne, Serre, Greenberg–Monsky, Hida and the main conjecture are taken from it, with section numbers), and Lemmermeyer [11] (arXiv; §8, Theorem 5, which states and sharpens Kida’s theorem [9]). Ribet’s paper [16], Deligne’s Bourbaki exposé [3], Mazur–Wiles [14] and Washington’s book [18] were *not* opened; their bibliographic data were verified on Crossref.

Conventions. $\Delta = q \prod (1 - q^n)^{24} = \sum \tau(n) q^n$; E_{12} is normalized here as $G_{12} = -\frac{B_{12}}{24} + \sum_{n \geq 1} \sigma_{11}(n) q^n$ (Mazur’s normalization [13, §5, item 6]), so that both forms have $a_1 = 1$ and $L(\bar{E}_{12}, s) = \zeta(s) \zeta(s - 11)$. ω is the Teichmüller (mod- p cyclotomic) character; for $p = 691$, $\omega^{-11} = \omega^{679}$. Local forms follow Lemma 65: for a degree-2 Euler factor $1 - a_p X + p^w X^2$ put

$Q_p(x, y) = x^2 + a_p xy + p^w y^2$ (this is $\det(x + y\pi)$ for a Frobenius π with trace a_p and determinant p^w). $\Lambda(\Delta, s) = (2\pi)^{-s} \Gamma(s) L(\Delta, s)$, $\Lambda(\Delta, 12 - s) = \Lambda(\Delta, s)$; the same completion is used for E_{12} . Scripts: `num/lemma66_{checks,zeros_h,hecke,figs}.py` $\rightarrow$ `num/166/`.

2. THE MULTIPLICATIVE SIDE: Δ AND ITS LOCAL FORMS

Proposition 2.1 (Local form $\Leftrightarrow$ temperedness). *Let $1 - aX + p^w X^2$ have real a and roots α, β of $T^2 - aT + p^w$. Then $Q_p = x^2 + axy + p^w y^2$ is positive definite $\Leftrightarrow |a| < 2p^{w/2} \Leftrightarrow \beta = \bar{\alpha}$ with $|\alpha| = |\beta| = p^{w/2}$. It is indefinite (signature $(1, 1)$) $\Leftrightarrow \alpha, \beta$ are real, distinct, with $\alpha\beta = p^w$. [THEOREM]*

Proof. Discriminant of Q_p is $a^2 - 4p^w$, which is also the discriminant of $T^2 - aT + p^w$. $\square$

For Δ , $w = 11$ and $a = \tau(p)$. $|\tau(p)| \leq 2p^{11/2}$ is Deligne's theorem [4] (via [3]) [BIB]; equality is impossible because $2p^{11/2} \notin \mathbb{Q}$, so every Q_p^Δ is positive definite [THEOREM] (given Deligne). For E_{12} , $a = 1 + p^{11}$ and

$$Q_p^{E_{12}}(x, y) = x^2 + (1 + p^{11})xy + p^{11}y^2 = (x + y)(x + p^{11}y), \quad \text{disc} = (1 - p^{11})^2 > 0,$$

so every $Q_p^{E_{12}}$ is hyperbolic [THEOREM]. In the variable $x_p = a_p/2p^{11/2}$: $|x_p^\Delta| < 1$, while $x_p^{E_{12}} = \cosh(\frac{11}{2} \log p) \geq 22.6$.

Numerics [NUMERICAL] (Fig. 1). For the 9592 primes $p < 10^5$: all $\text{disc}(Q_p^\Delta) < 0$; $\max |x_p| = 0.99808$ (at $p = 74471$); mean of x_p^2 is 0.24887 (the Sato–Tate value is $1/4$; shown for orientation only, no theorem is cited for it here).

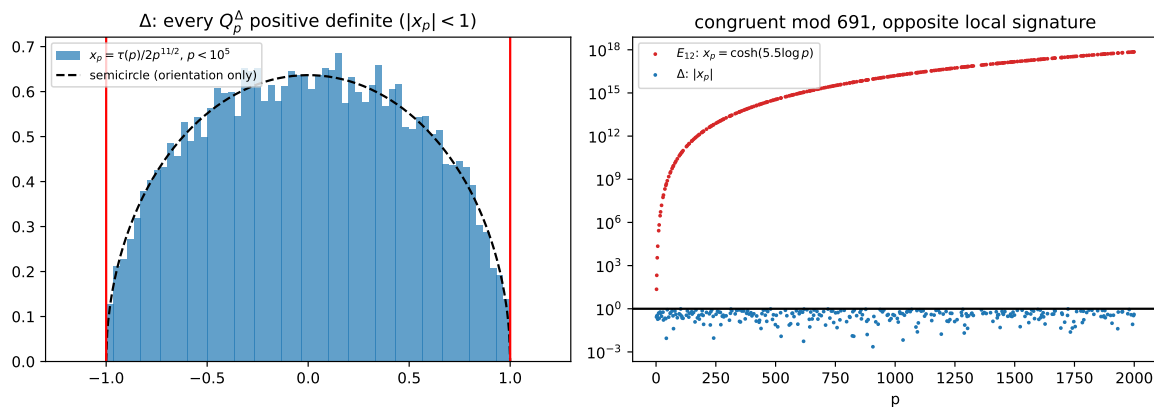


FIGURE 1. Left: $x_p = \tau(p)/2p^{11/2}$ for $p < 10^5$; all inside $(-1, 1)$ (positive definite Q_p^Δ). Right: $|x_p|$ for Δ (blue) and $x_p = \cosh(5.5 \log p)$ for E_{12} (red), $p < 2000$, log scale. The two forms are congruent mod 691 at every p .

3. THE ℓ -ADIC CHAIN (EXPOSITORY)

This section answers J.W.'s question "how does mod 691 connect to ℓ -adics?" Everything is [CITED-sec] from [13] unless marked.

(L1) *The ℓ -adic representation.* For every prime ℓ there is a continuous irreducible $\rho_{\Delta, \ell} : G_{\mathbb{Q}} \rightarrow \text{GL}_2(\mathbb{Q}_{\ell})$, unramified at $p \neq \ell$, with $\text{Tr } \rho_{\Delta, \ell}(\text{Frob}_p) = \tau(p)$ [13, Thm. 3], proof in [3]. The determinant is the character with value $p^{w-1} = p^{11}$ at Frob_p [13, Thm. 4 and following paragraph], i.e. $\det \rho_{\Delta, \ell} = \chi_{\ell}^{11}$ with χ_{ℓ} the ℓ -adic cyclotomic character. Geometrically, $\rho_{\Delta, \ell}$ is cut out of an H^{11} (étale cohomology of a variety) [13, §9]. So Frob_p acts on a 2-dimensional space with

characteristic polynomial $T^2 - \tau(p)T + p^{11}$, exactly the “single Frobenius” of Lemma 65 with p replaced by p^{11} [RESTATEMENT].

- (L2) *Lattices and reduction.* $\rho_{\Delta, \ell}$ preserves a $\mathbb{Z}_\ell$ -lattice M ; $\bar{\rho}_M = M/\ell M$ may depend on M , but its semisimplification does not (Brauer–Nesbitt), and it is determined by $p \mapsto \tau(p) \bmod \ell$ [13, §9].
- (L3) *Mod 691.* Ramanujan’s congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ [15] (for the analogous congruences modulo powers of 2, 3, 5, 7 and modulo 23, the other exceptional primes of Δ , see [17], as cited in [13, fn. 18]) gives $\tau(p) \equiv 1 + p^{11}$, hence $\bar{\rho}_{\Delta, 691}^{ss} \cong \mathbf{1} \oplus \omega^{11}$ for every lattice [13, §12]. The 691-adic representation is irreducible while its residual representation is reducible: Mazur calls such a degeneration *liminal* [13, §4.5, §19].
- (L4) *The non-split lattice.* Up to homothety there are exactly two stable lattices; on each the residual action is triangular and non-semisimple, with $\mathbf{1}$ and ω^{11} in the two orders (Monsky–Greenberg, [13, Prop. 1, §12]; unpublished). Greenberg’s remark [13, fn. 35] that this can be checked with T_2, T_3 alone (no a with $-24 \equiv 2^a + 2^{11-a}$ and $252 \equiv 3^a + 3^{11-a} \pmod{691^2}$) we re-checked: over all $a \in \mathbb{Z}/(690 \cdot 691)$ the $p = 2$ congruence has 6 solutions, the $p = 3$ congruence 2, and they share none [NUMERICAL]. Serre had predicted this in 1967/68 [13, §12].
- (L5) *Unramified at 691.* The forms are 691-ordinary (U_{691} -eigenvalue $\equiv 1$) [13, fn. 51]; local and global orderings of $\mathbf{1}, \omega^{11}$ differ (Ribet’s “wrench”), which forces the inertia at 691 to act semisimply, so the field cut out over $\mathbb{Q}(\mu_{691})$ is everywhere unramified [13, §§13, 18, 21].
- (L6) *Class field theory.* The unramified cyclic extension $L_{\{12\}}/\mathbb{Q}(\mu_{691})$ of degree 691 corresponds to a quotient of $\text{Cl}(\mathbb{Q}(\mu_{691}))$ on which $\text{Gal}(\mathbb{Q}(\mu_{691})/\mathbb{Q})$ acts by ω^{-11} [13, §5, items 2–3]. Greenberg and Monsky showed (unpublished) that the ω^{-11} -part of the 691-Hilbert class field lies in the field cut out by $\rho_{\Delta, 691}$ [13, §12].
- (L7) *Ribet’s general theorem* (Herbrand–Ribet, [13, Thm. 1]; [16, 7]). For $2 \leq k \leq p - 3$ even, $p \mid \text{num}(B_k) \iff \text{Cl}(\mathbb{Q}(\mu_p))[p]$ has a nonzero ω^{1-k} -eigenspace. Ribet’s proof, valid for every irregular p , uses weight-2 cuspidal eigenforms on $\Gamma_1(p)$ (abelian varieties in $J_1(p)$), not Δ [13, §6, “6 $\Rightarrow$ 5”, “5 $\Rightarrow$ 4”]. For 691 the extension was known before Ribet (691 is properly irregular; cyclotomic units) [13, §§4.6, 5].

Remark 3.1 (Two classes at 691). 691 divides $\text{num}(B_k)$ for exactly two even $k \leq 688$, namely $k = 12$ and $k = 200$ [NUMERICAL]; correspondingly $\text{Cl}(\mathbb{Q}(\mu_{691}))/691$ is 2-dimensional with eigencharacters ω^{-11} and ω^{-199} [13, §5][CITED-sec], the second coming from a weight-200 eigenform $\Delta_{\{200\}}$ with coefficients in a degree-16 field [13, fn. 19]. Our exact computation gives $v_{691}(h^-(\mathbb{Q}(\mu_{691}))) = 2$ [NUMERICAL] (§5).

Numerics [NUMERICAL]. $\tau(p) \equiv 1 + p^{11} \pmod{691}$ for all 9592 primes $p < 10^5$ (no exception); $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ for all $n \leq 20000$. Controls (diagnostic: the test can fail): with the wrong exponent, $\tau(p) \equiv 1 + p^{10}$ holds for a fraction 0.00156 of primes; with modulus 701, 0.00198; and modulo 691^2 the congruence $\tau(p) \equiv 1 + p^{11}$ holds for a fraction 0.00125 of primes, i.e. at the random rate $\approx 1/691$. The bridge is exactly first order in 691, which is the residual (mod- ℓ) level of the ℓ -adic chain.

Proposition 3.2 (Parity: why the class is in the minus part). $-I \in SL_2(\mathbb{Z})$ forces every nonzero level-one form to have even weight k . Hence the eigencharacter ω^{1-k} of Herbrand–Ribet is odd, and the class lies in the minus part Cl^- (complex conjugation acts by -1). Equivalently, $\det \bar{\rho}(c) = \omega^{k-1}(c) = -1$: the residual representation is odd, and its two constituents $\mathbf{1}, \omega^{11}$ are separated by complex conjugation. [THEOREM]

Proof. $f(z) = f(-I \cdot z) = (-1)^k f(z)$. $\omega(c) = -1$ for complex conjugation c , so $\omega^{1-k}(c) = (-1)^{1-k} = -1$ for k even. $\square$

Reading. In TAA terms complex conjugation is the real place, where the duality/polarization structure lives (Lemma 62’s helicity pairing $\rho \leftrightarrow \bar{\rho}$; Lemma 65’s J_p , whose $\pm i$ eigenlines are exchanged

by conjugation). The “minus” of the χ^{11} -minus class is the *one piece of the real-place structure that survives mod 691*: a sign, not a positivity [ANALOGY] (the parity statement itself is [THEOREM]).

4. WHAT THE BRIDGE DOES TO THE POLARIZATION

Lemma 65 built the triple (ω, J_p, g_p) from the single Frobenius of $E/\mathbb{F}_p$. The same algebra applies verbatim to any 2×2 Frobenius with $\text{Tr} = a$, $\det = p^w$; for Δ , $a = \tau(p)$, $w = 11$, and π acts on the 2-dimensional $\rho_{\Delta, \ell}$ (L1).

Theorem 4.1 (The polarization of Δ at p and its reduction mod 691). *Let π satisfy $\pi^2 - \tau\pi + p^{11} = 0$, $\tau = \tau(p)$, and put $d_p = \tau^2 - 4p^{11}$.*

- (a) (Over $\mathbb{R}$.) $(2\pi - \tau)^2 = d_p < 0$, so $J_p := (2\pi - \tau)/\sqrt{-d_p}$ satisfies $J_p^2 = -1$; $\omega(x, y) = \det(x, y)$ satisfies $\omega(\pi x, \pi y) = p^{11}\omega(x, y)$, and $g_p = \omega(\cdot, J_p \cdot)$ is definite. This is Lemma 65’s triple with $\mathbb{L}$ replaced by $\mathbb{L}^{11}$ [THEOREM] (positivity of g_p uses Deligne’s bound).
- (b) (Mod 691, $p \not\equiv 1 \pmod{691}$.) $d_p \equiv (1 - p^{11})^2 \not\equiv 0$. Hence 691 splits in $\mathbb{Q}(\sqrt{d_p})$ (the Frobenius eigenvalues become 1 and p^{11} in $\mathbb{F}_{691}$), and the reduction $\bar{\sigma}_p := (2\bar{\pi} - \bar{\tau})/(1 - p^{11})$ satisfies $\bar{\sigma}_p^2 = +1$: it is an involution, whose $+1/-1$ eigenlines are the p^{11} - and 1 -eigenlines of $\bar{\pi}$, i.e. the constituents ω^{11} and $\mathbf{1}$.
- (c) (The complex structure itself.) $-d_p \equiv -(1 - p^{11})^2$ is a non-square mod 691 because $691 \equiv 3 \pmod{4}$. So 691 is inert in $\mathbb{Q}(\sqrt{-d_p})$, the field of definition of J_p ; J_p has no reduction to $\mathbb{F}_{691}$, and its reduction to $\mathbb{F}_{691^2}$ is $\bar{J}_p = \pm i^{-1}\bar{\sigma}_p$ with $i^2 = -1$ in $\mathbb{F}_{691^2}$. In particular “ $\bar{g}_p$ positive” has no meaning.
- (d) (Mod 691, $p \equiv 1 \pmod{691}$.) $\bar{d}_p = 0$, $2\bar{\pi} - \bar{\tau}$ is nilpotent, and $\bar{\pi}$ has the repeated eigenvalue 1: on a non-split lattice this is the unipotent block $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ that carries the Ribet extension [THEOREM].

Proof. (a) is Lemma 65 Thm. “triple” with $p \mapsto p^{11}$ (Cayley–Hamilton gives $(2\pi - \tau)^2 = \tau^2 - 4p^{11}$). (b) Ramanujan’s congruence; $\bar{\sigma}_p^2 = (2\bar{\pi} - \bar{\tau})^2/(1 - p^{11})^2 = 1$; on the λ -eigenline of $\bar{\pi}$, $\bar{\sigma}_p = (2\lambda - 1 - p^{11})/(1 - p^{11})$, which is -1 for $\lambda = 1$ and $+1$ for $\lambda = p^{11}$; $p^{11} \equiv 1 \iff p \equiv 1 \pmod{691}$ because $\gcd(11, 690) = 1$. (c) $\left(\frac{-1}{691}\right) = -1$. (d) immediate. $\square$

Numerics [NUMERICAL]. For all 2262 primes $p < 20000$: $\left(\frac{d_p}{691}\right) = +1$ and $\left(\frac{-d_p}{691}\right) = -1$, except $p = 6911, 8293, 11057 \pmod{691}$, where $v_{691}(d_p) = 1$ (691 ramifies in $\mathbb{Q}(\sqrt{d_p})$).

Corollary 4.2 (“Polarization $\rightarrow \mathbf{1} \oplus \mathbb{L}^{11}$ ”, made precise). *Mod 691 the complex structure J_p of Δ degenerates into the splitting involution $\bar{\sigma}_p$ of $\mathbf{1} \oplus \mathbb{L}^{11}$ (times i^{-1} over $\mathbb{F}_{691^2}$). For E_{12} , whose motive is $\mathbf{1} \oplus \mathbb{Q}(-11)$ (its L -function is $\zeta(s)\zeta(s-11)$), the same formula $\sigma_p = (2\pi - a_p)/(p^{11} - 1)$ with $a_p = 1 + p^{11}$ is an honest involution over $\mathbb{Q}$, with eigenlines h^0 and $\mathbb{L}^{11}$ [THEOREM]. Ramanujan’s congruence says: the h^1 -type object Δ and the pole-type object $\mathbf{1} \oplus \mathbb{L}^{11}$ have the same Frobenius mod 691 [RESTATEMENT].*

Remark 4.3 (Link to Lemmas 64 and 65). For a curve, the pole terms of the explicit formula are the traces on $h^0 = \mathbf{1}$ and $h^2 = \mathbb{L}$ (Lemma 65) [TAA-L]; Lemma 64’s POLE quadratic form has signature $(1, 1)$ [TAA-L], matching the hyperbolic plane $\langle C_1, C_2 \rangle \subset \text{NS}(C \times C)$ [ANALOGY] (Lemma 65). Here the match is exact at every prime: the local form of $\mathbf{1} \oplus \mathbb{L}^w$ is $(x + y)(x + p^w y)$, signature $(1, 1)$ [THEOREM]. So, at the residual level, *the definite h^1 form is congruent to the hyperbolic pole form*. In Lemma 64’s three-ingredient reading (FE/ ω ; Euler product/Frobenius; positivity g), the bridge transports the first two ingredients (both $L(\Delta, s)$ and $L(E_{12}, s)$ have Euler product and the FE $s \mapsto 12 - s$) and *destroys the third* [THEOREM] (Thm. 4.1(c)).

5. THE CLASS-GROUP SIDE AND PRINCIPALIZATION

5.1. Exact minus class numbers. $h^-(\mathbb{Q}(\mu_p)) = 2p \prod_{\chi \text{ odd}} (-\frac{1}{2} B_{1,\chi})$, $B_{1,\chi} = \frac{1}{p} \sum_{a=1}^{p-1} a\chi(a)$ [18][BIB]. Writing $\chi(g) = \zeta_{p-1}^j$ (g a primitive root, j odd) and $F(x) = \sum_{i=0}^{p-2} (g^i \bmod p) x^i$, the product over odd χ is the resultant $\text{Res}(x^{(p-1)/2} + 1, F)$, an exact integer computation [THEOREM]. Results [NUMERICAL]: $h^- = 3, 8, 9, 37, 41241, 853513, 3547404378125$ for $p = 23, 29, 31, 37, 59, 67, 101$ (agreeing with the classical values); $v_p(h^-)$ equals the number of irregular indices for every prime $5 \leq p < 400$ ($= 2$ at $p = 157, 353, 379$); for $p = 691$, h^- has 218 digits, $h^- = 29138283829700137424 \dots$, and $v_{691}(h^-) = 2$, matching the two indices 12, 200.

Independent archimedean route [NUMERICAL]: with $L(1, \chi)$ computed by PARI's `lfun` for each odd character, $\log_{10}[2p(p/4\pi^2)^{(p-1)/4} \prod |L(1, \chi)|] = 1.568201724$ ($p = 37$) and 217.464463969 ($p = 691$), equal to $\log_{10} h^-$ from the resultant to all printed digits. This confirms the formula $h^- = 2p(p/4\pi^2)^{(p-1)/4} \prod_{\chi \text{ odd}} |L(1, \chi)|$, which follows from the one above and $|B_{1,\chi}| = \sqrt{p} |L(1, \bar{\chi})|/\pi$ for odd primitive χ [18][BIB].

5.2. Principalization: what can and cannot happen.

Theorem 5.1 (No 691-class capitulates in the cyclotomic $\mathbb{Z}_{691}$ -tower). *Let $k = \mathbb{Q}(\mu_{691})$, $k_n = \mathbb{Q}(\mu_{691^{n+1}})$.*

- (a) (*Minus part.*) $\ker(\text{Cl}^-(k) \rightarrow \text{Cl}^-(k_n))$ has trivial 691-part for all $n \geq 1$. [CITED-sec]: Lemmermeyer [11, §8, Thm. 5(ii)]: for an odd prime p and a cyclic p -extension L/k of CM fields with $\zeta_p \in k$ and $L = k(\zeta_{p^n})$, $n \geq 2$, the minus capitulation kernel is trivial; this sharpens Kida [9] (generalizing Moriya and Greenberg) as stated in [11]. Also quoted as “well known” in [2] (Lemma 60).
- (b) (*Plus part.*) 691 is properly irregular, i.e. $691 \nmid h^+(k)$ [13, §5][CITED-sec]. The 691-part of the plus class group of each k_n is then also trivial (a $\mathbb{Z}_p$ -extension totally ramified at the unique prime above p preserves $p \nmid h$; Washington Thm. 10.4 as quoted in [11, §6]) [CITED-sec] (secondary; the general statement was not checked in [18] itself).
- (c) Hence no 691-class of k (in particular neither the ω^{-11} nor the ω^{-199} class) becomes principal in any k_n . [THEOREM] (from (a), (b))
- (d) Every class of k capitulates in its Hilbert class field (principal ideal theorem, Lemma 60 (b)) [BIB], and in the degree-691 unramified extension $L_{\{12\}}$ the capitulation kernel has order divisible by 691 (Hilbert 94 and Suzuki, Lemma 60 Thm. “Classical capitulation theorems (a), (c)”) [TAA-L]. Which eigencomponents form that kernel is not determined here [OPEN].

This refines Lemma 60, Remark “The 691 class never principalizes” (which relied on [2]): the plus part was not addressed there.

What “principalization kills the discrete part” can mean. (i) $\text{Cl} = \pi_0(\text{Pic}^0)$ of the Arakelov Picard group, and principalization kills π_0 (Lemma 60) [TAA-L]. (ii) In the natural tower nothing is killed (Thm. 5.1). (iii) The precise version of the hunch: class field theory converts the discrete obstruction (the ω^{-11} class) into a *symmetry*, $\text{Gal}(L_{\{12\}}/k) \cong \mathbb{Z}/691$ with $G_{\mathbb{Q}}$ acting by ω^{-11} , and that symmetry is the off-diagonal entry $*$ of $\bar{\rho}_{\Delta, 691}$ [RESTATEMENT]. So the bridge is not “principalization kills the class”; it is “the class is the gluing datum $*$ of the degenerate polarization” (Thm. 4.1(d)) [RESTATEMENT].

5.3. The bridge as a point of $\text{Spec } \mathbb{T}$.

Proposition 5.2 (The Hecke algebra of weight k , level 1, $\dim S_k = 1$). *Let $k \in \{12, 16, 18, 20, 22, 26\}$, $M_k = \mathbb{C}G_k \oplus \mathbb{C}\Delta_k$. The Hecke operators act diagonally with eigenvalues $(\sigma_{k-1}(n), a_n(\Delta_k))$, so the $\mathbb{Z}$ -algebra $\mathbb{T}_k$ they generate is $\{(a, b) \in \mathbb{Z}^2 : a \equiv b \bmod N_k\}$, $N_k = \gcd_n(\sigma_{k-1}(n) - a_n(\Delta_k))$ [THEOREM]. Numerically $N_k = |\text{num}(B_k/2k)|$: $N_{12} = 691$, $N_{16} = 3617$, $N_{18} = 43867$, $N_{20} = 283 \cdot 617$, $N_{22} = 131 \cdot 593$, $N_{26} = 657931$ [NUMERICAL] (gcd over $n \leq 3000$; the gcd over all n divides this, and equals it by the Eisenstein congruences, which we verified for $p < 10^4$ only, Fig. 2). Thus $\text{Spec } \mathbb{T}_k$ is two*

copies of $\text{Spec } \mathbb{Z}$ (the Eisenstein and the cuspidal component) glued exactly at the primes dividing N_k . [THEOREM]/[NUMERICAL]

Proof. $\mathbb{T}_k \ni T_1 = (1, 1)$ and $T_n = (\sigma_{k-1}(n), a_n)$; the $\mathbb{Z}$ -span of $(1, 1)$ and $(\sigma - a, 0)$'s is the stated ring. For $k = 12$: $T_2 - T_1 \cdot (-24) = (2073, 0) = (3 \cdot 691, 0)$ and $T_3 - 252 T_1 = (176896, 0) = (256 \cdot 691, 0)$, $\gcd(3, 256) = 1$. $\square$

Reading. The 691 bridge is *the closed point (691) where the two components of $\text{Spec } \mathbb{T}_{12}$ meet*. A point of residue characteristic 691 carries the multiplicative data mod 691 and nothing archimedean. Zero location and positivity are properties of the components (their generic fibres over $\mathbb{R}, \mathbb{C}$), as §6 shows [HEURISTIC].

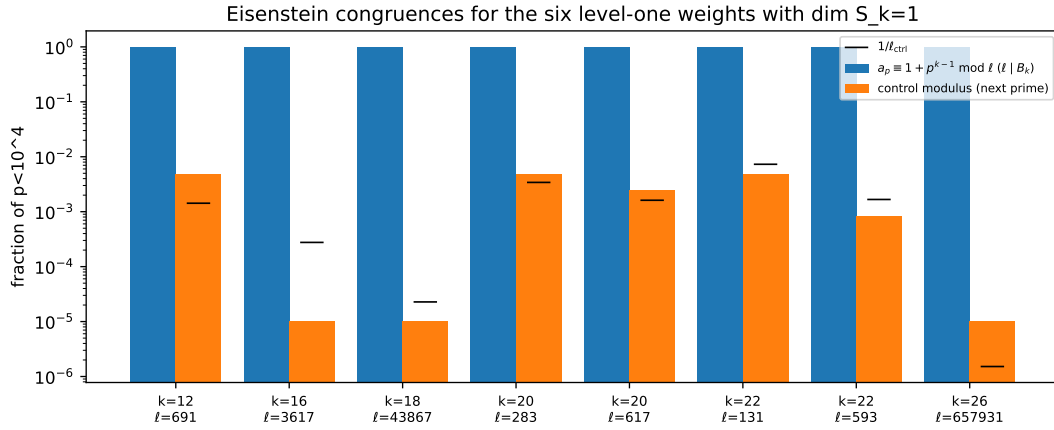


FIGURE 2. Fraction of primes $p < 10^4$ with $a_p(\Delta_k) \equiv 1 + p^{k-1} \pmod{\ell}$ for every $\ell \mid N_k$ with $\ell > k + 1$ (blue: 1 in all eight cases) versus a control modulus $\ell' = \text{next prime}$ (orange; zero bars drawn at 10^{-5}), with $1/\ell'$ marked. The primes $11 \mid B_{22}$ and $13 \mid B_{26}$ lie outside the Herbrand–Ribet range and cancel in $B_k/2k$.

6. AROUND THE BRIDGE: WHERE GHOSTS ENTER

J.W.'s steering (t253u) asks for the elements around the bridge at which zero location (ghosts) and positivity come into play. We take them in order of distance from the closed point (691).

6.1. The key example: Δ versus E_{12} .

Theorem 6.1 (E_{12} is all ghost). $\Lambda(E_{12}, s) = (2\pi)^{-s} \Gamma(s) \zeta(s) \zeta(s - 11)$ satisfies $\Lambda(E_{12}, 12 - s) = \Lambda(E_{12}, s)$ and has an Euler product of degree 2. Its zeros are exactly $s = 3, 5, 7, 9$ (simple) and $\rho, \rho + 11$ for the nontrivial zeros ρ of ζ . None lies on the axis $\text{Re } s = 6$: the real zeros are at distance 1 or 3, and the complex zeros at distance in $(5, 6)$, unconditionally; under RH, exactly $11/2$. [THEOREM]

Proof. $\Gamma(s)$ has no zeros and no poles in $\text{Re } s > 0$. In $0 < \text{Re } s < 12$, the zeros of $\zeta(s) \zeta(s - 11)$ are the nontrivial zeros of $\zeta(s)$, those of $\zeta(s - 11)$ (i.e. $\rho + 11$), and the trivial zeros $s - 11 \in \{-2, -4, -6, -8, -10\}$, i.e. $s \in \{9, 7, 5, 3, 1\}$; at $s = 1$ the zero of $\zeta(s - 11)$ cancels the pole of $\zeta(s)$ ($\zeta(-10) = 0$ simply), so $\Lambda(E_{12}, 1) \neq 0$. $\zeta(6) \zeta(-5) \neq 0$. For $\text{Re } s \leq 0$ or ≥ 12 use the functional equation, which is that of a level-one weight-12 modular form (or directly from those of ζ). $0 < \text{Re } \rho < 1$ gives the distances. $\square$

Numerics [NUMERICAL]. The functional equation residual $|\Lambda(12-s) - \Lambda(s)|/|\Lambda(s)| \leq 1.7 \times 10^{-30}$ at three test points (mpmath, 30 digits); $\Lambda(E_{12}, s) = 0$ at $s = 3, 5, 7, 9$ and $\Lambda(E_{12}, 6) = -7.87 \times 10^{-6}$, $\Lambda(E_{12}, 4) = \Lambda(E_{12}, 8) = 1.74 \times 10^{-5}$ (sign changes at each real zero); $|\Lambda(E_{12}, \rho_1)| = |\Lambda(E_{12}, \rho_1 + 11)| = 2.7 \times 10^{-36}$. For Δ , PARI's `lfunmf` (`lfuncheckfeq` = -67 , i.e. functional equation satisfied to 10^{-67}) and `lfunzeros` give the first zeros on $\text{Re } s = 6$ at $\text{Im } s = 9.2223793999, 13.9075498614, 17.4427769782, 19.6565131420, 22.3361036372, \dots$ (13 zeros with $\text{Im } s \leq 40$). `lfunzeros` locates sign changes on the line; it does not certify absence of zeros off the line.

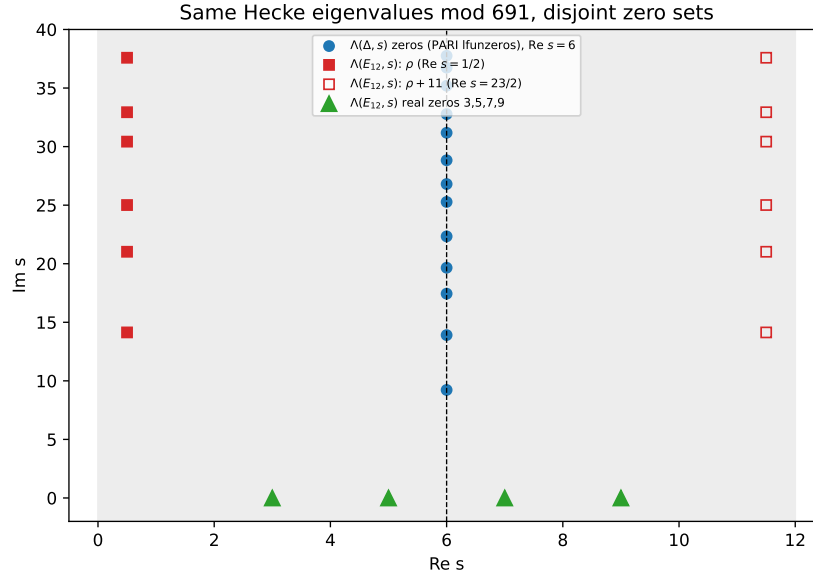


FIGURE 3. Zeros with $0 \leq \text{Im } s \leq 40$ in the strip $0 < \text{Re } s < 12$ (grey). Blue: $\Lambda(\Delta, s)$, all found on the axis $\text{Re } s = 6$ (dashed). Red: $\Lambda(E_{12}, s)$, on $\text{Re } s = \frac{1}{2}$ and $\frac{23}{2}$; green: its real zeros 3, 5, 7, 9. The two L -functions have Hecke eigenvalues congruent mod 691 at every prime.

Corollary 6.2 (Zero location is not a function of residual data). *There exist two degree-2 L -functions with Euler products and the same functional equation, whose Euler factors are congruent mod 691 at every prime, such that one has all its zeros off the symmetry axis and the other is conjectured (GRH) to have all its zeros on it. Hence no property of the mod- ℓ Euler data (or of the residual representation, or of the ℓ -part of a class group built from it) can decide zero location. [THEOREM] (the “all on it” half is GRH, [CONJECTURE]; the conclusion uses only the unconditional half and the fact that Δ and E_{12} are different L -functions with identical residual data)*

This is the sharp form of Grok Bot’s “ghost-blind” and it is *stronger* than the Lemma 64 ablations (Davenport–Heilbronn: FE without Euler product; random-sign Euler products: no FE): E_{12} has *both* FE and Euler product, and fails only the third ingredient, local positivity (Prop. 2.1). It fits Lemma 64’s ablation table as a new row: “EP + FE, all local forms hyperbolic $\Rightarrow$ 100% of zeros off-axis” [THEOREM]. It also explains why the Selberg class demands a Ramanujan-type bound (Lemma 64 §3 table) [ANALOGY].

6.2. Local positivity: the sign of the discriminant. *Correction of “ghost-blind” (Grok Bot chat, (d)).* The full ℓ -adic representation $\rho_{\Delta, \ell}$ determines every $\tau(p)$ (L1), hence $L(\Delta, s)$ and all its zeros; it is not ghost-blind in principle. Moreover the ℓ -adic machinery *proves local positivity*: Deligne’s purity of weights [4] gives $|\alpha_p| = p^{11/2}$, i.e. $Q_p^\Delta > 0$ at every p (Prop. 2.1) [BIB]. $E_{12} = \mathbf{1} \oplus \mathbb{L}^{11}$ is *mixed*,

of weights 0 and 22, and every $Q_p^{E_{12}}$ is hyperbolic. So purity = local definiteness [RESTATEMENT]. What no known ℓ -adic statement supplies is the *global* positivity (GRH for $L(\Delta, s)$) [OPEN]: this is Lemma 64’s “collective, not prime by prime” split, now with Deligne in the role of Hasse.

6.3. Unitarity: L^2 and the Petersson norm. Δ is a cusp form, hence in $L^2(SL_2(\mathbb{Z}) \backslash \mathbb{H})$ with Petersson norm $\langle \Delta, \Delta \rangle = 1.035362056804 \times 10^{-6}$ (PARI `mfpetersson`, its normalization) [NUMERICAL]; E_{12} is not square-integrable (its constant term does not decay) [THEOREM]. In representation-theoretic language Δ generates a unitary cuspidal automorphic representation, while E_{12} belongs to the continuous spectrum and its local components at finite p are principal series with Satake exponent $\pm \frac{11}{2}$ (analytic normalization), far outside the unitary range $|\operatorname{Re}| < \frac{1}{2}$ [1][BIB] (not opened; standard). The congruence therefore relates an L^2 (unitary) vector to a non- L^2 one [THEOREM]/[BIB]. For J.W.’s thesis “RH is a statement about unitarity”: along the bridge, *unitarity is exactly what is lost*, together with local definiteness and zero location [ANALOGY]. Caveat: unitarity of the cuspidal representation is proved and is *not* known to imply GRH; temperedness (Ramanujan) is stronger than unitarity and is what Deligne proved [BIB].

6.4. The archimedean size of the same class group. The 691-part of $\operatorname{Cl}(\mathbb{Q}(\mu_{691}))$ is the bridge; its *size* is archimedean:

$$h^-(\mathbb{Q}(\mu_p)) = 2p \left(\frac{p}{4\pi^2} \right)^{(p-1)/4} \prod_{\chi \text{ odd}} |L(1, \chi)|$$

(§5; formula [BIB], checked [NUMERICAL] at $p = 37, 691$). The values $L(1, \chi)$ are zero-sensitive: a real zero β of $L(s, \chi)$ close to 1 (a “Siegel zero”) forces $L(1, \chi)$ to be small, and conversely the integrality $h \geq 1$ bounds $1 - \beta$ from below; this is the classical class-number/Siegel-zero mechanism of Lemma 60 [TAA-L]. At $p = 691 \equiv 3 \pmod{4}$ the odd characters include the quadratic one, χ_{-691} , with $h(\mathbb{Q}(\sqrt{-691})) = 5$ and $L(1, \chi_{-691}) = 5\pi/\sqrt{691} = 0.59756$ [NUMERICAL]; separately, $v_5(h^-(\mathbb{Q}(\mu_{691}))) = 2$ [NUMERICAL]. Kummer conjectured that the residual product $\prod |L(1, \chi)| \rightarrow 1$; Granville showed this is false assuming the Elliott–Halberstam and a quantitative twin-prime conjecture [6][CITED] (abstract). So the size of the minus class group is governed by the distribution of primes in progressions mod p , i.e. by zeros of Dirichlet L -functions [HEURISTIC].

Diagnostic numerics [NUMERICAL] (Fig. 4). For the 76 primes $5 \leq p < 400$, the residual $\sum_{\chi \text{ odd}} \log |L(1, \chi)|$ lies in $[-0.36, 0.36]$; its mean is -0.0056 over the 20 irregular primes and -0.0101 over the 56 regular ones. So the archimedean, zero-sensitive part of h^- is uncorrelated (at this sample size) with p -divisibility, which is the bridge. At $p = 691$: $\sum \log |L(1, \chi)| = -0.2624$, mean -7.6×10^{-4} per character.

6.5. The Iwasawa side: a spectral statement without positivity. The main conjecture (Mazur–Wiles [14]; Wiles [19]) identifies the characteristic power series of the eigen-components of the minus Iwasawa module with Kubota–Leopoldt p -adic L -functions [10, 8] [BIB]; Herbrand–Ribet is its shadow at the bottom layer [BIB]. Mazur’s reformulation [13, §22][CITED-sec]: L_p vanishes at χ^* iff there is an Iwasawa representation of determinant χ , and there is exactly one iff the zero is *simple*. Whether all zeros of Kubota–Leopoldt L -functions are simple is unknown [13, §22, Sidenote][OPEN].

- (I1) *Hilbert–Pólya form.* With $\mu = 0$ [5][BIB], the module $X^{(i)} \otimes \mathbb{Q}_p$ is finite-dimensional (λ), and the zeros of the p -adic L -function in the open unit disc are the eigenvalues of $\gamma - 1$ on it (characteristic polynomial = distinguished polynomial, Weierstrass preparation) [RESTATEMENT] (given the main conjecture). This is a proved “zeros = spectrum of an operator” statement [THEOREM]/[BIB].
- (I2) *Free p -adic unitarity.* The eigenvalues of γ are automatically 1-units ($|\cdot|_p = 1$), since the roots of a distinguished polynomial lie in the open unit disc [THEOREM]. This is the p -adic analogue of Lemma 60’s “positivity is free”: it costs nothing and excludes nothing [ANALOGY].

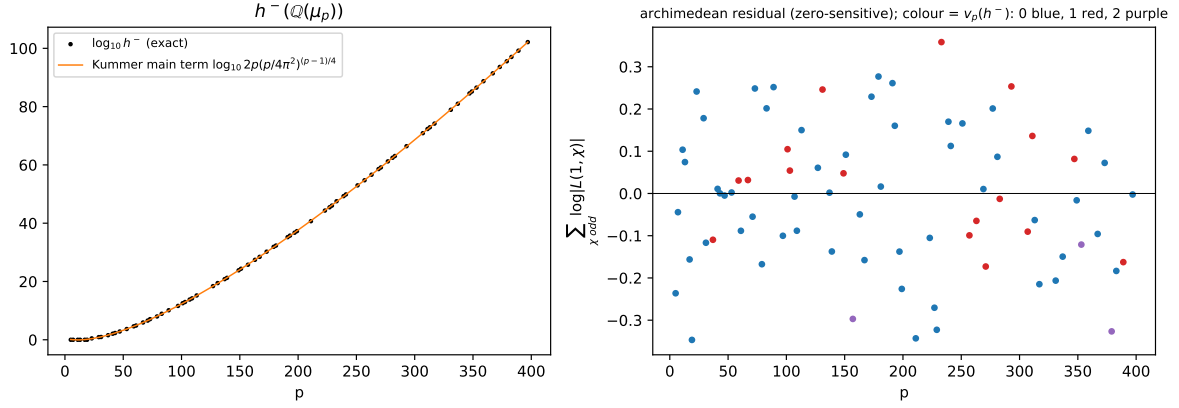


FIGURE 4. Left: exact $\log_{10} h^-(\mathbb{Q}(\mu_p))$ (resultant) and Kummer’s main term. Right: the residual $\sum_{\chi \text{ odd}} \log |L(1, \chi)|$ (archimedean, zero-sensitive) coloured by $v_p(h^-)$ (the p -adic bridge datum). No visible relation.

- (I3) *No axis, no positivity.* There is no functional equation with a symmetry axis for L_p , hence no “off-line” (Lemma 38(e), Lemma 60) [TAA-L]. $\mu = 0$ is a finiteness statement, not a sign; $\lambda \geq 0$ is trivial. We found no positivity on the Iwasawa-module side [OPEN] (we searched only the sources listed; this is a report of absence, not a theorem).

6.6. Summary: what crosses the bridge.

Datum	crosses mod 691?	status
Hecke eigenvalues mod 691 (multiplicativity)	yes	[THEOREM] (Ramanujan)
FE $s \mapsto 12 - s$, oddness / minus sign (duality, real place)	yes (both sides have it)	[THEOREM] (Prop. 3.2)
Splitting $\mathbf{1} \oplus \mathbb{L}^{11}$ and gluing class $*$	<i>created</i> by the bridge	[CITED-sec] (Ribet, Greenberg–Monsky)
Sign of disc Q_p , complex structure J_p	no (Thm. 4.1(c))	[THEOREM]
L^2 / unitarity / Petersson positivity	no	[THEOREM]/[BIB]
Zero location (ghosts)	no (Cor. 6.2)	[THEOREM]
Archimedean size $h^- (\prod L(1, \chi))$	independent of the 691-part	[NUMERICAL]

Verdict. J.W.’s reading “the 691 bridge is one piece of the positivity” is not right as stated: the bridge carries multiplicativity and the duality *sign*, and it is precisely the point where positivity is *lost* [THEOREM]. It is, however, a piece of the *structure* that positivity acts on: the residual splitting $\mathbf{1} \oplus \mathbb{L}^{11}$ is the pole-type (signature $(1, 1)$) part, and the class is the gluing datum. Ghosts enter at the real place of the two components: local signature (proved for Δ by Deligne), L^2 (proved), and global zero location (open), and on the class-group side through the archimedean size $\prod L(1, \chi)$. “Ghost killing is collective” (Lemma 64) is consistent with all of this: nothing at a single finite place, and nothing at the residual level, sees ghosts [HEURISTIC].

7. RANKED DIRECTIONS

Score = 0.6·significance+0.4·probability (both in $[0, 1]$); J.W.’s standing filter: does it shed light on the polarization J , beyond zero counting?

	Direction	sig.	prob.	score
H1	Lemma 64 anatomy for the pair (Δ, E_{12}) : Weil form of $L(\Delta, s)$ vs $L(E_{12}, s)$, prime-by-prime slack; where exactly E_{12} 's form goes negative, given congruent Euler data	.80	.80	.80
H2	Archimedean class-number functional $\sum_{\chi \text{ odd}} \log L(1, \chi) $ as a collective positivity over the family of odd characters mod p ($h^- \geq 1$ integrality), vs. the explicit formula	.60	.60	.60
H3	Petersson/Rankin–Selberg positivity $\langle \Delta, \Delta \rangle > 0$ as a candidate global g ; relation to Weil positivity for $L(\Delta, s)$	.80	.25	.58
H4	Eisenstein congruences at higher level (Mazur's Eisenstein ideal [12]) as the family generalization	.40	.80	.56
H5	Hida family through Δ at 691 (Φ_w , same residual representation, [13, §13]): local signature and zeros along the family	.60	.35	.50
H6	Search for a Hermitian/positive structure on Iwasawa modules (Kummer duality $X^- \leftrightarrow X^+$)	.70	.20	.50
H7	GRH numerics for $L(\Delta, s)$ to large height (PARI); confirmation only	.20	.90	.48

H1 is the natural next step: it is cheap (Lemma 64 code with two new Euler products), diagnostic (the congruence fixes the Euler data mod 691, so any difference in slack is archimedean by construction), and sheds light on J (Thm. 4.1).

8. CORRECTIONS TO GROK BOT'S CHAT CLAIMS

- K1. “ Q_p^Δ positive definite $\iff$ Ramanujan–Deligne bound (Deligne, Weil I)”: *correct*. Refinement: the inequality is automatically strict ($2p^{11/2} \notin \mathbb{Q}$); Deligne's proof combines [3] with [4].
- K2. “Mod 691 the polarization degenerates to $1 \oplus \mathbb{L}^{11}$; discriminant $\equiv (1-p^{11})^2$ a square”: *correct as a statement about the residual semisimplification*, imprecise about J . Precisely (Thm. 4.1): the Frobenius eigenvalues become $1, p^{11} \in \mathbb{F}_{691}$, J_p itself does *not* reduce to $\mathbb{F}_{691}$ ($691 \equiv 3 \pmod{4}$), over $\mathbb{F}_{691^2}$ it becomes i^{-1} times the splitting involution, and for $p \equiv 1 \pmod{691}$ the discriminant vanishes and the residual Frobenius is unipotent. The square is nonzero only for $p \not\equiv 1 \pmod{691}$.
- K3. “Ribet builds a non-split extension from this $[\Delta]$ ”: *misattributed*. For Δ itself the non-split lattice is due to Greenberg–Monsky (unpublished; predicted by Serre 1967/68), and the 691 extension was known before Ribet. Ribet's 1976 proof, valid for all irregular primes, uses weight-2 cusp forms on $\Gamma_1(p)$ [13, §§5, 6, 12].
- K4. “the class appears in the ω^{-11} (i.e. χ^{11} minus) eigenspace”: *correct*, with a notational caution: the class group eigencharacter is $\omega^{-11} = \omega^{1-12}$, while ω^{11} is the residual constituent of $\bar{\rho}_{\Delta, 691}$; both are odd. There is no single “eigenvalue”: σ_a acts on the eigenspace by the scalar $\omega^{-11}(a)$. There is also a second 691-eigenspace, ω^{-199} ($691 \mid B_{200}$), which the chat omitted.
- K5. “The class is in the minus part and never capitulates in the $\mathbb{Z}_{691}$ tower”: *correct*; precise citation is Kida's theorem in Lemmermeyer's form [11, Thm. 5(ii)] ([CITED-sec]; the request to cite “Iwasawa's theorem” could not be matched to a source we read; Washington Prop. 13.26 was seen only in a search snippet and is not cited). Added: the plus part has no 691-classes in any layer (Thm. 5.1(b)), so *no* 691-class capitulates.
- K6. “ghost-blind” (applied to the whole ℓ -adic picture): *wrong as stated*; see §6. The full ℓ -adic representation determines $L(\Delta, s)$ and proves local positivity; blind are the residual data, the class group's ℓ -part and Iwasawa invariants (Cor. 6.2).
- K7. (b) “ $\zeta(-11) = 691/32760$ ”: *correct* ($= -B_{12}/12$, $B_{12} = -691/2730$) [NUMERICAL]. “Kubota–Leopoldt and Mazur–Wiles link class groups to special values, never to zeros”: *half wrong*. The main conjecture links Iwasawa modules to the *zeros* of p -adic L -functions (their number is λ);

what is true is “never to complex zeros”. And the archimedean class number formula links h^- to $L(1, \chi)$, whose size *is* zero-sensitive.

- K8. (c) $\Delta \equiv E_{12}$, $L(E_{12}, s) = \zeta(s)\zeta(s-11)$, $Q_p^{E_{12}} = (x+y)(x+p^{11}y)$ indefinite, “positivity invisible mod ℓ ”: *correct* with Lemma 65’s sign convention ($a_p = 1 + p^{11}$ enters with $+$). Strengthened to Thm. 6.1: every zero of $\Lambda(E_{12}, s)$ is off the axis, including real zeros at 3, 5, 7, 9 which the chat did not mention. Lemma 64 contains no E_{12} remarks to reconcile (checked in `build64/`); E_{12} is a new ablation row.
- K9. (d) “ ℓ -adic cohomology plus weights proved local positivity; what the congruence and class-group data miss is global positivity”: *correct*, and adopted as the corrected version of K6.
- K10. (e) “Main conjecture as a proved Hilbert–Pólya-type statement with no unitarity or positivity”: *correct as an analogy*, sharpened in (I1)–(I3): it has a *free* p -adic unitarity (eigenvalues are 1-units), and $\mu = 0$ is not a positivity. “Herbrand–Ribet is the first case of Mazur–Wiles”: acceptable as “bottom-layer shadow” [BIB].
- K11. (Lemma 60 cross-check) Lemma 60’s Remark on the 691 class is confirmed; it addressed only the minus part (now completed by Thm. 5.1(b)).

9. SUMMARY OF CLAIMS

Claim	Label
Local form definite $\iff$ tempered (Prop. 2.1); $Q_p^{E_{12}} = (x+y)(x+p^{11}y)$	[THEOREM]
ℓ -adic chain (L1)–(L7)	[CITED-sec] [13]
Class is in the minus part because $-I \in SL_2(\mathbb{Z})$ (Prop. 3.2)	[THEOREM]
$J_p \rightarrow i^{-1}\bar{\sigma}_p$, involution of $\mathbf{1} \oplus \mathbb{L}^{11}$ (Thm. 4.1)	[THEOREM]
Definite h^1 form $\equiv$ hyperbolic pole form (Rem. 4.3)	[THEOREM]; link to L64/65 [ANALOGY]
No 691-class capitulates in the cyclotomic tower (Thm. 5.1)	[CITED-sec] + [THEOREM]
Class = gluing datum * of the degenerate polarization	[RESTATEMENT]
$T_k = \{a \equiv b \bmod N_k\}$, $N_k = \text{num}(B_k/2k) $ (Prop. 5.2)	[THEOREM]/[NUMERICAL]
E_{12} all ghost (Thm. 6.1); residual data zero-blind (Cor. 6.2)	[THEOREM]
Unitarity/ L^2 lost across the bridge	[THEOREM]/[BIB]; reading [ANALOGY]
h^- archimedean size zero-sensitive, independent of $v_p(h^-)$	[BIB]/[NUMERICAL]/[HEURISTIC]
Main conjecture = spectral statement with free p -adic unitarity	[RESTATEMENT]/[ANALOGY]
Bridge “is one piece of the positivity”	false as stated ([THEOREM]); structural reading [HEURISTIC]
RH; GRH for $L(\Delta, s)$	<i>not proven</i> [OPEN]

10. QUESTIONS FOR J.W.

- Q1. Do you accept the corrected reading of your hunch: the bridge carries multiplicativity and the duality *sign* (minus part $\leftarrow -I \in SL_2(\mathbb{Z})$), and it is exactly where positivity (J_p , L^2 , zero location) is lost, with the class as the gluing datum of $\mathbf{1} \oplus \mathbb{L}^{11}$ rather than something “killed” by principalization?
- Q2. Adopt Theorem 6.1 (“ E_{12} is all ghost; congruent to $\Delta \bmod 691$ ”) as the official TAA example that residual data cannot see ghosts, and add E_{12} as a new row of Lemma 64’s ablation table?
- Q3. Run H1 next (Weil-form anatomy of $L(\Delta, s)$ vs $L(E_{12}, s)$, slack prime by prime), or H2 (the archimedean class-number functional) first?
- Q4. Your thesis “RH is about unitarity”: the bridge relates an L^2 cusp form to a non- L^2 Eisenstein series. Should Lemma 67 make the L^2 /Petersson positivity (H3) the candidate global g , knowing that unitarity alone is not known to imply GRH?

Q5. Lemma 66 in Volume 4 with its four figures (approval needed; nothing is submitted without you)? And should Lemma 60's Remark on the 691 class get an erratum note adding the plus part (Thm. 5.1(b))?

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TAA LEMMA 67: THE ARCHIMEDEAN LIFT OF THE 691 CONGRUENCE. LOWER-WEIGHT EISENSTEIN SERIES, THE QUASI-MODULAR ANOMALY, THE RAMANUJAN–SERRE DERIVATIVE, AND WHERE GHOSTS ENTER

J.W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA. Nothing below is a step toward a proof that we can certify; every RH-equivalent statement is labelled [RESTATEMENT]. Tags: [THEOREM] proved here (elementary) or classical with proof indicated; [CITED] checked at the source ([lit/lemma67/verified_refs.md](#)); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (floating point or exact integer arithmetic; not certified unless said).*

Questions (J.W.). [t255u] “What we really need to find is what elements have a *size that adds up* to the congruences of mod 691 ... we need to be looking for multiplicativity and duality in the lower weight Eisenstein series, quasi-modular anomaly, and Serre derivatives.” [t256u] “Could the quasi-modular anomaly, and more precisely its fix, be a precise definition of positivity that we are looking for? If so we need to apply it to our evaluation of the prime side.” [t257u] Does the saddle/separatrix/Cassini discussion of Lemma 63 touch the anomaly, Δ , Ramanujan’s P, Q, R ? (preferred name: *Ramanujan–Serre derivative*).

Answer in brief. (1) *The archimedean lift exists and is Ramanujan’s own identity* (§2). $691E_{12} = 441E_4^3 + 250E_6^2$ and $E_4^3 - E_6^2 = 1728\Delta$ give the exact integer identity $91\sigma_{11}(n) + 600\tau(n) = 691b(n)$ with $b(n) = [E_4^3]_n/720 \in \mathbb{Z}$ [THEOREM], of which $\tau \equiv \sigma_{11} \pmod{691}$ is the reduction. Over $\mathbb{R}$ the two sides have *different sizes*: $\sigma_{11}(n) \asymp n^{11}$, $\tau(n) = O(n^{11/2+\epsilon})$. The object that measures the size of the cuspidal residue is the Petersson norm $y^6|\Delta|$, and the quasi-modular anomaly is exactly its logarithmic derivative: $E_2^* = E_2 - 3/(\pi y) = \frac{1}{\pi i} \partial_z \log(y^6|\Delta|)$ [THEOREM]. (2) *Is the anomaly fix the positivity?* (§6, central). It *is* a precise positivity: E_2^* is the difference between the holomorphic Ramanujan–Serre connection and the Chern connection of the Petersson metric, $\partial_{\bar{z}} E_2^* = \frac{3i}{2\pi y^2}$ is the hyperbolic area form, and the curvature of the weight- k Petersson bundle is $\frac{k}{4\pi} d\mu > 0$ [THEOREM]. What this positivity proves: Petersson positive definite, Hecke operators self-adjoint, real eigenvalues, Hecke’s bound $\tau(n) = O(n^6)$, Rankin–Selberg mean square, the valence formula, $\Delta_{\mathbb{H}} \geq 0$. What it does not prove: Ramanujan (needed Deligne) and anything at RH depth. We make the second half a theorem by writing the Selberg trace formula for $\mathrm{PSL}_2(\mathbb{Z})$ exactly (constants verified numerically to 10^{-16}) and isolating its ζ -term: it is Weil’s explicit formula evaluated on the *kinked* test functions $G(v) = e^{-|v|/2} g(2v)$, which lie in a cone $\mathcal{C}_{1/2}$ on which Weil positivity holds *unconditionally* (every zero with $0 < \beta < 1$ contributes positively) [THEOREM]. Introducing $\mathcal{C}_\theta = \{e^{-\theta|v|} P\}$ gives a *ladder*: zeros in $|\beta - \frac{1}{2}| < \theta \Rightarrow W \geq 0$ on $\mathcal{C}_\theta$ [THEOREM]; $\mathrm{RH} \iff W \geq 0$ on $\mathcal{C}_\theta$ for every $\theta > 0$ [RESTATEMENT]. Spectral positivity of the modular surface reaches exactly the rung $\theta = \frac{1}{2}$ (prime number theorem strength), never $\theta < \frac{1}{2}$. Verdict on [t256u]: *the fix is a genuine positivity, but at prime-number-theorem depth; applied to the prime side it reshuffles Weil’s form into a cone where positivity is automatic, so it neither implies Lemma 64’s $W_A \geq 0$ nor is circular.* (3) *Where ghosts enter* (§7). Only through the real-analytic $E(z, s)$: $\varphi(s) = \xi(2s-1)/\xi(2s)$ has poles at $s = \rho/2$; the scattering phase has Breit–Wigner peaks at $r = \gamma/2$ ([NUMERICAL], all 13 peaks in $[0, 30]$ within the grid step); the closed-horocycle error is an explicit formula over zeros with terms $y^{1-\rho/2}$ (Zagier 1981) and our exact computation reproduces it to $2 \cdot 10^{-6}$ with 100 zeros; a ghost $\beta > \frac{1}{2}$ would appear as $y^{1-\beta/2}$. All of these are [RESTATEMENT] of zero location. (4) *Saddles* (§8). Critical points of $y^6|\Delta| = \text{zeros of } E_2^*; \log(y^6|\Delta|)$ is strictly superharmonic ($\Delta_{\mathbb{H}} = -6$), so unlike $\log|\xi|$ (Lemma 63) it has maxima: ρ is the maximum, i a saddle, no minima, Poincaré–Hopf $1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = \chi$ [THEOREM]+[NUMERICAL]. The valence formula is Green’s identity for $\log(y^{k/2}|f|)$, i.e. Gauss–Bonnet for the Petersson curvature [THEOREM]. Δ has no zeros in $\mathbb{H}$: no ghosts there.

Date: October 22, 2026.

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1. SOURCES AND CONVENTIONS

Read for this lemma: STATE.md §§73–75 (Lemmas 64–66), Lemma 66 (**build66/**, read only; its §6 “Around the bridge” and §6.3 on L^2 /Petersson), Lemma 63 (saddle cross, dents), Lemma 60 (J_Q), Lemma 65 (triple (ω, J, g)), Lemma 61 (Jordan blocks). Primary sources downloaded and read: Ramanujan 1916 [1] (transcription, Table I(6), Table II(6), eq. (30)), Zagier 1981 [2] (pp. 275–280 and §4–5, page images checked), Zagier’s “1-2-3” notes [3] (Prop. 2, Prop. 6, eq. (21), (28), (51)–(54), Prop. 15), Ramakrishnan–Sahu–Singh [5] (arXiv 2303.02921). Sarnak [11], Iwaniec [14], Hejhal [15], Lax–Phillips [16], Serre [7, 8], Deligne [9], Rankin [10], El Basraoui–Sebbar [18], Verjovsky [13]: Crossref records checked, content as stated in the named secondary source or standard (see **verified_refs.md**). The Selberg trace formula constants were taken from the form quoted in Wong’s thesis [17] (search-engine extract of Theorem 3.1.1, the thesis PDF itself was blocked) and from Lax–Phillips [16, (8.12)] (extract), and then *verified numerically* (§6.3); this numerical verification is our primary evidence for them.

Conventions. $z = x + iy \in \mathbb{H}$, $q = e^{2\pi iz}$, $D = \theta = q d/dq = \frac{1}{2\pi i} \frac{d}{dz}$. $E_2 = 1 - 24 \sum \sigma_1(n)q^n$, $E_4 = 1 + 240 \sum \sigma_3(n)q^n$, $E_6 = 1 - 504 \sum \sigma_5(n)q^n$ (Ramanujan’s P, Q, R), $E_{12} = 1 + \frac{65520}{691} \sum \sigma_{11}(n)q^n$, $\Delta = q \prod (1 - q^n)^{24} = \sum \tau(n)q^n$. $\mathcal{F}$ the standard fundamental domain, $d\mu = dx dy/y^2$, $\operatorname{vol}(\mathcal{F}) = \pi/3$, $\Delta_{\mathbb{H}} = y^2(\partial_x^2 + \partial_y^2)$ (non-positive; the positive Laplacian is $-\Delta_{\mathbb{H}}$). $E(z, s) = \sum_{\Gamma_{\infty} \backslash \Gamma} \operatorname{Im}(\gamma z)^s$, $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ (pole at $s = 0, 1$), $\varphi(s) = \xi(2s - 1)/\xi(2s)$. Zeros $\rho = \frac{1}{2} + \delta + i\gamma$. Lemma 64’s Weil

form: for even G , $\hat{G}(t) = \int G(v) e^{itv} dv$,

$$W(G) = \text{POLE} + \text{ARCH} - \text{PRIME} = \sum_{\rho} \hat{G}(\gamma_{\rho} - i\delta_{\rho}), \quad \text{POLE} = \int G(v) 2 \cosh(v/2) dv,$$

$$\text{ARCH} = G(0)(\psi(\tfrac{1}{4}) - \log \pi) + \int_0^{\infty} \frac{2e^{-u/2}}{1 - e^{-2u}} (G(0) - G(u)) du, \quad \text{PRIME} = 2 \sum_n \frac{\Lambda(n)}{\sqrt{n}} G(\log n),$$

verified numerically here against 200 zeros ($W = 0.0024366248566$ for a Gaussian of width 0.25, difference $2.1 \cdot 10^{-14}$) [NUMERICAL]. Scripts: `num/lemma67_{qexp,anom,horo,horo100,selberg,cone,eis,extra,figs}.py` → `num/167/`.

2. THE ARCHIMEDEAN LIFT OF THE 691 CONGRUENCE

Theorem 2.1 (Ramanujan’s identity and its integer form). [THEOREM] (*identity: Ramanujan 1916, Table I(6) [CITED]*)

$$691 E_{12} = 441 E_4^3 + 250 E_6^2, \quad E_4^3 - E_6^2 = 1728 \Delta.$$

Consequently $691 E_{12} = 691 E_4^3 - 432000 \Delta$, and with $b(n) := [E_4^3]_n / 720 \in \mathbb{Z}$ ($n \geq 1$),

$$(1) \quad 91 \sigma_{11}(n) + 600 \tau(n) = 691 b(n) \quad (n \geq 1).$$

In particular $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$, because $600 \equiv -91 \pmod{691}$.

Proof. $M_{12}(\text{SL}_2(\mathbb{Z}))$ is spanned by E_4^3, E_6^2 ; comparing the constant terms ($441 + 250 = 691$) and the q -coefficients ($441 \cdot 720 - 250 \cdot 1008 = 65520$) gives the first identity; the second is the standard definition of Δ (Zagier [3, (22), (26)]). Substituting $E_6^2 = E_4^3 - 1728 \Delta$: $691 E_{12} = 691 E_4^3 - 250 \cdot 1728 \Delta$. Coefficient of q^n : $65520 \sigma_{11}(n) = 691 [E_4^3]_n - 432000 \tau(n)$. Every coefficient of $E_4^3 = (1 + 240A)^3$ for $n \geq 1$ is divisible by 720 ($720, 3 \cdot 240^2 = 240 \cdot 720, 240^3 = 19200 \cdot 720$), and $65520 = 720 \cdot 91$, $432000 = 720 \cdot 600$. Divide by 720. $\square$

[NUMERICAL] Exact integer arithmetic (Kronecker substitution, $n \leq 10^4$): both identities, (1), $720 \mid [E_4^3]_n$, and the congruence hold for all $n \leq 10^4$; $b(1..5) = 1, 249, 23548, 551353, 6434526$; $\tau(1..6) = 1, -24, 252, -1472, 4830, -6048$ (product formula computed independently via Jacobi’s $\prod(1 - q^n)^3$). Control: $\Delta E_4 \equiv \sum \sigma_{15}(n) q^n \pmod{3617}$ holds for $n \leq 10^4$; the analogous congruences modulo 7, 11, 13, 683, 701 fail for 146, 183, 185, 198, 198 of $n < 200$ (as they must). Ramanujan also records the derivative form $65520 \Phi_{1,12} = P(441 Q^3 + 250 R^2) - 691 Q^2 R$ [1, Table II(6)] [CITED].

Remark 2.2 (What has a “size that adds up”). (1) is an identity of integers, so it holds in $\mathbb{R}$ and in every $\mathbb{Z}/N$. Over $\mathbb{R}$ it says: the cuspidal part is the *small* residue of two large Eisenstein quantities, $600 \tau(n) = 691 b(n) - 91 \sigma_{11}(n)$, with $\sigma_{11}(n) \asymp n^{11}$ and $|\tau(n)| \leq d(n) n^{11/2}$ (Deligne [9]; $\leq$ verified for $n \leq 2000$, maximum of $|\tau(n)| / (d(n) n^{11/2})$ is 1 at $n = 1$) [NUMERICAL]. Mod 691 the residue is invisible ($\equiv$ the Eisenstein part). So the answer to [t255u] is: *the elements with a size are the same objects, read in $\mathbb{R}$; what measures the size of the cuspidal residue is the Petersson norm $\|\Delta\|^2 = \int_{\mathcal{F}} y^{12} |\Delta|^2 d\mu = 1.035362056804 \cdot 10^{-6}$ (Lemma 66 §6.3 [TAA-L]), finite because Δ is cuspidal, while $E_{12} \notin L^2$. Figure 1.*

3. MULTIPLICATIVITY AND DUALITY IN THE EISENSTEIN FAMILY; THE ANOMALY AS A CENTRAL POLE

Proposition 3.1. (a) [THEOREM] For even $k \geq 4$, E_k is a Hecke eigenform with eigenvalues $\sigma_{k-1}(n)$, and $\Lambda(E_k, s) := (2\pi)^{-s} \Gamma(s) L(E_k, s) = (2\pi)^{-s} \Gamma(s) c_k \zeta(s) \zeta(s - k + 1)$ ($c_k = -2k/B_k$) has an Euler product of degree 2 and the functional equation $\Lambda(E_k, k - s) = (-1)^{k/2} \Lambda(E_k, s)$, obtained by Mellin transform of $E_k(i/y) = (-1)^{k/2} y^k E_k(iy)$ (Hecke). Poles: $s = 0, k$ only.

(b) [THEOREM] For $k = 2$: $\Lambda(E_2, s) = -24(2\pi)^{-s} \Gamma(s) \zeta(s) \zeta(s - 1)$ still has the Euler product and $\Lambda(E_2, 2 - s) = -\Lambda(E_2, s)$ (this follows from ζ ’s functional equation alone), but it has a third

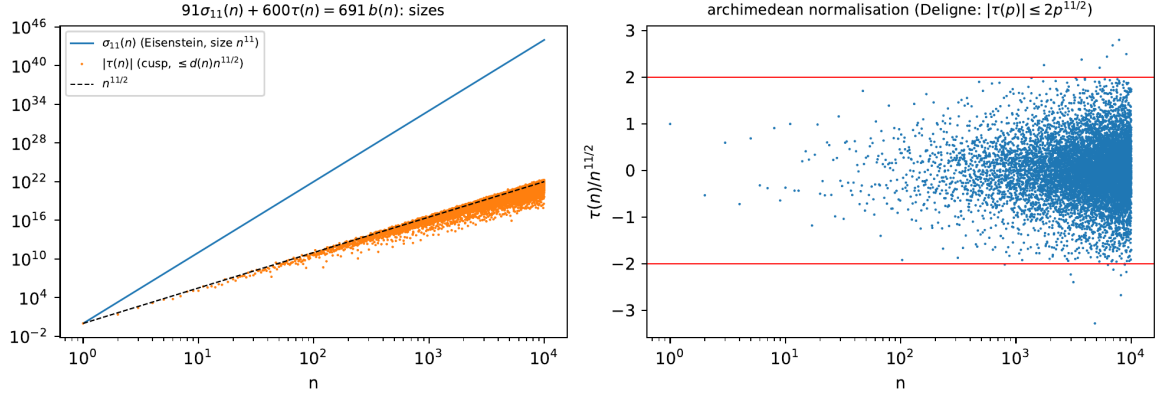


FIGURE 1. Left: $\sigma_{11}(n)$ vs $|\tau(n)|$, $n \leq 10^4$: the cusp form is the $n^{11/2}$ -size residue in $91\sigma_{11} + 600\tau = 691b$. Right: $\tau(n)/n^{11/2}$ (red: ± 2 , Deligne's bound at primes). [NUMERICAL]

pole, at the centre $s = 1$, with residue $6/\pi$. This central pole is the Mellin image of the anomaly: $E_2(i/y) = -y^2 E_2(iy) + 6y/\pi$.

(c) [NUMERICAL] $|\Lambda(E_2, s) + \Lambda(E_2, 2 - s)| \leq 8 \cdot 10^{-26}$ at three test points (and $|\Lambda - \Lambda(2 - \cdot)|$ is $O(1)$, so the sign is -1); residues at $s = 0, 1, 2$: -1.00000000000014 , $1.9098593171028 = 6/\pi$, -0.99999999999986 (step 10^{-12}); $\Lambda(E_4, 2) = -0.8\bar{3}$ finite (no central pole for $k = 4$).

Proof of (b). The Euler product and the functional equation follow from those of ζ and Legendre duplication, exactly as for $k \geq 4$; the residue at $s = 1$ is $-24(2\pi)^{-1}\Gamma(1)\zeta(0) = -24 \cdot (-\frac{1}{2})/(2\pi) = 6/\pi$. In Hecke's argument $\int_0^\infty (E_2(iy) - 1)y^{s-1}dy$ split at $y = 1$, the term $6y/\pi$ of the transformation law integrates to $\frac{6}{\pi} \frac{1}{s-1}$ -type terms, producing this pole. $\square$

Remark 3.2 (Reading). [HEURISTIC] Multiplicativity (Euler product) and duality (functional equation) both *survive* at weight 2; what the anomaly breaks is the absence of a central pole. The anomaly is an obstruction located at the centre of symmetry. Hecke's *converse* theorem goes the other way (functional equation + growth $\Rightarrow$ modularity); for E_2 the extra central pole is exactly what the converse theorem reads as “quasi-modular, not modular”. This is a precise sense in which “duality” at weight 2 needs a correction, but it is a statement about a pole of $\zeta(s)\zeta(s-1)$, not about zeros.

4. THE RAMANUJAN–SERRE DERIVATIVE

Theorem 4.1 (Ramanujan's system; Serre's derivative). [CITED] (Ramanujan 1916, eq. (30); Zagier 2008, Prop. 15) With $D = q d/dq$:

$$DE_2 = \frac{E_2^2 - E_4}{12}, \quad DE_4 = \frac{E_2 E_4 - E_6}{3}, \quad DE_6 = \frac{E_2 E_6 - E_4^2}{2}.$$

With $\vartheta_k f = Df - \frac{k}{12} E_2 f$: $\vartheta_4 E_4 = -\frac{1}{3} E_6$, $\vartheta_6 E_6 = -\frac{1}{2} E_4^2$, $\vartheta_{12} \Delta = 0$ (i.e. $D \log \Delta = E_2$), and $\vartheta_2 E_2 := DE_2 - \frac{1}{6} E_2^2 = -\frac{1}{12} (E_2^2 + E_4)$.

[NUMERICAL] All seven identities hold exactly for the first $10^4 + 1$ coefficients; the variants with the opposite signs ($3DE_4 = E_6 - E_2 E_4$, $2DE_6 = E_4^2 - E_2 E_6$) fail already at q^1 .

History and name (J.W.'s request). The system is Ramanujan's: [1, eq. (30)] reads $x \frac{dP}{dx} = \frac{P^2 - Q}{12}$, $x \frac{dQ}{dx} = \frac{PQ - R}{3}$, $x \frac{dR}{dx} = \frac{PR - Q^2}{2}$ [CITED]. The operator ϑ_k is due to Serre's p -adic work [7] (Lecture Notes in Math. 350, 1973) [CITED-sec]: we did not read Serre's text; Zagier [3, §5.1, (53)] says “sometimes called the Serre derivative” and attributes Prop. 15 (the system) to Ramanujan [CITED]. The name **Ramanujan–Serre derivative** is in the literature: Ramakrishnan–Sahu–Singh [5] (arXiv

2303.02921, abstract: “known as the Ramanujan–Serre derivative map”) [CITED], and Ganash–Sahu [6] (Ramanujan J., 2026; title verified by Crossref, text not read); the Lean `Mathlib` documentation also writes “(Ramanujan-)Serre derivative” (web page, checked). We adopt “Ramanujan–Serre derivative” in TAA, as J.W. prefers.

Proposition 4.2 (Two connections and their difference). *[THEOREM] Let $\partial_k f := Df - \frac{k}{4\pi y} f$ (the non-holomorphic derivative of [3, (55)]). Then for every f ,*

$$\partial_k f - \vartheta_k f = \frac{k}{12} E_2^* f, \quad E_2^* = E_2 - \frac{3}{\pi y}.$$

∂_k is the Chern connection of the Petersson metric $h_k = |y^k| \cdot |^2$: for holomorphic f , $\frac{1}{2\pi i} \partial_z \log(y^k |f|^2) = \frac{Df}{f} - \frac{k}{4\pi y} = \frac{\partial_k f}{f}$. ϑ_k is holomorphic and modular; ∂_k is modular and metric; neither is both, and the anomaly E_2^* is the difference.

Proof. $\partial_k f - \vartheta_k f = \frac{k}{12} E_2 f - \frac{k}{4\pi y} f = \frac{k}{12} (E_2 - \frac{3}{\pi y}) f$. For holomorphic f , $\partial_z \log |f|^2 = f'/f$ and $\partial_z \log y = \frac{1}{2iy}$, so $\frac{1}{2\pi i} \partial_z \log(y^k |f|^2) = Df/f + \frac{k}{2\pi i \cdot 2iy} = Df/f - \frac{k}{4\pi y}$. $\square$

5. THE ANOMALY AND ITS FIX

Proposition 5.1 (Anomaly, fix, and the Petersson norm). *[THEOREM] (transformation law: [3, Prop. 6, (19), (21)] [CITED])*

- (a) $E_2(-1/z) = z^2 E_2(z) + \frac{12z}{2\pi i} = z^2 E_2(z) - \frac{6iz}{\pi}$; $E_2^* := E_2 - \frac{3}{\pi y}$ transforms with weight 2.
- (b) $D \log \Delta = E_2$, and $E_2^* = \frac{1}{\pi i} \partial_z \log(y^6 |\Delta(z)|)$.
- (c) $\partial_z E_2^* = \frac{3i}{2\pi y^2}$: the non-holomorphy of the fix is the hyperbolic area density, and $\Delta_{\mathbb{H}} \log(y^6 |\Delta|) = -6$ (more generally $\Delta_{\mathbb{H}} \log(y^{k/2} |f|) = -k/2$ off the zeros of f).
- (d) (Kronecker limit formula, [BIB] for the formula, [NUMERICAL] for the constant.) With $C(z) := \lim_{s \rightarrow 1} (E(z, s) - \frac{3}{\pi(s-1)})$,

$$C(z) = \frac{6}{\pi} \left(\gamma - \log 2 - \frac{\zeta'(2)}{\zeta(2)} - \log(\sqrt{y} |\eta(z)|^2) \right), \quad E_2^*(z) = 2i \partial_z C(z) = 2i \lim_{s \rightarrow 1} \partial_z E(z, s).$$

Proof. (a) is [3, Prop. 6] with $E_2 = \frac{3}{\pi^2} G_2$. (b) $\Delta = q \prod (1 - q^n)^{24}$ gives $D \log \Delta = 1 - 24 \sum \frac{nq^n}{1 - q^n} = E_2$; then $\partial_z \log |\Delta| = \frac{1}{2} \Delta' / \Delta = \pi i E_2$ and $\partial_z \log y^6 = 6/(2iy) = -3i/y$; sum $= \pi i (E_2 - 3/(\pi y))$. (c) $\partial_z y = i/2$. $\log |\Delta|$ is harmonic (Δ has no zeros in $\mathbb{H}$) and $y^2 \partial_y^2 \log y^6 = -6$. (d) Differentiate: $2i \partial_z (-\frac{6}{\pi} \log(\sqrt{y} |\eta|^2)) = -\frac{12i}{\pi} \cdot \frac{\pi i}{12} E_2^* = E_2^*$ (using $\partial_z \log(\sqrt{y} |\eta|^2) = \frac{\pi i}{12} E_2^*$, which is (b) divided by 12). $\square$

[NUMERICAL] (num/167/anom.json): (a) residual $5.7 \cdot 10^{-16}$ at four points (the opposite sign gives 3.6); $E_2(i) = 0.954929658551372 = 3/\pi$; modularity of E_2^* to $4.6 \cdot 10^{-16}$; (b) by finite differences to $8 \cdot 10^{-11}$; (c) $\Delta_{\mathbb{H}} \log(y^6 |\Delta|) = -6.000008, -5.999997, -6.0000006, -6.000007$ (step 10^{-3}); (d) $E(z, s)$ from its Fourier expansion agrees with the direct lattice sum at $s = 3$ (2.31179481827 vs 2.31179481825, the latter truncated); $C(z) + \frac{6}{\pi} \log(\sqrt{y} |\eta|^2) = 0.8671324277207$ at two points vs the predicted constant 0.8671324277206646; $E_2^* = 2i \partial_z C$ to $5.7 \cdot 10^{-9}$ (finite differences in z and s).

Remark 5.2 (Where E_2^* meets $E(z, s)$, without overclaiming). (d) is the precise link requested in item (5) of the chat: E_2^* is $2i \partial_z$ of the constant term of $E(z, s)$ at its pole $s = 1$, equivalently $y E_2^* = R_0 E(\cdot, s)|_{s \rightarrow 1}$ up to the normalization of the Maass raising operator $R_0 = 2iy \partial_z$ (we checked only the ∂_z form). The pole of $E(z, s)$ at $s = 1$ is governed by $\xi(2s - 1)$ at $2s - 1 = 1$, i.e. by ζ at $s = 1$; no zero of ζ is involved in E_2^* [THEOREM].

6. IS THE ANOMALY FIX THE POSITIVITY?

This is J.W.'s central question [t256u]. We answer in four steps: what the fix proves (§6.1); where ζ enters the spectral theory it controls (§6.2); the exact trace formula and its prime side (§6.3); the cone theorem and the ladder (§6.4).

6.1. What the fix proves.

Theorem 6.1 (Positivity supplied by the anomaly fix). *[THEOREM] (classical; sources as indicated)*

- (i) The Petersson metric $h_k = y^k |\cdot|^2$ on weight- k forms has Chern curvature $-\partial\bar{\partial} \log y^k = \frac{k}{4y^2} dz \wedge d\bar{z}$, i.e. first Chern form $\frac{k}{4\pi} d\mu > 0$ (the line bundle is positive); its connection differs from the Ramanujan–Serre connection by $\frac{k}{12} E_2^*$ (Prop. 4.2).
- (ii) $\langle f, g \rangle = \int_{\mathcal{F}} f \bar{g} y^k d\mu$ is positive definite on cusp forms S_k ; the Hecke operators are self-adjoint for it, so their eigenvalues (e.g. $\tau(p)$) are real [3, §4.1] [CITED].
- (iii) Hecke's bound: $y^6 |\Delta|$ is bounded on $\mathbb{H}$, hence $|\tau(n)| \leq Cn^6$ [3, §2.4] [CITED]. Rankin–Selberg: $\sum_{n \leq x} \tau(n)^2 n^{-11} \sim \alpha x$ [10] [BIB] (record checked), i.e. $\tau(n) \approx n^{11/2}$ on average.
- (iv) $-\Delta_{\mathbb{H}} \geq 0$ on $L^2(\Gamma \backslash \mathbb{H})$, with spectrum $\{0\} \cup [\frac{1}{4}, \infty)$ (continuous, from $E(z, \frac{1}{2} + ir)$) $\cup \{\frac{1}{4} + r_j^2\}$ (Maass cusp forms, $r_1 = 9.53369526135 \dots$ [4], LMFDB [CITED]).

It does not prove Ramanujan's conjecture $|\tau(p)| \leq 2p^{11/2}$: that is Deligne's theorem [9], via the Weil conjectures (Zagier: “immeasurably deeper” than multiplicativity [3, §2.4]) [CITED].

So the fix is a precise definition of a positivity: positivity of the Hodge (Petersson) line bundle on the modular curve, equivalently of the hyperbolic area. In the language of Lemma 65 it is a g (§9). The question is what it says about zeros.

6.2. The Laplacian sees ζ only on the line $\operatorname{Re} s = 1$.

Proposition 6.2. *[THEOREM] (i) $E(z, s) = y^s + \varphi(s)y^{1-s} + \frac{4\sqrt{y}}{\xi(2s)} \sum_{n \geq 1} n^{s-1/2} \sigma_{1-2s}(n) K_{s-1/2}(2\pi n y) \cos 2\pi n x$ (cf. [2, (14)] [CITED]; normalization checked numerically above). (ii) $|\varphi(\frac{1}{2} + ir)| = 1$ for real r , unconditionally (functional equation and $\xi(\bar{s}) = \xi(s)$); [NUMERICAL] $||\varphi| - 1| = 1.7 \cdot 10^{-21}$ at $r = 3.3$. (iii) The poles of φ in $\operatorname{Re} s < \frac{1}{2}$ are at $s = \rho/2$ (zeros of $\xi(2s)$), i.e. on $\operatorname{Re} s = \beta/2 \in (0, \frac{1}{2})$, never on the spectrum: they are resonances, not eigenvalues. (iv) $-\frac{\varphi'}{\varphi}(\frac{1}{2} + ir) = 4 \operatorname{Re} \frac{\xi'}{\xi}(1 + 2ir)$; [NUMERICAL] residual $8 \cdot 10^{-31}$.*

Proof. (iv): $\frac{\varphi'}{\varphi}(s) = 2 \frac{\xi'}{\xi}(2s - 1) - 2 \frac{\xi'}{\xi}(2s)$; at $s = \frac{1}{2} + ir$, $\frac{\xi'}{\xi}(2ir) = -\frac{\xi'}{\xi}(1 - 2ir) = -\overline{\frac{\xi'}{\xi}(1 + 2ir)}$. $\square$

Remark 6.3 (Unitarity, for J.W.'s thesis). The scattering matrix of the modular surface is unitary whether or not RH holds (ii). RH is not the unitarity of this S -matrix; it is a statement about where its poles are (iii). Zagier [2, §4] [CITED] constructs a G -invariant space $\mathcal{V}$ spanned by the $E(\cdot, \rho)$ and observes: if $\mathcal{V}$ were unitarizable, RH (and simplicity of zeros) would follow; he argues this is unlikely to be possible *naturally*, because multiple zeros produce non-split Jordan-type pieces (compare Lemma 61's Jordan blocks [TAA-L]), and analogues for other fields have multiple zeros. This is the cleanest published form of “RH as unitarity” we found, and it is explicitly an [RESTATEMENT] (Zagier's own caveat included).

6.3. The Selberg trace formula for $\operatorname{PSL}_2(\mathbb{Z})$, exactly.

Theorem 6.4 (Selberg; constants verified). *[CITED-sec] (Selberg 1956; form of [14, Thm. 10.2], [15], as quoted in [17, Thm. 3.1.1] and [16, (8.12)]) + [NUMERICAL] (verified below). Let h be even,*

holomorphic in $|\operatorname{Im} r| \leq \frac{1}{2} + \varepsilon$, $h(r) \ll (1 + |r|)^{-2-\varepsilon}$, $g(u) = \frac{1}{2\pi} \int h(r) e^{-iru} dr$. Then

$$\begin{aligned} \sum_{j \geq 0} h(r_j) + \frac{1}{4\pi} \int_{\mathbb{R}} h(r) \left(-\frac{\varphi'}{\varphi}\right) \left(\frac{1}{2} + ir\right) dr &= \frac{1}{12} \int_{\mathbb{R}} r \tanh(\pi r) h(r) dr \\ &+ \frac{1}{8} \int_{\mathbb{R}} \frac{h(r)}{\cosh \pi r} dr + \frac{1}{3\sqrt{3}} \int_{\mathbb{R}} \frac{h(r) \cosh(\pi r/3)}{\cosh \pi r} dr \\ &+ \sum_{t \geq 3} \sum_{\substack{f^2 | t^2 - 4 \\ D_0 = (t^2 - 4)/f^2 \equiv 0, 1 \pmod{4}}} h^+(D_0) \frac{2 \log \varepsilon_{D_0}}{\sqrt{t^2 - 4}} g(2 \log \varepsilon_t) \\ &+ \frac{h(0)}{4} (1 - \varphi(\tfrac{1}{2})) - g(0) \log 2 - \frac{1}{2\pi} \int_{\mathbb{R}} h(r) \psi(1 + ir) dr, \end{aligned}$$

where $r_0 = i/2$ (constants), r_j ($j \geq 1$) the Maass cusp spectral parameters, $\varphi(\frac{1}{2}) = -1$, $\varepsilon_t = \frac{t + \sqrt{t^2 - 4}}{2}$, $h^+(D_0)$ the number of proper $(\mathrm{SL}_2(\mathbb{Z})\text{-})$ classes of primitive forms of discriminant D_0 and ε_{D_0} the least $\frac{t_0 + u_0 \sqrt{D_0}}{2} > 1$ with $t_0^2 - D_0 u_0^2 = 4$. The hyperbolic term uses the bijection between hyperbolic classes of trace t and classes of (not necessarily primitive) forms of discriminant $t^2 - 4$, $\gamma \mapsto cx^2 + (d - a)xy - by^2$ (cf. Sarnak [12] [BIB]).

[NUMERICAL] (num/167/selberg.json) With $h(r) = e^{-r^2/T^2}$ (classes $t \leq 60$; h^+ by cycle counting, tested: $h^+(5, 8, 12, 13, 21, 24, 40, 60, 65, 85, 96, 140) = 1, 1, 2, 1, 2, 2, 2, 4, 2, 2, 4, 4$):

T	discrete	continuous	identity	ell. 2	ell. 3	hyperbolic	parabolic	LHS–RHS
1.5	1.117519	−0.336114	0.181041	0.113858	0.193887	0.061581	0.231038	$1.1 \cdot 10^{-16}$
2.0	1.064494	−0.357884	0.326674	0.118175	0.204038	0.012826	0.044897	$−8.0 \cdot 10^{-17}$
2.5	1.040811	−0.368784	0.514076	0.120433	0.209693	0.001879	−0.174054	$−5.06 \cdot 10^{-11}$

The discrete column contains $h(i/2)$ and $h(r_1)$ only. The $T = 2.5$ residual is diagnostic: it equals $-h(r_2)$ for a second cusp parameter $r_2 = 2.5\sqrt{-\log(5.06 \cdot 10^{-11})} \approx 12.17$, i.e. the formula *detects* the next Maass eigenvalue we left out (we did not cross-check 12.17 against a tabulated value). Wrong constants (e.g. dropping the $\frac{h(0)}{4}(1 - \varphi(\frac{1}{2}))$ term) would produce $O(10^{-1})$ residuals. Figure 2 (right).

Theorem 6.5 (The prime side of Selberg’s formula). [THEOREM] (i) For h, g as above, $\frac{1}{\pi} \int_{\mathbb{R}} h(r) \operatorname{Re} \frac{\zeta'}{\zeta} (1 + 2ir) dr = \frac{h(0)}{2} - 2 \sum_{n \geq 2} \frac{\Lambda(n)}{n} g(2 \log n)$.

(ii) Hence (using $\psi(z) + \psi(z + \frac{1}{2}) = 2\psi(2z) - 2 \log 2$) the trace formula for $\mathrm{PSL}_2(\mathbb{Z})$ is equivalent to

$$\sum_{j \geq 0} h(r_j) = \operatorname{Id} + \operatorname{Ell} + \operatorname{Hyp} + g(0) \log(2\pi) - \frac{1}{\pi} \int_{\mathbb{R}} h(r) \operatorname{Re} \psi(1 + 2ir) dr + 2 \sum_{n \geq 2} \frac{\Lambda(n)}{n} g(2 \log n).$$

The $\frac{h(0)}{2}$ of (i) cancels the parabolic $\frac{h(0)}{4}(1 - \varphi(\frac{1}{2}))$.

(iii) With $G(v) := e^{-|v|/2} g(2v)$, the Selberg prime sum is exactly Lemma 64’s prime functional: $2 \sum \frac{\Lambda(n)}{n} g(2 \log n) = \operatorname{PRIME}(G)$.

Proof. (i) For $\sigma > 1$, $\int h(r) \frac{\zeta'}{\zeta} (\sigma + 2ir) dr = -2\pi \sum \Lambda(n) n^{-\sigma} g(2 \log n)$; letting $\sigma \downarrow 1$, the pole $-\frac{1}{s-1}$ contributes $-\frac{\pi}{2} h(0)$ (from $\frac{1}{\epsilon + 2ir} \rightarrow \operatorname{PV} + \frac{\pi}{2} \delta(r)$); take real parts. (ii) Insert (i) and Prop. 6.2(iv), $\frac{\zeta'}{\zeta}(s) = -\frac{1}{2} \log \pi + \frac{1}{2} \psi(\frac{s}{2}) + \frac{\zeta'}{\zeta}(s)$, into Thm. 6.4. (iii) $n^{-1/2} G(\log n) = n^{-1} g(2 \log n)$. $\square$

[NUMERICAL] (i): residual $\leq 1.5 \cdot 10^{-18}$ ($T = 1.5, 2, 2.5$); (ii): LHS 1.1175190687418637, RHS 1.1175190687418635 ($T = 1.5$); (iii): $\operatorname{PRIME}(G)$ computed by the Lemma 64 routine equals the Selberg prime sum 0.122804678069884 to $4 \cdot 10^{-17}$.

Corollary 6.6 (Direction of the inequality). *[THEOREM] Take $h = |\hat{k}|^2$ -type, i.e. $g = k * \tilde{k}$ (positive type). Every term of $\sum_j h(r_j)$ is ≥ 0 (r_j real for $j \geq 1$ since $\lambda_j \geq \frac{1}{4}$ for $\mathrm{SL}_2(\mathbb{Z})$; $h(i/2) = \hat{k}(i/2)^2 \geq 0$ for even real k). Selberg positivity therefore gives a lower bound*

$$\mathrm{PRIME}(G) \geq -(\mathrm{Id} + \mathrm{Ell} + \mathrm{Hyp} + \mathrm{Arch}_{\mathrm{Sel}})(g),$$

whereas Weil positivity (RH) is an upper bound $\mathrm{PRIME}(G) \leq \mathrm{POLE}(G) + \mathrm{ARCH}(G)$. Adding the two identities eliminates the primes:

$$\sum_j h(r_j) + W(G) = \mathrm{Id} + \mathrm{Ell} + \mathrm{Hyp} + \mathrm{Arch}_{\mathrm{Sel}} + \mathrm{POLE}(G) + \mathrm{ARCH}(G) \quad (\text{prime-free}).$$

[NUMERICAL] For $T = 1.5$: $W(G) = 0.0200123$ from the prime side; the sum over the first 300 zeros gives 0.0186586, and the predicted tail ($\hat{G}(t) \approx 0.423/t^2$, zero density $\frac{1}{2\pi} \log \frac{t}{2\pi}$, $\gamma > \gamma_{300} \approx 560$) is $\approx 1.32 \cdot 10^{-3}$, matching the difference $1.35 \cdot 10^{-3}$.

6.4. The cone theorem and the θ -ladder.

Definition 6.7. For $0 < \theta \leq \frac{1}{2}$ let $\mathcal{C}_\theta = \{G(v) = e^{-\theta|v|}P(v) : P \text{ continuous, even, compactly supported, of positive type}\}$. Selberg's prime side (Thm. 6.5(iii)) lives in $\mathcal{C}_{1/2}$ (with $P = g(2\cdot)$).

Theorem 6.8 (Cone positivity). *[THEOREM] (a) If every nontrivial zero satisfies $|\beta - \frac{1}{2}| < \theta$, then $W(G) > 0$ for every $0 \neq G \in \mathcal{C}_\theta$.*

(b) In particular $W > 0$ on $\mathcal{C}_{1/2} \setminus \{0\}$ unconditionally (the hypothesis is $0 < \beta < 1$, Hadamard-de la Vallée Poussin). The same argument applies verbatim to any function with a Weil-type explicit formula whose zeros all lie in the open strip $0 < \beta < 1$; that hypothesis is much weaker than RH, so positivity on $\mathcal{C}_{1/2}$ is RH-blind. (Caution: we do not claim this for the Davenport–Heilbronn function of Lemma 64; whether all its zeros lie in $0 < \beta < 1$ was not checked here, and a zero with $\beta \geq 1$ could make W negative on $\mathcal{C}_{1/2}$. Testing this is cheap with Lemma 64's DH engine and is part of direction D1.)

(c) [RESTATEMENT] $\mathrm{RH} \iff W \geq 0$ on $\mathcal{C}_\theta$ for every $\theta \in (0, \frac{1}{2}]$.

*(d) Every $0 \neq G \in \mathcal{C}_\theta$ has a corner at 0: $G'(0^+) \leq -\theta G(0) < 0$. Hence $\mathcal{C}_\theta$ contains no smooth G , in particular none of Lemma 64's test functions $f * \tilde{f}$ with f smooth.*

Proof. (a) $\hat{G} = \frac{1}{2\pi}(\frac{2\theta}{\theta^2 + t^2}) * \hat{P}$ with $\hat{P} \geq 0$. For a zero $\rho = \frac{1}{2} + \delta + i\gamma$ the explicit formula evaluates $\hat{G}$ at $t = \gamma - i\delta$, and with $u = \gamma - s$,

$$\mathrm{Re} \frac{1}{\theta^2 + (u - i\delta)^2} = \frac{\theta^2 + u^2 - \delta^2}{|\theta^2 + (u - i\delta)^2|^2} > 0 \quad \text{when } \delta^2 < \theta^2.$$

So $\mathrm{Re} \hat{G}(\gamma_\rho - i\delta_\rho) > 0$ for every zero, and $W(G) = \sum_\rho \hat{G}(\gamma_\rho - i\delta_\rho)$ is real (the zero set is symmetric), absolutely convergent ($\hat{G} = O(t^{-2})$), and positive. (b) is (a) with $\theta = \frac{1}{2}$; the argument used only the location of zeros in the open strip and the existence of the explicit formula. (c) $\Rightarrow$: (a). $\Leftarrow$: for $P = (k * \tilde{k})$, $e^{-\theta|v|}P \rightarrow P$ as $\theta \rightarrow 0$ with dominated convergence in each term of W ; so $W(k * \tilde{k}) \geq 0$ for all k , which is Weil's criterion (Lemma 64's (H)). (d) P of positive type gives $P(v) \leq P(0)$, so $P'(0^+) \leq 0$ where it exists; $G'(0^+) = -\theta P(0) + P'(0^+)$; and $P(0) = \frac{1}{2\pi} \int \hat{P} > 0$ unless $P \equiv 0$. $\square$

Remark 6.9 (Reading). [HEURISTIC] Multiplying by $e^{-\theta|v|}$ is Poisson-smoothing the spectral side at width θ . A ghost at distance $\delta < \theta$ from the line is washed out by the smoothing. The modular surface's continuous spectrum sits at $\xi(1+2ir)$, i.e. at distance $\frac{1}{2}$ from the critical line, so its positivity is Poisson smoothing at width $\frac{1}{2}$: exactly the rung that the prime number theorem (non-vanishing on $\mathrm{Re} s = 1$) already guarantees. Zagier's spectral proof of $\zeta(1+it) \neq 0$ [2, §1] [CITED] is the same phenomenon. Every rung $\theta < \frac{1}{2}$ is equivalent to a zero-free strip $|\beta - \frac{1}{2}| < \theta$ in the direction (a); whether the converse holds for a fixed θ is [OPEN].

6.5. Diagnostic in Lemma 64’s framework. [NUMERICAL] We computed the Gram matrix of W on $\mathcal{C}_{1/2}$ for k in an 8-dimensional basis on $[-a, a]$ ($k_i = (1 - (v/a)^2)^3 T_i(v/a)$, G supported in $[-a, a]$, i.e. Lemma 64’s $A = a$), with the same per-term ablations as Lemma 64 (float64; num/167/cone.json):

a	$\lambda_{\min}$	$\lambda_{\max}$	no primes	primes $\times \frac{1}{2}$	primes $\times 2$	no POLE	no ARCH
0.5	$1.35 \cdot 10^{-5}$	0.497	$1.35 \cdot 10^{-5}$	$1.35 \cdot 10^{-5}$	$1.35 \cdot 10^{-5}$	-0.187	$1.3 \cdot 10^{-8}$
1.0	$1.30 \cdot 10^{-5}$	0.450	$-6.2 \cdot 10^{-4}$	$-5.4 \cdot 10^{-7}$	$-1.7 \cdot 10^{-4}$	-1.25	$-2.9 \cdot 10^{-3}$
1.5	$8.1 \cdot 10^{-6}$	0.198	-0.095	-0.032	-0.132	-2.83	-0.060
2.0	$2.27 \cdot 10^{-5}$	0.156	-0.363	-0.148	-0.526	-4.94	-0.059
2.5	$1.07 \cdot 10^{-5}$	0.190	-0.752	-0.328	-1.26	-7.56	-0.0057

(At $a = 0.5$ no prime is in the window.) θ -ladder at $a = 2$: $\lambda_{\min} = 2.27, 2.24, 2.17, 1.80, 1.14, 0.70$ ($\times 10^{-5}$) and $2.4 \cdot 10^{-7}$ for $\theta = 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0$; $\lambda_{\min}$ decreases monotonically toward the RH regime $\theta = 0$.

What this diagnoses. (1) On $\mathcal{C}_{1/2}$ positivity is a theorem, yet removing or rescaling any in-window term breaks it: *collective cancellation (Lemma 64) already occurs at prime-number-theorem depth*, so collectiveness alone is not a sign of RH depth. This is a new entry for Lemma 64’s ablation table. (2) The margins on $\mathcal{C}_{1/2}$ are $O(10^{-5})$ in this small basis, against $\lambda_* \sim 10^{-33}$ at $A = 2.2$ in Lemma 64’s large basis; the two are *not* directly comparable (basis size, float64 here vs 256–460-bit there), and a matched-basis comparison is direction D1. (3) A float run of the full cone ($\theta = 0$, support $[-2a, 2a]$) gave $\lambda_{\min} = -1.8 \cdot 10^{-7}$ at $a = 2.5$: this is round-off, not a ghost (Lemma 64 certified positivity there at high precision); it is reported only to show that float64 cannot resolve the $\theta = 0$ end.

6.6. Verdict on [t256u].

- The anomaly fix *is* a precise positivity (Thm. 6.1): curvature of the Petersson bundle = hyperbolic area, and E_2^* is the connection correction that produces it [THEOREM].
- Applied to the prime side (Thm. 6.5): the spectral positivity of the modular surface controls Lemma 64’s prime functional only on the kinked cone $\mathcal{C}_{1/2}$, from below, and positivity of W on $\mathcal{C}_{1/2}$ is unconditional and RH-blind (Thm. 6.8) [THEOREM].
- It therefore *neither implies* $W_A \geq 0$ *nor is* circular: it is a *reshuffle* (prime-free identity, Cor. 6.6) into a strictly smaller cone where the answer is already known. To reach RH one must descend the ladder $\theta \rightarrow 0$ [RESTATEMENT]; no spectral problem we know has continuous spectrum at $\xi(1 - 2\theta + 2ir)$ with $\theta < \frac{1}{2}$ [OPEN].

7. WHERE GHOSTS ENTER: THE REAL-ANALYTIC EISENSTEIN SERIES

7.1. Resonances. [NUMERICAL] (num/167/eis.json) The local maxima of $-\varphi'/\varphi(\frac{1}{2} + ir)$ on $[0, 30]$ (grid step 0.025) are at 7.075, 10.5, 12.5, 15.225, 16.475, 18.8, 20.45, 21.65, 24.0, 24.875, 26.475, 28.225, 29.675; the values $\gamma_k/2$ are 7.0674, 10.5110, 12.5054, 15.2124, 16.4675, 18.7931, 20.4594, 21.6635, 24.0026, 24.8869, 26.4852, 28.2231, 29.6735: all thirteen agree within one grid step (Figure 2). This is a [RESTATEMENT]: by the Hadamard product,

$$4 \operatorname{Re} \frac{\xi'}{\xi}(1 + 2ir) = \sum_{\rho} \frac{4(\frac{1}{2} - \delta)}{(\frac{1}{2} - \delta)^2 + (2r - \gamma)^2} + (\text{pole and constant terms}),$$

a sum of Lorentzians of width $\frac{1}{2} - \delta$ in $2r$; a ghost $\delta > 0$ would make its peak sharper and taller (and its partner $1 - \bar{\rho}$ broader), a visible but RH-equivalent signature.

7.2. Closed horocycles.

Theorem 7.1 (Zagier 1981). [CITED] [2, p. 279–280] (*page images checked*). For $U \subset \Gamma \backslash \mathbb{H}$ open, $\frac{\operatorname{length}(\mathcal{C}_y \cap U)}{\operatorname{length}(\mathcal{C}_y)} = \frac{\operatorname{vol}(U)}{\operatorname{vol}(\Gamma \backslash \mathbb{H})} + O(y^{1/2-\varepsilon})$; if the error can be replaced by $O(y^{3/4-\varepsilon})$ for all U , then RH holds. For F twice differentiable (and suitably decaying), the constant term satisfies $C(F; y) = \kappa +$

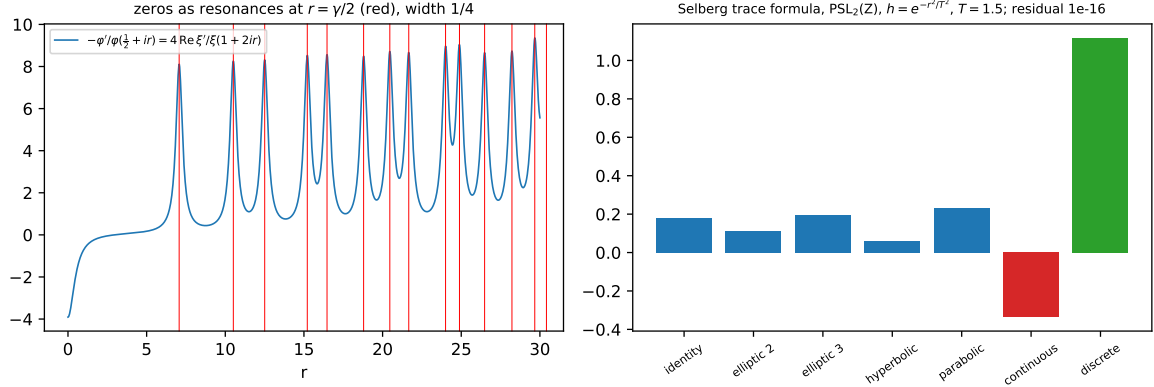


FIGURE 2. Left: scattering phase derivative $-\varphi'/\varphi(\frac{1}{2} + ir) = 4 \operatorname{Re} \xi'(1 + 2ir)$; red lines at $\gamma_k/2$. Every zero is a Breit–Wigner peak of width $\approx \frac{1}{4}$, a resonance at distance $\frac{1}{4}$ from the real r -axis (under RH). Right: the terms of the Selberg trace formula for $\operatorname{PSL}_2(\mathbb{Z})$ with $h = e^{-r^2/T^2}$, $T = 1.5$ (Thm. 6.4); residual $1.1 \cdot 10^{-16}$. [NUMERICAL]

$O(y^{1-\Theta/2-\varepsilon})$ with $\Theta = \sup \operatorname{Re} \rho$; for $F = \chi_U$ Zagier proves only $O(y^{1/2-\varepsilon})$, and $C(F; y) = \kappa + O(y^\alpha)$ for all U gives $\Theta \leq 2(1 - \alpha)$.

So for smooth test functions $\operatorname{RH} \Rightarrow O(y^{3/4-\varepsilon})$; the converse for smooth F follows from the exact formula below [THEOREM] (and is the form stated by Sarnak [11] and Verjovsky [13], [CITED-sec]: records checked, statements not read in the original).

Proposition 7.2 (Exact horocycle explicit formula). [THEOREM] Let $\psi \in C_c^\infty(1, \infty)$, $F_\psi(z) = \sum_{\Gamma_\infty \backslash \Gamma} \psi(\operatorname{Im} \gamma z)$, $\Psi(s) = \int_0^\infty \psi(t) t^{-s-1} dt$. The horocycle average $a_0(y) = \int_0^1 F_\psi(x + iy) dx$ satisfies

$$a_0(y) = \psi(y) + \sum_{c \geq 1} \phi(c) y g\left(\frac{1}{c^2 y}\right), \quad g(a) = \int_{\mathbb{R}} \psi\left(\frac{a}{1+t^2}\right) dt \quad (\text{unfolding}),$$

and, if all zeros are simple, $a_0(y) = \psi(y) + \frac{3}{\pi} \Psi(1) + \sum_\rho \Psi\left(\frac{\rho}{2}\right) \frac{\xi(\rho-1)}{2\xi'(\rho)} y^{1-\rho/2} + O(y^N)$ for every N (shift of contour in $a_0(y) = \frac{1}{2\pi i} \int_{(2)} \Psi(s) (y^s + \varphi(s) y^{1-s}) ds$; the only poles are $s = 1$, residue $\frac{3}{\pi} \Psi(1)$, and $s = \rho/2$). A zero with $\beta > \frac{1}{2}$ contributes $y^{1-\beta/2}$, larger than $y^{3/4}$.

[NUMERICAL] (ψ a standard bump on $(1.25, 3)$; num/167/horo100.json): main term $\frac{3}{\pi} \Psi(1) = 0.0606252$; $\max |a_0 - \frac{3}{\pi} \Psi(1)| = 6.1 \cdot 10^{-3}$ over $y \in [10^{-7}, 0.063]$ (60 points). Unfolded a_0 vs direct reduction of $8 \cdot 10^4/y$ horocycle points: agreement to $\leq 4.5 \cdot 10^{-10}$ at $y = 0.05, 0.01, 0.002$. Zero-sum residual: $1.9 \cdot 10^{-3}$ (1 zero), $3.3 \cdot 10^{-5}$ (40 zeros), $2.1 \cdot 10^{-6}$ (100 zeros): the error is the zero sum. Rates: $\max |\operatorname{err}|/y^{3/4} = 0.084$ overall and 0.073 for $y < 10^{-5}$ (bounded), whereas $|\operatorname{err}|/y^{1/2}$ falls from 0.028 to 0.0040. Diagnostic value: it confirms numerically that horocycle equidistribution is an explicit formula; it adds no information beyond zero location ([RESTATEMENT]). Figure 3.

8. SADDLES, SEPARATRICES AND THE PETERSSON LANDSCAPE

This answers [t257u].

Theorem 8.1. Let $u = \log(y^6 |\Delta(z)|)$, a Γ -invariant function on $\mathbb{H}$.

- (a) [THEOREM] $\partial_z u = \pi i E_2^*$, so the critical points of $y^6 |\Delta|$ are exactly the zeros of E_2^* .
- (b) [THEOREM] i and $\rho = e^{2\pi i/3}$ are forced critical points: E_2^* has weight 2 and $(cz + d)^2 \neq 1$ at the fixed points of S, ST ; so $E_2(i) = 3/\pi$, $E_2(\rho) = 2\sqrt{3}/\pi$.

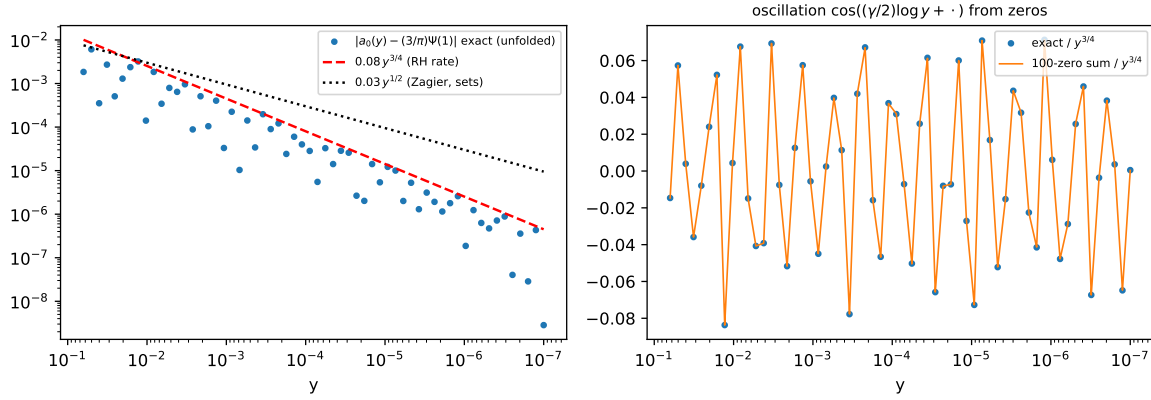


FIGURE 3. Closed-horocycle error for an incomplete Eisenstein series. Left: $|\text{error}|$ vs y with $y^{3/4}$ (RH rate, red) and $y^{1/2}$ (Zagier's unconditional rate for sets). Right: $\text{error}/y^{3/4}$ (dots) and the 100-zero prediction of Prop. 7.2 (line); the oscillation is $\cos(\frac{\gamma}{2} \log y + \cdot)$, sampled coarsely in $\log y$. [NUMERICAL]

- (c) [THEOREM] $\Delta_{\mathbb{H}} u = -6$ (constant negative curvature term; Prop. 5.1(c)). Hence u is strictly superharmonic: no local minima, maxima allowed. Contrast Lemma 63 [TAA-L]: $\log |\xi|$ is harmonic, so off the zeros only saddles occur, and they cross at 90° .
- (d) [THEOREM] At a nondegenerate saddle with Hessian eigenvalues $\lambda_+ > 0 > \lambda_-$, the two separatrix branches meet at angle $2 \arctan \sqrt{\lambda_+ / |\lambda_-|}$, which is 90° iff $\lambda_+ + \lambda_- = 0$, i.e. iff the Laplacian vanishes there. For u at i the angle is not 90° .
- (e) [NUMERICAL] Newton search from 625 seeds in $|x| \leq 0.6$, $0.55 \leq y \leq 2.5$ finds only (images of) i and ρ ; for $y > 2.5$, $|E_2^* - 1| < 0.4$, so there are no zeros there [THEOREM]. At ρ : Hessian $\approx -4I$ (trace $-8 = -6/y^2$), a maximum, $y^6 |\Delta| = 0.00202716775282$ (the grid maximum on $\mathcal{F}$ is 0.0020271589 at the grid point nearest ρ). At i : eigenvalues 1.789 (along x) and -7.789 (along y), a saddle, value 0.00178536985064; separatrix slopes $\pm \sqrt{1.789/7.789} = \pm 0.479$, angle 51.2° .
- (f) [THEOREM] (orbifold Poincaré–Hopf, Satake [19] [BIB]) On the compactified orbifold $\overline{X(1)}$ (cusp filled, cone points of orders 2, 3), $\chi^{\text{orb}} = 2 - \frac{1}{2} - \frac{2}{3} = \frac{5}{6}$. The gradient field of u has index +1 at the cusp ($u \rightarrow -\infty$), -1 at i (weight $\frac{1}{2}$), +1 at ρ (weight $\frac{1}{3}$): $1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. The numerically found critical set is therefore complete up to pairs (max, saddle) of total index 0.

Proof. (a)–(c): Prop. 5.1. (d): the level set through the saddle is $\lambda_+ X^2 + \lambda_- Y^2 = 0$ to second order. At ρ the stabilizer acts by rotation of order 3, and a rotation-invariant quadratic form is a multiple of the identity, so ρ is a non-degenerate extremum or degenerate; superharmonicity makes it a maximum. $\square$

Theorem 8.2 (Valence formula = Gauss–Bonnet for the Petersson metric). [THEOREM] For $0 \neq f \in M_k(\text{SL}_2(\mathbb{Z}))$: $\nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f) + \sum_P^* \nu_P(f) = \frac{k}{12}$ (Zagier [3, Prop. 2], proved there by the argument principle around $\partial\mathcal{F}$, the bottom arc contributing $\pi i k/6$ through $d \log f(Sz) = d \log f(z) + k dz/z$ [CITED]). Equivalent proof via the fix: $u_f = \log(y^{k/2}|f|)$ is Γ -invariant with $\Delta_{\text{eucl}} u_f = -\frac{k}{2y^2} + 2\pi \sum_P \nu_P \delta_P$. Green's identity on $\mathcal{F}_Y = \mathcal{F} \cap \{y \leq Y\}$: side contributions cancel by invariance, the top edge gives $\frac{k}{2Y} - 2\pi\nu_\infty + O(e^{-2\pi Y})$, and $\int_{\mathcal{F}_Y} \frac{k}{2y^2} dx dy = \frac{k}{2}(\frac{\pi}{3} - \frac{1}{Y})$. Hence $2\pi(\nu_\infty + \sum \nu_P/n_P) = \frac{k}{2} \cdot \frac{\pi}{3}$, i.e. zeros $= \frac{k}{4\pi} \text{vol}(\mathcal{F}) = \frac{k}{12}$; and $\text{vol}(\mathcal{F}) = \frac{\pi}{3} = -2\pi\chi^{\text{orb}}(Y(1))$ with $\chi^{\text{orb}}(Y(1)) = -\frac{1}{6}$ is Gauss–Bonnet.

So J.W.'s claim (4) is right in a precise sense: the argument-principle proof uses the *anomaly* of $d \log f$ (bottom arc), the Gauss–Bonnet proof uses the *curvature* of the fix, and Lemma 63's “Gauss–Bonnet = argument principle” is the harmonic special case where the curvature term vanishes.

Zeros of E_2 (not E_2^).* El Basraoui–Sebbar [18] [CITED] (abstract read; record checked): E_2 has infinitely many $\mathrm{SL}_2(\mathbb{Z})$ -inequivalent zeros in $\mathbb{H}$ but none in $\mathcal{F}$. These are not critical points of anything modular; the critical points of the Petersson landscape are the zeros of E_2^* ($E_2^*(i) = 0$ while $E_2(i) = 3/\pi \neq 0$).

Ghosts. Δ has no zeros in $\mathbb{H}$ (product formula), so the Petersson landscape of Δ contains no zero, hence no ghost [THEOREM]; zeta zeros enter only through $E(z, s)$ (§7); GRH for $L(\Delta, s)$ is open (Lemma 66 [TAA-L]).

Nodal maps of $E^(z, \frac{1}{2} + it)$.* [THEOREM] $E^*(z, s) = \xi(2s)E(z, s)$ is real-valued on $\mathrm{Re} s = \frac{1}{2}$ (functional equation $E^*(z, s) = E^*(z, 1-s)$ [2, (5)] plus conjugation); [NUMERICAL] $|\mathrm{Im}|/|\mathrm{Re}| \leq 3 \cdot 10^{-18}$ on our grids. Figure 4 shows its nodal lines for $t = 3$, $t = \gamma_1/2 = 7.0674$, $t = 10$. *Diagnostic value: low.* At $t = \gamma_1/2$ nothing singular happens on the line $\mathrm{Re} s = \frac{1}{2}$ (the resonance is at distance $\frac{1}{4}$); the nodal pattern changes smoothly with t . We claim no ghost link here.

9. CONNECTION TO (ω, J, g) (LEMMA 65) AND COLLECTIVE POSITIVITY (LEMMA 64)

Proposition 9.1. [THEOREM] (i) For $(\mathbb{R}^2, \omega)$ the complex structures J with $J^2 = -1$, $\det J = 1$ and $g = \omega(\cdot, J\cdot) > 0$ are parametrized by $\mathbb{H}$; in Lemma 60's notation $J_Q \leftrightarrow \tau_Q = \frac{-b+i\sqrt{|d|}}{2a}$ (the same root Zagier attaches to Q [2, Ex. 1] [CITED]). The hyperbolic metric is the $\mathrm{SL}_2(\mathbb{R})$ -invariant metric on this space of polarizations, and Prop. 4.2/Thm. 6.1 say: the anomaly fix is the canonical connection over the moduli of polarizations, and its positivity is the curvature of that moduli.

(ii) For $E/\mathbb{F}_p$, Lemma 65's J_p is the point $\tau_p = \frac{-a_p+i\sqrt{4p-a_p^2}}{2} \in \mathbb{H}$; Hasse's bound is $\mathrm{Im} \tau_p > 0$. For Δ , $\tau_p^\Delta = \frac{-\tau(p)+i\sqrt{4p^{11}-\tau(p)^2}}{2}$. Mod 691 the discriminant becomes a square (Lemma 66 [TAA-L]), i.e. the Frobenius point degenerates to the boundary $\mathbb{R} \cup \{\infty\}$ of $\mathbb{H}$, where the hyperbolic metric (and hence the fix) is singular [ANALOGY] for “ $E_{12} \notin L^2$ ”.

Reading. [ANALOGY]/[HEURISTIC] Local positivity (Lemma 64's Hasse forms) = each Frobenius point lies inside $\mathbb{H}$; the anomaly fix supplies the geometry of $\mathbb{H}$ itself. What is missing for ζ is not the metric on the space of J 's but a *global* J , a single point (or section) built from all primes at once, whose g is positive for a reason other than zero location (Lemma 65 [OPEN]). Lemma 64's “collective” positivity reappears on $\mathcal{C}_{1/2}$ (§6.5) at prime-number-theorem depth.

10. RANKED DIRECTIONS (SCORE = $0.6s + 0.4p$)

direction	s	p	score
D1 θ -ladder in Lemma 64's high-precision engine: $\lambda_*(A, \theta)$ for $\theta = \frac{1}{2} \rightarrow 0$ in the matched basis; locate the θ at which slack collapses to the 10^{-33} regime; test DH and $L(\chi)$ on $\mathcal{C}_{1/2}$	.85	.65	.77
D2 Petersson/Rankin–Selberg as global g (Lemma 66 H3): prove it reaches exactly $\theta = \frac{1}{2}$ (non-vanishing of $L(f \times \bar{f}, s)$ on $\mathrm{Re} s = 1$), or find where it goes further	.70	.60	.66
D3 A spectral problem whose continuous spectrum sits at $\xi(1 - 2\theta + 2ir)$, $\theta < \frac{1}{2}$ (higher-rank Eisenstein series, Connes' absorption picture of Lemma 61)	.95	.15	.63
D4 Ghost injection in the horocycle explicit formula (diagnostic display of $y^{1-\beta/2}$)	.35	.95	.59
D5 Zagier's unitarizability of $\mathcal{V}$ vs Lemma 61's Jordan blocks: is there a natural form?	.75	.25	.55
D6 Petersson landscapes for $E_4, E_6, \Delta E_4$; nodal statistics of $E^*(z, \frac{1}{2} + it)$	.25	.90	.51

11. CORRECTIONS TO GROK BOT'S CHAT CLAIMS

C1. (1) $691E_{12} = 441E_4^3 + 250E_6^2$, $E_4^3 - E_6^2 = 1728\Delta$ imply $\tau \equiv \sigma_{11} \pmod{691}$; $441 + 250 = 691$. **Correct** (Thm. 2.1). Additions: the identity is Ramanujan's own [1, Table I(6)]; the clean

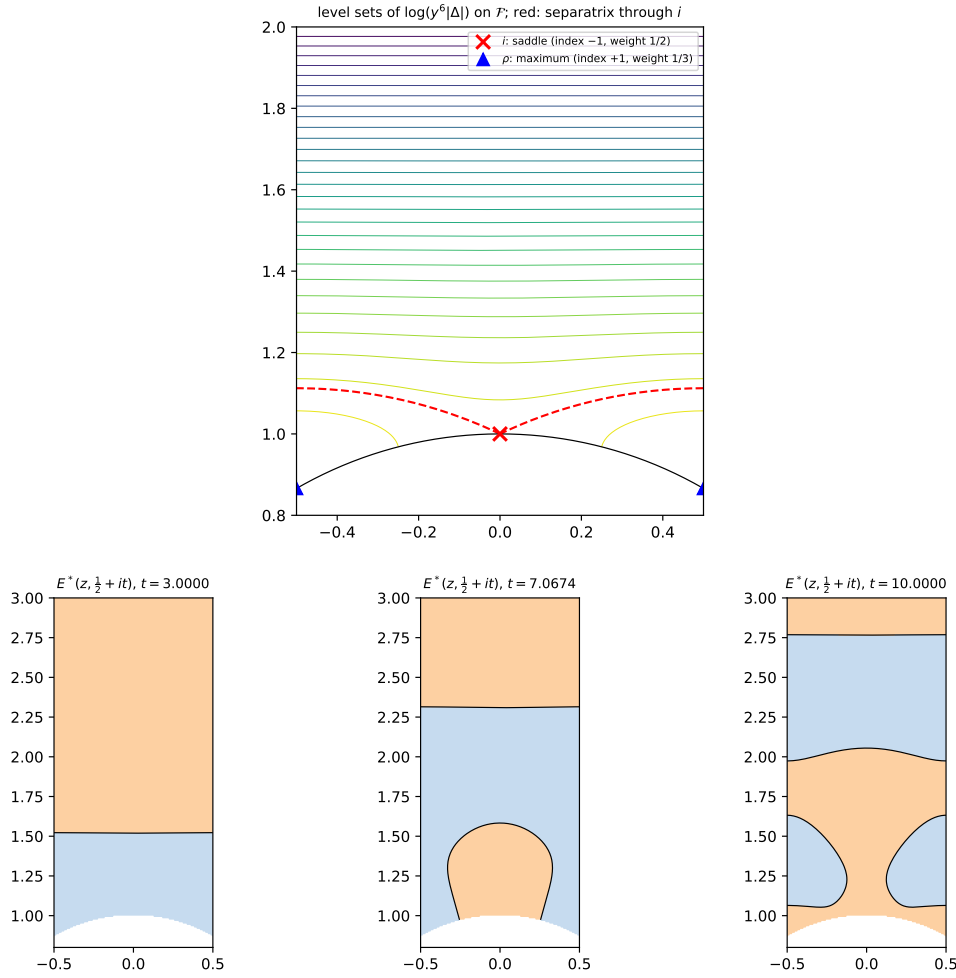


FIGURE 4. Top: level sets of $\log(y^6|\Delta|)$ on $\mathcal{F}$; red: the separatrix through the saddle i (branches at 51.2° , not 90°); ρ is the maximum. Bottom: sign domains and nodal lines of the real function $E^*(z, \frac{1}{2} + it)$ on $\mathcal{F}$ for $t = 3, \gamma_1/2, 10$. [NUMERICAL]

integer form is $91\sigma_{11} + 600\tau = 691b$; “the congruence is the mod-691 shadow of an archimedean identity” is accurate, provided “archimedean” means “an identity over $\mathbb{Z}$ read in $\mathbb{R}$ ”: the identity itself carries no inequality; size enters only through the Petersson norm (Remark 2.2).

- C2.** (2) *Multiplicativity = Hecke eigenforms with σ_{k-1} ; duality $= \tau \mapsto -1/\tau \leftrightarrow$ functional equation via Mellin (Hecke’s converse).* **Correct for $k \geq 4$; imprecise as stated.** The Mellin step is Hecke’s direct theorem; the *converse* theorem is the other direction. At $k = 2$ both multiplicativity and the functional equation survive; the anomaly appears as an extra *central pole* of $\Lambda(E_2, s)$, residue $6/\pi$ (Prop. 3.1).
- C3.** (3) *Ramanujan’s system and $\vartheta E_4 = -E_6/3$, $\vartheta E_6 = -E_4^2/2$.* **Correct**, signs verified exactly to q^{10^4} ; it is eq. (30) of Ramanujan 1916.
- C4.** (4) $E_2(-1/\tau) = \tau^2 E_2 + 12\tau/(2\pi i)$; E_2^* modular, needs $\text{Im } \tau$; $\frac{1}{2\pi i} \frac{d}{d\tau} \log \Delta = E_2$ links E_2^* to $y^6|\Delta|$ and Kronecker. **Correct**; made precise: $E_2^* = \frac{1}{\pi i} \partial_z \log(y^6|\Delta|)$ and $E_2^* = 2i\partial_z C(z)$ with the Kronecker constant 0.86713242772066 (Prop. 5.1). “Needs a size” is a [HEURISTIC]; the theorem is that $\partial_z E_2^*$ is the area density.

- C5.** (5) $E(z, s)$ has constant term $y^s + \varphi(s)y^{1-s}$, $\varphi = \xi(2s-1)/\xi(2s)$, “so zeta zeros are scattering poles”. **Correct with a precision:** the poles are at $s = \rho/2$ (zeros of $\xi(2s)$), not at ρ . Zagier: “ $RH \iff$ equidistribution at rate $O(y^{3/4-\varepsilon})$ ”. **Imprecise:** Zagier proves $O(y^{1/2-\varepsilon})$ for sets and “ $O(y^{3/4-\varepsilon})$ for all $U \Rightarrow RH$ ”; for smooth test functions the error is $O(y^{1-\Theta/2-\varepsilon})$, which gives the equivalence for smooth F (Thm. 7.1, Prop. 7.2). Citation corrected/confirmed: *Automorphic Forms, Representation Theory and Arithmetic* (Bombay Colloquium 1979), Tata/Springer 1981, pp. 275–301. Sarnak 1981: Comm. Pure Appl. Math. 34, 719–739 (record verified; content second-hand).
- C6.** [t256u](a) E_2^* is forced by the hyperbolic metric; Petersson positive definite; Hecke self-adjoint; real eigenvalues; not Ramanujan. **Correct;** sharpened: “forced by the metric” = Prop. 4.2 (difference of the Ramanujan–Serre and Chern connections). Omitted in chat: this positivity does give Hecke’s $O(n^6)$ and the Rankin–Selberg mean square.
- C7.** (b) Δ self-adjoint ≥ 0 ; zeros only through φ , as resonances at $s = \rho/2$; positivity of Δ does not control them. **Correct;** now a theorem in the form of Thm. 6.8. Added: $|\varphi(\frac{1}{2} + ir)| = 1$ unconditionally, so “unitarity of the scattering” is not RH (Remark 6.3).
- C8.** (c) “The continuous term contains $-\frac{1}{4\pi} \int h\varphi'/\varphi$, and φ'/φ contains ξ'/ξ , hence Weil’s prime sum.” **Partly wrong.** The sign convention is fine, but $\varphi'/\varphi(\frac{1}{2} + ir) = -4 \operatorname{Re} \frac{\xi'}{\xi}(1 + 2ir)$ evaluates ζ on the line $\operatorname{Re} s = 1$, not on the critical line. The primes enter with weight $\Lambda(n)/n$ at $2 \log n$ and with a *plus* sign, i.e. as Weil’s prime functional only for the kinked test functions $G = e^{-|v|/2}g(2v) \in \mathcal{C}_{1/2}$, where Weil positivity is automatic (Thms. 6.5, 6.8).
- C9.** [t257u](1) $\partial_\tau \log(y^6|\Delta|) = \pi i E_2^*$; zeros of $E_2^* =$ critical points. **Correct.** (2) i, ρ forced; $E_2(i) = 3/\pi$. **Correct;** also $E_2(\rho) = 2\sqrt{3}/\pi$. (3) not harmonic; maxima possible. **Correct:** $\Delta_{\mathbb{H}} = -6$; ρ is the maximum, i a saddle, no minima possible. Added: the separatrices at i cross at 51.2° , not 90° (Lemma 63’s right angle is special to harmonic functions). (4) valence = Gauss–Bonnet/argument principle. **Correct,** now proved both ways (Thm. 8.2). (5) Δ has no zeros, so no ghosts; ζ zeros only via $E(z, s)$. **Correct.** El Basraoui–Sebbar 2010: verified (Proc. AMS 138 (2010) 2289–2299); it concerns zeros of E_2 , not E_2^* .
- C10.** Name “Ramanujan–Serre derivative”. **In the literature** (Ramakrishnan–Sahu–Singh 2023, Ganash–Sahu 2026 (title); Mathlib). Serre’s 1973 paper was *not* read; attribution of ϑ to it is second-hand (Zagier 2008).

12. SUMMARY OF CLAIMS

#	claim	label
1	$91\sigma_{11}(n) + 600\tau(n) = 691b(n)$, $b = [E_4^3]/720$; congruence is its reduction	[THEOREM], [NUMERICAL] ($n \leq 10^4$)
2	$\Lambda(E_2, s)$: Euler product, FE sign -1 , central pole residue $6/\pi = \text{anomaly}$	[THEOREM], [NUMERICAL]
3	Ramanujan system, Ramanujan–Serre de- rivative, $\partial_k - \vartheta_k = \frac{k}{12}E_2^*$	[CITED], [THEOREM], [NUMERICAL]
4	$E_2^* = \frac{1}{\pi i} \partial_z \log(y^6 \Delta)$; $\partial_z E_2^* = \frac{3i}{2\pi y^2}$; Kro- necker form	[THEOREM], [NUMERICAL]
5	Fix = positive curvature of Petersson bun- dle; gives Hecke, RS, self-adjointness; not Ramanujan	[THEOREM]/[CITED]
6	$ \varphi(\frac{1}{2} + ir) = 1$ unconditionally; zeros = res- onances at $\rho/2$	[THEOREM], [NUMERICAL]
7	Selberg trace formula for $\text{PSL}_2(\mathbb{Z})$ with ex- plicit constants	[CITED-sec] + [NUMERICAL] (10^{-16})
8	Prime side of Selberg = Lemma 64’s PRIME on $G = e^{- v /2}g(2v)$; lower bound only	[THEOREM], [NUMERICAL]
9	Cone theorem: $W > 0$ on $\mathcal{C}_{1/2}$ uncondition- ally; ladder RH $\iff W \geq 0$ on all $\mathcal{C}_\theta$	[THEOREM], [RESTATEMENT]
10	Collective cancellation already at PNT depth (ablations on $\mathcal{C}_{1/2}$)	[NUMERICAL]
11	Horocycle error = explicit formula over ze- ros; RH rate $y^{3/4}$	[CITED], [THEOREM], [NUMERICAL], [RESTATEMENT]
12	Petersson landscape: ρ max, i saddle (51.2°), Poincaré–Hopf $\frac{5}{6}$; valence = Gauss– Bonnet	[THEOREM], [NUMERICAL]
13	$\mathbb{H}$ = moduli of J ; fix = connection on it; Frobenius points; mod 691 $\rightarrow$ boundary	[THEOREM], [ANALOGY]
14	Converse of the ladder at fixed θ ; a spectral problem at $\theta < \frac{1}{2}$	[OPEN]

RH is not proven. Nothing here was sent, posted or submitted.

13. QUESTIONS FOR J.W.

- Q1.** Adopt the answer to [t255u]: the archimedean lift of the 691 congruence is Ramanujan’s identity $91\sigma_{11} + 600\tau = 691b$, and the “size” of the cuspidal residue is measured by the Petersson norm (Thm. 2.1, Remark 2.2)?
- Q2.** Adopt the verdict on [t256u]: the anomaly fix is a genuine positivity (Petersson curvature), but applied to the prime side it reaches only the cone $\mathcal{C}_{1/2}$, where positivity is a prime-number-theorem fact, so it neither implies Lemma 64’s $W_A \geq 0$ nor is circular (Thms. 6.5, 6.8)?
- Q3.** Adopt the θ -ladder ($W \geq 0$ on $\mathcal{C}_\theta$) as a TAA counter next to κ , G_n , τ_n , and run D1 next (0.77) in Lemma 64’s high-precision engine, including the DH and $L(\chi)$ tests on $\mathcal{C}_{1/2}$?
- Q4.** Adopt the name “Ramanujan–Serre derivative” and the two-connection Proposition 4.2; and add to Lemma 63 the remark that the right-angle cross is special to harmonic functions (Petersson saddle at i : 51.2°)?
- Q5.** Include Lemma 67 (five figures) in Volume 4, and add the ablation row “collective cancellation already at PNT depth” to Lemma 64? (Nothing is submitted without your approval.)

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**TAA LEMMA 68: ROTATION VERSUS BOOST.
SPINORS, MULTIPLICATIVITY AND DUALITY FOR Δ AND E_{12}**

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

Date: October 22, 2026.

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA, and *the generalized Riemann hypothesis for $L(\Delta, s)$ is open*. Nothing below is a step toward a proof that we can certify; every RH-equivalent statement is labelled [RESTATEMENT]. Tags: [THEOREM] proved here (elementary) or classical with proof indicated; [CITED] checked at the source (lit/lemma68/verified_refs.md); [CITED-sec] via a named secondary source; [BIB] bibliographic record checked, content standard but not read by us; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (floating point or exact integer arithmetic; not certified unless said).

Questions (J.W.). [t258u] “Can a spinor be defined between the $s = 6$ L -function zeros [of E_{12}] that are not on the critical line and the zeta zeros that are on the critical line?” [t259u] “This congruent mod 691 opposite local signature graph feels like it has something to do with multiplicativity and duality.” [t262u] “Shouldn’t this alternating symplectic pairing like the h^1 of a curve ... relate back to our discussion on the Möbius function, fermion Fock space, and the Euler characteristic alternating sum?”

Answer in brief. (1) *Local spinor* (§2). Normalized Frobenius $A_p = F_p/p^{11/2} \in \mathrm{SL}_2$ acts on the spin- $\frac{1}{2}$ representation of $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{Spin}(2, 1)^0$. For Δ it is elliptic (a rotation; $|t_p| < 2$ strictly, Deligne) with a unique fixed point $z_p \in \mathbb{H}$, which is Lemma 67’s Frobenius point; for E_{12} it is hyperbolic (a boost of rapidity $11 \log p$ on the adjoint $\mathbb{R}^{2,1}$) with two fixed points $-p^{\pm 11/2}$ on $\partial\mathbb{H}$ [THEOREM]. Tempered versus non-tempered, non-unitary principal series [BIB]. The *sign* of $\tau(p)$ is spinor data, invisible in the vector (adjoint) representation [THEOREM]. Mod ℓ the real dichotomy has a precise finite-field analogue, split versus non-split torus (fixed points in $\mathbb{P}^1(\mathbb{F}_\ell)$ or in $\mathbb{P}^1(\mathbb{F}_{\ell^2}) \setminus \mathbb{P}^1(\mathbb{F}_\ell)$): mod 691 *every* Frobenius of Δ is split (or unipotent-type), as for E_{12} , while mod 701 about half are non-split [THEOREM]+[NUMERICAL]. This makes Lemma 67’s “ τ_p moves to the boundary” precise. (2) *Global doublet* (§3). Exactly, $\Lambda(E_{12}, s) = \frac{P(s)}{128\pi^6} \xi_{\mathbb{R}}(s) \xi_{\mathbb{R}}(s-11)$ with $P(s) = \prod_{j \text{ odd} \leq 11} (s-j)$ and $\xi_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$. For $\Psi(s) = (\xi_{\mathbb{R}}(s), \xi_{\mathbb{R}}(s-11))^T$ the functional equation is $\Psi(12-s) = \sigma_x \Psi(s)$ and $\Lambda(E_{12}, s) = \frac{P(s)}{256\pi^6} \Psi^T \sigma_x \Psi$, a hyperbolic quadratic form of signature (1,1) evaluated on the doublet [THEOREM] (checked to 10^{-40}). The “spinor” reading is the chiral (Weyl) pair of a 1+1-dimensional Dirac spinor with parity $= \sigma_x$: precise as algebra, [ANALOGY] as physics. $\mathrm{RH} \iff$ every doublet is centred on $\mathrm{Re} s = 6$ [RESTATEMENT]. There is no zero-by-zero map between Δ and ζ , but $\zeta(s-11)$ is a factor of $L(\Delta \otimes \Delta, s)$, coming from $\Lambda^2 V_\Delta$, the line carrying the symplectic form [THEOREM]. (3) *Multiplicativity and duality do not fix the sign* (§4, central). Every 2-dimensional V carries the equivariant alternating pairing $\Lambda^2 V = \det V$; the equivariant symmetric pairings $V \times V \rightarrow \det V$ are in bijection with $(\mathrm{ad}^0 V)^G$. So the true distinction is not “alternating versus hyperbolic” but *irreducible versus reducible*: Δ admits only the alternating pairing, E_{12} also admits the symmetric hyperbolic one, and the latter is Lemma 66’s involution σ [THEOREM]. The local sign is $\mathrm{sign} \det(2F_p - a_p) = \mathrm{sign}(4p^{11} - a_p^2)$ [THEOREM]. Mod 691 the semisimplification acquires the hyperbolic pairing but the non-split Ribet lattice does not [THEOREM]. The pole order of $L(V \otimes V, s)$ at $s = 12$ counts these pairings (1 for Δ , 2 for E_{12} , Ramanujan’s $\zeta(s-11)^2$), and for Δ the residue is the Petersson norm (checked to 10^{-11}) [THEOREM]+[BIB]+[NUMERICAL]. (4) *Parity* (§5, steering [t262u]). Cup product, fermion anticommutation and the $(-1)^i$ of Euler characteristics are one rule, the Koszul sign of super vector spaces [THEOREM]. $\det(1 - FT) = \mathrm{Str}_{\Lambda V}$, $\det(1 - FT)^{-1} = \mathrm{Tr}_{\mathrm{Sym} V}$; which one is “the” partition function is convention, exchanged by parity shift; what is invariant is *relative* parity, and for pure motives parity = weight mod 2 (canonical): Δ odd (weight 11), $E_{12} = \mathbb{Q}(0) \oplus \mathbb{Q}(-11)$ even (weights 0, 22) [THEOREM]. The Δ -Möbius function μ_Δ (Dirichlet inverse of τ) is the graded trace of the fermionic Fock space over primes with flavour V_Δ [THEOREM]. Mod 691, the mixed super Fock space $\mathrm{Sym}(V_\Delta) \otimes \Lambda(V_{E_{12}})$ has graded trace $\equiv 1$: all coefficients of $\zeta(s)\zeta(s-11)/L(\Delta, s)$ beyond the first are divisible by $691^{\omega(n)}$ [THEOREM]+[NUMERICAL]. This is exactly Ramanujan’s congruence [RESTATEMENT] and, like every supertrace, it sees only the class in K_0 , never the Ribet gluing in Ext^1 [THEOREM].

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1. SOURCES, CONVENTIONS, AND WHAT IS REUSED

$\Delta = q \prod (1 - q^n)^{24} = \sum \tau(n) q^n$, $E_{12} = 1 + \frac{65520}{691} \sum \sigma_{11}(n) q^n$; Hecke-normalized, $a_p(\Delta) = \tau(p)$, $a_p(E_{12}) = 1 + p^{11}$. $L(f, s) = \sum a_n n^{-s} = \prod_p (1 - a_p p^{-s} + p^{11-2s})^{-1}$, so $L(E_{12}, s) = \zeta(s) \zeta(s - 11)$; $\Lambda(f, s) = (2\pi)^{-s} \Gamma(s) L(f, s)$, $\Lambda(f, 12 - s) = \Lambda(f, s)$ for both. V_Δ is Deligne's 2-dimensional ℓ -adic representation (trace $\tau(p)$, determinant χ_ℓ^{11} at $p \neq \ell$; absolutely irreducible) [CITED-sec] (Mazur [7], Thms. 3–4, as recorded in Lemma 66); $V_{E_{12}} = \mathbb{K} \oplus \chi_\ell^{11}$, i.e. $\mathbb{K} \oplus \mathbb{L}^{11}$, $\mathbb{L} = \mathbb{Q}(-1)$. F_p denotes (geometric or arithmetic, it does not matter for traces and determinants) Frobenius at p , $w = 11$, $t_p = a_p/p^{w/2}$. Reused, not redone [TAA-L]: Lemma 64 (collective positivity; the POLE form has signature $(1, 1)$), Lemma 65 (triple (ω, J_p, g_p) , $J_p = (2\pi - a_p)/\sqrt{4p - a_p^2}$; Fock factorization; two $\mathbb{Z}/2$ gradings; $Z(C) = \mathrm{Str}$ over $\mathrm{Sym} H^0 \otimes \Lambda H^1 \otimes \mathrm{Sym} H^2$), Lemma 66 ($J_p \rightarrow$ involution $\bar{\sigma}_p$ mod 691; E_{12} is all ghost; $Q_p^\Delta \equiv (x + y)(x + p^{11}y)$), Lemma 67 (Frobenius point $\tau_p \in \mathbb{H}$; Petersson curvature), Lemma 43 (elliptic centres versus source/sink pairs; Krein signs). Numerics: `num/lemma68_{traces,doublet,parity,rankin,omega,figs}.py`, outputs in `num/168/`; PARI/GP via `cypari2`, `mpmath` at 40 digits.

2. THE LOCAL SPINOR: ROTATION VERSUS BOOST

2.1. Normalized Frobenius in SL_2 . For a 2-dimensional Frobenius with $\mathrm{Tr} F_p = a_p$, $\det F_p = p^w$, put $A_p = p^{-w/2} F_p$ (over $\mathbb{R}$ or $\mathbb{C}$, choosing $p^{w/2} > 0$). Then $\det A_p = 1$ and $\mathrm{Tr} A_p = t_p$. Recall $\mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Spin}(3, \mathbb{C})$, $\mathrm{SU}(2) = \mathrm{Spin}(3)$, $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{Spin}(2, 1)^0$; the defining 2-dimensional

representation is the spin- $\frac{1}{2}$ representation, and the adjoint representation on $\mathfrak{sl}_2 \cong \mathbb{R}^{2,1}$ (Killing form, signature (2, 1)) is the vector representation [BIB].

Theorem 2.1 (Rotation versus boost). (a) (Δ .) $|t_p| < 2$ for every p , strictly. Hence A_p is elliptic: conjugate in $\mathrm{SL}_2(\mathbb{R})$ (in its companion realization) to the rotation $\begin{pmatrix} \cos \theta_p & -\sin \theta_p \\ \sin \theta_p & \cos \theta_p \end{pmatrix}$, $t_p = 2 \cos \theta_p$, with eigenvalues $e^{\pm i\theta_p}$ on the unit circle; on the vector representation it is a rotation by $2\theta_p$ in a compact $\mathrm{SO}(2) \subset \mathrm{SO}(2, 1)^0$.
 (b) (E_{12} .) $t_p = p^{11/2} + p^{-11/2} > 2$. Hence A_p is hyperbolic: conjugate to the boost $\mathrm{diag}(p^{11/2}, p^{-11/2})$, with real eigenvalues; on the vector representation it is a Lorentz boost of rapidity $11 \log p$ (adjoint eigenvalues $p^{11}, 1, p^{-11}$).
 (c) (Spinor data.) The adjoint trace is $t_p^2 - 1$; it does not see the sign of t_p . The sign of $\tau(p)$ is therefore information carried by the spin- $\frac{1}{2}$ representation only; $-A_p$ and A_p have the same image in $\mathrm{SO}(2, 1)^0$.

Proof. (a) Deligne [3][BIB] gives $|\tau(p)| \leq 2p^{11/2}$; equality would force $\tau(p)^2 = 4p^{11}$, impossible since p^{11} is not a square. An element of $\mathrm{SL}_2(\mathbb{R})$ with $|\mathrm{Tr}| < 2$ is elliptic. (b) Arithmetic-geometric mean, $p \geq 2$. The adjoint eigenvalues of $\mathrm{diag}(\lambda, \lambda^{-1})$ are $\lambda^2, 1, \lambda^{-2}$. (c) $\mathrm{Tr} \mathrm{ad}(A) = \mathrm{Tr}(A)^2 - 1$ for $A \in \mathrm{SL}_2$. $\square$

Proposition 2.2 (Satake parameters; tempered versus non-tempered). The normalized Satake parameters at p are $\{e^{i\theta_p}, e^{-i\theta_p}\}$ for Δ (tempered, unitary unramified principal series) and $\{p^{11/2}, p^{-11/2}\}$ for E_{12} , i.e. the local component of E_{12} is the normalized induced representation $\mathrm{Ind}(|\cdot|^{11/2} \otimes |\cdot|^{-11/2})$ of $\mathrm{GL}_2(\mathbb{Q}_p)$, and globally E_{12} corresponds to the isobaric sum $|\cdot|^{11/2} \boxplus |\cdot|^{-11/2}$; in the unitary normalization $L(E_{12}, s + \frac{11}{2}) = \zeta(s + \frac{11}{2})\zeta(s - \frac{11}{2})$ [THEOREM] (identity of Euler products). The local representation is irreducible (the ratio $|\cdot|^{11} \neq |\cdot|^{\pm 1}$) and lies outside both the tempered range and the complementary series ($|\text{exponent}| < \frac{1}{2}$), hence is not unitarizable [BIB] (standard classification of unitary unramified representations of GL_2 , e.g. Bump [1]; not re-read).

2.2. Fixed points: Lemma 67's Frobenius point made exact. The companion matrix $C = \begin{pmatrix} 0 & -p^w \\ 1 & a_p \end{pmatrix}$ acts on $\mathbb{P}^1$ by $z \mapsto -p^w/(z + a_p)$; its fixed points solve $z^2 + a_p z + p^w = 0$, i.e. the zeros of the local form $Q_p(z, 1) = z^2 + a_p z + p^w$ of Lemmas 64–66.

Proposition 2.3 (Fixed points). [THEOREM] (a) Δ : C has a unique fixed point in $\mathbb{H}$, $\tau_p^\Delta = \frac{-\tau(p) + i\sqrt{4p^{11} - \tau(p)^2}}{2}$, Lemma 67's Frobenius point; after normalization $z_p = \tau_p^\Delta / p^{11/2} = e^{i(\pi - \theta_p)}$ lies on the geodesic $|z| = 1$. (b) E_{12} : $z^2 + (1 + p^{11})z + p^{11} = (z + 1)(z + p^{11})$; the two fixed points $-1, -p^{11}$ (normalized $-p^{\mp 11/2}$) lie on $\partial\mathbb{H} = \mathbb{P}^1(\mathbb{R})$, and the axis of the boost is the geodesic joining them. (c) Mod ℓ : the fixed points of $\bar{C}$ lie in $\mathbb{P}^1(\mathbb{F}_\ell)$ iff $d_p = a_p^2 - 4p^w$ is a square mod ℓ (split torus; two points if $d_p \not\equiv 0$, one if $d_p \equiv 0$), and in $\Omega_\ell := \mathbb{P}^1(\mathbb{F}_{\ell^2}) \setminus \mathbb{P}^1(\mathbb{F}_\ell)$ iff d_p is a non-square (non-split torus).

Ω_ℓ is the finite-field counterpart of $\mathbb{H} = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$; for $\mathrm{SL}_2(\mathbb{F}_\ell)$ the split/non-split dichotomy of regular semisimple elements is the counterpart of hyperbolic/elliptic [THEOREM] (elementary; the analogy with $\mathbb{H}$ is [ANALOGY]). “Positive” has no meaning in $\mathbb{F}_\ell$ (Lemma 66), but “fixed point off the boundary” does.

Theorem 2.4 (Mod 691 every Frobenius of Δ is boundary-type). For $p \neq 691$: $d_p \equiv (1 - p^{11})^2 \pmod{691}$, a square; it is $\not\equiv 0$ unless $p \equiv 1 \pmod{691}$. So every $\bar{F}_p$ of $\bar{\rho}_\Delta$ has its fixed points in $\mathbb{P}^1(\mathbb{F}_{691})$ (two of them, -1 and $-p^{11}$, if $p \not\equiv 1$; one, with unipotent-type reduction, if $p \equiv 1$), exactly as for E_{12} over $\mathbb{R}$. For the weight-16 control, the same holds mod 3617. [THEOREM] (Ramanujan's congruence; its weight-16 analogue as in Lemma 66, Prop. “Hecke” [TAA-L])

Numerics [NUMERICAL] (primes $p < 10^4$, 1229 primes). $\max_p |t_p| = 1.96053$ (at $p = 7589$), mean $t_p^2 = 0.98337$ (Sato–Tate predicts 1); $\min_p (p^{11/2} + p^{-11/2}) = 45.277$ ($p = 2$); $\tau(p) < 0$ for

49.9% of p . Mod 691: split 0.99837, non-split 0, $d_p \equiv 0 \pmod{691}$ (0.00163 ($p = 6911, 8293$)). Controls: mod 701 split 0.493, non-split 0.507; mod 13: 0.492/0.427/0.081; mod 19: 0.476/0.471/0.054. Weight 16 ($\Delta_{16} = E_4\Delta$): $\max |t_p| = 1.99562$; mod 3617 split 1.000, non-split 0; mod 3613 0.510/0.490. $|z_p| = 1$ to $2.2 \cdot 10^{-16}$. (For orientation only: mod 23, an exceptional prime for Δ (Mazur [7], footnote 18, citing [15]; [CITED-sec]), the proportions 0.511/0.332/0.157 are visibly non-generic.)

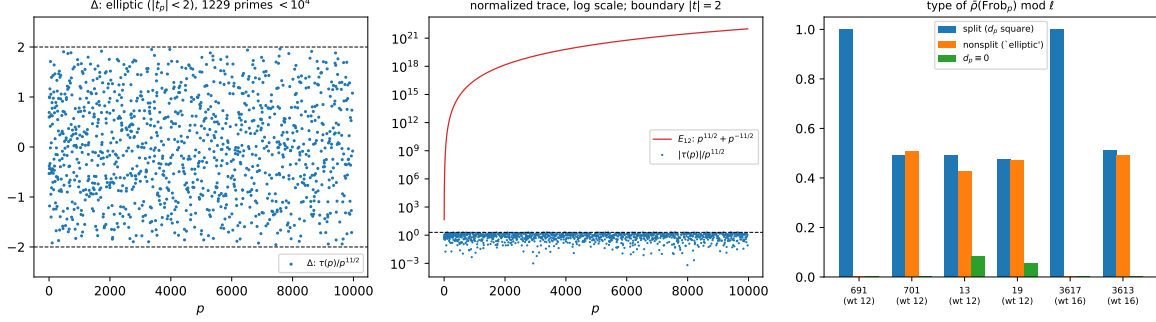


FIGURE 1. Left: $t_p = \tau(p)/p^{11/2}$ for $p < 10^4$, all inside the elliptic band $|t| < 2$. Middle: $|t_p|$ for Δ (blue) versus $p^{11/2} + p^{-11/2}$ for E_{12} (red), log scale; the dashed line is the parabolic boundary $|t| = 2$. Right: finite-field type of $\bar{\rho}(\text{Frob}_p) \bmod \ell$: split (fixed points in $\mathbb{P}^1(\mathbb{F}_\ell)$, the “boundary”), non-split (fixed points in Ω_ℓ , “elliptic”), and $d_p \equiv 0$. At the Eisenstein primes 691 (weight 12) and 3617 (weight 16) the non-split class is empty; at the neighbouring controls it has density about $\frac{1}{2}$.

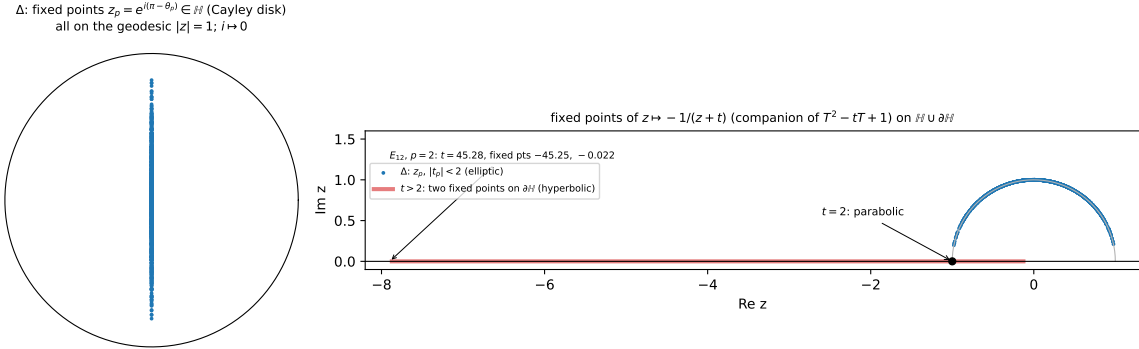


FIGURE 2. Left: the normalized Frobenius points z_p of Δ ($p < 10^4$) in the Cayley disk; they lie on the geodesic $|z| = 1$ (the diameter), distributed by Sato–Tate. Right: fixed points of $z \mapsto -1/(z+t)$ as t varies: for $|t| < 2$ a point of $\mathbb{H}$ on $|z| = 1$ (blue, the Δ data), at $t = 2$ the parabolic boundary point -1 , and for $t > 2$ two points on $\partial\mathbb{H}$ (red). E_{12} sits far out on the red branch already at $p = 2$.

Remark 2.5 (Krein reading, Lemma 43). In Lemma 43 an elliptic equilibrium is a centre with a definite (Krein-signed) energy, and an off-line pair is a source/sink. Here: A_p elliptic preserves the positive form g_p (Lemma 65), so it lies in a compact $\text{SO}(2)$; A_p hyperbolic preserves only the indefinite form $(x+y)(x+p^{11}y)$ and lies in the non-compact $\text{SO}(1,1)$, whose two null lines (the eigenlines $\mathbb{K}, \mathbb{L}^{11}$) are the source and sink directions [THEOREM] (linear algebra) / [ANALOGY] (to Lemma 43’s flows).

3. THE GLOBAL DOUBLET

Correction first (chat claim (A), confirmed): it is E_{12} , not Δ , whose zeros miss $\operatorname{Re} s = 6$ (Lemma 66, Thm. “ E_{12} is all ghost” [TAA-L]). The computed zeros of $\Lambda(\Delta, s)$ (9.2224, 13.9075, 17.4428, ...) lie on $\operatorname{Re} s = 6$; that *all* do is GRH for $L(\Delta, s)$, which is open.

Theorem 3.1 (The doublet factorization). *Let $\xi_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ (so $\xi_{\mathbb{R}}(1-s) = \xi_{\mathbb{R}}(s)$) and $P(s) = (s-1)(s-3)(s-5)(s-7)(s-9)(s-11)$. Then*

$$\Lambda(E_{12}, s) = \frac{P(s)}{128\pi^6} \xi_{\mathbb{R}}(s) \xi_{\mathbb{R}}(s-11).$$

Put $\Psi(s) = \begin{pmatrix} \xi_{\mathbb{R}}(s) \\ \xi_{\mathbb{R}}(s-11) \end{pmatrix}$ and $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\Psi(12-s) = \sigma_x \Psi(s), \quad P(12-s) = P(s), \quad \Lambda(E_{12}, s) = \frac{P(s)}{256\pi^6} \Psi(s)^T \sigma_x \Psi(s).$$

So the functional equation of E_{12} is the off-diagonal swap of the two components, and Λ is the hyperbolic form $2xy$ (signature $(1, 1)$; the same form as the local pole form $(x+y)(x+p^{11}y)$ in null coordinates) evaluated on Ψ . [THEOREM]

Proof. Legendre duplication $\Gamma(s) = \pi^{-1/2}2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})$ and $\Gamma(\frac{s+1}{2}) = \Gamma(\frac{s-11}{2})\prod_{k=0}^5(\frac{s-11}{2} + k) = \Gamma(\frac{s-11}{2})P(s)/2^6$ give $(2\pi)^{-s}\Gamma(s) = \frac{\pi^{-s-1/2}}{128}\Gamma(\frac{s}{2})\Gamma(\frac{s-11}{2})P(s)$; and $\pi^{-s/2}\pi^{-(s-11)/2} = \pi^{-s+11/2}$. The swap: $\xi_{\mathbb{R}}(12-s) = \xi_{\mathbb{R}}(1-(12-s)) = \xi_{\mathbb{R}}(s-11)$ and $\xi_{\mathbb{R}}((12-s)-11) = \xi_{\mathbb{R}}(1-s) = \xi_{\mathbb{R}}(s)$. P is symmetric because $\{12-j\} = \{j\}$ for j odd ≤ 11 and P has even degree. $\square$

Numerics [NUMERICAL] (40 digits). Factorization relative error $\leq 4.8 \cdot 10^{-41}$ at five test points; swap error $\leq 1.7 \cdot 10^{-41}$; $|\Lambda(E_{12}, \rho_n)|$ and $|\Lambda(E_{12}, \rho_n + 11)|$ relative to nearby values $\leq 4.0 \cdot 10^{-39}$ for the first 20 zeros. Independently, PARI’s `lfunmf` on `mfEk(12)` (normalized $a_0 = 1$, hence a factor $65520/691 = 94.819$ against $\zeta(s)\zeta(s-11)$, checked at $s = 14$) has `lfuncheckfeq` = -67 and $|\Lambda| \leq 1.7 \cdot 10^{-7}$ at $\frac{1}{2} + i\gamma_n$, $\leq 1.1 \cdot 10^{-13}$ at $\frac{23}{2} + i\gamma_n$, $n \leq 3$ (limited by the 15-digit γ_n passed in). Poles: P cancels the poles of $\xi_{\mathbb{R}}(s)$ at $s = 1$ and of $\xi_{\mathbb{R}}(s-11)$ at $s = 11$; the poles at $s = 0$ and $s = 12$ survive, so $\Lambda(E_{12}, s)$ has simple poles at $0, 12$ while $\Lambda(\Delta, s)$ is entire. The real zeros $3, 5, 7, 9$ are the remaining zeros of P .

Corollary 3.2 (Zero map and centring). [THEOREM] *Each nontrivial zero $\rho = \beta + i\gamma$ of ζ gives the two zeros ρ (component Ψ_1) and $\rho + 11$ (component Ψ_2) of $\Lambda(E_{12})$, at centred positions $\beta - 6 + i\gamma$ and $\beta + 5 + i\gamma$. The swap $s \mapsto 12 - s$ sends the Ψ_1 -zero ρ to the Ψ_2 -zero $12 - \rho = (1 - \rho) + 11$ at height $-\gamma$; composed with complex conjugation, $s \mapsto 12 - \bar{s}$ sends it to $(1 - \bar{\rho}) + 11$ at height $+\gamma$. The doublet $\{\rho, \rho + 11\}$ has midpoint $\rho + \frac{11}{2}$, which lies on the axis $\operatorname{Re} s = 6$ iff $\beta = \frac{1}{2}$. Hence $\text{RH} \iff$ every E_{12} doublet is centred on E_{12} ’s symmetry axis, at displacement exactly $\pm \frac{11}{2}$ [RESTATEMENT].*

So the chat’s “each ζ zero becomes a pair at $\pm \frac{11}{2} + i\gamma$ ” is right *under RH* (and for every zero verified to lie on the line, e.g. the 20 checked here); unconditionally the displacements are $\beta - 6$ and $\beta + 5$, and the functional-equation partner of ρ is built from $1 - \rho$, which equals $\bar{\rho}$ only on the line.

3.1. How far is the doublet a spinor? *Precise content.* (i) A 2-component object Ψ ; (ii) an involution σ_x implementing the functional equation; (iii) an invariant bilinear $\Psi^T \sigma_x \Psi$ of signature $(1, 1)$, whose isometry group $O(1, 1)$ contains the “boosts” $\Psi \mapsto \operatorname{diag}(f, f^{-1})\Psi$. In $1+1$ dimensions, in the chiral basis, a Dirac spinor is a pair of Weyl components, boosts act as $\operatorname{diag}(e^{\eta/2}, e^{-\eta/2})$, parity acts as $\gamma^0 = \sigma_x$ and swaps the chiralities, and the scalar bilinear is built from σ_x [BIB]. All three structures match [THEOREM] (as algebra). *Where it stops* [ANALOGY]. Nothing acts on s as a Lorentz group: the boost $\operatorname{diag}(f, f^{-1})$ preserves $\Psi_1\Psi_2$ for any function f , so it is a gauge redundancy of the factorization, not a symmetry of E_{12} ; and the “chiralities” $\xi_{\mathbb{R}}(s)$, $\xi_{\mathbb{R}}(s-11)$ are the *same* function translated by the weight 11. The doublet is the global shadow of the local boost (Thm. 2.1(b)): the components are the two Satake lines $\mathbb{K}$ and $\mathbb{L}^{11}$, the swap is $\mathbb{K} \leftrightarrow \mathbb{L}^{11}$ under

duality, and the off-axis displacement $\pm \frac{11}{2}$ is the boost exponent of Prop. 2.2. That identification is [THEOREM] at the level of Euler factors and [ANALOGY] beyond.

No zero-by-zero pairing for Δ . For Δ there is no factorization of $\Lambda(\Delta)$ and we know no map from zeros of $L(\Delta)$ to zeros of ζ ; we do not claim none can exist ([OPEN], low expectation). The precise place where ζ meets Δ is the Rankin–Selberg square (Thm. 4.4 below): $L(V_\Delta \otimes V_\Delta, s) = \zeta(s - 11) L(\text{Sym}^2 \Delta, s)$, the ζ factor being the L -function of $\Lambda^2 V_\Delta = \det V_\Delta \cong \mathbb{L}^{11}$, i.e. of the symplectic form itself. Since $L(\text{Sym}^2 \Delta, s)$ is entire [BIB] (Shimura [13]), every $\rho + 11$ is a zero of $L(V_\Delta \otimes V_\Delta, s)$ [THEOREM] (given that input).

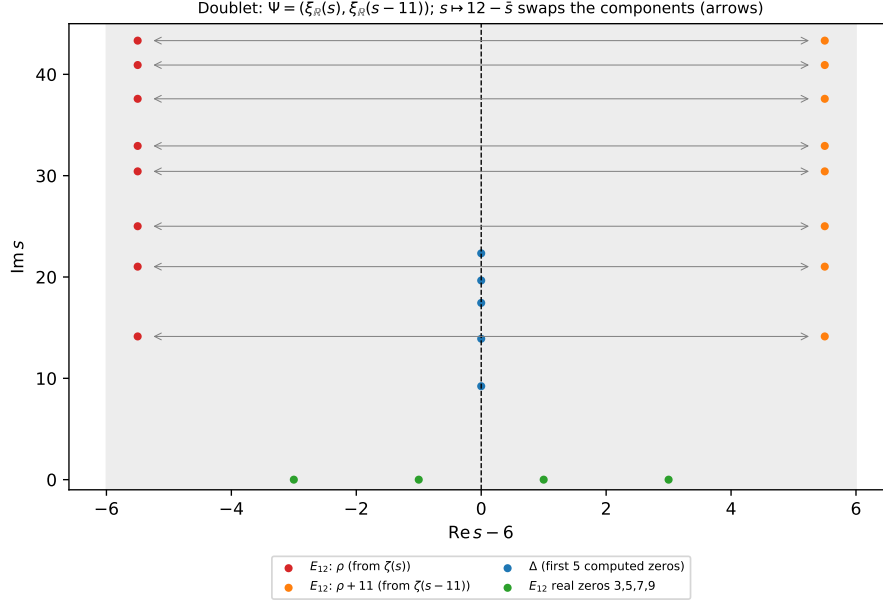


FIGURE 3. Centred coordinates $\text{Re } s - 6$. Red/orange: the doublets $\{\rho, \rho + 11\}$ of $\Lambda(E_{12})$ from the first 12 zeros of ζ , at $\mp \frac{11}{2}$; arrows: the swap $s \mapsto 12 - \bar{s}$ exchanging the components $\xi_{\mathbb{R}}(s)$, $\xi_{\mathbb{R}}(s - 11)$. Blue: the first computed zeros of $\Lambda(\Delta)$, on the axis (Lemma 66). Green: real zeros 3, 5, 7, 9. The grey band is $0 < \text{Re } s < 12$.

4. MULTIPLICATIVITY AND DUALITY DO NOT FIX THE SIGN

J.W.'s [t259u]: Δ and E_{12} share *multiplicativity* (Hecke eigenforms; Euler products of degree 2 with the same shape $1 - a_p X + p^{11} X^2$) and *duality* (the functional equation $s \leftrightarrow 12 - s$ with sign +1, coming from $V^\vee \cong V \otimes \det^{-1}$), they are congruent mod 691, and their local signatures are opposite. So multiplicativity + duality do not determine the sign; this is the sharpest instance of Lemma 64's “where, not mechanism” [THEOREM]. (One refinement: their completed L -functions differ in the *polar part*, $\Lambda(E_{12})$ having poles at 0, 12, the traces of $\mathcal{K}$ and $\mathbb{L}^{11}$; Lemma 66 [TAA-L].) The question is what *does* distinguish them.

4.1. Equivariant pairings on a 2-dimensional representation. Let G be a group, k a field of characteristic $\neq 2$, V a 2-dimensional $k[G]$ -module, $\psi = \det V$. Let $\omega : V \times V \rightarrow \psi$ be the determinant pairing; it is G -equivariant and alternating, and induces $V \cong V^\vee \otimes \psi$.

Theorem 4.1 (Classification of equivariant pairings). [THEOREM] (a) Every G -equivariant bilinear $b : V \times V \rightarrow \psi$ is $b(x, y) = \omega(x, \phi y)$ for a unique $\phi \in \text{End}_G(V)$; b is alternating iff ϕ is scalar,

symmetric iff $\text{Tr } \phi = 0$. Hence

$$\text{Hom}_G(V \otimes V, \psi) = k\omega \oplus (\text{ad}^0 V)^G, \quad \text{ad}^0 V \cong \text{Sym}^2 V \otimes \psi^{-1}.$$

(b) If V is absolutely irreducible, the only equivariant pairings are the multiples of ω (alternating).
(c) If $V = \chi_1 \oplus \chi_2$ with $\chi_1 \neq \chi_2$, then $(\text{ad}^0 V)^G = k\sigma$ with $\sigma = \text{diag}(-1, 1)$, and $b_\sigma(x, y) = \omega(x, \sigma y)$ is the symmetric hyperbolic form $x_1 y_2 + x_2 y_1$ (two null lines χ_1, χ_2 ; over $\mathbb{R}$ signature $(1, 1)$). (d) If V is a non-split extension $0 \rightarrow \chi_1 \rightarrow V \rightarrow \chi_2 \rightarrow 0$ with $\chi_1 \neq \chi_2$, then $\text{End}_G(V) = k$ and again only ω survives.

Proof. (a) ω is nondegenerate, so $b(x, \cdot) = \omega(x, \phi \cdot)$ defines ϕ ; equivariance of b and ω forces $\phi g = g\phi$. b symmetric $\iff \omega(x, \phi y) = \omega(y, \phi x) = -\omega(\phi x, y) \iff \phi \in \mathfrak{sp}(\omega) = \mathfrak{sl}(V)$; alternating $\iff \omega(x, \phi x) = 0 \forall x \iff \phi$ scalar (for 2-dimensional V). Every ϕ splits as scalar + traceless. (b) Schur. (c) $\text{End}_G(V) =$ diagonal matrices. (d) $\phi \in \text{End}_G V$ preserves the unique stable line χ_1 and acts on it and on the quotient by scalars a, b ($\text{Hom}(\chi_i, \chi_j) = 0$ for $i \neq j$). Then $\phi - a$ kills χ_1 , so factors through $\chi_2 \rightarrow V$; its image is a stable line isomorphic to χ_2 or zero, and the unique stable line is $\chi_1 \not\cong \chi_2$; so $\phi = a$. $\square$

Corollary 4.2 (Δ versus E_{12} , and the bridge). *[THEOREM] (with [CITED-sec] for irreducibility). (i) V_Δ is absolutely irreducible, so its only Galois-equivariant pairing is the symplectic $\omega : \Lambda^2 V_\Delta \cong \mathbb{L}^{11}$. (ii) $V_{E_{12}} = \mathbb{K} \oplus \mathbb{L}^{11}$ admits ω and the symmetric hyperbolic pairing b_σ , where σ is Lemma 66's involution $\sigma_p = (2F_p - a_p)/(p^{11} - 1)$; it is the same σ for every p (it is global, a G -endomorphism). (iii) Mod 691: on the semisimplification $\bar{p}_\Delta^{ss} = \mathbb{K} \oplus \omega_{691}^{11}$ the hyperbolic pairing appears; on a non-split (Ribet/Greenberg-Monsky) lattice it does not (Thm. 4.1(d)). “Mod 691 the irreducible piece breaks into two” is true for the semisimplification and false for the glued lattice.*

Proof. (i) Mazur [7], Thm. 3 (Deligne) gives irreducibility over $\mathbb{Q}_\ell$. Absolute irreducibility: complex conjugation c has $\det = \chi_\ell^{11}(c) = -1$, so eigenvalues ± 1 , with eigenlines defined over $\mathbb{Q}_\ell$; $\text{End}_G V$ commutes with c , so sits inside the diagonal algebra $\mathbb{Q}_\ell \times \mathbb{Q}_\ell$, and being a division algebra (Schur) it is $\mathbb{Q}_\ell$. If $V \otimes \bar{\mathbb{Q}}_\ell$ had a stable line, either it is semisimple (then End would be larger) or the stable line is unique, hence $\text{Gal}(\bar{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ -stable, hence defined over $\mathbb{Q}_\ell$: contradiction. (ii),(iii) Thm. 4.1(c),(d); $\omega_{691}^{11} \neq \mathbb{K}$. $\square$

So the chat's sentence “for Δ duality pairs an irreducible piece with itself (alternating), for E_{12} it swaps two pieces (hyperbolic)” needs one correction: *both* carry the alternating ω (every 2-dimensional representation does; this is why both have the same functional equation). What distinguishes E_{12} is the *additional* symmetric hyperbolic pairing, which exists iff $(\text{ad}^0 V)^G \neq 0$, i.e. iff the representation is reducible-and-split. Lemma 66's “ J degenerates into an involution” is this statement at the residual level.

4.2. The local sign.

Proposition 4.3 (Local signature = elliptic/hyperbolic). *[THEOREM] Let F be a Frobenius on a real 2-dimensional V , $\text{Tr } F = a$, $\det F = p^w$, and $\phi_F = 2F - a$ (traceless, $\phi_F^2 = d \cdot 1$, $d = a^2 - 4p^w$). Then $b_F(x, y) = \omega(x, \phi_F y)$ is a symmetric F -semi-invariant form ($b_F(Fx, Fy) = p^w b_F(x, y)$) with $\det \phi_F = -d = 4p^w - a^2$, and b_F is definite iff $\det \phi_F > 0$ (elliptic), of signature $(1, 1)$ iff $\det \phi_F < 0$ (hyperbolic). In the basis $(1, F)$ of the companion model, $b_F(x + yF, x + yF)$ is, up to the factor ± 2 and $y \mapsto -y$, Lemma 64's form $x^2 + axy + p^w y^2$. For Δ , b_F is g_p (Lemma 65's $J_p = \phi_F/\sqrt{-d}$); for E_{12} , $\phi_F = (p^{11} - 1)\sigma$ and b_F is the global hyperbolic b_σ up to scale.*

Proof. ϕ_F commutes with F and is traceless, so b_F is symmetric (Thm. 4.1(a)) and $b_F(Fx, Fy) = \omega(Fx, F\phi_F y) = p^w b_F(x, y)$. For traceless ϕ , $\phi^2 = -\det \phi$; $q(x) = \omega(x, \phi x)$ has discriminant $-4\det \phi$ in any symplectic basis, so it is definite iff $\det \phi > 0$. The companion computation is Lemma 65's [TAA-L]. $\square$

Interpretation. The local positive form exists for Δ at every p , but it is built from F_p : it is Frob_p -invariant, not G -invariant (Cor. 4.2(i) forbids a global symmetric equivariant form). E_{12} 's symmetric form is global and equivariant, and it is hyperbolic. So: *Galois-equivariant symmetric pairings, when they exist, are hyperbolic and signal reducibility; positivity for Δ is local (Frobenius-by-Frobenius) or archimedean (Petersson), never Galois-equivariant [THEOREM] (for these two forms).*

4.3. Pole order, Rankin–Selberg, and the Petersson residue.

Theorem 4.4 (Rankin–Selberg counts the pairings). (a) [THEOREM] Locally, $\sum_{k \geq 0} h_k(\alpha, \beta)^2 X^k = \frac{1 - \alpha^2 \beta^2 X^2}{(1 - \alpha^2 X)(1 - \alpha \beta X)^2(1 - \beta^2 X)}$ (h_k complete symmetric), i.e. $\sum a_n^2 n^{-s} = L(V \otimes V, s) / \zeta(2s - 22)$ with $L(V \otimes V) = L(\text{Sym}^2 V) L(\Lambda^2 V)$ and $L(\Lambda^2 V, s) = \zeta(s - 11)$. (b) E_{12} : $\sum \sigma_{11}(n)^2 n^{-s} = \zeta(s) \zeta(s - 11)^2 \zeta(s - 22) / \zeta(2s - 22)$ (Ramanujan [9][BIB]), so $L(V \otimes V, s)$ has a double pole at $s = 12$: $\dim \text{Hom}_G(V \otimes V, \mathbb{L}^{11}) = 2$ [THEOREM]. (c) Δ : $L(V \otimes V, s) = \zeta(s - 11) L(\text{Sym}^2 \Delta, s)$ with $L(\text{Sym}^2 \Delta, s)$ entire [13][BIB] and $L(\text{Sym}^2 \Delta, 12) \neq 0$ (numerically 0.63179, below), so a simple pole: $\dim = 1$, the alternating ω [THEOREM] (given [BIB]). Its residue is Rankin's

$$\text{res}_{s=12} \sum \tau(n)^2 n^{-s} = \frac{L(\text{Sym}^2 \Delta, 12)}{\zeta(2)} = \frac{3}{\pi} \frac{(4\pi)^{12}}{11!} \langle \Delta, \Delta \rangle > 0$$

(unfolding against $E(z, s)$, residue $3/\pi$; Rankin [10][BIB]).

Numerics [NUMERICAL]. (a) verified symbolically to X^8 (sympy); (b) exact for all $n \leq 3000$; (c) PARI `lfuncreate` for $L(\text{Sym}^2 \Delta, s)$ with Γ -factor $\Gamma_{\mathbb{R}}(s) \Gamma_{\mathbb{R}}(s + 1) \Gamma_{\mathbb{R}}(s - 10)$, weight 23, conductor 1 (`lfuncheckfeq` = -27; the shift -11 gives -2, i.e. wrong), 10^4 coefficients: $L(\text{Sym}^2 \Delta, 12) = 0.631792945721$; `mfpetersson` gives $\langle \Delta, \Delta \rangle = 1.035362056804 \cdot 10^{-6}$ (Lemma 66's value), and the right-hand side times $\zeta(2)$ is 0.631792945728: relative difference $1.1 \cdot 10^{-11}$.

Reading [ANALOGY]/[HEURISTIC]. The line that carries the symplectic form, $\Lambda^2 V_{\Delta} = \mathbb{L}^{11}$, contributes the factor $\zeta(s - 11)$, and its pole has residue $\propto \langle \Delta, \Delta \rangle > 0$. So the Petersson positivity (Lemma 66 H3, Lemma 67 D2: the candidate global g) is visible in an L -function precisely as the residue of the pole attached to ω ; for E_{12} the pole is double and there is no finite norm ($E_{12} \notin L^2$). This is a genuine global positivity, but it is a statement about one residue, not about zeros of $L(\Delta, s)$ [THEOREM]; turning it into a Weil-type positivity for $L(\Delta, s)$ is direction P2.

4.4. The chain in M_{12} .

Proposition 4.5. *For the two Hecke eigenforms of $M_{12}(\text{SL}_2(\mathbb{Z}))$ the following are equivalent, and hold for Δ and fail for E_{12} : (1) square-integrable for the Petersson norm; (2) cuspidal; (3) V_f irreducible; (4) the only Galois-equivariant pairing $V_f \otimes V_f \rightarrow \det$ is alternating; (5) $L(V_f \otimes V_f, s)$ has a simple pole at $s = 12$; (6) every local Frobenius is elliptic (tempered); (7) $\Lambda(f, s)$ is entire. [THEOREM] (by inspection of the two cases, using [CITED-sec] for (3) (and Ribet [11][BIB] for irreducibility of V_f in general) and [BIB] for (5) and (6)'s input, Deligne)*

Not on the list: (8) all zeros of $\Lambda(f, s)$ on $\text{Re } s = 6$. It fails for E_{12} (Lemma 66), and for Δ it is GRH, [OPEN]/[CONJECTURE]. Prop. 4.5 is an enumeration in a 2-dimensional space, not an implication toward (8); in general cuspidality $\Rightarrow$ GRH is the Selberg-class [CONJECTURE], and no item (1)–(7) is known to imply it. Multiplicativity and duality are shared by both columns; the sign is decided by irreducibility, which is the third ingredient in Lemma 64's reading (positivity g) at the local level only. Proposed new Lemma 64 ablation row: “EP + FE + congruent residual data; reducible (hyperbolic, global symmetric pairing) versus irreducible (elliptic, alternating only): zero location opposite.”

5. PARITY: KOSZUL SIGN, FOCK SPACE, EULER FACTORS

J.W.'s [t262u] asks whether the symplectic h^1 -type pairing of Δ is the same structure as the Möbius function, the fermionic Fock space and the alternating Euler-characteristic sum. It is, through one sign rule; but there are *three* different $\mathbb{Z}/2$ gradings in play (prime-side fermion number, cohomological/weight parity, and below in §6 the q -expansion fermion number), and they must not be conflated.

5.1. One sign rule.

Theorem 5.1 (Koszul sign). *[THEOREM] (standard; [8][TAA-L] via Lemma 65 for the cohomological facts) (a) On a Weil cohomology of a smooth projective X of dimension d , $a \cup b = (-1)^{ij} b \cup a$ for $a \in H^i$, $b \in H^j$. Hence the Poincaré pairing $H^d \times H^d \rightarrow H^{2d}$ on the middle degree is $(-1)^d$ -symmetric, and with a Lefschetz class L the pairing $\langle a, b \rangle = \int a \cup L^{d-i} b$ on H^i is $(-1)^i$ -symmetric: alternating for odd i . (b) In the symmetric monoidal category of $\mathbb{Z}/2$ -graded vector spaces with the Koszul rule $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$: the graded-symmetric algebra of an odd space is the exterior algebra (fermions), the categorical trace of an endomorphism is the supertrace $\text{Tr}(\Phi|_{\text{even}}) - \text{Tr}(\Phi|_{\text{odd}})$, and the categorical dimension is the super-dimension. (c) So the sign in (a), the anticommutation of fermion modes, and the $(-1)^i$ of $\chi(X) = \sum (-1)^i b_i$ and of the Lefschetz trace formula are the same symmetry isomorphism, applied to cohomology regarded as a super vector space by degree parity.*

Correction to chat claim (1). “Graded commutativity makes the duality on odd-degree H^i alternating” is precise only in the *middle* degree $i = d$ (for curves, H^1), or for H^i paired with itself through L^{d-i} . For $i \neq d$, Poincaré duality pairs H^i with the different space H^{2d-i} , where “alternating” has no meaning. The rest of (1) is correct.

5.2. Conventions: which side is fermionic.

Proposition 5.2 (Two dual partition functions). *[THEOREM] For any 2-dimensional V with Frobenius F , eigenvalues α, β :*

$$\text{Str}_{\Lambda^\bullet V}(F T^N) = 1 - (\alpha + \beta)T + \alpha\beta T^2 = \det(1 - FT), \quad \text{Tr}_{\text{Sym}^\bullet V}(F T^N) = \det(1 - FT)^{-1},$$

*and their product is 1, the Witten index of $\text{Sym } V \otimes \Lambda V$ (checked symbolically to T^8). For Δ : the 0-, 1-, 2-fermion states give 1, $-\tau(p)$, $\det F = p^{11}$, and the top state $\Lambda^2 V_\Delta \cong \mathbb{L}^{11}$ is exactly the line in which the symplectic form ω takes values (chat claim (2): **correct**).*

Convention. With the standard definition $L(M, s) = \prod_p \det(1 - F_p p^{-s} | M)^{-1}$, the L -function of a single motive is the *bosonic* trace $\text{Tr}_{\text{Sym } M}$; thus $L(\Delta, s)$ is bosonic and $1/L(\Delta, s)$ fermionic. For a variety, $\zeta(X, s) = \prod_i L(h^i X, s)^{(-1)^i}$, so odd-degree pieces enter *inverted*, as Str_Λ , in the numerator: this is Lemma 65's $Z(C, T) = P_1(T)/((1-T)(1-qT))$ [TAA-L]. Δ is fermionic in this second sense: V_Δ occurs in H^{11} of the Kuga–Sato variety (Deligne [2], Scholl [12], [BIB]), an odd degree, so in the zeta function of that variety it contributes $\det(1 - F_p p^{-s} | V_\Delta)$ to the numerator. *What is invariant:* not the absolute assignment (a global parity shift Π exchanges Sym and Λ and inverts every factor; Lemma 65, Thm. “two Fock spaces”(d)) but the *relative* parity of pieces inside one zeta function.

Proposition 5.3 (Parity = weight mod 2). *[THEOREM] (given [BIB] inputs) A pure motive of weight w is realized in H^w of some variety (up to Tate twist $\mathbb{Q}(-1) = h^2(\mathbb{P}^1)$, which raises weight and degree by 2), so its degree parity is $w \bmod 2$, independent of choices. $h(\Delta)$ (Scholl's motive) has rank 2, weight 11, Hodge types $(11, 0), (0, 11)$: odd. $h(E_{12}) = \mathbb{Q}(0) \oplus \mathbb{Q}(-11)$ has weights 0 and 22, Hodge types $(0, 0), (11, 11)$: both even. A pure motive of weight w carries a polarization $M \otimes M \rightarrow \mathbb{Q}(-w)$ that is $(-1)^w$ -symmetric: alternating for Δ (this is ω); E_{12} is not pure and has no polarization of a single weight, only the swap duality $M^\vee(-11) \cong M$, $\mathbb{Q}(0) \leftrightarrow \mathbb{Q}(-11)$.*

Caveat on degree placement [BIB]. “Even” for E_{12} is canonical by weight, *not* by cohomological degree: in the Eichler–Shimura isomorphism $H^1(\text{SL}_2(\mathbb{Z}), \text{Sym}^{10} \mathbb{C}) \cong S_{12} \oplus \overline{S}_{12} \oplus \mathbb{C} E_{12}$ the Eisenstein class sits in degree 1, next to Δ . So “ E_{12} is bosonic” is a statement about weights (Deligne's theory

of weights makes it intrinsic), and chat claim (4) holds in that sense only. On the level of Hodge–Riemann: for the pure weight-11 Hodge structure of Δ , $i^{p-q}Q(x, \bar{x}) > 0$ on $H^{11,0}$, and up to a nonzero constant this form is the Petersson norm ([BIB]; standard via Eichler–Shimura, constant not checked by us). That is the symplectic-plus-positive structure (Lemma 65’s (ω, J, g)) at the archimedean place, globally.

5.3. The Möbius function of Δ and Spector’s fermions.

Proposition 5.4 (μ_Δ as a fermionic Fock space over primes). *[THEOREM] Let μ_Δ be multiplicative with $\mu_\Delta(p) = -\tau(p)$, $\mu_\Delta(p^2) = p^{11}$, $\mu_\Delta(p^k) = 0$ for $k \geq 3$. Then $\sum \mu_\Delta(n)n^{-s} = 1/L(\Delta, s)$ and $\mu_\Delta * \tau = \delta$. It is the graded trace of $\bigotimes_p \Lambda^\bullet(V_{\Delta,p})$ (two fermion modes per prime, the Satake lines, energies $\log p$), with sign $(-1)^{\# \text{fermions}}$. For $V = \mathbb{K}$ this is Spector’s $\mu = (-1)^F$ [14][TAA-L]; for V_Δ the prime fermions carry a 2-dimensional “flavour” and the 2-fermion state at p carries the symplectic line $\mathbb{L}^{11}$.*

Numerics [NUMERICAL]: $\mu_\Delta * \tau = \delta$ exactly for $n \leq 5000$.

Relation to chat claim (3), without conflation. Locally at one prime the two pictures coincide: $\det(1 - F_p T)$ is both the prime-side fermion supertrace and the cohomological Λ -supertrace of V_Δ [THEOREM]. Globally they differ exactly as in Lemma 65: the prime-side space $\bigotimes_p \Lambda(V_{\Delta,p})$ is a restricted tensor product over primes (Thm. “Fock factorization”), whereas a cohomological Fock space on whose global H^1 Frobenius would act with eigenvalues the zeros of $L(\Delta, s)$ is [CONJECTURE] (Deninger-type), exactly as for ζ . The bridge is a Lefschetz formula, which exists for curves (Lemma 65 [THEOREM]) and, for Δ , prime by prime on the Kuga–Sato variety over $\mathbb{F}_p$ (Deligne [BIB]), not as a global trace formula over $\mathbb{Q}$ ([OPEN]). The two gradings are related only through graded traces, never state by state [TAA-L].

5.4. Fermion–boson confusion mod 691.

Theorem 5.5 (Graded cancellation mod 691). *[THEOREM] Let $c = \mu_\Delta * \sigma_{11}$ and $c' = \tau * \mu_E$, i.e. the Dirichlet coefficients of $\zeta(s)\zeta(s-11)/L(\Delta, s)$ and of $L(\Delta, s)/(\zeta(s)\zeta(s-11))$. These are the graded traces of the mixed super Fock spaces $\Lambda(V_\Delta) \otimes \text{Sym}(V_{E_{12}})$ and $\text{Sym}(V_\Delta) \otimes \Lambda(V_{E_{12}})$. Then $c(1) = c'(1) = 1$, c, c' are multiplicative, and $691^{\omega(n)} \mid c(n), c'(n)$ for all n . Each of these statements is equivalent to Ramanujan’s congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ for all n [RESTATEMENT].*

Proof. Locally $1 - \tau(p)T + p^{11}T^2 \equiv (1 - T)(1 - p^{11}T) \pmod{691}$ (chat claim (5): **correct**); both have constant term 1, so their ratio is a power series in $\mathbb{Z}[[T]]$ congruent to 1, i.e. every coefficient of T^e , $e \geq 1$, is divisible by 691. Multiplicativity gives $691^{\omega(n)}$. Conversely $c \equiv \delta$ gives $\sigma_{11} \equiv \tau * \delta = \tau$. $\square$

Numerics [NUMERICAL] ($n \leq 10^4$): all $c(n), c'(n)$ ($n \geq 2$) divisible by 691, $v_{691}(c(n)) \geq \omega(n)$ always, with equality for 99.96% of n ; $c(2) = 3 \cdot 691$, $c(3) = 256 \cdot 691$, $c(4) = 6147 \cdot 691$.

Proposition 5.6 (What the supertrace cannot see). *[THEOREM] Every graded trace above is a function of the class of the representation in the Grothendieck group K_0 (characteristic polynomials are additive in short exact sequences). Mod 691, $[\bar{\rho}_\Delta] = [\mathbb{K}] + [\omega_{691}^{11}]$ in K_0 for every lattice (Brauer–Nesbitt), so the virtual super-object $\bar{V}_\Delta \ominus (\mathbb{K} \oplus \omega_{691}^{11})$ has class 0 and graded trace 1. The Ribet/Greenberg–Monsky extension class in $\text{Ext}^1(\omega_{691}^{11}, \mathbb{K})$, which produces the ω^{-11} class-group element (Lemma 66 [TAA-L]), is invisible to every supertrace and to every Witten index.*

So “fermion–boson confusion mod 691” is precise as follows [ANALOGY] (qualified): mod 691 the odd-weight object V_Δ and the even-weight object $\mathbb{K} \oplus \mathbb{L}^{11}$ have the same class in K_0 , so no Euler-characteristic-type invariant distinguishes odd from even placement; but they are *not* isomorphic on the glued lattice, and the gluing is exactly the arithmetic content (class group). Parity (weight mod 2) is not even defined mod 691 for these residual representations, since $\omega_{691}^{11} = \omega_{691}^{11+690k}$ forgets the weight.

6. LOWER-WEIGHT EISENSTEIN SERIES, THE ANOMALY, AND THE RAMANUJAN–SERRE DERIVATIVE

J.W.’s [t263u]: how does the fermion–boson confusion relate to E_4, E_6 , the quasi-modular anomaly of E_2 , its correction E_2^* , and the Ramanujan–Serre derivative? Notation as in Lemma 67: $\theta = q d/dq$, $\vartheta_k f = \theta f - \frac{k}{12} E_2 f$, $E_2^* = E_2 - \frac{3}{\pi y}$, and $\partial_k - \vartheta_k = \frac{k}{12} E_2^*$ (Lemma 67 [TAA-L]).

6.1. Over $\mathbb{Q}$ the Eisenstein and cusp lines split; over $\mathbb{Z}$ they are glued with index 691.

Theorem 6.1 (Integral gluing). *[THEOREM] Let $M_{12}(\mathbb{Z})$ be the forms in $M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ with integral q -expansion. Then $M_{12}(\mathbb{Z}) = \mathbb{Z}E_4^3 \oplus \mathbb{Z}\Delta$, and the primitive integral vectors on the two Hecke eigenlines are $691E_{12} = 691 + 65520 \sum \sigma_{11}(n)q^n$ (primitive since $\gcd(691, 65520) = 1$) and Δ . In the basis (E_4^3, Δ) , $691E_{12} = 691E_4^3 - 432000\Delta$, so*

$$[M_{12}(\mathbb{Z}) : \mathbb{Z}691E_{12} \oplus \mathbb{Z}\Delta] = 691.$$

Equivalently (Lemma 66, Prop. “Hecke” [TAA-L]), the Hecke algebra is $\mathbb{T}_{12} = \{(a, b) \in \mathbb{Z}^2 : a \equiv b \pmod{691}\}$ (Eisenstein and cuspidal eigenvalue), with congruence module $\mathbb{Z}^2/\mathbb{T}_{12} \cong \mathbb{Z}/691$. Also $691E_{12} = 441E_4^3 + 250E_6^2 = 691E_6^2 + 762048\Delta$ and $E_4^3 - E_6^2 = 1728\Delta$; mod 691, $762048 \equiv 65520 \equiv 566$, which is Ramanujan’s congruence again.

Proof. $E_4^3 = 1 + 720q + \dots$ and $\Delta = q + \dots$; any $f \in M_{12}(\mathbb{Z})$ equals $a_0E_4^3 + (a_1 - 720a_0)\Delta$ with integer coefficients, so $\{E_4^3, \Delta\}$ is a $\mathbb{Z}$ -basis. The q^1 coefficient of $691E_{12}$ is $65520 = 691 \cdot 720 - 432000$. The determinant of $\begin{pmatrix} 691 & -432000 \\ 0 & 1 \end{pmatrix}$ is 691. The other identities: compare q^0, q^1 (two-dimensional space). $\square$

Numerics [NUMERICAL]: all three expressions for $691E_{12}$ and $E_4^3 - E_6^2 = 1728\Delta$ agree coefficientwise to q^{400} ; $441 \cdot 1728 = 762048$. The general notion (congruence module, Eisenstein ideal) is Mazur’s [6][BIB]; the computation here is self-contained.

Chat claim (1): correct, now with the precise index. The “bosons” E_4, E_6 and their monomials span $M_{12}(\mathbb{Z})$; the “fermion” Δ is the difference $(E_4^3 - E_6^2)/1728$ of two bosonic monomials; the Eisenstein line E_{12} needs the denominator 691. Over $\mathbb{Q}$ (or $\mathbb{R}$) the two eigenlines split, and the splitting is Petersson-orthogonal (Hecke self-adjointness, i.e. the Petersson metric whose curvature is E_2^* , Lemma 67 [TAA-L]; [16][CITED] for self-adjointness); over $\mathbb{Z}$ they do not split, with index 691; mod 691 they become congruent. This is the cleanest form of the “confusion”: *the archimedean metric separates boson and fermion, the integral lattice glues them, and the residue field confuses them* [THEOREM] (each clause) / [HEURISTIC] (as a slogan).

6.2. Signed products: Δ as a fermionic q -Fock space.

Proposition 6.2 (Product identities). *[THEOREM] (a) $\prod_{n \geq 1} (1 - q^n) = \mathrm{Str}_{\Lambda(\oplus_n \mathbb{C}e_n)}(q^N) = \sum_{\lambda \text{ distinct parts}} (-1)^{\ell(\lambda)} q^{|\lambda|} = \sum_{k \in \mathbb{Z}} (-1)^k q^{k(3k-1)/2}$ (Euler’s pentagonal theorem; Franklin’s sign-reversing involution is an explicit boson-free “supersymmetric” pairing of all non-pentagonal states [BIB]). (b) $\Delta/q = \prod (1 - q^n)^{24}$ is the graded trace of 24 such fermion species, and $q/\Delta = \prod (1 - q^n)^{-24} = \sum p_{24}(n)q^n$ is the partition function of 24 bosonic oscillator species ($p_{24} = 1, 24, 324, 3200, 25650, 176256, 1073720, \dots$). (c) Their product is 1 (Witten index of 24 boson + 24 fermion species).*

Numerics [NUMERICAL]: pentagonal identity to q^{400} ; $\Delta \cdot (1/\Delta) = 1$ to q^{400} . *Labels and the parallel with μ .* The identities are [THEOREM]. The reading of $1/\eta^{24}$ as the oscillator partition function of the 24 transverse bosons of the bosonic string is standard physics [ANALOGY] (not sourced here). The parallel with Spector is exact in form, $\prod_n (1 - q^n) \leftrightarrow \prod_p (1 - p^{-s}) = 1/\zeta$, but the modes differ: *additive* modes n with weight q^n versus *multiplicative* modes p with weight p^{-s} . This q -side fermion number is a *third* $\mathbb{Z}/2$ grading, distinct from the prime-side grading of Prop. 5.4 and from the weight parity of Prop. 5.3; the Hecke multiplicativity of τ is not visible in it. *Correction to chat claim (2):* Euler’s pentagonal theorem gives the signs of $\prod (1 - q^n)$, not of $\tau(n)$; the signs of $\tau(n)$ (about half negative, §2) come from the 24th power and are governed by Sato–Tate, not by any pentagonal rule.

Proposition 6.3 (Discriminant as a squared Vandermonde; η^{12} is the antisymmetric root). *[THEOREM] For the lattice $\mathbb{Z} + \mathbb{Z}\tau$, with $g_2 = \frac{4\pi^4}{3}E_4$, $g_3 = \frac{8\pi^6}{27}E_6$ and e_1, e_2, e_3 the roots of $4x^3 - g_2x - g_3$ (the values of $\wp$ at the half-periods):*

$$(2\pi)^{12}\Delta = g_2^3 - 27g_3^2 = 16 \prod_{i < j} (e_i - e_j)^2, \quad \eta^{12} = \frac{(e_1 - e_2)(e_1 - e_3)(e_2 - e_3)}{16\pi^6}$$

with $e_1 = \frac{\pi^2}{3}(\vartheta_3^4 + \vartheta_4^4)$, $e_2 = \frac{\pi^2}{3}(\vartheta_2^4 - \vartheta_4^4)$, $e_3 = -\frac{\pi^2}{3}(\vartheta_2^4 + \vartheta_3^4)$ (Jacobi thetas). η^{12} transforms under $\mathrm{SL}_2(\mathbb{Z})$ with the character $T \mapsto -1$, $S \mapsto -1$, which is the sign character of $\mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{SL}_2(\mathbb{F}_2) \cong S_3$ permuting the e_i (both T and S map to transpositions).

Proof. $g_2^3 - 27g_3^2 = \frac{64\pi^{12}}{27}(E_4^3 - E_6^2) = 4096\pi^{12}\Delta$. The monic cubic $x^3 - \frac{g_2}{4}x - \frac{g_3}{4}$ has discriminant $\frac{g_2^3 - 27g_3^2}{16}$. $e_1 - e_2 = \pi^2\vartheta_4^4$, $e_1 - e_3 = \pi^2\vartheta_3^4$, $e_2 - e_3 = \pi^2\vartheta_2^4$, and $\vartheta_2\vartheta_3\vartheta_4 = 2\eta^3$. $\eta(\tau + 1) = e^{\pi i/12}\eta$, $\eta(-1/\tau) = \sqrt{-i\tau}\eta$, so $\eta^{12} \mapsto -\eta^{12}$, $-\tau^6\eta^{12}$. $\square$

Numerics [NUMERICAL] (40 digits, three sample τ): all identities, the cubic, and both transformation signs to $\leq 1.2 \cdot 10^{-39}$. *Chat claim (2')*: constants **correct**. “Slater-determinant shape”: the Vandermonde is $\det(e_i^{j-1})$, the Slater determinant of three fermions in the orbitals $1, x, x^2$ evaluated at e_1, e_2, e_3 [THEOREM] (as algebra) / [ANALOGY] (as physics). The precise fermionic content is the last sentence of the proposition: Δ is the square of an S_3 -antisymmetric object, exactly as the symplectic $\omega \in \Lambda^2$ is antisymmetric and its “square” (the norm g_p) symmetric.

6.3. The Ramanujan–Serre flow: bosons move, Δ is conserved.

Theorem 6.4 (Δ is the Hamiltonian of Ramanujan’s system). *[THEOREM] $\theta \log \Delta = E_2$ (from the product), hence $\vartheta_{12}\Delta = 0$. With $\vartheta_4E_4 = -\frac{1}{3}E_6$, $\vartheta_6E_6 = -\frac{1}{2}E_4^2$ (Ramanujan [9], Lemma 67 [TAA-L]) and ϑ a derivation of $M_* = \mathbb{C}[E_4, E_6]$, the restriction of ϑ to the generators is the vector field*

$$\dot{x} = -\frac{1}{3}y = \frac{\partial H}{\partial y}, \quad \dot{y} = -\frac{1}{2}x^2 = -\frac{\partial H}{\partial x}, \quad H(x, y) = \frac{x^3 - y^2}{6} = 288\Delta,$$

i.e. the Hamiltonian vector field of 288Δ for the symplectic form $dE_4 \wedge dE_6$. In particular Δ is conserved ($\vartheta\Delta = 0$), and the “bosons” E_4, E_6 flow along its level sets $E_4^3 - E_6^2 = \text{const}$.

Numerics [NUMERICAL]: $12\vartheta_{12}\Delta = 0$, $12\vartheta_4E_4 = -4E_6$, $12\vartheta_6E_6 = -6E_4^2$, $(E_4^3 - E_6^2)/6 = 288\Delta$, exactly to q^{400} . *Chat claim (3)* first half: **correct**, and sharper: Δ is not only flat for ϑ , it generates the flow. “Bosons rotate” should read “bosons flow along the level curves of the discriminant”; those curves are affine cuspidal-type cubics $x^3 - y^2 = c$, not circles, so “rotate” is [ANALOGY]. Flatness is for the *holomorphic* (Ramanujan–Serre) connection only: for the Chern connection of the Petersson metric, $\partial_{12}\Delta = \vartheta_{12}\Delta + E_2^*\Delta = E_2^*\Delta \neq 0$ (Lemma 67’s two-connection proposition [TAA-L]). The difference is the anomaly correction, i.e. the curvature, i.e. the positivity.

6.4. The anomaly as a Casimir-type log-derivative.

Proposition 6.5. *[THEOREM] Let $Z(q) = 1/\Delta = q^{-1} \prod (1 - q^n)^{-24}$. Then $-\theta \log Z = E_2$, with constant term 1. Writing $Z = \mathrm{Tr} q^{L_0 - c/24}$ for 24 free oscillator species with $c = 24$, the constant term of $-\theta \log Z$ is $c/24 = 1$, and the rest is minus the thermal expectation of L_0 . Z has modular weight -12 ; its log-derivative is therefore quasi-modular, with anomaly $\frac{12}{2\pi i}c(\tau + d)$ inherited from the weight. The modular combination is $E_2^* = \frac{1}{\pi i}\partial_\tau \log(y^6|\Delta|)$ (Lemma 67), i.e. $-\frac{1}{2\pi i}\partial_\tau \log Z_{\mathrm{inv}}$ with $Z_{\mathrm{inv}} = (y^6|\Delta|)^{-2} = y^{-12}|\eta|^{-48}$.*

Source check for the physics reading. Dijkgraaf [4][CITED] (arXiv text read): torus partition functions are traces $\mathrm{Tr} q^{L_0 - c/24} \bar{q}^{L_0 - c/24}$ (his (3.20)); for the free boson the propagator is $-\wp(z) + \frac{\pi^2}{3}E_2^*$ and the non-holomorphic τ_2 -dependence comes “because of the zero mode contribution” (his (4.44)–(4.48)); E_2 ’s modular anomaly is his (4.57), and quasi-modular forms are credited to Kaneko–Zagier

[5][CITED-sec]. So “the $-3/(\pi y)$ correction is the zero-mode term that restores modular invariance” is [CITED-sec] (Dijkgraaf, for the free boson). “ E_2 is the vacuum/Casimir energy $-c/24$ with $c = 24$ ” is *our* reading of the elementary Prop. 6.5; Dijkgraaf does not state it in these words: [ANALOGY], standard lore, not sourced. *Chat claim (3)* second half: identity [THEOREM], physics [ANALOGY] with the partial source above.

6.5. Weights 4–10 and 14: no fermion, no denominator.

Proposition 6.6. [THEOREM] $\dim S_k = 0$ for $k = 4, 6, 8, 10, 14$ and $S_{12} = \mathbb{C}\Delta$ [16][CITED]. The numerators of $B_k/2k$ are $-1, 1, -1, 1, -691, 1, -3617$ for $k = 4, \dots, 16$; so $E_k \in \mathbb{Z}[[q]]$ with $a_0 = 1$ exactly for $k = 4, 6, 8, 10, 14$ in that range. Thus weight 12 is the first weight with a cusp form (the first odd-weight motive at level one, weight 11) and the first with an Eisenstein denominator (691), and $k = 14$ is the next case with neither.

Numerics [NUMERICAL]: for even $k \leq 1000$, the k with $\text{numerator}(B_k/2k) = \pm 1$ are exactly $\{2, 4, 6, 8, 10, 14\}$, which are exactly the k with $\dim S_k = 0$. We do not claim this coincidence for all k from a source ([OPEN] for us; we believe it classical but did not locate a reference). The mechanism behind “denominator only when there is a cusp form to be congruent to” is the Herbrand–Ribet/Eisenstein-ideal picture of Lemma 66 [TAA-L]; for $k = 12$ it is Thm. 6.1. *Chat claim (4)*: **correct** (“first fermion” in the weight-parity sense of Prop. 5.3).

6.6. Synthesis for [t263u]. [HEURISTIC]/[ANALOGY] except where tagged. (i) The confusion is integral and residual: Thm. 6.1 (index 691) and Thm. 5.5. (ii) What separates the two lines is archimedean: Hecke self-adjointness for the Petersson metric, whose curvature is the anomaly correction E_2^* ; Δ has finite norm, E_{12} does not [THEOREM]. (iii) The Ramanujan–Serre derivative is the holomorphic connection that *forgets* the metric: for it Δ is flat and in fact is the Hamiltonian (Thm. 6.4), so the holomorphic world cannot see the size of Δ ; only $\partial_k = \vartheta_k + \frac{k}{12}E_2^*$ does. (iv) The “fermionic” sign structures attached to Δ are three: q -side (η -product, pentagonal), S_3 -antisymmetry of η^{12} (Vandermonde), and weight parity / symplectic ω . Only the last is the h^1 -type structure of Lemma 65; the first two are properties of the particular product formula and do not transfer to L -functions [THEOREM] (they are not Hecke-equivariant). None of this bears on zeros of $L(\Delta, s)$ or ζ .

7. SYNTHESIS: WHAT THE FOUR QUESTIONS COME TO

	Δ	E_{12}
local Frobenius $A_p \in \text{SL}_2$	elliptic: rotation, $ t_p < 2$ [THEOREM]	hyperbolic: boost, rapidity $11 \log p$ [THEOREM]
fixed points	one, $z_p \in \mathbb{H}$ (Lemma 67’s τ_p)	two, $-p^{\pm 11/2} \in \partial \mathbb{H}$
Satake / local rep.	tempered, unitary	$\text{Ind} \cdot ^{\pm 11/2}$, not unitarizable [BIB]
equivariant pairings	ω only (alternating)	ω and hyperbolic b_σ [THEOREM]
pole of $L(V \otimes V)$ at 12	simple, residue $\propto \langle \Delta, \Delta \rangle > 0$	double; $E_{12} \notin L^2$
weight parity	odd (11): “fermionic”	even (0, 22): “bosonic”
global object	no factorization; GRH [OPEN]	doublet Ψ , FE = σ_x , $\Lambda \propto \Psi^T \sigma_x \Psi$ [THEOREM]
zeros mod 691	computed ones on $\text{Re } s = 6$ split / unipotent; same K_0 class	none on $\text{Re } s = 6$ [THEOREM] split

Answer to [t258u]. A spinor is defined, precisely, *locally*: Frobenius in the spin- $\frac{1}{2}$ representation of SL_2 , rotation for Δ , boost for E_{12} . *Globally* the doublet of E_{12} is an exact 2-component object with the FE as σ_x ; its components are the two Satake lines and its displacement $\pm \frac{11}{2}$ is the boost exponent. The “spinor” name is [ANALOGY] for the global object; there is no comparable pairing for Δ ’s zeros. *Answer to [t259u].* Multiplicativity and duality are identical for the two; the sign

is decided by irreducibility, equivalently by the absence of a Galois-equivariant symmetric pairing, equivalently (for these two) by cuspidality and square-integrability; mod 691 the semisimplification acquires the hyperbolic pairing while the glued lattice does not. *Answer to [t262u]*. Yes, by the Koszul rule, with the corrections on middle degree and on which side is fermionic; parity = weight mod 2 is the invariant notion. *Answer to [t263u]*. The confusion is integral (index 691) and residual; the anomaly correction is the archimedean metric that separates the two lines; the Ramanujan–Serre derivative is blind to it (Δ is its conserved Hamiltonian).

Circularity and metaphor audit. No statement here assumes RH or GRH. “Spinor”, “boost”, “fermion”, “boson”, “Casimir” and “Slater” are used only where an exact algebraic statement stands behind them (Thms. 2.1, 3.1, 5.1, Props. 5.2, 6.2, 6.3, 6.5); their physical readings are [ANALOGY]. Prop. 4.5 is an enumeration, not an implication toward GRH. *RH is not proven. GRH for $L(\Delta, s)$ is open.*

8. CORRECTIONS TO GROK BOT’S CHAT CLAIMS

- K1. (A) “ E_{12} , not Δ , has zeros off $\operatorname{Re} s = 6$; Δ ’s computed zeros are on the line; GRH open”: **correct**.
- K2. (B) Spinor/elliptic/hyperbolic: **correct**, with precisions: $|t_p| < 2$ is strict (non-square p^{11}); the vector representation sees rotation angle $2\theta_p$ and loses the sign of $\tau(p)$; “mod 691 they coincide” means equal characteristic polynomials over $\mathbb{F}_{691}$ (the real elliptic/hyperbolic dichotomy has no meaning there; its finite-field analogue split/non-split does, and all of Δ ’s Frobenii become split or unipotent-type). The local component of E_{12} is irreducible and not unitarizable [BIB]. Lemma 67’s “moves to the boundary” is now Prop. 2.3 and Thm. 2.4 (no longer only [ANALOGY]).
- K3. (C) “Each ζ zero $\frac{1}{2} + i\gamma$ gives $\pm\frac{11}{2} + i\gamma$ ”: **correct under RH**; unconditionally $\beta - 6$ and $\beta + 5$. “The FE swaps the two copies”: it maps $\rho \mapsto (1 - \rho) + 11$ at height $-\gamma$; at the same height it is $s \mapsto 12 - \bar{s}$. “No natural pairing between Δ ’s zeros and ζ ’s”: no zero-by-zero map is known, but $\zeta(s - 11)$ divides $L(\Delta \otimes \Delta, s)$ through $\Lambda^2 V_\Delta$ (§3.1).
- K4. (D) “Duality pairs an irreducible piece with itself (alternating) vs swaps two pieces (hyperbolic)”: **imprecise**. Both carry the alternating ω ; E_{12} *additionally* carries a symmetric hyperbolic equivariant pairing, which exists iff $(\operatorname{ad}^0 V)^G \neq 0$ (Thm. 4.1). The caution “equivariant pairings on a 2-dim representation are automatically alternating” is **false for reducible V** . “Mod 691 the irreducible piece breaks into two”: true for the semisimplification, false for the non-split lattice. The two L -functions also differ in their polar parts.
- K5. (1) Koszul: **correct**, but “odd-degree duality is alternating” only for middle degree or with L^{d-i} .
- K6. (2) $\det(1 - FT) = \operatorname{Str}_\Lambda$, top state = $\mathbb{L}^{11}$: **correct**. Convention: $L(\Delta, s)$ itself is the bosonic Sym-trace; Δ is “fermionic” as an odd-weight piece of ζ (Kuga–Sato); only relative parity is invariant.
- K7. (3) $1/\zeta$ = fermionic Fock space: **correct** (Lemma 65); the Δ version is μ_Δ with 2-flavour prime fermions; local agreement, global bridge only conjectural.
- K8. (4) Hodge types and $E_{12} = \mathbb{Q}(0) \oplus \mathbb{Q}(-11)$: **correct**; “even” is canonical by weight, not by cohomological degree (Eichler–Shimura places E_{12} ’s class in degree 1).
- K9. (5) Mod 691 factorization: **correct**; made precise as Thm. 5.5 ($691^{\omega(n)}$ divisibility, equivalent to Ramanujan’s congruence) and Prop. 5.6 (supertraces see K_0 , not Ext^1). “Fermion–boson confusion” stays [ANALOGY], qualified.
- K10. ([t263u] 1) Index: **correct**, $[M_{12}(\mathbb{Z}) : \mathbb{Z}691E_{12} \oplus \mathbb{Z}\Delta] = 691$, congruence module $\mathbb{Z}/691$.
- K11. ([t263u] 2) Products: identities **correct**; but the pentagonal theorem gives the signs of $\prod(1 - q^n)$, **not** of $\tau(n)$. (2’) Constants **correct**: $(2\pi)^{12}\Delta = g_2^3 - 27g_3^2 = 16 \prod(e_i - e_j)^2$.
- K12. ([t263u] 3) $\vartheta_{12}\Delta = 0$, $\vartheta E_4 = -E_6/3$, $\vartheta E_6 = -E_4^2/2$: **correct**; sharpened: ϑ is the Hamiltonian flow of 288Δ . “Bosons rotate” is [ANALOGY] (level curves are cubics). “ E_2 = Casimir energy

$-c/24$, $c = 24$: identity [THEOREM], physics [ANALOGY]; Dijkgraaf supports the zero-mode origin of $-3/(\pi y)$ for the free boson, not the Casimir wording.

- K13. ([t263u] 4) No cusp forms in weights 4–10; weight 12 first cusp form and first denominator 691: **correct**; weight 14 also has neither.

9. RANKED DIRECTIONS

Score = 0.6·significance+0.4·probability of a clean result.

	direction	sig.	prob.	score
P1	Lemma 64 engine on Δ vs E_{12} (L66 H1) with the pole/doublet decomposition of Thm. 3.1: which terms carry the sign?	0.70	0.80	0.74
P2	Rankin–Selberg: Weil-type form for $L(\Delta \otimes \Delta)$ whose pole term is $\propto \langle \Delta, \Delta \rangle$ (the $\Lambda^2 = \mathbb{L}^{11}$ line) as the candidate global g (L66 H3, L67 D2)	0.80	0.45	0.66
P3	Finite-field “upper half-plane” Ω_ℓ : split/non-split statistics vs ℓ and weight, images of $\bar{\rho}$, exceptional primes	0.40	0.90	0.60
P4	Parity bookkeeping across all three $\mathbb{Z}/2$ gradings (prime-side, weight, q -side) in one table with exact identities; Eisenstein-cohomology degree vs weight	0.50	0.70	0.58
P5	Sign of $\tau(p)$ as spinor data: Sato–Tate on $SU(2)$ vs $SO(3)$ statistics, and the η^{12} sign character	0.30	0.90	0.54
P6	Derived/glued super Fock space whose differential is the Ribet class, so that an Euler-type invariant sees Ext^1 (Bloch–Kato/Selmer viewpoint)	0.70	0.30	0.54
P7	Ramanujan–Serre Hamiltonian flow vs the Chern connection: does the metric correction act on the level curves of Δ as a positive (gradient-like) perturbation?	0.50	0.50	0.50
P8	1+1 Dirac reading of the doublet: any role for the gauge “boost” $\text{diag}(f, f^{-1})$ in Weil’s form for E_{12} ?	0.40	0.50	0.44

10. QUESTIONS FOR J.W.

- Q1. Adopt the answer to [t258u]: the spinor is the local Frobenius in SL_2 (rotation vs boost, Satake, fixed points, split/non-split mod ℓ), and the global object is the E_{12} doublet theorem with $FE = \sigma_x$, its spinor reading labelled [ANALOGY]?
- Q2. Adopt the answer to [t259u]: multiplicativity and duality are shared; the distinguishing datum is irreducibility $\Leftrightarrow$ no equivariant symmetric pairing (Thm. 4.1); add the proposed Lemma 64 ablation row?
- Q3. Adopt the parity section: Koszul rule (with the middle-degree correction), parity = weight mod 2 as the invariant notion, the graded-cancellation theorem and “supertraces see K_0 , not Ext^1 ” as the precise form of the fermion–boson confusion?
- Q4. Adopt the [t263u] section: index 691 / congruence module, Δ as the Hamiltonian of the Ramanujan–Serre flow, the three distinct sign structures, and “metric separates, lattice glues, residue field confuses”?
- Q5. Next: P1 (engine anatomy Δ vs E_{12}) or P2 (Rankin–Selberg pole as global g)? And Lemma 68 into Volume 4 with its three figures (needs your approval)?

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TAA LEMMA 69: CURVATURE, THE EISENSTEIN POLE, AND GHOSTS. HOW FAR DOES THE POSITIVITY OF THE ANOMALY REACH?

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA. Nothing below is a step toward a proof that we can certify. Every RH-equivalent statement is labelled [RESTATEMENT], every RH-consequence used as a test is flagged as such, and every metaphor is labelled [ANALOGY] or [HEURISTIC].* Tags: [THEOREM] proved here (elementary) or classical with the proof indicated; [CITED] checked at the source (`lit/lemma69/verified_refs.md`); [CITED-sec] via a named secondary source (second-hand); [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (floating point or mpmath; not certified).

Question (J.W., [t265u]). The quasi-modular anomaly and its fix are curvature terms, so they should relate directly to ghosts (the Lemma 63 “denting”: a local change with a global effect on curvature). Linked to this: “no global boost without a local one”.

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ANSWER IN BRIEF

(1) *Anomaly = residue = 1/area* (§2). For $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $\mathrm{Res}_{s=1} E(z, s) = \mathrm{Res}_{s=1} \varphi(s) = 3/\pi = 1/\mathrm{vol}(\Gamma \backslash \mathbb{H})$ and $d(E_2^* dz) = -\frac{1}{\mathrm{vol}} d\mu = \frac{1}{\mathrm{vol}} K_{\mathrm{Gauss}} d\mu$ [THEOREM]. Stokes’ theorem forces the coefficient to be $1/\mathrm{vol}$, because the constant term of E_2 is 1 [THEOREM]. So the anomaly coefficient is the *reciprocal* of the Gauss–Bonnet area, not the area itself. It is the residue at the *same* pole of ξ ($w = 1$) that produces Weil’s POLE term, but its *value* $1/(2\xi(2))$ is not Weil’s POLE coefficient.

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Area enters the trace formula through the identity (Weyl) term, the analogue of ARCH [THEOREM]/[ANALOGY]. (2) *Ghosts never touch the curvature; they dent the screen* (§5). The curvature form $-d\mu/\text{vol}$ is geometric. A ghost model can only violate its normalization $\text{Res } \varphi = 1/\text{vol}$, by $Q(1)/Q(2) - 1 \approx 10^{-11}$ at height 630 [THEOREM]+[NUMERICAL]. Zeros enter through the scattering phase. Exactly, $-\frac{\varphi'}{\varphi}(\frac{1}{2} + ir) = 4 \sum_{\rho} P_{1-\beta}(2r - \gamma) - 4P_1(2r)$ with $P_a(u) = a/(a^2 + u^2)$: each zero is seen through a Poisson kernel whose width is its distance $1 - \beta$ to the screen $\text{Re } w = 1$ [THEOREM] (checked to $4 \cdot 10^{-10}$ [NUMERICAL]). A “dent” is a change of width at fixed mass. The global flux is conserved ($\int P_a = \pi$ for every $a > 0$, the argument principle) [THEOREM], and numerically the mass changes by $\leq 2 \cdot 10^{-7}$ [NUMERICAL]. This is the precise content of “local change, global effect” here: the effect is *local in height and zero in total*. (3) *Maass–Selberg reaches exactly $\text{Re } s = 1$* (§6). The Maass–Selberg relation (Sarnak 2004, eq. (25) [CITED]; verified to $2 \cdot 10^{-15}$ [NUMERICAL]) plus Bessel’s inequality gives $S(t) \leq |\zeta(1 + 2it)|^2 \text{RHS}(t)$, and hence $\zeta(1 + it) \neq 0$ [THEOREM]+[CITED]. The reason, in screen language: a zero *on* the screen has width 0, i.e. a δ -mass, and a finite L^2 norm cannot afford one. The reach is the rung $\theta = \frac{1}{2}$ of Lemma 67’s cone ladder, *exactly*: the cone kernel at $\theta = \frac{1}{2}$ is $(P_{\beta} + P_{1-\beta})$, the same widths as the screen [THEOREM]. (4) *How much zero-free region* (§7). With only the $m = 1$ coefficient, $1 - \beta \gg 1/(\gamma \log^4 \gamma)$; with primes, $\gamma^{-1/2} \log^{-5} \gamma$; with Sarnak’s sieve, $c/\log^5 \gamma$, against de la Vallée Poussin’s $c/\log \gamma$ [CITED]/[THEOREM]. In our dictionary a zero-free region is a *tapering* rung $\theta(\gamma) = \frac{1}{2} - \eta(\gamma)$, never a fixed $\theta < \frac{1}{2}$ [HEURISTIC]. Obstructions: Poisson kernels are positive for every width [THEOREM]; for generic Γ there is no pole-free region beyond $\frac{1}{2} - c \log \gamma / \gamma$ (conditional, Sarnak) [CITED]; Beurling systems have zeros on $\sigma = 1 - a/\log t$ (Diamond–Montgomery–Vorhauer) [CITED](abstract). (5) *Which positivity breaks first under a ghost* (§8). Screen, trace formula spectral side and cone $\mathcal{C}_{1/2}$ never break, for any $0 < \beta < 1$ [THEOREM]. Maass–Selberg breaks only in the *frozen* model (local Hecke data kept, zeros moved); there the break is triggered by the inconsistency of local and global data, not by the displacement (it fires at $\delta = 0$ for collisions; the $m = 1$ floor fires for $\beta > 0.971$ at the Lehmer pair $n = 363$) [NUMERICAL]. In the *consistent* model it never fires [NUMERICAL]. Weil’s W_A (Lemma 64) detects every tested $\delta \geq 0.002$ once the window is large enough, $A^*(\delta) \in [0.70, 1.125]$ at height 17.6 [NUMERICAL]; Lemma 63’s κ detects every strip ghost at all tested heights [NUMERICAL]. Both are RH-level ([RESTATEMENT], resp. an RH-consequence), so their sensitivity is not evidence. (6) *“No global boost without a local one”* (§9). The Satake boost of $E(\cdot, s)$ is $\text{Re } s - \frac{1}{2}$. At the resonance $s = \rho/2$ it equals $(1 - \beta)/2$ in absolute value, and $\text{RH} \iff$ every resonance has boost $\frac{1}{4}$ [RESTATEMENT]. The heuristic is *right* as a consistency principle: the frozen model is precisely “global boost, no local boost”, and Maass–Selberg rejects it [NUMERICAL]. It is *empty* as a proof principle beyond $\beta < 1$: the obstructions in (4) are counterexamples to every version that uses positivity alone [CITED] [HEURISTIC].

(7) *The anomaly term and the reflection* (§3, J.W. [t266u]). J.W.’s feeling is right in a precise sense. $-3/(\pi y)$ is the $s \rightarrow 1$ limit of ∂_y of the *reflected* half $\varphi(s)y^{1-s}$ of the constant term of $E(z, s)$, and the holomorphic 1 comes from y^s [THEOREM] (checked to $2 \cdot 10^{-10}$ [NUMERICAL]). It cancels E_2 ’s anomaly exactly under every $\gamma \in \text{SL}_2(\mathbb{Z})$ [THEOREM] (sympy). It equals E_2 at the elliptic fixed points: $E_2(i) = 3/\pi$ and $E_2(\rho) = 2\sqrt{3}/\pi$ [THEOREM] (to 10^{-31}). Under Mellin, $\text{Res}_{s=1} \Lambda(E_2, s) = 6/\pi = 2E_2(i)$: the two halves are the anomaly term at iy and at i/y , which meet at the fold $y = 1$ [THEOREM]. A ghost, however, does *not* show up as a failure of a fixed-point balance. $\varphi(\frac{1}{2}) = -1$ and the Maass–Selberg balance at $s = \frac{1}{2}$ hold identically for every FE-symmetric ghost configuration [THEOREM]+[NUMERICAL]. The first-order datum at the centre is Li’s $\lambda_1 > 0$, which holds automatically [THEOREM]. Ghosts appear only in the full Taylor germ, where positivity of all λ_n is RH [RESTATEMENT] (Li 1997 [CITED](record)): $\lambda_n < 0$ first at $n = 6753$ for a ghost with $\delta = 0.49$ at height 17.6 [NUMERICAL]. (8) *The norm in the automorphy factor* (§4, [t267u]). Yes: $\text{Im}(\gamma\tau) = y/|\tau + d|^2$, and $|\tau + d|^2$ is a positive definite binary form in (c, d) with discriminant $-4y^2$. Normalized, $g_{\tau} = |\tau + d|^2/y$ is the unimodular positive form of the point τ , so $1/\text{Im}(\gamma\tau) = g_{\tau}(c, d)$ [THEOREM]. The Hasse form is $Q_p(m, n) = |m + n\alpha_p|^2 = m^2 + a_p mn + pn^2$, and Lemma 65’s $g_p = g_{\alpha_p}$ [THEOREM]. Local only: no global τ over $\mathbb{Q}$ is known [OPEN].

1. SETUP AND NORMALIZATIONS

Throughout, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acting on $\mathbb{H}$, $z = x + iy$, $d\mu = dx dy/y^2$, and $\mathrm{vol} = \mathrm{vol}(\Gamma \backslash \mathbb{H}) = \pi/3$. The Eisenstein series is

$$E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s \quad (\Gamma_\infty \ni -I),$$

with constant term $y^s + \varphi(s)y^{1-s}$ and

$$\varphi(s) = \frac{\xi(2s-1)}{\xi(2s)}, \quad \xi(w) = \pi^{-w/2} \Gamma(w/2) \zeta(w) = \xi(1-w).$$

This is Sarnak's $\varphi_{\mathrm{SL}(2, \mathbb{Z})}(s) = \sqrt{\pi} \Gamma(s - \frac{1}{2}) \zeta(2s-1) / (\Gamma(s) \zeta(2s))$ [8, (21)] [CITED]. Note that ξ here has poles at $w = 0, 1$ with residues $-1, +1$. Riemann's entire $\xi_R(w) = \frac{1}{2} w(w-1) \xi(w)$ is used only in §3. For $m \neq 0$ the Fourier coefficient is $2|m|^{s-1/2} \sigma_{1-2s}(|m|) \sqrt{y} K_{s-1/2}(2\pi|m|y) / \xi(2s)$ [8, (22)] [CITED]. We write $\Delta_{\mathbb{H}} = y^2(\partial_x^2 + \partial_y^2)$, so $\Delta_{\mathbb{H}} E = s(s-1)E$. $E_2 = 1 - 24 \sum \sigma(n) q^n$, $E_2^* = E_2 - \frac{3}{\pi y}$, and $P_a(u) = a/(a^2 + u^2)$ for $a > 0$, so that $\int_{\mathbb{R}} P_a = \pi$.

Other normalizations [THEOREM]. If the sum runs over coprime (c, d) *without* the factor $\frac{1}{2}$, every residue below doubles $(6/\pi)$. For $G(z, s) = \sum'_{(m,n)} y^s |mz + n|^{-2s} = 2\zeta(2s)E(z, s)$ the residue is $2\zeta(2) \cdot 3/\pi = \pi$. Any claim of the form “residue = area” has to name its normalization.

Scripts: `num/lemma69_*.py` (const, ms, ghost_ms, ms_large, weil, kappa, poisson, reflect, li, norm, disc, figs) $\rightarrow$ `num/169/*.json`. The Lemma 64 engine (`num/lemma64_engine.py`) is imported read-only; nothing in build63–68 was modified.

2. THE ANOMALY IS THE RESIDUE, AND THE RESIDUE IS 1/AREA

Theorem 2.1 (Anomaly = residue = 1/vol). (a) $\mathrm{Res}_{s=1} E(z, s) = \mathrm{Res}_{s=1} \varphi(s) = \frac{1}{2\xi(2)} = \frac{3}{\pi} =$

$\frac{1}{\mathrm{vol}}$, and $2\xi(2) = \pi/3 = \mathrm{vol} = 2\pi |\chi^{\mathrm{orb}}|$ with $\chi^{\mathrm{orb}} = -\frac{1}{6}$ [THEOREM].

(b) (Kronecker limit formula, [BIB] for the formula, [NUMERICAL] for the constant.) $E(z, s) = \frac{3}{\pi(s-1)} + K(z) + O(s-1)$ with $K(z) = C - \frac{1}{2\pi} \log(y^6 |\Delta(z)|)$ and $C = \frac{6}{\pi}(\gamma - \log 2) - \frac{36\zeta'(2)}{\pi^3} = 0.86713242772066455517 \dots$

(c) $\Delta_{\mathbb{H}} K = 1/\mathrm{vol}$, and $E_2^* = 2i \partial_z K$ [THEOREM].

(d) $\partial_{\bar{z}} E_2^* = \frac{3i}{2\pi y^2}$, i.e. $d(E_2^* dz) = -\frac{1}{\mathrm{vol}} d\mu = \frac{1}{\mathrm{vol}} K_{\mathrm{Gauss}} d\mu$ with $K_{\mathrm{Gauss}} \equiv -1$ [THEOREM].

(e) (Stokes forces the coefficient.) Let F_T be the standard fundamental domain truncated at $y = T$. Then $\int_{F_T} d(E_2^* dz) = -\int_{-1/2}^{1/2} E_2^*(x + iT) dx = -1 + \frac{3}{\pi T}$, because the side pairings cancel ($E_2^* dz$ is Γ -invariant) and the constant term of E_2 is 1. Hence the coefficient of the area form in $d(E_2^* dz)$ must be $-1/\mathrm{vol}(F)$ [THEOREM].

Proof. (a) $\xi(w) \sim 1/(w-1)$ at $w = 1$ and $w = 2s-1$, so $\mathrm{Res}_{s=1} \xi(2s-1) = \frac{1}{2}$; $\xi(2) = \pi^{-1} \cdot \frac{\pi^2}{6} = \frac{\pi}{6}$. The residue of E is the residue of its constant term, because the Fourier coefficients $m \neq 0$ are regular at $s = 1$ (the denominator $\xi(2)$ is finite). (c) Apply $\Delta_{\mathbb{H}}$ to the Laurent expansion: $s(s-1)E = \frac{3}{\pi}s + O(s-1)$. Alternatively, $\log |\Delta|$ is harmonic and $-\frac{6}{2\pi} y^2 \partial_y^2 \log y = \frac{3}{\pi}$. $2i \partial_z = i \partial_x + \partial_y$ applied to K gives $E_2 - \frac{3}{\pi y}$ (since $\frac{1}{2\pi i} \frac{d}{dz} \log \Delta = E_2$). (d) $\partial_{\bar{z}} y = \frac{i}{2}$ gives $\partial_{\bar{z}}(-\frac{3}{\pi y}) = \frac{3i}{2\pi y^2}$, and $d(E_2^* dz) = \partial_{\bar{z}} E_2^* d\bar{z} \wedge dz = \frac{3i}{2\pi y^2} \cdot 2i dx \wedge dy = -\frac{3}{\pi} d\mu$. (e) Stokes on F_T . $E_2^*(\gamma z) d(\gamma z) = E_2^*(z) dz$ because E_2^* has weight 2, so the two vertical sides and the two halves of the bottom arc cancel. On the top edge (oriented right to left) $\int E_2^* dz = -\int (1 - \frac{3}{\pi T}) dx$, since the q -terms integrate to 0. Then $-\frac{3}{\pi} \mathrm{vol}(F_T) = -1 + \frac{3}{\pi T}$ with $\mathrm{vol}(F_T) = \frac{\pi}{3} - \frac{1}{T}$. $\square$

[NUMERICAL] (`num/169/const.json`). $\text{Res } \varphi$ by finite difference 0.954929658552239 vs $3/\pi = 0.954929658551372$ ($9 \cdot 10^{-13}$, step artefact); $2\xi(2) = \pi/3$ to 30 digits; $\Delta_{\mathbb{H}} K = 0.9549296653$ (finite differences); $\partial_z E_2^* = 3i/(2\pi y^2)$ and $2i\partial_z K = E_2^*$ to $\sim 10^{-8}$. The Kronecker constant agrees with the direct evaluation to ~ 20 digits at 50-digit precision; at 30 digits a $4.7 \cdot 10^{-12}$ discrepancy was a precision artefact, which we fixed. The Fourier expansion of $E(z, 2)$ agrees with the direct lattice sum to $1.2 \cdot 10^{-5}$, the truncation tail of the latter. All of this matches the checks of Lemma 67 (its Prop. “Anomaly, fix, and the Petersson norm”) [TAA-L].

Remark 2.2 (What the chat claim gets right and wrong). “Anomaly coefficient = Gauss–Bonnet area = Eisenstein pole residue $\leftrightarrow$ Weil POLE.” (i) *Coefficient = residue*: correct (Thm. 2.1(a),(d)) [THEOREM]. (ii) = *area*: wrong. It is $1/\text{area}$, and (e) explains why: the curvature integral is normalized to the constant term 1 of E_2 . Equivalently, $\text{Res} \times \text{vol} = 1$ is the statement that the residual spectrum is the constant function normalized in L^2 , $\|\text{vol}^{-1/2}\|^2 = 1$ [THEOREM]. (iii) $\leftrightarrow$ *Weil POLE*: half right. The pole of φ at $s = 1$ is the pole of ξ at $w = 1$, *the same pole* that produces Weil’s POLE term [THEOREM]. In the Selberg trace formula the residual eigenvalue $\lambda_0 = 0$ contributes $h(\frac{i}{2})$, which is the evaluation point of Weil’s POLE term $\hat{g}(\pm \frac{i}{2})$ [THEOREM] (Lemma 67’s arithmetic trace formula [TAA-L]). But the *value* $3/\pi = 1/(2\xi(2))$ is not Weil’s POLE coefficient (which is 1), and the area enters the trace formula through the identity (Weyl) term $\frac{\text{vol}}{4\pi} \int rh(r) \tanh(\pi r) dr$, the analogue of ARCH, not of POLE [THEOREM]/[ANALOGY]. (iv) *Ghosts and the curvature*. The curvature form $-d\mu/\text{vol}$ is geometric: no ghost can change it [THEOREM]. A ghost model $\xi \mapsto \xi Q$ (with $Q(w) = Q(1-w)$ a rational factor that moves zeros) changes $\text{Res } \varphi$ by the factor $Q(1)/Q(2)$. So a ghost model is *inconsistent* with Thm. 2.1(a) unless $Q(1) = Q(2)$. The size of this inconsistency is tiny and is dominated by the vertical move. For the pair $n = 1, 2$ collapsed to $\bar{\gamma} = 17.58$: $Q(1)/Q(2) - 1 = 1.570 \cdot 10^{-3}$ at $\delta = 0$ and $1.599 \cdot 10^{-3}$ at $\delta = 0.49$. For the Lehmer pair $n = 363$: $2.1 \cdot 10^{-12}$ and $2.0 \cdot 10^{-11}$ [NUMERICAL]. The curvature “sees” a ghost only as a violated normalization, at order $\bar{\gamma}^{-4}$, like the frozen-model tests of §8. This is the precise version of J.W.’s “curvature relates directly to ghosts”: the relation exists but is a consistency constraint, not a positivity [THEOREM]+[NUMERICAL].

3. THE ANOMALY TERM AND THE REFLECTION

J.W. [t266u]: “the $-3/(\pi y)$ term is the object that undergoes the reflective symmetry.” We verify the four chat claims and then test the fixed-point idea against ghosts.

Theorem 3.1 (Exact cancellation). *For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$:*

$$\begin{aligned} E_2(\gamma\tau) &= (c\tau + d)^2 E_2(\tau) + \frac{12c(c\tau + d)}{2\pi i} = (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau + d), \\ -\frac{3}{\pi \text{Im}(\gamma\tau)} &= -\frac{3|c\tau + d|^2}{\pi y} = (c\tau + d)^2 \left(-\frac{3}{\pi y}\right) + \frac{6i}{\pi} c(c\tau + d), \end{aligned}$$

so the two non-modular pieces cancel exactly and $E_2^(\gamma\tau) = (c\tau + d)^2 E_2^*(\tau)$. For S ($c = 1, d = 0$) this is J.W.’s claim (1): $E_2(-1/\tau) = \tau^2 E_2(\tau) - 6i\tau/\pi$ and $-3|\tau|^2/(\pi y) = \tau^2(-3/(\pi y)) + 6i\tau/\pi$ [THEOREM].*

Proof. $|c\tau + d|^2 - (c\tau + d)^2 = (c\tau + d)(c\bar{\tau} - c\tau) = -2iyc(c\tau + d)$. Multiplying by $-3/(\pi y)$ gives $+6ic(c\tau + d)/\pi$. The transformation law of E_2 is standard [10, (21)] [CITED]. $\square$

[NUMERICAL] `sympy (num/169/norm.json)`: the gap identity, the split, the covariance residual of E_2^* , and $\text{Im}(\gamma\tau) = y/|c\tau + d|^2$ all simplify to 0. Numerically (`reflect.json`, 30 digits, three points τ) the transformation law and the identity hold to $\leq 2 \cdot 10^{-31}$.

Theorem 3.2 (Elliptic fixed points). (a) $S\tau = -1/\tau$ has order 2 in $\text{PSL}_2(\mathbb{Z})$ and fixes i , with automorphy factor $i^2 = -1$; so $E_2^*(i) = -E_2^*(i) = 0$ and $E_2(i) = \frac{3}{\pi}$ [THEOREM]. (b) $\tau \mapsto -1/(\tau + 1)$ (i.e. ST or TS according to convention; matrix $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$) has order 3 in $\text{PSL}_2(\mathbb{Z})$ and fixes $\rho = e^{2\pi i/3}$,

with factor $(\rho + 1)^2 = e^{2\pi i/3} \neq 1$; so $E_2^*(\rho) = 0$ and $E_2(\rho) = \frac{3}{\pi \cdot \sqrt{3}/2} = \frac{2\sqrt{3}}{\pi}$ [THEOREM]. At both points the holomorphic part equals the curvature term: $E_2 = 3/(\pi y)$.

[NUMERICAL] $E_2(i) = 0.954929658551372014613302580235 = 3/\pi$ and $E_2(\rho) = 1.10265779084358409902265299663 = 2\sqrt{3}/\pi$ (all 30 digits); $|E_2^*(\rho)| = 1.4 \cdot 10^{-31}$; $|-1/(\rho + 1) - \rho| = 3.6 \cdot 10^{-31}$.

Theorem 3.3 (The anomaly term is the reflected half of the constant term). $E_2^*(z) = \lim_{s \rightarrow 1} 2i \partial_z E(z, s)$. On the constant term,

$$2i \partial_z (y^s + \varphi(s) y^{1-s}) = \underbrace{s y^{s-1}}_{\rightarrow 1} + \underbrace{(1-s)\varphi(s) y^{-s}}_{\rightarrow -\text{Res } \varphi/y = -3/(\pi y)}.$$

So the holomorphic constant term 1 of E_2 comes from y^s , and the anomaly term $-3/(\pi y)$ comes from the reflected term $\varphi(s) y^{1-s}$, the image of y^s under $s \mapsto 1-s$. Its coefficient is minus the residue [THEOREM].

Proof. $2i \partial_z = i \partial_x + \partial_y$. The limit exists because $2i \partial_z$ kills the z -independent pole $\frac{3}{\pi(s-1)}$, and it equals $2i \partial_z K = E_2^*$ by Thm. 2.1(c). On the constant term, $(1-s)\varphi(s) \rightarrow -\text{Res}_{s=1} \varphi = -3/\pi$. $\square$

[NUMERICAL] At $s = 1 + 10^{-10}$ and $y = 0.9, 1.7, 3.2$ the constant-term expression equals $1 - 3/(\pi y)$ to $2 \cdot 10^{-11}, 1.3 \cdot 10^{-10}, 2.2 \cdot 10^{-10}$ (first order in $s - 1$, as expected).

This is the precise sense in which “ $-3/(\pi y)$ undergoes the reflective symmetry” [THEOREM]: it is the reflected partner $\varphi(s) y^{1-s}$, which survives at $s = 1$ only through the pole of φ .

Theorem 3.4 (Mellin: the central pole is the anomaly at the fold). Let $\Lambda(E_2, s) = \int_0^\infty (E_2(iy) - 1) y^{s-1} dy = -24(2\pi)^{-s} \Gamma(s) \zeta(s) \zeta(s-1)$. Then, for all s ,

$$\Lambda(E_2, s) = \int_1^\infty (E_2(iy) - 1) (y^{s-1} - y^{1-s}) dy - \frac{1}{s-2} + \frac{6/\pi}{s-1} - \frac{1}{s},$$

so $\Lambda(E_2, 2-s) = -\Lambda(E_2, s)$, with simple poles at $0, 1, 2$ [11, Prop. md] [TAA-L]. The central residue is

$$\text{Res}_{s=1} \Lambda(E_2, s) = \frac{6}{\pi} = \frac{3}{\pi} + \frac{3}{\pi} = 2E_2(i).$$

The two halves are the anomaly term evaluated at the two S -related points iy and i/y [THEOREM].

Proof. Substitute $y \mapsto 1/y$ on $(0, 1)$ and use $E_2(i/y) = -y^2 E_2(iy) + \frac{6y}{\pi}$ (Thm. 3.1 at $\tau = iy$). The polynomial terms give $-\frac{1}{s-2} + \frac{6/\pi}{s-1} - \frac{1}{s}$. The $\frac{6y}{\pi}$ splits as follows. From $E_2^*(i/y) = -y^2 E_2^*(iy)$ we get $E_2(i/y) - \frac{3y}{\pi} = -y^2 E_2(iy) + \frac{3y}{\pi}$. One $\frac{3y}{\pi}$ is the anomaly term $3/(\pi \text{Im}(i/y))$ of the image point; the other is $-y^2 \cdot (-3/(\pi y))$, the anomaly term of iy carried by the weight. At the fold $y = 1$ (the fixed point i of S) both equal $3/\pi = E_2(i)$. Antisymmetry about $s = 1$ forces either a zero or an odd pole at the fixed point $s = 1$, and the anomaly makes it a pole. $\square$

[NUMERICAL] The split formula agrees with $-24(2\pi)^{-s} \Gamma(s) \zeta(s) \zeta(s-1)$ to relative $3 \cdot 10^{-31}$ at $s = 2.7, 1.3 + 0.8i, 0.4 + 2.1i, 1 + 10^{-6}$. $10^{-8} \Lambda(1 + 10^{-8}) = 1.90985931710274423$ vs $6/\pi = 1.90985931710274403$. $|\Lambda(s) + \Lambda(2-s)| = 3 \cdot 10^{-31}$ at $s = 0.7 + 1.1i$.

Remark 3.5 (Correction to claim (3)). “Via Mellin, $S \leftrightarrow$ the reflection $s \leftrightarrow 2-s$, and the anomaly = the central pole” is correct, with one precision. The reflection $s \leftrightarrow 2-s$ is *exact for E_2 itself*: it follows from ζ ’s functional equation alone (Lemma 67 [TAA-L]). The anomaly does not repair a broken functional equation. It is the *residue at the fixed point* of an exact reflection (Hecke’s argument: modularity with the anomaly term $\Leftrightarrow$ functional equation + this pole structure) [THEOREM]. Likewise, for ζ the reflection $w \mapsto 1-w$ of ξ is exact with no correction term, so “the curvature term makes the reflection exact” is true for $\tau \mapsto -1/\tau$ at weight 2 but not for $s \mapsto 1-s$ [THEOREM].

3.1. Does a ghost break a fixed-point balance? J.W. [t264u] and claim (4): $\text{RH} \iff$ every zero is a fixed cell of $\sigma : s \mapsto 1 - \bar{s}$ (equivariant Morse). As a statement about the zero set this is a [RESTATEMENT]: the zero set is σ -invariant, σ fixes the critical line pointwise, and RH says every orbit has length 1. The precise question is whether an injected ghost shows up as a failure of a balance *at* the fixed point. Ghost models: $\xi \mapsto \xi Q$ with

$$Q(w) = \prod_{\text{new}} (w - \rho') / \prod_{\text{removed}} (w - \rho),$$

which replaces the zeros $\frac{1}{2} \pm i\gamma_a, \frac{1}{2} \pm i\gamma_b$ by the quartet $\frac{1}{2} \pm \delta \pm i\bar{\gamma}$. Then $Q(w) = Q(1 - w)$, so the functional equation is untouched.

Theorem 3.6 (Fixed-point balances are ghost-blind). (a) $\varphi(\frac{1}{2}) = -1$, forced by the poles of ξ alone ($\xi(2\epsilon)/\xi(1+2\epsilon) \rightarrow (-1)/(+1)$). For every ghost model, $\varphi_Q(\frac{1}{2}) = \varphi(\frac{1}{2}) Q(0)/Q(1) = -1$ [THEOREM]. Hence $E(z, \frac{1}{2}) \equiv 0$ for ζ and for the model.
 (b) The Maass–Selberg relation (1) at the centre reduces to $0 = 0$ identically, for any φ with $\varphi(\frac{1}{2}) = -1$ and $|\varphi| = 1$ on the line [THEOREM]. Expanding $\varphi(\frac{1}{2} + it) = -(1 + itc + O(t^2))$ with $c = \varphi'/\varphi(\frac{1}{2})$ gives $2 \log T - c - (2 \log T - c) = 0$.
 (c) The first-order datum is $-\varphi'(\frac{1}{2}) = 4\lambda_1 - 4$, where $\lambda_1 = \sum_{\rho} \frac{1}{\rho} = 1 + \frac{\gamma}{2} - \frac{1}{2} \log 4\pi = 0.0230957 \dots$ is Li’s first coefficient. For every configuration in the strip, $\lambda_1 = \sum \beta/|\rho|^2 > 0$ [THEOREM].
 (d) Write $\xi_R = \frac{1}{2}w(w-1)\xi$. Then $\log \varphi(\frac{1}{2} + \epsilon) = \log \xi_R(1-2\epsilon) - \log \xi_R(1+2\epsilon) + \log \frac{1+2\epsilon}{2\epsilon-1}$. So the Taylor germ of the constant term at the fixed point carries only the odd power sums $\sigma_k = \sum_{\rho} \rho^{-k}$, k odd. Li’s coefficients $\lambda_n = \sum_{k=1}^n \binom{n}{k} (-1)^{k-1} \sigma_k$ (binomial expansion of $1 - (1 - 1/\rho)^n$) need the even ones too. Those sit in the non-constant Fourier coefficients, through the denominator $\xi(2s) = \xi(1+2\epsilon)$ [THEOREM].

[NUMERICAL] (reflect.json) For the pair $n = 1, 2$ collapsed to $\bar{\gamma} = 17.58$ and $\delta = 0, 0.1, 0.3, 0.49$: $\varphi_Q(\frac{1}{2} + 10^{-12}) = -1.0000000000039$ in all four cases (the $3.9 \cdot 10^{-12}$ is the offset). The centre Maass–Selberg residual is $6.8 \cdot 10^{-12}$ in all four, i.e. zero up to the finite-difference step. $-\varphi'_Q/\varphi_Q(\frac{1}{2}) = -3.910790, -3.910793, -3.910813, -3.910850$, vs -3.907617 for ζ ($= 4\lambda_1 - 4$; the direct evaluation agrees to $2 \cdot 10^{-11}$). So the centre value moves by $3.2 \cdot 10^{-3}$ because of the vertical collision and only by $6 \cdot 10^{-5}$ because of the horizontal displacement, and it stays > -4 , i.e. $\lambda_1 > 0$.

[NUMERICAL] Li ghost test (num/lemma69_li.py; 10^5 Odlyzko zeros plus a smooth tail; λ_1 reproduced as 0.0230951 vs 0.0230957). With the same ghost quartet at $\bar{\gamma} = 17.58$, $\lambda_n < 0$ first at $n = 6753$ ($\delta = 0.49$), 12914 ($\delta = 0.3$), 20332 ($\delta = 0.2$). There is no sign change for $n \leq 3 \cdot 10^4$ at $\delta \leq 0.1$. The growth rate per n is $\log |1 - 1/(1 - \rho')| \approx \delta/\bar{\gamma}^2$ ($1.58 \cdot 10^{-3}$ at $\delta = 0.49$). Li’s criterion, $\text{RH} \iff \lambda_n \geq 0$ for all n [5] [CITED](record), makes this an RH-level detector [RESTATEMENT], of the same type as W_A and κ (§8).

Verdict on claim (4). [THEOREM]+[ANALOGY]. The curvature term *does* sit at the fixed points ($E_2(i) = 3/\pi$, $\text{Res}_{s=1} \Lambda(E_2) = 2E_2(i)$, $-3/(\pi y)$ = the reflected half). But a ghost does *not* appear as the failure of a fixed-point balance: every balance at the fixed point that the reflection itself imposes ($\varphi(\frac{1}{2}) = -1$, Maass–Selberg at $s = \frac{1}{2}$, $\lambda_1 > 0$) holds identically for ghost configurations. This is because ghosts respect the reflection: they are orbits of length 2, not violations of the symmetry. Detecting them takes the *whole* Taylor germ at the fixed point (Li), and that is RH-equivalent [RESTATEMENT]. “Equivariant Morse” is an [ANALOGY]: the fixed-point index counts orbits of the reflection, and RH is the statement that all orbits are fixed. No index theorem forces that here [OPEN].

4. THE NORM IN THE AUTOMORPHY FACTOR

J.W. [t267u]: *is the $|\tau|^2$ in $\text{Im}(-1/\tau) = y/|\tau|^2$ precisely our positive-definite norm?*

Theorem 4.1 (The automorphy norm). (a) $\text{Im}(\gamma\tau) = y/|c\tau + d|^2$. So $1/y$ transforms with the hermitian factor $|c\tau + d|^2 = (c\tau + d)(c\bar{\tau} + d)$ (weight $(1, 1)$), while E_2 transforms with the holomorphic $(c\tau + d)^2$. Their gap $-2iyc(c\tau + d)$, times $-3/(\pi y)$, is $+6ic(c\tau + d)/\pi$, which cancels E_2 's anomaly (Thm. 3.1) [THEOREM].

(b) As a binary quadratic form in (c, d) , $N_\tau(c, d) := |c\tau + d|^2 = |\tau|^2 c^2 + 2x cd + d^2$ has discriminant $(2x)^2 - 4|\tau|^2 = -4y^2$. It is positive definite iff $y \neq 0$; τ and $\bar{\tau}$ give the same form, so $y > 0$ only fixes the orientation [THEOREM]. The normalized form $g_\tau := N_\tau/y$ has discriminant -4 , and $\tau \mapsto g_\tau$ is the classical identification of $\mathbb{H}$ with positive definite forms of determinant 1 (classical, [THEOREM]). Then

$$\frac{1}{\text{Im}(\gamma\tau)} = g_\tau(c, d), \quad -\frac{3}{\pi \text{Im}(\gamma\tau)} = -\frac{3}{\pi} g_\tau(c, d) :$$

along the orbit, the anomaly term is the positive definite norm g_τ evaluated on the bottom row of γ [THEOREM].

(c) Hasse. For $E/\mathbb{F}_p$ with Frobenius eigenvalue $\alpha_p = \frac{a_p + i\sqrt{4p - a_p^2}}{2}$:

$$Q_p(m, n) = |m + n\alpha_p|^2 = m^2 + a_p mn + pn^2 = N_{\alpha_p}(n, m), \quad \text{Im } \alpha_p = \frac{1}{2}\sqrt{4p - a_p^2}.$$

This is Lemma 65's degree form $Q_p(x, y) = \deg(x + y\pi) = x^2 + a_p xy + py^2$, with the same sign [TAA-L] + [THEOREM]. Lemma 65's metric $g_p = 2Q_p/\sqrt{|d|}$ with $|d| = 4p - a_p^2 = 4(\text{Im } \alpha_p)^2$ is $Q_p/\text{Im } \alpha_p = g_{\alpha_p}$, exactly the normalized automorphy norm at the point α_p [THEOREM]. Hasse's bound $\iff Q_p$ positive definite $\iff \text{Im } \alpha_p \neq 0 \iff$ the Frobenius point lies in $\mathbb{H}$ (Lemma 67's "Hasse $\iff \text{Im } \tau_p > 0$ ") [THEOREM].

[NUMERICAL] sympy (norm.json): $\text{disc} = -4y^2$; $|m + n\alpha_p|^2$ expands to $a_p mn + m^2 + pn^2$; the residuals of (a) are 0.

Remark 4.2 (Convention note, a correction to the chat statement). The chat writes the Hasse form as $|m + n\pi_p|^2$ with $\text{Im } \pi_p = \sqrt{4p - a_p^2}/2$. That is correct, with $\pi_p = \alpha_p$. However, the point that Lemma 60's rule $\tau_Q = \frac{-b + i\sqrt{|d|}}{2a}$ attaches to $Q_p = [1, a_p, p]$ is $\tau_p = \frac{-a_p + i\sqrt{4p - a_p^2}}{2} = -\bar{\alpha}_p$ (Lemmas 65, 67 [TAA-L]). This is the mirror image of α_p under $x \mapsto -x$. In general Lemma 60 attaches $-\bar{\tau}$ to $|x + y\tau|^2$. Imaginary parts, positivity and the metric are unaffected; only the sign of the real part depends on the convention. A second precision: positive definiteness of $|c\tau + d|^2$ is equivalent to $y \neq 0$, not to $y > 0$ (see (b)) [THEOREM].

Theorem 4.3 (Norm versus gap (J.W. [t268u])). For a Hecke eigenvalue pair $\pi, \bar{\pi}$ of weight k (roots of $X^2 - a_p X + p^{k-1}$):

- (a) The norm $\pi\bar{\pi} = \det = p^{k-1}$ is the same for Δ and E_{12} (p^{11}), and for every $E/\mathbb{F}_p$ it is p . It is blind to the type [THEOREM].
- (b) The type is decided by the discriminant = squared gap: $a_p^2 - 4p^{k-1} = (\pi - \bar{\pi})^2$. For the automorphy norm, $\text{disc } N_\tau = (\tau - \bar{\tau})^2 = -4y^2$. There are three regimes [THEOREM]: (i) conjugate pair in $\mathbb{H}$, $\text{disc} < 0$: definite form, elliptic, rotation (Δ , by Deligne; all $E/\mathbb{F}_p$, by Hasse). (ii) a double real root, $\text{disc} = 0$: rank-1 degenerate form, parabolic (Hasse equality). This cannot occur for Δ or for $E/\mathbb{F}_p$ with p prime, since $4p^{k-1}$ with $k-1$ odd is not a square; it occurs over $\mathbb{F}_{p^2}$ (e.g. supersingular $a = \pm 2p$). (iii) distinct real roots, $\text{disc} > 0$: indefinite form of signature $(1, 1)$, hyperbolic, boost (E_{12} : roots $1, p^{11}$, $\text{disc} = (1 - p^{11})^2$).
- (c) The curvature term is the inverse gap: $-\frac{3}{\pi y} = -\frac{6i}{\pi(\tau - \bar{\tau})}$ [THEOREM]. It blows up exactly where the gap closes ($y \rightarrow 0$, $\partial\mathbb{H}$, regimes (ii), (iii)).

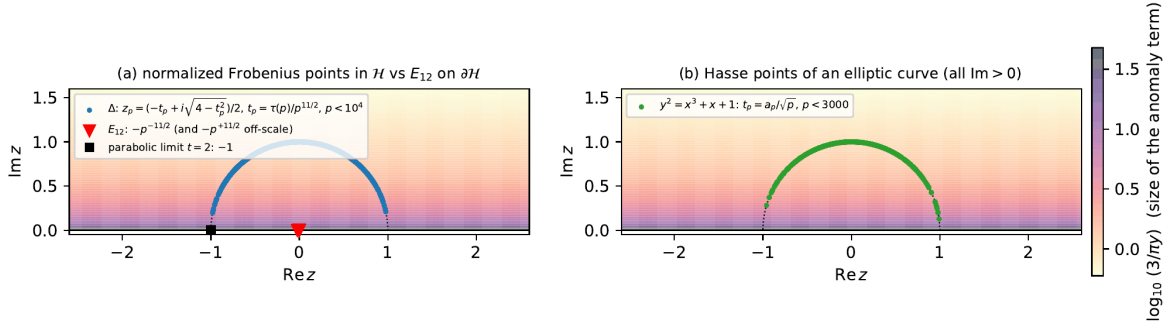


FIGURE 1. Normalized Frobenius points in $\mathbb{H}$ (Lemma 67/68 convention $z_p = -\bar{\alpha}_p/p^{(k-1)/2}$), over the heat map of the anomaly-term size $\log_{10}(3/\pi y)$. (a) Δ : $t_p = \tau(p)/p^{11/2}$ for the 1229 primes $p < 10^4$ (data `num/168/traces.npz`, read only); all points on $|z| = 1$, $\max |t_p| = 1.96053$. E_{12} : its fixed points $-p^{\pm 11/2}$ lie on $\partial\mathbb{H}$ (red, $p \leq 11$; $-p^{+11/2}$ off scale), where $3/(\pi y) = \infty$. The parabolic limit $t = 2$ is -1 . (b) The Lemma 65 curve $y^2 = x^3 + x + 1$, 428 good primes $3 \leq p < 3000$, $\max |a_p|/\sqrt{p} = 1.98334$ (Lemma 65: $\max |a_p|/2\sqrt{p} = 0.9917$ for $p < 4000$, consistent). All points have $\text{Im} > 0$: local positivity is everywhere [NUMERICAL].

(d) Since $\tau(p) \equiv 1 + p^{11} \pmod{691}$, the discriminants are congruent: $\tau(p)^2 - 4p^{11} \equiv (1 + p^{11})^2 - 4p^{11} \equiv (1 - p^{11})^2 \pmod{691}$. Mod 691 the gap of Δ becomes a square, i.e. type (iii) (Lemma 66 [TAA-L]; Lemma 68's "split mod 691" [TAA-L] [THEOREM].

[NUMERICAL] (`num/lemma69_disc.py`, `disc.json`). $\tau(n) \pmod{691}$ from η^{24} , computed exactly mod 691 for $n < 10^4$, with exact integers for $p < 200$ as a control (they agree; Lemma 68's t_p reproduced exactly). The congruence (d) and $\tau(p) \equiv 1 + p^{11}$ hold for all 1229 primes $p < 10^4$, with no failures. Examples: $p = 2$: $\text{disc}_\Delta = -7616$, $\text{disc}_{E_{12}} = 4190209$, both $\equiv 676$; $p = 13$: -6834860379504 and $3211838877951270784369296$, both $\equiv 641 \pmod{691}$. Sympy: (c) simplifies to 0.

Answer to [t268u] [THEOREM]. Degeneracy lives in the *gap*, not in the norm. The norm $\pi\bar{\pi} = p^{k-1}$ (or $|\tau|^2$ in $\text{Im}(-1/\tau) = y/|\tau|^2$, which for $\gamma = S$ is $N_\tau(1, 0)$) is the value of the form at a vector. The *type* of the form is its discriminant $(\tau - \bar{\tau})^2$, and the anomaly term $-3/(\pi y)$ is, up to $6i/\pi$, the reciprocal of that gap.

Caveat ([OPEN], [HEURISTIC]). Everything here is *local*: one point $\alpha_p \in \mathbb{H}$ (one norm g_{α_p}) per prime. Positivity is Hasse/Deligne [BIB], and Fig. 1 shows it holds at every prime. What RH would need is a *global* τ over $\mathbb{Q}$, a single positive form built from all primes at once, whose positivity is the Weil functional (Lemmas 64, 65, 67 [TAA-L]). No such object is known. The automorphy norm makes precise *what kind* of object is missing (a point of a period domain, or a positive hermitian form), not that it exists. In the language of §9: every local boost is zero for Δ (all z_p in $\mathbb{H}$, elliptic), the E_{12} points are pure boosts on $\partial\mathbb{H}$, and the global boost of a ghost is invisible to every individual z_p .

5. THE SCREEN: HOW ZEROS ENTER THE EISENSTEIN SERIES

Theorem 5.1 (Screen decomposition). *For real $r \neq 0$,*

$$-\frac{\varphi'}{\varphi}\left(\frac{1}{2} + ir\right) = 4 \operatorname{Re} \frac{\xi'}{\xi}(1 + 2ir) = 4 \sum_{\rho} P_{1-\beta}(2r - \gamma) - 4P_1(2r),$$

where the sum runs over all nontrivial zeros $\rho = \beta + i\gamma$ (symmetric summation) [THEOREM]. Each zero is seen through a Poisson kernel whose width is its distance $1 - \beta$ to the screen $\operatorname{Re} w = 1$ (variable

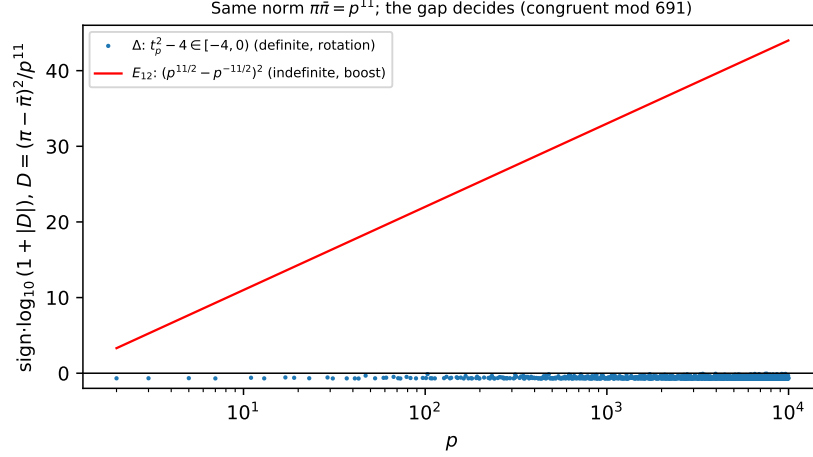


FIGURE 2. Normalized discriminant $D = (\pi - \bar{\pi})^2/p^{11}$, plotted as $\text{sign}(D) \log_{10}(1 + |D|)$: $t_p^2 - 4 \in [-4, 0)$ for Δ (definite, 1229 primes $p < 10^4$) vs $(p^{11/2} - p^{-11/2})^2$ for E_{12} (indefinite). The norm p^{11} is common to both; only the gap distinguishes them [NUMERICAL].

$w = 2s$). The two poles of ξ ($w = 0, 1$) enter with the opposite sign and width 1. Every kernel has total mass π , whatever its width, so

moving a zero horizontally (a “dent”) changes the shape of the density, never its integral [THEOREM].

Proof. $\varphi'/\varphi(s) = 2\frac{\xi'}{\xi}(2s-1) - 2\frac{\xi'}{\xi}(2s)$, and $\frac{\xi'}{\xi}(w) = -\frac{\xi'}{\xi}(1-w)$, so on $s = \frac{1}{2} + ir$ we get $-\varphi'/\varphi = 4 \text{Re} \frac{\xi'}{\xi}(1+2ir)$. Hadamard (with the symmetric pairing $\rho \leftrightarrow 1-\rho$, which removes the constant): $\frac{\xi'}{\xi}(w) = \sum_{\rho} \frac{1}{w-\rho} - \frac{1}{w} - \frac{1}{w-1}$. Take real parts at $w = 1+2ir$: $\text{Re} \frac{1}{1+2ir-\rho} = P_{1-\beta}(2r-\gamma)$, and $\text{Re}(-\frac{1}{1+2ir} - \frac{1}{2ir}) = -P_1(2r)$. $\square$

[NUMERICAL] (poisson.json). With 10^5 zeros plus a tail estimate, the decomposition matches the exact value to $4.2 \cdot 10^{-10}$ at $r = 0.3, 1, 3, 7.07, 8.79, 20, 50$. Ghost quartets (pair n replaced by $\frac{1}{2} \pm \delta \pm i\bar{\gamma}$): the windowed mass change is $1.9 \cdot 10^{-7}$ ($n = 1$) and $\leq 3.3 \cdot 10^{-9}$ ($n = 363$), i.e. zero up to the windowing error, for $\delta = 0.1, \dots, 0.49$. The local peak grows like $1/(1-\beta)$: 16.3, 24.7, 83.9, 403.7 at $\delta = 0.1, 0.3, 0.45, 0.49$ for $n = 1$.

Remark 5.2 (Denting, made precise). [THEOREM]/[ANALOGY]. Lemma 63’s “denting” (a local change with a global effect on curvature) has an exact counterpart here. Moving a zero toward $\text{Re } w = 1$ sharpens its Poisson bump (local), while the total flux $\int (-\varphi'/\varphi) dr$, i.e. the winding of the scattering phase, is unchanged (global, by the argument principle; the Gauss-law form of “Gauss–Bonnet = argument principle” from Lemma 63 [TAA-L]). So the global effect of a dent is *zero in total and local in height*. The curvature form itself (Thm. 2.1) does not move at all. “Curvature” in J.W.’s sense should therefore be read as the *scattering density* $-\varphi'/\varphi$, not as the Gauss curvature of $\Gamma \backslash \mathbb{H}$. That reading is an [ANALOGY] made into a theorem by Thm. 5.1.

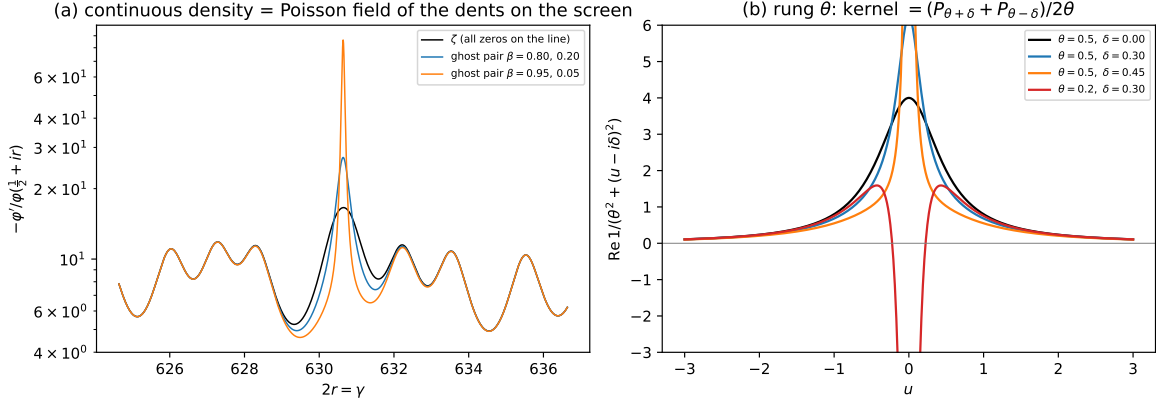


FIGURE 3. (a) The screen density $-\varphi'/\varphi(\frac{1}{2} + ir)$ near the Lehmer pair $n = 363$ ($2r = \gamma$), for ζ and for ghost quartets with $\beta = 0.8$ and 0.95 . The bump sharpens and the mass is conserved [NUMERICAL]. (b) Lemma 67's cone kernel $\text{Re}(\theta^2 + (u - i\delta)^2)^{-1} = (P_{\theta+\delta} + P_{\theta-\delta})/2\theta$. It is positive iff $|\delta| < \theta$; at $\theta = \frac{1}{2}$ the widths are $\beta, 1 - \beta$, the screen widths of Thm. 5.1 [THEOREM].

6. MAASS-SELBERG POSITIVITY AND $\zeta(1 + it) \neq 0$

Theorem 6.1 (Maass-Selberg relation [8, (25)]). *Let $\Lambda^T E$ be the truncation at height $T \geq 1$ (Sarnak's E_A , (26)). For $s = \sigma + it$, $\sigma \neq \frac{1}{2}$, $t \neq 0$:*

$$\|\Lambda^T E(\cdot, s)\|^2 = \frac{T^{2\sigma-1} - |\varphi(s)|^2 T^{1-2\sigma}}{2\sigma-1} + \frac{\overline{\varphi(s)} T^{2it} - \varphi(s) T^{-2it}}{2it},$$

and on the line $\sigma = \frac{1}{2}$

$$(1) \quad \|\Lambda^T E(\cdot, \tfrac{1}{2} + it)\|^2 = 2 \log T - \frac{\varphi'}{\varphi}(\tfrac{1}{2} + it) - \frac{\text{Im}(\varphi(\tfrac{1}{2} + it) T^{-2it})}{t}$$

[CITED] (Sarnak's display (25); the off-line form is the standard one, [BIB] via Iwaniec, and is checked numerically).

[NUMERICAL] (ms_check.json). Direct numerical integration of $|\Lambda^T E|^2$ over the truncated domain (Fourier expansion with exact divisor sums, plus the constant-term region) vs the formula: relative errors $1.9 \cdot 10^{-15}$ (s, T) = $(\frac{1}{2} + 3i, 1)$; $2 \cdot 10^{-16}$ ($\frac{1}{2} + 3i, 2.5$); $3 \cdot 10^{-16}$ ($\frac{1}{2} + 7.1i, 1.7$); $3 \cdot 10^{-16}$ ($0.8 + 2.3i, 1.5$); $7.7 \cdot 10^{-16}$ ($0.3 + 4i, 1.2$).

Proposition 6.2 (Where the positivity has teeth). [THEOREM]. (a) *On the line, the right side of (1) is nondecreasing in $u = \log T$: its derivative is $2 + 2 \text{Re}(\varphi e^{-2itu}) \geq 0$ since $|\varphi| = 1$. The left side's non-constant part over $y \geq 1$ does not depend on T . So the best inequality is at $T = 1$; optimizing over T or s gives nothing more on the line.* (b) *For $\sigma < \frac{1}{2}$ the first term is positive for $T > 1$ large, so positivity is automatic. For $\sigma > \frac{1}{2}$ the term $-|\varphi|^2 T^{1-2\sigma}/(2\sigma-1)$ is negative, so this is where positivity could bite. But φ has no poles in $\sigma > \frac{1}{2}$ except $s = 1$ (the residual spectrum), and its resonances sit at $s = \rho/2$, with $\text{Re} = \beta/2 < \frac{1}{2}$.*

Theorem 6.3 (Sarnak's inequality and $\zeta(1 + it) \neq 0$). Put $J_m(t) = \int_{2\pi m}^{\infty} |K_{it}(x)|^2 \frac{dx}{x}$ and $S(t) = 8 \cosh(\pi t) \sum_{m \geq 1} |\sigma_{-2it}(m)|^2 J_m(t)$. Bessel's inequality on $y \geq 1$ with $T = 1$ gives

$$S(t) \leq |\zeta(1 + 2it)|^2 \text{RHS}(t), \quad \text{RHS}(t) = 4 \text{Re} \frac{\xi'}{\xi}(1 + 2it) - \frac{\text{Im} \varphi(\frac{1}{2} + it)}{t}$$

[THEOREM] (from Thm. 6.1; this is Sarnak’s route [8, (27)–(36)] [CITED]). Consequently $\zeta(1 + 2it_0) \neq 0$.

Proof. $|\xi(1 + 2it)|^2 = |\zeta(1 + 2it)|^2 / \cosh(\pi t)$, so the m -th coefficient squared, integrated over $y \geq 1$, $|x| \leq \frac{1}{2}$, is $4|\sigma_{-2it}(m)|^2 \cosh(\pi t) J_m / |\zeta(1 + 2it)|^2$ (both signs of m give the factor 8). The truncated norm dominates the non-constant part. Suppose $\zeta(1 + 2it_0) = 0$. In screen language this is a zero on the screen: its Poisson kernel has width 0, so $4 \operatorname{Re} \frac{\xi'}{\xi}(1 + 2it)$ stays bounded near t_0 (the polar part $\frac{1}{2i(t-t_0)}$ is purely imaginary). Then $|\zeta|^2 \text{RHS} \rightarrow 0$, while $S(t_0) \geq 8 \cosh(\pi t_0) J_1(t_0) > 0$, a contradiction. $\square$

History ([CITED]/[CITED-sec]). The qualitative argument is what Sarnak calls “the simple magical and standard derivation” [8, p. 9 of the PDF], going back to the Jacquet–Shalika method [4] [CITED-sec]. Sarnak’s contribution is the *effective* version [8], and its generalization is [2] (“Comparing the two by Bessel’s inequality gives a coarse lower bound for zeta”).

[NUMERICAL] *Saturation* (Fig. 4; `ms_slack.out`, `ms_ghost1_*.json`, `ms_large.json`). LHS/RHS lies in $[0.727, 0.953]$ for $t \in [10, 60]$ (64 values of $t \in [0.5, 60]$ computed exactly); it is 0.961 at $t = 315.32$ and 0.947 at $t = 375.41$ (exact S); the WKB average $J_m \approx \frac{\pi}{2t} \operatorname{arccosh} \frac{t}{2\pi m}$ (relative error $3.9 \cdot 10^{-4}$ and $7.2 \cdot 10^{-4}$ against exact S at those two points) gives 0.943–0.950 at six heights up to $\gamma = 7 \cdot 10^4$. The $m = 1$ term alone gives 0.06–0.93 ($t \geq 10$). For small t the inequality is far from sharp (10^{-5} at $t = 0.5$). [HEURISTIC]: near-saturation is explained by Ingham’s identity $\sum |\sigma_{it}(n)|^2 n^{-s} = \zeta^2(s) \zeta(s+it) \zeta(s-it) / \zeta(2s)$ [8, (54)], which makes S itself carry the factor $|\zeta(1 + 2it)|^2$. The inequality is “ $|\zeta|^2 \times$ (nearly the same thing)”.

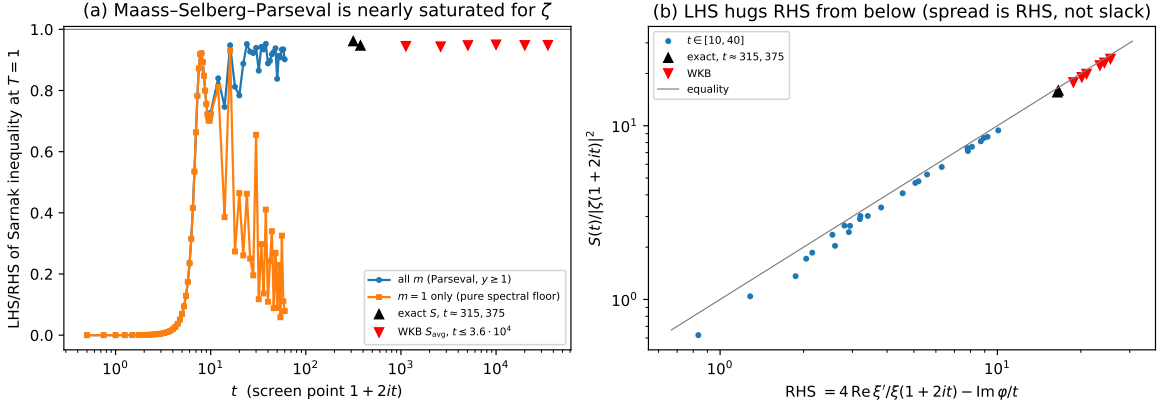


FIGURE 4. (a) The ratio LHS/RHS of Thm. 6.3 at $T = 1$: all m (blue), $m = 1$ only (orange), exact S near the Lehmer pairs (black), WKB S_{avg} at large height (red). (b) $S/|\zeta(1 + 2it)|^2$ against RHS: the points hug the diagonal from below, so the spread is in RHS and the slack is small [NUMERICAL].

7. HOW FAR THE REACH GOES: ZERO-FREE REGIONS AND THE $\theta \leftrightarrow \beta$ DICTIONARY

Proposition 7.1 (From a lower bound to a zero-free region). [THEOREM] (elementary). If $\zeta(\beta + i\gamma) = 0$ and $1 - \beta = \eta \leq 1$, then $|\zeta(1 + i\gamma)| \leq \eta \max_{[\beta, 1]} |\zeta'(\sigma + i\gamma)| \ll \eta \log^2 \gamma$. Hence $|\zeta(1 + i\gamma)| \geq L(\gamma)^{-1}$ implies $\eta \gg 1/(L(\gamma) \log^2 \gamma)$.

Theorem 7.2 ($\theta = \frac{1}{2}$ is the Maass–Selberg rung, exactly). Lemma 67’s cone $\mathcal{C}_\theta$ has kernel [TAA-L]

$$\operatorname{Re} \frac{1}{\theta^2 + (u - i\delta)^2} = \frac{P_{\theta+\delta}(u) + P_{\theta-\delta}(u)}{2\theta} \quad (\delta = \beta - \tfrac{1}{2}),$$

method	lower bound for $ \zeta(1 + it) $	zero-free width $\eta(\gamma) = 1 - \beta$	status
MS, $m = 1$ only	$\gg 1/(t \log^2 t)$ [8, (36)]	$\gg 1/(\gamma \log^4 \gamma)$	[CITED]+[THEOREM]
MS, primes ($ \sigma(p) ^2 + \sigma(p^2) ^2 \gg 1$)	$\gg 1/(\sqrt{t} \log^3 t)$ [8, (38)]	$\gg \gamma^{-1/2} \log^{-5} \gamma$	[CITED]+[THEOREM]
MS + Bombieri–Davenport sieve	$\gg 1/\log^3 t$ [8, (53)]	$c/\log^5 \gamma$ (Sarnak’s statement)	[CITED]
de la Vallée Poussin	$\geq c/\log t$ [2, (1)]	$c/\log \gamma$ [8, (1)]	[CITED]
explicit dVP	—	$1/(5.573412 \log \gamma)$, $\gamma \geq 2$ [7]	[CITED](abstract)
Vinogradov	—	$c_\alpha/\log^\alpha \gamma$, any $\alpha > \frac{2}{3}$ [8, p. 2]	[CITED]
generic Γ , spectral only	—	no pole-free region beyond $\frac{1}{2} - c \log \gamma/\gamma$ [8, (18)–(20)]	[CITED], conditional

TABLE 1. The reach of Eisenstein/Maass–Selberg positivity compared with classical zero-free regions. The generic- Γ line is conditional on Luo’s nonvanishing [6] [CITED-sec] and “a suitable form of a standard multiplicity bound conjecture” [8, p. 8].

which is ≥ 0 for all u iff $|\delta| < \theta$ (at $u = 0$ it equals $\frac{1}{\theta^2 - \delta^2}$, and $P_{-a} = -P_a$) [THEOREM]. At $\theta = \frac{1}{2}$ the widths are β and $1 - \beta$, the screen widths of Thm. 5.1. So Maass–Selberg/screen positivity and $W \geq 0$ on $C_{1/2}$ are the same positivity: both hold iff $0 < \beta < 1$ [THEOREM].

Dictionary ([HEURISTIC], except the $\theta = \frac{1}{2}$ entry, which is [THEOREM]). Rung $\theta \leftrightarrow$ a screen at $\text{Re } w = \frac{1}{2} + \theta$, through which a zero at $\frac{1}{2} + \delta$ is seen with widths $\theta \pm \delta$. A zero-free region $\eta(\gamma)$ guarantees $|\delta| < \frac{1}{2} - \eta(\gamma)$, i.e. it corresponds to a *tapering* rung $\theta(\gamma) = \frac{1}{2} - \eta(\gamma)$, which is not a fixed cone:

$$\theta_{\text{MS},1} = \frac{1}{2} - \frac{c}{\gamma \log^4 \gamma}, \quad \theta_{\text{sieve}} = \frac{1}{2} - \frac{c}{\log^5 \gamma}, \quad \theta_{\text{dVP}} = \frac{1}{2} - \frac{1}{5.573412 \log \gamma}, \quad \theta_{\text{Vin}} = \frac{1}{2} - \frac{c_\alpha}{\log^\alpha \gamma}.$$

All of them tend to $\frac{1}{2}$. No known method reaches a fixed $\theta < \frac{1}{2}$. That would be a zero-free strip $\beta < 1 - \eta_0$, a quasi-RH, open [OPEN]. A fixed θ for every $\theta > 0$ is RH (Lemma 67 [RESTATEMENT]).

The obstruction, in three precise forms.

- (O1) [THEOREM]. $P_a > 0$ for every $a > 0$. Any positivity statement that is linear with non-negative weights in the screen density holds for *every* configuration with $0 < \beta < 1$, so it cannot exclude a single strip ghost. Quantitative information can only come from *size* (the $|\varphi|^2$ term, L^2 norms of Fourier coefficients). Size is decisive only as the width $1 - \beta \rightarrow 0$, i.e. near the screen. This is why every row of Table 1 is an edge statement.
- (O2) [CITED] (conditional). For generic cofinite Γ the nonvanishing on $\sigma = 1$ survives but the pole-free region does not: “one cannot apply this Eisenstein series method ... to get zero free regions at least if all one uses is the general spectral theory” [8, p. 8]. Any improvement must use the arithmetic of $\text{SL}_2(\mathbb{Z})$, i.e. the Hecke data in the Fourier coefficients (this is what Sarnak’s sieve does).
- (O3) [CITED](abstract). Beurling systems: Diamond–Montgomery–Vorhauer construct generalized primes with $N_B(x) = \kappa x + O(x^\theta)$, $\frac{1}{2} < \theta < 1$, whose zeta function has infinitely many zeros on $\sigma = 1 - a/\log t$ and none to the right [1, Thm. 1]. So positivity of the prime measure plus PNT-type regularity cannot beat the dVP shape. Anything better must use structure that Beurling systems lack (functional equation, automorphy) [HEURISTIC].

8. GHOST INJECTION: WHICH POSITIVITY BREAKS FIRST?

Models ([NUMERICAL]; all models are hypothetical functions, not ζ). A ghost replaces the on-line zeros $\frac{1}{2} \pm i\gamma_a, \frac{1}{2} \pm i\gamma_b$ (consecutive zeros $n, n+1$) by $\frac{1}{2} \pm \delta \pm i\bar{\gamma}$, $\bar{\gamma} = \frac{\gamma_a + \gamma_b}{2}$, through $\xi \mapsto \xi Q$ (§3.1). Write $\eta = \frac{1}{2} - \delta = 1 - \beta$ for the distance to the edge. Two ways to treat the *local* data:

- *frozen*: the Hecke data $(\sigma_{-2it}(m))$, hence $S(t)$ are kept while ζ is replaced by ζQ . This is “global change, no local change”;
- *consistent*: S is rescaled by $|Q(1+2it)|^2$, as Ingham’s identity would require if the local data moved coherently. This is not realizable by an Euler product (it is Beurling-like) [HEURISTIC].

Heights: $n = 1$ ($\bar{\gamma} = 17.58$), 10 (51.37), 29 (100.07), 79 (199.64), the Lehmer pairs 363 (630.47/630.81) and 453 (750.66/750.97), and six WKB heights up to $7 \cdot 10^4$.

positivity	logical status	response to a strip ghost	first break
screen density / TF spectral side $h(r_j) \geq 0$	[THEOREM] (any $0 < \beta < 1$)	bump sharpens, mass conserved ($\leq 2 \cdot 10^{-7}$); Maass spectrum unchanged; TF residual local in height (Gaussian at 0 sees $\sim 10^{-11}$ at the 630 pair)	never
cone $\mathcal{C}_{1/2}$ (L67)	[THEOREM]	same kernel as the screen (Thm. 7.2)	never
curvature normal-ization Res $\varphi = 1/\text{vol}$	[THEOREM] for ζ	violated by $Q(1)/Q(2) - 1$: $1.6 \cdot 10^{-3}$ ($n = 1$), $2 \cdot 10^{-11}$ ($n = 363$); not a positivity	consistency only
fixed-point bal-ances $(\varphi(\frac{1}{2}) = -1$, MS at $s = \frac{1}{2}$, $\lambda_1 > 0$)	[THEOREM]	identically satisfied	never
MS-Sarnak, frozen	[THEOREM] for ζ ; models [NUMERICAL]	flags the <i>collision</i> already at $\delta = 0$ ($n = 1$: ratio 58; $n = 10$: 13; $n = 29$: 7.5; $n = 79$: 17); Lehmer pairs all- m : flagged at $\eta = \frac{1}{2}$; $m = 1$ floor flags $\beta > 0.971$ ($n = 363$), $\beta > 0.9907$ ($n = 453$); large heights, horizontal move: $\beta > 0.60$	local/global in-consistency, not displacement
MS-Sarnak, consistent	[NUMERICAL]	ratio decreases: 0.85, 0.59, 0.34, 0.039, 0.004 at $\eta = 0.4, 0.2, 0.1, 0.01, 0.001$ ($n = 363$; $n = 453$ alike)	never
Weil W_A (L64 engine)	RH $\iff W \geq 0$ [RESTATEMENT]	$\lambda_{\min} < 0$ for every tested $\delta \geq 0.002$ once $A \geq A^*(\delta)$	first, at a cost
Li λ_n	[RESTATEMENT] [5]	$\lambda_n < 0$ from $n = 6753$ ($\delta = 0.49$), 20332 ($\delta = 0.2$) at $\bar{\gamma} = 17.6$	late
L63 curvature κ	RH $\Rightarrow \kappa > 0$	visible iff $\delta < \delta_\kappa \approx \sqrt{2/B}$; $\delta_\kappa > \frac{1}{2}$ at all tested heights	immediately ($\delta = 0^+$)

TABLE 2. Which positivity breaks first (num/169/).

[NUMERICAL] *Details.* (1) *Weil* (weil_astar_n1.json, $K = 20$, float64; the engine was validated against 1000 zeros to 0.032, which is exactly the predicted truncation tail $\approx \frac{4(2j+1)}{2\pi A} \frac{\log(T/2\pi)+1}{T}$). At $\bar{\gamma} = 17.58$: $A^*(\delta) = 1.125, 1.05, 1.025, 0.975, 0.875, 0.825, 0.75, 0.725, 0.70$ for $\delta = 0.002, 0.005, 0.01, 0.02, 0.05, 0.1, 0.2, 0.3, 0.49$. At $\bar{\gamma} = 51.37$ the float run sees nothing for $A \leq 1.4$. At 256 bits with $K = 56$: $\delta = 0.05$ is detected at $A = 1.6$ ($\lambda_{\min} = -3.2 \cdot 10^{-19}$, against $\lambda_\zeta = 1.7 \cdot 10^{-17}$); $\delta = 0.01$ at $A = 1.8$ ($-1.6 \cdot 10^{-15}$; $\lambda_\zeta = 7.4 \cdot 10^{-23}$); at $A = 2.0$, $\delta = 0.49$ gives -0.186 and $\lambda_\zeta = 1.7 \cdot 10^{-26}$. The $\delta = 0$ (collision) baseline stays positive. The detection is tautological in the sense that W is the Gram form of the zero configuration; it measures the window needed, not a mechanism. (2) κ (kappa_*.json). $\delta_\kappa = 5.79, 2.93, 2.45, 1.89, 1.33, 1.45$ at $\bar{\gamma} = 17.6, 51.4, 100.1, 199.6, 630.6, 750.8$, against $\sqrt{2/B} = 5.77, 2.92, 2.45, 1.89, 1.32, 1.45$, where B is the background κ at $\bar{\gamma}$ with the pair removed. κ could miss strip ghosts only where $B > 8$. (3) *Large heights* (ms_large.json). At $\bar{\gamma} = 2254.7, 5229.2, 10113.9, 20002.0, 40095.0, 70149.2$: ratio 0.945, 0.944, 0.948, 0.950, 0.948, 0.948; frozen

horizontal threshold $\eta_h^* = 0.392, 0.392, 0.397, 0.402, 0.401, 0.403$. For comparison, the Mossinghoff–Trudgian width $1/(5.573412 \log \gamma) = 0.023, \dots, 0.016$.

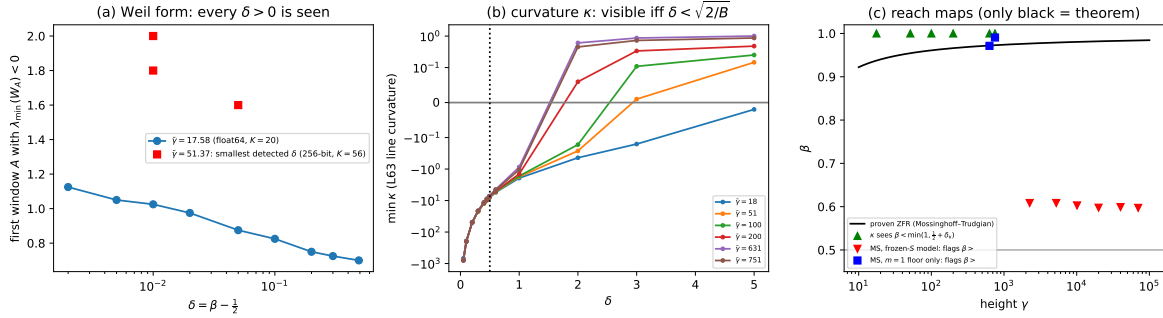


FIGURE 5. (a) Weil: the first window A with $\lambda_{\min}(W_A) < 0$ as a function of δ . (b) κ : the minimum of Lemma 63's line curvature near the ghost, against δ (dotted: the strip edge $\delta = \frac{1}{2}$). (c) Reach maps: only the black curve (the proven zero-free region, Mossinghoff–Trudgian) is a theorem; the triangles and squares are model thresholds [NUMERICAL].

Reading ([HEURISTIC], carefully). The informative failure is the frozen Maass–Selberg test. It fires for the wrong reason (the collision, at $\delta = 0$), and that is the point. Because the inequality is nearly saturated (Fig. 4), it detects any mismatch between the local Hecke data and the global ζ , not off-line displacement as such. Once the local data are allowed to follow the global change (consistent model), the ghost *lowers* the ratio and is never flagged. Its only genuine bite is near $\beta \rightarrow 1$, through the $m = 1$ floor or a sieve, exactly as in Sarnak's proof. Positivities that see bulk ghosts (W , λ_n , κ) are RH-equivalent or RH-consequences, so their sensitivity is *not* evidence for RH (circularity flag).

9. “NO GLOBAL BOOST WITHOUT A LOCAL ONE”

Precise local boost ([THEOREM]). The Satake parameters of $E(\cdot, s)$ at p are $p^{\pm(s-\frac{1}{2})}$, so its local boost (rapidity per $\log p$) is $\operatorname{Re} s - \frac{1}{2}$: 0 on the unitary line, $\frac{1}{2}$ at the residue $s = 1$. At a resonance $s = \rho/2$ it equals $-(1-\beta)/2$. Hence

RH $\iff$ every resonance of φ has local boost of size exactly $\frac{1}{4}$ [RESTATEMENT].

The proven statement $0 < \beta < 1$ is “local boost strictly between 0 and $\frac{1}{2}$ ”, i.e. strictly inside the unitary range of the complementary series (Thm. 7.2) [THEOREM].

Evaluation.

- (i) *As a consistency principle: right* ([NUMERICAL]+[HEURISTIC]). The frozen model is literally “a global boost (a zero moves) with no local boost (Hecke data fixed)”. Maass–Selberg, being nearly saturated, rejects it. Even the curvature normalization rejects it, at order $\bar{\gamma}^{-4}$ (Rem. 2.2(iv)). Global data cannot move unless local data move with them.
- (ii) *As a proof principle: empty beyond $\beta < 1$* ([THEOREM]/[CITED]). In the consistent model the local data do move, and nothing is flagged. (O1)–(O3) of §7 show that *every* version using positivity alone stops at the edge: Beurling systems [1] have positive local data and zeros at $1 - a/\log t$.
- (iii) *The norm picture* (§4). For Δ every local boost is 0 (all z_p elliptic, Deligne [BIB]). For E_{12} every local datum is a pure boost (on $\partial\mathbb{H}$). A ghost of ζ would be a global boost $\frac{1}{4} - \frac{1-\beta}{2} \neq 0$ invisible to each individual prime. So the heuristic, read literally (“each local boost is 0, hence the global boost is 0”), is exactly the missing local-to-global step: [HEURISTIC], not a theorem, and equivalent in strength to RH for automorphic L -functions [RESTATEMENT].

- (iv) *Gauss-law version: a theorem* (Thm. 5.1). The total flux of the screen density is conserved under every dent. “No global *flux* change without a local one” is true, but it constrains mass, not position.

10. SYNTHESIS: THE DIAMOND

[HEURISTIC] (with the [THEOREM] ingredients marked above). (1) *Curvature is a screen*. The Gauss curvature of $\Gamma \backslash \mathbb{H}$ is fixed, and its only arithmetic content is $\text{Res } \varphi = 1/\text{vol}$, the pole sitting at the fixed point of the reflection. (2) *Proven positivities are Poisson positivities*. Maass–Selberg, $h(r_j) \geq 0$ in the trace formula, the cone $\mathcal{C}_{1/2}$ and the L^2 norm of $\Lambda^T E$ all see each zero through $P_{1-\beta}$ and hold iff $0 < \beta < 1$. Their only quantitative bite is at the edge, and that bite is a zero-free region. (3) *Positivities that see bulk ghosts* (W_A , Li’s λ_n, κ) are RH-equivalent or RH-consequences, so they cannot be used to prove RH without circularity. (4) *What an RH-level screen would need*: a screen at $\text{Re } w = \frac{1}{2}$ through which every zero is seen as a *residue* (width 0), as the pole at $w = 1$ is seen by the anomaly. That is a spectral realization of the zeros, the Hilbert–Pólya requirement restated [RESTATEMENT]/[OPEN]. J.W.’s anomaly is the one place where a zero-width object (a pole on its screen) is already handled exactly; the zeros of ζ are not.

11. CORRECTIONS TO THE CHAT CLAIMS

- (C1) “Anomaly coefficient = Gauss–Bonnet area”: **wrong**. It is $1/\text{area}$: $d(E_2^* dz) = -d\mu/\text{vol}$, $3/\pi = 1/(\pi/3)$ (Thm. 2.1).
 (C2) “= Eisenstein pole residue”: **correct**, with the normalization $\Gamma_\infty \ni -I$. Other conventions give $6/\pi$ or π (§1).
 (C3) “ $\leftrightarrow$ Weil POLE”: **same pole, different value**. The area sits in the identity (ARCH-like) term (Rem. 2.2).
 (C4) “Curvature terms relate directly to ghosts”: **only as a normalization**. The curvature does not move; ghosts move the scattering density (Thm. 5.1). Reading “curvature” as the screen density is an [ANALOGY] made precise.
 (C5) “Maass–Selberg positivity gives $\zeta(1+it) \neq 0$ (Sarnak 2004)”: **correct, attribution refined**. The qualitative argument is the standard Jacquet–Shalika-type one (Sarnak: “simple magical and standard”); Sarnak 2004 makes it *effective* (Thm. 6.3).
 (C6) “The reach is $\text{Re } s = 1$, parallel to L67’s $\theta = \frac{1}{2}$ ”: **correct and sharper than a parallel**. The kernels are identical (Thm. 7.2).
 (C7) Optimizing T (or s on the line) in Maass–Selberg: **collapses to $T = 1$** (Prop. 6.2).
 (C8) The inequality bounds the **distance to the edge** $1 - \beta$, not the distance to $\frac{1}{2}$ (Table 1).
 (C9) “Curvature positivity” is $L^2/\text{spectral}$ positivity of a norm, with the curvature only fixing its normalization ([ANALOGY]).
 (C10) The Kronecker constant is **exact**: $C = \frac{6}{\pi}(\gamma - \log 2) - 36\zeta'(2)/\pi^3$; a 10^{-12} discrepancy was a precision artefact (Thm. 2.1(b)).
 (C11) [t266u] (1), (2): **correct** (Thms. 3.1, 3.2). (3): correct, but the reflection $s \leftrightarrow 2 - s$ is exact for E_2 without the anomaly; the anomaly is the central residue $6/\pi = 2E_2(i)$ (Rem. 3.5). (4): a ghost does **not** break a fixed-point balance; it shows only in the full Taylor germ (Li), which is RH-equivalent (Thm. 3.6).
 (C12) [t267u]: **correct**, with two convention notes. $\pi_p = \alpha_p$ is the mirror $-\bar{\tau}_p$ of L67’s Frobenius point, and definiteness of $|c\tau + d|^2$ is equivalent to $y \neq 0$ (Rem. 4.2).
 (C13) [t268u]: **correct** as recorded in Thm. 4.3. The norm p^{11} is blind, the type is the discriminant $(\pi - \bar{\pi})^2$, $-3/(\pi y) = -6i/(\pi(\tau - \bar{\tau}))$, and the congruence mod 691 is verified for all $p < 10^4$.

12. RANKED DIRECTIONS

Score = 0.6·significance+0.4·probability of a definite result in one lemma (both in $[0, 1]$; judgments, [HEURISTIC]).

	direction	sig.	prob.	score
1	D1 Tapering ladder. Make the dictionary of §7 a theorem: zero-free region $\eta(\gamma) \iff$ positivity of a family of height-localized Weil functionals with kernels $P_{\theta(\gamma)\pm\delta}$ (L67 D3, screen at $\operatorname{Re} w = \frac{1}{2} + \theta$).	0.6	0.65	0.62
2	D6 Zero-width screen. A screen on which every zero is a residue (Hilbert–Pólya, §10(4)).	1.0	0.01	0.60
3	D3 Beurling stress test. Run every TAA positivity (screen, $\mathcal{C}_\theta$, MS-type, λ_1 , κ) on a DMV-type system to sort which ones use automorphy.	0.7	0.4	0.58
4	D2 Sieve constants. Make Sarnak’s (39)–(53) explicit and compare the resulting c with Mossinghoff–Trudgian’s R_0 .	0.4	0.8	0.56
5	D8 Global norm. Look for a single positive form g (point of a period domain) assembling the local g_{α_p} of §4; its positivity would be Weil’s.	0.9	0.05	0.56
6	D4 κ at large height. Locate where $B > 8$, the heights where κ starts missing strip ghosts.	0.3	0.9	0.54
7	D5 Matched Weil cone. Smallest θ with $W _{\mathcal{C}_\theta}$ detecting a ghost at δ ; test $\theta^* = \delta$ numerically.	0.4	0.7	0.52
8	D7 Fixed-point germ. The even power sums live in the non-constant Fourier coefficients (Thm. 3.6(d)); look for a positivity at the fixed point $s = \frac{1}{2}$ that uses them, and check whether it is Li in disguise.	0.5	0.5	0.50

13. QUESTIONS FOR J.W.

- Q1. Do you accept the correction “anomaly = $1/\text{area}$ ” (Stokes forces $\operatorname{Res} \times \text{vol} = 1$), together with $\operatorname{Res}_{s=1} \Lambda(E_2, s) = 2E_2(i) = 6/\pi$ and “ $-3/(\pi y)$ is the reflected half $\varphi(s)y^{1-s}$ ” (Thms. 2.1, 3.3, 3.4), as the official form of your [t265u]/[t266u] intuition?
- Q2. Should “denting” officially mean a width change at fixed mass on the screen $\operatorname{Re} w = 1$ (Thm. 5.1), with the Gauss curvature itself recognized as ghost-blind?
- Q3. Do you accept the reach verdict: Eisenstein/Maass–Selberg positivity is exactly the rung $\theta = \frac{1}{2}$ and yields only tapering rungs (zero-free regions), never a fixed $\theta < \frac{1}{2}$ (Thm. 7.2, (O1)–(O3))?
- Q4. Do you accept that ghosts respect the reflection (orbits of length 2) and therefore never break a fixed-point balance, so that the fixed-point route to RH must go through the full germ (Li), i.e. is RH-equivalent (Thm. 3.6)? And should $g_\tau = |c\tau + d|^2/y$, with “degeneracy = gap” (Thm. 4.3), be the official “norm” uniting the anomaly term, Hasse’s form and L65’s g_p , with the local-only caveat?
- Q5. Next lemma: D1 (tapering ladder), D3 (Beurling stress test), or D8 (global norm)?

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TAA LEMMA 70: THE WEIGHT-2 EISENSTEIN CORE AND THE CUSP. CAN THE 691 AND MAZUR CONGRUENCES BE READ OFF THE LOW-TEMPERATURE EXPANSION?

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Nothing below is a step toward a proof that we can certify. Every RH-equivalent statement is labelled [RESTATEMENT], and every metaphor is labelled [ANALOGY] or [HEURISTIC]. Tags: [THEOREM] proved here (elementary) or classical with the proof indicated; [CITED] checked at the source (`lit/lemma70/verified_refs.md`); [CITED-sec] via a named secondary source (second-hand); [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (PARI/GP 2.17.2 exact arithmetic or mpmath; finite ranges, not proofs).

Question (J.W., [t269u]–[t273u]). Is the lower-weight Eisenstein structure (E_2 , its quasi-modular anomaly, and the correction $-3/(\pi y)$) the irreducible core? Can the 691 congruence (weight 12, level 1) and Mazur’s Eisenstein congruences (weight 2, prime level N , modulus $\text{num}((N-1)/12)$) be read off the low-temperature ($q \rightarrow 0$, cusp) expansion alone? **Steering [t274u]:** add “The Weil light cone as $\mathbf{1} \oplus \mathbb{L}$ ”.

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ANSWER IN BRIEF

- (1) *Yes for existence, no for identity, and never for positivity.* In both cases the congruence *modulus* is the numerator of a constant term, that is, of an Euler-modified value of ζ at a negative integer: $\text{num}(\zeta(1-k)/2) = \text{num}(B_k/2k)$ at level 1, and $n_N = \text{num}((1-N)\zeta(-1)) = \text{num}((N-1)/12)$ at prime level N [THEOREM]/[CITED] (§7). Mazur proves that $T/\mathfrak{I} \cong \mathbb{Z}/n_N$ and that the series $\delta = \sum \sigma'(n)q^n$ is a cusp form mod m exactly when $m \mid n_N$ ([7, II (5.12), (9.7)], read [CITED]). Numerically, every one of the 255 pairs (N, ℓ) with $N \leq 997$ prime and $\ell \mid n_N$ has a congruent newform, and every one of the 120 pairs (k, ℓ) with $12 \leq k \leq 100$ and $\ell \mid \text{num}(B_k/2k)$ has exactly one congruent prime λ [NUMERICAL].
- (2) *The cusp ∞ alone is not enough. The two cusps together are.* The constant term $(N-1)/24$ of $E_2^{(N)}$ at ∞ gives $\text{num}((N-1)/24)$. This differs from n_N by a factor 2 exactly when $N \equiv 1 \pmod{8}$ (37 of the 168 primes $N \leq 997$) [THEOREM]+[NUMERICAL]. For $N \equiv 9 \pmod{16}$ the cusp ∞ misses the prime 2 entirely, yet the mod-2 congruence exists (e.g. $N = 41$). Mazur's proof recovers the factor from the residue at the *other* cusp $0 = w(\infty)$, together with the fact that constants are weight-2 forms mod m only for $m \mid 12$ [7, II (5.11)]. So the right object is the pair of constant terms exchanged by the Atkin–Lehner reflection w . This is the $\Gamma_0(N)$ analogue of $s \leftrightarrow 1-s$.
- (3) *Three mechanisms, only one of them a pure constant term.* Numerically $\tau(p) \equiv p^a + p^b \pmod{\ell}$ for all $p \leq 300$ exactly for $\ell = 2 (1+p^{11}), 3 (1+p^{11}), 5 (p+p^2), 7 (p+p^4)$ and $691 (1+p^{11})$; there is no such pair for $\ell = 11, 13, 23$ [NUMERICAL]. The reasons differ. Mod 691 it is the numerator of $\zeta(-11)/2$. The primes 2, 3, 5, 7, 13 are exactly the primes of the *denominator* 65520 of $B_{12}/24$ ($\ell-1 \mid 12$). For those primes constants are weight-12 forms mod ℓ ($E_4^3 \equiv 1 \pmod{2, 3, 5}$; $E_6^2 \equiv 1 \pmod{2, 3, 7}$; $E_{12} \equiv 1 \pmod{13}$), so the constant term carries *no* information there [THEOREM]. The actual answer is untwisted at 2, 3, twisted at 5, 7 ($\Delta \equiv \theta E_2 \pmod{5}$; $\Delta \equiv \theta E_4/240 \pmod{7}$, with $\theta = q d/dq$ killing the constant term), and none at 13. So E_2 *itself* is the mod-5 shadow of Δ (classical; re-verified for $n \leq 400$ [NUMERICAL]). This partly supports J.W.'s hunch that lower weight is where the structure lives, though not through positivity.
- (4) *Low temperature suffices to certify, not to explain.* Sturm's bound makes the congruence a statement about *finitely many* low-order coefficients. Mod 691, the coefficients a_0, a_1 of $\Delta - E_{12}$ suffice. At level N , the first $\lfloor (N+1)/6 \rfloor$ suffice [BIB]+[NUMERICAL]. But the theorem that these suffice is global (the q -expansion principle, Deligne–Serre lifting, Mazur's $T/\mathfrak{I}$). The slogan “read off the cusp alone” is a [HEURISTIC] unless that global input is stated.
- (5) *Thermal reading (§8).* With $q = e^{-\beta}$, $U = -E_2$ is the mean energy of 24 bosons, the Casimir value is -1 , and $U^* = U + 6/\beta = -E_2^*$ transforms with weight 2 under $\beta \leftrightarrow 4\pi^2/\beta$. All of this is checked to 10^{-30} [THEOREM]+[NUMERICAL]. The moduli appear thermally *only* as zero-point data: $691/65520 = \zeta(-11)/2$, a regularized vacuum sum, and the Casimir gap $N-1$ of $Z(\beta)/Z(N\beta) = \Delta(N\tau)/\Delta(\tau)$, reduced by the 12-torsion of the cuspidal divisor [7, Thm. (1)]. They do not appear in the shape of $U(\beta)$.
- (6) *The Weil light cone as $\mathbf{1} \oplus \mathbb{L}$ (§4, steering [t274u]).* On $C \times C$ with $C = X_0(11)$, the span of e_1, e_2, Γ_F has signature (1, 2) and determinant $2q$, and adding Δ gives (1, 3), for all 13 primes tested [NUMERICAL]. The complement of the (1, 1) plane $\langle e_1, e_2 \rangle$ carries $-D'^2 = 2Q_p(m, n)$, exactly Lemma 65's form; it is negative definite iff $a_p^2 < 4p$ [THEOREM]. For E_2 's “surface” $\mathbb{P}^1 \times \mathbb{P}^1$ the complement is 0 and the bound degenerates to the equality $N_1 = q+1$ [THEOREM]. Over $\mathbb{Q}$, “RH $\Leftrightarrow$ the complement of the pole plane is definite” is a [RESTATEMENT] of Weil positivity, not a theorem.

- (7) *Verdict on “ E_2 is the irreducible core” [HEURISTIC].* What holds: the congruence modulus at weight 2 is built from $\zeta(-1) = -1/12$, the same $1/12$ behind the anomaly ($E_2 = 1 - 24 \sum \dots$, $3/\pi = 1/\text{vol}$, Lemma 69) and behind $\text{num}((N-1)/12)$. What fails: congruences live in $\mathbb{F}_\ell$, and *positivity is archimedean*. The congruent pair (timelike f , spacelike $E_2^{(N)}$) agrees mod λ while sitting on opposite sides of the light cone, so the cusp data cannot see the sign that RH needs.

1. SETUP AND NORMALIZATIONS

$q = e^{2\pi i \tau}$ and $\tau = x + iy$. $E_k = 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n$; the integrally normalized form is $\tilde{E}_k = -\frac{B_k}{2k} + \sum \sigma_{k-1}(n) q^n$, with constant term $\zeta(1-k)/2$. $E_2 = 1 - 24 \sum \sigma_1(n) q^n$, $E_2^* = E_2 - 3/(\pi y)$ (Lemma 69). For prime N ,

$$E_2^{(N)}(\tau) := -\frac{1}{24}(E_2(\tau) - NE_2(N\tau)) = \frac{N-1}{24} + \sum_{n \geq 1} \sigma'_N(n) q^n, \quad \sigma'_N(n) = \sum_{d|n, N \nmid d} d.$$

$n_N := \text{num} \frac{N-1}{12}$. $\mathbb{T}$ is the Hecke algebra of $S_2(\Gamma_0(N))$ and $\mathfrak{I}$ the Eisenstein ideal, generated by $1 + \ell - T_\ell$ ($\ell \neq N$) and $1 + w$ [7, II §9] [CITED]. $\delta := \sum_{n \geq 1} \sigma'_N(n) q^n$ is Mazur’s series. Binary forms $[A, B, C] = Am^2 + Bmn + Cn^2$ have discriminant $B^2 - 4AC$. Lemma 65’s Hasse form is $Q_p(m, n) = m^2 + a_p mn + pn^2$ (sign convention of Lemma 69, §4).

2. E_2 AS $\mathbf{1} \oplus \mathbb{L}$: CHAT CLAIM (A)

Proposition 2.1 ([THEOREM]). (i) $T_p \tilde{E}_k = \sigma_{k-1}(p) \tilde{E}_k$ with $\sigma_{k-1}(p) = 1 + p^{k-1}$, and $L(\tilde{E}_k, s) = \zeta(s)\zeta(s-k+1)$; this includes $k = 2$, where E_2 is quasi-modular but still a formal T_p -eigenseries (Lemma 67). (ii) For $k = 2$ the “Frobenius” has eigenvalues $1, p$, trace $p + 1 = \#\mathbb{P}^1(\mathbb{F}_p)$ and determinant p ; these are the eigenvalues of Frobenius on H^0, H^2 of $\mathbb{P}^1$, i.e. the ℓ -adic realization of $h(\mathbb{P}^1) = \mathbf{1} \oplus \mathbb{L}$. (iii) The discriminant of $x^2 - (1+p)x + p$ is $(p-1)^2 > 0$ for $p \neq 1$. After normalization by $\sqrt{\det} = \sqrt{p}$ the eigenvalues are $p^{\mp 1/2} = e^{\mp \frac{1}{2} \log p}$: a boost of rapidity $\frac{1}{2} \log p$ (weight $k: \frac{k-1}{2} \log p$). (iv) $1 + p \geq 2\sqrt{p}$, with equality iff $p = 1$ (AM-GM).

Proof. (i) is classical (Hecke). The Euler product follows from $\sum \sigma_{k-1}(n) n^{-s} = \zeta(s)\zeta(s-k+1)$. (ii)–(iv) are linear algebra. $\square$

Remark 2.2 (What is a Restatement, what is a Heuristic). “ $E_2 = \mathbf{1} \oplus \mathbb{L}$ ” is a statement about realizations (traces of Frobenius, L -function), not an identity of motives. E_2 is not a modular form, and the Eisenstein “motive” is just the (ℓ -adic) realization $\mathbf{1} \oplus \mathbb{L}$ [RESTATEMENT]. “ $E_2 = \text{Weil’s pole term}$ ” is the same [RESTATEMENT]: the pole term in Weil’s explicit formula comes from the poles of ζ at $0, 1$, i.e. from H^0, H^2 (Lemma 65). The $\mathbb{F}_1$ reading of $p = 1$ is [HEURISTIC]. **Correction to (a):** “ $p \mapsto 1/p$ swaps the eigenvalues” is true for the *normalized* pair $p^{\mp 1/2}$. The unnormalized pair $\{1, p\}$ goes to $\{1, 1/p\}$, which is the swapped pair rescaled by $1/p$ (the functional equation’s twist by $\mathbb{L}^{-1}$).

3. THE BINARY-FORM LIGHT CONE: CHAT CLAIM (C)

Proposition 3.1 ([THEOREM]). (i) On $V = \{[A, B, C]\} \cong \mathbb{R}^3$, $\text{disc} = B^2 - 4AC$ has signature $(2, 1)$. In the coordinates $t = \frac{A+C}{2}$, $x = \frac{A-C}{2}$, $z = \frac{B}{2}$ one has $\frac{1}{4} \text{disc} = x^2 + z^2 - t^2$. (ii) $\text{SL}_2(\mathbb{R})$ acts by substitution, preserving disc, and this gives $\text{SL}_2(\mathbb{R})/\{\pm 1\} \cong \text{SO}(2, 1)^0$; so $\text{SL}_2(\mathbb{R}) \cong \text{Spin}(2, 1)$. (iii) Definite forms ($\text{disc} < 0$) are timelike, indefinite ones spacelike, degenerate ones null. The forms $A|m - \tau n|^2$ with $AC - B^2/4 = 1$, $A > 0$ make up the upper sheet of the unit hyperboloid, and $\tau \mapsto \text{form}$ identifies it with $\mathbb{H}$. (iv) The forms divisible by a fixed real linear factor L make up the plane tangent to the null cone along L^2 .

Proof. (i) is direct. (ii) is the standard exceptional isomorphism (the adjoint action on $\mathfrak{sl}_2$, whose Killing form is a multiple of disc). (iii) is the definition plus $\tau \leftrightarrow$ its root. (iv): $L \cdot M$ with M variable is a linear 2-space containing the null vector L^2 , and on it $\text{disc}(LM) = \text{res}(L, M)^2 \geq 0$, vanishing only on the line $M \propto L$. So it is the tangent plane along L^2 . $\square$

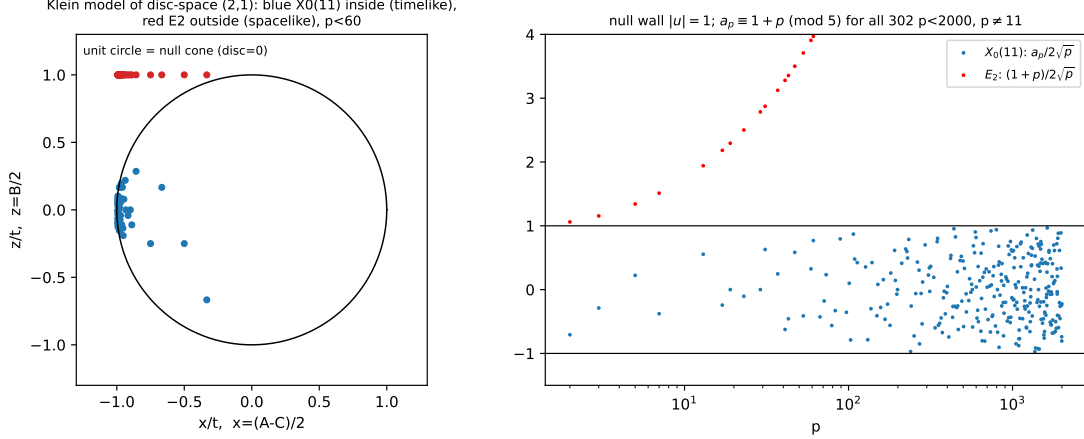


FIGURE 1. Left: Klein model $(x/t, z/t)$ of the $(2, 1)$ form space; the unit circle is the null cone. Blue: normalized Hasse forms $Q_p/\sqrt{p}$ of $X_0(11)$ (11a1), all inside (timelike). Red: E_2 's forms $(m+n)(m+pn)/\sqrt{p}$, all on the line $z/t = 1$. That line is the tangent to the cone at $(0, 1)$, the image of the null ray $(m+n)^2$ (Prop. 3.1(iv)), and the touching point is the wall $p = 1$. Right: normalized trace $u_p = a_p/2\sqrt{p}$ against $(1+p)/2\sqrt{p}$. The null wall is $|u| = 1$, and $a_p \equiv 1+p \pmod{5}$ holds for all 302 primes $p < 2000$, $p \neq 11$ [NUMERICAL].

Remark 3.2 (E_2 is tangent to the cone [THEOREM], reading [HEURISTIC]). Every E_2 form $(m+n)(m+pn)$ contains the common null factor $m+n$. So the whole E_2 family lies in a *single* tangent plane of the null cone, and it touches the cone only in the limit $p \rightarrow 1$. In the language of [t271u], E_2 is “spacelike but grazing”. Its boost is built on a null direction shared by all p , which is the **1** (trivial) factor. The physical Lorentz reading remains [HEURISTIC], for the reasons given in chat: $\text{SO}(2, 1)$ and not $\text{SO}(3, 1)$, no dynamics, and Lorentzian signature is generic for index-one forms.

4. THE WEIL LIGHT CONE AS $\mathbf{1} \oplus \mathbb{L}$ (STEERING [T274U])

4.1. $(\mathbf{1}) \oplus$ and traces.

Proposition 4.1 ([THEOREM]). *In an additive category, $\oplus$ is the biproduct: simultaneously product and coproduct, with ι_j, π_j such that $\pi_j \iota_j = 1$ and $\sum \iota_j \pi_j = 1$. Chow motives with $\mathbb{Q}$ -coefficients form such a category, pseudo-abelian and $\otimes$ -symmetric [10] [BIB]. For any realization with an endomorphism F , $\text{Tr}(F | M \oplus M') = \text{Tr}(F | M) + \text{Tr}(F | M')$ and $\text{Tr}(F | M \otimes M') = \text{Tr}(F | M) \text{Tr}(F | M')$. Since $\text{Tr}(F | \mathbb{L}) = p$, $\text{Tr}(F | \mathbf{1} \oplus \mathbb{L}^{\otimes 11}) = 1 + p^{11} = \sigma_{11}(p)$.*

Proof. Block-diagonal and Kronecker-product traces. $\square$

4.2. (2) The hyperbolic plane on $C \times C$ and the Hodge index. Conventions follow [8, §1] (read): $C_1 = C \times \text{pt}$ and $C_2 = \text{pt} \times C$ satisfy $C_1^2 = C_2^2 = 0$ and $C_1 \cdot C_2 = 1$; for a divisor D , $d_1 = D \cdot C_1$ and $d_2 = D \cdot C_2$. For the graph of $f : C \rightarrow C$, $d_1 = \deg f$ and $d_2 = 1$, and adjunction gives $\Gamma_f^2 = \deg f (2 - 2g)$, so $\text{def}(\Gamma_f) = 2g \deg f$ [8, Ex. 1.7]. Castelnuovo–Severi, $D^2 \leq 2d_1 d_2$, is

proved there by applying the Hodge index theorem to $D - d_2C_1 - d_1C_2 \perp (C_1 + C_2)$ [8, Thm. 1.5]. The consequence $|(\Delta \cdot \Gamma_\pi) - q - 1| \leq 2gq^{1/2}$ is stated there with $\text{def}(\Delta) = 2g$ and $\text{def}(\Gamma_\pi) = 2gq$ [CITED]. We write $e_1 = C_1$ and $e_2 = C_2$.

Theorem 4.2 ([THEOREM], using [8]). (i) $\langle e_1, e_2 \rangle$ is a hyperbolic plane of signature $(1, 1)$ with null basis. Its orthogonal complement in $\text{NS}(C \times C)_\mathbb{R}$ is negative definite (Hodge index). (ii) $\Gamma_F \cdot e_1 = q$ and $\Gamma_F \cdot e_2 = 1$, and the projections are $\Gamma' = \Gamma_F - e_1 - qe_2$ and $\Delta' = \Delta - e_1 - e_2$. (iii) $\Delta'^2 = -2g$, $\Gamma'^2 = -2gq$ and $\Gamma' \cdot \Delta' = N_1 - q - 1 = -\text{Tr}(F | h^1)$. For $g = 1$, $-(m\Delta' + n\Gamma')^2 = 2(m^2 + amn + qn^2) = 2Q_q(m, n)$, which is Lemma 65's $\text{def}(m\Delta + n\Gamma) = 2Q_p$. (iv) Negative definiteness of the complement (Castelnuovo–Severi $D'^2 \leq 0$) gives $|\text{Tr}(F | h^1)| \leq 2g\sqrt{q}$, i.e. $|a| \leq 2\sqrt{q}$ for $g = 1$.

Proof. (ii)–(iii) by bilinear expansion with the numbers above. (iv) is [8, 1.6–1.7]. $\square$

Remark 4.3 (Künneth reading [BIB]/[RESTATEMENT]). For a curve, the Künneth projectors π_0, π_2 are algebraic, represented by $\text{pt} \times C$ and $C \times \text{pt}$ (which is which depends on the composition convention), and $\pi_1 = \Delta - \pi_0 - \pi_2 = \Delta'$. So $h(C) = \mathbf{1} \oplus h^1(C) \oplus \mathbb{L}$. The $(1, 1)$ plane is the $\mathbf{1} \oplus \mathbb{L}$ part: Γ_F 's components $(1, q)$ are the traces of F on h^0 and h^2 , and Γ' carries h^1 . In this sense J.W.'s “ $\mathbf{1} \oplus \mathbb{L}$ light cone” is the hyperbolic plane, with e_1, e_2 its two null rays.

4.3. **(3)** E_2, E_{12}, Δ . $E_2 \leftrightarrow h(\mathbb{P}^1) = \mathbf{1} \oplus \mathbb{L}$ and $E_{12} \leftrightarrow \mathbf{1} \oplus \mathbb{L}^{11}$ hold at the level of realizations and L -functions (Prop. 2.1) [RESTATEMENT]. Δ : Deligne attached the ℓ -adic representation using the cohomology of a Kuga–Sato variety (second-hand via [6], Thm. 3; Lemma 66). Scholl constructed the Chow motive [9] (Crossref record: Invent. Math. 100 (1990) 419–430; *text not read*, so its content is [BIB]: the motive is cut out of h^{11} of the 10-fold fibre product of the universal elliptic curve). For weight 12 the Hodge-index analogue is the Hodge–Riemann positivity on H^{11} , equivalently the Petersson norm (Lemma 68, [TAA-L]). In characteristic p it would be Grothendieck's Hodge standard conjecture, of which “almost nothing is known except for surfaces” [8, §3] [CITED]. Deligne's proof of the Weil conjectures did not go through the standard conjectures: Milne writes that “Deligne recognized the intractability of the standard conjectures and looked for other approaches” [8, §5] [CITED]. So the steering claim “Deligne avoided it” is **correct**.

4.4. **(4) Over $\mathbb{Q}$.** “RH $\Leftrightarrow$ the zero part is definite on the complement of the $(1, 1)$ pole plane” is a [RESTATEMENT] of Weil's positivity criterion in the Lemma 64/65 normalization $W = \text{POLE} + \text{ARCH} - \text{PRIME}$. It is not an independent statement, and the global surface it presupposes is the missing object (Lemma 65: $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z} = \text{Spec } \mathbb{Z}$) [OPEN].

4.5. **(5) Numerics.** Table 1 lists, for $C = X_0(11)$ (11a1, $y^2 + y = x^3 - x^2 - 10x - 20$) over $\mathbb{F}_p$, the Gram matrix on $(e_1, e_2, \Gamma_F, \Delta)$ with entries $[[0, 1, q, 1], [1, 0, 1, 1], [q, 1, 0, N_1], [1, 1, N_1, 0]]$.

Remark 4.4 (Diagnostic value and circularity flag). The table uses only a_p (computed by PARI) and the intersection numbers above. Its signature is therefore *equivalent* to Hasse for each p ($\det M = 4q - a^2$), so it checks the algebra and the conventions, not geometry. The informative parts are three. (a) $\det_3 = 2q$ shows that $\langle e_1, e_2, \Gamma_F \rangle$ is never degenerate, whatever a_p . (b) Adding Δ gives a nondegenerate rank-4 lattice, consistent with $\rho(\bar{C} \times \bar{C}) = 2 + \text{rk End}(\bar{E})$, which equals 4 for ordinary reduction (no claim is made for supersingular p , where $\rho = 6$). (c) The eigenvalue -2 at $a_p = 0$ ($p = 19, 29$) is where Δ' and Γ' are orthogonal: the “rotation by 90° ” of supersingular Frobenius.

Proposition 4.5 ($\mathbb{P}^1 \times \mathbb{P}^1$: E_2 's surface [THEOREM]). $\text{NS}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, with $\Gamma_F \equiv e_1 + qe_2$ ($\Gamma_F^2 = 2q$) and $\Delta \equiv e_1 + e_2$ ($\Delta^2 = 2$), so $\Gamma_F \cdot \Delta = 1 + q = N_1$. The complement of the pole plane is 0, so $\Gamma' = \Delta' = 0$ and the bound reads $|N_1 - q - 1| \leq 0$, i.e. equality. E_2 is all pole plane: no h^1 , nothing definite, nothing to bound. This is the geometric form of Lemma 66's “ E_{12} is all ghost”.

p	a_p	N_1	$\text{sig}(e_1, e_2, \Gamma)$	$\text{sig}(\text{all } 4)$	$\det_3$	$\text{eig}(M) \text{ on } \langle \Delta', \Gamma' \rangle$	$4p - a_p^2$	$5 \mid N_1$
2	-2	5	(1,2)	(1,3)	4	-5.236, -0.764	4	yes
3	-1	5	(1,2)	(1,3)	6	-6.236, -1.764	11	yes
5	1	5	(1,2)	(1,3)	10	-10.123, -1.877	19	yes
7	-2	10	(1,2)	(1,3)	14	-14.325, -1.675	24	yes
13	4	10	(1,2)	(1,3)	26	-26.649, -1.351	36	yes
17	-2	20	(1,2)	(1,3)	34	-34.125, -1.875	64	yes
19	0	20	(1,2)	(1,3)	38	-38.000, -2.000	76	yes
23	-1	25	(1,2)	(1,3)	46	-46.023, -1.977	91	yes
29	0	30	(1,2)	(1,3)	58	-58.000, -2.000	116	yes
31	7	25	(1,2)	(1,3)	62	-62.806, -1.194	75	yes
97	-7	105	(1,2)	(1,3)	194	-194.255, -1.745	339	yes
101	2	100	(1,2)	(1,3)	202	-202.020, -1.980	400	yes
1009	-10	1020	(1,2)	(1,3)	2018	-2018.050, -1.950	3936	yes

TABLE 1. Weil cone of $X_0(11) \times X_0(11)$ over $\mathbb{F}_p$ [NUMERICAL] (num/lemma70_cone.py). Here $\det_3 = 2q$ exactly; M is the Gram matrix of $\langle \Delta', \Gamma' \rangle$, equal to $-[[2, a], [a, 2q]]$. The last column is the mod-5 Eisenstein congruence: $N_1 = p + 1 - a_p \equiv 0$ (rational 5-torsion).

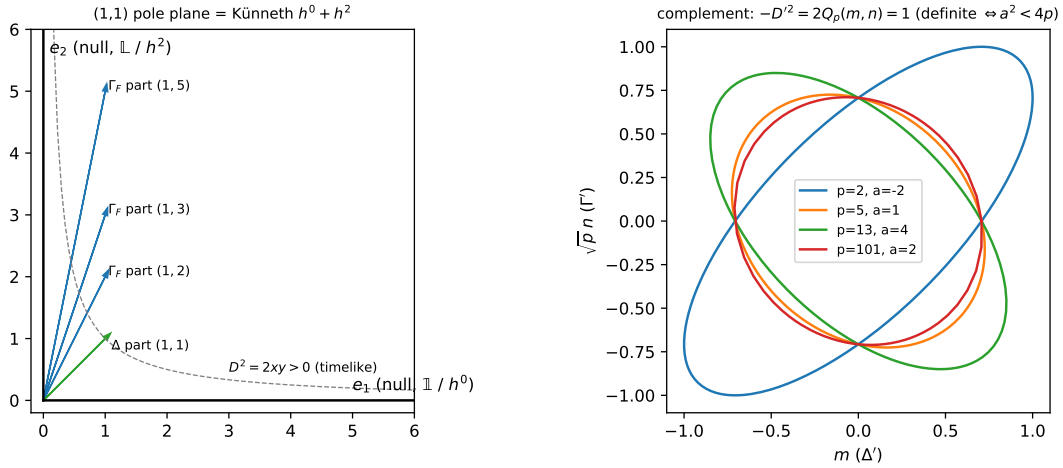


FIGURE 2. Left: the (1,1) pole plane $\langle e_1, e_2 \rangle$, whose two null rays are $\mathbf{1}$ (h^0) and $\mathbb{L}$ (h^2); Γ_F projects to $(1, q)$ and Δ to $(1, 1)$, both timelike ($D^2 = 2d_1d_2 > 0$). For $E_2(\mathbb{P}^1 \times \mathbb{P}^1)$ this is everything. Right: the complement for $X_0(11)$, the unit ellipses $-D'^2 = 2Q_p(m, n) = 1$ for $p = 2, 5, 13, 101$; these are ellipses (definite) exactly because $a_p^2 < 4p$.

5. MAZUR'S EISENSTEIN CONGRUENCES AT PRIME LEVEL: DIAGNOSTIC 1

5.1. What Mazur proves (read at source).

Theorem 5.1 (Mazur 1977 [CITED]). *Let $N \geq 5$ be prime and $n = n_N = \text{num } \frac{N-1}{12}$ [7, Conventions]. (i) [7, II (9.7)]: $\mathbb{T}/\mathfrak{I} \cong \mathbb{Z}/n\mathbb{Z}$. The Eisenstein primes correspond one-to-one to the primes $\ell \mid n$, with $\mathbb{T}/\mathfrak{P} = \mathbb{F}_\ell$. Mazur adds (just before (9.8)) that $n > 1$ iff the genus of $X_0(N)$ is > 0 . (ii) [7, II (5.12)]: for m prime to N , δ is the q -expansion of a holomorphic weight-2 form mod m iff $m \mid \frac{N-1}{2}$. For any m , δ is a parabolic (cusp) form mod m iff $m \mid n$. (iii) [7, II (5.11)(ii)]: if $1 + a_1q^N + \dots$*

with integral coefficients is a weight-2 form on $\Gamma_0(N) \bmod m$ (m prime to N), then $m \mid 12$. (iv) [7, Intro, Thm. (1)] (conjecture of Ogg): the torsion of $J_0(N)(\mathbb{Q})$ is cyclic of order n , generated by the class of $(0) - (\infty)$.

The proof of (ii)(iii) is the key to the “cusp-only” question. Mazur writes $e' = (N - 1) + 24\delta$. He uses (iii) to bound the holomorphic modulus by $\frac{N-1}{2}$, and then, for parabolicity, computes residues of the differential at *both* cusps ∞ and 0 , which the involution w swaps [7, proof of (5.12)] [CITED]. So the modulus is read from constant-term data at the pair of cusps exchanged by w , together with the global fact (iii).

5.2. Numerical census, $N \leq 997$ (num/lemma70_mazur.py). For all 168 primes $N \leq 997$ we split $S_2^{\text{new}}(\Gamma_0(N))$ into Galois orbits (PARI `mfsplit`). For every $\ell \mid n_N$ and every prime $\lambda \mid \ell$ of the Hecke field (`nfin` maximal at ℓ) we tested $a_p \equiv 1 + p \pmod{\lambda}$ for all primes $p \leq 1000$, $p \neq N$, and $a_n \equiv \sigma'_N(n) \pmod{\lambda}$ for all $n \leq 300$. Results [NUMERICAL]:

- $n_N = 1$ iff $\dim S_2(\Gamma_0(N)) = 0$: true for all 168 ($N = 2, 3, 5, 7, 13$ only).
- All 255 pairs (N, ℓ) have a congruent orbit, 260 congruent orbits in total. In every one $a_N = +1$, matching the Eisenstein U_N -eigenvalue $\sigma'_N(N) = 1$ and $1 + w \in \mathfrak{I}$. There are no failures. (Primes $\ell \nmid n_N$ were not tested, so the census says nothing about “false positives”; by Thm. 5.1(i) there are none.)
- The q -expansion congruence $a_n \equiv \sigma'_N(n)$ for $1 \leq n \leq 300$ held for every congruent (orbit, λ). Since $300 \geq \lfloor (N + 1)/6 \rfloor$ for all $N \leq 997$, this exceeds Sturm’s bound (§7).
- *Depth.* 46 pairs have $\ell^2 \mid n_N$. In 4 of them the ℓ -part is shared by several orbits $((113, 2), (353, 2), (487, 3), (593, 2))$, and in 6 by two primes λ in one orbit $((257, 2), (271, 3), (337, 2), (577, 2), (751, 5), (881, 2))$. In the remaining 36 a single λ in a single orbit carries it, often ramified ($e > 1$). We did not compute the depth of the congruence except for rational orbits. For $N = 17$ the rational form 17a satisfies $a_p \equiv 1 + p \pmod{4}$ for all $p \leq 1000$: the full $\mathbb{Z}/4$ in a single form. For 113, 353 and 593 the rational orbit reaches only ℓ^1 , and the rest lies in other orbits.
- Rational congruent forms occur only for $N = 11, 17, 19, 37, 73, 89, 113, 233, 353, 593$. In general the congruent orbit has a large Hecke field (Fig. 3, right), and the constant term does not predict it.
- Hasse/Deligne: every embedding of every a_p ($p \leq 500$) of every newform satisfies $a_p^2 < 4p$ (§10). All congruent forms are timelike, and $E_2^{(N)}$ is spacelike.

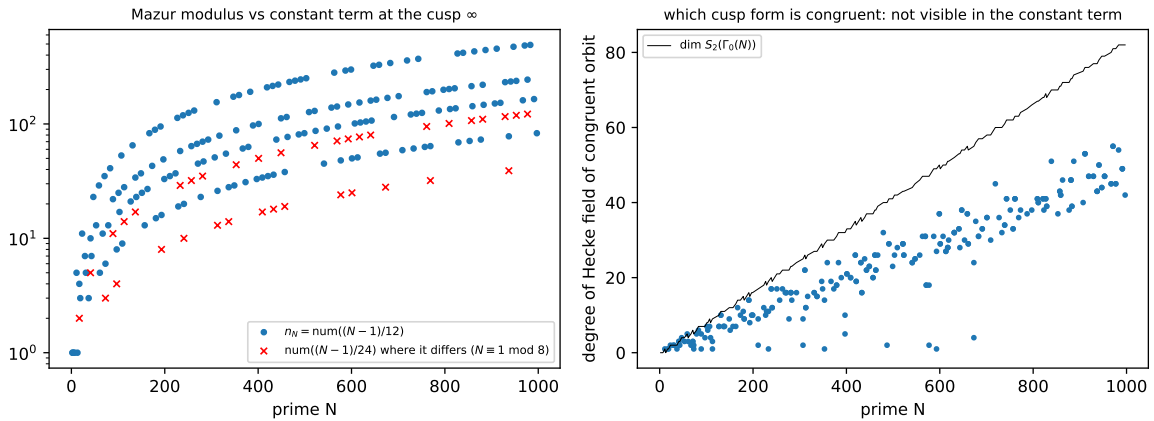


FIGURE 3. Left: n_N (dots) and, where different, the numerator of the constant term $(N - 1)/24$ at ∞ (crosses). They differ exactly for $N \equiv 1 \pmod{8}$ (37 primes ≤ 997). Right: degree of the Hecke field of the congruent orbit(s) against $\dim S_2$.

5.3. The cusp ∞ alone: an informative failure.

Proposition 5.2 ([THEOREM]). $\text{num } \frac{N-1}{24} = n_N$ unless $N \equiv 1 \pmod{8}$, in which case $n_N = 2 \text{ num } \frac{N-1}{24}$. If $N \equiv 9 \pmod{16}$, then $2 \mid n_N$ but $2 \nmid \text{num } \frac{N-1}{24}$.

Proof. v_3 agrees, since both divide by 3 once. $v_2(n_N) = \max(v_2(N-1) - 2, 0)$ and $v_2(\text{num } \frac{N-1}{24}) = \max(v_2(N-1) - 3, 0)$, which differ iff $v_2(N-1) \geq 3$. $\square$

So for $N = 41$ ($n = 10$) the constant term $5/3$ at ∞ is odd, but the mod-2 Eisenstein prime exists (Thm. 5.1(i)), and the census finds it: the cubic orbit, in whose field 2 is totally ramified ($e = 3$, $f = 1$). The factor 2 comes from the second cusp. The Atkin–Lehner involution w , which swaps the cusps, is the level- N incarnation of a reflection. This connects to J.W.’s reflection theme (Lemma 69) only as [ANALOGY].

6. LEVEL ONE: 691 AND THE BERNOULLI NUMERATORS

Census [NUMERICAL] (`num/lemma70_level1.py`). (a) For even $k \leq 2000$, $\text{num}(B_k/2k) = 1$ exactly for $k = 2, 4, 6, 8, 10, 14$, and these are exactly the $k \leq 2000$ with $\dim S_k = 0$. So “constant-term numerator = 1 $\Leftrightarrow$ no cusp forms” holds in this range. This is a numerical coincidence check, not a proof for all k . (b) For $12 \leq k \leq 100$ ($k \neq 14$) there are 120 pairs (k, ℓ) with $\ell \mid \text{num}(B_k/2k)$. Each has exactly one prime λ in exactly one Hecke orbit of $S_k(1)$ with $a_p \equiv 1 + p^{k-1} \pmod{\lambda}$ for all $p \leq 300$, and each is Sturm-certified (coefficients $n \leq \lfloor k/12 \rfloor$ agree mod λ). All exponents are 1. The only case with $\ell < k$ is $(68, 37)$ (37 irregular, $68 \equiv 32 \pmod{36}$).

Theorem 6.1 ([THEOREM] with cited inputs). *Let $k \geq 12$ be even and $\ell \geq 5$ with $(\ell - 1) \nmid k$. Then a normalized cusp eigenform $f \in S_k(1)$ with $a_p(f) \equiv 1 + p^{k-1} \pmod{\lambda}$ for all p exists iff $\ell \mid \text{num}(B_k/2k)$.*

Proof. ($\Leftarrow$) With the Miller basis of $M_k(\mathbb{Z})$, $\tilde{E}_k - c_0 F_0 \in S_k(\mathbb{Z}_{(\ell)})$ reduces to $\tilde{E}_k \pmod{\ell}$ when $\ell \mid c_0 = -B_k/2k$; Deligne–Serre’s lifting lemma [4] [BIB] then gives an eigenform. ($\Rightarrow$) If f is congruent, then $\tilde{E}_k - f \equiv c_0 \pmod{\lambda}$ is a constant weight- k form mod ℓ . By Swinnerton-Dyer’s structure theorem (the kernel of reduction is generated by $E_{\ell-1} - 1$) [13] [BIB], this forces $c_0 \equiv 0$ unless $(\ell - 1) \mid k$. $\square$

The exceptional primes (`num/lemma70_twist.py`, $n \leq 400$) [NUMERICAL]. $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$; $\pmod{2^8}$ (n odd); $\pmod{3}$ ($3 \nmid n$). $\tau(n) \equiv n\sigma_1(n) \pmod{5}$, i.e. $\Delta \equiv \theta E_2$. $\tau(n) \equiv n\sigma_3(n) \pmod{7}$, i.e. $\Delta \equiv \theta E_4/240$. No pair (a, b) gives $\tau(p) \equiv p^a + p^b \pmod{11, 13, 23}$. These are Ramanujan’s congruences, attributed via [6, fn. 18] to [13] (second-hand). The primes 2, 3, 5, 7, 13 are those of the denominator 65520 of $B_{12}/24$, where constants are weight-12 forms mod ℓ . There Theorem 6.1 is silent, and the truth varies: untwisted at 2, 3, θ -twisted at 5, 7, none at 13.

7. THE CORE TEST: WHAT THE CUSP DETERMINES

Theorem 7.1 (Unified modulus [THEOREM]/[CITED]). *Let $\zeta^{(N)}(s) = (1 - N^{-s})\zeta(s)$ (the Euler factor at N removed; $\zeta^{(1)} = \zeta$). (i) The constant term at ∞ of the integral Eisenstein series is $\zeta(1-k)/2$ at level 1 and $\zeta^{(N)}(-1)/2 = \frac{N-1}{24}$ for $E_2^{(N)}$. (ii) The Eisenstein modulus is $\text{num } \zeta(1-k)$ at level 1 (for $\ell \geq 5$, $(\ell - 1) \nmid k$; Thm. 6.1) and $n_N = \text{num } \zeta^{(N)}(-1) = \text{num } \frac{N-1}{12}$ at level N (Thm. 5.1). (iii) At level 1, $\text{num}(\zeta(1-k)/2) = \text{num } \zeta(1-k)$ because $2 \mid \text{den } B_k$. At level N the halving matters exactly when $N \equiv 1 \pmod{8}$ (Prop. 5.2). So the modulus is the numerator of the (Euler-depleted) L -value of the Eisenstein series at the edge, and not always of its constant term at one cusp.*

Proof. (i) $-B_k/2k = \zeta(1-k)/2$; $(1-N)(-\frac{1}{12})/2 = \frac{N-1}{24}$. (ii) by the cited theorems. (iii) von Staudt–Clausen and Prop. 5.2. $\square$

7.1. Sturm truncation: finitely many low-temperature coefficients.

Proposition 7.2 ([BIB] + [NUMERICAL]). *Sturm [12] [BIB] (the characteristic-0 form, with proof, in [1, Thm. 1] [CITED-sec]): if $f \in M_k(\Gamma)$ has λ -integral coefficients with $a_n \equiv 0 \pmod{\lambda}$ for $n \leq km/12$, $m = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$, then $f \equiv 0$. Consequences: (a) $k = 12$, $m = 1$: $\Delta \equiv \tilde{E}_{12} \pmod{691}$ is certified by a_0, a_1 alone: $a_0(\tilde{E}_{12} - \Delta) = \frac{691}{65520}$ and $a_1 = 1 - 1 = 0$. (b) $\Gamma_0(N)$, $m = N + 1$: $\lfloor (N+1)/6 \rfloor$ coefficients suffice, provided the constant term vanishes mod λ . For $N = 11$, a_0, a_1, a_2 of $E_2^{(N)} - f_{11a}$ are $\frac{5}{12}, 0, 3 - (-2) = 5$, all $\equiv 0 \pmod{5}$. (c) In the census every congruent pair agreed for $n \leq 300$, which exceeds every Sturm bound at $N \leq 997$ (≤ 166) and $k \leq 100$ (≤ 8) [NUMERICAL].*

Remark 7.3 (Three distinctions, as diagnostics). The question “constant term alone vs. the whole form” splits into three parts.

- (a) *Existence of a congruence at ℓ* : decided by the constant-term numerator for $\ell \geq 5$, $(\ell - 1) \nmid k$ (level 1), and by the two-cusp numerator n_N (level N). The cusp ∞ alone fails at $\ell = 2$ for $N \equiv 1 \pmod{8}$, and the constant term is *silent* at the primes where constants are themselves modular ($\ell - 1 \mid k$ at level 1; $\ell \mid 12$ at level N , Thm. 5.1(iii)). There the truth must be found elsewhere, and it varies (Δ : yes at 2, 3, 5, 7, no at 13).
- (b) *Which form, how many, what depth*: not determined by the constant term. The Hecke field degree, the number of orbits and λ 's, and ramification all vary (Fig. 3).
- (c) *Certification*: finitely many low-order coefficients (Sturm). But that this *suffices* is global: the q -expansion principle (used by Mazur as [3, VII (3.9)] via [7], second-hand), modularity of the comparison form, Deligne–Serre lifting, and Mazur’s computation of $\mathbb{T}/\mathfrak{I}$ via the Néron model at N .

Verdict [HEURISTIC] for the slogan, [THEOREM] for its parts: the low-temperature/cusp expansion *encodes* the congruence modulus, as a zeta value at a negative integer. Reading it off *correctly* requires both cusps (the w -pair), and the proof that the encoded number is the true modulus is global. Positivity is not encoded at all: f (timelike) and $E_2^{(N)}$ (spacelike) agree mod λ .

8. THERMAL READING: CHAT CLAIM (E), DIAGNOSTIC 3

On $\tau = iy$: $q = e^{-2\pi y} = e^{-\beta}$, $\beta = 2\pi y$. $Z(\beta) = 1/\Delta = q^{-1} \prod (1 - q^n)^{-24} = \sum p_{24}(n+1)q^n$ (with $p_{24}(0) = 1$), the generating function of 24-coloured partitions, shifted by the vacuum q^{-1} [THEOREM].

Proposition 8.1 ([THEOREM], checked [NUMERICAL] with `num/lemma70_thermal.py`). (i) $q \frac{d}{dq} \log \Delta = E_2$, hence $U(\beta) := -\partial_\beta \log Z = -E_2$; $U(\infty) = -1 = 24 \cdot \frac{1}{2} \zeta(-1) = 24 \cdot (-\frac{1}{24})$ (error 10^{-31} at $\beta = 20\pi$). (ii) S : $\Delta(i/y) = y^{12} \Delta(iy)$, so $\beta \leftrightarrow 4\pi^2/\beta$, with self-dual point $\beta = 2\pi$; $y^6 |\Delta|$ is S -invariant (residual $2.4 \cdot 10^{-30}$ on $y \in [0.3, 3]$). (iii) $E_2(i/y) = -y^2 E_2(iy) + 6y/\pi$ (residual $1.6 \cdot 10^{-30}$), while $U^* := U + 6/\beta = -E_2^*$ satisfies $U^*(4\pi^2/\beta) = -y^2 U^*(\beta)$ (residual $1.6 \cdot 10^{-30}$). (iv) $E_2^*(i) = 0$, so $E_2(i) = 3/\pi = 0.954929658551372 \dots$ (30 digits) and $U(2\pi) = -3/\pi$.

Remark 8.2 (Where the anomaly sits thermally [THEOREM], reading [ANALOGY]). $-3/(\pi y) = -6/\beta$. The correction is ∂_β of $6 \log \beta$, the logarithmic prefactor $y^6 = (\beta/2\pi)^6$ that makes $y^6 |\Delta|$ (Lemma 67) invariant. Read physically, $6/\beta = 12 \cdot \frac{1}{2\beta}$ is the classical equipartition energy of $12 = k/2$ quadratic degrees of freedom. This is an [ANALOGY] only: nothing identifies those degrees of freedom. The Cardy-type use of S (high- T asymptotics from the vacuum) is the standard physics reading [2] [BIB].

Do the moduli appear thermally? [THEOREM]+[NUMERICAL]. (a) 691: only as $\zeta(-11)/2 = 691/65520$ (checked to 30 digits), the zeta-regularized zero-point sum $\frac{1}{2} \sum n^{11}$ of a would-be field with spectrum n^{11} . It is a *vacuum* quantity, absent from $Z = 1/\Delta$ itself. (b) Mazur: the ratio $Z(\beta)/Z(N\beta) = \Delta(N\tau)/\Delta(\tau) = q^{N-1} \prod(\dots)$ has Casimir gap $N - 1$ (numerically 9.99999999,

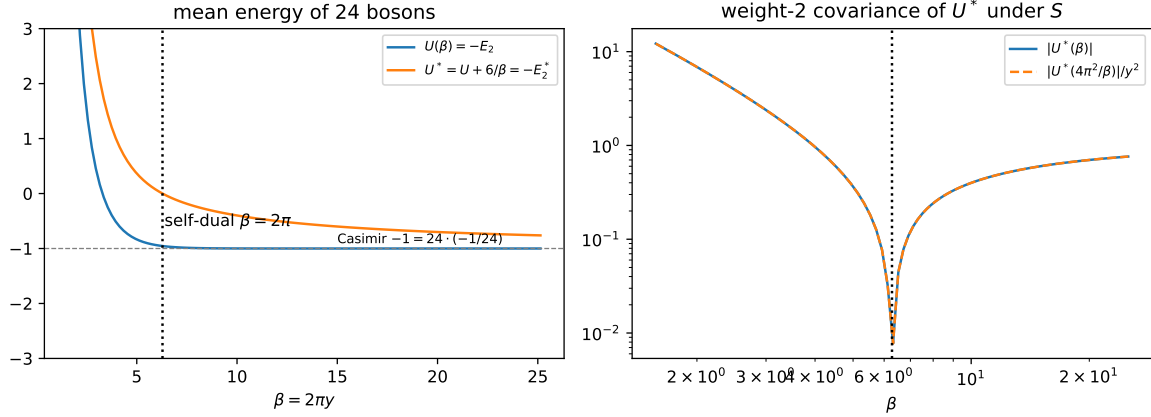


FIGURE 4. Left: $U = -E_2$ and $U^* = U + 6/\beta$ against β ; Casimir floor -1 , self-dual $\beta = 2\pi$ where $U^* = 0$. Right: $|U^*(\beta)|$ and $|U^*(4\pi^2/\beta)|/y^2$ coincide (weight-2 covariance); the cusp at $\beta = 2\pi$ is the zero $E_2^*(i) = 0$.

15.99999999, 35.99999999 at $y = 3$ for $N = 11, 17, 37$, approaching $N - 1$ as $y \rightarrow \infty$). Its divisor on $X_0(N)$ is $(N - 1)((\infty) - (0))$, and the order of $(0) - (\infty)$ in $J_0(N)$ is $(N - 1)/\gcd(N - 1, 12) = n_N$ (Thm. 5.1(iv)). The gcd with 12 is the largest root of the η -quotient that is still a function on $X_0(N)$ (Ogg’s η -quotient construction; not checked at source, [BIB]). So the Mazur number *is* a thermal quantity: the vacuum-energy gap between temperatures β and $N\beta$, divided by the 12 that also sits in $\zeta(-1) = -1/12$. (c) Neither modulus appears in the *shape* of $U(\beta)$ or its S -behaviour. Honest answer: *only via zero-point (constant-term) data*.

9. THE TATE CURVE: CHAT CLAIM (D)

Proposition 9.1 ([THEOREM]/[BIB]). (i) $|q| < 1 \Leftrightarrow y > 0$. $|q| = 1$ is $y = 0$, where $\tau - \bar{\tau} = 0$ (the null wall of Lemma 69). (ii) Over $\mathbb{Z}((q))$ the Tate curve $\mathbb{G}_m/q^{\mathbb{Z}}$ is the formal neighbourhood of the cusp, and its fibre at $q = 0$ is a nodal cubic [11] [BIB]. (iii) For a p -adic Tate curve E_q/K , $K/\mathbb{Q}_p$ finite, $|q|_p < 1$, and $\ell \neq p$: $0 \rightarrow \mathbb{Z}_\ell(1) \rightarrow T_\ell E_q \rightarrow \mathbb{Z}_\ell \rightarrow 0$, non-split since q is not a unit. The semisimplification is $1 \oplus \chi_\ell$ [BIB].

Corrections to (d). (1) The extension in (iii) is a *local* Galois representation at p attached to a p -adic q , whereas the thermal q is complex. The two are the same formal parameter, which is what (ii) says, but “its Galois representation is the Eisenstein shape” is a statement about the semisimplification of the local representation, not about the global one of any curve. (2) At multiplicative reduction the actual local factor sees only the inertia invariants: $a_p = +1$ (split) or -1 (non-split), not $1 + p$ (as the parent warned). Example: $X_0(11)$ at $p = 11$ is split multiplicative with $a_{11} = 1$, matching $\sigma'_{11}(11) = 1$. All 260 congruent orbits in the census have $a_N = +1$ [NUMERICAL]. (3) The Eisenstein shape $1 + p$ at *good* primes appears for a cusp form only *mod* λ (Thm. 5.1). So “Tate curve $\Rightarrow$ Eisenstein shape” is a correct local-at-the-cusp statement [THEOREM], and its use as the bridge between the Frobenius p and the nome q remains [HEURISTIC].

10. HASSE CHECK FOR ALL CONGRUENT FORMS

Exact check [NUMERICAL] (num/lemma70_hasse.py). For every newform orbit on $\Gamma_0(N)$, $N \leq 997$ prime, and every prime $p \leq 300$, $p \neq N$, we computed the minimal polynomial of a_p . Sturm’s theorem (PARI polsturm) then showed that all of its roots are real and lie in $(-2\sqrt{p}, 2\sqrt{p})$. That is 387176 conjugates with 0 violations. The smallest normalized margin $(4p - a^2)/4p$ among orbits

of degree ≤ 10 is $1.5 \cdot 10^{-3}$ ($N = 491$, $p = 61$). This is Deligne/Eichler–Shimura’s theorem, so the check is a sanity check of the pipeline. *Failures were informative.* A first floating-point evaluation of high-degree embeddings gave a spurious margin of $-4 \cdot 10^{10}$ (catastrophic cancellation in degree ~ 50). A first exact run with `charpoly` reported 242 false “violations”, caused by repeated roots when a_p generates a proper subfield (`polsturm` counts distinct roots). Using `minpoly` removed both. Every congruent form is timelike, and its Eisenstein partner is spacelike.

11. SYNTHESIS: IS WEIGHT 2 THE IRREDUCIBLE CORE?

- (S1) *The 12 is one number* [THEOREM]. $\zeta(-1) = -\frac{1}{12}$ is the orbifold Euler characteristic of $\mathrm{SL}_2(\mathbb{Z})$ (Harder’s Gauss–Bonnet [5] [BIB]). The area of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ is $\pi/3 = 4\pi \cdot \frac{1}{12}$, so the anomaly coefficient is $3/\pi = 1/\text{area}$ (Lemma 69). The same number gives $E_2 = 1 - 24 \sum \sigma_1 q^n$ ($24 = -2/\zeta(-1)$), the Casimir energy $\frac{1}{2}\zeta(-1)$ per boson, the Mazur modulus $\text{num}((1-N)\zeta(-1))$, and Mazur’s “constants are weight-2 forms only mod divisors of 12”. This is the precise sense in which the weight-2 anomaly, the thermal vacuum and the Eisenstein ideal share an origin: they are all $\zeta(-1)$, i.e. by the functional equation, $\zeta(2)$.
- (S2) *Weight 2 is all pole plane* [THEOREM]. $E_2 \leftrightarrow \mathbf{1} \oplus \mathbb{L} = h(\mathbb{P}^1)$, whose “surface” has no h^1 and therefore no definite complement (§4). Its forms lie in one tangent plane of the cone (Rem. 3.2). The level- N correction removes the anomaly ($E_2^* - NE_2^*(N \cdot)$ is holomorphic) and leaves the number n_N , which measures how much $h^1(X_0(N))$ is congruent to $\mathbf{1} \oplus \mathbb{L}$.
- (S3) *Lower weight controls Δ at small primes* [THEOREM]. $\Delta \equiv \theta E_2 \pmod{5}$. This is J.W.’s hunch in a precise form, with one caveat: θ (not the Ramanujan–Serre derivative) is what kills the constant term, and the statement is purely mod 5.
- (S4) *But congruences are blind to positivity* [THEOREM]. A timelike f and the spacelike $E_2^{(N)}$ agree mod λ , and so do the discriminants: $a_p^2 - 4p \equiv (p-1)^2 \pmod{\lambda}$ whenever $a_p \equiv 1+p$. Over $\mathbb{F}_\ell$ there is no sign (Lemma 66). Every congruence in this lemma is therefore RH-neutral. No circularity arises, but nothing about RH is gained.

Answer to J.W. The cusp ($q \rightarrow 0$) expansion does carry the moduli, as negative zeta values, and it certifies congruences with finitely many coefficients. It needs both cusps, and global theorems to justify it. The weight-2 level is where the anomaly, the Casimir energy and the Eisenstein ideal visibly share the single number $\zeta(-1)$ [THEOREM]. Calling it the “irreducible core” of positivity is not supported: weight 2 is exactly the part with *no* definite piece [HEURISTIC].

12. CORRECTIONS TO THE CHAT CLAIMS

- (C1) (a) $\sigma_{k-1}(p) = 1 + p^{k-1}$, trace $\#\mathbb{P}^1(\mathbb{F}_p)$, $\det p$, $L(E_2, s) = \zeta(s)\zeta(s-1)$: **correct**. “ $E_2 = \text{motive } \mathbf{1} \oplus \mathbb{L}$ ”: **correct as a realization/ L -function statement** ([RESTATEMENT]). E_2 is quasi-modular, so there is no motive in the strict sense.
- (C2) (a) “ $p \mapsto 1/p$ swaps the eigenvalues”: **correct only after normalizing** by $\sqrt{p}$. $\{1, p\} \mapsto \{1, 1/p\}$ is the swap rescaled by $\mathbb{L}^{-1}$. Rapidity $\frac{1}{2} \log p$ and AM–GM: correct.
- (C3) (b) Cancellation of $-3/(\pi y)$ in $E_2^*(\tau) - NE_2^*(N\tau)$, holomorphy on $\Gamma_0(N)$, constant term $1 - N$: **correct**.
- (C4) (b) “Mazur’s Eisenstein ideal has index $\text{num}((N-1)/12)$ ”: **correct**, verified at source [7, II (9.7)]. Precisely, $\mathbb{T}/\mathfrak{I} \cong \mathbb{Z}/n$ for $N \geq 5$ prime, where $\mathfrak{I}$ also contains $1+w$.
- (C5) (b) “ $a_p \equiv 1+p \pmod{5}$ for all 301 primes below 2000 ($p \neq 2, 11$)”: **correct but the exclusion of $p = 2$ was unnecessary**. It holds for all 302 primes $p < 2000$, $p \neq 11$ ($a_2 = -2 \equiv 3$). At $p = 11$, $a_{11} = 1 = \sigma'_{11}(11)$.
- (C6) [t269u] “the 12 in the denominator is the same 12 behind the anomaly coefficient”: **correct, and sharpened**. Both are $\zeta(-1) = -\frac{1}{12} = \chi_{\text{orb}}(\mathrm{SL}_2(\mathbb{Z}))$, and $\text{area} = \pi/3 = 4\pi/12$ (S1).
- (C7) (c) Signature $(2, 1)$, $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{Spin}(2, 1)$, timelike/spacelike/null: **correct**, with the precise form $\mathrm{SL}_2(\mathbb{R})/\pm 1 \cong \mathrm{SO}(2, 1)^0$. The physics reading stays [HEURISTIC].

- (C8) (d) Tate curve: **correct locally, overstated as a bridge**. The non-split extension is a local representation at p (for p -adic q), and the actual $a_p = \pm 1$. The “1, p ” live on the semisimplification (§9).
- (C9) (e) “leading 1 = Casimir $24 \cdot (1/24)$ ”: **sign**. The vacuum energy is $U(\infty) = 24 \cdot (-\frac{1}{24}) = -1$, and E_2 ’s leading 1 is $-U(\infty)$. The rest of (e) is correct and checked to 10^{-30} (Prop. 8.1).
- (C10) [t272u] “Eisenstein Frobenius never becomes timelike for any real $p > 0$ ”: **correct**, and null only at $p = 1$. New: all E_2 forms lie in one tangent plane (Rem. 3.2).
- (C11) [t272u] “check whether the 691 and Mazur congruences can be read off the low-temperature expansion alone”: **answered** (§7). Yes as zeta values, with both cusps needed, global justification needed, and positivity invisible.
- (C12) [t274u] (1)–(4): **correct** with these refinements. The intersection numbers were checked in [8] *instead of* Hartshorne V Ex. 1.9–1.10, whose exercise numbers we could not check. “Signature (1, 2)” holds for $\langle e_1, e_2, \Gamma_F \rangle$, and with Δ it is (1, 3). Which of e_1, e_2 is π_0 versus π_2 is convention-dependent. Scholl’s Kuga–Sato construction is [B1B] (Crossref only).

13. RANKED NEXT DIRECTIONS (0.6 s + 0.4 p)

	Direction	s	p	score
E1	Two-cusp reflection: constant terms at all cusps of $\Gamma_0(N)$, N squarefree, and whether the Eisenstein modulus is the numerator of the Atkin–Lehner-symmetrized depleted ζ -value (the w -pair as level- N functional equation)	.60	.80	.68
E2	L64 Weil engine on a congruent pair (11a vs $E_2^{(11)}$, Δ vs E_{12}): where does positivity diverge between the two members of a congruence (= L66 H1 / L68 P1)	.70	.60	.66
E3	$\Delta \equiv \theta E_2 \bmod 5$: $\theta E_2 = (E_2^2 - E_4)/12$ (Ramanujan). Follow the anomaly through the θ -twisted congruences and the Ramanujan–Serre derivative mod ℓ	.55	.80	.65
E4	Congruence primes vs Petersson norm (Hida): $\ell \mid$ congruence module $\leftrightarrow \ell \mid L(\text{ad } f, 1)$, i.e. a positive archimedean quantity that “sees” congruences (toward D8 global norm)	.80	.40	.64
E6	Genus-2 curve over $\mathbb{F}_{11}$ (L64/L65 G1): rank- > 2 definite complement	.50	.70	.58
E5	Census to $N \leq 5000$ and composite N	.30	.90	.54
E7	Thermal ghost test: does any $Z(\beta)$ observable feel an off-line zero	.40	.50	.44

14. QUESTIONS FOR J.W.

- Q1.** Do we adopt “the congruence modulus is the numerator of the Euler-depleted $\zeta(1 - k)$, read at both cusps” as the official TAA form of “the cusp knows the congruence”?
- Q2.** Do we adopt “weight 2 is all pole plane” ($\mathbf{1} \oplus \mathbb{L}$, $\mathbb{P}^1 \times \mathbb{P}^1$, tangent to the cone) as the meaning of “irreducible core”? Or must a core contain a definite part?
- Q3.** The missing factor 2 comes from the second cusp via the Atkin–Lehner w . Should we pursue w as the level- N reflection (E1)?
- Q4.** Do you accept that every congruence is RH-neutral (a timelike f is congruent to a spacelike E), so that congruences can organize but never supply positivity?
- Q5.** Which comes next: E1, E2, E3, E4, or one of Lemma 69’s D1/D3/D8?

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TAA LEMMA 71: THE KUSTAAANHEIMO–STIEFEL LIFT OF THE LOCAL TRIPLE. FROBENIUS AS A NULL VECTOR, THE WEIL LIGHT CONE ON $E \times E$, ORBITS, HAMILTONIANS, AND PRINCIPALIZATION

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ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA, and neither is GRH for any $L(E, s)$ or $L(\Delta, s)$. Nothing below is a step toward a proof that we can certify. Every RH-equivalent statement is labelled [RESTATEMENT], every metaphor [ANALOGY] or [HEURISTIC]. Tags: [THEOREM] proved here (elementary) or classical with the proof indicated; [CITED] checked at the source (`lit/lemma71/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (PARI/GP 2.17.2 via `cypari2`, `sympy`, `mpmath`, `numpy`; finite ranges, not proofs).

Question (J.W., [t281u]). Treat each local triple (ω, J_p, g_p) (Lemma 65) as a 3D Kepler-like problem, find its “orbit”, and lift it by a Kustaanheimo–Stiefel (KS) type lift to 4D, where space and time are scale-free and coordinate-free. Related: [t275u] (does fermionic structure force $(3, 1)$?). **Steering [t283u]:** add “Principalization as Hamiltonization” to the Hamiltonian search.

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ANSWER IN BRIEF

- (1) *The lift exists and is exact* [THEOREM] (§2). The 3D space is the $(2, 1)$ space of real binary quadratic forms (Lemma 70). The 4D space is Herm_2 , the 2×2 Hermitian matrices with $\det$ of signature $(1, 3)$, i.e. Minkowski space. The KS/Hopf map $\ell \mapsto \ell\ell^\dagger$ sends the Frobenius spinor $\ell_p = (1, \pi_p)$ to the null vector $\mathbf{X}_\pi = \begin{pmatrix} 1 \\ \pi \\ p \end{pmatrix}$, with light-cone coordinates $x^+ = 1$ ($\mathbf{1}$), $x^- = p$ ($\mathbb{L}$) and transverse part $x_\perp = \pi_p$ (h^1). Dropping $\text{Im } x_\perp$ is the same as *restricting the Hermitian form to $\mathbb{R}^2$* , and it returns Lemma 65's Hasse form $Q_p = [1, a_p, p]$, whose $(2, 1)$ norm is $(\text{Im } \pi_p)^2 = (4p - a_p^2)/4$. Spacelike forms (Eisenstein) are *not* shadows of any null vector.
- (2) *The lift is the Weil light cone, literally* [THEOREM] (§3, the main new result). On $E \times E$ over $\mathbb{F}_p$, the graph of an endomorphism φ has class $\Gamma_\varphi = e_1 + \deg \varphi e_2 + D_\varphi$ with D_φ linear in φ and $D_\varphi \cdot D_\psi = -\text{Tr}(\varphi\psi)$. So $\varphi \mapsto \Gamma_\varphi$ is exactly the KS parametrization $(1, |\varphi|^2, \varphi)$ of the null cone, the intersection form is $2 \det$ on Herm_2 , the diagonal is $\mathbf{X}_1$ (the “ $p = 1$ ” point), and $\Gamma_{\pi^n} \cdot \Delta = |1 - \pi^n|^2 = \#E(\mathbb{F}_{p^n})$. The hidden fourth coordinate $\text{Im } \pi_p$ is the rank of $\mathbb{Z}[\pi]$ beyond $\mathbb{Z}$. The $(3, 1)$ signature is the Hodge index theorem on $E \times E$.
- (3) *Orbits* [THEOREM]+[NUMERICAL] (§4). Frobenius iterates trace a null helix: a loxodromy with rapidity $\frac{1}{2} \log p$ along the $\mathbf{1}$ – $\mathbb{L}$ axis and rotation θ_p in the transverse plane. For p prime the orbit closes *iff* E is supersingular at p (periods 4; 8 at $p = 2$; 12 at $p = 3$). For Δ it closes *iff* $\tau(p) = 0$, which is Lehmer's question [OPEN]. Census of five curves to 10^6 and of Δ to 10^4 : CM supersingular fraction 0.5009, 0.5002; non-CM ≤ 0.0013 ; angles match Sato–Tate ($\sqrt{n}D \leq 1.03$) [NUMERICAL].
- (4) *Kepler and KS: a genuine Hamiltonian, but a degenerate orbit* (§6). Normalized Frobenius is the time- θ_p map of the oscillator $H_p = \frac{1}{2}g_p$ on (h^1, ω) , with $F/\sqrt{p} = \exp(\theta_p J_p)$ [THEOREM]. Gauge-fixed, the scale-free orbit is an isotropic 4D oscillator orbit satisfying the KS bilinear constraint. Its KS image is a *circular* Kepler orbit (eccentricity $< 2 \cdot 10^{-11}$; a generic control orbit gives $e = 0.6008$) [THEOREM]+[NUMERICAL]. The unnormalized helix is *not* a KS-regularized Kepler orbit for any energy and any Sundman time [THEOREM]. $H = 288\Delta$ (Lemma 68) does not commute with Hecke ($\vartheta E_{10} = -\frac{5}{6}E_{12} + \frac{38016}{691}\Delta$) [THEOREM]. Along $u = \log p$ there are no dynamics (consecutive-prime correlation ≤ 0.016) [NUMERICAL]. “Kepler” remains [ANALOGY] beyond the local circular orbit.
- (5) *Principalization as Hamiltonization* (§7, [t283u]). For curves, $\text{Pic}^0 = E$ and $\#\text{Pic}^0(\mathbb{F}_{p^n}) = \Gamma_{\pi^n} \cdot \Delta$: the class group is read off by pairing the null helix with the diagonal [THEOREM]+[NUMERICAL]. For $\text{Spec } \mathbb{Z}$ the finite class group is trivial and what survives is $C_{\mathbb{Q}} \cong \mathbb{R}_{>0}^\times \times \hat{\mathbb{Z}}^\times$. The $\mathbb{R}_{>0}^\times$ part is Connes' scaling flow, and its log-time $u = \log p$ is exactly twice the boost rapidity of the KS null vector [THEOREM] (identity) / [ANALOGY] (identification with Connes' flow).
- (6) *Global question* (§8). A positive combination of future null vectors is timelike, so *every* weighted sum of local lifts is timelike whatever the zeros do [THEOREM]. That property is RH-blind (true data, random Sato–Tate data and $\theta \equiv 0$ all give positive norms) [NUMERICAL]. The explicit formula of $L(11a, s)$ is a weighted sum of the *transverse projections* $\cos k\theta_p$ (residual 10^{-17}) [NUMERICAL]. A global null configuration whose shadow is Weil positivity would have to encode the zeros, which is circular [HEURISTIC].

1. SETUP AND CONVENTIONS

$E/\mathbb{Q}$ is an elliptic curve and $p \nmid N_E$ a good prime with $a_p = p + 1 - \#E(\mathbb{F}_p)$. The Frobenius eigenvalue is $\pi_p = \frac{a_p + i\sqrt{4p - a_p^2}}{2}$, chosen with $\text{Im } \pi_p > 0$, so $\pi_p \bar{\pi}_p = p$ and $\pi_p + \bar{\pi}_p = a_p$. This is Lemma 69's α_p . Lemma 67's Frobenius point is its mirror $\tau_p = -\bar{\pi}_p$ [TAA-L]. The normalized angle is $\theta_p \in (0, \pi)$ with $a_p = 2\sqrt{p} \cos \theta_p$. For Δ , $\pi_p = \frac{\tau(p) + i\sqrt{4p^{11} - \tau(p)^2}}{2}$ and $|\pi_p|^2 = p^{11}$ (Deligne [BIB]). The local triple is Lemma 65's (ω, J_p, g_p) on $h_{\mathbb{R}}^1$: ω the symplectic form, $J_p = (2F - a_p)/\sqrt{4p - a_p^2}$, $g_p = \omega(\cdot, J_p \cdot)$ [TAA-L].

Two quadratic spaces. $V_3 = \{[A, B, C]\}$ is the space of real binary forms $Am^2 + Bmn + Cn^2$, with the $(2, 1)$ norm $N_3 = AC - B^2/4 = -\text{disc}/4$, positive on definite forms (“timelike”) (Lemma 70, Prop. 3.1) [TAA-L]. $V_4 = \text{Herm}_2$ is the space of Hermitian 2×2 matrices

$$X = \begin{pmatrix} x^+ & \bar{c} \\ c & x^- \end{pmatrix} = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix}, \quad \det X = x^+x^- - |c|^2 = t^2 - x^2 - y^2 - z^2,$$

which is Minkowski space with $x^\pm = t \pm z$ and transverse coordinate $c = x + iy = x_\perp$. $\text{SL}_2(\mathbb{C})$ acts by $X \mapsto gXg^\dagger$ and preserves $\det$, which is the classical covering $\text{SL}_2(\mathbb{C}) \rightarrow \text{SO}(3, 1)^0$ [BIB]. The real symmetric matrices $V_3 \subset V_4$ ($y = 0$) form the 3D slice, and $\text{SL}_2(\mathbb{R})$ preserves it (Lemma 70) [TAA-L]. The identification is $[A, B, C] \leftrightarrow \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$, under which $N_3 = \det$.

Polar form. $\langle X, Y \rangle := \frac{1}{2}(\det(X+Y) - \det X - \det Y)$. For null vectors $X = uu^\dagger$, $Y = ww^\dagger$ one has $\langle X, Y \rangle = \frac{1}{2}|\det[uw]|^2 \geq 0$ [THEOREM] (sympy, `sym.json`). So two future null vectors always have nonnegative product.

2. THE 3D SPACE AND THE KS/HOPF LIFT: CHAT CLAIMS (A)–(C)

Proposition 2.1 (the lift [THEOREM]). (i) *The Hasse form $Q_p = [1, a_p, p]$ is timelike in V_3 , with $N_3(Q_p) = p - \frac{a_p^2}{4} = (\text{Im } \pi_p)^2 > 0$.*

(ii) *(Hopf/KS map.) $\kappa : \mathbb{C}^2 \rightarrow V_4$, $\ell \mapsto \ell\ell^\dagger$, lands on the future null cone ($\det = 0$, $t = \frac{1}{2}|\ell|^2 \geq 0$), and it is the quotient by the phase $\ell \mapsto e^{i\phi}\ell$. Restricted to $|\ell|^2 = 2$ it is the Hopf fibration $S^3 \rightarrow S^2$ [BIB].*

(iii) *For $\ell_p = (1, \pi_p)$: $X_\pi := \kappa(\ell_p) = \begin{pmatrix} 1 & \bar{\pi}_p \\ \pi_p & p \end{pmatrix}$, so $x^+ = 1$, $x^- = p$, $x_\perp = \pi_p = (\frac{a_p}{2}, \text{Im } \pi_p)$, $t = \frac{1+p}{2} = \frac{1}{2}\sigma_1(p)$ (half the E_2 eigenvalue), $z = \frac{1-p}{2}$. The null condition $x^+x^- = |x_\perp|^2$ is $\pi_p\bar{\pi}_p = p$.*

(iv) *(Shadow = restriction.) Dropping $y = \text{Im } x_\perp$ sends X_π to $\frac{1}{2}(X_\pi + X_{\bar{\pi}}) = \begin{pmatrix} 1 & a_p/2 \\ a_p/2 & p \end{pmatrix} = Q_p$. Equivalently, X_π^T is the Gram matrix of the Hermitian form $|m + \pi_p n|^2$ on $\mathbb{C}^2$, and its restriction to $(m, n) \in \mathbb{R}^2$ is $Q_p(m, n) = m^2 + a_p mn + pn^2$.*

(v) *(Which forms are shadows.) The image of the future null cone under $(t, x, y, z) \mapsto (t, x, z)$ is the closed solid future cone $\{N_3 \geq 0, t \geq 0\}$ of V_3 . A shadow has $N_3 = y^2$. So a form is the shadow of a null vector iff it is timelike or null, and the missing coordinate is then $y = \pm\sqrt{N_3}$, the gap. Spacelike forms have no null lift.*

Proof. (i) is Lemma 65/70. (ii): $\det(\ell\ell^\dagger) = 0$ because the rank is 1, $\text{Tr} = |\ell|^2$, and $\ell\ell^\dagger$ determines ℓ up to phase. (iii) is direct multiplication. (iv): $X_\pi + X_{\bar{\pi}}$ has off-diagonal entries $\pi + \bar{\pi} = a_p$. The Gram statement is $|m + \pi n|^2 = m^2 + (\pi + \bar{\pi})mn + pn^2$ (sympy, residual 0). (v): $N_3 = t^2 - x^2 - z^2 = \det + y^2 = y^2$ on the null cone. $\square$

Remark 2.2 (conventions: π versus $\bar{\pi}$; the chat matrix). The chat’s $\ell\ell^\dagger = \begin{pmatrix} 1 & \bar{\pi} \\ \pi & p \end{pmatrix}$ is correct as the null vector of the spinor $(1, \pi)$. As a *Gram matrix* (in $v^\dagger X v$) it is the form $|m + \bar{\pi}n|^2$, i.e. the transpose (mirror $y \mapsto -y$) of the Hasse Hermitian form. Both restrict to the same Q_p on $\mathbb{R}^2$, so nothing in (a)–(c) depends on the choice. The celestial point of $(1, \pi_p)$ is $\zeta = \pi_p$ (Lemma 69’s α_p). Lemma 67’s $\tau_p = -\bar{\pi}_p$ is the celestial point of $(1, -\bar{\pi}_p)$, whose real restriction is the characteristic-polynomial form $[1, -a_p, p]$ (Lemma 65’s $y \mapsto -y$) [THEOREM].

Proposition 2.3 (Eisenstein: the honest picture [THEOREM]). *For a split pair of real eigenvalues $\lambda_1 \neq \lambda_2$ (for E_2 : $1, p$; for E_{12} : $1, p^{11}$), there is no complex spinor. The natural objects are the two real null vectors $N_i = \kappa(1, \lambda_i) = (m + \lambda_i n)^2$, which lie in the slice V_3 on its null cone. The Eisenstein form $Q_E = (m + \lambda_1 n)(m + \lambda_2 n) = [1, \lambda_1 + \lambda_2, \lambda_1 \lambda_2]$ is their polar: $\langle Q_E, N_1 \rangle = \langle Q_E, N_2 \rangle = 0$ and $N_3(Q_E) = -(\lambda_1 - \lambda_2)^2/4 < 0$. For the elliptic case the same construction gives the complex-conjugate null pair $X_\pi, X_{\bar{\pi}}$ off the slice, whose polar in the slice is the timelike Q_p . This is the pole–polar duality of the Klein model: a chord between two real boundary points has its pole outside the disc (spacelike); a pair of conjugate imaginary points has its pole inside (timelike).*

Proof. With the polar form of N_3 , $\langle [A, B, C], [A', B', C'] \rangle = \frac{1}{2}(AC' + A'C) - \frac{1}{4}BB'$. Then $\langle [1, \lambda_1 + \lambda_2, \lambda_1\lambda_2], [1, 2\lambda_1, \lambda_1^2] \rangle = \frac{1}{2}(\lambda_1^2 + \lambda_1\lambda_2) - \frac{1}{2}\lambda_1(\lambda_1 + \lambda_2) = 0$ (sympy, `eis_polar`). For E_2 , $N_3 = p - \frac{(1+p)^2}{4} = -\frac{(p-1)^2}{4}$. $\square$

So the chat’s “Eisenstein spinor is real, hence the lift lies in the 3D slice” is right in the following precise form. Eisenstein data live entirely in the slice $y = 0$: two real null rays on its boundary, and a spacelike polar. Cusp-form data need the fourth coordinate. Equivalently, Eisenstein Frobenius is a hyperbolic element of $\mathrm{SL}_2(\mathbb{R})$ with real fixed points, and cusp-form Frobenius is elliptic with conjugate fixed points (Lemma 68) [TAA-L].

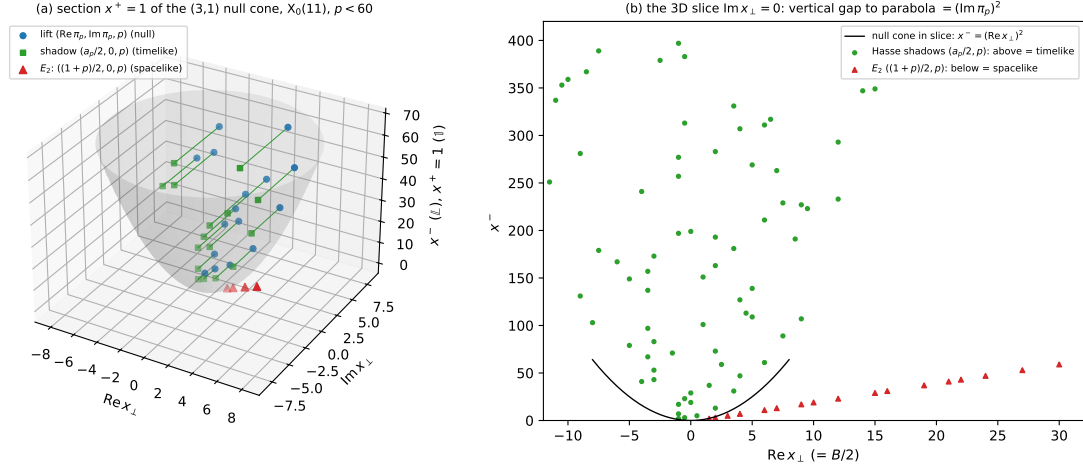


FIGURE 1. (a) The section $x^+ = 1$ of the future null cone of V_4 is the paraboloid $x^- = |x_\perp|^2$. Blue: KS lifts X_π of $X_0(11)$ (11a1) for $p < 60$, all on the paraboloid (relative $|\det| \leq 2.0 \cdot 10^{-16}$ for $p < 10^5$). Green: their shadows $(a_p/2, 0, p)$ in the slice, joined by the dropped coordinate $\mathrm{Im} \pi_p$. Red: E_2 ’s $((1+p)/2, 0, p)$. (b) The slice $y = 0$: the null cone is the parabola $x^- = (\mathrm{Re} x_\perp)^2$. Hasse shadows ($p < 400$) lie above it at height $(\mathrm{Im} \pi_p)^2$ (timelike); E_2 lies below it (spacelike) and has no null lift (Prop. 2.1(v)) [NUMERICAL].

[NUMERICAL] (`census.json`). For 11a1, 37a1, 32a2 ($y^2 = x^3 - x$, CM by $\mathbb{Z}[i]$), 27a1 ($y^2 + y = x^3 - 7$, CM by $\mathbb{Z}[\zeta_3]$) and 5077a1 (rank 3), all 78497 good primes $p < 10^6$ satisfy $4p - a_p^2 > 0$ (exact integers). The relative nullness defect of X_π in double precision is $\leq 1.98 \cdot 10^{-16}$ for $p < 10^5$. For Δ , the 1229 primes $p < 10^4$ give $4p^{11} - \tau(p)^2 > 0$ exactly and a defect $\leq 4.7 \cdot 10^{-16}$. These floating-point checks only test rounding; the identities are exact (Prop. 2.1).

3. THE LIFT IS THE WEIL LIGHT CONE ON $E \times E$

This section upgrades the chat’s “the lift is Weil’s light cone exactly” from an analogy to a theorem. Let $S = E \times E$ over $\mathbb{F}_p$, $e_1 = E \times \{O\}$, $e_2 = \{O\} \times E$, and for $\varphi \in \mathrm{End}_{\mathbb{F}_p}(E)$ let $\Gamma_\varphi = \{(x, \varphi x)\}$. We use the intersection numbers

$$(1) \quad e_1^2 = e_2^2 = 0, \quad e_1 e_2 = 1, \quad \Gamma_\varphi e_1 = \deg \varphi, \quad \Gamma_\varphi e_2 = 1, \quad \Gamma_\varphi^2 = 0, \quad \Gamma_\varphi \Gamma_\psi = \deg(\varphi - \psi) \quad (\varphi \neq \psi).$$

The first four are Milne’s conventions for $C_1 \times C_2$ ([8, §1, Ex. 1.7], read for Lemma 70) [CITED]. $\Gamma_\varphi^2 = 0$ is adjunction ($\Gamma_\varphi \cong E$ has genus 1 and $K_S = 0$). $\Gamma_\varphi \Gamma_\psi = \# \ker(\varphi - \psi)$ with multiplicity is standard [BIB]; its special case $\Gamma_\pi \cdot \Delta = \deg(1 - \pi)$ is in [8, §1] [CITED]. Lemma 70 checked the resulting Gram matrices numerically [TAA-L].

Theorem 3.1 (KS parametrization of the Weil cone [THEOREM]). Put $D_\varphi := \Gamma_\varphi - e_1 - \deg \varphi e_2 \in \text{NS}_{\text{num}}(S)_{\mathbb{R}}$. Then:

- (i) $D_\varphi \perp e_1, e_2$ and $D_\varphi \cdot D_\psi = -\text{Tr}(\varphi\hat{\psi})$, where $\text{Tr}(\varphi\hat{\psi}) = \varphi\hat{\psi} + \psi\hat{\varphi} \in \mathbb{Z}$. In particular $D_\varphi^2 = -2 \deg \varphi$.
- (ii) $\varphi \mapsto D_\varphi$ is additive, so it extends to an $\mathbb{R}$ -linear isometry $(\text{End}(E)_{\mathbb{R}}, -\text{Tr}) \hookrightarrow \langle e_1, e_2 \rangle^\perp$.
- (iii) On $\mathbb{Z}[\pi]_{\mathbb{R}} \cong \mathbb{C}$ (via $\pi \mapsto \pi_p$) one has $\deg \varphi = |\varphi|^2$, and the linear map

$$\Psi : \alpha e_1 + \beta e_2 + D_c \mapsto \begin{pmatrix} \alpha & \bar{c} \\ c & \beta \end{pmatrix} \in \text{Herm}_2$$

satisfies $A \cdot B = 2\langle \Psi A, \Psi B \rangle$ (intersection form = $2 \times$ Minkowski) and $\Psi(\Gamma_\varphi) = \kappa(1, \varphi)$. Thus $\Gamma_\pi \mapsto X_\pi$ (the KS lift), $\Delta = \Gamma_1 \mapsto X_1$, $e_1 \mapsto$ the $\mathbf{1}$ ray, $e_2 \mapsto$ the $\mathbb{L}$ ray, and $\Gamma_{\pi^n} \mapsto X_{\pi^n}$ (the null helix of §4).

- (iv) $\Gamma_{\pi^n} \cdot \Delta = 2\langle X_{\pi^n}, X_1 \rangle = |1 - \pi_p^n|^2 = \#E(\mathbb{F}_{p^n})$.

Proof. (i) $D_\varphi e_1 = \deg \varphi - 0 - \deg \varphi = 0$ and $D_\varphi e_2 = 1 - 1 - 0 = 0$. Expanding with (1) gives $D_\varphi D_\psi = \Gamma_\varphi \Gamma_\psi - \deg \varphi - \deg \psi$. With $\deg(\varphi - \psi) = (\varphi - \psi)(\hat{\varphi} - \hat{\psi}) = \deg \varphi + \deg \psi - \text{Tr}(\varphi\hat{\psi})$ this is $-\text{Tr}(\varphi\hat{\psi})$. For $\varphi = \psi$, $\Gamma_\varphi^2 = 0$ gives the same answer. (ii) $R = D_{\varphi+\psi} - D_\varphi - D_\psi$ is orthogonal to e_1, e_2 and, by (i) and bilinearity of Tr , to every D_χ . So R lies in the span W of these classes and is orthogonal to W . By the Hodge index theorem on S , $\langle e_1, e_2 \rangle^\perp$ is negative definite [8, §1] [CITED], so $R \cdot R = 0$ forces $R = 0$ in NS_{num} . (iii) For $\varphi \in \mathbb{Z}[\pi]$, $\hat{\varphi} = \bar{\varphi}$ (since $\hat{\pi} = p/\pi = \bar{\pi}$), so $\deg \varphi = |\varphi|^2$ and $\text{Tr}(\varphi\hat{\psi}) = 2 \text{Re}(\varphi\bar{\psi})$. The polar form of $\det$ is $\frac{1}{2}(\alpha\beta' + \alpha'\beta) - \text{Re}(c\bar{c}')$, which is half the intersection number $\alpha\beta' + \alpha'\beta - 2 \text{Re}(c\bar{c}')$. Finally $\Gamma_\varphi = e_1 + |\varphi|^2 e_2 + D_\varphi \mapsto \begin{pmatrix} 1 & \bar{\varphi} \\ \varphi & |\varphi|^2 \end{pmatrix} = \kappa(1, \varphi)$. (iv) By (iii) and the polar identity of §1, $2\langle \kappa(1, \pi^n), \kappa(1, 1) \rangle = |\det(\begin{smallmatrix} 1 & 1 \\ \pi^n & 1 \end{smallmatrix})|^2 = |1 - \pi_p^n|^2$. Also $\Gamma_{\pi^n} \cdot \Delta = \deg(1 - \pi^n) = \#E(\mathbb{F}_{p^n})$, the Lefschetz count [BIB]. (sympy: `NS_null=0, NS_deg_diff=0, Np_from_cone=1-a+p`.) $\square$

Corollary 3.2 (what the fourth coordinate is [THEOREM]). (i) Over $\mathbb{F}_p$ with p prime, $\pi \notin \mathbb{Z}$, so $W \supseteq \langle e_1, e_2, \Delta, \Gamma_\pi \rangle$ is 4-dimensional, of signature $(1, 3)$. The extra transverse direction is $\text{Im } \pi_p$, i.e.

the direction of J_p in $\mathbb{Z}[\pi]_{\mathbb{R}} = \mathbb{R}[J_p] \cong \mathbb{C}$ (since $\pi = \frac{a_p + \sqrt{4p - a_p^2} J_p}{2}$). The $(3, 1)$ signature of the KS space is the Hodge index theorem on $E \times E$, and the fourth dimension is the complex structure J_p .

(ii) If $\pi \in \mathbb{Z}$ (e.g. E supersingular over $\mathbb{F}_{p^2}$ with $\pi = \pm p$), then $\Gamma_\pi = e_1 + p^2 e_2 \pm p D_1$ lies in the 3-space $\langle e_1, e_2, \Delta \rangle =$ the slice. “Degeneracy = the lift falls into the 3D slice” is therefore exactly “ $\mathbb{Z}[\pi] = \mathbb{Z}$ ”. (Hasse is then an equality: $|a| = 2\sqrt{q}$.)

(iii) Weil/Hasse restated: $a_p = -D_\pi \cdot D_1 \leq \sqrt{D_\pi^2 D_1^2} = 2\sqrt{p}$ by Cauchy-Schwarz in the definite plane.

In KS language: $x_\perp$ is a vector of Euclidean length $\sqrt{x^+ x^-} = \sqrt{p}$, and $a_p/2$ is its projection. This is Lemma 70’s complement Gram $-2Q_p$ written in light-cone coordinates [TAA-L].

Remark 3.3 (rank and the CM/supersingular cases [BIB]). $\text{End}_{\mathbb{F}_p}(E)$ is an order in $\mathbb{Q}(\pi)$, so over $\mathbb{F}_p$ the span W is all of $\text{NS}_{\text{num}}(S)$ (rank 4). Over $\overline{\mathbb{F}_p}$ the rank is $2 + \text{rank End}$: 4 if E is ordinary and 6 if supersingular (quaternion order), with signature $(1, 5)$. We use only the $\mathbb{F}_p$ -rational 4-space, and these rank statements are standard (Tate, Deuring) and not re-proved.

Remark 3.4 (answer to [t275u] in this setting [THEOREM]/[HEURISTIC]). Earlier chat said $V \otimes \bar{V}$ gives $(3, 1)$, $\text{Sym}^2 V$ gives $(2, 1)$, and $\Lambda(V)$ stays in $(2, 1)$. Here this becomes concrete. The $(2, 1)$ slice is $\langle e_1, e_2, \Delta \rangle$, the part of NS that sees only $\mathbb{Z}$. The fourth direction appears exactly because h^1 carries the complex structure J_p , so End acts through $\mathbb{C}$ rather than $\mathbb{R}$. Whether this deserves the word “fermionic” is [HEURISTIC]. The precise statement is that the transverse plane is (h^1, J_p) viewed as a complex line, and $X_\pi = \ell \ell^\dagger$ is a spinor times its conjugate.

[NUMERICAL] (`global.json`). For 11a1, brute-force counts of $y^2 + y = x^3 - x^2 - 10x - 20$ over $\mathbb{F}_p$ for all 45 primes $p < 200$, $p \neq 11$, agree with $p + 1 - a_p$, $\det(1 - F)$ for the companion matrix, and

$|1 - \pi_p|^2$. Over $\mathbb{F}_{p^n}$, $p \in \{2, 3, 5, 7\}$, $n \leq 6$, PARI's `ellcard` agrees with $|1 - \pi^n|^2$. For example $p = 2$ gives 5, 5, 5, 25, 25, 65, and the period-8 pattern of the supersingular prime 2 is visible.

4. ORBITS: THE NULL HELIX, CLOSURE, AND THE CENSUS (CLAIM (D))

Proposition 4.1 (the helix is a loxodromy [THEOREM]). *Let $w_p = \log \pi_p = \frac{1}{2} \log p + i\theta_p$. Then $\ell^{(n)} = (1, \pi_p^n) = g_p^n(1, 1)$ with $g_p = \text{diag}(1, e^{w_p}) \in \text{GL}_2(\mathbb{C})$, and $X_{\pi^n} = g_p^n X_1 g_p^{\dagger n}$. In light-cone coordinates $x^+ = 1$, $x^- = p^n$, $x_\perp = p^{n/2} e^{in\theta_p}$. After the scale-free normalization $\tilde{X} = X/\sqrt{x^+ x^-}$ (projectivization), g_p acts as the Lorentz loxodromy with boost rapidity $\frac{1}{2} \log p$ along the $1\text{--}\mathbb{L}$ axis and rotation θ_p in the transverse plane. All g_p for all p commute: they lie in the Cartan subgroup $\mathbb{C}^\times$ that fixes the two null rays $1, \mathbb{L}$. The normalized lifts are $\tilde{X}_{\pi_p} = \Lambda(w_p) X_1$, where $\Lambda(w)$ is the loxodromy with complex rapidity w (numerical check: max error $8.9 \cdot 10^{-16}$, `ks.json`).*

Theorem 4.2 (closure $\Leftrightarrow$ supersingular [THEOREM]). *Let p be prime and $|a| < 2\sqrt{p}$. The normalized transverse orbit $e^{i\theta}$ is periodic iff $e^{i\theta} = \pi/\sqrt{p}$ is a root of unity. This happens iff $a^2/p \in \{0, 1, 2, 3\}$, i.e. iff $a = 0$ (period 4), or $(p, a) = (2, \pm 2)$ (period 8), or $(p, a) = (3, \pm 3)$ (period 12). Equivalently: iff E is supersingular at p ($p \mid a_p$). For Δ the orbit closes iff $\tau(p) = 0$.*

Proof. If $e^{i\theta}$ is a root of unity, then $2 \cos 2\theta = a^2/p - 2$ is a rational algebraic integer, hence an integer in $[-2, 2]$. So $a^2/p \in \{0, 1, 2, 3, 4\}$. The value 4 violates the strict Hasse inequality. $a^2 = p$ is impossible for p prime. $a^2 = 2p$ forces $p = 2$, $a = \pm 2$, and $a^2 = 3p$ forces $p = 3$, $a = \pm 3$. Conversely $\theta = \pi/2$, $\pi/4$ or $3\pi/4$, $\pi/6$ or $5\pi/6$ give periods 4, 8, 12. For $p \geq 5$, $|a| < 2\sqrt{p} < p$, so $p \mid a \Leftrightarrow a = 0$. At $p = 2, 3$ the allowed a are $\{0, \pm 2\}$, $\{0, \pm 3\}$, all divisible by p . For Δ : $\tau^2/p^{11} \in \{1, 2, 3, 4\}$ would need $p^{11} \cdot k$ to be a square, which is impossible for $k \in \{1, 4\}$, and for $k \in \{2, 3\}$ needs $p = k$ with $\tau(p)^2 = p^{12}$, i.e. $\tau(2) = \pm 64$ or $\tau(3) = \pm 729$. But $\tau(2) = -24$ and $\tau(3) = 252$. So only $\tau(p) = 0$ remains. $\square$

[NUMERICAL] (`sym.json`). An independent brute-force search over all Hasse-admissible (p, a) with $p < 1000$, testing $(\pi/\sqrt{p})^k = 1$ for $k \leq 120$, finds exactly the 168 pairs with $a = 0$ (all of period 4) plus $(2, \pm 2)$ (period 8) and $(3, \pm 3)$ (period 12).

Lehmer. “A closed orbit for Δ ” is therefore exactly “ $\tau(p) = 0$ ”. Whether this ever happens is Lehmer’s question [OPEN]. $\tau(p) \neq 0$ for all 1229 primes $p < 10^4$ [NUMERICAL] (much larger ranges are in the literature, not re-checked here). The set of p with $\tau(p) = 0$ has density 0 (Serre 1981, [11] [BIB]).

Remark 4.3 (sources for the densities). CM: E with CM by K is supersingular at $p \nmid N$ iff p is inert or ramified in K (Deuring), so the density is $1/2$ [BIB]. Non-CM: infinitely many supersingular primes (Elkies 1987, [4]: Crossref record checked, content [BIB]), density 0 (Serre 1981, [11]: [BIB]), conjectured count $\sim c_E \sqrt{x}/\log x$ (Lang–Trotter) [CONJECTURE]. Sato–Tate for non-CM $E/\mathbb{Q}$ is a theorem (Barnet-Lamb–Gerahty–Harris–Taylor, [BIB], not opened). The table is consistent with all of these, and it proves none of them.

Remark 4.4 (“precession” is a rotation number, not Kepler precession). In the normalized picture the continuous curve $s \mapsto e^{is\theta_p}$ is always a closed circle. What fails to close for ordinary primes is the *sampling at integer n* : the rotation number $\theta_p/2\pi$ is irrational. This is unlike a perturbed Kepler orbit, where the continuous orbit itself precesses (Bertrand’s theorem; Runge–Lenz vector not conserved). The chat’s “like a perturbed Kepler orbit, since a closed Kepler orbit needs its hidden $\text{SO}(4)$ symmetry” is therefore a misleading analogy (correction C5).

5. THE CELESTIAL SPHERE (CLAIM (E))

Proposition 5.1 ([THEOREM], standard). *Projectivizing the future null cone of V_4 gives $\mathbb{CP}^1 \cong S^2$ via $[\ell] \mapsto \zeta = \ell_2/\ell_1$ and $X/t \mapsto \frac{(2\text{Re}\zeta, 2\text{Im}\zeta, 1-|\zeta|^2)}{1+|\zeta|^2}$. Lorentz transformations act by Möbius maps,*

curve	N	CM	$ss < 10^3$	$ss < 10^4$	$ss < 10^5$	$ss < 10^6$	non-4 periods ($p < 10^4$)	$\sqrt{n}D_{ST}$ ($p < 10^5$)
11a1	11	–	6	16	42	99	2:8	0.488
37a1	37	–	7	11	33	84	2:8, 3:12	0.652
5077a1	5077	–	6	15	41	94	2:8, 3:12	1.028
32a2	32	$\mathbb{Z}[i]$	87	619	4808	39322	–	24.74
27a1	27	$\mathbb{Z}[\zeta_3]$	87	617	4807	39266	–	24.77

TABLE 1. Orbit census [NUMERICAL] (`census.json`). “ss” = supersingular = closed orbits (Thm. 4.2, checked independently for $p < 10^4$: the closed set equals the supersingular set for all five curves). 11a1 below 400: $p = 2, 19, 29, 199$, confirming the chat. CM fractions at 10^6 are 0.5009, 0.5002 of the 78497 good primes. Non-CM fractions are 0.0013, 0.0011, 0.0012. For comparison $\sqrt{x}/\log x = 72.4$ at $x = 10^6$, and the non-CM counts are 1.16–1.37 times that, in line with the Lang–Trotter shape [CONJECTURE]. $\sqrt{n}D_{ST}$ is the Kolmogorov–Smirnov statistic of θ_p against Sato–Tate (5% critical value ≈ 1.36). For CM curves the ordinary angles against the uniform law give 0.539 and 0.417. Δ ($p < 10^4$): $\sqrt{n}D_{ST} = 0.561$, mean $\cos^2 \theta_p = 0.2458$ (Sato–Tate: $1/4$).

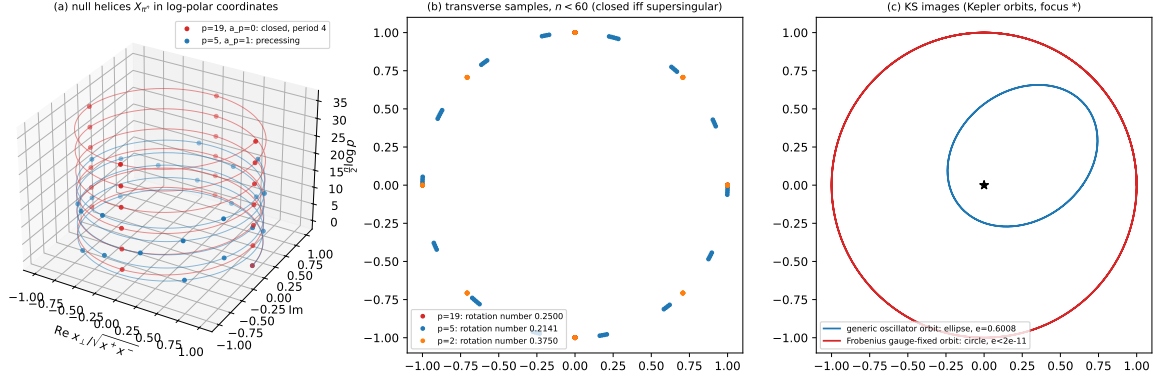


FIGURE 2. (a) Null helices $\tilde{X}_{\pi^n}$ of 11a1 in log-polar coordinates (transverse direction $e^{in\theta_p}$, height = rapidity $\frac{n}{2} \log p$). $p = 19$ ($a_{19} = 0$, supersingular) returns to the same transverse direction every 4 steps; $p = 5$ ($a_5 = 1$) precesses. (b) Transverse samples $n < 60$: $p = 19$ (period 4; its four points coincide with four of the $p = 2$ points), $p = 2$ ($a_2 = -2$, period 8), $p = 5$ (rotation number 0.2141, dense). (c) KS images (§6): a generic constraint-satisfying 4D oscillator orbit maps to a Kepler ellipse ($e = 0.6008$, conserved to $1.5 \cdot 10^{-11}$); the gauge-fixed Frobenius orbit maps to a circle ($e < 2 \cdot 10^{-11}$) [NUMERICAL].

which is Penrose’s celestial sphere ([10] [BIB]). In these coordinates: $\mathbf{1} = \{\zeta = 0\}$ and $\mathbb{L} = \{\zeta = \infty\}$ are antipodal poles. The wall $\{\zeta \in \mathbb{R} \cup \infty\}$ (degenerate forms, $\text{Im } \pi = 0$) is a great circle through both poles. $\mathbb{H} = \{\text{Im } \zeta > 0\}$ is one of the two hemispheres it bounds. The circle $|\zeta| = 1$ ($x^+ = x^-$) is the great circle orthogonal to the $\mathbf{1}$ – $\mathbb{L}$ axis.

Where things sit. The unnormalized Frobenius points $\zeta = \pi_p$ lie in $\mathbb{H}$ at $|\zeta| = \sqrt{p}$, drifting toward the $\mathbb{L}$ pole as p grows. Normalized ($\zeta = \pi_p / \sqrt{p} = e^{i\theta_p}$) they lie on the half of the circle $|\zeta| = 1$ inside $\mathbb{H}$, at longitude θ_p . The Eisenstein points $\zeta = p^{\pm 1/2}$ lie on the wall, off that circle. The two circles meet at $\zeta = \pm 1$, and $\zeta = 1$ is $X_1 = \Delta$, the “ $p = 1$ ” point. So the cusp-form/Eisenstein dichotomy is “on the $|\zeta| = 1$ circle, off the wall” versus “on the wall, off the circle” [THEOREM]. Iterates do *not*

stay in $\mathbb{H}$: π^n leaves the upper half-plane whenever $\sin n\theta < 0$. The $\mathbb{H}$ -picture is per-Frobenius, not per-orbit.

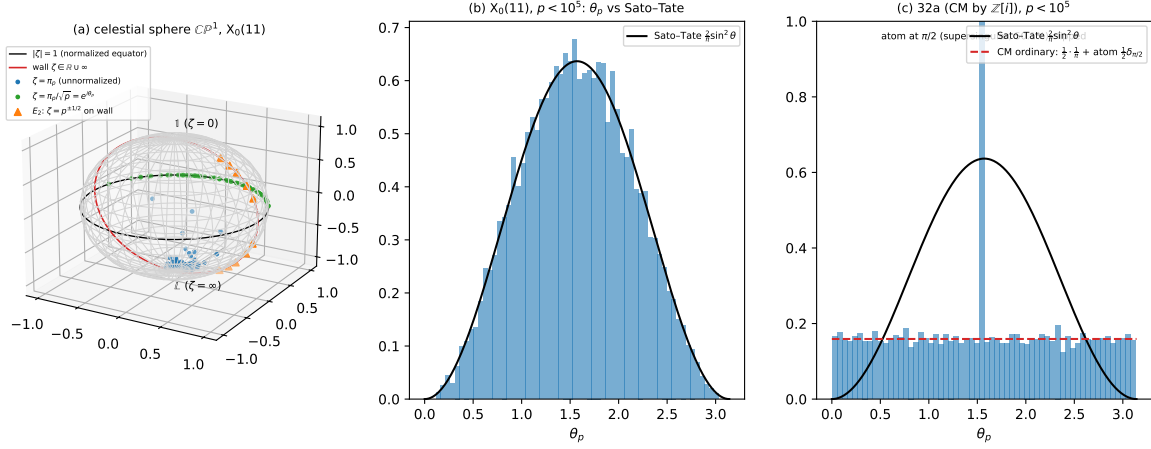


FIGURE 3. (a) Celestial sphere for 11a1, $p < 300$: unnormalized $\zeta = \pi_p$ (blue, toward $\mathbb{L}$), normalized $e^{i\theta_p}$ (green, on the circle $|\zeta| = 1$, black), E_2 's $p^{\pm 1/2}$ (orange, on the wall, red). (b) Histogram of θ_p , $p < 10^5$ (9591 primes) against the Sato–Tate density $\frac{2}{\pi} \sin^2 \theta$. (c) Same for the CM curve 32a2: an atom of mass 0.501 at $\pi/2$ (supersingular, clipped) plus a uniform ordinary part of density $\frac{1}{2\pi}$ [NUMERICAL].

6. WHAT KS ACTUALLY DOES, AND THE HAMILTONIAN SEARCH (CLAIM (F))

The KS theorem. Kustaanheimo and Stiefel introduced spinor regularization of Kepler motion in 1965 ([6]: Crossref record checked; the paper itself not opened, [BIB]). We use the modern statement read at source in [12] [CITED-sec]. The Hopf map $z \mapsto \bar{z}iz$, $\mathbb{H} \rightarrow \text{Im } \mathbb{H}$, has S^1 fibres. The KS map $(z, w) \mapsto (Q, P) = (\bar{z}iz, \bar{z}iw/2|z|^2)$ restricted to the zero set Σ_1 of the bilinear function $BL(z, w) = \text{Re}(\bar{z}iw)$ (the moment map of the S^1 -action) is symplectic onto $T^*(\text{Im } \mathbb{H} \setminus 0)$ after S^1 -reduction [12, Lemma 1]. Zero-energy orbits of the 4D Hooke Hamiltonian $\frac{|w|^2}{8} + f|z|^2 + m$ in Σ_1 go to zero-energy orbits of the Kepler Hamiltonian $\frac{|P|^2}{2} + \frac{m}{|Q|} + f$ after the time reparametrization $dt = |Q| ds$ (Sundman) [12, Prop. 2]. So KS converts a fixed-energy Kepler problem into an isotropic oscillator. A bound Kepler energy $-f < 0$ corresponds to oscillator frequency $\propto \sqrt{f}$.

[NUMERICAL] control (ks.json). In our normalization ($\mathbf{x}$ = spatial part of $uu^\dagger$, $r = |u|^2/2$, $dt = |u|^2 ds$) a generic oscillator orbit $u = A \cos \omega s + B \sin \omega s$ with $\text{Im}(A^\dagger B) = 0$, $\omega = 0.7$, maps to a Kepler orbit. The fitted μ is constant to $2.7 \cdot 10^{-8}$, the residual of $\ddot{\mathbf{x}} + \mu \mathbf{x}/r^3$ is $1.5 \cdot 10^{-7}$ (relative), the eccentricity $e = 0.60083$ (Runge–Lenz) is conserved to $1.5 \cdot 10^{-11}$, and the energy is $-0.24500 = -\omega^2/2$. So our implementation reproduces KS, and the results below are meaningful.

Theorem 6.1 (the local triple is an oscillator [THEOREM]). *On $(h_{\mathbb{R}}^1, \omega)$, $F/\sqrt{p} = \cos \theta_p + \sin \theta_p J_p = \exp(\theta_p J_p)$. J_p is the Hamiltonian vector field of $H_p(v) = \frac{1}{2} g_p(v, v) = \frac{1}{2} \omega(v, J_p v)$ (up to the sign convention for $\iota_X \omega = \pm dH$). So normalized Frobenius is the time- θ_p map of the harmonic oscillator whose symplectic form, complex structure and Hamiltonian are exactly Lemma 65's triple (ω, J_p, g_p) .*

Proof. $J_p = (2F - a_p)/\sqrt{4p - a_p^2}$ gives $F = \frac{a_p}{2} + \frac{\sqrt{4p - a_p^2}}{2} J_p$, i.e. $F/\sqrt{p} = \cos \theta_p + \sin \theta_p J_p$, and $J_p^2 = -1$ (sympy: residual matrix 0). Since J_p is ω -compatible, $dH_p(\xi) = \omega(\xi, J_p v)$. $\square$

Proposition 6.2 (KS image of the scale-free orbit: a circle [THEOREM]+[NUMERICAL]). *The normalized spinor orbit, in the gauge $u(s) = (e^{-i\theta_p s/2}, e^{i\theta_p s/2})$ (KS fibre phase removed), equals $A \cos \frac{\theta_p s}{2} + B \sin \frac{\theta_p s}{2}$ with $A = (1, 1)$, $B = (-i, i)$. It satisfies $u'' = -(\theta_p/2)^2 u$ and the bilinear constraint $\text{Im}(u^\dagger u') = 0$ (sympy: 0), and it passes through the Frobenius iterates at $s = n$. Its KS image is the circle $\mathbf{x}(s) = (\cos \theta_p s, \sin \theta_p s, 0)$: a Kepler orbit with $\mu_p = \theta_p^2/4$, energy $E_p = -\theta_p^2/8$ and eccentricity 0. Numerically for $p = 2$ (11a1, $\theta = 3\pi/4$): $\mu = 1.387913$ ($= \theta^2/4$), $E = -0.693957$ ($= -\theta^2/8$), $e < 2 \cdot 10^{-11}$.*

Theorem 6.3 (no-go for the unnormalized helix [THEOREM]). *For $p > 1$ and $\pi \notin \mathbb{R}$ there is no KS-regularized orbit ($u'' = cu$, $c \in \mathbb{R}$, any energy) and no choice of Sundman times s_n and fibre phases ϕ_n with $u(s_n) = e^{i\phi_n}(1, \pi^n)$ for all $n \geq 0$.*

Proof. $|u(s_n)|^2 = 1 + p^n \rightarrow \infty$. If $c < 0$ then u is bounded, a contradiction. If $c = \kappa^2 > 0$ then $u_1 = \alpha e^{\kappa s} + \beta e^{-\kappa s}$, and $|u_1(s_n)| = 1$ along $s_n \rightarrow \pm\infty$ forces $u_1 \equiv 0$, contradicting $|u_1(s_n)| = 1$. If $c = 0$ then u is affine, and $|u_1(s_n)| = 1$ with s_n unbounded forces $u_1 \equiv \alpha$, so $e^{i\phi_n} = \alpha$ for all n . Then $\alpha\pi^n = u_2(s_n)$ lies on a line, so $1, \pi, \pi^2$ are collinear, i.e. $(\pi^2 - 1)/(\pi - 1) = \pi + 1 \in \mathbb{R}$, which contradicts $\pi \notin \mathbb{R}$. (Numerically $\text{Im}[(\pi - 1)(\pi^2 - 1)] \neq 0$ for 11a1, $p \leq 31$.) $\square$

So the Kepler/KS picture applies *only after* removing the boost (scale-free normalization), and then it gives the degenerate circular orbit. The boost $\frac{1}{2} \log p$ is the part that is not Kepler. In the language of Lemma 68: the rotation (elliptic) part is an oscillator, and the boost (Eisenstein-like, hyperbolic) part is not [THEOREM].

candidate H	what its flow does	commutes with Frobenius?	verdict
$\frac{1}{2}g_p$ on (h^1, ω)	generates $F/\sqrt{p}$ $\exp(\theta_p J_p)$	= yes (generates it)	[THEOREM]. KS image circular Kepler, $\mu = \theta_p^2/4$, $e = 0$; local, one per prime
geodesic flow on $\text{SL}_2(\mathbb{R})/\text{SO}(2)$	hyperbolic translations (boosts) along a geodesic	cuspidal forms: (elliptic F)	no [THEOREM]. Fits <i>Eisenstein</i> : E_k Frobenius is hyperbolic, a pure boost of rapidity $\frac{k-1}{2} \log p$ with real fixed points (Lemma 68)
$H = 288\Delta$ (Ramanujan–Serre, Lemma 68)	ϑ on $\mathbb{C}[E_4, E_6]$	no	[THEOREM]. $\vartheta_{10} E_{10} = -\frac{5}{6} E_{12} + \frac{38016}{691} \Delta$ is not a T_2 -eigenform (exact, $n < 40$)
log-time $u = \log p$	boost K_z of rapidity $u/2$	trivially (same Cartan)	[THEOREM] as kinematics. No dynamics in p : $\text{corr}(\cos \theta_p, \cos \theta_{p'}) = 0.0099, -0.0158, -0.0086$ for consecutive primes (11a1, 5077a1, 32a2, $p < 10^5$) [NUMERICAL]
Sundman time $ds = dt/r$	$r = t$ -component $= \frac{1+p}{2}$	–	with “ $t = p$ ”, $s = 2 \log(1 + p) \approx 2u$: the log scale appears as Sundman time [HEURISTIC]

TABLE 2. Hamiltonian candidates. Only $\frac{1}{2}g_p$ is a genuine Hamiltonian for Frobenius, and it is local and degenerate.

Conserved quantities (Runge–Lenz diagnostic). *Along n (fixed p):* the axis $\{\mathbf{1}, \mathbb{L}\}$ and the complex rapidity w_p are conserved, and the KS Runge–Lenz vector is 0 (circular orbit) [THEOREM]+[NUMERICAL]. *Along p :* the only common structure is the stabilizer $\mathbb{C}^\times$ of the two null rays, an abelian group. Nothing like a hidden $\mathrm{SO}(4)$ or $\mathrm{SO}(3, 1)$ relates different primes. The full Lorentz group acts on Herm_2 but moves e_1, e_2 , which are arithmetic classes ($E \times \mathrm{pt}, \mathrm{pt} \times E$) and cannot be moved [THEOREM].

Proposition 6.4 (transpose = duality reflection [THEOREM]). *The transpose correspondence ${}^t\Gamma_\varphi$ has class $\deg \varphi e_1 + e_2 + D_{\hat{\varphi}}$, so $\Psi({}^t\Gamma_\pi) = \begin{pmatrix} p & \pi \\ \pi & 1 \end{pmatrix} = \sigma_x \mathbf{X}_\pi \sigma_x = \kappa(\pi, 1)$. On $\mathbb{CP}^1$ this is $\zeta \mapsto 1/\bar{\zeta}$. Composed with the mirror $\zeta \mapsto \bar{\zeta}$ (Rosati/complex conjugation, Lemma 69’s $\pi \leftrightarrow \bar{\pi}$) it is $\zeta \mapsto 1/\zeta$, whose fixed set is exactly the circle $|\zeta| = 1$. So: normalized cusp-form points are fixed by “duality $\circ$ conjugation”; Eisenstein points $p^{\pm 1/2}$ are swapped (Lemma 70’s “ $p \mapsto 1/p$ after normalization”).*

Proof. ${}^t\Gamma_\varphi \cdot e_1 = \Gamma_\varphi \cdot e_2 = 1$, ${}^t\Gamma_\varphi \cdot e_2 = \deg \varphi$, and ${}^t\Gamma_\varphi \cdot \Gamma_\chi = \deg(1 - \varphi\chi) = 1 + \deg \varphi \deg \chi - \mathrm{Tr}(\varphi\chi)$. Comparing with $(\deg \varphi e_1 + e_2 + D') \cdot \Gamma_\chi = \deg \varphi \deg \chi + 1 + D' \cdot D_\chi$ gives $D' \cdot D_\chi = -\mathrm{Tr}(\varphi\chi) = D_{\hat{\varphi}} \cdot D_\chi$. Then $(D' - D_{\hat{\varphi}})^2 = D'^2 - 2D' \cdot D_{\hat{\varphi}} + D_{\hat{\varphi}}^2 = -2\deg \varphi + 4\deg \varphi - 2\deg \varphi = 0$ (using ${}^t\Gamma_\varphi^2 = 0$), and $D' - D_{\hat{\varphi}} \perp e_1, e_2$, so $D' = D_{\hat{\varphi}}$ in $\mathrm{NS}_{\mathrm{num}}$ by Hodge index. For $\varphi = \pi$, $\hat{\pi} = \bar{\pi}$. $\square$

Reading [RESTATEMENT]/[HEURISTIC]. “ $|\pi| = \sqrt{p}$ ” (RH for $E/\mathbb{F}_p$) is “the normalized point is fixed by duality $\circ$ conjugation”. That is the curve version of “zeros on the fixed line of $s \mapsto 1 - \bar{s}$ ” (Lemma 64’s Morse picture). It is a restatement: the fixed-point property is equivalent to RH for the curve and is proved by Hodge index (Cor. 3.2(iii)), not by the reflection.

7. PRINCIPALIZATION AS HAMILTONIZATION (STEERING [T283U])

J.W.’s phrase from [t222u]/[t275u] was “principalization as Hamiltonization”. The chat’s five claims are checked one by one. The phrase has an exact meaning for curves over $\mathbb{F}_q$ and only a partial one for $\mathrm{Spec} \mathbb{Z}$.

(1) Curves: Pic^0 as the state space.

Proposition 7.1 ([THEOREM], classical; $X_0(11)$ checked). *For $E/\mathbb{F}_p$, $\mathrm{Pic}^0(E) = \mathrm{Div}^0/\mathrm{Prin} \cong E$ (Abel–Jacobi [BIB]). $T_\ell \mathrm{Pic}^0$ is free of rank 2, and $\det(1 - FT | V_\ell) = 1 - a_p T + pT^2$, so $\#\mathrm{Pic}^0(\mathbb{F}_{p^n}) = \det(1 - F^n) = |1 - \pi^n|^2$. The zeros of $Z(E, p^{-s})$ are the s with $p^s \in \{\pi, \bar{\pi}\}$ (Weil; [8, §1] [CITED]). By Thm. 3.1(iv), $\#\mathrm{Pic}^0(\mathbb{F}_{p^n}) = \Gamma_{\pi^n} \cdot \Delta = 2\langle \mathbf{X}_{\pi^n}, \mathbf{X}_1 \rangle$: the class number of the curve is the Minkowski pairing of the null helix with the diagonal $\mathbf{X}_1$.*

[NUMERICAL]: confirmed for 11a1 by brute force at 45 primes and along the helix over $\mathbb{F}_{p^n}$ (§3). For $p = 2, 3, 5$ the group is $\mathbb{Z}/5$, consistent with the rational 5-torsion of $X_0(11)$.

What is a Theorem and what is a Heuristic. “ $T_\ell \mathrm{Pic}^0$ is the Hilbert–Pólya space” is [HEURISTIC]. T_ℓ is a $\mathbb{Z}_\ell$ -module with no inner product. The positivity that forces $|\pi| = \sqrt{p}$ comes from the Rosati involution (Weil’s second proof) or Hodge index (Cor. 3.2(iii)), i.e. from the definite complement in $\mathrm{NS}(E \times E)$, not from T_ℓ itself. With that addition the slogan is exact for curves [THEOREM]: *state space* = Pic^0 (principalization), *evolution* = $F/\sqrt{p} = \exp(\theta_p J_p)$ generated by $H_p = \frac{1}{2}g_p$ (Thm. 6.1, Hamiltonization), *positivity* = Hodge index.

(2) Iwasawa main conjecture. Mazur–Wiles proved the main conjecture over $\mathbb{Q}$ ([9]: Crossref record checked; text not read) and Wiles extended it to totally real fields (Wiles 1990, [BIB]). In the form read at source in [7, §22] [CITED-sec]: the Kubota–Leopoldt p -adic L -function vanishes at χ^* iff there is an Iwasawa representation of determinant χ , and simplicity of the zero corresponds to uniqueness. The usual formulation, that the characteristic ideal of the χ -part of $X_\infty = \varprojlim \mathrm{Cl}(\mathbb{Q}(\mu_{p^n})) [p^\infty]$ is generated by the power series interpolating L_p , is [BIB] here. Precisions: (a) it is a characteristic *power series* (or ideal) of a torsion Λ -module, not a characteristic polynomial of an operator on a Hilbert space. The eigenvalues of γ are p -adic 1-units, and there is no positivity (Lemma 66 [TAA-L]).

“ p -adic Hilbert–Pólya” is [ANALOGY]. (b) X_∞ is a *projective* limit along norms, while capitulation is about the kernels of the *extension* maps $\mathrm{Cl}(K_n) \rightarrow \mathrm{Cl}(K_m)$. “Capitulation shrinks the state space” conflates the two and is [HEURISTIC]. For the relevant minus parts no class capitulates in the cyclotomic tower (Lemma 66: Kida via Lemmermeyer, [CITED-sec]). (c) Greenberg’s conjecture ($\lambda = \mu = 0$ for totally real fields) is in [5] (Crossref record checked: Amer. J. Math. 98 (1976) 263). Its relation to capitulation is [BIB] second-hand; we did not read the paper and state no precise criterion.

(3) $\mathbb{Z}$ is a PID; Connes’ idele class space.

Proposition 7.2 ([THEOREM], standard). $\mathrm{Cl}(\mathbb{Z}) = 1$, but the idele class group is not trivial: $C_{\mathbb{Q}} = \mathbb{A}^\times / \mathbb{Q}^\times \cong \mathbb{R}_{>0}^\times \times \hat{\mathbb{Z}}^\times$, with $\mathbb{R}_{>0}^\times$ the idelic norm and $\hat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ by class field theory [BIB]. The idele with p at the place p and 1 elsewhere has norm p^{-1} . Its $\mathbb{R}_{>0}^\times$ -component therefore has log-norm $-\log p = -u_p$.

Connes [3] [CITED] (introduction read): $C_k = \mathrm{GL}_1(\mathbb{A})/\mathrm{GL}_1(k)$ acts on the adèle class space $X = \mathbb{A}/k^\times$. The critical zeros appear “as an absorption spectrum, i.e. as missing spectral lines”, and noncritical zeros enter as resonances. The construction first needs a Sobolev-type parameter δ . Connes then shows, “by proving positivity of the Weil distribution”, that the δ -independent global trace formula “is equivalent to the Riemann Hypothesis for all L -functions with Grössencharakter”. So the chat’s “positivity missing = RH” is right as an *equivalence* ([RESTATEMENT]); nothing is proved about RH. In the function-field case Connes notes one has “not a flow but rather a single transformation” (Frobenius) [3, p. 8] [CITED]. For $\mathrm{Spec} \mathbb{Z}$, principalization kills no finite group. What survives is the continuous $\mathbb{R}_{>0}^\times$ (the scaling flow, the arithmetic degree) and the compact $\hat{\mathbb{Z}}^\times$ (Galois). The state space cannot be a Pic^0 (it is trivial). It must be a function space on $C_{\mathbb{Q}}$ or X , which is Connes’ choice.

(4) **Bost–Connes and CMR.** BC [1] (Crossref checked; content via [2, §3] [CITED-sec]): $Z(\beta) = \mathrm{Tr} e^{-\beta H} = \sum k^{-\beta} = \zeta(\beta)$; a unique KMS state for $0 < \beta \leq 1$; for $\beta > 1$ extremal KMS states indexed by $\hat{\mathbb{Z}}^\times$; ground states take algebraic values on a rational subalgebra, which generate $\mathbb{Q}^{ab}$, with Galois acting through $\hat{\mathbb{Z}}^\times$. CMR [2] [CITED] (Intro, table, Thm. 5.1 read): for imaginary quadratic K , the partition function is $\zeta_K(\beta)$; there is a unique KMS state for $0 < \beta \leq 1$; for $\beta > 1$, $\mathcal{E}_\beta \simeq \mathbb{A}_{K,f}^\times / K^\times$; the symmetry group is $\mathrm{Gal}(K^{ab}/K)$, and the class group $\mathrm{Cl}(\mathcal{O})$ acts by “symmetries by endomorphisms”, “so that our results hold for any class field”. **Correction:** the chat said “(Hilbert class field)”. The values generate the *maximal abelian* extension K^{ab} , which contains the Hilbert class field. **“Spectrum $\log n$ is RH-blind”** [HEURISTIC], made precise. The spectrum determines ζ (via $Z(\beta)$ and analytic continuation), so it is not information-blind. What is blind is the *positivity structure* ($H \geq 0$, KMS, Fock factorization). The same structure exists for any Beurling system (free Fock space over generalized primes, [THEOREM]), and Beurling zetas can violate RH (DMV06, abstract only via Lemma 69, [CITED-sec]).

(5) Does the scaling flow / $u = \log p$ connect to the KS lift?

Proposition 7.3 ([THEOREM] identities; [ANALOGY] identification). (i) The normalized KS lift is $\tilde{X}_{\pi_p} = \Lambda(\frac{1}{2}u_p + i\theta_p)\mathbf{X}_1$ (Prop. 4.1). The boost rapidity is $\frac{1}{2}u_p = \frac{1}{2}\log p$, and $|u_p|$ is the log-norm of the $\mathbb{R}_{>0}^\times$ -component of the idele class of p (Prop. 7.2). (ii) The rotation θ_p is not GL_1 data: it is the Satake angle of a GL_2 representation and has no image in $C_{\mathbb{Q}}$. (iii) The boost part alone is exactly the Eisenstein (pole-plane) data. $\Lambda(\frac{1}{2}u)$ moves $\mathbf{X}_1$ along the wall to $\kappa(1, \sqrt{p})$, and this has nothing to do with h^1 .

Verdict. The $\mathbb{R}_{>0}^\times$ scaling of Connes and BC matches the *boost* of the KS picture, and the match is exact at the level of the numbers $\log p$ [THEOREM]. That Connes’ flow “is” this Lorentz boost

is [ANALOGY]: no theorem maps one onto the other. The structural lesson is consistent with Lemmas 64–70. GL_1 scaling sees only the $\mathbf{1} \oplus \mathbb{L}$ axis, and the rotation in the transverse plane (where curve-RH lives, Cor. 3.2) is invisible to it. The Sundman time $s = 2\log(1+p) \approx 2u$ (Table 2) is a curiosity, [HEURISTIC]. “Principalization as Hamiltonization” is therefore: exact for curves ($\text{Pic}^0 + \frac{1}{2}g_p + \text{Hodge index}$, [THEOREM]); for $\text{Spec } \mathbb{Z}$, principalization leaves only the continuous scaling, the BC Hamiltonian $\log n$ has the right partition function, and the positivity that would matter is equivalent to RH (Connes), [RESTATEMENT].

8. THE GLOBAL QUESTION: ONE NULL CONFIGURATION FOR ALL PRIMES?

Theorem 8.1 (convexity [THEOREM]). *If X_i are future null vectors and $w_i \geq 0$ (not all 0), then $\sum w_i X_i$ is future timelike unless all X_i with $w_i > 0$ are proportional. Hence for any local data with $|\pi_p| = \sqrt{p}$ (true Frobenius, random Sato–Tate angles, angles 0), every positively weighted sum of lifts is timelike.*

Proof. $\det(X + Y) = \det X + \det Y + 2\langle X, Y \rangle$, and $2\langle X, Y \rangle = |\det[uv]|^2 \geq 0$, with equality iff $u \parallel w$. Induct. $\square$

X	10^2	10^3	10^4	10^5
true 11a1 angles	0.400	0.543	0.647	0.717
random Sato–Tate angles	0.473	0.560	0.653	0.719
all $\theta_p = 0$	0.246	0.393	0.517	0.610

TABLE 3. $\det(S_X)/(x^+x^-)$ for $S_X = \sum_{p < X} \frac{\log p}{\sqrt{p}} \tilde{X}_{\pi_p}$ (Weil weights). All positive, as Thm. 8.1 requires, and the true data are indistinguishable from random angles [NUMERICAL].

Where the zeros actually enter. The explicit formula for $L(E, s)$ (analytic normalization, conductor 11, Γ -factor $(2\pi)^{-s}\Gamma(s + \frac{1}{2})$; standard form, [BIB]) with $h(r) = e^{-r^2/T^2}$, $T = 4$, reads

$$\sum_{\gamma} h(\gamma) = \frac{1}{2\pi} \int h(r) \left[\log 11 + 2 \operatorname{Re} \frac{\Gamma'}{\Gamma}(1 + ir) - 2 \log 2\pi \right] dr - 2 \sum_{p,k} \frac{\log p}{p^{k/2}} \underbrace{2 \cos k\theta_p}_{\text{transverse projection}} \hat{h}(k \log p)$$

(with the bad-prime term $11^{-k} \log 11$). The prime side is a weighted sum of the *transverse projections* $\operatorname{Re} x_{\perp}^{(k)}/\sqrt{x^+x^-}$ of the null helix. Numerically, with the 26 zeros of $L(11a, s)$ up to height 40 (first 6.3626, 8.6035, 10.0355): zero side 0.1831364159, archimedean -0.0523729 , primes $+0.2355093$, residual $1.0 \cdot 10^{-17}$ [NUMERICAL]. A ghost quadruple $\frac{1}{2} \pm \delta \pm 10i$ changes the zero side by 0.00767 ($\delta = 0.1$) or 0.00723 ($\delta = 0.3$), i.e. about 4%, with the prime side fixed. So it is inconsistent with the given primes, which is a consistency statement (Lemma 69), not positivity. Nothing in Table 3 changes.

Verdict [THEOREM]/[HEURISTIC]. (i) “Timelike” for a global sum is automatic (convexity), so it is RH-blind. (ii) The RH-relevant information is in the *signed transverse projections*, and Weil positivity is a quadratic condition on test functions, not a cone condition on these vectors. (iii) The archimedean place has no null vector here: there is no Frobenius at ∞ , and the Γ -term enters only as a weight [OPEN]. (iv) A single null configuration whose shadows reproduce Weil positivity for all test functions would have to encode the zeros, which is *circular*. Over $\mathbb{F}_p$ the configuration exists (Thm. 3.1); over $\mathbb{Q}$ it would be the missing surface (Lemmas 65, 70), [OPEN].

9. SYNTHESIS: THE ANSWER TO J.W.

- (S1) *Yes, there is an exact $3D \rightarrow 4D$ lift, and it is not a metaphor [THEOREM].* The 3D space is V_3 (binary forms, $(2, 1)$). The 4D space is Herm_2 (Minkowski). The lift is the Hopf/KS map of the Frobenius spinor, and on $E \times E$ it is the Néron–Severi lattice: $\Gamma_\varphi \mapsto \kappa(1, \varphi)$, intersection $= 2 \det$, $(3, 1) = \text{Hodge index}$, fourth axis $= J_p$.
- (S2) *Scale-free, coordinate-free [THEOREM].* Projectivizing gives the celestial sphere. Normalizing removes exactly the boost $\frac{1}{2} \log p$, and what remains of each prime is a point $e^{i\theta_p}$ on the circle fixed by duality $\circ$ conjugation. The $\mathbf{1}$ and $\mathbb{L}$ poles, the wall and the circle are intrinsic: they are the classes e_1, e_2 , the real locus, and the fixed set of the transpose.
- (S3) *The orbit [THEOREM].* The orbit is a null helix: boost plus rotation, closed iff supersingular. Its scale-free part is a uniform circular motion, which KS turns into a *circular* Kepler orbit for the Hamiltonian $\frac{1}{2}g_p$ of the local triple. So the triple (ω, J_p, g_p) is literally (symplectic form, generator, Hamiltonian) of an oscillator, and that is the precise content of “Kepler-like”. Anything more (eccentricity, Runge–Lenz, a hidden $\text{SO}(4)$ across primes, a global Sundman time) is [ANALOGY]: we found nothing of the kind.
- (S4) *Positivity [RESTATEMENT].* Locally, positivity is “the shadow has a nonzero fourth coordinate” $= \mathbb{Z}[\pi] \neq \mathbb{Z} = \text{Hodge index on } E \times E$. Globally, sums of lifts are timelike automatically (RH-blind), and the RH-relevant content is in signed transverse projections, i.e. the explicit formula itself. The global object is again the missing surface over $\mathbb{Q}$ [OPEN].

RH is not proven. Every positivity statement above is either local (proved, Hasse/Deligne/Hodge index on $E \times E$) or an equivalent restatement (Connes, Weil).

10. CORRECTIONS TO THE CHAT CLAIMS

- (C1) (a) V_3 is $(2, 1)$ and Q_p is timelike with norm $p - a_p^2/4$: **correct** (norm $AC - B^2/4$, positive on definite forms).
- (C2) (b) KS = Hopf, $\ell\ell^\dagger$ null, light-cone coordinates, $\pi\bar{\pi} = p$, $t = \frac{1+p}{2}$ (half the E_2 eigenvalue $\sigma_1(p)$), $z = \frac{1-p}{2}$: **correct**. Convention: $(\frac{1}{\pi} \frac{\bar{\pi}}{p})$ is the Gram matrix of $|m + \bar{\pi}n|^2$, the transpose of the Hasse Hermitian form. Lemma 67’s $\tau_p = -\bar{\pi}_p$ belongs to the spinor $(1, -\bar{\pi}_p)$ and the charpoly form $[1, -a_p, p]$ (Rem. 2.2).
- (C3) (c) Shadow = Hasse, norm = gap^2 , positivity = nonzero fourth coordinate: **correct**, and sharpened. The shadow is the *restriction to $\mathbb{R}^2$* (equivalently the average of X_π and $X_{\bar{\pi}}$). Only timelike or null forms have null lifts. The fourth coordinate exists iff $\mathbb{Z}[\pi] \neq \mathbb{Z}$ (Cor. 3.2).
- (C4) (c) “Degeneracy = the lift falls into the 3D slice”: **correct**, and now a theorem: $\Gamma_\pi \in \langle e_1, e_2, \Delta \rangle$ iff $\pi \in \mathbb{Z}$.
- (C5) (d) “closed iff $a_p^2 \in \{0, p, 2p, 3p\}$; supersingular plus a few tiny cases”: **imprecise**. For p prime $a^2 = p$ is impossible, and the remaining cases $(2, \pm 2)$ and $(3, \pm 3)$ are *also supersingular*. So: closed $\Leftrightarrow$ supersingular, with periods 4, 8 ($p = 2$), 12 ($p = 3$) (Thm. 4.2). For Δ : closed $\Leftrightarrow \tau(p) = 0$ (Lehmer, [OPEN]). The chat’s numbers (null to $2 \cdot 10^{-16}$; 77 primes < 400 ; closed at 2, 19, 29, 199, with $p = 2$ of period 8) are **confirmed**.
- (C6) (d) “precesses forever, like a perturbed Kepler orbit, since a closed Kepler orbit needs its hidden $\text{SO}(4)$ ”: **misleading**. The continuous scale-free orbit is always a closed circle. Non-closure is an irrational *rotation number* under integer sampling, not Kepler precession (Rem. 4.4).
- (C7) (e) “ $\mathbb{H}$ = upper hemisphere, degenerate wall = equator”: **needs care**. In coordinates where $\mathbf{1}, \mathbb{L}$ are the poles, the wall $\mathbb{R} \cup \infty$ is a great circle *through* the poles, and $\mathbb{H}$ is a hemisphere bounded by it. The circle $|\zeta| = 1$ orthogonal to the axis is where the *normalized* Frobenius points lie. Iterates leave $\mathbb{H}$ (Prop. 5.1).
- (C8) (f) “Kepler is an Analogy until a Hamiltonian and Sundman time exist”: **correct, and now partly resolved**. A Hamiltonian exists ($\frac{1}{2}g_p$, Thm. 6.1). It yields only the circular orbit,

and the unnormalized helix is provably not KS-Kepler (Thm. 6.3). “Kepler” beyond that stays [ANALOGY].

- (C9) “The KS lift is exactly the $(2, 1) \rightarrow (3, 1)$ step” and “the lift is Weil’s light cone exactly”: **correct**, upgraded from analogy to Thm. 3.1.
- (C10) [t283u] (1) “Hilbert–Pólya space = Tate module of Pic^0 ”: **the counts are correct; the name is heuristic**. T_ℓ has no inner product, and positivity comes from Rosati/Hodge index. “Eigenvalues = zeta zeros”: precisely, zeros at $p^s \in \{\pi, \bar{\pi}\}$.
- (C11) [t283u] (2) “char poly of γ on $X = p$ -adic L -function”: **imprecise**. It is a characteristic ideal (power series) of a torsion Λ -module, and the statement was read only second-hand via [7]. “Capitulation shrinks the state space” conflates the projective limit X with the extension maps ([HEURISTIC]). Greenberg–capitulation: bibliographic only.
- (C12) [t283u] (3) PID, $C_{\mathbb{Q}} = \text{ideles mod principal ideles}$, absorption spectrum: **correct** (Connes read at source). Two precisions: Connes proves an *equivalence* (trace formula $\Leftrightarrow$ RH via Weil positivity), and $C_{\mathbb{Q}} \cong \mathbb{R}_{>0}^\times \times \hat{\mathbb{Z}}^\times$ is not trivial.
- (C13) [t283u] (4) BC ($H = \log n, \zeta(\beta), \beta = 1, \mathbb{Q}^{ab}$): **correct** (via CMR). CMR “(Hilbert class field)”: **too weak**, the values generate K^{ab} . “Spectrum RH-blind”: true of the positivity structure, not of the spectrum’s information ([HEURISTIC]).
- (C14) [t283u] (5) $u = \log p \leftrightarrow$ KS: **exact as kinematics**. $\frac{1}{2}u_p$ is the boost rapidity, and u_p is the idelic log-norm. Identifying Connes’ flow with the Lorentz boost is [ANALOGY]. The scaling sees only $1 \oplus \mathbb{L}$, never θ_p .

11. RANKED NEXT DIRECTIONS (0.6 s + 0.4 p)

	Direction	s	p	score
F1	Genus 2 / abelian surfaces: the KS parametrization $\Gamma_\varphi \mapsto (1, \deg \varphi, \varphi)$ for $J \times J$ with $\text{End}(J)_{\mathbb{R}}$ of rank > 2 . Is the transverse space the Rosati-positive part, signature $(1, 1 + \text{rk})$? (= L65 G1 / L70 E6)	.70	.70	.70
F2	Explicit formula as intersection with the diagonal: write the prime side of $L(E, s)$ as $\sum_{p,k} w \Gamma_{\pi_p^k} \cdot \Delta$ -type pairings in $\bigoplus_p \text{NS}(E_p \times E_p)$, and locate exactly where Weil positivity is <i>not</i> a cone condition (toward the global surface)	.85	.35	.65
F3	Bianchi $(3, 1)$ (queued since [t275u]): Hermitian forms over $\mathbb{Q}(i), \mathbb{Q}(\sqrt{-3})$, Eisenstein congruences, time-like/spacelike split	.60	.60	.60
F4	Idelic dictionary: the Weil group $W_{\mathbb{Q}} \rightarrow C_{\mathbb{Q}}$ image of Frob_p as the boost, and the archimedean $W_{\mathbb{R}}$ as a would-be null vector at ∞ (the missing Γ -term vector, §8(iii))	.70	.35	.56
F5	Closure census: supersingular counts vs Lang–Trotter constants for many curves; Lehmer as “no closed Δ orbit” to 10^7	.30	.90	.54
F6	$[\vartheta, T_p]$: structure of the failure of Hecke-equivariance of the Ramanujan–Serre flow	.40	.70	.52

Still open from earlier lemmas: L70 E1 (Atkin–Lehner two-cusp reflection, .68), E2 (L64 engine on a congruent pair, .66), E3 ($\Delta \equiv \theta E_2 \pmod{5}$, .65).

12. QUESTIONS FOR J.W.

- Q1.** Do we adopt Thm. 3.1 (“the KS lift of the local triple is the Néron–Severi light cone of $E \times E$; $(3, 1) =$ Hodge index; fourth axis $= J_p$ ”) as the official TAA form of [t281u]?
- Q2.** Do you accept that the only natural Hamiltonian is $\frac{1}{2}g_p$, that it gives circular orbits, and that “Kepler” beyond this stays an Analogy (Thm. 6.3)?

- Q3.** Do we replace the chat’s closure list by “closed $\Leftrightarrow$ supersingular” and record “closed Δ orbit $\Leftrightarrow \tau(p) = 0$ (Lehmer)”?
- Q4.** For “principalization as Hamiltonization”: adopt the curve version ($\text{Pic}^0 + \frac{1}{2}g_p + \text{Hodge index}$, all theorems) and record for $\text{Spec } \mathbb{Z}$ that only the $\mathbb{R}_{>0}^\times$ boost survives and that it is blind to θ_p ?
- Q5.** Which comes next: F1 (genus 2), F2 (explicit formula as intersection with the diagonal), F3 (Bianchi), or L70’s E1–E3?

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TAA LEMMA 72: PURE 3D GRAVITY, ζ ZEROS, AND NEGATIVE DENSITIES OF STATES. WHERE EXACTLY DO THE ZEROS MEET THE NON-UNITARITY OF THE MALONEY–WITTEN–KELLER PARTITION FUNCTION?

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Nothing below is a step toward a proof that we can certify. Every RH-equivalent statement is labelled [RESTATEMENT], and every metaphor is labelled [ANALOGY] or [HEURISTIC]. Tags: [THEOREM] proved here (elementary) or classical with the proof indicated; [CITED] checked at the source (`lit/lemma72/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (mpmath, 90–170 digits; finite ranges, not proofs).

Question (J.W., [t287u]). Find a direct, checkable meeting point between the nontrivial zeros of ζ and the non-unitarity of the Maloney–Witten–Keller (MWK) pure AdS_3 gravity partition function (negative densities, continuous spectrum, the -6), under the TAA thesis “RH is a statement about unitarity”. Context: [t284u]–[t286u] (Einstein equations, cosmological constant, principalization).

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ANSWER IN BRIEF

- (1) Where $\varphi(s) = \xi(2s-1)/\xi(2s)$ enters (§2). The whole spin-0 MWK density is built from the Eisenstein scattering coefficient φ sampled at the half-integers $s = m + \frac{1}{2}$: each scalar seed of reduced energy E contributes $\sum_{m \geq 0} \frac{(2\pi(-E))^m}{m!} \varphi(m + \frac{1}{2}) y^{1/2-m}$, i.e. the reflected halves $\varphi(s)y^{1-s}$ of the constant terms of $E(\tau, m + \frac{1}{2})$ [THEOREM] (a re-reading of [11, (3.40)], [10, (47)–(49), (72)]). The famous -6 is $-6 = 2\varphi(\frac{1}{2}) - 2 \cdot 2\sigma_0(1) = -2 - 4$: twice the centre value $\varphi(\frac{1}{2}) = -1$ (Lemma 69’s ghost-blind fixed-point value; $E(\tau, \frac{1}{2}) \equiv 0$) and twice the residue-matching value 2 of the two spinning seeds $(1-q)(1-\bar{q})$ [THEOREM]+[NUMERICAL].
- (2) Where the zeros enter (§3). (a) In the MWK density itself, at the physical regularization point $s = \frac{1}{2}$, nowhere: written as a Rademacher sum over c it contains only Kloosterman (Ramanujan) sums, and in closed form only $\zeta(2m)$, $\zeta(2m+1)$ ($m \geq 1$), $\zeta(0) = -\frac{1}{2}$ and the residue of ζ at 1 [THEOREM]. As a function of the KM regulator s the coefficients have poles at $s = \rho/2 - m$; the physical point $s = \frac{1}{2}$ is $2s = 1$, the edge of the critical strip, i.e. Lemma 69’s screen. It is safe because $\zeta(1+it) \neq 0$ (prime number theorem level), not because of RH [THEOREM]. (b) The zeros do appear when the reflected-half series is resummed at high temperature. For $X > 0$,

$$S(X) := \sum_{m \geq 0} \frac{(-X)^m}{m!} \varphi(m + \frac{1}{2}) = -\frac{6}{\sqrt{\pi}} \sqrt{X} + \sum_{\rho} \Gamma(\frac{1}{2} - \frac{\rho}{2}) \text{Res}_{s=\rho/2} \varphi X^{\rho/2-1/2} + O(X^{-A})$$

for every A (§3.3, Thm. 3.3) [THEOREM], checked to 10^{-36} for $100 \leq X \leq 200$ with 30 zero pairs [NUMERICAL]. The leading coefficient is $\Gamma(-\frac{1}{2}) \text{Res}_{s=1} \varphi = -2\sqrt{\pi} \cdot \frac{3}{\pi}$: Lemma 69’s anomaly $3/\pi = 1/\text{vol}$. $S(X)$ is the reflected part of the spin-0 mode of a heavy ($E > 0$) scalar Poincaré seed at temperature $\propto 1/y$, with $X = 2\pi E/y$.

- (3) Literature check (§3.4). The claim “later work expresses deviations from semiclassical gravity as a series over the nontrivial zeros” is *partly true*. Benjamin–Chang [1] write the scalar partition function of $U(1)^c$ CFTs as Cardy term + correction + $\sum_k T^{c/2-1+z_k/2}(\dots)$ and restate RH as the envelope $T^{c/2-3/4}$ [CITED]; they only test a Virasoro version on Z^{MWK} ($c = 3$). Godet [9] writes the Hartle–Hawking state of toroidal cosmology as a sum over zeros (RH $\Leftrightarrow$ fluctuations of order $G^{1/3}$ in $d = 3$) [CITED]. We found no paper writing the MWK density itself as a zero series. The formula in (2b) is derived here.
- (4) Numerics reproduce the source results (§4). The -6 to 25 digits; the $j = 0$ continuum (two independent assemblies agree to 10^{-90}), positive for $\xi = \frac{c-1}{24} \geq \frac{1}{2}$ and negative at small e for $\xi < \frac{1}{2}$; the large-odd-spin near-extremal negativity of BOSW [4, (6.14)] to 10 digits ($\rho_9 = -6.296258 \cdot 10^{30}$ at $\xi = 2$, $\bar{h} - \xi = 10^{-40}$), with window width $\approx 10^{-1.9} e^{-2\pi\sqrt{\xi j}}$ for $3 \leq j \leq 25$ [NUMERICAL].
- (5) Ghost tests: is positivity sensitive to RH? (§5). **No.** (G1) In the MWK density a “ghost” can only be inserted by editing the values $\zeta(2m+1)$ in the closed forms; this breaks the exact Ramanujan-sum identity that produces the $s = 1$ cancellation, and a perturbation of relative size 10^{-8} ($\delta = 0.001$) already flips the sign of the window. The window is an *arithmetic cancellation*, not a zero effect, and the experiment is ill-posed as an RH test [NUMERICAL]+[THEOREM]. (G2) In the resummed reflected half, an off-line pair $\beta = \frac{1}{2} + \delta$ changes the decay of the oscillating zero part from $X^{-1/4}$ to $X^{(\beta-1)/2}$ (fitted slopes in Fig. 5). It never changes the sign, which is fixed by the anomaly term, since the zero part is $\lesssim 10^{-5.6}$ of it for $X \geq 10$ [NUMERICAL]. (G3) The -6 is exactly ghost-invariant once the ghost keeps $\text{Res}_{w=1} \zeta = 1$ [THEOREM].
- (6) Verdict [HEURISTIC]. In MWK, **negativity is RH-blind** and **RH is negativity-blind**. The negativities come from (i) the centre value $\varphi(\frac{1}{2}) = -1$ and the vacuum null states (the -6) and (ii) Kloosterman phases $S(j, \cdot; 2) = \pm(-1)^j$ (the odd-spin window). The zeros sit in the *size of fluctuations* around the semiclassical (anomaly/Cardy) term: RH $\Leftrightarrow$ the zero part of every such resummation decays like $X^{-1/4}$ [RESTATEMENT] (modulo convergence, Rem. 3.4). So the

meeting point exists, but it reads “RH is a statement about the *size of the non-semiclassical fluctuations*”, i.e. square-root cancellation, as in $\pi(x) - \text{Li}(x) = O(\sqrt{x} \log x)$. It does not read “RH is the positivity of a density”. The TAA slogan “RH is unitarity” is *not* supported by this example [HEURISTIC].

- (7) *Einstein-equation claims* (§6.1). Hyperbolic quotients are Einstein with $\Lambda < 0$ in dimension 3 ($\text{Ric} = -2g$, MWK’s saddles). In dimension 2 the Einstein tensor vanishes identically, so $\text{Ric} = \lambda g$ is *not* “Einstein’s equation with a cosmological constant” (the trace forces $\Lambda = 0$). Humbert’s Bianchi volume $|d_K|^{3/2} \zeta_K(2)/(4\pi^2)$ is verified (source and 10 digits). Corrections in §7.

1. SETUP: THE MWK PARTITION FUNCTION AS A POINCARÉ SUM

Conventions follow [10]. $\tau = x + iy$, $q = e^{2\pi i \tau}$, $\xi := \frac{c-1}{24}$ (the letter ξ also denotes the completed zeta function $\xi(w) = \pi^{-w/2} \Gamma(\frac{w}{2}) \zeta(w)$; the meaning is always clear from the argument). A Virasoro primary of weights $(h, \bar{h})$ has reduced energy $E = h + \bar{h} - 2\xi$ and spin $J = h - \bar{h}$. The vacuum character with its null states removed is $q^{-\xi} \bar{q}^{-\xi} (1-q)(1-\bar{q})$, so the vacuum is *four* seeds:

$$(1) \quad (E, J; \pm) \in \{(-2\xi, 0; +), (-2\xi + 1, 1; -), (-2\xi + 1, -1; -), (-2\xi + 2, 0; +)\}.$$

Each seed $y^{1/2} q^\Delta \bar{q}^{\bar{\Delta}}$ is summed over $\Gamma_\infty \backslash \text{SL}_2(\mathbb{Z})$ after the regularization $y^{1/2} \rightarrow y^s$, continued to $s = \frac{1}{2}$ [10, (36)–(39)], [11, §3] [CITED]. Poisson resummation gives the Fourier modes [10, (41)–(44)]

$$(2) \quad E_j(s, E, J) = \sum_{m \geq 0} I_{j,m}(s, E, J) y^{1-m-s} Z_{j,J}(m+s), \quad Z_{j,J}(w) = \sum_{c \geq 1} c^{-2w} S(j, J; c),$$

with the Kloosterman sums $S(j, J; c) = \sum_{d \in (\mathbb{Z}/c)^*} e^{2\pi i(jd + J\bar{d})/c}$. The density of new primaries $\rho_j(e) = \sum_m \rho_{j,m}(e)$ is the inverse Laplace transform of $y^{-1/2} E_j$ [10, (65)–(68)]. For $j \neq 0$,

$$(3) \quad \rho_{j,m}(e) = Z_{j,J}(m + \tfrac{1}{2}) c_m |j|^{m-1} \nu_{j,m}(e/|j|), \quad c_m = \frac{2^{3m+1} \pi^{2m}}{(2m)!},$$

where $\nu_{j,m} = D_t^m f_m$, $f_k(t) = \cosh(k \cosh^{-1} t) / \sqrt{t^2 - 1}$ for $t > 1$ (and 0 for $t < 1$), and $D_t f_k = -E f_k - J f_{k-1} - \frac{J}{2}(1 - \frac{m}{k})(f_{k+1} - f_{k-1})$ [10, (73)–(85)], [4, (6.6)–(6.10)] [CITED]. Cosmic censorship $\rho_j(e) = 0$ for $e < |j|$ is built in. $t = e/|j| \rightarrow 1$ is extremality, $\bar{h} \rightarrow \xi$.

The three non-unitarities [CITED].

- (N1) $\rho_0(e) \ni -6\delta(e)$: “the number of states at that energy is not a positive integer ... but rather -6 ” [11, (3.39)], [10, (93)].
- (N2) Power-law (not exponential) corrections and a continuous spectrum: the partition function is not $\text{Tr } e^{-\beta H}$ for a discrete H [11, §2.3, (2.31), (3.47)].
- (N3) Large odd spin near extremality: $\rho_j(e) < 0$ in a window $\bar{h} - \xi \lesssim e^{-2\pi\sqrt{\xi j}}$, dominated by the $c = 2$ Kloosterman term [4, (6.12)–(6.15)]; restated as $\int_0^{t_0} \rho_{\text{MWK},j} \sim (-1)^j e^{\pi\sqrt{\xi j}}$, $t_0 \sim e^{-2\pi\sqrt{\xi j}}$ in [8, (1)].

What the cure looks like [CITED]. BOSW add N copies of a twist- $\frac{c-1}{16}$ scalar and $N+6$ copies of the $c = 1$ self-dual boson [4, (6.16)]. Benjamin–Collier–Maloney add conical-defect states [3] (abstract only). Di Ubaldo–Perlmutter add two “string” states of spin $\propto c$ and obtain a unitary partition function with gap $\frac{c-1}{12}$ [8]. In every cure the negativity is removed by *adding light states*. None of the cures involves ζ .

2. WHERE $\varphi(s) = \xi(2s-1)/\xi(2s)$ ENTERS THE MWK DENSITY

Recall $E(\tau, s) = \sum_{\Gamma_\infty \backslash \Gamma} \text{Im}(\gamma\tau)^s = y^s + \varphi(s)y^{1-s} + \sum_{j \neq 0} a_j(s, y)e^{2\pi i j x}$ with
(4)

$$\varphi(s) = \frac{\xi(2s-1)}{\xi(2s)} = \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2}) \zeta(2s-1)}{\Gamma(s) \zeta(2s)}, \quad a_j(s, y) = \frac{2 |j|^{s-1/2} \sigma_{1-2s}(|j|)}{\xi(2s)} \sqrt{y} K_{s-1/2}(2\pi |j| y)$$

(classical; [2, (2.8)–(2.9)] in the normalization $\Lambda(s) = \pi^{-s} \Gamma(s) \zeta(2s)$) [THEOREM]/[CITED]. Lemma 69 called $\varphi(s)y^{1-s}$ the *reflected half* of the constant term, and identified $-3/(\pi y)$ in E_2^* with its $s \rightarrow 1$ limit, $\text{Res}_{s=1} \varphi = 3/\pi = 1/\text{vol}$ [TAA-L].

Theorem 2.1 (The spin-0 MWK density is φ at the half-integers). *Let $(E, 0)$ be a scalar seed. Then:*

(a) *Its regularized Poincaré sum is $\sum_{m \geq 0} \frac{(-2\pi E)^m}{m!} E(\tau, m+s)$, evaluated at $s = \frac{1}{2}$. The $m = 0$ term is $E(\tau, \frac{1}{2}) \equiv 0$.*

(b) *The non-identity part of its spin-0 mode is*

$$(5) \quad y^{1/2} \sum_{m \geq 0} \frac{(2\pi(-E))^m}{m!} \varphi(m + \frac{1}{2}) y^{-m},$$

and its spin-0 density of new primaries is

$$(6) \quad \rho_0(e) = \varphi(\frac{1}{2}) \delta(e) + \sum_{m \geq 1} \frac{(2\pi(-E))^m}{m!} \varphi(m + \frac{1}{2}) \frac{(2\pi)^m e^{m-1}}{(m-1)!}, \quad \varphi(\frac{1}{2}) = -1.$$

(c) *For $j \neq 0$ its spin- j density is $\sum_{m \geq 1} \frac{\sigma_{2m}(|j|)}{|j|^{2m} \zeta(2m+1)} c_m |j|^{m-1} (-E)^m f_m(e/|j|)$, i.e. it is built from the Fourier coefficients $a_j(m + \frac{1}{2}, \cdot)$ of (4), whose only ζ -dependence is $1/\zeta(2m+1)$. The $m = 0$ term vanishes because $1/\zeta(1) = 0$.*

Proof. (a) Expand $e^{-2\pi E y} = \sum_m (-2\pi E)^m y^m / m!$ inside the Poincaré sum (absolutely convergent for $\text{Re } s > 1$, then continue) [2, (E.3)] [CITED]. $E(\tau, \frac{1}{2}) \equiv 0$: its constant term is $(1 + \varphi(\frac{1}{2}))y^{1/2} = 0$ and its other coefficients carry $1/\xi(1) = 0$ (Lemma 69, Thm. fpblind) [TAA-L]. (b) KM's (47) reads $E_{0,m}(s, E, 0) = \frac{\zeta(2(m+s)-1)}{\zeta(2(m+s))} \frac{2^m \pi^{m+1/2} \Gamma(m+s-\frac{1}{2})}{m! \Gamma(m+s)} (-E)^m y^{1-m-s}$ [10] [CITED]. With $\sigma = m+s$ the right side is $\frac{(2\pi(-E))^m}{m!} \varphi(\sigma) y^{1-\sigma}$ by (4). At $s = \frac{1}{2}$ this is (5). Apply $\mathcal{L}^{-1}(y^{-m}) = (2\pi)^m e^{m-1} / (m-1)!$ [10, (71)]. Near $s = \frac{1}{2}$, $\Gamma(s - \frac{1}{2}) \sim (s - \frac{1}{2})^{-1}$ and $\zeta(2s) \sim (2(s - \frac{1}{2}))^{-1}$, so $\varphi(s) \rightarrow 2\zeta(0) = -1$. (c) is [10, (53)–(54), (74)] together with $Z_{j,0}(w) = \sigma_{1-2w}(|j|)/\zeta(2w)$ (Ramanujan sums, [10, (B3)], [11, (3.49)]) [CITED]. $\square$

[NUMERICAL] (`mwk_checks.json`) $\varphi(\frac{1}{2} + 10^{-30}) = -1.0000\dots$ (25 digits). The coefficient of (6) equals $2^{m-1} c_m \frac{\zeta(2m)}{\zeta(2m+1)} (-E)^m e^{m-1}$, KM's (72). MW's $w_m = \frac{2^m \pi^{m+1/2} \zeta(2m)}{m \Gamma(m+1/2) \zeta(2m+1)}$ [11, (3.40)] is $\frac{(2\pi)^m}{m!} \varphi(m + \frac{1}{2})$ [THEOREM].

Proposition 2.2 (Anatomy of the -6). *For a spinning seed (E, J) , $J \neq 0$, the $j = 0$, $m = 0$ coefficient is $\lim_{s \rightarrow 1/2} \frac{\sqrt{\pi} \Gamma(s - \frac{1}{2}) \sigma_{2s-1}(|J|)}{\Gamma(s) |J|^{2s-1} \zeta(2s)} = 2\sigma_0(|J|)$. For the vacuum (1),*

$$\rho_0^{\text{vac}} \ni [(+1)\varphi(\frac{1}{2}) + (+1)\varphi(\frac{1}{2}) + (-1)2\sigma_0(1) + (-1)2\sigma_0(1)]\delta(e) = (-1 - 1 - 2 - 2)\delta(e) = -6\delta(e),$$

independently of c [THEOREM]. The two scalar seeds give -2 through the centre value of the reflected half. The two null descendants L_{-1} , $\bar{L}_{-1}$ give -4 through the residue matching $\Gamma(s - \frac{1}{2})/\zeta(2s) \rightarrow 2$, which uses $\text{Res}_{w=1} \zeta(w) = 1$. MW identify both mechanisms ([11, (3.35)–(3.36)]: “negative because $\zeta(0) = -\frac{1}{2}$ ”; “ $\Gamma(s - \frac{1}{2})/\zeta(2s) \rightarrow 2$ ”) [CITED]. The φ -reading is TAA's.

[NUMERICAL] $2\varphi(\frac{1}{2}) - 2 \cdot 2\sigma_0(1) = -6.000\dots$ (25 digits; `num/lemma72_mwk.py`).

Remark 2.3 (The -6 is RH-blind). $\varphi(\frac{1}{2}) = -1$ follows from the functional equation and the simple poles of ξ alone, and holds for every FE-symmetric ghost $\xi \rightarrow \xi Q$ (Lemma 69) [TAA-L]. The value $2\sigma_0(J)$ uses only $\text{Res}_{w=1} \zeta = 1$ and $\text{Res} \Gamma = 1$. A ghost model that keeps $\text{Res}_{w=1} = 1$ leaves -6 exactly invariant. A raw ghost $\zeta \rightarrow \zeta Q$ turns it into $-2 - 4/Q(1)$, which measures the model's violation of $\text{Res}_{w=1} = 1$, not RH (§5) [THEOREM]+[NUMERICAL].

Theorem 2.4 (In Rademacher form the MWK density contains no ζ). *For $j \neq 0$, the $m \geq 1$ part of the MWK density is*

$$(7) \quad \rho_j^{(m \geq 1)}(e) = \sum_{(E,J,\pm)} \pm \sum_{c \geq 1} \frac{S(j, J; c)}{c} \sum_{m \geq 1} c^{-2m} c_m |j|^{m-1} \nu_{j,m}^{E,J}(e/|j|),$$

an absolutely convergent double sum. The only arithmetic input is the Kloosterman sums. ζ appears only when the c -sum is done in closed form for $J = 0$ ($\sigma_{2m}(j)/(j^{2m}\zeta(2m+1))$) or $j = 0$ [THEOREM].

Proof. $|S(j, J; c)| \leq c$ gives $\sum_c |S| c^{-2m-1} \leq \zeta(2m) \leq \zeta(2)$ for $m \geq 1$, and $\sum_m c_m |j|^{m-1} |\nu_{j,m}(t)| < \infty$ for fixed t (the coefficients of $\nu_{j,m}$ are polynomials of degree m in E, J , and c_m decays like $1/(2m)!$). Fubini then applies. $\square$

The $m = 0$ terms are 0 ($J = 0, j \neq 0$), $-\delta$ ($J = 0, j = 0$), $2\sigma_0(J)\delta$ ($J \neq 0, j = 0$), and, for $J, j \neq 0$, $Z_{j,J}(\frac{1}{2})c_0|j|^{-1}f_0(e/|j|)$. The last needs the analytic continuation of the Kloosterman zeta to $s = \frac{1}{2}$ (KM App. B.1, after Selberg). KM bound it by $|Z_{j,J}(\frac{1}{2})| \leq |j|^2 K_J$ [10, (89)] and use that for $\text{SL}_2(\mathbb{Z})$, $\frac{1}{2}$ is not a discrete eigenvalue of the Laplacian [CITED]. *We do not compute these terms.* They are polynomial in j with a fixed positive shape f_0 , and they are negligible against the $e^{2\pi\sqrt{\xi}j}$ scale of (N3) for the j used below. The input to this continuation is the Maass spectrum (Selberg's eigenvalue bound), not the zeros of ζ . That the continuous-spectrum part involves only the Eisenstein coefficients on $\text{Re } s = \frac{1}{2}$, i.e. $1/\zeta(1+2ir)$, is standard spectral theory, which we did not re-verify at a source here [BIB].

3. WHERE THE ZEROS OF ζ ENTER

3.1. Primitivity is where $1/\zeta$ comes from.

Theorem 3.1 (Coprimality = division by $\zeta(2s)$). $\Gamma_\infty \backslash \text{SL}_2(\mathbb{Z}) \leftrightarrow \{(c, d) = 1\}/\pm$, and $E(\tau, s) = \frac{1}{2\zeta(2s)} \sum_{(m,n) \neq 0} \frac{y^s}{|m\tau+n|^{2s}}$. *Hence the poles of $E(\tau, s)$ in $0 < \text{Re } s < \frac{1}{2}$ are at $s = \rho/2$, and every MWK coefficient obtained by summing a seed over $\Gamma_\infty \backslash \Gamma$ in closed form carries $1/\zeta(2(m+s))$ [THEOREM] (classical; residues $\text{Res}_{s=z/2} E_s$ written out in [1, (3.14)] [CITED]).*

Geometrically, $(c, d) = 1$ labels the primitive cycle of the boundary torus that becomes contractible in the solid torus $M_{c,d}$ (the $\text{SL}_2(\mathbb{Z})$ family of BTZ saddles) [11, §2] [CITED]. So the zeros of $\zeta(2s)$ enter MWK through the *primitivity* constraint, by Möbius inversion. Godet says exactly this for the Hartle–Hawking state: “its origin lies in the relatively prime condition ... coming from the no-boundary condition” [9, §3.2] [CITED]. In the language of [t286u] [ANALOGY]: restricting to *primitive* (“principal”) vectors is what introduces $1/\zeta$ and hence the zeros. This is the opposite of the class-group picture of [t286u], where the restriction to principal ideals is trivial for $\mathbb{Z}$.

3.2. The regulator plane: MWK sits on the screen.

Theorem 3.2 (MWK is evaluated on the line $2s = 1$). *For $J = 0$ seeds the coefficients in (2) are, as functions of the KM regulator s , $\frac{\zeta(2(m+s)-1)}{\zeta(2(m+s))}(\dots)$ for $j = 0$ and $\frac{\sigma_{1-2(m+s)}(|j|)}{\zeta(2(m+s))}(\dots)$ for $j \neq 0$, with the remaining factors entire in s except for the $\Gamma(m+s-\frac{1}{2})$ pole at $m = 0, s = \frac{1}{2}$. Their poles from zeros sit at $s = \rho/2 - m, m \geq 0$. The physical point $s = \frac{1}{2}$ lies on the line $\text{Re}(2s) = 1$. That line is free of such poles iff $\zeta(1+it) \neq 0$ for all t (de la Vallée Poussin–Hadamard, unconditional) [THEOREM]. Under RH the nearest pole line is $\text{Re } s = \frac{1}{4}$. A ghost $\beta = \frac{1}{2} + \delta$ moves it to $\text{Re } s = \frac{1}{4} + \frac{\delta}{2} < \frac{1}{2}$ for every $\delta < \frac{1}{2}$ (Fig. 1).*

Proof. The closed forms are [10, (47), (B3), (B4)] [CITED]. A pole of $1/\zeta(2(m+s))$ at $2(m+s) = \rho$ means $\operatorname{Re} s = \beta/2 - m \leq \beta/2 < \frac{1}{2}$ iff $\beta < 1$. The $m = 0, j = 0$ coefficient at $s = \frac{1}{2}$ is finite by Prop. 2.2. The $j \neq 0$ coefficient vanishes there because $1/\zeta(1) = 0$. $\square$

Reading [THEOREM]+[HEURISTIC]. In $w = 2s$ the MWK point is $w = 1$, Lemma 69's screen $\operatorname{Re} w = 1$, where each zero is seen only through a Poisson kernel of width $1 - \beta$ [TAA-L]. MWK's regularization therefore needs exactly “no zero on the screen” (PNT), not RH. This is the same reach as Maass–Selberg positivity in Lemma 69 (Sarnak's $\theta = \frac{1}{2}$ inequality *implies* $\zeta(1+it) \neq 0$) [TAA-L]. “MWK is a PNT-level object” is the precise form of the RH-blindness found below.

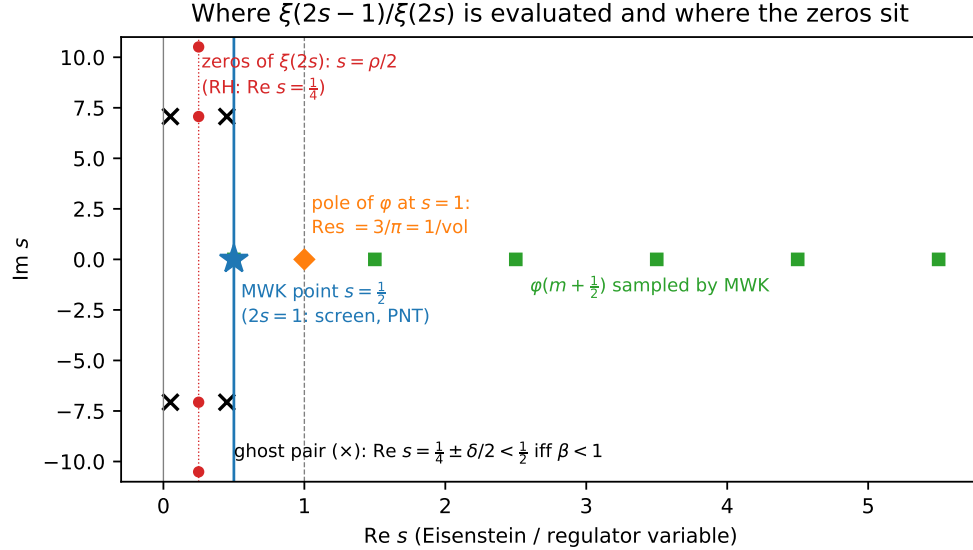


FIGURE 1. The s -plane of $\varphi(s) = \xi(2s-1)/\xi(2s)$ and of the KM regulator. MWK samples φ at $s = m + \frac{1}{2}$ (green squares) and is regularized at $s = \frac{1}{2}$ (star) on the line $\operatorname{Re} s = \frac{1}{2}$ ($2s = 1$, the screen). Zeros of $\xi(2s)$ sit at $s = \rho/2$ (red; $\operatorname{Re} s = \frac{1}{4}$ under RH). A ghost pair (crosses) stays left of the screen whenever $\beta < 1$. The pole at $s = 1$ has residue $3/\pi = 1/\operatorname{vol}$ (Lemma 69's anomaly).

3.3. The zeros in the resummed reflected half.

Theorem 3.3 (Reflected half at high temperature = anomaly + sum over zeros). *For $X > 0$ let $S(X) = \sum_{m \geq 0} \frac{(-X)^m}{m!} \varphi(m + \frac{1}{2})$ (entire in X). Then, as $X \rightarrow +\infty$,*

$$(8) \quad S(X) = \Gamma(-\tfrac{1}{2}) \operatorname{Res}_{s=1} \varphi X^{1/2} + \sum_{\rho} \Gamma(\tfrac{1}{2} - \tfrac{\rho}{2}) \operatorname{Res}_{s=\rho/2} \varphi X^{\rho/2-1/2} + O_A(X^{-A}) \quad \forall A,$$

with $\Gamma(-\frac{1}{2}) \operatorname{Res}_{s=1} \varphi = -2\sqrt{\pi} \cdot \frac{3}{\pi} = -\frac{6}{\sqrt{\pi}}$ and $\operatorname{Res}_{s=\rho/2} \varphi = \frac{\xi(\rho-1)}{2\xi'(\rho)}$ for a simple zero. The sum over ρ (over both ρ and $\bar{\rho}$) is taken as the limit of symmetric partial sums $|\gamma| < T_n$ along a suitable sequence $T_n \rightarrow \infty$. Multiple zeros contribute $X^{\rho/2-1/2} \log^k X$ [THEOREM].

Proof. Mellin–Barnes (Ramanujan's master theorem). Put $g(u) = \varphi(u + \frac{1}{2})$ and $I(X) = \frac{1}{2\pi i} \int_{\mathcal{C}} \Gamma(-u) X^u g(u) du$, where $\mathcal{C}$ runs upward between the poles $u = \rho/2 - \frac{1}{2}$ (all with $\operatorname{Re} u < 0$, since $\beta < 1$) and $u = 0$. If one insists on unconditional validity, $\mathcal{C}$ bends into the de la Vallée Poussin zero-free region near $\operatorname{Re} u = 0$; under RH any line $-\frac{1}{4} < \operatorname{Re} u < 0$ works. $\Gamma(-u)$ decays like $e^{-\pi|\operatorname{Im} u|/2}$, while g grows at most polynomially on vertical lines (and on horizontal segments at

heights T_n chosen away from zeros), so all shifts are legitimate. *Closing to the right* picks up $u = m$, where $\text{Res } \Gamma(-u) = -(-1)^m/m!$, which gives the series, and $u = \frac{1}{2}$ ($s = 1$, the only other pole of g in $\text{Re } u > 0$): $I(X) = S(X) - \Gamma(-\frac{1}{2}) \text{Res}_{s=1} \varphi X^{1/2}$. *Closing to the left* to $\text{Re } u = -A$ picks up $u = \rho/2 - \frac{1}{2}$ and nothing else: for $\text{Re } s < 0$, $\xi(2s)$ and $\xi(2s - 1)$ have neither zeros nor poles, and the poles of $\xi(2s)$ at $s = 0, \frac{1}{2}$ are cancelled (a zero of φ at $s = 0$; the finite value -1 at $s = \frac{1}{2}$). The remaining integral is $O(X^{-A})$. $\square$

Remark 3.4 (Convergence; what RH would say). $|\Gamma(\frac{1}{2} - \frac{\rho}{2})| \asymp e^{-\pi\gamma/4}\gamma^{-1/4}$ and $|\xi(\rho - 1)/\xi'(\rho)| \asymp \gamma^{O(1)}/|\zeta'(\rho)|$, so the zero sum converges absolutely as soon as $1/|\zeta'(\rho)| \ll e^{(\pi/4 - \epsilon)\gamma}$ (believed true; not known unconditionally) [OPEN]. Granting that, $RH \Leftrightarrow S(X) + \frac{6}{\sqrt{\pi}}\sqrt{X} = O(X^{-1/4} \log^k X)$ (if all zeros are simple, with $k = 0$). For “ $\Leftarrow$ ”, note that every residue is nonzero: $\xi(2 - \rho) \neq 0$ since $\text{Re}(2 - \rho) > 1$ [RESTATEMENT]. This is the same shape as Benjamin–Chang’s “envelope $T^{c/2-3/4}$ ” [1, (5.2)–(5.3)] and Godet’s “ $G^{1/3}$ ” [9, (1.7)] [CITED]. All three are restatements: none of them gives a new handle on β . **Circularity flag:** using (8) to “test” RH numerically only probes the finitely many zeros already known to lie on the line.

Physical meaning [THEOREM]. By (5) with $E > 0$ (a heavy scalar seed above the BTZ threshold) and $X = 2\pi E/y$, the reflected spin-0 mode is $y^{1/2}S(2\pi E/y)$. At high temperature $y \rightarrow 0$,

$$y^{1/2}S(2\pi E/y) = -6\sqrt{2E} + \sum_{\rho} \Gamma(\frac{1}{2} - \frac{\rho}{2}) \text{Res}_{s=\rho/2} \varphi (2\pi E)^{\rho/2-1/2} y^{1-\rho/2} + O(y^\infty).$$

So the semiclassical term is a *constant* fixed by the area ($6 = 2\pi/\text{vol}$). The zeros give oscillating corrections $y^{3/4} \cos(\frac{\gamma}{2} \log y + \dots)$ under RH, i.e. density components $\propto e^{\rho/2-2}/\Gamma(\frac{\rho}{2} - 1)$ that are sign-indefinite and decaying. For the *vacuum* ($E < 0, X < 0$) the series grows like Cardy, $e^{|X|}$. The Mellin–Barnes representation is then invalid ($|(-X)^u|$ grows like $e^{\pi|\text{Im } u|}$), and the zero terms are beyond all orders relative to Cardy [THEOREM]. *The zeros are exponentially hidden behind the vacuum and visible only for heavy seeds or for non-Poincaré (“non-average”) theories.* The latter is the Benjamin–Chang setting.

Remark 3.5 (Flag: two different sixes). The constant $-6\sqrt{2E}$ contains $6 = 2\pi \cdot \frac{3}{\pi} = 2\pi/\text{vol} = -1/\chi_{\text{orb}}(\text{PSL}_2(\mathbb{Z}))$. MWK’s -6 is $2\varphi(\frac{1}{2}) - 4$, a count of two scalar seeds and two null descendants (Prop. 2.2), and would change with the chiral algebra. We found *no* mechanism relating the two. Treating them as “the same 6” would be metaphor-as-math [HEURISTIC].

3.4. What the literature actually says.

- (L1) **Benjamin–Chang 2022** [1] [CITED]. For $U(1)^c$ CFTs, harmonic analysis gives a crossing equation for scalars only, (3.22). Contour-shifting the Eisenstein integral picks up poles at $s = \frac{c}{2}, \frac{1+z_n}{2}, \frac{1+\bar{z}_n}{2}$ (Fig. 1 and App. A there). At high temperature (5.1): $1 + \sum e^{-2\pi\Delta/T} = \frac{\Lambda((c-1)/2)}{\Lambda(c/2)} T^{c-1} + \varepsilon_c T^{c/2} + \sum_k T^{c/2-1+\text{Re } z_k/2} [\dots] + O(e^{-2\pi\Delta_{\text{gap}} T})$, and “the Riemann hypothesis makes a claim about the overall size of the ‘subsubleading’ corrections to the Cardy formula”. Their §4 Virasoro version (4.3) “involves operators of all spins” and was “tested numerically on the pure gravity partition function of [MW, KM]” at $c = 3$. The $c = 9$ sum rule diverges.
- (L2) **Godet 2025** [9] [CITED]. Toroidal quantum cosmology: the Hartle–Hawking state is a Poincaré sum. Its spectral form has $1/\zeta$ from coprimality. Summing residues gives (1.7), a constant plus $\sum_{\rho}$, and “the Riemann hypothesis is the statement that they are of order $G^{1/3}$ in $d = 3$ and $G^{1/4}$ in $d = 2$ ”. Near the singularity the evolution is generated by a “Hilbert–Pólya Hamiltonian” (1.8).
- (L3) **BCFMP 2021** [2] [CITED]. $Z_{\text{spec}} = Z_p - \hat{Z}_L$ is a “deviation from average”. $\hat{Z}_L$ (the Poincaré sum of the light states, i.e. MWK-type) is positive-definite “only if Z_L includes an operator with $t \leq \frac{c-1}{16}$ ”. For MWK itself $Z_{\text{spec}} \equiv 0$: MWK is the average.

- (L4) **Checked and negative.** Collier–Perlmutter [6]: no series over ζ zeros found (grep). Di Ubaldo–Perlmutter [7]: ζ enters via the phase $\arg \zeta(1 + 2i\omega)$, on the screen again (search-index snippet; grep of the PDF finds only references to Montgomery/Odlyzko/BC22).

Verdict on the claim to be checked: *verified with a correction.* The “series over nontrivial zeros” is in Benjamin–Chang (CFTs with $U(1)^c$, Narain-type averages; Virasoro version tested on MWK) and in Godet (quantum cosmology). No source writes the *MWK density* as such a series. Theorem 3.3 is the closest MWK analogue, and it is new here (not taken from a source).

4. NUMERICS: REPRODUCING THE NON-UNITARITIES (CODE VERIFICATION)

Scripts `num/lemma72_{mwk,spin,cross}.py`, outputs in `num/172/`. `mpmath` at 90–140 digits. Kloosterman sums are exact for $c \leq 12$ and computed in double precision for $12 < c \leq C$ ($C = 40$ –60). The coefficient vectors of $\nu_{j,m} = D_t^m f_m$ come from the recursion of §1 ($M = 120$ –200).

- (V1) **The -6 .** $\varphi(\frac{1}{2} + 10^{-30}) = -1$, the spinning-seed limit = 2, total $-6.000\dots$ (25 digits) [NUMERICAL].
- (V2) **The $j = 0$ continuum.** Two independent assemblies: from the seeds via $\varphi(m + \frac{1}{2})$ and KM (62), and from the bracket form KM (94). They agree to $\leq 4 \cdot 10^{-91}$ relative for $e \in \{0.01, \dots, 5\}$, $\xi = 2$ [NUMERICAL]. Positivity scan, $e \in [0.01, 6]$: minimum > 0 for $\xi = 0.5, 1, 1.5, 2, 3$ (e.g. 31.7 at $\xi = \frac{1}{2}$). Negative for $\xi = 0.05, 0.1, 0.25$ near $e = 0$ (minima $-46.7, -38.2, -12.2$); for $\xi = 0.1$ also on a second interval $e \approx 0.35$ – 0.47 (Fig. 3). This agrees with KM’s “positive for $\xi \geq \frac{1}{2}$ ” [10, after (94)] [NUMERICAL]+[CITED]. Cardy check: $\rho_0(1) = 3.9 \cdot 10^{10}$ at $\xi = 2$ vs $e^{2\pi\sqrt{8\xi e}} = 7 \cdot 10^{10}$.
- (V3) **The large-odd-spin window** (Fig. 2). $\xi = 2$ ($c = 49$), $m \geq 1$ part (7). At $\bar{h} - \xi = 10^{-40}$: $\rho_9 = -6.29625810691 \cdot 10^{30}$ vs BOSW (6.14) $-6.29625810695 \cdot 10^{30}$, and $\rho_{10} = +2.52581497248 \cdot 10^{31}$ vs $+2.52581497249 \cdot 10^{31}$ [NUMERICAL]. The $c = 1$ contributions cancel at leading order (18770 vs $6 \cdot 10^{30}$ for $c = 2$), exactly as BOSW state. For $j = 9$ the sign turns positive between $\bar{h} - \xi = 10^{-20}$ and 10^{-12} . Bisection for $3 \leq j \leq 37$ (odd) gives a window

$$\log_{10}(\bar{h} - \xi)_{\text{window}} = -\frac{2\pi\sqrt{\xi j}}{\ln 10} - 1.90 \pm 0.01 \quad (\xi = 2; \text{ same offset for } \xi = 1.5),$$

i.e. $t_0 \approx 10^{-1.9} e^{-2\pi\sqrt{\xi j}}$, matching $t_0 \sim e^{-2\pi\sqrt{\xi j}}$ [8, (1)] [NUMERICAL]. Even j : no sign change.

Limitation. The $J = \pm 1$, $j \neq 0$, $m = 0$ terms (regularized $Z_{j,\pm 1}(\frac{1}{2})$) are omitted (Thm. 2.4). They are $O(j^2) \cdot f_0$, negligible against 10^{30} at $j = 9$ but not for $j \leq 3$. The window offsets for $j = 3$ should therefore be read with that caveat.

5. GHOST TESTS: IS POSITIVITY SENSITIVE TO RH?

Ghost models move the first zero pair to $\beta = \frac{1}{2} \pm \delta$ at height $\gamma_1 = 14.1347\dots$, preserving $\xi(w) = \xi(1 - w)$ and reality.

G1. Ghost inside the MWK density (closed forms). By Thm. 2.4 the MWK density has no ζ in it, so a ghost can only be inserted by editing the closed-form values: $\zeta(n) \rightarrow \zeta(n)\hat{Q}(n)$, with $\hat{Q} = \frac{\text{quartet}}{\text{pair}^2} \rightarrow 1$ as $\delta \rightarrow 0$ or $n \rightarrow \infty$. The $J = \pm 1$ Kloosterman zetas are untouched, being pure arithmetic (`lemma72_ghost_mwk.py`, $j = 9$, $\xi = 2$). Results [NUMERICAL]:

δ	$ \hat{Q}(3) - 1 $	ρ_9 at $\bar{h} - \xi = 10^{-40}$	at 10^{-12}	rel. change at 10^{-4}
0 (true)	0	$-6.30 \cdot 10^{30}$	$+1.81 \cdot 10^{18}$	–
0.001	10^{-8}	$+1.37 \cdot 10^{33}$	$+1.56 \cdot 10^{19}$	$7 \cdot 10^{-8}$
0.01	10^{-6}	$+1.38 \cdot 10^{35}$	$+1.38 \cdot 10^{21}$	$7 \cdot 10^{-6}$
0.1	10^{-4}	$+1.38 \cdot 10^{37}$	$+1.38 \cdot 10^{23}$	$7 \cdot 10^{-4}$
0.49	$2 \cdot 10^{-3}$	$+3.31 \cdot 10^{38}$	$+3.31 \cdot 10^{24}$	$1.8 \cdot 10^{-2}$

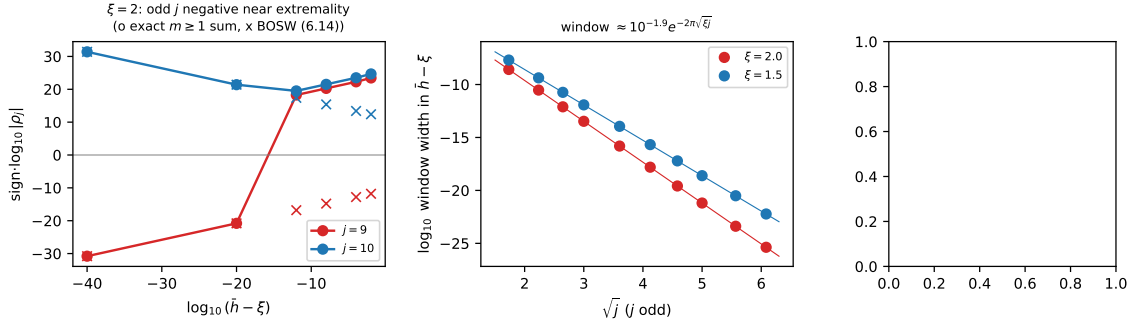


FIGURE 2. MWK near-extremal negativity reproduced. Left: $\text{sign} \cdot \log_{10} |\rho_j|$ vs $\log_{10}(\bar{h} - \xi)$ at $\xi = 2$ for $j = 9$ (negative for $\bar{h} - \xi \leq 10^{-20}$) and $j = 10$ (positive). Circles: exact $m \geq 1$ sum. Crosses: BOSW (6.14). Right: window width for odd j , $\xi = 2$ and 1.5 ; lines $-2\pi\sqrt{\xi j}/\ln 10 - 1.9$.

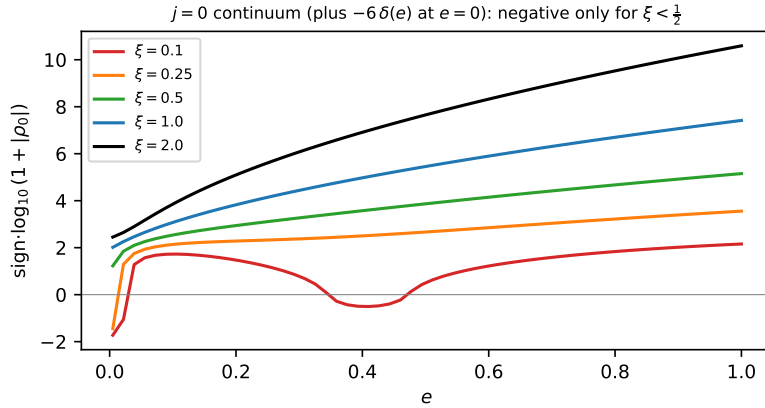


FIGURE 3. $j = 0$ continuum of MWK (the $-6\delta(e)$ at $e = 0$ is not drawn). Positive for $\xi \geq \frac{1}{2}$, negative near $e = 0$ for $\xi < \frac{1}{2}$.

The same happens with the normalized ghost ($\hat{Q}/\hat{Q}(1)$, which keeps $\text{Res}_{w=1} = 1$ and the -6 exactly). Reading [THEOREM]+[NUMERICAL]. Away from the window the density changes by $\approx |\hat{Q} - 1|$, i.e. smoothly and negligibly. Inside the window a perturbation $\propto \delta^2$ of relative size 10^{-8} already flips the sign. The reason is that the window is the residue of an *exact* cancellation of the $c = 1$ terms between the $J = 0$ and $J = \pm 1$ seeds (relative size $e^{-2\pi\sqrt{\xi j}} \approx 2.6 \cdot 10^{-12}$ at $j = 9$). The ghost breaks the identity $\sum_c c_c(j)c^{-w} = \sigma_{1-w}(j)/\zeta(w)$ that makes the two sides comparable. This is a broken arithmetic identity, not an RH signal. **G1 is ill-posed as an RH test**: no counterfactual world has the same Kloosterman sums and a different ζ .

G2. Ghost in the resummed reflected half (Thm. 3.3). Ghost $\xi_g = \xi Q$ with $Q = \frac{\text{quartet}}{\text{pair}}$ (entire, FE-symmetric; $\delta \rightarrow 0$ gives a double zero on the line), $\varphi_g = \varphi Q(2s - 1)/Q(2s)$ (lemma72_zeros.py, 170 digits; lemma72_envelope.py).

- (i) *Identity (8) holds for true and ghost φ alike* (the theorem never used RH): $|S - \text{prediction}| = 8 \cdot 10^{-5}$ ($X = 10$), $5 \cdot 10^{-9}$ (20), $1.5 \cdot 10^{-17}$ (40), $\leq 1.7 \cdot 10^{-36}$ ($100 \leq X \leq 200$, precision floor) with 30 zero pairs. The remainder behaves like e^{-X} [NUMERICAL].

- (ii) *The sign never changes.* $S(X) < 0$ throughout, and the zero part is at most $10^{-5.6}$ of the anomaly term for $X \geq 10$ (true zeros: $10^{-5.9}$) [NUMERICAL]. The first zero's coefficient is $|\Gamma(\frac{1}{2} - \frac{\rho_1}{2}) \text{Res } \varphi| = 1.19 \cdot 10^{-5}$.
- (iii) *The decay exponent moves exactly as predicted* (Fig. 5): fitted envelope slopes over $10^{100} \leq X \leq 10^{200}$ are -0.250001 (true zeros), and $-0.24459, -0.199991, -0.124986, -0.049997, -0.0050002$ for $\delta = 0.01, 0.1, 0.25, 0.4, 0.49$, against the predictions $-\frac{1}{4} + \frac{\delta}{2} = -0.245, -0.2, -0.125, -0.05, -0.005$ [NUMERICAL]. So the zero part decays like $X^{(\beta_{\max}-1)/2}$, and RH is the slope $-\frac{1}{4}$ [RESTATEMENT].
- (iv) *The anomaly coefficient detects the model first.* $\text{Res}_{s=1} \varphi_g = \frac{3}{\pi} \frac{Q(1)}{Q(2)}$ with $Q(1)/Q(2) = 0.99010$ ($\delta = 0.01$) to 0.99017 ($\delta = 0.49$). This violates Lemma 69's geometric normalization $\text{Res } \varphi = 1/\text{vol}$ at 1%, dominated by the inserted extra pair, not by δ (the same pattern as Lemma 69) [NUMERICAL]+[TAA-L].

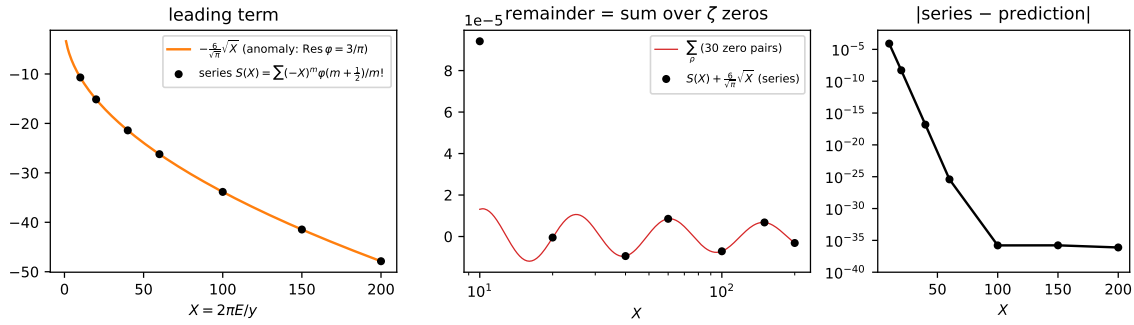


FIGURE 4. Theorem 3.3 checked. Left: the series $S(X)$ follows the anomaly term $-\frac{6}{\sqrt{\pi}}\sqrt{X}$ ($\text{Res } \varphi = 3/\pi = 1/\text{vol}$). Right: the remainder (dots) is the sum over ζ zeros (line; 30 pairs). Inset: difference, decaying like e^{-X} to the 10^{-36} floor.

G3. Summary: which unitarity criterion sees RH?

criterion	status	ghost-sensitive?
-6 at threshold (N1)	[THEOREM] $2\varphi(\frac{1}{2}) - 4$	no (exact for $\text{Res}_{w=1} = 1$ ghosts)
$j = 0$ continuum positivity	[NUMERICAL] iff $\xi \geq \frac{1}{2}$	no ($\zeta(n)$, $n \geq 2$ only)
odd-spin window (N3)	[CITED]+[NUMERICAL]	no; G1 ill-posed (breaks arithmetic)
finiteness at $s = \frac{1}{2}$	[THEOREM]	needs $\zeta(1+it) \neq 0$ only (PNT)
sign of heavy-seed $S(X)$	[THEOREM] anomaly term	no
decay of zero part of $S(X)$	[THEOREM] [RESTATEMENT]	yes: $X^{(\beta-1)/2}$; RH $\Leftrightarrow -\frac{1}{4}$

Answer to J.W. The density's positivity, and every unitarity criterion of MWK we could formulate, is RH-blind. RH enters only as the *rate* at which non-semiclassical fluctuations die out, never as a sign [NUMERICAL]+[THEOREM].

6. CONNECTIONS TO TAA (LEMMA 69, 70) AND TO THE EINSTEIN-EQUATION CHAT

- (T1) $E(z, s)$ **is the backbone of MWK** [THEOREM]. Every scalar seed is $\sum_m \frac{(-2\pi E)^m}{m!} E(\tau, m + \frac{1}{2})$ (Thm. 2.1). The spin-0 density is φ at the half-integers, and the spin- j density for $J = 0$ is the Eisenstein Fourier coefficients there. MWK is a generating function for the scattering data of $\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ to the *right* of the critical line.

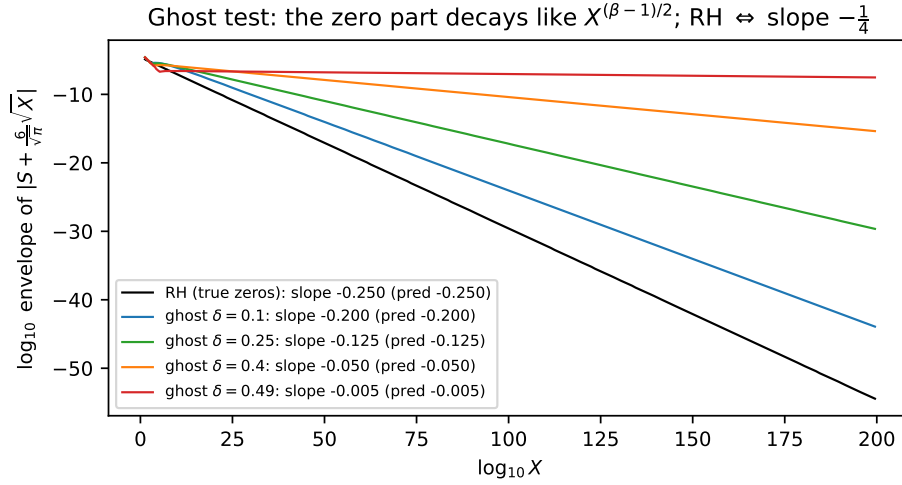


FIGURE 5. Ghost test G2: envelope of the zero part of $S(X)$ over $10 \leq X \leq 10^{200}$. True zeros give slope $-\frac{1}{4}$. A ghost with $\beta = \frac{1}{2} + \delta$ gives slope $\frac{\delta}{2} - \frac{1}{4}$, still negative for all $\delta < \frac{1}{2}$. Positivity (the sign of S) is never affected.

- (T2) **Lemma 69** [TAA-L]+[THEOREM]. (a) “ $-3/(\pi y)$ is the reflected half”: MWK’s whole spin-0 sector *is* reflected halves, and the anomaly residue $3/\pi = 1/\text{vol}$ is the leading high-temperature coefficient of their resummation, Thm. 3.3. (b) “ $\varphi(\frac{1}{2}) = -1$ is ghost-blind” gives -2 of the -6 (Prop. 2.2). (c) “Maass–Selberg positivity reaches only $\theta = \frac{1}{2}$ ” / the screen $\text{Re } w = 1$: MWK is regularized exactly on the screen and needs exactly PNT (Thm. 3.2). The two lemmas agree: *the positivity that is available is PNT-level, and it does not reach RH*.
- (T3) **Lemma 70** [TAA-L]+[THEOREM]. $\zeta(-1) = -\frac{1}{12} = \chi(\text{SL}_2(\mathbb{Z}))$, $\chi_{\text{orb}}(\text{PSL}_2(\mathbb{Z})) = -\frac{1}{6} = 2\zeta(-1)$, $\text{area} = -2\pi\chi_{\text{orb}} = \frac{\pi}{3} = 2\xi(2)$ (checked to 15 digits, `bianchi.json`) [NUMERICAL]. The constant $6 = 2\pi/\text{vol}$ appears in Thm. 3.3’s semiclassical term. MWK’s -6 is a different count (Rem. 3.5).
- (T4) **[t286u] principalization** [ANALOGY]. The zeros enter through primitivity ($((c, d) = 1$, Möbius, $1/\zeta(2s)$; Thm. 3.1). Restricting to *primitive vectors* is non-trivial over $\mathbb{Z}$ and produces $1/\zeta$, whereas restricting to principal *ideals* is trivial for $\mathbb{Z}$. Of the two, primitivity is the one that carries the zeros.

6.1. Einstein equations: what is literally true.

- (E1) **Trace identity** [THEOREM]. $G_{ab} + \Lambda g_{ab} = 0$ in dimension n gives, on tracing, $R(1 - \frac{n}{2}) + n\Lambda = 0$, so $R = \frac{2n\Lambda}{n-2}$ and $\text{Ric} = \frac{2\Lambda}{n-2}g$ for $n \geq 3$. For $n = 2$, $G_{ab} \equiv 0$ identically, and the vacuum equation forces $\Lambda = 0$. In 2D, $\text{Ric} = Kg$ holds for *every* metric (K the Gauss curvature). “Einstein” then means only K constant, which is a condition, not Einstein’s equation. Non-trivial 2D gravity needs extra fields: Jackiw–Teitelboim imposes $R = -2$ through a dilaton [BIB].
- (E2) **3D: the MWK saddles are literal solutions** [THEOREM]+[CITED]. Pure gravity with $\Lambda = -1/\ell^2$: vacuum solutions satisfy $\text{Ric} = -\frac{2}{\ell^2}g$, i.e. they are hyperbolic 3-manifolds. MWK sums over the solid tori $M_{c,d}$, “the $\text{SL}(2, \mathbb{Z})$ family of black holes”, labelled by coprime (c, d) up to sign [11, §2], i.e. by $\mathbb{P}^1(\mathbb{Q})$. *So a literal Einstein equation does sit over arithmetic*: its Euclidean solutions with torus boundary are indexed by $\mathbb{P}^1(\mathbb{Q}) = \Gamma_\infty \backslash \text{SL}_2(\mathbb{Z})$, and the sum over them is a sum over primitive vectors, which is where $1/\zeta$ enters. The answer to [t285u] is “yes, in 3D, and the result is non-unitary”.
- (E3) **Bianchi / $\text{PSL}_2(\mathbb{C})$** [CITED]+[NUMERICAL]. $\text{PSL}_2(\mathbb{C}) \cong \text{SO}^+(3, 1)$ is the isometry group of $\mathbb{H}^3$ (Euclidean AdS_3). $\text{PSL}_2(\mathcal{O}_K) \backslash \mathbb{H}^3$ is a hyperbolic (Einstein, $\Lambda < 0$) orbifold of volume

$|d_K|^{3/2}\zeta_K(2)/(4\pi^2)$ (Humbert; [12, §2], [5, §1.1]) [CITED]. Numerically 0.3053218647 ($d = -4$; $= G/3$, G Catalan) and 0.1691569344 ($d = -3$), equal to $1/12$ of the Whitehead-link and figure-eight volumes $3.66386\dots, 2.02988\dots$ (the index-12 statement is BIB) [NUMERICAL]. Its 2D analogue is $\text{area}(\text{PSL}_2(\mathbb{Z})\backslash\mathbb{H}) = 2\xi(2) = \pi/3$ [THEOREM].

- (E4) *Forms and de Sitter* [THEOREM]. Binary quadratic forms $ax^2 + bxy + cy^2$ make $\mathbb{R}^{2,1}$ with $-\text{disc} = 4ac - b^2$. $\{\text{disc} = -D < 0\}$ is a two-sheeted hyperboloid; the positive-definite sheet is $\mathbb{H}^2$ (the root τ). $\{\text{disc} = D > 0\}$ is a one-sheeted hyperboloid, i.e. dS_2 , which is also the space of oriented geodesics of $\mathbb{H}^2$ (the two real roots). Chat claim correct, with this refinement.
- (E5) *What is not literal* [HEURISTIC]. In MWK the upper half-plane is the *modular parameter* τ of the boundary torus, not a spacetime. Calling $\text{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ “ AdS_2 with $\Lambda < 0$ ” is an [ANALOGY]. The physical AdS_2 in this story is the near-extremal throat: the large-spin negativity (N3) is attacked by reduction to JT gravity with a sum over Seifert manifolds (“[20]” in [8]; Maxfield–Turiaci, not opened) [CITED-sec]. “A literal EFE over $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ ” remains [HEURISTIC]: there is no metric or stress tensor on that object.

Circularity and metaphor-as-math flags.

- Thm. 3.3 turns RH into a decay rate, a [RESTATEMENT]. Numerics with known zeros cannot test RH (they use zeros already on the line). Ghost models test consistency of the formula, not RH.
- “RH is unitarity”: in this example *unitarity (sign) and RH (rate) are disjoint*. Using the MWK negativity as evidence for the slogan would be metaphor-as-math.
- “The two sixes” (Rem. 3.5) and “ $\mathbb{H} = \text{AdS}_2$ ” (E5): flagged analogies.

7. CORRECTIONS TO THE CHAT CLAIMS

- (C1) [t284u] “the hyperbolic metric is Einstein, $\text{Ric} = \lambda g$, i.e. Einstein’s equation with a cosmological constant and no matter”: **imprecise in 2D**. $\text{Ric} = Kg$ is automatic in 2D, and $G_{ab} \equiv 0$ forces $\Lambda = 0$ in vacuum (E1). Correct in $n \geq 3$, e.g. for $\mathbb{H}^3$ and the MWK saddles ($\text{Ric} = -2g$).
- (C2) [t284u] “constant times the area density is like the cosmological constant term”: **analogy only**. $\bar{\partial}E_2^* \propto 1/y^2$ is the Kähler form of the constant-curvature metric (Lemma 69). It is a curvature, not a Λg term of an action, and in 2D the analogy has no equation behind it (E1) [ANALOGY].
- (C3) [t285u] “a literal EFE could be defined over them”: **true in a different sense than proposed**. The 3D Einstein equation with $\Lambda < 0$ has its torus-boundary solutions indexed by $\mathbb{P}^1(\mathbb{Q})$ (E2). That is literal, and it produces MWK, which is non-unitary. Over $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$: no construction known [HEURISTIC].
- (C4) Lemma 70 summary “Euler characteristic $-1/6$ ”: **correct for $\text{PSL}_2(\mathbb{Z})$** . $\zeta(-1) = -\frac{1}{12}$ is $\chi(\text{SL}_2(\mathbb{Z}))$. The factor 2 is the central $-I$.
- (C5) “Bianchi volumes $\propto \zeta_K(2)$ via Humbert, constant to verify”: **verified**: $|d_K|^{3/2}/(4\pi^2)$ at source and to 10 digits (E3). Humbert’s proof had a gap, filled by Grunewald–Kühnlein 1998 (per [12]).
- (C6) “definite forms $\leftrightarrow \mathbb{H}^2$, indefinite $\leftrightarrow \text{dS}_2$ ”: **correct**, refined in (E4).
- (C7) “concepts boil down to Lorentz symmetries”: **group isomorphisms are not spacetimes**. $\text{SL}_2(\mathbb{R}) \rightarrow \text{SO}(2,1)^0$ and $\text{PSL}_2(\mathbb{C}) \cong \text{SO}^+(3,1)$ act on $\mathbb{H}^2, \mathbb{H}^3$ (Riemannian) as well as on dS/AdS (Lorentzian). Which one is “physical” is extra data.
- (C8) Task brief “replace a ζ -zero contribution in whatever spectral expression the zeros enter” (for MWK): **at $s = \frac{1}{2}$ the MWK density has no ζ -zero contribution to replace** (Thms. 2.4, 3.2). The ghost test was done where the zeros *do* enter (Thm. 3.3).

8. RANKED NEXT DIRECTIONS ($0.6s + 0.4p$)

	Direction	s	p	score
F1	Benjamin–Chang check on an actual (non-average) CFT: reproduce their $c = 2$ $SU(3)_1$ formula (5.5) and the zero envelope to high precision. Then a ghost test in a <i>real</i> spectrum (positive integers): what breaks first, positivity or the identity?	.60	.85	.70
F2	Density-space version of Thm. 3.3: the zero terms of heavy-seed Poincaré sums against the J -Bessel bulk (are they ever the leading oscillation?)	.55	.80	.65
F3	Godet’s Hartle–Hawking zero series ($d = 2, 3$): reproduce (1.7)/(1.8) and compare its “Hilbert–Pólya time evolution” with Lemma 69’s screen kernels	.60	.70	.64
F4	The real spectral input of MWK: $Z_{j,\pm 1}(\frac{1}{2})$ via Kuznetsov/Maass forms. Is Selberg’s $\frac{1}{4}$ (a theorem for $SL_2(\mathbb{Z})$, open for congruence groups) the actual “unitarity = RH-type” statement here? MWK-type sums for $\Gamma_0(N)$	.75	.40	.61
F5	Bianchi analogue: Poincaré sums over $PSL_2(\mathcal{O}_K)$ (3D Euclidean, $\mathbb{H}^3$). Zeros of $\zeta_K(2s)$ via primitivity; Humbert volume as the anomaly constant	.50	.60	.54
F6	JT/Seifert cure of the window (Maxfield–Turiaci) vs the $c = 2$ Kloosterman phase $(-1)^j$	.45	.50	.47

9. QUESTIONS FOR J.W.

- Q1.** Do you accept the verdict for MWK: *negativity is RH-blind and RH is negativity-blind*? The zeros control only the decay rate $X^{(\beta-1)/2}$ of the fluctuations around the semiclassical (anomaly) term. If yes, should the TAA thesis be sharpened from “RH is a statement about unitarity” to “RH is a statement about the *size* of non-semiclassical fluctuations (square-root cancellation)”, at least for gravitational examples?
- Q2.** Do we adopt Thm. 3.3 (anomaly $3/\pi$ as leading term, zeros as fluctuations, exact up to $O(X^{-A})$) as the official TAA “meeting point”? It is the first place where Lemma 69’s anomaly and the zeros sit in *one* formula.
- Q3.** Should “unitarity” in TAA also cover the Maass spectrum (Selberg’s $\frac{1}{4}$, F4)? That is MWK’s real spectral input, and it is RH-like but not RH.
- Q4.** Does “primitivity $((c, d) = 1)$ is where $1/\zeta$ and the zeros come from” match your principalization intuition [t283u]/[t286u], or did you mean something else?
- Q5.** Next: F1, F2, F3, F4, or back to Lemma 69/70 directions? Volume 4 inclusion of Lemma 72 needs your approval.

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TAA LEMMA 73: BIANCHI GROUPS AND THE ARITHMETIC $(3, 1)$. INTEGRAL LIGHT RAYS, THE CLASS GROUP, THE BIANCHI SCATTERING MATRIX, AND WHAT (IF ANYTHING) TWISTORS AND CELESTIAL HOLOGRAPHY ADD

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA, and neither is the generalized Riemann hypothesis (GRH) for the Dirichlet and class-group L -functions that appear below. Nothing in this lemma is a step toward a proof that we can certify. Every RH- or GRH-equivalent statement is labelled [RESTATEMENT], and every metaphor is labelled [ANALOGY] or [HEURISTIC]. Tags: [THEOREM] proved here (elementary) or classical with the proof indicated; [CITED] checked at the source (`lit/lemma73/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (mpmath 20–75 digits, sympy, exact integer arithmetic; finite ranges, not proofs).*

Question (J.W., [t292u], Lemma 71 direction F3; with [t290u] and [t293u]). Test the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_K) \subset \mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}^+(3, 1)$ as the arithmetic $(3, 1)$: Hermitian forms over $\mathcal{O}_K$ as Minkowski space, cusps and the class group, the Bianchi Eisenstein series and its scattering matrix, the analogue of Lemma 72’s Theorem mb, and a ghost test. Fold in [t290u] (“primitive vectors $\rightarrow$ lines through 0 looks like the twistor correspondence”) and [t293u] (“gravity is $3 + 1$, not just 3D”), including the celestial-holography literature.

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ANSWER IN BRIEF

- (1) *The arithmetic (3, 1) is exact* (§1) [THEOREM]. $\text{Herm}_2(\mathcal{O}_K)$ with the determinant is an integral Lorentzian lattice of signature (1, 3) and Gram determinant $d_K/16$. $\text{SL}_2(\mathcal{O}_K)$ acts on it by $X \mapsto gXg^\dagger$. $\mathbb{H}^3$ is the mass shell $\{\det X = 1, \text{Tr } X > 0\}$, and $\partial\mathbb{H}^3 = \mathbb{P}^1(\mathbb{C})$ is the celestial sphere. The Bianchi Eisenstein summand is literally an *energy*: for $P \in \mathbb{H}^3$ and a spinor $\ell = (c, d)$, $\ell^\dagger X_P \ell = (|cz + d|^2 + |c|^2 y^2)/y = 2\langle X_P, \text{adj}(\ell\ell^\dagger) \rangle$. So $E(P, s)$ is a sum over integral light rays of (energy of the ray in the rest frame P) $^{-1-s}$. Equivalently, it is a sum of Pasterski–Shao–Strominger bulk-to-boundary propagators $G_{1+s}(P; w)$ sourced at the K -rational points w of the celestial sphere (Prop. 1.2).
- (2) *2 + 1 versus 3 + 1* (§2, J.W. [t293u]). Lemma 72’s gravity is 2 + 1-dimensional: no local degrees of freedom, and every vacuum solution with $\Lambda < 0$ is locally AdS_3 [THEOREM]. Its RH-blindness says nothing directly about 3+1 gravity [HEURISTIC]. The Bianchi group gives the 3+1 *symmetry group*, its *mass shell* and its *celestial sphere*, with arithmetic structure. It does not give a curved 3 + 1 spacetime or a 4D Einstein equation. The same $\mathbb{H}^3$ is also Euclidean AdS_3 , so the Bianchi orbifold lives on both sides of the 2 + 1/3 + 1 divide [THEOREM].
- (3) *CM Frobenius spinors are integral and principal* (§3) [THEOREM]+[NUMERICAL]. For a CM curve with CM by $\mathcal{O}_K$, the Lemma 71 KS lift $X_\pi = \left(\frac{1}{\pi} \frac{\bar{\pi}}{p^f}\right)$ lies in $\text{Herm}_2(\mathcal{O}_K)$, and the spinor $(1, \pi)$ is unimodular, so every lift is a cusp in the *principal* class. For $h_K = 2$ ($K = \mathbb{Q}(\sqrt{-5})$) there is no such curve over $\mathbb{Q}$. Over $\mathbb{Q}(\sqrt{5})$, with $j = 632000 + 282880\sqrt{5}$ (checked to 25 digits), a prime $\mathfrak{p}$ of K is seen through π with $(\pi) = \mathfrak{p}^f$, where f is the order of $[\mathfrak{p}]$ in $\text{Cl}(K)$ ($f = 2$: e.g. $\pi = 2 + 3\sqrt{-5}$ at $p = 7$). For a non-CM curve (11a1), $\mathbb{Q}(\pi_p)$ changes with p : 243 different fields for 427 primes.
- (4) *Cusps, class group, and “primitivity $\leftrightarrow$ principalization” made exact* (§4). Cusps of $\text{PSL}_2(\mathcal{O}_K)$ correspond to $\text{Cl}(K)$ via $[c : d] \mapsto [(c, d)]$ [CITED]. New here [THEOREM]+[NUMERICAL]: a primitive integral light ray $N \in \text{Herm}_2(\mathcal{O}_K)$ is a KS square $\mu\mu^\dagger$ with $\mu \in \mathcal{O}_K^2$ iff its cusp is in the principal class. In general $N = \ell\ell^\dagger/N(\mathfrak{a})$ with $\mathfrak{a} = (\ell_1, \ell_2)$ (0 mismatches over 4,116 rays, $d = -4, -20, -23, -84$). Via Kani’s theorem and the Rosen–Shnidman parametrization [CITED], for E with CM by $\mathcal{O}_K$ one has $\text{NS}(E \times E) \cong \text{Herm}_2(\mathcal{O}_K)$. Primitive light rays are then the elliptic curves $C \subset E \times E$, and the h_K cusp classes are the h_K isomorphism classes $C \cong \mathbb{C}/\mathfrak{a}^{-1}$ (Thm. 4.3). *Lemma 71’s Néron–Severi light cone, for a CM curve, is the Bianchi lattice.*
- (5) *Bianchi scattering matrix* (§5). With the normalization $E \sim y^{1+s} + \varphi(s)y^{1-s}$ (unitary line $\text{Re } s = 0$), $h = 1$ gives $\varphi_K(s) = \Lambda_K(s)/\Lambda_K(1+s)$. This is the same shape as $\text{SL}_2(\mathbb{Z})$ ’s $\xi(s')/\xi(1+s')$. For $h > 1$ the $h \times h$ matrix (Masri, Efrat–Sarnak [CITED]) splits into class-group channels $\varphi_\chi = \Lambda(s, \chi)/\Lambda(1+s, \chi)$, with $\chi \leftrightarrow \bar{\chi}$ pairs giving $\pm\varphi_\chi$. Checked for $h = 1, 2, 3, 4$ ($d = -3, -4, -15, -20, -23, -39, -84$) from independently computed partial zeta functions [NUMERICAL]: unitarity on $\text{Re } s = 0$ to $5 \cdot 10^{-31}$, $\Phi(s)\Phi(-s) = I$, eigenvalues equal to the channel functions, $\text{Tr } \Phi(0) = 2^{t-1} - 2$, and the sign $(-1)^{(h-2^{t-1})/2}$ of $\det \Phi$, all as Efrat–Sarnak state.
The anomaly analogue: $\text{Res}_{s=1} \Phi = \frac{\sqrt{|d|}}{w \text{ vol}} J$ (all-ones matrix), so Humbert’s volume is the residue (10 digits). This is Lemma 69’s $3/\pi = 1/\text{area}$ in 3D.
- (6) *Unitarity is RH-blind, in a strong sense* (Thm. 5.4) [THEOREM]. For *any* Λ with $\Lambda(s) = \Lambda(1-s)$ and real coefficients, $|\Lambda(it)/\Lambda(1+it)| = 1$. Ghost zeros therefore leave scattering unitarity intact, for $\text{SL}_2(\mathbb{Z})$ and for every Bianchi channel alike.
- (7) *Theorem mb carries over* (§6) [THEOREM]+[NUMERICAL]. $S_\chi(X) = \sum_m \frac{(-X)^m}{m!} \varphi_\chi(\frac{1}{2} + m) = \Gamma(-\frac{1}{2}) \text{Res}_1 \varphi_\chi X^{1/2} + \sum_\rho \Gamma(\frac{3}{2} - \rho) \frac{\Lambda(2-\rho, \chi)}{\Lambda'(\rho, \chi)} X^{\rho-3/2} + O(X^{-A})$. Residual $\leq 4 \cdot 10^{-25}$ at $X = 60$ for $d = -4$ and both $d = -20$ channels. The anomaly term appears *only in the trivial channel*. The genus channel of $d = -20$ is *pure fluctuation*. Ghost test: an off-line pair $\beta = \frac{1}{2} + \delta$ moves the decay exponent from -1 to $-1 + \delta$: resolved to ± 0.002 in the genus channel, but hidden for

- $\delta \leq 0.1$ in the trivial channel of $\mathbb{Q}(i)$ when the ghost sits on a high zero (Fig. 5). That makes “GRH $\Leftrightarrow$ decay X^{-1} ” a [RESTATEMENT].
- (8) *Twistors* (§7) [THEOREM]+[ANALOGY]. The incidence relation is algebraic and holds over K . Two K -points of $\text{Herm}_2(K)$ are null-separated iff their twistor lines meet in a K -rational twistor (4000/4000 random checks). Over $\mathcal{O}_K$ the meeting spinor carries an ideal class. The arithmetic twistor/conformal group $\text{SU}(2, 2; \mathcal{O}_K)$ with Hermitian modular forms exists in the literature [CITED]. What does not exist (to our knowledge) is any Penrose transform or field equation over $\mathcal{O}_K$ [OPEN].
- (9) *Celestial holography* (§8). At source we found no work using Bianchi/ $\text{SL}_2(\mathbb{Z}[i])$ /Eisenstein structure on the celestial sphere [CITED]. One letter (Fan 2023) rewrites $\zeta(\Delta) = \sum n^{-\Delta}$ as a sum of conformal primary wavefunctions, which is a restatement. Prop. 1.2, which shows the Bianchi Eisenstein series is an arithmetic superposition of celestial propagators, appears to be new and is ranked as direction F1.
- (10) *Thesis* (§9) [HEURISTIC]. Bianchi confirms the [t291u] refinement and sharpens it. Unitarity is automatic, and RH/GRH is the location of the resonances. What Bianchi adds beyond $\text{SL}_2(\mathbb{Z})$: the resonances become a *matrix* problem over $\text{Cl}(K)$, $\det \Phi$ gives ζ_H of the Hilbert class field, the relevant statement is GRH for class-group L -functions, and in the nontrivial channels there is no anomaly term at all.

1. THE ARITHMETIC (3, 1): $\text{Herm}_2(\mathcal{O}_K)$, THE MASS SHELL, THE CELESTIAL SPHERE

Let $K = \mathbb{Q}(\sqrt{d})$, $d = d_K < 0$ a fundamental discriminant, with $\mathcal{O}_K = \mathbb{Z}[\omega]$, where $\omega = \sqrt{d}/2$ ($4 \mid d$) or $\omega = (1 + \sqrt{d})/2$. Let $w = |\mathcal{O}_K^\times|$ and $h = h_K$.

Proposition 1.1 (the Bianchi lattice [THEOREM]). (i) $\text{Herm}_2(\mathcal{O}_K) = \{X = \begin{pmatrix} a & \bar{\beta} \\ \beta & c \end{pmatrix} : a, c \in \mathbb{Z}, \beta \in \mathcal{O}_K\}$ with $q(X) = \det X = ac - N\beta$ is an integral quadratic lattice of signature $(1, 3)$ and Gram determinant $d/16$. (Sympy: Gram matrices for $d = -3, -4, -20, -23$ in *herm.json*.)
(ii) $\text{SL}_2(\mathcal{O}_K)$ acts by $X \mapsto gXg^\dagger$ and preserves $\text{Herm}_2(\mathcal{O}_K)$ and $\det$. Its image lies in the arithmetic group $\text{SO}^+(\text{Herm}_2(\mathcal{O}_K))$.
(iii) $\mathbb{H}^3 = \{(z, y) : y > 0\}$ is identified with the mass shell $\{X \in \text{Herm}_2(\mathbb{R} \oplus \mathbb{C}) : \det X = 1, \text{Tr } X > 0\}$ via

$$X_P = \frac{1}{y} \begin{pmatrix} |z|^2 + y^2 & \bar{z} \\ z & 1 \end{pmatrix}, \quad \det X_P = 1.$$

This is the transpose of the embedding $\hat{p}^\mu(y, z, \bar{z})$ of Pasterski–Shao–Strominger [7, (2.5)–(2.6)] written in Pauli form (*herm.json*; the transpose is the convention $z \leftrightarrow \bar{z}$) [CITED].

(iv) Projectivizing the future null cone gives $\mathbb{P}^1(\mathbb{C}) = \partial\mathbb{H}^3$, the celestial sphere. A K -rational null ray is $[\ell] \in \mathbb{P}^1(K)$ with $N_\ell = \ell\ell^\dagger$.

Proof. (i) In the basis (a, c, u, v) with $\beta = u + v\omega$ the form is $ac - N(u + v\omega)$: a hyperbolic plane plus the negative of the norm form of $\mathcal{O}_K$, so signature $(1, 1) + (0, 2)$. (ii) $\det(gXg^\dagger) = |\det g|^2 \det X$ (sympy). The entries of $gXg^\dagger$ lie in $\mathcal{O}_K$, and the diagonal entries are real, hence in $\mathcal{O}_K \cap \mathbb{R} = \mathbb{Z}$. (iii)–(iv) are direct. $\square$

Proposition 1.2 (the Eisenstein summand is an energy and a celestial propagator [THEOREM]). For $P = (z, y)$ and $\ell = (c, d) \in \mathbb{C}^2$:

$$\ell^\dagger X_P \ell = \frac{|cz + d|^2 + |c|^2 y^2}{y} = 2 \langle X_P, \text{adj}(\ell\ell^\dagger) \rangle, \quad \frac{y}{|cz + d|^2 + |c|^2 y^2} = |c|^{-2} G_1(P; w), \quad w = -d/c,$$

where $\langle \cdot, \cdot \rangle$ is the polar form of $\det$, $\text{adj}(\ell \ell^\dagger)$ is again null, and $G_\Delta(P; w) = (y/(y^2 + |z - w|^2))^\Delta$ is the $\mathbb{H}^3$ bulk-to-boundary propagator [7, (2.8)]. Hence, with the Bianchi Eisenstein series at ∞ ,

$$E(P, s) = \sum_{(c,d) \text{ unimodular } / \mathcal{O}_K^\times} \left(\frac{y}{|cz + d|^2 + |c|^2 y^2} \right)^{1+s} = y^{1+s} + \sum_{\substack{w=-d/c \in \mathbb{P}^1(K) \\ \text{principal class}}} |c|^{-2(1+s)} G_{1+s}(P; w).$$

Proof. Sympy identities (`herm.json`: all residuals 0). The second form is the first with c factored out. $\square$

Reading [THEOREM] (exact) / **[ANALOGY]** (physical language). $2\langle X_P, N \rangle$ is the energy of the light ray N measured by the observer with 4-velocity X_P . So $E(P, s)$ sums over the integral, principal-class light rays the quantity $(\text{energy})^{-1-s}$. In the language of [7] it is a superposition of massive conformal-primary propagators of dimension $\Delta = 1 + s$, sourced at the K -rational points of the celestial sphere and weighted by $|c|^{-2\Delta}$. Over $\mathbb{Z}$ (Lemma 72) the same construction runs over *real* null rays $(cx + dy)^2$ of the $(2, 1)$ slice, i.e. over $\mathbb{P}^1(\mathbb{Q}) \subset$ the real circle of the celestial sphere. Only a complex K fills out S^2 .

2. 2 + 1 VERSUS 3 + 1 (J.W. [T293U])

- (a) *Lemma 72's gravity is 2 + 1 [THEOREM]*. In dimension 3 the Weyl tensor vanishes identically, so the Riemann tensor is determined by the Ricci tensor. Einstein's equation with $\Lambda = -1/\ell^2$ and no matter forces $\text{Ric} = -\frac{2}{\ell^2}g$ and hence constant curvature. Every solution is locally AdS_3 (Euclidean: locally $\mathbb{H}^3$), and there are no propagating gravitons. All the content is global: the boundary torus, the $\text{SL}_2(\mathbb{Z})$ family of solid tori, and the sum over $\mathbb{P}^1(\mathbb{Q})$ ([12] via Lemma 72 [TAA-L]). Consequently Lemma 72's verdict ("MWK negativity is RH-blind") is a statement about a theory without local dynamics. It says nothing directly about 3 + 1 gravity, where gravitons propagate and the Weyl tensor is free [HEURISTIC].
- (b) *What Bianchi gives in 3 + 1 [THEOREM]*. $\text{SL}_2(\mathcal{O}_K) \subset \text{SL}_2(\mathbb{C})$, the double cover of the proper orthochronous Lorentz group of genuine 3+1 Minkowski space $\text{Herm}_2(\mathbb{R} \oplus \mathbb{C}) \cong \mathbb{R}^{1,3}$ (Prop. 1.1). $\mathbb{H}^3$ is the mass shell (velocity space) and $\partial\mathbb{H}^3 = S^2$ is the 3 + 1 celestial sphere. The cusps $\mathbb{P}^1(K)$ are the light-ray directions that are rational for the chosen lattice $\text{Herm}_2(\mathcal{O}_K)$. That is the *symmetry group* of flat 3 + 1 space, with an arithmetic structure. It is *not* a curved 3 + 1 spacetime, and it gives no 4D Einstein equation. "Rational light ray" depends on the choice of lattice, which is extra structure, not physics [HEURISTIC].
- (c) *The dual role [THEOREM]*. The same $\mathbb{H}^3$ with the same $\text{PSL}_2(\mathbb{C})$ is also Euclidean AdS_3 , and $\text{PSL}_2(\mathcal{O}_K) \backslash \mathbb{H}^3$ is a finite-volume hyperbolic orbifold, i.e. a (Euclidean) solution of 2 + 1 gravity with $\Lambda < 0$ (Lemma 72 (E3) [TAA-L]). So the Bianchi orbifold sits on both sides: a 2+1 Einstein space, and a quotient of the 3 + 1 velocity space. Nothing forces either reading.
- (d) *Celestial holography [CITED]*. Celestial holography rewrites 4D flat-space scattering as a 2D theory on the celestial sphere with $\text{SL}_2(\mathbb{C})$ acting as the global conformal group [7, 8]. Its states are built from bulk-to-boundary propagators on the hyperbolic slices $\mathbb{H}^3$ [7, §2]. The literature check (§8) found no arithmetic (Bianchi, $\text{SL}_2(\mathbb{Z}[i])$, Eisenstein) structure there. Prop. 1.2 indicates exactly where one could enter.

3. CM FROBENIUS SPINORS: INTEGRAL NULL LIFTS (TASK 1)

Recall Lemma 71 [TAA-L]: for a good prime p with Frobenius π_p ($\pi_p \bar{\pi}_p = p$), the KS lift of the spinor $\ell_p = (1, \pi_p)$ is the future null vector $X_\pi = \ell_p \ell_p^\dagger = \begin{pmatrix} 1 & \bar{\pi}_p \\ \pi_p & p \end{pmatrix}$. On $E \times E$ over $\mathbb{F}_p$ it is the Néron–Severi class of the graph of Frobenius.

Theorem 3.1 (CM lifts are $\mathcal{O}_K$ -integral and principal [THEOREM]). *Let E have CM by $\mathcal{O}_K$ and let $\mathfrak{P}$ be a prime of good reduction of a field of definition, with residue field $\mathbb{F}_q$, $q = p^{f'}$, and Frobenius $\pi \in \mathcal{O}_K$ (Deuring; [BIB]). Then $X_\pi \in \text{Herm}_2(\mathcal{O}_K)$ is primitive and null, and $[1 : \pi] \in \mathbb{P}^1(K)$ is a*

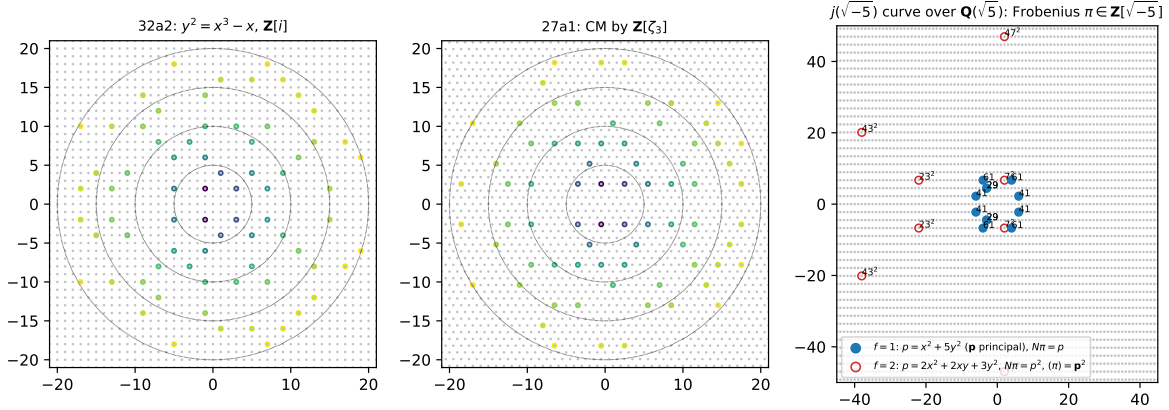


FIGURE 1. CM Frobenius as lattice points. Left, middle: $\pm\pi_p$ for ordinary $p < 400$ (colour $\log p$) on the lattice $\mathcal{O}_K$ (grey) with norm circles $|\pi| = \sqrt{p}$. Every point is a lattice point (Thm. 3.1). Right: the $j(\sqrt{-5})$ curve over $\mathbb{Q}(\sqrt{5})$. Filled: $f = 1$, $\mathfrak{p}$ principal, $N\pi = p$. Open: $f = 2$, $\mathfrak{p}$ non-principal, $N\pi = p^2$, $(\pi) = \mathfrak{p}^2$. The class group appears as the residue degree.

cusps of $\mathrm{PSL}_2(\mathcal{O}_K)$ in the principal class, i.e. $\mathrm{SL}_2(\mathcal{O}_K)$ -equivalent to ∞ . In particular the KS lifts of a CM curve never reach the non-principal cusps.

Proof. X_π has entries $1, \pi, \bar{\pi}, q$ in $\mathcal{O}_K$. The ideal $(1, \pi) = \mathcal{O}_K$ is principal, and Thm. 4.1 below then gives the class. $\square$

Where the class group shows up instead [THEOREM]+[BIB]+[NUMERICAL]. If E has CM by $\mathcal{O}_K$ with $h_K > 1$, then $j(E)$ has degree h_K over $\mathbb{Q}$ (CM theory [BIB]), so there is *no such curve over $\mathbb{Q}$* . For $K = \mathbb{Q}(\sqrt{-5})$ ($h = 2$) we use the curve with $j = j(\sqrt{-5})$ over $\mathbb{Q}(\sqrt{5})$. Numerically $1728 J(\sqrt{-5}) = 1264538.909475140509320227 = 632000 + 282880\sqrt{5}$, and $j(\frac{1+\sqrt{-5}}{2}) = 632000 - 282880\sqrt{5}$, each to 25 digits (cm.json) [NUMERICAL]. Counting points on $y^2 = x^3 + 3j(1728 - j)x + 2j(1728 - j)^2$ over $\mathbb{F}_p$ ($p \equiv \pm 1 \pmod{5}$) and $\mathbb{F}_{p^2}$ ($p \equiv \pm 2 \pmod{5}$), for $7 \leq p < 80$:

- p split in K with $\mathfrak{p}$ principal ($p = x^2 + 5y^2$: 29, 41, 61): residue degree $f = 1$ and $\pi = a/2 + v\sqrt{-5}$ with $N\pi = p$, e.g. $\pi = -3 + 2\sqrt{-5}$ at $p = 29$.
- p split with $\mathfrak{p}$ non-principal ($p = 2x^2 + 2xy + 3y^2$: 7, 23, 43, 47): these p are inert in $\mathbb{Q}(\sqrt{5})$, so $f = 2$ and $\pi \in \mathbb{Z}[\sqrt{-5}]$ with $N\pi = p^2$, $\pi \neq \pm p$. Explicitly $2 + 3\sqrt{-5}$, $-22 + 3\sqrt{-5}$, $-38 + 9\sqrt{-5}$, $2 + 21\sqrt{-5}$. So $(\pi) = \mathfrak{p}^2$ or $\bar{\mathfrak{p}}^2$.
- p inert in K : supersingular ($a = 0$ at $f = 1$; $\pi = -p$ at $f = 2$).

This is Artin reciprocity for the Hilbert class field $H = K(i)$ in action: $\mathfrak{p}$ splits completely in H iff it is principal, and Frobenius sees $\mathfrak{p}$ only through $\mathfrak{p}^f$, with f the order of $[\mathfrak{p}]$ in $\mathrm{Cl}(K)$ [BIB]. The non-principal prime $\mathfrak{p} = (7, 3 + \sqrt{-5})$ *does* define a cusp, through its own spinor $(7, 3 + \sqrt{-5})$, and that cusp is non-principal. That spinor is not a Frobenius spinor.

Census [NUMERICAL] (cm.json, all good $p < 3000$). 32a2 ($\mathbb{Z}[i]$), 27a1 ($\mathbb{Z}[\zeta_3]$), 49a1 ($\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$): every ordinary prime is split in K , with $\pi_p = (a_p + v\sqrt{d})/2 \in \mathcal{O}_K$ (211, 207, 210 primes); every supersingular prime is inert (217, 221, 217); 0 violations. All ordinary lifts are cusps of *one* Bianchi group. For the non-CM curve 11a1, $\mathbb{Q}(\pi_p) = \mathbb{Q}(\sqrt{a_p^2 - 4p})$ runs through 243 different imaginary quadratic fields for 427 primes, so each prime's lift is a rational null direction of its *own* Bianchi group. Fig. 1.

4. CUSPS, THE CLASS GROUP, AND INTEGRAL LIGHT RAYS (TASK 2)

Theorem 4.1 (cusps $\leftrightarrow \text{Cl}(K)$ [CITED]). α/β and α'/β' ($\alpha, \beta, \alpha', \beta' \in \mathcal{O}_K$) are $\text{PSL}_2(\mathcal{O}_K)$ -equivalent iff the ideals (α, β) , (α', β') lie in the same class. So $\text{PSL}_2(\mathcal{O}_K)$ has exactly h_K cusps [2, §1], [6, Prop. 2.2] (proof in Siegel, [BIB]).

[NUMERICAL] (cusps.json). Class groups via reduced forms and ideal arithmetic: $h = 1$ ($d = -3, -4, -7, -8, -11, -19, -43, -67, -163$), $h = 2$ ($-15, -20, -24, -35, -40$), $h = 3$ ($-23, -31, -59$), $h = 4$ ($-39, -55, -56, -68$ cyclic; -84 is $(\mathbb{Z}/2)^2$). The map $[c : d] \mapsto [(c, d)]$ was checked to be invariant under random words of length 12 in $\text{SL}_2(\mathcal{O}_K)$: 900 tests, 0 violations (a code check of the elementary half). Fig. 2.

Theorem 4.2 (light rays and KS squares: primitivity = principalization [THEOREM]). Let $\ell \in \mathcal{O}_K^2 \setminus 0$ and $\mathbf{a} = (\ell_1, \ell_2)$. Then $N_{[\ell]} := \ell \ell^\dagger / N(\mathbf{a})$ is the unique primitive vector of $\text{Herm}_2(\mathcal{O}_K)$ on the future null ray $[\ell]$. It is a KS square $\mu \mu^\dagger$ with $\mu \in \mathcal{O}_K^2$ iff $\mathbf{a}$ is principal, i.e. iff the cusp $[\ell]$ lies in the principal class.

Proof. The entries $\ell_i \bar{\ell}_j$ generate $\mathbf{a} \bar{\mathbf{a}} = (N \mathbf{a})$, so $N_{[\ell]}$ is integral and its entries generate $\mathcal{O}_K$, hence it is primitive. If $\mathbf{a} = (\alpha)$, take $\mu = \ell/\alpha$. Conversely, if $N = \mu \mu^\dagger$ with $\mu \in \mathcal{O}_K^2$ then $\mu = \lambda \ell$ with $\lambda \in K$ and $|\lambda|^2 = N(\mathbf{a})^{-1}$. So $\lambda \mathbf{a} = (\mu_1, \mu_2)$ is integral of norm 1, i.e. $\lambda \mathbf{a} = \mathcal{O}_K$. $\square$

[NUMERICAL]: all primitive rank-one $N \in \text{Herm}_2(\mathcal{O}_K)$ with diagonal entries ≤ 40 were enumerated, and KS-representability was tested by brute force: $d = -20$: 499 principal rays (all KS), 536 non-principal (none KS); $d = -23$: 425/394/394; $d = -84$: 167/160/200/128; $d = -4$: 1213, all KS. 0 mismatches with Thm. 4.2.

What this makes precise ([t286u]/[t290u]). Over $\mathbb{Z}$ every primitive vector is a spinor and every ideal is principal, so “primitivity” (dividing out gcd, producing $1/\zeta(2s)$, Lemma 72) and “principalization” are invisible as separate steps. Over $\mathcal{O}_K$ they separate. *Primitive* light rays are classified by $\text{Cl}(K)$. *Principal* ones are exactly the KS squares of integral spinors, and they are what the Eisenstein series at ∞ sums over (Prop. 1.2). The class group is the obstruction to writing an integral light ray as the square of an integral spinor [THEOREM].

Theorem 4.3 (Lemma 71’s light cone is the Bianchi lattice; cusps = elliptic curves on $E \times E$). Let $E/\mathbb{C}$ have $\text{End}(E) = \mathcal{O}_K$ (or $E/\bar{\mathbb{F}}_p$ ordinary with $\text{End}(E) = \mathcal{O}_K$).

(i) [CITED]+[TAA-L]. $\text{NS}(E \times E) = \mathbb{Z}v \oplus \text{End}(E) \oplus \mathbb{Z}h$ with $(X_\beta \cdot X_\gamma) = -\text{Tr}(\hat{\gamma}\beta)$, $X_\beta \perp v, h$, and $\frac{1}{2}L^2 = ac - \deg \beta$ for $L = av + X_\beta + ch$ [10, §5]. Hence Lemma 71’s map Ψ (Thm. NS there) is an isometry $(\text{NS}(E \times E), \cdot) \cong (\text{Herm}_2(\mathcal{O}_K), 2 \det)$ [THEOREM].

(ii) [CITED]. Elliptic subgroups $C \leq E \times E$ correspond bijectively to primitive classes D with $D^2 = 0$ and $D \cdot \Theta > 0$ [5, Thm. 1.1], i.e. to primitive future light rays of $\text{Herm}_2(\mathcal{O}_K)$.

(iii) [THEOREM] (proof sketch; standard facts [BIB]). If $C = \mu(E)$ with $\mu = (\mu_1, \mu_2) \in \mathcal{O}_K^2$ and $\mathbf{a} = (\mu_1, \mu_2)$, then $\ker \mu = E[\mathbf{a}]$ and $C \cong E/E[\mathbf{a}] \cong \mathbb{C}/\mathbf{a}^{-1}$. So $C \cong E$ iff $\mathbf{a}$ is principal. The h_K cusp classes are the h_K isomorphism classes of elliptic curves with CM by $\mathcal{O}_K$, realized as curves on $E \times E$. Consistency check: $C \cdot (\{x\} \times E) = \deg(\text{pr}_1|_C) = N \mu_1 / N \mathbf{a} = (N_{[\mu]})_{11}$ and $C^2 = 0 = \det N_{[\mu]}$.

Reading [THEOREM] (given (i)–(iii)). $\mathbb{H}^3$ is the space of normalized real ample classes of $E \times E$ (its positive cone, divided by scaling). $\text{SL}_2(\mathcal{O}_K) \subset \text{Aut}(E \times E)$ acts on it, and $E(P, s)$ sums $(C \cdot H_P)^{-1-s}$ over the elliptic curves $C \cong E$ on $E \times E$ (each counted once modulo units). The class H_P is the polarization of self-intersection 2 in direction P , and $C \cdot H_P = 2\langle X_P, N_C \rangle$ up to the adj convention of Prop. 1.2, which replaces $\mathbf{a}$ by $\bar{\mathbf{a}}$. We have not found this “Eisenstein series = height zeta function of elliptic curves on $E \times E$ ” reading in the sources we opened. It may well be classical, so treat novelty as unverified [OPEN].

5. THE BIANCHI EISENSTEIN SERIES AND ITS SCATTERING MATRIX (TASK 3)

Normalization. We write $E(P, s) \sim y^{1+s} + \varphi(s)y^{1-s}$, so that $\Delta E = (1 - s^2)E$. The unitary (continuous-spectrum) line is $\text{Re } s = 0$ and the centre is $s = 0$. In [2] the exponent is y^s (unitary line

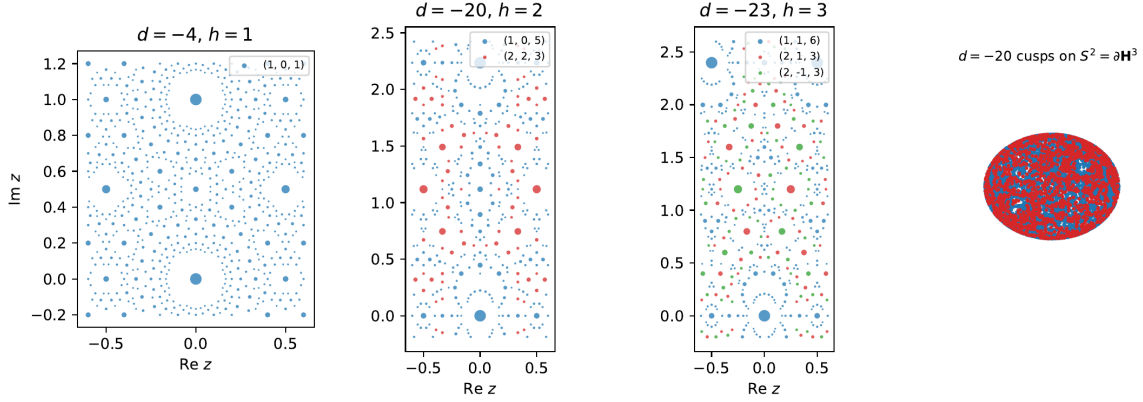


FIGURE 2. K -rational cusps $z = d/c$ (marker size $\propto N(\mathfrak{a})/N(c)$), coloured by the ideal class of (c, d) , i.e. by cusp class (Thm. 4.1), labelled by reduced forms. $d = -4$: one class. $d = -20$: two. $d = -23$: three. Right: $d = -20$ cusps with c, d in a box, mapped to the celestial sphere $S^2 = \partial\mathbb{H}^3$ by stereographic projection. Both classes are dense.

$\text{Re } s = 1$). In [6] it is $N(v)^{-s}$ with v^2 (unitary line $\text{Re } s = \frac{1}{2}$). The dictionary is $s_{\text{ES}} = 1 + s = 2s_{\text{M}}$. Put $\Lambda_K(s) = (2\pi)^{-s}|d|^{s/2}\Gamma(s)\zeta_K(s)$, so that $\Lambda_K(s) = \Lambda_K(1-s)$.

Theorem 5.1 ($h = 1$ [THEOREM]). For $h_K = 1$, $\varphi_K(s) = \frac{2\pi}{s\sqrt{|d|}} \frac{\zeta_K(s)}{\zeta_K(1+s)} = \frac{\Lambda_K(s)}{\Lambda_K(1+s)}$.

Proof. For $c \neq 0$ write $d = d_0 + c\mu$, $\mu \in \mathcal{O}_K$. Unfolding over $\mathbb{C}/\mathcal{O}_K$ (area $A = \sqrt{|d|}/2$) and using $\int_{\mathbb{C}} (|u|^2 + y^2)^{-1-s} du = \pi y^{-2s}/s$, the y^{1-s} coefficient is $\frac{\pi}{sA} \sum_{c \neq 0/\mathcal{O}_K^\times} N(c)^{-1-s} \#(\mathcal{O}_K/c)^\times = \frac{2\pi}{s\sqrt{|d|}} \frac{\zeta_K(s)}{\zeta_K(1+s)}$. The second form uses $\Gamma(s)/\Gamma(1+s) = 1/s$. $\square$

Unified shape [THEOREM]. For $\text{SL}_2(\mathbb{Z})$, $\varphi(s_{72}) = \xi(2s_{72} - 1)/\xi(2s_{72})$. With $s' = 2s_{72} - 1$ this reads $\xi(s')/\xi(1+s')$, the same ratio $\Lambda_F(s)/\Lambda_F(1+s)$ of completed zeta functions of the ground field $F = \mathbb{Q}$ or K . That is Langlands' constant-term formula for SL_2 over F [BIB], here verified in both cases. Poles: $s = 1$ (from $\Lambda_F(s)$) and $s = \rho - 1$ (zeros of $\Lambda_F(1+s)$); nothing else. $\text{RH}/\text{GRH} \Leftrightarrow$ all resonances on $\text{Re } s = -\frac{1}{2}$ [RESTATEMENT].

Theorem 5.2 ($h > 1$: Masri's matrix and its channels). (i) [CITED] [6, Thm. 1.1], transported to our normalization (the factor 2^{2s} and $D(Q^{-1}) = I$ on the determinant-one locus): with $Z(s)_{ij} = \zeta_{[\mathfrak{a}_i^{-1}\mathfrak{a}_j]}(s)$ (partial zeta functions of classes) and P the permutation $[\mathfrak{a}] \mapsto [\mathfrak{a}]^{-1}$,

$$\Phi(s) = \left(\frac{2\pi}{\sqrt{|d|}} \right)^{2s} \frac{\Gamma(1-s)}{\Gamma(1+s)} Z(1+s)^{-1} P Z(1-s).$$

(ii) [THEOREM] (derived here; consistent with [2]). For a class-group character χ let $L_K(s, \chi) = \sum_C \chi(C) \zeta_C(s)$, $\Lambda(s, \chi) = (2\pi)^{-s}|d|^{s/2}\Gamma(s)L_K(s, \chi)$, and $\varphi_\chi(s) = \Lambda(s, \chi)/\Lambda(1+s, \chi)$. Since $\zeta_C = \zeta_{C^{-1}}$ we have $L_K(s, \chi) = L_K(s, \bar{\chi})$. The characters diagonalize the convolution matrices Z , and P swaps $\chi \leftrightarrow \bar{\chi}$. So the spectrum of $\Phi(s)$ is $\{\varphi_\chi : \chi = \bar{\chi}\} \cup \{\pm\varphi_\chi : \{\chi, \bar{\chi}\} \text{ pairs}\}$. Consequently $\det \Phi = (-1)^{(h-m)/2} \prod_\chi \varphi_\chi$, with $m = \#\text{Cl}(K)[2] = 2^{t-1}$, and $\prod_\chi L_K(s, \chi) = \zeta_H(s)$ for the Hilbert class field H . This is Efrat-Sarnak's Theorem 1 [2] [CITED]. Also $\varphi_{\chi_0}(0) = -1$ and $\varphi_\chi(0) = +1$ for $\chi \neq \chi_0$ (because $L_K(0, \chi) = 0$, the root number is 1, and $\Lambda(0, \chi) = \Lambda(1, \chi)$), so $\text{Tr } \Phi(0) = (m-1) - 1 = 2^{t-1} - 2$: Efrat-Sarnak's Theorem 2 [CITED], Masri's Theorem 1.6.

(iii) [THEOREM]. For genus characters, $L_K(s, \chi) = L(s, \chi_{d_1})L(s, \chi_{d_2})$ with $d = d_1d_2$ (Kronecker [BIB]; e.g. $d = -20$: $L(\chi_{-4})L(\chi_5)$, checked to $4 \cdot 10^{-30}$ at $s = \frac{1}{2} + 14i$). For χ_0 , $L_K = \zeta L(\chi_d)$.

Numerical verification [NUMERICAL] (`lemma73_scatter.py` $\rightarrow$ `scatter.json`, 30 digits). The partial zetas were computed from an Epstein expansion derived from scratch (`lemma73_lib.epstein`; it agrees with brute force at $s = 2$ and with $4\zeta(2)G$ for $x^2 + y^2$ to 30 digits). The ideal-class arithmetic was done independently with HNF ideals.

d	h	m	$\max \ \Phi^\dagger \Phi - I\ $ on $\operatorname{Re} s = 0$	$\Phi(s)\Phi(-s) = I$	eig = channels	$\operatorname{Tr} \Phi(0)$ (ES)	$\det \Phi / \prod \varphi_\chi$ (ES)	Res_1 entry
-4	1	1	$3 \cdot 10^{-31}$	$3 \cdot 10^{-31}$	yes	-1 (-1)	1 (1)	1.637616096
-3	1	1	$2 \cdot 10^{-31}$	$2 \cdot 10^{-31}$	yes	-1 (-1)	1 (1)	1.706552177
-20	2	2	$1 \cdot 10^{-31}$	$1 \cdot 10^{-31}$	yes	0 (0)	1 (1)	0.5318944644
-15	2	2	$5 \cdot 10^{-31}$	$5 \cdot 10^{-31}$	yes	0 (0)	1 (1)	0.616989454
-23	3	1	$3 \cdot 10^{-31}$	$3 \cdot 10^{-31}$	yes	-1 (-1)	-1 (-1)	0.3718164517
-39	4	2	$3 \cdot 10^{-31}$	$3 \cdot 10^{-31}$	yes	0 (0)	-1 (-1)	0.2263012456
-84	4	4	$3 \cdot 10^{-31}$	$3 \cdot 10^{-31}$	yes	2 (2)	1 (1)	0.1358684921

Φ is exactly symmetric in all cases, so on $\operatorname{Re} s = 0$ the two middle columns coincide ($\Phi(-it) = \overline{\Phi(it)} = \Phi(it)^\dagger$). “eig = channels” means each eigenvalue of $\Phi(it)$ ($t = 0.7, 3.3, 9.1, 17.9$) equals some $\pm \varphi_\chi(it)$ to double precision. For $d = -23$ the cubic channel was additionally checked against its functional equation (relative error $\leq 1.5 \cdot 10^{-30}$), and its coefficients against $\eta(z)\eta(23z)$ for $n < 40$ (the weight-one form of this L -function, [BIB]). A first run with a mislabelled class table gave $\operatorname{Tr} \Phi(0) = 1$ for $d = -23$ against Efrat–Sarnak’s -1 . The ES trace formula is what caught the bug.

Theorem 5.3 (the anomaly in 3D: residue = Humbert [THEOREM]+[NUMERICAL]). $\operatorname{Res}_{s=1} \Phi(s) = \frac{\sqrt{|d|}}{w \operatorname{vol}(\operatorname{PSL}_2(\mathcal{O}_K) \backslash \mathbb{H}^3)} J$, where J is the all-ones $h \times h$ matrix and $\operatorname{vol} = |d|^{3/2} \zeta_K(2) / (4\pi^2)$ (Humbert, [11, §2] [CITED-sec], Lemma 72). For $h = 1$, $\operatorname{Res}_1 \varphi_K = \frac{\sqrt{|d|}}{w \operatorname{vol}} = \frac{\operatorname{area}(\Gamma_\infty \backslash \mathbb{C})}{\operatorname{vol}}$, since Γ_∞ acts on $\mathbb{C}$ through $\mathcal{O}_K$ -translations and the rotations $z \mapsto \varepsilon^2 z$ ($[\mathcal{O}_K^\times : \pm 1] = w/2$), so the area is $A/(w/2) = \sqrt{|d|}/w$.

Proof. In (i), only $\Gamma(1-s)$ has a pole at $s = 1$, with residue -1 . Every $\zeta_C(0) = -1/w$, so $Z(0) = -J/w$. $PJ = J$, and $Z(2)^{-1}J = J/\zeta_K(2)$ because each row of $Z(2)$ sums to $\zeta_K(2)$. Hence $\operatorname{Res} = \frac{4\pi^2}{|d|} \frac{1}{w \zeta_K(2)} J = \frac{\sqrt{|d|}}{w \operatorname{vol}} J$. The table’s last column agrees with $\sqrt{|d|}/(w \operatorname{vol})$ to all 10 printed digits. Volumes: 0.305321864726 (-4 , $= G/3$), 0.169156934402 (-3), 4.20396925948 (-20), 6.44919220406 (-23), 33.7280235021 (-84). $\square$

So Lemma 69’s anomaly “ $3/\pi = 1/\operatorname{area} = \operatorname{Res} E(z, s)$ ” has an exact 3D analogue: cusp cross-section area over hyperbolic volume. It is rank one: every cusp class sees the same residue, and only the trivial channel carries it ($L_K(s, \chi)$ is entire for $\chi \neq \chi_0$).

Theorem 5.4 (scattering unitarity is RH-blind [THEOREM]). Let Λ be any meromorphic function with $\Lambda(s) = \Lambda(1-s)$ and $\overline{\Lambda(\bar{s})} = \Lambda(s)$, and put $\varphi(s) = \Lambda(s)/\Lambda(1+s)$. Then $|\varphi(it)| = 1$ for real t , and $\varphi(s)\varphi(-s) = 1$. In particular, replacing Λ by ΛQ with $Q(s) = \frac{((s-\beta)^2 + \gamma^2)((s-1+\beta)^2 + \gamma^2)}{(s-\frac{1}{2})^2 + \gamma^2}$ (a ghost quartet in place of a zero pair) preserves unitarity for every β . The same holds for $\operatorname{SL}_2(\mathbb{Z})$ ’s $\xi(s')/\xi(1+s')$ and for every class-group channel.

Proof. $\Lambda(1+it) = \Lambda(-it) = \overline{\Lambda(it)}$. The ghost Q satisfies $Q(1-s) = Q(s)$ and has real coefficients. $\square$

[NUMERICAL]: $||\varphi^{\text{ghost}}(it)| - 1| \leq 5 \cdot 10^{-41}$ (the working precision, 40 digits) for $\delta = 0.05$ – 0.4 , $t \in \{1.3, 7.7, 21.1\}$ (`mb.json`). So the unitarity of the scattering matrix carries no information about the location of zeros. It is a consequence of the functional equation alone. This is the precise sense of [t291u]’s “already-unitary” scattering process.

6. THEOREM MB FOR BIANCHI CHANNELS, AND THE GHOST TEST (TASK 4)

As in Lemma 72 we resum the reflected halves of a heavy Poincaré seed. On $\mathbb{H}^3$, the seed $y^{1+\sigma} e^{-2\pi E y}$, unfolded against the Eisenstein series of each channel, gives the reflected part

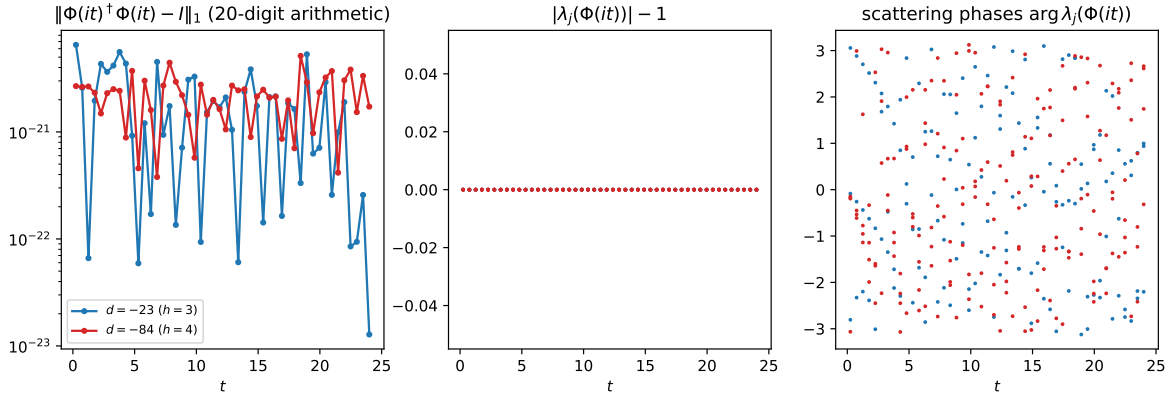


FIGURE 3. Bianchi scattering matrices on the unitary line (20-digit run of `lemma73_figs.py`). Left: $\|\Phi(it)^\dagger \Phi(it) - I\|_1$ for $d = -23$ ($h = 3$) and $d = -84$ ($h = 4$), at the working-precision floor. Middle: moduli of eigenvalues minus 1. Right: scattering phases $\arg \lambda_j$. These are the arithmetic content, and the zeros sit in their analytic continuation, not in their modulus.

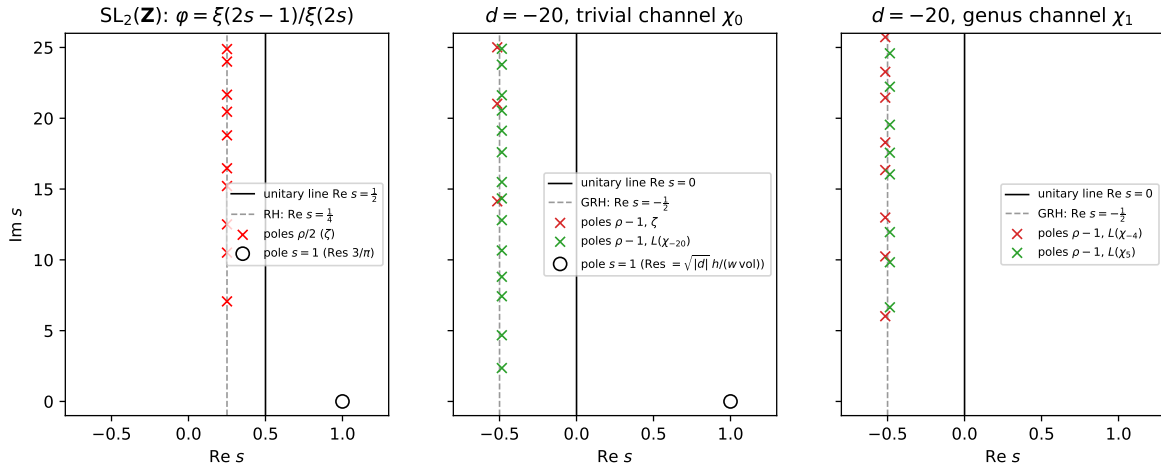


FIGURE 4. Resonance maps. Left: $\mathrm{SL}_2(\mathbb{Z})$ (Lemma 72 normalization), poles of φ at $\rho/2$, unitary line $\mathrm{Re} s = \frac{1}{2}$. Middle/right: $\mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{-5})})$, $h = 2$. Trivial channel: poles at $\rho - 1$ for zeros of ζ and $L(\chi_{-20})$, plus the anomaly pole at $s = 1$. Genus channel: zeros of $L(\chi_{-4})$ and $L(\chi_5)$, no pole at $s = 1$. The dashed lines are where RH/GRH puts the resonances.

$y^{1-\sigma} S_\chi(X)$, $X = 2\pi E/y$, with

$$S_\chi(X) := \sum_{m \geq 0} \frac{(-X)^m}{m!} \varphi_\chi(\sigma + m).$$

This is termwise and formal, exactly as in Lemma 72 (Rem. conv there). We take $\sigma = \frac{1}{2}$. At $\sigma = 0$ the sample point $s = 1$ would collide with the anomaly pole.

Theorem 6.1 (Mellin–Barnes, Bianchi [THEOREM]). *For $X \rightarrow \infty$ and every $A > 0$,*

$$S_\chi(X) = \Gamma(\sigma - 1) \operatorname{Res}_{s=1} \varphi_\chi X^{1-\sigma} + \sum_{\rho} \Gamma(1 + \sigma - \rho) \frac{\Lambda(2 - \rho, \chi)}{\Lambda'(\rho, \chi)} X^{\rho-1-\sigma} + O(X^{-A}),$$

the sum running over the nontrivial zeros ρ of $L_K(s, \chi)$ (simple zeros assumed; true for all zeros used). $\operatorname{Res}_{s=1} \varphi_{\chi_0} = \frac{h\sqrt{|d|}}{w \operatorname{vol}}$ and $\operatorname{Res}_{s=1} \varphi_\chi = 0$ for $\chi \neq \chi_0$.

Proof. $S_\chi(X) = \frac{1}{2\pi i} \int \Gamma(-u) \varphi_\chi(\sigma + u) X^u du$ along a contour enclosing $u = 0, 1, 2, \dots$ clockwise. Shift it left. The poles of $\varphi_\chi(\sigma + u)$ are at $\sigma + u = 1$ (χ_0 only) and at $\sigma + u = \rho - 1$. $\Lambda(1 + s, \chi)$ has no other zeros: the Γ -poles are cancelled by trivial zeros. $\Lambda(s, \chi)$ has poles only at $s = 0, 1$, and the one at $s = 0$ cancels against $\Lambda(1 + s)$. The residue at $s = \rho - 1$ is $\Lambda(\rho - 1)/\Lambda'(\rho) = \Lambda(2 - \rho)/\Lambda'(\rho)$ by the functional equation. Stirling bounds the shifted integral as in Lemma 72. $\square$

Exponents [THEOREM]/[RESTATEMENT]. Leading (anomaly) term: $X^{1-\sigma} = X^{1/2}$, coefficient $\Gamma(-\frac{1}{2}) \frac{h\sqrt{|d|}}{w \operatorname{vol}} = -2\sqrt{\pi} \cdot (\text{Humbert residue})$, the 3D twin of Lemma 72's $-6/\sqrt{\pi} = -2\sqrt{\pi} \cdot 3/\pi$. Zero terms: $X^{\operatorname{Re} \rho - 3/2}$, i.e. X^{-1} on GRH. An off-line zero β gives $X^{\beta-3/2}$. “GRH for $L_K(s, \chi) \Leftrightarrow$ the zero part of S_χ is $O(X^{-1+\epsilon})$ ” is a [RESTATEMENT] (modulo the convergence caveat of Lemma 72).

Numerical check [NUMERICAL] (lemma73_mb.py, mb.json; direct series at $\lceil X/2.3 \rceil + 45$ digits; all zeros up to height 75: 18 of ζ , 34 of $L(\chi_{-4})$, 37 of $L(\chi_5)$, 53 of $L(\chi_{-20})$, matching the Riemann–von Mangoldt counts):

channel	X	$S_\chi(X)$	zero part	$ S - \text{lead} - \text{zeros} $
$d = -4, \chi_0$ ($\zeta L(\chi_{-4})$)	20	-25.96164664894	$-1.236 \cdot 10^{-5}$	$1.3 \cdot 10^{-8}$
	40	-36.71527026738	$+2.506 \cdot 10^{-5}$	$5.3 \cdot 10^{-17}$
	60	-44.96687910629	$-9.455 \cdot 10^{-6}$	$1.7 \cdot 10^{-25}$
$d = -20, \chi_0$ ($\zeta L(\chi_{-20})$)	60	-29.21040017146	$-9.977 \cdot 10^{-5}$	$3.7 \cdot 10^{-25}$
$d = -20, \chi_1$ ($L(\chi_{-4})L(\chi_5)$)	20	$3.8406 \cdot 10^{-5}$	$3.8378 \cdot 10^{-5}$	$2.9 \cdot 10^{-8}$
	60	$1.13649965949 \cdot 10^{-5}$	$1.13649965949 \cdot 10^{-5}$	$3.7 \cdot 10^{-25}$

The residue at $s = 1$ matches Humbert: 1.63761609555586 ($d = -4$) and 1.06378892874135 = 2×0.5318944644 ($d = -20$, = $h \times$ the matrix entry of Thm. 5.3). The remainder falls roughly like Xe^{-X} , the same in both $d = -20$ channels. It is exponentially small, so the $O(X^{-A})$ is very conservative [NUMERICAL]. *The genus channel has no anomaly term:* S_{χ_1} is its zero sum, to 25 digits.

Ghost test [NUMERICAL]. Replace one zero pair by an off-line quartet: $\tilde{\Lambda} = \Lambda \cdot Q$ with Q as in Thm. 5.4, $\beta = \frac{1}{2} + \delta$. The ghost channel $\tilde{\varphi} = \tilde{\Lambda}(s)/\tilde{\Lambda}(1 + s)$ is still exactly unitary ($\leq 5 \cdot 10^{-41}$). Thm. 6.1 still holds for it, with the same accuracy as for the true data: residual $5.6 \cdot 10^{-17}$ ($d = -4$) and $1.3 \cdot 10^{-16}$ ($d = -20, \chi_1$) at $X = 40$. Both statements are RH-blind. What changes is the *exponent*:

fitted envelope slope, $X \in [10^{20}, 10^{60}]$	$\delta = 0$	0.05	0.10	0.25	0.40
prediction $-1 + \delta$	-1	-0.95	-0.90	-0.75	-0.60
$d = -4, \chi_0$, ghost at $\gamma_1(\zeta) = 14.13$	-1.0010	-1.0007	-0.9736	-0.7581	-0.5997
$d = -20, \chi_1$, ghost at $\gamma_1(L(\chi_{-4})) = 6.02$	-0.9912	-0.9517	-0.9020	-0.7518	-0.6011

Diagnostic reading. In the genus channel the ghost replaces the lowest zero, and every δ is resolved to ± 0.002 . In the trivial channel of $\mathbb{Q}(i)$ the ghost sits on ζ 's first zero ($\gamma = 14.13$). That zero is outweighed by the $L(\chi_{-4})$ zero at 6.02 by the Stirling factor $|\Gamma(\frac{3}{2} - \rho)|$, about $e^{-\pi(14.13-6.02)/2} \approx 3 \cdot 10^{-6}$. So a ghost becomes visible only once $X^\delta \gtrsim 10^5$ – 10^6 : invisible at $\delta = 0.05$, partly visible at 0.1, clean at ≥ 0.25 . This is the Bianchi form of Lemma 72's “high zeros are exponentially hidden” [HEURISTIC]. In the ghost model the leading coefficient rescales by $Q(1)/Q(2) = 0.9901$ ($\delta = 0.05$, $d = -4$). This is an artefact of the ad hoc model: vol is fixed geometrically, and $Q\Lambda$ is not an L -function. The genus channel has no leading term to rescale. Remainder: $|S - \text{lead} - \text{zeros}|/(Xe^{-X}) =$

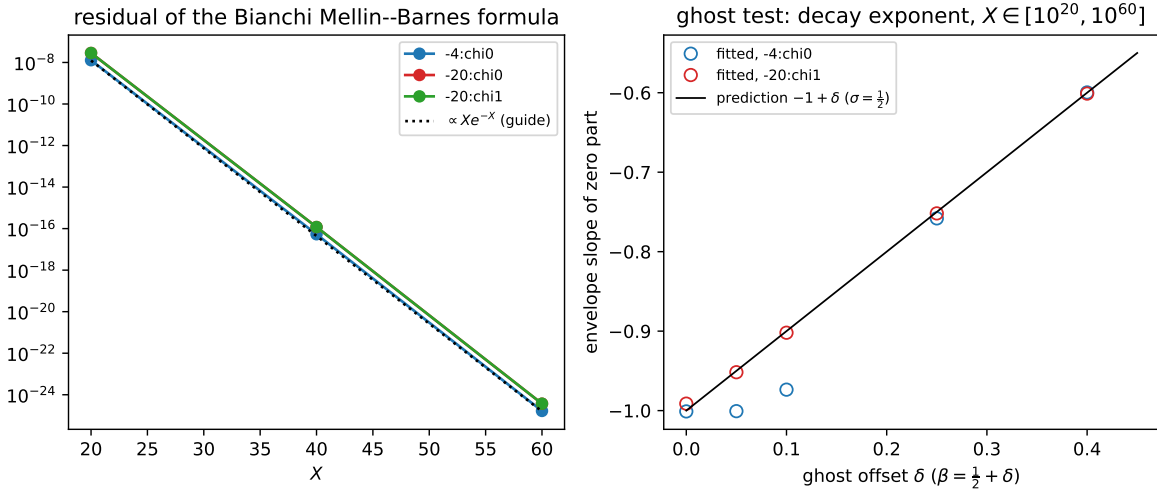


FIGURE 5. Left: residual of Thm. 6.1 against the direct series ($d = -4$ trivial channel; $d = -20$ trivial and genus channels; the last two coincide). Right: ghost test. Fitted envelope slopes of the zero part over $X \in [10^{20}, 10^{60}]$ against the prediction $-1 + \delta$, for a ghost replacing the first zero of ζ ($d = -4$, χ_0 ; hidden for small δ by the lower $L(\chi_{-4})$ zeros) or the first zero of $L(\chi_{-4})$ ($d = -20$, genus channel; a GRH-type ghost). $\delta = 0$ is the true data.

0.31, 0.31, 0.32 ($d = -4$) and 0.69, 0.70, 0.71 ($d = -20$) at $X = 20, 40, 60$. So the remainder is numerically $\approx cXe^{-X}$ [NUMERICAL]. An exact formula for it is [OPEN]. Its natural source is the e^{-X} -type contribution of the cancelled pole at $s = 0$ ([HEURISTIC]).

What this is not. Thm. 6.1 is an equivalence between GRH for $L_K(s, \chi)$ and a decay rate, not a proof of either. The decay of the zero part is read from the zeros. The heavy-seed sum itself has no independent physical bound in this setting. Exactly as in Lemma 72, a bound on S_χ coming from Bianchi “gravity” would be needed and is [OPEN].

7. TWISTORS OVER $\mathcal{O}_K$ (TASK 5, [T290U])

Standard facts [BIB] (Penrose–Rindler, not opened). Projective twistor space is $\mathbb{P}^3$ with twistors $Z = (\omega, \pi) \in \mathbb{C}^2 \times \mathbb{C}^2$. A point x of complexified Minkowski space corresponds to the line $L_x = \{(ix\pi, \pi)\} \cong \mathbb{P}^1$. Real points correspond to lines in the null quadric $\mathbb{P}\mathbb{N}$. L_x and $L_{x'}$ meet iff $x - x'$ is null. The twistor group is $SU(2, 2)$, the double cover of the conformal group $SO(4, 2)$.

Proposition 7.1 (incidence over K [THEOREM]). *Let $x, x' \in \text{Herm}_2(K)$ (drop the conventional i , or absorb it in the Hermitian structure). Then $L_x \cap L_{x'} \neq \emptyset$ over $\mathbb{C}$ iff $\det(x - x') = 0$. In that case the intersection point is the K -rational twistor $(x\pi, \pi)$ with π spanning $\ker(x - x') \cap K^2$. For $x, x' \in \text{Herm}_2(\mathcal{O}_K)$, scale π to be primitive. Its ideal (π_1, π_2) has a class in $\text{Cl}(K)$, and the separation is $x - x' = nN_{[\pi]}$ with $n \in \mathbb{Z}$ and N the primitive light ray of Thm. 4.2 (π' the adjugate direction).*

Proof. $(x\pi, \pi) = (x'\pi', \pi')$ forces $\pi = \pi'$ and $(x - x')\pi = 0$. A 2×2 matrix over K with $\det = 0$ has a K -rational kernel. The rest is Thm. 4.2. $\square$

[NUMERICAL] (`cusps.json`): 4000 random pairs each for $d = -20$ and $d = -4$, half of them null-separated by construction: “ $\det = 0 \iff$ rational meeting twistor” held 4000/4000. For $d = -20$ the meeting spinors were principal 1505 times and non-principal 527 times.

Theorem/Analogy split.

- [THEOREM]: the spinor/celestial-sphere layer (Prop. 1.1, Thm. 4.2) and the incidence layer (Prop. 7.1) are algebraic and hold over K and $\mathcal{O}_K$. The class group enters both as the obstruction

to integral spinors. So [t290u]’s “points of $\mathbb{Z}^2 \rightarrow$ lines through 0” is, over $\mathcal{O}_K$, literally “integral light rays $\rightarrow K$ -rational null directions”, with $\text{Cl}(K)$ measuring the failure of integral spinor lifts.

- [CITED]: the arithmetic twistor (conformal) group exists and is studied. Hermitian modular forms of degree 2 live on the Hermitian half-space $\mathcal{H}_2$ (the tube over the future cone of Herm_2 , i.e. complexified Minkowski space’s future tube) under $\text{SU}_{2,2}(\mathcal{O}_K)$, with Hermitian Eisenstein series $E_k^{(K)}$ (Braun; Krieg) [4]. This is the genuine “arithmetic group acting on (complexified) $3 + 1$ Minkowski space by conformal maps”. We read only the abstract and introduction. Its relevance to TAA is a direction (F3), not a result.
- [OPEN]/[ANALOGY]: we know of no Penrose transform, sheaf cohomology description of massless fields, or field equation over $\mathcal{O}_K$, and found none at source. “Twistor lines are spacetime points” over $\mathcal{O}_K$ therefore remains an [ANALOGY] beyond the incidence statement. Our searches for “integral/arithmetic twistors” turned up nothing we could verify, and none is cited.

8. CELESTIAL HOLOGRAPHY: LITERATURE CHECK ([T293U] (C))

Read at source (`lit/lemma73/`):

- Pasterski–Shao–Strominger [7]: $\text{SL}_2(\mathbb{C})$ acts as global conformal group of the celestial sphere; massive conformal primary wavefunctions are $\int_{\mathbb{H}^3} G_\Delta(\hat{p}; w) e^{\pm i m \hat{p} \cdot X}$ (their (2.8)–(2.10)), with $\mathbb{H}^3$ the mass shell. [CITED]. No discrete subgroups.
- Pasterski, lectures [8]: celestial amplitudes as $\mathcal{S}$ -matrices in a boost-eigenstate basis, global conformal $\text{SL}_2(\mathbb{C})$. [CITED] (abstract and introduction). No arithmetic structure.
- Atanasov–Ball–Melton–Raclariu–Strominger [1]: in $(2, 2)$ Klein space the celestial sphere becomes a Lorentzian torus with $\text{SL}_2(\mathbb{R})_L \times \text{SL}_2(\mathbb{R})_R$, and single-valuedness forces integral or half-integral weights (“a discrete character”). [CITED] (abstract, §1). Discreteness here is of weights, not of the group. There is no $\text{SL}_2(\mathbb{Z})$ or Bianchi group.
- Fan [3]: for celestial boundary dimension $d > 2$, sums of conformal primary wavefunctions over a chain with $-q_n \cdot X = n$ equal $(\mp i)^\Delta \Gamma(\Delta) \zeta(\Delta)$, and the zeros are reached by continuation to $d = 1$. [CITED]. This is $\zeta(\Delta) = \sum n^{-\Delta}$ rewritten. The continuation in d is acknowledged by the author as lacking a physical picture. It is a [RESTATEMENT] with no constraint on the zeros.
- Remmen [9] (flat-space amplitudes, not celestial): an amplitude with exchanges of masses $m_n^2 = \mu_n^2$, $\zeta(\frac{1}{2} \pm i\mu_n) = 0$. Real masses $\Leftrightarrow$ RH by construction ([RESTATEMENT]), and unitarity bounds (positivity of odd moments of $1/\mu_n^2$) are *implied by* RH, not equivalent to it [CITED]. Relevant to the TAA slogan: in this model unitarity is a weaker consequence of RH, not RH.

Verdict. We found no work at source that puts Bianchi/ $\text{SL}_2(\mathbb{Z}[i])$ or Eisenstein-series structure on the celestial sphere. Search-engine summaries agreed, but those are not evidence and we do not rely on them. Prop. 1.2 shows the Bianchi Eisenstein series *is* a weighted sum of PSS propagators $G_{1+s}(P; w)$ over $w \in \mathbb{P}^1(K)$, i.e. an $\text{SL}_2(\mathcal{O}_K)$ -invariant “celestial wavefunction”. Its Fourier transform over the mass shell would be an $\text{SL}_2(\mathcal{O}_K)$ -invariant solution of the Klein–Gordon equation on $\mathbb{R}^{1,3}$, with the resonances $s = \rho - 1$ appearing as conformal dimensions $\Delta = \rho$. That is direction F1 (§12), unexplored [OPEN].

9. THE SHARPENED THESIS: WHAT BIANCHI ADDS (TASK 6)

- (a) *Confirmed* [THEOREM]. Thm. 5.4: unitarity on the unitary line follows from the functional equation alone and survives every ghost. Thm. 6.1: the anomaly (residue) term is ghost-blind in sign and shape. Ghosts change only the decay exponent. In $\text{SL}_2(\mathbb{Z})$ and in all Bianchi channels, *RH/GRH is the statement that the resonances of an automatically unitary scattering matrix sit on $\text{Re } s = -\frac{1}{2}$ (equivalently $\text{Re } s = \frac{1}{4}$ in Lemma 72’s normalization)* [RESTATEMENT]. This is a restatement, not a route to proof.
- (b) *Sharpened* [THEOREM]. “Fluctuations around the anomaly term” is right for the trivial channel only. For $h > 1$ the non-trivial channels have *no* anomaly term: $S_\chi(X)$ is pure fluctuation

(Fig. 5, $d = -20$ genus channel). The cusp-basis series $S_{\infty\infty} = \frac{1}{h} \sum_{\chi} S_{\chi}$ dilutes the anomaly by $1/h$.

- (c) *New beyond* $\mathrm{SL}_2(\mathbb{Z})$. (1) The resonance problem is a matrix problem over $\mathrm{Cl}(K)$, with $\det \Phi$ governed by ζ_H of the Hilbert class field [CITED]. (2) The relevant hypothesis is GRH for $\zeta_H = \prod_{\chi} L_K(s, \chi)$, i.e. for Dirichlet $L(\chi_{d_i})$ (genus characters) and for weight-one dihedral L -functions (non-genus characters, e.g. $\eta(z)\eta(23z)$) [RESTATEMENT]. (3) Primitivity and principalization separate (Thm. 4.2), and the class group becomes geometric: elliptic curves on $E \times E$ (Thm. 4.3). (4) The pairing $\chi \leftrightarrow \bar{\chi}$ gives eigenvalues $\pm \varphi_{\chi}$. The trace at the centre counts 2-torsion in $\mathrm{Cl}(K)$, a genus-theory invariant inside a scattering matrix.
- (d) *Not added* [HEURISTIC]. Nothing here constrains the location of any zero. Bianchi multiplies the number of L -functions whose GRH is restated. It does not supply a positivity.

10. CONNECTIONS TO TAA, CIRCULARITY AND METAPHOR FLAGS

- (T1) **Lemma 71** [TAA-L]+[THEOREM]. The KS lift's Minkowski space Herm_2 acquires the integral structure $\mathrm{Herm}_2(\mathcal{O}_K)$ exactly when E has CM by $\mathcal{O}_K$, and then it *is* $\mathrm{NS}(E \times E)$ (Thm. 4.3(i)). Lemma 71's "fourth axis = J_p " is the $\sqrt{d}$ -direction of $\mathcal{O}_K$. For non-CM curves each prime's lift lives in its own $\mathrm{Herm}_2(\mathcal{O}_{\mathbb{Q}(\pi_p)})$.
- (T2) **Lemma 72** [TAA-L]+[THEOREM]. Same skeleton: $\varphi = \Lambda_F(s)/\Lambda_F(1+s)$; anomaly = residue = cusp area/volume ($3/\pi$ becomes $\sqrt{|d|}/(w \text{ vol})$); zeros enter as resonances through primitivity ($1/\zeta_K(1+s)$, or $Z(1+s)^{-1}$ for $h > 1$); Thm. mb carries over with exponent $\rho - 1 - \sigma$ in place of $\rho/2 - \frac{1}{2}$ (only the sampling convention differs).
- (T3) **Lemma 69** [TAA-L]. $\varphi(\text{centre}) = -1$ in the trivial channel (ghost-blind fixed-point value, as for $\mathrm{SL}_2(\mathbb{Z})$). The non-trivial real channels have $+1$. The centre value is an arithmetic sign.
- (T4) **Lemma 70** [TAA-L]+[HEURISTIC]. $\mathrm{Tr} \Phi(0) = 2^{t-1} - 2$ is a genus-theory count, i.e. congruence/2-torsion data. This is RH-neutral, like Lemma 70's congruences.
- (T5) **[t286u] principalization** [THEOREM]. Over $\mathcal{O}_K$ the class group is the obstruction to integral spinor square roots of light rays (Thm. 4.2). In the CM picture it is the set of elliptic curves on $E \times E$ (Thm. 4.3). Over $\mathbb{Z}$ this obstruction is empty, which is the "missing h^1 " in yet another guise. Note: $\mathrm{Cl}(K)$ here is finite, and the zeros still come from the infinite product, not from the class group.

Flags. (i) Every "RH/GRH $\Leftrightarrow$ resonances on a line / decay X^{-1} " is a [RESTATEMENT] and circular as a proof strategy. (ii) Numerics with known zeros use zeros already on the line. Ghosts test consistency of the formulas, not RH. (iii) "Rational light rays", "energy", "celestial wavefunction" are exact identities with physical names ([ANALOGY] for the names). No $3+1$ dynamics is implied. (iv) " $\mathbb{H}^3$ = space of polarizations of $E \times E$ " is exact (given Thm. 4.3). "Eisenstein series = height zeta of elliptic curves" is exact but its novelty is unverified. (v) The Hermitian-modular/twistor link (F3) and the celestial link (F1) are directions, not results.

11. CORRECTIONS TO THE CHAT CLAIMS

- (C1) [t290u] "a primitive (c, d) is a projective spinor, i.e. a null direction on the celestial sphere (Theorem)": **correct, with a restriction**. Over $\mathbb{Z}$ the null direction is *real*: $[c : d] \in \mathbb{P}^1(\mathbb{Q})$ lies on the real circle of S^2 (the $(2, 1)$ slice of Lemma 71), not on the full sphere. Only complex K fills S^2 (Fig. 2).
- (C2) [t290u] "the full twistor point $\leftrightarrow$ line correspondence is only an Analogy over $\mathbb{Z}$ ": **too pessimistic for the incidence part**. The incidence relation is algebraic and holds over $K/\mathcal{O}_K$, including the class-group refinement (Prop. 7.1). What stays [ANALOGY]/[OPEN] is the Penrose transform and field content.
- (C3) [t290u] "cusps of $\mathrm{PSL}_2(\mathcal{O}_K)$ correspond to $\mathrm{Cl}(K)$ (h_K cusps)": **correct** [CITED]. *Refined*: the class of $[c : d]$ is $[(c, d)]$. Primitive integral light rays are KS squares iff principal (Thm. 4.2). For CM E , cusp classes = elliptic curves on $E \times E$ up to isomorphism (Thm. 4.3).

- (C4) [t290u] “the Bianchi Eisenstein series divides by $\zeta_K(2s)$ ”: **correct only for $h = 1$, and normalization-dependent**. In Masri’s normalization the denominator is $\zeta_K(2s)$, in ours $\zeta_K(1+s)$, in Efrat–Sarnak’s $\zeta_K(s)$. For $h > 1$ there is no single denominator: one has the matrix $Z(1+s)^{-1}$ of partial zetas, diagonalized by class-group characters with $\chi \leftrightarrow \bar{\chi}$ pairing (Thm. 5.2). The determinant involves ζ_H of the Hilbert class field, not just ζ_K .
- (C5) [t290u] “in 3D gravity, primitive (c, d) = the simple closed curve that becomes contractible”: **correct for $\mathrm{SL}_2(\mathbb{Z})$** (Lemma 72 [TAA-L]). *No Bianchi analogue claimed*: $(c, d) \in \mathcal{O}_K^2$ does not label a curve on a torus, and we know of no gravitational sum over $\Gamma_\infty \backslash \mathrm{PSL}_2(\mathcal{O}_K)$ [OPEN].
- (C6) Humbert vol = $|d|^{3/2} \zeta_K(2)/(4\pi^2)$: **confirmed again**, and now tied to the scattering residue $\sqrt{|d|}/(w \text{ vol})$ (Thm. 5.3).
- (C7) [t291u] “RH = where the resonances of an already-unitary scattering matrix sit; $|\varphi| = 1$ on the unitary line; poles at zeros”: **correct and strengthened**. Unitarity follows from the functional equation alone and holds for ghosts (Thm. 5.4). *Refined*: “fluctuation size around the anomaly term” holds only in the trivial channel. Non-trivial class-group channels are pure fluctuation. Still a [RESTATEMENT].
- (C8) [t293u], parent “MWK gravity is $2+1$: Riemann determined by Ricci, no local degrees of freedom, vacuum $\Lambda < 0$ is locally AdS_3 ; its RH-blindness says nothing directly about $3+1$ ”: **correct** (§2(a)).
- (C9) [t293u], parent “ $\mathrm{SL}_2(\mathcal{O}_K) \subset \mathrm{SL}_2(\mathbb{C})$, Lorentz group of $3+1$; $\mathbb{H}^3$ mass shell; S^2 celestial sphere; cusps = rational light-ray directions; symmetry group only, not curved spacetime or 4D EFE”: **correct**, with two refinements. “Rational” depends on the chosen lattice $\mathrm{Herm}_2(\mathcal{O}_K)$. The same $\mathbb{H}^3$ is also Euclidean AdS_3 , so the Bianchi orbifold is also a $2+1$ Einstein space (§2(c)).
- (C10) [t293u], parent “celestial holography (Strominger, Pasterski et al.) might involve arithmetic/Bianchi/Eisenstein structure”: **not found at source**. The $\mathrm{SL}_2(\mathbb{C})/\mathbb{H}^3$ machinery is confirmed [CITED]. The only zeta-related celestial paper we found is a restatement (§8).
- (C11) Task brief “CM curves with CM by $\mathcal{O}_K$ for $\mathbb{Q}(\sqrt{-5})$ ($h = 2$)”: **no such curve over $\mathbb{Q}$** . We used the $j(\sqrt{-5})$ curve over $\mathbb{Q}(\sqrt{5})$. Its lifts are still $\mathcal{O}_K$ -integral and principal, and the class group appears as residue degree (Thm. 3.1).
- (C12) Lemma 71’s direction F3 (“Eisenstein congruences, timelike/spacelike split” for Bianchi): **not done here**. The Bianchi Eisenstein *congruences* (Bianchi analogue of Lemma 70) remain open [OPEN].

12. RANKED NEXT DIRECTIONS (0.6 s + 0.4 p)

	Direction	s	p	score
F1	<i>Arithmetic celestial wavefunction:</i> Fourier-transform $E(P, s)$ (Prop. 1.2) over the mass shell to an $\mathrm{SL}_2(\mathcal{O}_K)$ -invariant Klein–Gordon solution on $\mathbb{R}^{1,3}$. Do the resonances appear as celestial dimensions $\Delta = \rho$? Is there a conformal-primary (Mellin) reading of Thm. mb?	.65	.60	.63
F2	<i>Height zeta of elliptic curves on $E \times E$:</i> prove/locate Thm. 4.3’s reading in the literature (Kani, Rosen–Shnidman). Count curves of bounded degree w.r.t. a polarization ($\sim$ residue = 1/vol); the class-group channels as twisted counts	.60	.65	.62
F3	<i>Hermitian modular group $\mathrm{SU}(2, 2; \mathcal{O}_K)$</i> (the genuine arithmetic twistor/conformal group): constant terms of Hermitian Eisenstein series (products of ζ , $L(\chi_d)$ values) and their scattering matrix on the 3 + 1 future tube	.70	.45	.60
F4	<i>Selberg-type eigenvalue bound for Bianchi groups</i> ($\lambda_1 \geq 1$, the continuous-spectrum threshold of $\mathbb{H}^3$): the one “unitarity”-type statement that is not automatic. Read Sarnak 1983 at source	.75	.35	.59
F5	Ghost in a non-genus (weight-one dihedral) channel, $d = -23$, via $\eta(z)\eta(23z)$; GRH-sensitivity of $\mathrm{Tr} \Phi(0)$ (none expected)	.40	.80	.56
F6	Bianchi Eisenstein congruences (Lemma 70 analogue): numerators of $L_K(-1, \chi)$, Bianchi cusp forms, the $h > 1$ channel structure	.55	.55	.55

Still open from earlier: Lemma 72 F1 (Benjamin–Chang real CFT, .70), Lemma 71 F1 (genus-2 KS, .70), Lemma 70 E1–E3.

13. QUESTIONS FOR J.W.

- Q1.** Do we adopt “*the class group is the obstruction to writing an integral light ray as the square of an integral spinor*” (Thm. 4.2) as the official TAA form of “primitivity $\leftrightarrow$ principalization”? Together with its geometric version: cusp classes = elliptic curves on $E \times E$.
- Q2.** Do you accept that *scattering unitarity is automatic* (it follows from the functional equation and survives every ghost), so that the TAA thesis should read “RH/GRH = location of resonances of an automatically unitary scattering matrix”, with the explicit label [RESTATEMENT]?
- Q3.** For $h > 1$ the non-trivial channels carry no anomaly term. Should the thesis say “fluctuations around the anomaly” only for the trivial channel, and “pure resonance” for class-group channels?
- Q4.** On 3 + 1 ([t293u]): is the celestial route (F1: Bianchi Eisenstein series as a sum of celestial propagators over K -rational light rays) the 3 + 1 direction you want next, or the Hermitian modular/twistor route (F3)?
- Q5.** What comes next: F1, F2, F3, F4, or an older direction (Lemma 72 F1, Lemma 71 F1, Lemma 70 E1–E3)? Any publication of Lemma 73 needs your approval.

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**TAA LEMMA 74: HOROCYCLES, HEIGHTS, AND THE HYPERBOLIC
LEGENDRE TRANSFORM.
THE MWK DISK, BUSEMANN FUNCTIONS AS HEIGHTS ON $\mathbb{P}_{\mathbb{Z}}^1$, THE
TROPICAL LIMIT OF $E(\tau, s)$, AND THE THERMODYNAMICS OF THE
HEIGHT COUNT**

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA. Nothing in this lemma is a step toward a proof that we can certify. Every RH-equivalent statement is labelled [RESTATEMENT], and every metaphor is labelled [ANALOGY] or [HEURISTIC].* Tags: [THEOREM] proved here (elementary) or classical with the proof indicated; [CITED] checked at the source (`lit/lemma74/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, content standard; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (mpmath 20–40 digits, numpy, exact integer arithmetic; finite ranges, not proofs).

Question (J.W., [t294u]–[t297u]). (A) Is the disk bounded by the contractible (primitive) cycle of a Maloney–Witten–Keller (MWK) saddle a Poincaré disk, and does “primitive $\Rightarrow$ disk $\Rightarrow$ Poisson kernel $\Rightarrow$ maximum principle” explain where the zeros come from? (B) Are the level sets of $\text{Im}(\gamma\tau)$ horocycles, is $-\log \text{Im } \gamma\tau$ a Busemann function, and is the horocycle average the “ $\oplus$ ” projection? Check the pasted AI list and its links. (C) Is a primitive $(c : d)$ a section of $\mathbb{P}_{\mathbb{Z}}^1$, is E the height zeta function of $\mathbb{P}^1(\mathbb{Q})$, and what is the Arakelov geometry of $\mathbb{P}_{\mathbb{Z}}^1$? (D) Is there a Legendre-type transform for horocycles: Helgason/Radon, horospherical support functions, the tropical limit $E^{1/s}$, and the thermodynamic Legendre transform of $\log E$ against the height count?

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Date: October 23, 2026.

ANSWER IN BRIEF

- (1) *The Eisenstein series is a height zeta function, and its summands are horocycle data* (§1) [THEOREM]. For $x = [c : d] \in \mathbb{P}^1(\mathbb{Q})$ with $\gcd(c, d) = 1$, put $H_\tau(x) = Q_\tau(c, d) = |c\tau + d|^2/y$. Then $\text{Im}(\gamma\tau) = 1/H_\tau(x)$, and

$$\log \text{Im}(\gamma\tau) = \langle \tau, \xi \rangle_i - 2h_{\text{FS}}(\xi), \quad \xi = -d/c, \quad h_{\text{FS}}(\xi) = \frac{1}{2} \log(c^2 + d^2),$$

where $\langle \tau, \xi \rangle_i = -B_{\xi,i}(\tau)$ is the horocyclic (Busemann) coordinate. So *Busemann function* = twice the *Fubini–Study height*, up to a horocyclic shift. $E(\tau, s) = \sum_x H_\tau(x)^{-s}$ is the anticanonical height zeta function of $\mathbb{P}^1(\mathbb{Q})$, with the archimedean metric given by τ (Franke–Manin–Tschinkel normalization $\zeta_{V,H}(s) = E_P^G((2s-1)\rho_P)$ checked at source [CITED]). Primitivity is exactly the statement that the finite-place local heights vanish.

- (2) *The anomaly $3/\pi$ is Peyre’s constant* (Prop. 1.3) [THEOREM]+[CITED-sec]. $3/\pi = \alpha(\mathbb{P}^1) \cdot \omega_\infty \cdot \prod_p (1-p^{-1})^{\frac{\#\mathbb{P}^1(\mathbb{F}_p)}{p}} = \frac{1}{2} \cdot \pi \cdot \frac{6}{\pi^2}$. The point count $\#\mathbb{P}^1(\mathbb{F}_p) = 1+p$ (the $\mathbb{K} \oplus \mathbb{L}$ of [t296u]) enters here, as the Euler factor $1-p^{-2}$. This is the same $1/\zeta(2)$ that primitivity produces.
- (3) *Arakelov geometry of $\mathbb{P}_{\mathbb{Z}}^1$* (§2) [THEOREM]+[NUMERICAL]. $\widehat{\mathcal{O}(1)}^2 = \frac{1}{2}$ for the Fubini–Study metric of every Q_τ (15 digits, three τ). The finite intersection of two sections is $(x \cdot y)_{\text{fin}} = \log |cd' - c'd|$, and this equals half the signed hyperbolic distance between the Ford horoballs at x and y (Penner’s λ -length). Ford circles are tangent iff the sections are disjoint (Farey neighbours). $\mathbb{P}_{\mathbb{Z}}^1$ is the analogue of the ruled surface $\mathbb{P}^1 \times C$, not of $C \times C$: genus 0, motive $\mathbb{K} \oplus \mathbb{L}$, no h^1 , and the arithmetic Hodge index theorem is vacuous here.
- (4) *The MWK disk* (§3) [THEOREM]. The meridian disk of a thermal-AdS saddle (constant-time hemisphere) or BTZ saddle (the cigar) is a totally geodesic $\mathbb{H}^2$, i.e. a Poincaré disk. “Primitive $\Leftrightarrow$ bounds an embedded disk” is a theorem. *Correction*: the Poisson kernel in the summand, $\text{Im } \gamma\tau = c^{-2}P(\tau, -d/c)$, lives on the *moduli* plane $\mathbb{H}_\tau$, not on the bulk disk. And P^s is subharmonic only for $s \geq 1$: at MWK’s weight $s = \frac{1}{2}$ it is *superharmonic*. The chain is [ANALOGY] at the link “disk $\Rightarrow$ Poisson”. The zeros-as-defects statement $\Delta \log |\xi| = 2\pi \sum \delta_\rho$ is exact but RH-blind (Jensen holds verbatim for a ghost, [NUMERICAL]).
- (5) *The $\oplus$, corrected* (§4) [THEOREM]+[NUMERICAL]. The horocycle average is $y^s + \varphi(s)y^{1-s}$ (checked from a lattice sum to $7 \cdot 10^{-11}$). This split is the archimedean Weyl-group split (incoming/reflected). *Both* pieces have the same Hecke eigenvalue $p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s}$. The $\mathbb{K} \oplus \mathbb{L}$ split lives *inside* the Hecke operator, as the Bruhat cells $\mathbb{P}^1(\mathbb{F}_p) = \{\infty\} \sqcup \mathbb{A}^1(\mathbb{F}_p)$, giving $1 + p^{k-1}$ for E_k . The limit set of $\text{SL}_2(\mathbb{Z})$ is all of $\partial\mathbb{H}$, $\delta = 1$ is the anomaly pole, and $\lambda_0 = \delta(1-\delta) = 0$ [CITED].
- (6) *The hyperbolic Legendre transform* (§5). Helgason’s transform at curvature -1 , $\hat{f}(\lambda, b) = \int f e^{(-i\lambda + \frac{1}{2})(z, b)} dz$, equals the Fourier transform in t of $e^{t/2} \times$ (horocycle Radon transform), with inversion density $\lambda \tanh(\pi\lambda)/(2\pi)$ (checked to $6 \cdot 10^{-8}$). Its Maslov dequantization is the *horospherical Legendre transform*

$$m(\tau) := \lim_{s \rightarrow \infty} \frac{1}{s} \log E(\tau, s) = \sup_{\xi \in \mathbb{P}^1(\mathbb{Q})} [\langle \tau, \xi \rangle_i - 2h_{\text{FS}}(\xi)] = \log \max_{\gamma} \text{Im } \gamma\tau$$

(Thm. 5.2). Rate: $O(1/s)$ with $s(E^{1/s}/M - 1) \rightarrow \log m$ on the tie locus ($m = 2$ on $\Gamma\text{-arc}$, $m = 3$ at ρ ; $s = 2000$: 0.69327, 1.09891). Off the tie locus the rate is exponential, $\sim r_2^s$ (fitted slope = $\log r_2$ to 10^{-6}). The corner locus is the dual Farey tree. The tropical limit is MW10’s “tessellation of the upper half plane into phases”.

- (7) *Thermodynamics of the height count* (§6) [THEOREM]+[NUMERICAL]. With $\Theta_\tau(T) = \sum_x e^{-H_\tau(x)/T}$ we prove

$$\Theta_\tau(T) = \frac{3}{\pi}T + \sum_{n \geq 0} \frac{(-1)^n E(\tau, -n)}{n! T^n} + 2 \text{Re} \sum_{\rho} \frac{\Gamma(\rho/2) Z_{Q_\tau}(\rho/2)}{4 \zeta'(\rho)} T^{\rho/2}$$

(simple zeros assumed), with *no Maass cusp forms*. That is an incomplete Eisenstein series, which corrects the brief. It is checked against an independent Möbius–theta evaluation for $\tau = i, i\sqrt{2}, \rho$ and $T \leq 10^{10}$: $2 \cdot 10^{-2}$ residual without zeros, 10^{-15} (the float floor) with 20. Sharp count: $N_\tau(T) \sim \frac{3}{\pi}T$ (exact integers to $T \approx 10^{13}$; N/T matches $3/\pi$ to $3 \cdot 10^{-10}$). Its error has $\text{RMS} \asymp T^{0.27}$ and is dominated by RH-blind lattice-point noise. A surgery ghost at γ_1 with $\delta \leq 0.1$ stays masked up to 10^{13} ; $\delta = 0.2$ becomes visible near $10^{9.5}$. Legendre: $S_L(\mathcal{E}) - \log N(e^\mathcal{E}) = 1 + \log(\mathcal{E} + c_1)$ with $c_1 \rightarrow \pi C_0/3$. The Kronecker constant C_0 , and with it the L69 anomaly $\partial_\tau C_0 = -\frac{i}{2}E_2^*$ (10 digits), enters only the canonical ensemble.

- (8) *Verdict on the polarization J* [THEOREM]+[HEURISTIC]. The exponents $\rho/2$ are τ -independent, and J_τ enters only the amplitudes $Z_{Q_\tau}(\rho/2)$ and $C_0(\tau)$. The horocycle/height/Legendre picture is therefore *RH-blind for J* . Its gain is structural: one dictionary (Busemann = height, Ford tangency = disjoint sections, λ -length = arithmetic intersection, tropical = dominant saddle, $3/\pi$ = Peyre). It gives no new handle on the zeros.

1. HEIGHTS ON $\mathbb{P}^1(\mathbb{Q})$, HOROCYCLES AND BUSEMANN FUNCTIONS

Conventions. $\tau = x + iy \in \mathbb{H}$. $\Gamma = \text{SL}_2(\mathbb{Z})$, Γ_∞ the stabilizer of ∞ . For $(c, d) \in \mathbb{Z}^2$ put $Q_\tau(c, d) = |c\tau + d|^2/y = (c, d)M_\tau(c, d)^\top$, where $M_\tau = \frac{1}{y} \begin{pmatrix} |\tau|^2 & x \\ x & 1 \end{pmatrix}$ and $\det M_\tau = 1$. $P(\tau, \xi) = y/|\tau - \xi|^2$ is the Poisson kernel of $\mathbb{H}$ at $\xi \in \mathbb{R}$, and $P(\tau, \infty) = y$. We use

$$E(\tau, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \text{Im}(\gamma\tau)^s = \frac{1}{2} \sum_{\text{gcd}(c,d)=1} Q_\tau(c, d)^{-s}, \quad Z_Q(s) = \sum_{v \neq 0} Q(v)^{-s} = 2\zeta(2s)E(\tau, s).$$

$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$, and $\varphi(s) = \xi(2s-1)/\xi(2s)$, so $\text{Res}_{s=1} E = 3/\pi$ (Lemma 69 [TAA-L]). Scripts: `num/lemma74_*.py`. Output: `num/174/*.json`.

Definition 1.1 (height). For $x = [c : d] \in \mathbb{P}^1(\mathbb{Q})$ with $\text{gcd}(c, d) = 1$ put $H_\tau(x) = Q_\tau(c, d)$ and $h_{\text{FS}}(x) = \frac{1}{2} \log(c^2 + d^2)$. Then $H_i(x) = e^{2h_{\text{FS}}(x)}$. Primitive coordinates are unique up to sign, so H_τ is well defined. It is the exponential height attached to $\mathcal{O}(2) = \omega_{\mathbb{P}^1}^{-1}$ with the trivial (max) metric at every finite place and the metric M_τ at ∞ (§2).

Proposition 1.2 (horocycles and Busemann functions [THEOREM]+[NUMERICAL]). *Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$, and $\xi = \gamma^{-1}\infty = -d/c$.*

- (i) $\text{Im}(\gamma\tau) = c^{-2}P(\tau, \xi) = 1/H_\tau([c : d])$.
- (ii) *The level set $\{\text{Im} \gamma\tau = h\}$ is the horocycle at ξ , i.e. the circle tangent to $\mathbb{R}$ at ξ with Euclidean diameter $1/(c^2h)$. The Ford circle at p/q is $\{H_\tau([q : -p]) = 1\}$ read as a set of τ : the height-one horoball. The horoball $\{\text{Im} \gamma\tau \geq 1\}$ is the Γ -image of $\{y \geq 1\}$.*
- (iii) *Put $\langle \tau, \xi \rangle_i := \log \frac{P(\tau, \xi)}{P(i, \xi)} = -B_{\xi, i}(\tau)$, where $B_{\xi, i}(\tau) = \lim_{t \rightarrow \infty} [d(\tau, \sigma(t)) - t]$ for the unit-speed geodesic σ from i to ξ . Then*

$$(1) \quad \log \text{Im}(\gamma\tau) = \langle \tau, \xi \rangle_i - 2h_{\text{FS}}(\xi), \quad E(\tau, s) = \sum_{\xi \in \mathbb{P}^1(\mathbb{Q})} e^{-s(B_{\xi, i}(\tau) + 2h_{\text{FS}}(\xi))}.$$

Proof. (i) $|c\tau + d|^2 = c^2|\tau - \xi|^2$. (ii) $y/|\tau - \xi|^2 = c^2h$ is the circle through ξ of diameter $1/(c^2h)$. For $(c, d) = (q, -p)$, i.e. $\xi = p/q$, the diameter at $h = 1$ is $1/q^2$, which is Ford's circle. (iii) $P(i, \xi) = 1/(1 + \xi^2)$, so $\log P(\tau, \xi) - 2 \log |c| = \langle \tau, \xi \rangle_i - \log(c^2(1 + \xi^2)) = \langle \tau, \xi \rangle_i - \log(c^2 + d^2)$. The Busemann identity $\log(P(\tau, \xi)/P(i, \xi)) = -B_{\xi, i}$ is standard in curvature -1 ([6], Def. 1.1, defines horoballs as Busemann sublevel sets [CITED]). Numerically [NUMERICAL] (`horo.json`, (a)): $-B$ from the geodesic limit ($t = 25$) equals $\log(P/P(i))$ to 15 digits for three (τ, c, d) , and the two sides of (1) agree to 15 digits. $\square$

Reading [THEOREM]. The Busemann function of the cusp ξ , normalized at i , differs from $-\log \text{Im} \gamma\tau$ by exactly $2h_{\text{FS}}(\xi)$. So *the height is the horocyclic offset*. The height- T horoball at ξ is $\{\tau : \langle \tau, \xi \rangle_i \geq 2h_{\text{FS}}(\xi) + \log T\}$, and arithmetic complexity of a cusp becomes the depth of its horoball (Fig. 1, left).

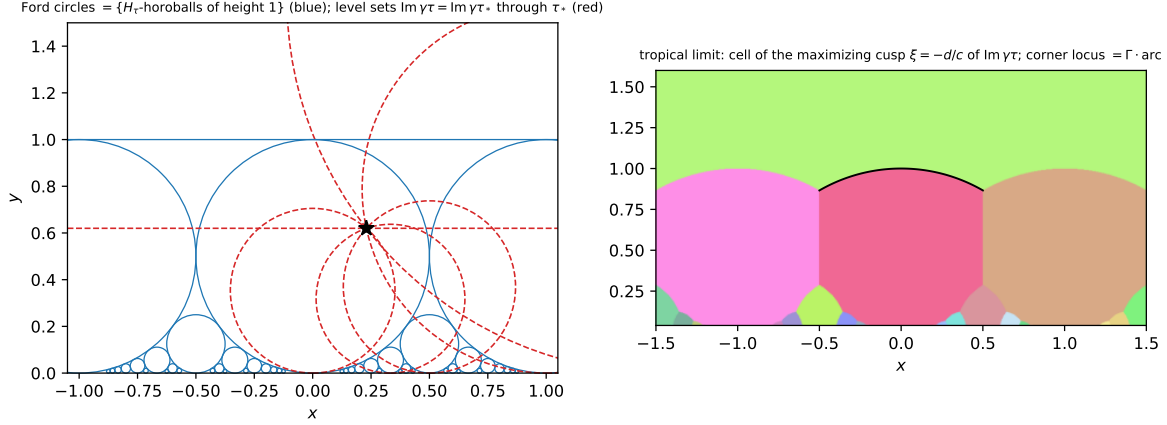


FIGURE 1. Left: horocycles at cusps. In blue, Ford circles = height-one horoballs $\{H(x) \leq 1\}$; their tangencies are Farey neighbours, i.e. disjoint sections of $\mathbb{P}_{\mathbb{Z}}^1$ (Prop. 2.1). In red, the level sets $\text{Im } \gamma\tau = \text{Im } \gamma\tau_*$ through $\tau_* = 0.23 + 0.62i$ for six cusps $-d/c$. Right: the tropical limit (Thm. 5.2). Each colour is the cell where a given cusp maximizes $\text{Im } \gamma\tau$ (the dominant MWK saddle). Cell walls are the corner locus $\Gamma \cdot \{|\tau| = 1\}$ (black: the arc), i.e. the dual Farey tree. (num/lemma74_figs.py)

Proposition 1.3 (the height zeta function and Peyre’s constant [THEOREM]+[CITED]+[CITED-sec]). (i) $\mathbb{P}_{\mathbb{Z}}^1(\mathbb{Z}) = \mathbb{P}^1(\mathbb{Q})$: a point $[c : d]$ with $\gcd(c, d) = 1$ is the section $\text{Spec } \mathbb{Z} \rightarrow \mathbb{P}_{\mathbb{Z}}^1$. Its local height at p is $\log \max(|c|_p, |d|_p) = 0$. Hence the global height has only the archimedean term, and

$$E(\tau, s) = \zeta_{\mathbb{P}^1, H_\tau}(s) = \sum_{x \in \mathbb{P}^1(\mathbb{Q})} H_\tau(x)^{-s}$$

is the anticanonical height zeta function. This is the case $G = \text{SL}_2$, $P = B$ of Franke–Manin–Tschinkel, $\zeta_{V, H}(s) = E_P^G((2s - 1)\rho_P, e)$, H relative to ω_V^{-1} ([10], §2(1) [CITED]). Since $\rho_B = \alpha/2$, $E_B((2s - 1)\rho_B)$ is the classical $E(\cdot, s)$.

(ii) $N_\tau(T) := \#\{x : H_\tau(x) \leq T\} \sim \frac{3}{\pi}T$, independently of τ , and

$$\frac{3}{\pi} = \alpha(\mathbb{P}^1) \cdot \tau_H(\mathbb{P}^1), \quad \alpha(\mathbb{P}^1) = \frac{1}{2}, \quad \tau_H = \omega_\infty \prod_p (1 - p^{-1}) \frac{\#\mathbb{P}^1(\mathbb{F}_p)}{p} = \pi \cdot \prod_p (1 - p^{-2}) = \frac{6}{\pi},$$

with $\omega_\infty = \int_{\mathbb{R}} \frac{dt}{Q_\tau(t, 1)} = \pi$ for every τ (Peyre’s constant, [4], §§1.2–1.4 [CITED-sec]).

Proof. (i) is the valuative criterion plus $\gcd = 1$. (ii) Directly: $\{v \in \mathbb{R}^2 : Q_\tau(v) \leq T\}$ is an ellipse of area $\pi T / \sqrt{\det M_\tau} = \pi T$. Primitive vectors have density $6/\pi^2$, and $\pm v$ give the same x , so $N \sim \frac{1}{2} \cdot \frac{6}{\pi^2} \cdot \pi T$. For Peyre’s form: $\omega_\infty = \int dt y / |t\tau + 1|^2 = \pi$ by residues; the factor at p is $(1 - p^{-1})(p + 1)/p = 1 - p^{-2}$; and $\alpha(\mathbb{P}^1) = 1/2$ because $-K = 2H$. $\square$

Reading [THEOREM]. The two places where “ $1/\zeta(2)$ ” appears in TAA coincide. In Lemma 72, primitivity (Möbius) gives $E = Z_Q/(2\zeta(2s))$. In Peyre’s constant it is the product of the local densities $\#\mathbb{P}^1(\mathbb{F}_p)/p \cdot (1 - p^{-1})$. The point count $1 + p = \#\mathbb{P}^1(\mathbb{F}_p)$, i.e. the motive $\mathcal{K} \oplus \mathbb{L}$, is what turns $1 - p^{-1}$ into $1 - p^{-2}$. The anomaly $3/\pi$ (Lemma 69) is therefore $\frac{1}{2} \times \pi \times \frac{6}{\pi^2}$: the effective-cone constant, times the archimedean volume of $\mathbb{P}^1(\mathbb{R})$, times the finite Tamagawa factor.

2. ARAKELOV GEOMETRY OF $\mathbb{P}_{\mathbb{Z}}^1$

Proposition 2.1 (self-intersection, pole plane, Ford tangency [THEOREM]+[NUMERICAL]). *Let $\widehat{\mathcal{O}(1)}_{\tau}$ be $\mathcal{O}(1)$ on $\mathbb{P}_{\mathbb{Z}}^1$ with the Fubini–Study metric of the Hermitian extension of M_{τ} , $\|X_j\|^2(v) = |v_j|^2/v^*M_{\tau}v$.*

- (i) $\widehat{\mathcal{O}(1)}_{\tau}^2 = \frac{1}{2}$ for every τ . Numerically $h(\text{div} X_1) + \int -\log \|X_0\| c_1 = 0.5$ to 15 digits for $\tau = i, 0.3 + 1.7i, -0.45 + 0.6i$ (`horo.json(b)`, and $\int c_1 = 1$). At $\tau = i$ this is the BGS/Tamvakis value $h(\mathbb{P}^n) = \frac{1}{2} \sum_{k \leq n} H_k$ at $n = 1$ ([11], citing [14, §3.3.1] [CITED-sec]).
- (ii) With $\widehat{\mathcal{O}(c)}$ the trivial bundle with constant metric e^{-c} , the classes $\{\widehat{\mathcal{O}(1)}, \widehat{\mathcal{O}(1)}\}$ have Gram matrix $\begin{pmatrix} 1/2 & 1 \\ 1 & 0 \end{pmatrix}$, which has signature $(1, 1)$: a hyperbolic plane. Classes of generic degree 0 are multiples of $\widehat{\mathcal{O}(1)}$, isotropic. So the arithmetic Hodge index theorem (Faltings–Hriljac) holds trivially. It has no content in genus 0 [BIB].
- (iii) For sections $x = [c : d]$, $y = [c' : d']$, $(x \cdot y)_{\text{fin}} = \sum_p v_p (cd' - c'd) \log p = \log |cd' - c'd|$. Let $\delta(x, y)$ be the signed hyperbolic distance between the Ford horoballs at x, y (negative if they overlap). Then

$$(x \cdot y)_{\text{fin}} = \frac{1}{2} \delta(x, y),$$

and the Ford circles are tangent iff x, y are disjoint sections iff x, y are Farey neighbours.

Proof. (i) Use $\hat{c}_1(\hat{L})^2 = h_{\hat{L}}(\text{div } s) + \int_{\mathbb{P}^1(\mathbb{C})} -\log \|s\| c_1(\hat{L})$ [14] with $s = X_0$, whose divisor is the section $[0 : 1]$. Its height, via the section X_1 , is $\frac{1}{2} \log(M_{\tau})_{22} = -\frac{1}{2} \log y$. The integral is computed in `lemma74_horo.py`. Analytically, $M_{\tau} = g^{\top} g$ with $g \in \text{SL}_2(\mathbb{R})$, and the change of metric by g changes the integral by $+\frac{1}{2} \log y$, cancelling the finite part. (ii) $\widehat{\mathcal{O}(c)} \cdot \hat{L} = c \deg L_{\mathbb{Q}}$, and $\widehat{\mathcal{O}(c)}^2 = 0$. (iii) Two sections meet over p iff $p \mid cd' - c'd$, transversally with multiplicity v_p . For the horoballs, send $x_1 = p/q$ to ∞ by $z \mapsto -1/(z - x_1)$. A horocycle at x_2 of diameter D_2 goes to one of diameter $D_2/(x_2 - x_1)^2$ (from $\text{Im } g(z)/|g(z) - g(x_2)|^2 = (x_2 - x_1)^2 \text{Im } z/|z - x_2|^2$). The Ford horoball at x_1 goes to $\{y \geq 1/D_1\}$. Hence $\delta = \log \frac{(x_2 - x_1)^2}{D_1 D_2}$. With $D = 1/q^2$ this gives $\delta = 2 \log |pq' - p'q|$. This is Penner’s λ -length $e^{\delta/2}$ [BIB]. $\square$

$\mathbb{P}_{\mathbb{Z}}^1$ versus $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ ([THEOREM], correcting [t296u]). $\mathbb{P}_{\mathbb{Z}}^1 = \mathbb{P}^1 \times \text{Spec } \mathbb{Z}$ is the arithmetic analogue of a ruled surface $\mathbb{P}^1 \times C$, not of $C \times C$. Its fibres have genus 0, its motive is $\mathcal{K} \oplus \mathbb{L}$, and $\zeta_{\mathbb{P}_{\mathbb{Z}}^1}(s) = \zeta(s)\zeta(s-1)$ has no h^1 term. Weil’s proof for curves over $\mathbb{F}_q$ uses the Hodge index theorem on $C \times C$ applied to graphs of Frobenius. On $\mathbb{P}_{\mathbb{Z}}^1$ there is no h^1 to bound, and the zeros of ζ seen by E enter through the Möbius inversion $1/\zeta(2s)$ over all sections, not through any cohomology of $\mathbb{P}_{\mathbb{Z}}^1$ [HEURISTIC]. The $\mathbb{F}_1$ picture “ $q + 1 \rightarrow 2$ ”, $\mathbb{P}^1(\mathbb{F}_1) = \{0, \infty\}$, is [ANALOGY]. Its arithmetic content is the Euler factor of Prop. 1.3(ii).

Busemann = energy of an integral light ray ([THEOREM]; the $2 + 1$ form of Lemma 73’s PSS energy). $Q_{\tau}(c, d) = \text{Tr}(M_{\tau} N)$ with $N = (c, d)^{\top}(c, d)$, $\det N = 0$. Here M_{τ} is a unit timelike vector of $(\text{Sym}_2(\mathbb{R}), \det) \cong \mathbb{R}^{1,2}$ and N is an integral null vector. So $H_{\tau}(x)$ is the energy of the light ray N_x in the rest frame M_{τ} , and E sums (energy) $^{-s}$ over primitive integral light rays. The tropical limit (§5) selects the lowest-energy ray.

3. (A) THE MWK DISK, POISSON KERNELS AND THE MAXIMUM PRINCIPLE

Set-up (MW10 via Lemma 72 [TAA-L]). Thermal AdS_3 with boundary modulus τ is $\mathbb{H}^3 / \langle (w, t) \mapsto (qw, |q|t) \rangle$ with $q = e^{2\pi i \tau}$, in upper half-space coordinates (w, t) . The saddle $M_{c,d}$ is the solid torus in which the cycle (c, d) of the boundary torus becomes contractible. Its action has $I_{\gamma} \propto 2\pi i(\gamma\tau - \gamma\bar{\tau})$, so the dominant saddle maximizes $\text{Im } \gamma\tau$ ([9], (4.11) $I_{\gamma} = 2\pi i(\gamma\tau - \gamma\bar{\tau})$, and “the upper half plane is tessellated in phases dominated by different classical solutions” [CITED]). The pure-gravity partition function is a Poincaré series over relatively prime (c, d) , i.e. over $\Gamma_{\infty} \backslash \Gamma$. Its basic building block is

$E(\tau; \kappa, \mu) = \sqrt{\text{Im } \tau} \sum_{c,d} |c\tau + d|^{-1} \exp\{2\pi\kappa \text{Im } \gamma\tau + 2\pi i\mu \text{Re } \gamma\tau\}$, which for $\kappa = \mu = 0$ is “a non-holomorphic Eisenstein series of weight $1/2$ ”, i.e. $\sum \text{Im}(\gamma\tau)^{1/2}$ ([9], (3.4)–(3.7) [CITED]). MW10 note that $\Delta(y^{1/2}) = \frac{1}{4}y^{1/2}$ in their sign convention, which is $s(s-1) = -\frac{1}{4}$ in ours (Lemma 72’s Theorem mb uses the s -deformation).

Proposition 3.1 (the meridian disk is a Poincaré disk [THEOREM]). (i) A nonzero class $(c, d) \in H_1(T^2, \mathbb{Z})$ is represented by a simple closed curve iff $\gcd(c, d) = 1$. A class bounds an embedded disk in a solid torus iff it is $\pm$ the meridian, which is primitive.

(ii) In thermal AdS the hemisphere $S = \{|w|^2 + t^2 = 1\}$ is a totally geodesic $\mathbb{H}^2 \subset \mathbb{H}^3$ whose ideal boundary is the circle $|w| = 1$, the meridian. It embeds in the quotient (it lies in the fundamental shell $1 \leq |w|^2 + t^2 < |q|^{-2}$) as a constant-global-time slice. Intrinsically it is a full Poincaré disk.

(iii) For $M_{c,d} = \gamma \cdot (\text{thermal AdS})$ the meridian disk is the image of S , again a totally geodesic Poincaré disk. For BTZ $((c, d) = (1, 0))$ it is the Euclidean cigar $\{\phi = \text{const}\}$ in Schwarzschild coordinates.

Proof. (i) A simple closed essential curve on T^2 has primitive class (classification of curves on the torus). If $k(c, d)$ with $k > 1$ were simple it would be isotopic to k times a simple curve, which is impossible. Dehn’s lemma identifies compressing disks of the solid torus with the meridian. (ii) Hemispheres orthogonal to $\{t = 0\}$ are totally geodesic, being fixed sets of inversions in $\mathbb{H}^3$, a local property. The deck transformation moves S to the disjoint hemisphere of radius $|q|$, so S embeds. (iii) is (ii) transported by γ . $\square$

Proposition 3.2 (where the Poisson kernel lives, and with which sign [THEOREM]). (i) $\text{Im}(\gamma\tau)^s = c^{-2s}P(\tau, -d/c)^s$ is a power of the Poisson kernel of the moduli plane $\mathbb{H}_\tau$ (the Teichmüller space of the boundary torus), at the cusp $-d/c$. It is the Helgason plane wave $e^{s(\tau, \xi)}$ (§5). This kernel exists at every cusp. It is the same for a non-primitive multiple $k(c, d)$, which contributes $k^{-2s}c^{-2s}P^s$. The bulk meridian disk of Prop. 3.1 has its own Poisson kernel, on a different hyperbolic plane, and it does not appear in the summand.

(ii) $\Delta_{\mathbb{H}}P^s = s(s-1)P^s$ with $\Delta_{\mathbb{H}} = y^2(\partial_x^2 + \partial_y^2)$. So P^s is subharmonic ($\Delta \geq 0$) for real $s \geq 1$ or $s \leq 0$, and superharmonic for $0 < s < 1$. On bounded domains, $E(\cdot, s)$ for $s > 1$ obeys the maximum principle (sum of positive subharmonic functions). At MWK’s weight $s = \frac{1}{2}$ every summand is superharmonic and only the minimum principle holds.

(iii) (Poisson–Jensen, Poincaré–Lelong.) For entire f and a disk D , $\log |f(z)| = \int_{\partial D} \log |f| d\omega_z - \sum_{\rho \in D} G_D(z, \rho)$, and $\Delta \log |f| = 2\pi \sum_{\rho} \delta_{\rho}$. This holds for every entire f , in particular for ξ and for a ghost $\xi \cdot Q$.

Proof. (i) is Prop. 1.2(i). (ii) is a direct computation, with $\Delta y^s = s(s-1)y^s$ and Γ -invariance. (iii) is classical. Numerically [NUMERICAL] (horo.json(f)): on the disk $|s - (\frac{1}{2} + 20i)| < 9$, the circle mean of $\log |\Xi|$ minus the Jensen sum equals $\log |\Xi(\text{centre})|$ to 14 digits for the Riemann Ξ (three zeros inside). It does the same for the ghost $\Xi \cdot Q_{0,2}$ (the pair ρ_1, ρ_2 replaced by the quartet $\frac{1}{2} \pm 0.2 + i\gamma_1$). $\square$

Verdict on [t294u] [THEOREM] at each link, [ANALOGY] for the chain. (1) MWK disk = Poincaré disk: *correct*, for both thermal AdS (constant-time hemisphere) and BTZ (cigar). (2) Primitive $\Leftrightarrow$ disk: *correct* (Prop. 3.1(i)). (3) Disk $\Rightarrow$ Poisson/maximum principle: *metaphor-as-math*. The Poisson kernel in E is the $\mathbb{H}_\tau$ kernel at a cusp. Its existence does not use primitivity, and its sign at $s = \frac{1}{2}$ is the wrong one for a maximum principle. (4) Minimum modulus $\Leftrightarrow \Delta \log |\xi| = 2\pi \sum \delta_{\rho}$: *correct and RH-blind*. Zeros are exactly the defects of harmonicity of $\log |\xi|$ wherever they sit, on the line or off it. (5) “The maximum principle and primitivity are not restatements of each other”: *correct*. The maximum principle holds equally for the non-primitive sum $Z_Q = 2\zeta(2s)E$. Primitivity is what produces the factor $1/\zeta(2s)$, and with it the poles at $\rho/2$ (Lemma 72 [TAA-L]). There are *three* planes in play: the bulk meridian disk ($\mathbb{H}^2 \subset \mathbb{H}^3$), the moduli plane $\mathbb{H}_\tau$ (cusps, horocycles, Ford circles, heights), and the spectral s -plane (zeros, not hyperbolic).

4. (B) THE HOROCYCLE AVERAGE, HECKE OPERATORS AND THE LIMIT SET

Proposition 4.1 (constant term = horocycle average [THEOREM]+[NUMERICAL]). $\int_0^1 E(x + iy, s) dx = y^s + \varphi(s)y^{1-s}$, $\varphi(s) = \xi(2s-1)/\xi(2s)$. In Langlands' form this is $\sum_{w \in W} M(w, s) y^{w \cdot s}$ with $W = \{1, w\}$ the Weyl group of SL_2 , $M(1) = 1$, and $M(w) = \varphi(s) = \frac{\xi_\infty(2s-1)}{\xi_\infty(2s)} \prod_p \frac{1-p^{-2s}}{1-p^{1-2s}}$ [BIB].

Numerics [NUMERICAL] (horo.json(d), Fig. 2). Average the lattice sum $E = \sum_x H^{-s}$ over 64 (or 32) equally spaced points of the horocycle $\mathrm{Im} \tau = y$. This is exact for trigonometric polynomials of degree < 64 ; the nonconstant terms are $O(e^{-2\pi \cdot 64y})$. Truncate at $H \leq Q_{\max}$ with tail $\frac{3}{\pi} Q_{\max}^{1-s}/(s-1)$. For $s = 2, 3$ and $y \in [0.2, 2.5]$ the result agrees with $y^s + \varphi(s)y^{1-s}$ to relative $6.9 \cdot 10^{-11}$ ($s = 2$) and $6.6 \cdot 10^{-14}$ ($s = 3$). For $y < 1$ the reflected piece φy^{1-s} dominates.

Proposition 4.2 (the two $\oplus$'s [THEOREM]). (i) With $T_p f(z) = p^{-1/2}[f(pz) + \sum_{j=0}^{p-1} f(\frac{z+j}{p})]$, both y^s and y^{1-s} are eigenfunctions of T_p with the same eigenvalue $p^{s-\frac{1}{2}} + p^{\frac{1}{2}-s}$. So the constant-term split $y^s \oplus \varphi y^{1-s}$ is not a Hecke-eigenvalue split. (ii) The $p+1$ cosets of T_p are indexed by $\mathbb{P}^1(\mathbb{F}_p) = \{\infty\} \sqcup \mathbb{A}^1(\mathbb{F}_p)$: the coset $z \mapsto pz$ is the point ∞ , and the p cosets $z \mapsto (z+j)/p$ are the affine line. On y^s the two cells contribute $p^{-1/2}p^s$ and $p^{-1/2} \cdot p \cdot p^{-s}$. For holomorphic E_k , with $T_p f = p^{k-1}f(pz) + p^{-1} \sum_j f((z+j)/p)$, they contribute p^{k-1} and $p \cdot p^{-1} = 1$ on the constant term $\zeta(1-k)/2$. This gives the eigenvalue $1 + p^{k-1} = \sigma_{k-1}(p)$, the point count of $\mathbb{P}^1$ twisted by weight. For holomorphic E_k the partner y^{1-k} in the constant term vanishes. (iii) Both splits are indexed by W . The archimedean one is W acting on the exponent ($y^s \leftrightarrow y^{1-s}$, the intertwining operator φ). The p -adic one is the Bruhat decomposition $G(\mathbb{F}_p) = B \sqcup BwB$, i.e. $\mathbb{P}^1(\mathbb{F}_p) = \{\infty\} \sqcup \mathbb{A}^1$, inside T_p .

Proof. (i) $T_p y^s = p^{-1/2}(p^s + p \cdot p^{-s})y^s$, and $s \mapsto 1-s$ leaves the eigenvalue invariant. (ii) The same computation, cell by cell. (iii) is the identification of cosets with $\mathbb{P}^1(\mathbb{F}_p)$. $\square$

Verdict on [t295u] “the horocycle average is the $\oplus$ projection”: *half right*. The horocycle average is a projection (onto Γ_∞ -invariants, i.e. the constant term), and its image is 2-dimensional, spanned by y^s and y^{1-s} . But this $\oplus$ is the Weyl-group (incoming $\oplus$ reflected) split. The $\mathbb{K} \oplus \mathbb{L}$ split (E_k eigenvalue $1 + p^{k-1}$) is a different $\oplus$: the cell decomposition of $\mathbb{P}^1(\mathbb{F}_p)$ inside the Hecke operator. The zeros sit in φ , i.e. in the *reflected* channel (φ has poles at $\rho/2$), and not in either cell.

Proposition 4.3 (limit set and critical exponent [CITED]+[THEOREM]). $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ is a lattice, hence of the first kind: $\Lambda(\Gamma) = \partial\mathbb{H}$, and the convex core is all of $\Gamma \backslash \mathbb{H}$ (no funnels) ([1], first kind $\Leftrightarrow \Lambda = \partial\mathbb{H} \Leftrightarrow C(X) = X$ [CITED]). The Poincaré series $\sum_{\Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma\tau)^s = E(\tau, s)$ converges for $s > 1$ and has a pole at $s = 1$ with residue $3/\pi$. So the critical exponent is $\delta = 1$, the Patterson–Sullivan measure is Lebesgue (harmonic) measure, and the bottom of the spectrum is $\lambda_0 = \delta(1-\delta) = 0$ (the constant function) [BIB]. “ $\delta = 1$ at the anomaly pole” is exact: the anomaly residue is the Patterson–Sullivan normalization $1/\mathrm{vol}$.

The pasted AI list ([t295u]) at source [CITED]. All four links exist and were downloaded (lit/lemma74/). (a) arXiv:2607.27120 (Dai–Ennis–Nguyen–Wei, Aug. 2026) is about first Dirichlet eigenfunctions on horoconvex domains in $\mathbb{H}^n$. Its Def. 1.1 defines horoballs/horospheres as Busemann sub-/level sets, which *supports* the “Busemann level sets” item. (b) The note.com page (Japanese) summarizes Thurston, *Three-dimensional geometry and topology*, Ch. 2 (models, sphere at infinity). It does *not* use “absolute set” or “directrix”; it mentions only the absolute unit of length (*zettai-teki na nagasa no tan'i*). (c) Waterloo PMATH 321 Ch. 4 (Poincaré disc) defines horocycles (4.30) and horolations (4.39). No “absolute”. (d) arXiv:1508.02280 (Basmajian–Šarić) treats convex cores and funnels and proves first kind $\Leftrightarrow \Lambda = \partial\mathbb{H} \Leftrightarrow C(X) = X$. *Verdict*: the list's mathematics is standard and correct. Its terminology “absolute set/directrix” is not supported by the linked sources (the Cayley–Klein “absolute” is the boundary conic, which is standard but not what these links say).

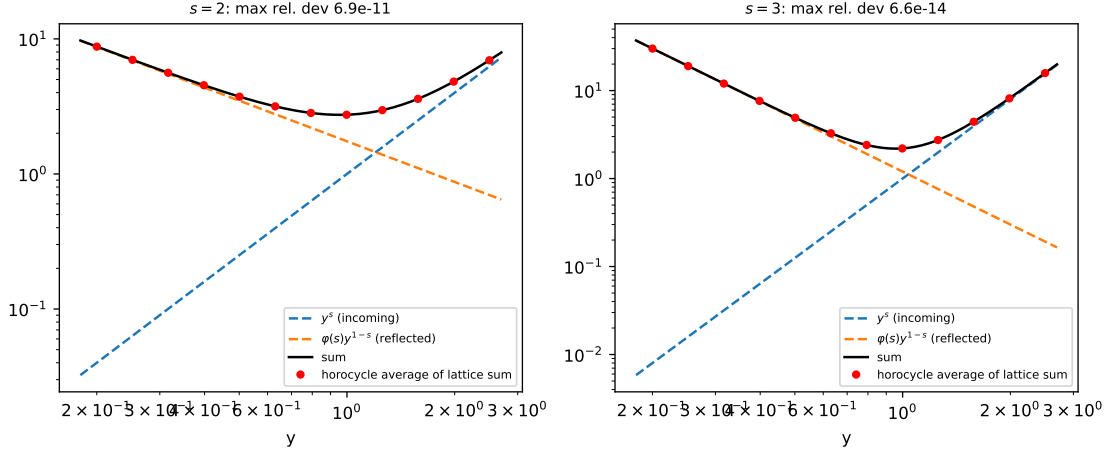


FIGURE 2. The constant-term projection. Red dots: horocycle averages of the lattice sum $\sum_x H_\tau(x)^{-s}$ at height y . Curves: the incoming piece y^s , the reflected piece $\varphi(s)y^{1-s}$, and their sum. The maximal relative deviation is in the title.

5. (D I–III) LEGENDRE TRANSFORMS FOR HOROCYCLES

5.1. (D i) Helgason’s Fourier transform and the horocycle Radon transform. For z in the disk model and $b \in \partial\mathbb{D}$ let $\langle z, b \rangle$ be the signed distance from 0 to the horocycle through z at b . In curvature -1 , $e^{\langle z, b \rangle} = P(z, b) = \frac{1-|z|^2}{|z-b|^2}$.

Proposition 5.1 (Helgason–Radon at curvature -1 [BIB]+[NUMERICAL]). (i) $\tilde{f}(\lambda, b) = \int_{\mathbb{H}} f(z) e^{(-i\lambda + \frac{1}{2})\langle z, b \rangle} dz$ and $f(z) = \frac{1}{4\pi} \int_{\mathbb{R}} \int_{\partial\mathbb{D}} \tilde{f}(\lambda, b) e^{(i\lambda + \frac{1}{2})\langle z, b \rangle} \lambda \tanh(\pi\lambda) d\lambda db$ (with db of total mass 1).

(ii) For radial f : $f(0) = \frac{1}{2\pi} \int_0^\infty \hat{f}(\lambda) \lambda \tanh(\pi\lambda) d\lambda$, where $\hat{f}(\lambda) = 2\pi \int_0^\infty f(r) \varphi_\lambda(r) \sinh r dr$ and $\varphi_\lambda(z) = \int_{\partial\mathbb{D}} e^{(i\lambda + \frac{1}{2})\langle z, b \rangle} db = {}_2F_1(\frac{1}{2} + i\lambda, \frac{1}{2} - i\lambda; 1; -\sinh^2 \frac{r}{2})$.

(iii) (Fourier slice.) In horocyclic coordinates (t, u) at b ($t = \langle z, b \rangle$, $u = \text{arclength along the horocycle}$) one has $dz = dt du$, hence

$$\tilde{f}(\lambda, b) = \int_{\mathbb{R}} e^{(-i\lambda + \frac{1}{2})t} (\mathcal{R}f)(b, t) dt, \quad (\mathcal{R}f)(b, t) = \int_{\langle z, b \rangle = t} f du.$$

Status. (i) is Helgason’s theorem transcribed to curvature -1 . The curvature -4 version, $\tilde{f} = \int f e^{(-i\lambda + 1)\langle z, b \rangle}$ with density $\lambda \tanh(\pi\lambda/2)/(4\pi)$, is quoted from Helgason in [5] [CITED-sec]. Helgason’s book itself was not opened. We therefore checked the transcription numerically [NUMERICAL] (horo.json(c)). The boundary average of $P^{1/2+i\lambda}$ equals the hypergeometric φ_λ to 12 digits ($\lambda = 0.7$, $|z| = 0.45$: 0.839606588752). For $f = \cosh^{-4} r$, the inversion integral (ii) returns 0.99999994 against $f(0) = 1$ (quadrature error). (iii): at $b = \infty$, $z = x + iy$, $t = \log y$, $du = dx/y$, $dz = dx dy/y^2 = du dt$. $\square$

Reading. $\tilde{f}$ is the Euclidean Fourier transform in t of $e^{t/2} \mathcal{R}f$. The Eisenstein summand $P(\cdot, \xi)^s$ is the plane wave $e^{s\langle \cdot, \xi \rangle}$ with $s = \frac{1}{2} + i\lambda$, so E is a Γ -periodized plane wave summed over rational directions with weights $e^{-2sh_{\text{FS}}(\xi)}$ (Prop. 1.2).

5.2. (D ii) Horospherical support functions. For an h-convex compact $\Omega \subset \mathbb{H}^{n+1}$ (hyperboloid model) the horospherical support function is $u(x) = \inf\{s : \Omega \subset B_x(s)\} = \sup\{\log(-\langle X, (x, 1) \rangle) : X \in \Omega\}$, $x \in S^n$, and $\Omega = \bigcap_x B_x(u(x))$ ([3], (2.1) [CITED]; also [2, 7] [CITED]). The idempotent Fourier transform is the Legendre transform, $\tilde{\varphi}(\xi) = \sup_x (\xi \cdot x + \varphi(x))$ ([8], §7 (9) [CITED]). *Definition*

(ours). The *horospherical Legendre transform* of a function h on a set of boundary points $\Xi \subset \partial\mathbb{H}$ is

$$h^\natural(\tau) := \sup_{\xi \in \Xi} [\langle \tau, \xi \rangle_i - h(\xi)].$$

It is the Legendre transform with the Euclidean pairing replaced by the Busemann pairing. It is dual to u : u takes a sup over *points* for a fixed direction, $h^\natural$ a sup over *directions* for a fixed point. Its sublevel sets are complements of unions of horoballs: $\{h^\natural \leq c\} = \mathbb{H} \setminus \bigcup_{\xi} \{\langle \tau, \xi \rangle_i > h(\xi) + c\}$.

5.3. (D iii) The tropical limit.

Theorem 5.2 (tropical limit of E [THEOREM]+[NUMERICAL]). *Let $M(\tau) = \max_x 1/H_\tau(x) = \max_\gamma \text{Im } \gamma\tau$, attained by $m(\tau)$ points x , and let $r_2 < 1$ be the ratio of the second-largest value to M .*

(i) $\lim_{s \rightarrow \infty} \frac{1}{s} \log E(\tau, s) = \log M(\tau) = (2h_{\text{FS}})^\natural(\tau) = \sup_{\xi \in \mathbb{P}^1(\mathbb{Q})} [\langle \tau, \xi \rangle_i - 2h_{\text{FS}}(\xi)]$.

(ii) (Rate.) For $s \geq s_0 > 1$: $M \leq E^{1/s} \leq M^{1-s_0/s} E(\tau, s_0)^{1/s}$, and

$$s \left(\frac{E(\tau, s)^{1/s}}{M} - 1 \right) = \log m(\tau) + O\left(\frac{1}{s}\right) + O(r_2^s).$$

So the convergence is $O(1/s)$ on the tie locus $\{m \geq 2\}$ and exponential, $\sim \frac{m_2}{s} r_2^s$, off it.

(iii) (Corner locus.) $\{m \geq 2\} = \Gamma \cdot \{|\tau| = 1\}$, the dual tree of the Farey tessellation. $m = 2$ on its edges and $m = 3$ at $\Gamma\rho$ (cusps $\infty, 0, -1$ for $\rho = e^{2\pi i/3}$). The cell of the cusp ∞ is $\{|\tau - n| \geq 1 \ \forall n \in \mathbb{Z}\} = \bigcup_n (\mathcal{F} + n)$, and the other cells are its Γ -images. These are exactly MW10's phases.

Proof. (i)–(ii): $E = M^s(m + \sum_{x \notin \max} (MH(x))^{-s})$. The sum is $\leq \sum (MH)^{-s_0} r_2^{s-s_0}$ for $s \geq s_0$, which gives the bound and $E^{1/s}/M = m^{1/s}(1 + O(r_2^s))$, hence $s(m^{1/s} - 1) = \log m + \frac{(\log m)^2}{2s} + \dots$. The first equality in (i) is Prop. 1.2(iii). (iii): $\text{Im } \tau$ and $\text{Im}(-1/\tau) = y/|\tau|^2$ tie iff $|\tau| = 1$; transport by Γ . At ρ , $Q_\rho(c, d) = \frac{2}{\sqrt{3}}(c^2 - cd + d^2)$ is minimized by $(0, 1), (1, 0), (1, 1)$. $\square$

Numerics [NUMERICAL] (tropical.json, Fig. 3). The upper bound slack is ≥ 0 for all $s \geq 2$ and all six test points. Tie locus: at $s = 2000$, $s(E^{1/s}/M - 1) = 0.693267$ ($\tau = i, e^{1.3i}$, and $0.4 + 0.2i = Si - 2$) against $\log 2 = 0.693147$. The difference $1.20 \cdot 10^{-4} = (\log 2)^2/(2s)$, as predicted. At ρ : 1.098914 against $\log 3 = 1.098612$, difference $3.0 \cdot 10^{-4} = (\log 3)^2/(2s)$. Generic: fitted slope of $\log(s \cdot \text{excess})$ against s is -0.570415 against $\log r_2 = -0.570414$ ($\tau = 0.23 + 1.31i$), and -2.19882 against -2.19834 ($\tau = 0.1 + 3i$).

Readings. (a) [THEOREM]: $E^{1/s} \rightarrow M$ is Maslov dequantization with $\hbar = 1/s$: the sum over cusps becomes a max over cusps. The Fourier–Hegason picture (plane waves P^s) degenerates to the Legendre picture (Busemann pairing minus height). (b) [THEOREM]+[ANALOGY]: in MW10's language the tropical limit keeps the dominant saddle only. The corner locus is the set of Hawking–Page-type transitions, and $m = 3$ at ρ is a triple point. (c) [THEOREM]: in the language of §2, the tropical limit picks the lowest-energy integral light ray N_x in the frame M_τ . (d) *RH-blindness* [THEOREM]: $M(\tau)$ depends only on the reduction theory of Γ (the Farey tessellation), and its correction terms are finitely many explicit cusps. The zeros sit at finite s (poles $\rho/2$, $\text{Re} \leq \frac{1}{2}$), where the tropical expansion has no information.

6. (D IV) THE THERMODYNAMIC LEGENDRE TRANSFORM AND THE HEIGHT COUNT

Write $\mathcal{E} = \log T$ (“energy”) and s for the inverse temperature. Then $E(\tau, s) = \int e^{-s\mathcal{E}} dN_\tau(e^\mathcal{E})$ is the canonical partition function of the height count, the density of states grows like $e^\mathcal{E}$, and the Hagedorn point is $s = \delta = 1$ (Prop. 4.3) [THEOREM] (names [ANALOGY]).

Theorem 6.1 (smoothed height count [THEOREM]+[NUMERICAL]). *Let $\Theta_\tau(T) = \sum_{x \in \mathbb{P}^1(\mathbb{Q})} e^{-H_\tau(x)/T}$. Assume the zeros ρ of ζ are simple. Then, with the zero sum taken*

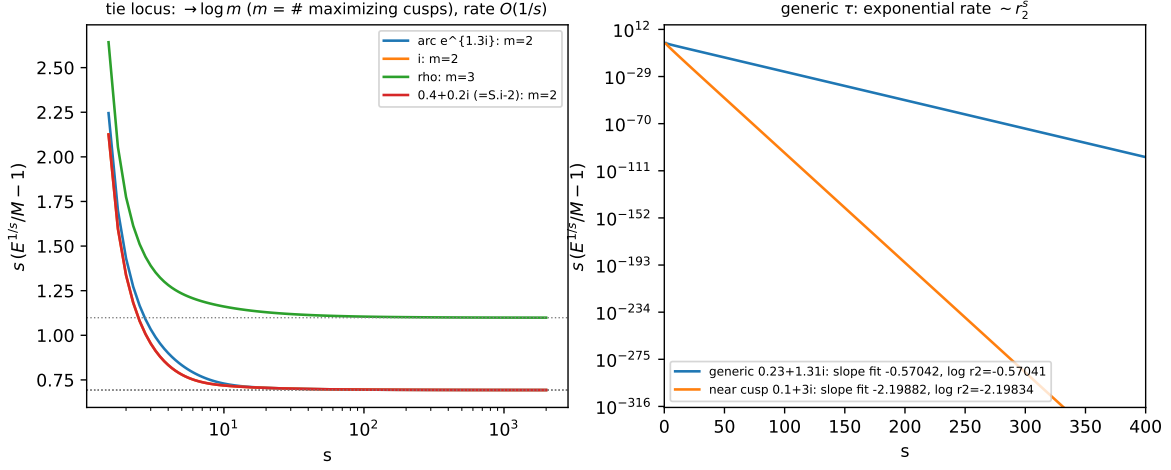


FIGURE 3. Tropical convergence $E(\tau, s)^{1/s} \rightarrow \max_{\gamma} \text{Im } \gamma\tau$. Left: on the tie locus, $s(E^{1/s}/M-1) \rightarrow \log m$ (dotted lines $\log 2, \log 3$). Right: off the tie locus the excess decays like r_2^s ; the fitted slope equals $\log r_2$.

over $|\gamma| < T_k$ for a suitable sequence $T_k \rightarrow \infty$,

$$(2) \quad \Theta_{\tau}(T) = \frac{3}{\pi}T + \sum_{n \geq 0} \frac{(-1)^n E(\tau, -n)}{n!} T^{-n} + 2 \text{Re} \sum_{\gamma > 0} \frac{\Gamma(\rho/2) Z_{Q_{\tau}}(\rho/2)}{4\zeta'(\rho)} T^{\rho/2}, \quad E(\tau, 0) = 1.$$

No Maass cusp form contributes.

Proof. $\Theta_{\tau}(T) = \frac{1}{2\pi i} \int_{(2)} \Gamma(s) T^s E(\tau, s) ds$. Shift the contour left: residues at $s = 1$ ($\frac{3}{\pi}T$), at $s = \rho/2$ ($\text{Res } E = Z_Q(\rho/2)/(4\zeta'(\rho))$ from $E = Z_Q/(2\zeta(2s))$), and at $s = -n$ (poles of Γ ; E is finite there because $Z_Q(-n) = 0$ for $n \geq 1$ cancels $\zeta(-2n) = 0$). $E(\tau, \frac{1}{2}) = 0$ is regular. Exponential decay of Γ on vertical lines beats the polynomial growth of E away from the zeros (horizontal segments at heights T_k , standard [BIB]). No Maass forms: $\Theta_{\tau}(T) = \sum_{\Gamma_{\infty} \backslash \Gamma} \psi_T(\text{Im } \gamma\tau)$ with $\psi_T(y) = e^{-1/(Ty)}$ is an incomplete Eisenstein series, and unfolding gives $\langle \Theta, u_j \rangle = \int_0^{\infty} \psi_T(y) a_0(u_j)(y) y^{-2} dy = 0$ for every cusp form u_j [15] [BIB]. $\square$

Numerics [NUMERICAL] (lemma74_smooth.py, smooth.json). The left side is computed *without zeros*: $2\Theta(T) = \sum_k \mu(k)(F(T/k^2) - 1)$ with the full-lattice theta $F(X) = \sum_{v \in \mathbb{Z}^2} e^{-Q(v)/X}$, using theta inversion $F(X) = \pi X F(1/(\pi^2 X))$ for $X \geq 1/\pi$ (all three forms are unimodular and self-dual; inversion checked to 10^{-40}). Epstein zetas are exact for the class-number-one forms $\tau = i$ ($4\zeta L_{-4}$), $i\sqrt{2}$ ($2\lambda^{-s}\zeta L_{-8}$), ρ ($6\lambda^{-s}\zeta L_{-3}$). The residual LHS–RHS with N_z zeros:

τ, T	$N_z = 0$	1	5	20	60
$i, 10^2$	$-3.2 \cdot 10^{-4}$	$2.0 \cdot 10^{-7}$	$7.8 \cdot 10^{-14}$	$-4.6 \cdot 10^{-17}$	same
$i, 10^{10}$	$2.3 \cdot 10^{-2}$	$6.9 \cdot 10^{-5}$	$1.7 \cdot 10^{-10}$	$7.6 \cdot 10^{-15}$	same
$i\sqrt{2}, 10^{10}$	$1.4 \cdot 10^{-2}$	$-1.4 \cdot 10^{-4}$	$7.8 \cdot 10^{-11}$	$6.4 \cdot 10^{-15}$	same
$\rho, 10^{10}$	$-3.6 \cdot 10^{-2}$	$-1.2 \cdot 10^{-4}$	$-1.2 \cdot 10^{-10}$	$3.1 \cdot 10^{-15}$	same

The floor $\sim 10^{-15}$ is the float64 part of the Möbius sum. For comparison, a Maass term would have size $\asymp T^{1/2} |\Gamma(\frac{1}{2} + ir_1)| \approx 10^5 \cdot 7.9 \cdot 10^{-7} \approx 0.08$ at $T = 10^{10}$ ($r_1 = 9.5337$), times a form-dependent coefficient. None is seen. This corrects the brief: *cusp forms do not enter the height count* (Fig. 4, bottom right).

The sharp count [NUMERICAL] (lemma74_count.py, count.json). $N_{\tau}(T) = \frac{1}{2} \sum_k \mu(k) A(\lfloor n_0/k^2 \rfloor)$ is computed in exact integers, with A the full lattice count (direct isqrt

sums above $4 \cdot 10^6$, cumulative tables below), and validated against brute force for $T \leq 3 \cdot 10^4$. Results:

- Leading constant: $N_i(9.092 \cdot 10^{12}) = 8\,682\,311\,344\,668$, $N/T = 0.95492965882$, against $3/\pi = 0.95492965855$; $N_\rho(9.874 \cdot 10^{12}) = 9\,428\,587\,256\,058$. The constant $3/\pi$ is the same for both τ (Prop. 1.3).
- Error $e(T) = N - \frac{3}{\pi}T - 1$: per decade 10^4 – 10^{13} (40 random T each), $\text{RMS}(e)/T^{1/4} \in [1.0, 2.1]$, with fitted RMS exponent 0.268 (i) and 0.266 (ρ). Also $\max |e|/T^{1/2}$ decreases from 0.24 to 0.003. This is the primitive circle problem; its exponent is RH-conditional only through the Möbius sum (Wu 2002: 221/608 under RH [BIB]; unconditional $\frac{1}{2}$ minus logs; conjectured $\frac{1}{4}$).
- The formal first-zero term $a(\rho_1) = Z_Q(\rho_1/2)T^{\rho_1/2}/(2\rho_1\zeta'(\rho_1))$ has $\text{RMS } 2\text{Re } a/T^{1/4} \approx 0.5$. That is the *same order* as the RH-blind lattice noise, so the zeros are not individually resolved in the sharp count.

Ghost tests. *Surgery* (sharp count; not circular in the masking question). Replace $2\text{Re } a(\rho_1)$ by $\text{Re}[a(\beta_+) + a(\beta_-)]$ with $\beta_\pm = \frac{1}{2} \pm \delta + i\gamma_1$, the same $\zeta'(\rho_1)$, and half weight each (continuous at $\delta = 0$). Ratio $\text{RMS}(\text{ghost increment})/\text{RMS}(e)$ at the top decade (10^{12} – 10^{13}), $\tau = i$: $\delta = 0.05$: 0.087; 0.1: 0.39; 0.2: 2.5; 0.3: 11; 0.45: 97 (ρ similar). The ratio crosses 1 at $T \approx 10^{9.5}$ for $\delta = 0.2$, at $10^{5.5}$ – $10^{6.5}$ for $\delta = 0.3$, and at 10^4 for $\delta = 0.45$. Ghosts with $\delta \leq 0.1$ stay masked to 10^{13} . RMS exponents with surgery: 0.268 (true), 0.272 ($\delta = 0.1$), 0.316 (0.2), 0.383 (0.3), 0.470 (0.45), against the naive $\frac{1}{4} + \frac{\delta}{2}$. *Smoothed* (circular). In (2) a ghost pair gives exponent $\frac{1}{4} + \frac{\delta}{2}$ by construction. The explicit side to $T = 10^{40}$ (Fig. 4, bottom left) only reads off the input. It shows that the smoothed count is $\Gamma(\rho/2)$ -suppressed ($|\text{fluct}|/T^{1/4} \sim 10^{-4}$), so the zeros are *cleanly separated* there but tiny. “RH $\Leftrightarrow \Theta_\tau(T) - \frac{3}{\pi}T - \sum_n \dots = O(T^{1/4+\epsilon})$ ” is a [RESTATEMENT].

Proposition 6.2 (canonical versus microcanonical; where the L69 anomaly enters [THEOREM]+[NUMERICAL]). *Let $E(\tau, s) = \frac{3/\pi}{s-1} + C_0(\tau) + O(s-1)$, and let $S_L(\mathcal{E}) = \inf_{s>1} [s\mathcal{E} + \log E(\tau, s)]$ be the Legendre transform of $\log E$.*

- (i) (Kronecker.) $C_0(\tau) = \frac{6}{\pi}(\gamma - \log 2 - \frac{\zeta'(2)}{\zeta(2)} - \log(\sqrt{y}|\eta(\tau)|^2))$, hence $\partial_\tau C_0 = -\frac{i}{2}E_2^*(\tau)$, $E_2^* = E_2 - \frac{3}{\pi y}$.
- (ii) $S_L(\mathcal{E}) = \mathcal{E} + 1 + \log \frac{3}{\pi} + \log(\mathcal{E} + c_1) + o(1)$ with $c_1 = \pi C_0/3$, whereas $\log N_\tau(e^\mathcal{E}) = \mathcal{E} + \log \frac{3}{\pi} + O(\mathcal{E}e^{-\mathcal{E}/2})$ (from $N = \frac{3}{\pi}T + O(T^{1/2} \log T)$, Möbius over the circle problem).

Proof. (i) Kronecker’s limit formula for $Z_Q = \sum_{v \neq 0} \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log \sqrt{y}|\eta|^2) + O(s-1)$ [BIB]. Dividing by $2\zeta(2s)$ adds $-\pi\zeta'(2)/\zeta(2)^2 = 1.0885453\dots$. Then $\partial_\tau \log \eta = \frac{\pi i}{12}E_2$ and $\partial_\tau \log y = \frac{1}{2iy}$. (ii) Minimize $u\mathcal{E} + \log(R/u + C_0)$ with $u = s-1$, $R = 3/\pi$: $u_* = \mathcal{E}^{-1}(1 + O(\mathcal{E}^{-1}))$, and expand. $\square$ *Numerics* [NUMERICAL] (`horo.json(e)`, `thermo.json`). The Fourier expansion agrees with the Epstein value at ($i, 1.3$) to 18 digits. $C_0 - \frac{6}{\pi}(\gamma - \log 2 - \log \sqrt{y}|\eta|^2) = 1.0885453$ at all three test τ , i.e. the term $-\frac{6}{\pi} \frac{\zeta'(2)}{\zeta(2)}$. This corrects the formula in our plan, which omitted it. A finite difference of C_0 (computed from the Fourier expansion) matches $-\frac{i}{2}E_2^*$ to 10 digits at $0.23+1.31i$ and $-0.3+0.97i$; at $\tau = i$ both vanish. Against the *exact* counts ($\tau = i$, $\mathcal{E} \leq 29.3$), the fitted $c_1(\mathcal{E}) = e^{S_L - \log N - 1} - \mathcal{E}$ rises to 1.901 and extrapolates (in $1/\mathcal{E}$) to 1.956, against $\pi C_0/3 = 1.963$.

Reading [THEOREM]+[HEURISTIC]. The canonical/microcanonical mismatch $1 + \log \mathcal{E}$ is the signature of a simple pole at the Hagedorn point $\delta = 1$. The τ -dependence of the canonical entropy enters through $c_1 = \pi C_0(\tau)/3$, whose τ -derivative is the Lemma 69 anomaly E_2^* . The microcanonical count sees no τ -dependence at leading order ($3/\pi$), and at lower order only through the amplitudes $Z_{Q\tau}(\rho/2)$. *The anomaly lives in the canonical ensemble, the zeros in the microcanonical fluctuations.* The Legendre transform is where the two meet, and it exchanges them only through the RH-blind constant C_0 .

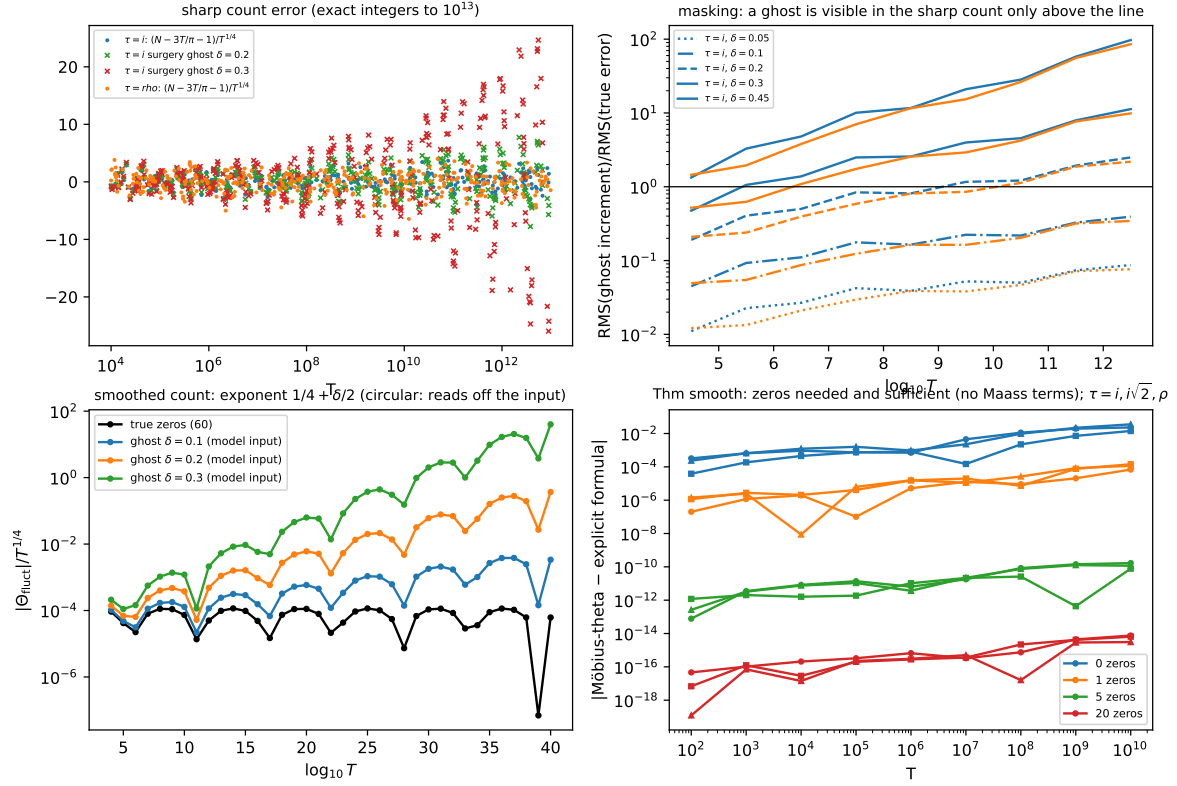


FIGURE 4. Counting and ghosts. Top left: sharp-count error $/T^{1/4}$ (exact integers) for $\tau = i, \rho$, with surgery ghosts $\delta = 0.2, 0.3$ at γ_1 . Top right: ghost visibility ratio per decade; a ghost is visible only above the line. Bottom left: smoothed count, explicit side with 60 true zeros against ghosts (circular, reads off the input exponent). Bottom right: Theorem 6.1, Möbius-theta minus explicit formula against the number of zeros included, for $\tau = i, i\sqrt{2}, \rho$.

Verdict on the polarization J . For every τ (every polarization J_τ of $\mathbb{R}^2$, equivalently every metric M_τ) the exponents in (2) and in the sharp count are 1, $-n$ and $\rho/2$, independent of τ [THEOREM]. J_τ enters only (a) the amplitudes $Z_{Q_\tau}(\rho/2)$ (amplitudes.log: $|2\Gamma(\rho_1/2)Z_Q(\rho_1/2)/(4\zeta'(\rho_1))| = 1.16, 0.45, 1.25 (\times 10^{-4})$ for $\tau = i, i\sqrt{2}, \rho$), and (b) $C_0(\tau)$. Since $E(\tau, \rho/2)$ has a pole for every τ , no choice of J moves or removes a zero. *The horocycle/height/Legendre picture is RH-blind for J* [THEOREM]. A zero would be invisible at a given τ only if $Z_{Q_\tau}(\rho/2) = 0$; whether that ever happens is [OPEN]. It is J -specific, but it does not concern the location of ρ .

7. CONNECTIONS TO TAA, CIRCULARITY AND METAPHOR FLAGS

- (T1) **Lemma 69** [TAA-L]+[THEOREM]. The constant term $y^s + \varphi y^{1-s}$ is the horocycle average (Prop. 4.1). The anomaly $3/\pi$ is Peyre's constant (Prop. 1.3). The anomaly $E_2^* = \partial_\tau$ of the Kronecker constant and appears in the *canonical* entropy only (Prop. 6.2). The Poisson widths of Lemma 69 are the Poisson kernels $P(\tau, -d/c)$ of Prop. 3.2(i) on the moduli plane.
- (T2) **Lemma 72** [TAA-L]+[THEOREM]. Primitivity $\leftrightarrow 1/\zeta(2s)$ is here “finite local heights vanish” (Prop. 1.3(i)). MWK's dominant saddle is the tropical limit (Thm. 5.2). MWK's weight $s = \frac{1}{2}$ is superharmonic (Prop. 3.2(ii)).

- (T3) **Lemma 73** [TAA-L]+[THEOREM]. “Eisenstein summand = (energy of an integral light ray)^{−s}” has its 2+1 form $H_\tau(x) = \text{Tr}(M_\tau N_x)$ (§2). The Bianchi analogue of Prop. 1.3 (Peyre’s constant of $\mathbb{P}_{\mathcal{O}_K}^1 \leftrightarrow$ Humbert’s volume residue) and of Prop. 2.1(iii) (finite intersections valued in ideals) is direction G2.
- (T4) [t288u]–[t289u] (“primitivity is where the zeros come from”; “anomaly as leading term, zeros as fluctuations”) [THEOREM]. Both are confirmed in the height-count form: leading term $\frac{3}{\pi}T$ (Peyre), fluctuations $T^{\rho/2}$ coming from the Möbius inversion over sections. *Refined*: in the sharp count the zero terms are of the same order as RH-blind lattice noise. Only after smoothing are they cleanly separated, and then they are tiny ($\Gamma(\rho/2)$).

Flags. (i) Every “RH $\Leftrightarrow$ fluctuation exponent $\frac{1}{4}$ ” (smoothed or sharp) is a [RESTATEMENT] and circular as a proof strategy. Ghost numerics in (2) read off their own input. Only the *masking* measurement in the sharp count has independent content. (ii) All numerics with zeros use known zeros, on the line. (iii) “Poincaré disk $\Rightarrow$ maximum principle $\Rightarrow$ zeros”, “ $\oplus$ projection = $\mathbb{K} \oplus \mathbb{L}$ ”, “ $\mathbb{P}_{\mathbb{Z}}^1 \approx \text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ ” and “ $\mathbb{F}_1: q+1 \rightarrow 2$ ” are metaphor-as-math. Each is corrected to its exact content above. (iv) “Energy”, “Hagedorn”, “canonical/microcanonical”, “light ray” are exact identities with physical names ([ANALOGY] for the names). (v) The term “hyperbolic/horospherical Legendre transform” is our definition. We found it at source only as the idempotent (Litvinov) Legendre transform and as the horospherical support function, not under this name.

8. CORRECTIONS TO THE CHAT CLAIMS AND THE BRIEF

- (C1) [t294u] “The MWK saddle disk is a Poincaré disk”: **correct** (thermal AdS: constant-time hemisphere; BTZ: cigar; totally geodesic $\mathbb{H}^2$, Prop. 3.1).
- (C2) [t294u] “primitive $\Rightarrow$ disk $\Rightarrow$ Poisson $\Rightarrow$ maximum principle”: **first link correct, second is metaphor**. The Poisson kernel in E is that of the moduli plane at the cusp $-d/c$. It exists for all (c, d) , primitive or not, and is unrelated to the bulk disk.
- (C3) [t294u] “Eisenstein summand = power of the Poisson kernel at $-d/c$ ”: **correct with a factor**: $\text{Im}(\gamma\tau)^s = c^{-2s}P(\tau, -d/c)^s$. The factor $c^{-2s}(1+\xi^2)^{-s}$ is $e^{-2sh_{\text{FS}}(\xi)}$, the height.
- (C4) [t294u] maximum principle: **sign error at MWK’s weight**. P^s is subharmonic only for $s \geq 1$ (or $s \leq 0$). At $s = \frac{1}{2}$ it is superharmonic.
- (C5) [t294u] “minimum modulus $\leftrightarrow \Delta \log |\xi| = 2\pi \sum \delta_\rho$ ”: **correct, RH-blind** (Jensen holds verbatim for a ghost; `horo.json(f)`). “Maximum principle and primitivity are not restatements”: **correct**.
- (C6) [t295u] “level sets of $\text{Im } \gamma\tau$ are horocycles; $-\log \text{Im } \gamma\tau$ is a Busemann function; $E = \sum e^{-sB}$ ”: **correct up to the additive constant $2h_{\text{FS}}(\xi)$** , which is the height itself (Prop. 1.2). “Two distinct hyperbolic planes”: **correct**. There is also a third, non-hyperbolic plane (the s -plane).
- (C7) [t295u] “the horocycle average is the $\oplus$ projection” (with E_k eigenvalue $1+p^{k-1}$): **conflates two splits**. $y^s \oplus \varphi y^{1-s}$ is the archimedean Weyl split, and both pieces have the same Hecke eigenvalue. $1+p^{k-1}$ comes from the Bruhat cells $\{\infty\} \sqcup \mathbb{A}^1$ of $\mathbb{P}^1(\mathbb{F}_p)$ inside T_p (Prop. 4.2).
- (C8) [t295u] “limit set = whole circle; PS $\delta = 1$ at the anomaly pole; $\lambda_0 = \delta(1-\delta)$ ”: **correct** (Prop. 4.3). Pasted links: **all four exist**. The mathematics is supported, but “absolute set/directrix” does not appear in the linked sources (§4).
- (C9) [t296u] “primitive $(c:d) =$ section of $\mathbb{P}_{\mathbb{Z}}^1$; primitivity kills finite heights; $E =$ height zeta (FMT)”: **correct**. Normalization at source: $\zeta_{V,H}(s) = E_P^G((2s-1)\rho_P)$, anticanonical H , which is the classical $E(\tau, s)$ with $\tau =$ metric at ∞ .
- (C10) [t296u] “ $\mathbb{P}_{\mathbb{Z}}^1$ as a bridge to $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}/\text{Hodge index}$ ”: **no**. $\mathbb{P}_{\mathbb{Z}}^1 = \mathbb{P}^1 \times \text{Spec } \mathbb{Z}$ is a ruled-surface analogue with no h^1 , and Hodge index is vacuous (Prop. 2.1(ii)). The $\mathbb{F}_1$ count $q+1 \rightarrow 2$ is [ANALOGY]. Its real content is the Euler factor $1-p^{-2}$ in Peyre’s constant.

- (C11) Arakelov self-intersection: $\widehat{\mathcal{O}(1)}^2 = \frac{1}{2}$, **verified, and τ -independent**. New: $(x \cdot y)_{\text{fin}} = \frac{1}{2}\delta(\text{Ford horoballs})$, Penner's λ -length (Prop. 2.1(iii)).
- (C12) (D i) Helgason at curvature -1 : **as planned**, with density $\lambda \tanh(\pi\lambda)/(2\pi)$ (radial), verified numerically. Helgason's book not opened.
- (C13) (D iii) tropical limit: **proved**, with rate $O(1/s)$ on the tie locus (limit $\log m$) and exponential off it.
- (C14) Brief (D iv) "error terms from cusp/Maass forms": **wrong**. The height count is an incomplete Eisenstein series, orthogonal to cusp forms. Only 1 , $-n$ and $\rho/2$ occur (Thm. 6.1, residual 10^{-15}).
- (C15) Plan (D iv) Kronecker constant: **missing term**. $C_0 = \frac{6}{\pi}(\gamma - \log 2 - \zeta'(2)/\zeta(2) - \log \sqrt{y}|\eta|^2)$, because primitivity divides by $2\zeta(2s)$. $\partial_\tau C_0 = -\frac{i}{2}E_2^*$ is unaffected.
- (C16) Brief "state what is RH-blind": the leading constant $3/\pi$, C_0 , the tropical limit, Jensen/Poisson, unitarity-type identities and all exponents' τ -independence are RH-blind. Only the fluctuation exponents $(\rho/2)$ carry RH, as a [RESTATEMENT].

9. RANKED NEXT DIRECTIONS (0.6 s + 0.4 p)

	Direction	s	p	score
G1	<i>λ-lengths as arithmetic intersections:</i> Penner's decorated Teichmüller coordinates (Ptolemy relation) versus Arakelov intersection on $\mathbb{P}_{\mathbb{Z}}^1$. Is the Ptolemy relation $\lambda_{13}\lambda_{24} = \lambda_{12}\lambda_{34} + \lambda_{14}\lambda_{23}$ the Plücker relation for sections? Extend to $\mathcal{O}_K$ (ideal-valued).	.65	.65	.65
G2	<i>Peyre's constant for $\mathbb{P}_{\mathcal{O}_K}^1 = \text{Bianchi residue}$:</i> identify Lemma 73's $\sqrt{ d }/(w \text{ vol})$ with $\alpha\tau_H$ for $\mathbb{P}^1$ over K , and the class-group channels with height zetas of twisted points	.60	.70	.64
G3	<i>Optimal smoothing for ghost visibility:</i> the kernel w minimizing lattice noise over zero signal in $\sum_x w(H(x)/T)$ (between sharp and e^{-u}). Quantify the masking threshold $\delta^*(T)$ as an RH-blind "resolution" curve	.50	.80	.62
G4	<i>Tropical MWK:</i> the corner locus Γ -arc as Hawking–Page walls for all MWK weights; subleading tropical corrections as one-loop terms; compare MW10 §6	.50	.70	.58
G5	<i>Canonical anomaly for Bianchi/Hilbert:</i> $c_1 = C_0/\text{Res}$ and its derivative (Kronecker–Hecke) in Lemma 73's setting	.45	.70	.55
G6	<i>Horoconvex domains:</i> $\mathbb{H} \setminus \bigcup(\text{height-}T \text{ horoballs})$ and its first Dirichlet eigenvalue ([6]), as a function of T	.40	.60	.48

Still open from earlier: Lemma 73 F1 (celestial wavefunction, .63), Lemma 72 F1 (.70), Lemma 71 F1 (.70).

10. QUESTIONS FOR J.W.

- Q1.** Do we adopt " $E(\tau, s)$ is the anticanonical height zeta function of $\mathbb{P}^1(\mathbb{Q})$ with archimedean metric τ , and $3/\pi$ is Peyre's constant $\frac{1}{2} \cdot \pi \cdot \frac{6}{\pi^2}$ " as the official TAA reading of the anomaly?
- Q2.** Do you accept the correction that the $\mathbb{K} \oplus \mathbb{L}$ split is the Bruhat-cell split inside T_p , and not the constant-term split $y^s \oplus \varphi y^{1-s}$, which is the Weyl/reflection split and carries the zeros through φ ?
- Q3.** Do you accept that the "disk $\Rightarrow$ maximum principle" chain is metaphor (wrong plane, and the wrong sign at $s = \frac{1}{2}$), while "zeros = defects of harmonicity of $\log |\xi|$ " is exact but RH-blind?
- Q4.** On J : the picture is RH-blind for the polarization (exponents τ -independent, J only in amplitudes). Do we close the polarization line for $\mathbb{P}_{\mathbb{Z}}^1$ and move the open question to "can $Z_{Q_\tau}(\rho/2)$ vanish?" ([OPEN])?

Q5. Next: G1 (λ -lengths/Ptolemy as arithmetic intersection), G2 (Peyre = Bianchi residue), G3 (optimal smoothing), or an older direction? Any publication needs your approval.

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TAA LEMMA 75: MAYER–HENSLEY.
LOCALIZING THE ζ ZEROS IN THE CUSP TAIL OF THE GAUSS TRANSFER
OPERATOR: PARITY, THE $\delta \rightarrow 1$ LIMIT, THE TAIL HOMOTOPY, AND A
GHOST TEST FOR POSITIVITY

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA. Nothing in this lemma is a step toward a proof that we can certify. Every RH-equivalent statement is labelled [RESTATEMENT]; every metaphor [ANALOGY] or [HEURISTIC].* Tags: [THEOREM] proved here (elementary) or classical with proof indicated; [CITED] checked at the source (`lit/lemma75/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical. **All numerics in this lemma are uncertified Galerkin truncations:** for each truncation size N the $N \times N$ determinant is enclosed rigorously by ball arithmetic (Arb, $\geq 3N + 128$ bits), but the passage $N \rightarrow \infty$ is not certified; we only show convergence in N .

Question (J.W., [t303u], approving the limit-set review’s proposal). Write the “Mayer–Hensley” lemma: read Mayer’s transfer-operator formula $Z_{\mathrm{PSL}_2(\mathbb{Z})}(s) = \det(1 - \mathcal{L}_s^2)$ and its parity refinement at source; settle which factor $\det(1 \mp \mathcal{L}_s)$ carries the scattering zeros $\rho/2$ and the even/odd Maass parameters; reproduce Hensley’s $\delta_A \rightarrow 1$ asymptotic; switch on the exact Hurwitz tail continuously and watch the zeros appear; inject a ghost and ask whether anything positive at real s notices; connect to the polarization J , the anomaly $3/\pi$, Lemma 72’s Theorem mb and the missing h^1 of [t301u]–[t302u].

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ANSWER IN BRIEF

- (1) *Sources* (§1) [CITED]. Mayer (Bull. AMS 1991, Thm. 2): $Z(s) = \det(1 - \mathcal{L}_s) \det(1 + \mathcal{L}_s)$, $(\mathcal{L}_s f)(z) = \sum_{n \geq 1} (z+n)^{-2s} f(\frac{1}{z+n})$ on $|z-1| < \frac{3}{2}$, meromorphic in s with simple poles at $s_k = \frac{1-k}{2}$ and rank-one residues $N_k f = \frac{1}{2k!} f^{(k)}(0)$ (Thm. 1). Efrat (Invent. 1993): $\det(1 - \mathcal{L}_s) = Z_+(s)$, $\det(1 + \mathcal{L}_s) = Z_-(s)$, and the Corollary “ $\det(1 - \mathcal{L}_s) = 0$ iff s is even; $\det(1 + \mathcal{L}_s) = 0$ iff s is odd”, where *even* includes $\zeta^*(2s) = 0$. Lewis–Zagier (Annals 2001, Ch. IV §3) restate this ($Z_+ =$ Selberg zeta of $\mathrm{PGL}_2(\mathbb{Z})$) and construct the eigenfunctions, e.g. $h(z) = 1/(z+1)$ at $s = 1$ and Zagier’s $\psi_s^+(z+1)$ when $\zeta(2s) = 0$.
- (2) *The reviewer’s OCR worry is resolved* (§1) [CITED]. The clean page image of Mayer 1991, p. 59, shows a *question*, not a conjecture: “One can then ask if it is generally true that all the eigenvalues of $-\Delta$ are determined by the factor $\det(1 - \mathcal{L}_s)$ whereas the nontrivial zeros of Riemann’s zeta function are determined by $\det(1 + \mathcal{L}_s)$.” Efrat’s paper opens by saying it answers “a question raised by D. Mayer in [M3]” (= that Bull. AMS note), and the answer is *no*: the ζ zeros sit in $\det(1 - \mathcal{L}_s)$, together with the even Maass forms.
- (3) *Parity settlement* (§3) [NUMERICAL]. With a Taylor–Galerkin discretization ($N = 24 \dots 192$, Hurwitz continuation), all 12 scattering zeros $\rho_k/2$, $k \leq 12$ (heights up to 28.2), are zeros of $\det(1 - \mathcal{L}_s)$: $|\det(1 - \mathcal{L}_{s,N})|$ falls by 56 to 66 orders between $N = 24$ and $N = 192$, while $|\det(1 + \mathcal{L}_s)| \in [1.07, 17.4]$ is stable to ≥ 6 digits. Newton-located zeros agree with $\rho_k/2$ to $< 10^{-38}$. The first 12 Maass forms (LMFDB, $R \leq 21.32$) split exactly by LMFDB parity: the 4 even ones (including $R = 13.7797513519$, under-resolved in the review) are zeros of $\det(1 - \mathcal{L}_s)$, the 8 odd ones of $\det(1 + \mathcal{L}_s)$; the located R agree with LMFDB and are stable to 40 digits. This confirms Efrat and Lewis–Zagier and agrees with the review’s probe.
- (4) *Anomaly in operator form* (§2) [THEOREM]+[NUMERICAL]. At $s = 1$: eigenvalue 1 with eigenfunction $1/(1+z)$ (Gauss density), $\lambda_2(1) = -0.30366300289873265859744812190$ (Gauss–Kuzmin–Wirsing; agrees with OEIS A038517 to 30 digits). Hensley’s constant $6/\pi^2 = 1/\zeta(2)$ is (Gauss mass of the cusp region)/(Lyapunov exponent $\zeta(2)/\log 2$), and $1/\zeta(2) = 2/(\pi \text{ vol})$ ties it to $\text{Res}_{s=1} E = 3/\pi = 1/\text{vol}$.
- (5) *Hensley* (§4) [NUMERICAL]. δ_A computed for $A = 2, \dots, 10^5$ (accuracy $\sim 10^{-17}$; δ_2 agrees with Jenkinson–Pollicott’s rigorous 100-digit value to 30 digits). Hensley’s $1 - \frac{6}{\pi^2 A} - \frac{72 \log A}{\pi^4 A^2}$ is confirmed, and the $O(A^{-2})$ term is *numerically* $+c/A^2$ with $c = 1.02068(2)$, followed by $(\log A)^2/A^3$ terms. We propose no closed form for c .
- (6) *The cusp tail* (§5) [NUMERICAL]+[THEOREM]. Under $\mathcal{L}_{s, \leq A} + \tau T_{s, > A}$ the zero starting at $\rho_k/2$ leaves $\text{Re } s = \frac{1}{4}$ *immediately* ($|ds/d\tau|$ at $\tau = 1$ is 0.27–1.39, growing like $\sqrt{A}$, as does the tail itself) and ends at $\tau = 0$ near $\text{Re } s \approx 0.37$ –0.41 for $A \geq 10$: truncation never converges (distance 0.15–0.20 for $A \leq 1280$). *But the tail carries no arithmetic*. Replacing it by its Euler–Maclaurin expansion around the parabolic pole (pole term $a^{1-\sigma}/(\sigma-1)$ plus K Bernoulli terms) reproduces $\rho_k/2$ with error $\asymp A^{-2K-3/2}$ (down to $9 \cdot 10^{-22}$ at $A = 1280$, $K = 3$). So the zeros are produced by *the cancellation between the finite parabolic head $\sum_{n \leq A}$ and a universal pole counterterm*: an operator form of the approximate functional equation. This corrects the review’s “zeros carried entirely by the tail”.
- (7) *The positivity barrier* (§6) [THEOREM]+[NUMERICAL]. $\mathcal{L}_s$ is a positive operator only for real $s > \frac{1}{2} = \delta_P$, the critical exponent of the cusp’s parabolic subgroup; at $s = \frac{1}{2}$ it has Mayer’s rank-one pole $f \mapsto f(0)/2$ (evaluation at the cusp point); for real $s < \frac{1}{2}$ the continued operator sends 1 to $\zeta(2s, z+1) < 0$, and $\text{tr } \mathcal{L}_s^2 < 0$ at $s \leq 0.3$. The zeros $\rho/2$ live at $\text{Re } s = \frac{1}{4} = \delta_P/2$, strictly on the far side of the barrier.

- (8) *Ghost test and J verdict* (§7) [NUMERICAL]+[THEOREM]. G1: positive, s -independent branch weights that keep $\delta = 1$ (eigenvalue 1 at $s = 1$, positive eigenfunction) move $\rho_1/2$ to $0.28 - 0.40 + 7.0 - 7.2i$. G2: an s -dependent tail reweighting $1 + \epsilon(s)$ with $\epsilon(1) = 0$ and $|\epsilon| \leq 1.4 \cdot 10^{-24}$ (resp. $9.2 \cdot 10^{-24}$) on $[\frac{1}{2}, \infty)$ places a zero exactly at $0.30 + i\gamma_1/2$ (resp. $0.40 + i\gamma_1/2$). Meanwhile every positive quantity at real s changes by $\leq 3.3 \cdot 10^{-23}$: the PF eigenvalue, $\text{tr } L$, and $\det(1 \pm L)$. Verdict: **no-go** (Prop. 7.2): no functional of the real- s positive structure can detect off-line zeros. What the ghosts break is (i) the functional equation (no partner at $\frac{1}{2} - \bar{s}$) and (ii) translation invariance of the cusp weights, which is exactly the Lewis three-term identity. *A J must use a cancellation identity tied to the parabolic generator $z \mapsto z + 1$, not positivity*; whether that is circular is [OPEN].
- (9) *Program* (§8) [HEURISTIC]. Three exponents organize everything: the anomaly at $\delta = 1$ (PF eigenvalue 1, $\text{Res } E = 3/\pi$), the cusp's parabolic pole at $\delta_P = \frac{1}{2}$, and the zeros at $\frac{1}{4}$. Lemma 72's Theorem mb has the same shape (smooth anomaly term + zero fluctuations). Mayer's $\det(1 - \mathcal{L}_s)$ is an infinite-dimensional, nuclear, s -dependent family; Deninger's h^1 wants $\det(s - \Theta)$ for one generator Θ of a flow. The gap between the two is a precise version of obstruction 3 of [t302u] [OPEN].

1. THE OPERATOR, AT SOURCE

1.1. **Mayer 1991.** Let $D = \{z : |z - 1| < \frac{3}{2}\}$ and $\mathcal{A}_\infty(D)$ the Banach space of functions holomorphic on D and continuous on $\bar{D}$. For $\text{Re } s > \frac{1}{2}$,

$$(1) \quad (\mathcal{L}_s f)(z) = \sum_{n \geq 1} \left(\frac{1}{z + n} \right)^{2s} f\left(\frac{1}{z + n} \right).$$

Theorem 1.1 (Mayer [1, Thm. 1, Thm. 2, Prop. 1] [CITED]). (a) For $\text{Re } s > \frac{1}{2}$, $\mathcal{L}_s$ is nuclear of order zero and $Z_n(T_G, A_s) = \text{tr } \mathcal{L}_s^n - \text{tr}(-\mathcal{L}_{s+1})^n$, where Z_n is the partition function of the Gauss map $T_G x = \{1/x\}$ over its n -periodic points. (b) $s \mapsto \mathcal{L}_s$ extends meromorphically to $\mathbb{C}$ with simple poles at $s_k = \frac{1-k}{2}$, $k = 0, 1, \dots$, whose residues are the rank-one operators $N_k f(z) = \frac{1}{2} \frac{1}{k!} f^{(k)}(0)$. (c) The Selberg zeta function of $\text{PSL}_2(\mathbb{Z})$ is $Z(s) = \det(1 - \mathcal{L}_s) \det(1 + \mathcal{L}_s) = \det(1 - \mathcal{L}_s^2)$, meromorphic on $\mathbb{C}$ with possibly removable singularities at the s_k .

The residue formula (b) was read from the page image (the OCR in the review garbles it). Note that $N_0 f = \frac{1}{2} f(0)$ is *evaluation at the point $z = 0$* , the limit of the inverse branches $1/(z + n)$ as $n \rightarrow \infty$, i.e. the cusp (infinite partial quotient).

What Mayer said about parity [CITED]. Page 59, from the page image: “The nontrivial zeros of $Z(s)$ are hence given by those s -values for which $\mathcal{L}_s$ has $\lambda = 1$ or $\lambda = -1$ as an eigenvalue. The trace formula approach shows [V] that these zeros are either $\frac{1}{2}$ times the nontrivial zeros of Riemann's zeta function or they determine via the formula $s = \frac{1}{2} + ir$ the eigenvalues $\rho = \frac{1}{4} + r^2$ of the Laplacian $-\Delta$ on the modular surface. The value $s = 1$ is such a zero since $\det(1 - L_1) = 0$ [...] One can then ask if it is generally true that all the eigenvalues of $-\Delta$ are determined by the factor $\det(1 - L_s)$ whereas the nontrivial zeros of Riemann's zeta function are determined by $\det(1 + L_s)$. [...] In this case the Riemann Hypothesis would be equivalent to the operator $\mathcal{L}_s$ having eigenvalue $\lambda = -1$ only for s -values on the line $\text{Re } s = \frac{1}{4}$.” So Mayer posed a *question*, with the guess “zeros in $\det(1 + L_s)$ ”. The review's reading (“he conjectured they come from $\det(1 + L_s)$ ”) overstated it: it was a question, and the answer turned out to be the opposite.

1.2. Efrat 1993.

Theorem 1.2 (Efrat [2, pp. 207–210] [CITED]). Let $\Gamma' = \text{PGL}_2(\mathbb{Z})$ act on $\mathbb{H}$ with determinant -1 elements acting by $z \mapsto (a\bar{z} + b)/(c\bar{z} + d)$. Define

$$Z_\pm(s) = \prod_{\{g\}, \det g=1} \prod_{\ell \geq 0} (1 - N(g)^{-s-\ell}) \prod_{\{h\}, \det h=-1} \prod_{\ell \geq 0} (1 \mp (-1)^\ell N(h)^{-s-\ell}).$$

Then $\det(1 - \mathcal{L}_s) = Z_+(s)$ and $\det(1 + \mathcal{L}_s) = Z_-(s)$. Corollary: call s (with $\operatorname{Re} s > 0$) even if $s(1-s)$ is an eigenvalue of an even Maass form or $\zeta^*(2s) = 0$, and odd if $s(1-s)$ is an eigenvalue of an odd one. Then $\det(1 - \mathcal{L}_s) = 0$ iff s is even, and $\det(1 + \mathcal{L}_s) = 0$ iff s is odd.

Efrat adds: “while the nontrivial zeros of $\det(1 + L_s)$ are all on the line $\operatorname{Re}(s) = \frac{1}{2}$, those of $\det(1 - L_s)$ are both on and off this line” and “The statement of the corollary was conjectured by Mayer around the time we proved it.” The paper opens: “The aim of this article is to answer a question raised by D. Mayer in [M3]”, and [M3] is the Bull. AMS note. The proof of the corollary goes through Venkov’s trace formula for L_{odd}^2 , in which the Eisenstein series (being even) does not appear.

1.3. Lewis–Zagier and Chang–Mayer. Lewis–Zagier [3, Ch. IV §3] [CITED] reprove Mayer’s theorem via reduced representatives of hyperbolic classes, and state: “the function $Z(s)$ has a natural splitting $Z_+(s)Z_-(s)$ where $Z_+(s)$ is the Selberg zeta function of $\Gamma^+ = \operatorname{PGL}(2, \mathbb{Z})$ [...] and $Z_+(s)$ and $Z_-(s)$ have zeros at the spectral parameters of Γ corresponding to even and odd Maass forms, respectively, and one in fact has $Z_{\pm}(s) = \det(1 \mp L_s)$.” Their Corollary (for $\operatorname{Re} s > 0$, $s \neq \frac{1}{2}$): (a) $L_s h = -h$ has a nonzero solution iff s is the spectral parameter of an odd Maass form; (b) $L_s h = h$ has one iff s is the parameter of an even Maass form, or $2s$ is a zero of ζ , or $s = 1$. They give the eigenfunctions: the period function of the Maass form (with $h(0) = 0$); Zagier’s function $\psi_s^+(z+1)$ when $\zeta(2s) = 0$; and $h(z) = 1/(z+1)$ at $s = 1$. The bridge is the identity

$$(2) \quad (\mathcal{L}_s h)(z-1) = (\mathcal{L}_s h)(z) + z^{-2s} h(1/z),$$

which turns $\mathcal{L}_s h = \pm h$ into the even/odd three-term functional equation $\psi(z) = \psi(z+1) \pm (z+1)^{-2s} \psi(\frac{z}{z+1})$ for $\psi(z) = h(z-1)$. Chang–Mayer [4] (abstract read) identify the period function of the non-holomorphic Eisenstein series with $B(\frac{1}{2}, s + \frac{1}{2})\psi_s(z)$, $\psi_s(z) = \sum_{m,n \geq 1} (mz+n)^{-2s} + \frac{1}{2}\zeta(2s)(1+z^{-2s})$; when $\zeta(2s) = 0$ only the positive-quadrant lattice sum survives. The long survey Chang–Mayer [5] was seen only as an abstract [BIB].

Remark 1.3 (RH in this language [RESTATEMENT]). By Thm. 1.2 and the Selberg/Venkov theory (no exceptional eigenvalues for $\operatorname{PSL}_2(\mathbb{Z})$), RH is equivalent to: *every s with $0 < \operatorname{Re} s < \frac{1}{2}$ at which $\mathcal{L}_s$ has eigenvalue $+1$ lies on $\operatorname{Re} s = \frac{1}{4}$.* This is Mayer’s reformulation with the sign corrected. It is a restatement, circular as a proof strategy.

1.4. Numerical method. Expand $f(z) = \sum_k a_k (z-1)^k$. Since $(z+n)^{-2s-j} = \sum_m \binom{-2s-j}{m} (n+1)^{-2s-j-m} (z-1)^m$, the matrix of $\mathcal{L}_s$ in the Taylor basis is

$$(3) \quad M_{mk}(s) = \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \binom{-2s-j}{m} W(2s+j+m), \quad W(\sigma) = \sum_{n \geq 1} w_n (n+1)^{-\sigma},$$

with $w_n \equiv 1$, so $W(\sigma) = \zeta(\sigma, 2)$ (Hurwitz). The Hurwitz continuation of each entry is exactly Mayer’s continuation; poles at $2s + q = 1$ reproduce the s_k . Head/tail splits use $W = \sum_{n \leq A} (n+1)^{-\sigma} + \tau \zeta(\sigma, A+2)$. We truncate to $0 \leq m, k < N$ and compute $\det(1 \mp M_N)$ in Arb ball arithmetic (`python-flint`), $\geq 3N + 128$ bits. **Uncertified:** the error of N -truncation is not bounded; it is only observed to decay (at $s = 1$, $|\det(1 - M_N)| = 1.5 \cdot 10^{-6}, 4.0 \cdot 10^{-9}, 1.2 \cdot 10^{-11}, 1.2 \cdot 10^{-16}, 1.4 \cdot 10^{-21}$ for $N = 16, 24, 32, 48, 64$, i.e. roughly 2^{-N} ; all inverse branches $z \mapsto 1/(z+n)$ map D into the smaller closed disc $|z-1| \leq 1$, which is what makes $\mathcal{L}_s$ nuclear). Rigorous truncation bounds of Jenkinson–Pollicott type [8] would certify these numbers; we have not done this (direction D4). This scheme is far more efficient than the review’s collocation on a circle: at $\rho_1/2$ and $N = 32$ it gives $|\det(1 - M_N)| = 1.4 \cdot 10^{-11}$ against the review’s $4.4 \cdot 10^{-6}$. Scripts: `num/lemma75_lib.py` and the files below.

2. THE ANOMALY IN OPERATOR FORM

Proposition 2.1 ([THEOREM] classical; [NUMERICAL] check). (a) $\mathcal{L}_1 h = h$ for $h(z) = 1/(1+z)$, and $\int_0^1 \mathcal{L}_1 f dx = \int_0^1 f dx$; so $\frac{1}{\log 2} \frac{dx}{1+x}$ is the invariant (Gauss) measure, the PF eigenvalue of $\mathcal{L}_1$ is 1, and $\det(1 - \mathcal{L}_1) = 0$ (Lewis–Zagier’s case (c)). (b) [NUMERICAL]: $\lambda_1(1) = 1 - 1.9 \cdot 10^{-31}$ at $N = 96$; the eigenfunction values at $x = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ are 2, 1.6, 1.3333, 1.142857, 1 times $h(1)$; and $\lambda_2(1) = -0.303663002898732658597448121901$ ($N = 96$; $N = 64$ agrees to 20 digits), equal to minus the Gauss–Kuzmin–Wirsing constant $0.303663002898732658597448121901556 \dots$ (OEIS A038517) to 30 digits. (c) The derivative of the pressure at $s = 1$ is $-\int_0^1 \log |T'_G| d\mu_G = -\frac{\pi^2}{6 \log 2} = -\frac{\zeta(2)}{\log 2}$ (the Lyapunov exponent of T_G ; Abramov’s formula with return time $2 \log(1/x)$ gives entropy 1 for the geodesic flow).

Dictionary with $T1/T4$ of the review ([THEOREM] for the identities, [HEURISTIC] for the reading). The anomaly of Lemma 69, $\text{Res}_{s=1} E(z, s) = 3/\pi = 1/\text{vol}$, is the critical-exponent pole of the Patterson–Sullivan series over cusps at $\delta = 1$. In operator form it is the PF eigenvalue 1 of $\mathcal{L}_1$: the zero of Z at $s = 1$ that sits in $\det(1 - \mathcal{L}_s)$. Both are the same $\zeta(2)$: $\text{vol}(\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}) = \pi/3 = 2\zeta(2)/\pi$, and the Lyapunov exponent is $\zeta(2)/\log 2$. Hence $1/\zeta(2) = 2/(\pi \text{vol})$, and this is the constant that reappears in Hensley’s formula (§4).

3. PARITY SETTLEMENT

Proposition 3.1 ([NUMERICAL], uncertified in N). For $k = 1, \dots, 12$ the point $s = \rho_k/2 = \frac{1}{4} + \frac{i}{2}\gamma_k$ is a zero of the Galerkin limit of $\det(1 - \mathcal{L}_s)$ and not of $\det(1 + \mathcal{L}_s)$ (Table 1). For the first 12 Maass cusp forms of level 1 (LMFDB labels 1.1–1.12, $R \leq 21.316$), $s = \frac{1}{2} + iR$ is a zero of $\det(1 - \mathcal{L}_s)$ for the four even forms and of $\det(1 + \mathcal{L}_s)$ for the eight odd forms (Table 2).

Method. For each target we evaluate $\det(1 \mp M_N(s))$ for $N \in \{24, 32, 40, 48, 64, 80, 96, 128, 160, 192\}$ (Arb balls, radii $< 10^{-29}$ relative), and we run a secant/Newton search for the zero of the vanishing factor at $N = 64, 96, 128, 160, 192$. For the zeta zeros, s is $\rho_k/2$ to working precision (Arb’s `zeta_zero`). For Maass forms, LMFDB gives R to 9 digits, so $|\det|$ at the LMFDB point plateaus at $\sim 10^{-9}$ – 10^{-7} , and the convergence test is the Newton-located zero.

TABLE 1. Scattering zeros $s = \rho_k/2$. Columns N : $|\det(1 - M_N(s))|$. Last two columns: $|\det(1 + M_{192}(s))|$, and $|s_* - \rho_k/2|$ for the zero s_* located at $N = 192$.

k	$\gamma_k/2$	$N = 24$	48	96	128	192	$ \det(1 + L) $	dist
1	7.06736257086	$7.4 \cdot 10^{-8}$	$5.5 \cdot 10^{-19}$	$1.8 \cdot 10^{-37}$	$1.3 \cdot 10^{-48}$	$2.1 \cdot 10^{-68}$	1.680946	$1.6 \cdot 10^{-68}$
2	10.5110198193	$3.0 \cdot 10^{-3}$	$6.2 \cdot 10^{-13}$	$5.9 \cdot 10^{-33}$	$5.4 \cdot 10^{-46}$	$7.5 \cdot 10^{-69}$	2.050078	$2.4 \cdot 10^{-69}$
3	12.5054287900	$2.8 \cdot 10^{-1}$	$4.8 \cdot 10^{-10}$	$2.4 \cdot 10^{-29}$	$4.0 \cdot 10^{-42}$	$4.0 \cdot 10^{-67}$	1.975512	$1.2 \cdot 10^{-67}$
4	15.2124380629	$1.5 \cdot 10^2$	$4.3 \cdot 10^{-6}$	$2.5 \cdot 10^{-24}$	$9.6 \cdot 10^{-37}$	$2.7 \cdot 10^{-61}$	4.173413	$3.9 \cdot 10^{-62}$
5	16.4675307938	$2.6 \cdot 10^3$	$1.7 \cdot 10^{-4}$	$3.4 \cdot 10^{-22}$	$2.0 \cdot 10^{-34}$	$8.9 \cdot 10^{-59}$	1.070118	$9.6 \cdot 10^{-60}$
6	18.7930890794	$1.0 \cdot 10^5$	$2.9 \cdot 10^{-2}$	$5.5 \cdot 10^{-19}$	$7.1 \cdot 10^{-31}$	$8.7 \cdot 10^{-55}$	7.613466	$1.2 \cdot 10^{-55}$
7	20.4593595060	$1.9 \cdot 10^7$	$3.4 \cdot 10^0$	$3.4 \cdot 10^{-16}$	$8.0 \cdot 10^{-28}$	$2.1 \cdot 10^{-51}$	8.478911	$1.9 \cdot 10^{-52}$
8	21.6635366404	$5.0 \cdot 10^8$	$3.0 \cdot 10^1$	$1.0 \cdot 10^{-14}$	$3.7 \cdot 10^{-26}$	$1.7 \cdot 10^{-49}$	5.402325	$1.6 \cdot 10^{-50}$
9	24.0025754405	$1.6 \cdot 10^{12}$	$4.6 \cdot 10^3$	$1.5 \cdot 10^{-11}$	$1.4 \cdot 10^{-22}$	$1.9 \cdot 10^{-45}$	11.389210	$1.9 \cdot 10^{-46}$
10	24.8869162388	$1.7 \cdot 10^{13}$	$6.3 \cdot 10^4$	$4.9 \cdot 10^{-10}$	$6.2 \cdot 10^{-21}$	$1.3 \cdot 10^{-43}$	10.633144	$6.3 \cdot 10^{-45}$
11	26.4851607388	$7.4 \cdot 10^{14}$	$3.5 \cdot 10^5$	$1.2 \cdot 10^{-8}$	$2.8 \cdot 10^{-19}$	$1.3 \cdot 10^{-41}$	9.800732	$8.5 \cdot 10^{-43}$
12	28.2231238485	$9.6 \cdot 10^{17}$	$7.3 \cdot 10^7$	$2.2 \cdot 10^{-6}$	$1.0 \cdot 10^{-16}$	$1.1 \cdot 10^{-38}$	17.434407	$4.4 \cdot 10^{-40}$

Diagnostic value. (i) The factor assignment is unambiguous at every target. For the zeta zeros the vanishing factor is 10^{-38} to 10^{-69} at $N = 192$. For the Maass forms it reaches the input-error plateau, and the located zero is stable. In both cases the other factor is 0.2 to 17 and stable to ≥ 6 digits. (ii) The onset N grows with height: $N \approx 48$ suffices at $\gamma_1/2 \approx 7$, but $\gamma_{12}/2 \approx 28$ needs $N \gtrsim 96$. This is

TABLE 2. Maass forms $s = \frac{1}{2} + iR$. R = Newton-located zero of the vanishing factor at $N = 192$ (first 21 digits); parity from LMFDB; columns N : $|\det(\text{vanishing factor})|$ at the LMFDB value of R (plateau = the 9-digit input error); |other| at $N = 192$; last column: change of the located R between $N = 160$ and $N = 192$.

label	R (located)	LMFDB factor	$N = 24$	48	192	other	ΔR	
1.1	9.53369526135355755434	odd	$\det(1 + L_s)$	$5.3 \cdot 10^{-5}$	$2.1 \cdot 10^{-9}$	$2.1 \cdot 10^{-9}$	1.811186	$< 10^{-40}$
1.2	12.1730083246796778495	odd	$\det(1 + L_s)$	$7.3 \cdot 10^{-2}$	$7.6 \cdot 10^{-8}$	$7.6 \cdot 10^{-8}$	0.916989	$< 10^{-40}$
1.3	13.7797513518907389442	even	$\det(1 - L_s)$	$1.1 \cdot 10^0$	$1.4 \cdot 10^{-7}$	$1.3 \cdot 10^{-7}$	2.190164	$< 10^{-40}$
1.4	14.3585095182598127798	odd	$\det(1 + L_s)$	$9.3 \cdot 10^0$	$1.7 \cdot 10^{-7}$	$7.9 \cdot 10^{-8}$	1.382636	$< 10^{-40}$
1.5	16.1380731715210305801	odd	$\det(1 + L_s)$	$3.7 \cdot 10^1$	$5.5 \cdot 10^{-6}$	$1.5 \cdot 10^{-7}$	1.668307	$< 10^{-40}$
1.6	16.6442592018998199435	odd	$\det(1 + L_s)$	$8.5 \cdot 10^1$	$1.9 \cdot 10^{-5}$	$4.3 \cdot 10^{-9}$	1.438743	$< 10^{-40}$
1.7	17.7385633810573778932	even	$\det(1 - L_s)$	$3.5 \cdot 10^3$	$1.0 \cdot 10^{-3}$	$3.9 \cdot 10^{-7}$	2.405336	$< 10^{-40}$
1.8	18.1809178345307038603	odd	$\det(1 + L_s)$	$2.8 \cdot 10^4$	$5.9 \cdot 10^{-3}$	$2.5 \cdot 10^{-7}$	1.444040	$< 10^{-40}$
1.9	19.4234814708282551916	even	$\det(1 - L_s)$	$1.2 \cdot 10^5$	$3.4 \cdot 10^{-2}$	$2.4 \cdot 10^{-7}$	0.358993	$< 10^{-40}$
1.10	19.4847138547410133641	odd	$\det(1 + L_s)$	$1.3 \cdot 10^5$	$4.1 \cdot 10^{-2}$	$3.1 \cdot 10^{-7}$	0.217328	$< 10^{-40}$
1.11	20.1066946825523128559	odd	$\det(1 + L_s)$	$1.0 \cdot 10^6$	$3.8 \cdot 10^{-1}$	$5.2 \cdot 10^{-7}$	1.526483	$1.7 \cdot 10^{-41}$
1.12	21.3157959402037187433	even	$\det(1 - L_s)$	$7.6 \cdot 10^6$	$1.6 \cdot 10^0$	$1.3 \cdot 10^{-7}$	1.386023	$1.1 \cdot 10^{-39}$

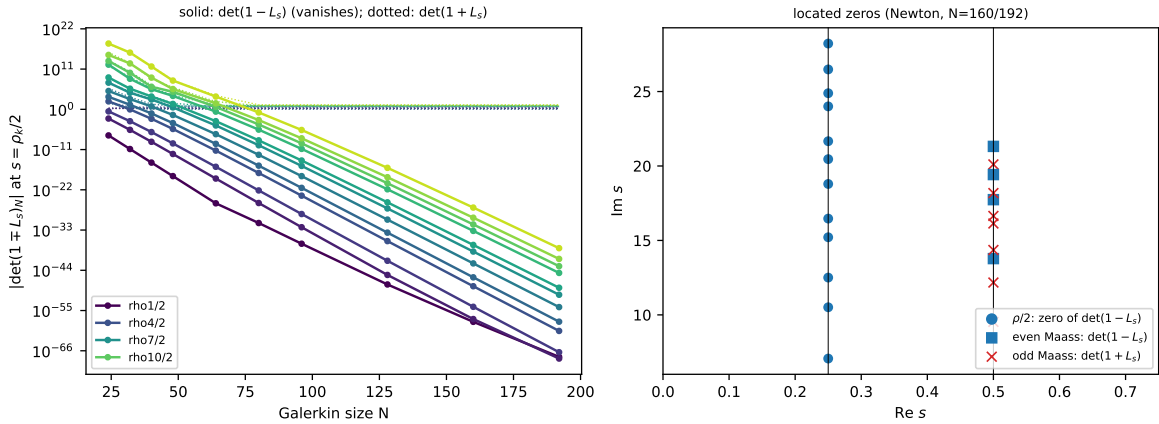


FIGURE 1. Left: $|\det(1 - M_N)|$ (solid) and $|\det(1 + M_N)|$ (dotted) at $s = \rho_k/2$, $k = 1..12$, against N . The vanishing factor decays geometrically in N , with onset later for higher zeros. The other factor is flat. Right: the located zeros: $\rho/2$ on $\text{Re } s = \frac{1}{4}$ and even Maass parameters on $\text{Re } s = \frac{1}{2}$, all zeros of $\det(1 - \mathcal{L}_s)$; odd Maass parameters zeros of $\det(1 + \mathcal{L}_s)$.

the reason the review's even-form test at $R = 13.78$ (collocation, $N = 32$) was inconclusive. At $N = 24$ the vanishing factor is 1.07 and the other 4.56. At $N = 32$ they are $3.9 \cdot 10^{-3}$ and 2.19, at $N = 40$ $6.7 \cdot 10^{-6}$ and 2.19, so the assignment is already clear at $N = 40$. (iii) The located Maass R reproduce LMFDB's 9 digits and are stable to 40 digits; e.g. $R_{1,3} = 13.779751351890738944243673281 \dots$. This is an independent (transfer-operator) computation of these eigenvalues, uncertified.

Verdict [NUMERICAL]+[CITED]. Parity is settled: **the ζ zeros sit in the $+1$ factor $\det(1 - \mathcal{L}_s) = Z_+(s)$, together with the even Maass forms and $s = 1$; the odd forms sit in $\det(1 + \mathcal{L}_s) = Z_-(s)$.** This agrees with Efrat's Corollary and Lewis–Zagier, and with the review's probe. Mayer's 1991 question is answered in the negative, as Efrat already did. The structural reason: the Eisenstein series is even under $x + iy \mapsto -x + iy$, so the continuous spectrum, and with it the scattering poles $\rho/2$, belongs to the even sector [2, p. 209].

4. HENSLEY'S $\delta_A \rightarrow 1$ LIMIT

Let $E_A \subset (0, 1)$ be the set of numbers whose continued-fraction digits all lie in $\{1, \dots, A\}$, and $\delta_A = \dim_H E_A$. Then δ_A is the unique $s > 0$ at which the PF eigenvalue of $\mathcal{L}_{s, \leq A} = \sum_{n \leq A} \mathcal{L}_n$ equals 1 (Bowen's formula), i.e. the largest real zero of $\det(1 - \mathcal{L}_{s, \leq A})$.

Theorem 4.1 (Hensley 1992 [CITED-sec]). $\delta_A = 1 - \frac{6}{\pi^2 A} - \frac{72 \log A}{\pi^4 A^2} + O\left(\frac{1}{A^2}\right)$.

Source status: the formula is read in Bourgain–Kontorovich [7, (1.12)] and in the abstract of [6] as reproduced by an indexing site. The paper itself was not accessible (paywall) [CITED-sec]. Note that Hensley's operators carry $(k+t)^{-s}$, so his s is our $2s$. *Rigorous anchor:* Jenkinson–Pollicott [8, Thm. 1] [CITED] prove $\dim_H E_2 = 0.53128050627720514162446864736847178549305910901839 \dots$ (100 digits).

Proposition 4.2 ([NUMERICAL]). (a) Our $\delta_2 = 0.531280506277205141624468647368$ agrees with [8] to 30 digits ($N = 64$; $|\delta^{(48)} - \delta^{(64)}| = 7 \cdot 10^{-30}$). (b) For A up to 10^5 (Table 3; truncation change $N = 48 \rightarrow 64$ at most $4.3 \cdot 10^{-17}$), $(1 - \delta_A)A \rightarrow 6/\pi^2 = 0.6079271$: it equals 0.6339 at $A = 100$, 0.6121 at 10^3 , 0.6085 at 10^4 , 0.60800 at 10^5 . (c) The remainder $r_A := A^2(1 - \delta_A - \frac{6}{\pi^2 A} - \frac{72 \log A}{\pi^4 A^2})$ converges. A least-squares fit $r_A = c_0 + c_1 \frac{\log A}{A} + c_2 \frac{1}{A} + c_3 \frac{(\log A)^2}{A}$ on $A \geq 200$ gives $c_0 = -1.020683$, $c_1 = -4.00$, $c_2 = 9.58$, $c_3 = 1.4222$ (max residual $2.9 \cdot 10^{-6}$); the fits on $A \geq 100$ and $A \geq 500$ give $c_0 = -1.020666$ and -1.020684 . Hence, numerically,

$$\delta_A = 1 - \frac{6}{\pi^2 A} - \frac{72 \log A}{\pi^4 A^2} + \frac{1.02068(2)}{A^2} - \frac{1.422(\log A)^2 - 4.0 \log A + 9.6}{A^3} + \dots$$

TABLE 3. δ_A ($N = 64$, 19 significant digits shown; the $N = 48 \rightarrow 64$ change is the third column), $(1 - \delta_A)A$, and r_A .

A	δ_A	N -change	$(1 - \delta_A)A$	r_A
2	0.53128050627720514	$7.2 \cdot 10^{-30}$	0.937438987	0.146684
3	0.70566090802873823	$7.8 \cdot 10^{-26}$	0.883017276	0.013230
5	0.83682944368120882	$1.5 \cdot 10^{-22}$	0.815852782	-0.149989
10	0.92573759114676577	$6.4 \cdot 10^{-20}$	0.742624089	-0.354988
20	0.96539326352521640	$1.6 \cdot 10^{-18}$	0.692134729	-0.530145
50	0.98696856765705365	$1.2 \cdot 10^{-17}$	0.651571617	-0.709349
100	0.99366111081062813	$2.3 \cdot 10^{-17}$	0.633888919	-0.807733
200	0.99688443636648922	$3.1 \cdot 10^{-17}$	0.623112727	-0.879130
500	0.99876953722073885	$3.8 \cdot 10^{-17}$	0.615231390	-0.941388
1000	0.99938793789808543	$4.1 \cdot 10^{-17}$	0.612062102	-0.970872
2000	0.99969487939874194	$4.2 \cdot 10^{-17}$	0.610241203	-0.990011
5000	0.99987820295793109	$4.3 \cdot 10^{-17}$	0.608985210	-1.004947
10000	0.99993914932497840	$4.3 \cdot 10^{-17}$	0.608506750	-1.011346
20000	0.99996958788251254	$4.3 \cdot 10^{-17}$	0.608242350	-1.015212
50000	0.99998783866619520	$4.3 \cdot 10^{-17}$	0.608066690	-1.018028
100000	0.99999391997991891	$4.3 \cdot 10^{-17}$	0.608002008	-1.019162

Where $6/\pi^2$ comes from ([THEOREM] first order, standard perturbation theory; [HEURISTIC] as a reading). Removing the digits $n > A$ removes the Gauss mass $\mu_G([0, \frac{1}{A+1})) = \log_2(1 + \frac{1}{A+1}) \sim \frac{1}{A \log 2}$. Dividing by $-P'(1) = \zeta(2)/\log 2$ (Prop. 2.1(c)) gives $1 - \delta_A \sim \frac{1}{\zeta(2)A} = \frac{6}{\pi^2 A}$. So the leading rate at which $\delta_A \rightarrow 1$ is (Gauss mass of the cusp neighbourhood)/(Lyapunov exponent), and $1/\zeta(2) = 2/(\pi \text{ vol})$ is the anomaly's constant. *Diagnostic value:* all of this is RH-blind. δ_A and its expansion are real- s PF data, and they converge smoothly to the anomaly ($\delta = 1$, Gauss measure) with no

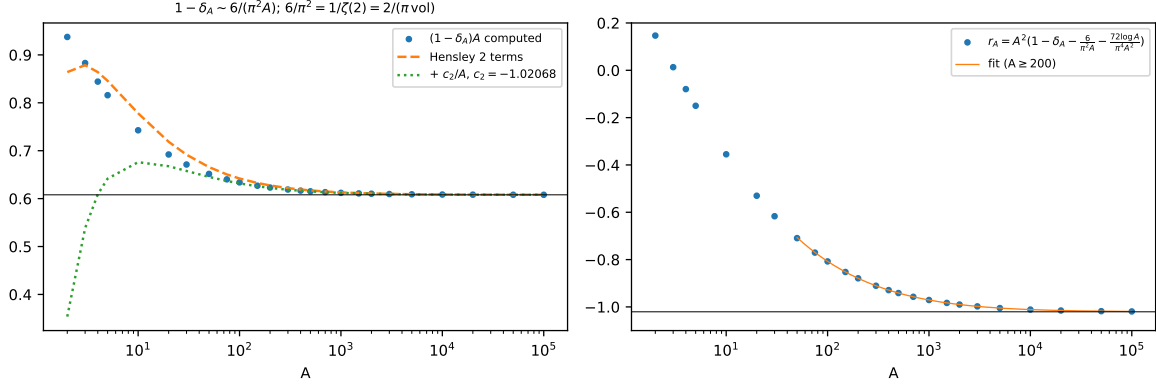


FIGURE 2. Left: $(1 - \delta_A)A$ against Hensley's two-term formula and with the fitted A^{-2} term. Right: the remainder r_A and its fit, converging to $c_0 = -1.0207$.

trace of the zeros. The review's collocation values ($N = 20$) were accurate only to $\sim 2 \cdot 10^{-7}$; for example $\delta_{10} = 0.9257375911$, not 0.9257377775 (correction C9).

5. THE CUSP TAIL: HOMOTOPY AND THE EULER-MACLAURIN COUNTERTERM

Split $\mathcal{L}_s = \mathcal{L}_{s, \leq A} + T_{s, > A}$, with $T_{s, > A}f(z) = \sum_{n > A} (z+n)^{-2s} f(\frac{1}{z+n})$. For $\text{Re } s > \frac{1}{2}$ this is a genuine splitting. For $\text{Re } s < \frac{1}{2}$ only the head is a finite sum, and T is *defined* by Hurwitz continuation: its matrix (3) uses $\zeta(\sigma, A+2)$. Consider

$$\mathcal{L}_s^{(\tau)} = \mathcal{L}_{s, \leq A} + \tau T_{s, > A}, \quad \tau \in [0, 1],$$

and follow, by Newton continuation in steps of 0.01 ($N = 64$), the zero of $\det(1 - \mathcal{L}_s^{(\tau)})$ that sits at $\rho_k/2$ at $\tau = 1$.

Proposition 5.1 ([NUMERICAL]). *For $A \in \{5, 10, 20, 40, 80, 160\}$ and $k = 1, 2$ (Fig. 3, left and middle): (a) The zero leaves the line at once: $|ds/d\tau|_{\tau=1} = 0.32, 0.39, 0.51, 0.71, 0.99, 1.39$ ($k = 1$) and $0.27, 0.32, 0.43, 0.60, 0.84, 1.18$ ($k = 2$), growing by $\approx \sqrt{2}$ per doubling of A . (b) Along $\tau \in [0, 1]$ the zero is off $\text{Re } s = \frac{1}{4}$ except at isolated crossings. $\max_\tau |\text{Re } s - \frac{1}{4}| = 0.06\text{--}0.19$. (c) At $\tau = 0$ (pure truncation) the zero sits at $0.37\text{--}0.41 + i(7.0\text{--}7.2)$ for $k = 1$, $A \geq 10$, and the distance to $\rho_k/2$ does not decrease with A : it is $0.15\text{--}0.24$ for all $A \leq 1280$ and both k (one Newton run, $A = 320$, $k = 2$, jumped to an unrelated zero). This confirms the review. (d) At $s = \rho_k/2$ the head and the tail are of comparable size: e.g. $|W_{\text{tail}}(2s)| = |\zeta(2s, A+2)| = 0.22, 0.26, 0.33, 0.46, 0.64, 0.90$ ($k = 1$), $\sim \sqrt{A}/|2s - 1|$. They cancel to $W(2s) = \zeta(2s, 2) = \zeta(2s) - 1 = -1$ exactly ([THEOREM]).*

So the tail is not a small perturbation at $\text{Re } s = \frac{1}{4}$; it grows like $A^{1-2\text{Re } s} = A^{1/2}$, and truncation is not a continuous deformation there (the review's T5 point, confirmed). What does the tail actually contain?

Theorem 5.2 (Euler-Maclaurin for the tail [THEOREM]). *For $a = A + 2$ and any $K \geq 0$,*

$$\zeta(\sigma, a) = \frac{a^{1-\sigma}}{\sigma-1} + \frac{a^{-\sigma}}{2} + \sum_{j=1}^K \frac{B_{2j}}{(2j)!}(\sigma)_{2j-1} a^{-\sigma-2j+1} + O_{K,\sigma}(a^{-\text{Re } \sigma - 2K-1}),$$

uniformly for σ in compacts. Inserted into (3), the tail operator is, up to $O(A^{-2K-1-\text{Re } 2s})$, an explicit smooth function of (s, A) . Its leading piece $\frac{a^{1-\sigma}}{\sigma-1}$ is the pole of the parabolic sum $\sum_n n^{-\sigma}$ at $\sigma = 1$, i.e. Mayer's pole at $s_0 = \frac{1}{2}$ (Thm. 1.1(b)).

Proposition 5.3 ([NUMERICAL]). *Replace $T_{s,>A}$ by its Euler–Maclaurin truncation (pole term only; $+\frac{1}{2}a^{-\sigma}$; $+K = 1, 2, 3$ Bernoulli terms) and locate the zero near $\rho_k/2$ ($N = 64$, Fig. 3 right). The distance to $\rho_1/2$ is*

A	pole only	$+\frac{1}{2}a^{-\sigma}$	$K = 1$	$K = 2$	$K = 3$	no tail
10	1.2	$4.6 \cdot 10^{-2}$	$1.0 \cdot 10^{-3}$	$3.7 \cdot 10^{-5}$	$1.5 \cdot 10^{-6}$	0.18
40	0.10	$6.4 \cdot 10^{-3}$	$1.3 \cdot 10^{-5}$	$3.8 \cdot 10^{-8}$	$1.3 \cdot 10^{-10}$	0.20
160	$7.4 \cdot 10^{-2}$	$8.8 \cdot 10^{-4}$	$1.1 \cdot 10^{-7}$	$2.2 \cdot 10^{-11}$	$5.0 \cdot 10^{-15}$	0.19
640	$5.8 \cdot 10^{-2}$	$1.1 \cdot 10^{-4}$	$9.2 \cdot 10^{-10}$	$1.2 \cdot 10^{-14}$	$1.6 \cdot 10^{-19}$	0.19
1280	$1.7 \cdot 10^{-2}$	$4.0 \cdot 10^{-5}$	$8.2 \cdot 10^{-11}$	$2.6 \cdot 10^{-16}$	$9.2 \cdot 10^{-22}$	0.20

and similarly for $\rho_2/2$ ($9 \cdot 10^{-14}$ at $A = 160$, $K = 3$). The error decays like $A^{-2K-3/2}$, which is the first omitted Euler–Maclaurin term at $\text{Re } \sigma = \frac{1}{2}$. With only the pole term it decays slowly and erratically, like $A^{-1/2}$.

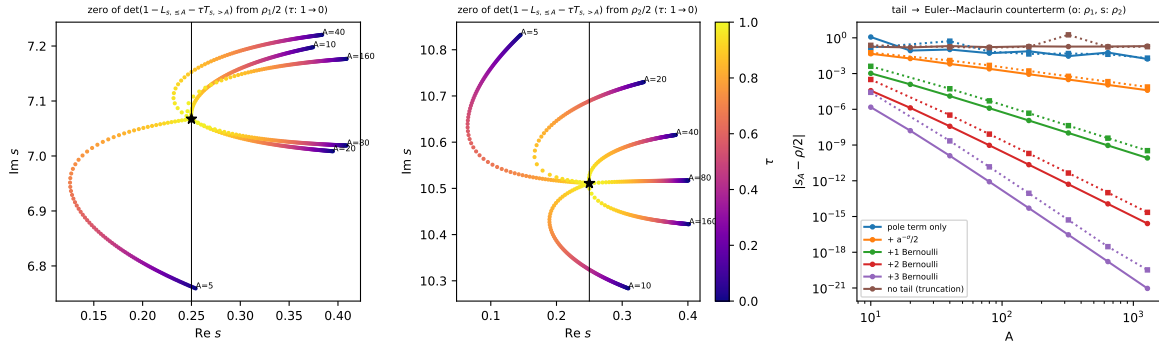


FIGURE 3. Left, middle: paths of the zero of $\det(1 - \mathcal{L}_{s, \leq A} - \tau T_{s, > A})$ from $\rho_1/2$, $\rho_2/2$ ($\star$, $\tau = 1$) to $\tau = 0$, colour = τ , for $A = 5 \dots 160$. Right: distance of the zero to $\rho_k/2$ when the tail is replaced by its Euler–Maclaurin expansion (colours), and with no tail (brown).

Corollary 5.4 (What generates the zeros [NUMERICAL]+[HEURISTIC]). *The zeros $\rho/2$ of $\det(1 - \mathcal{L}_s)$ are limits, as $A \rightarrow \infty$, of zeros of*

$$\det(1 - \mathcal{L}_{s, \leq A} - \mathcal{P}_{s, A}^{(K)}),$$

where $\mathcal{P}_{s, A}^{(K)}$ is the explicit Euler–Maclaurin counterterm (a universal smooth expression around the parabolic pole). The tail contributes no arithmetic information beyond this counterterm. All the dependence on the integers sits in the finite head $\sum_{n \leq A}$, i.e. in the first A translates by the parabolic generator $z \mapsto z + 1$. The zeros arise from the cancellation of a head of size $\asymp A^{1/2}$ against the counterterm. This is the transfer-operator form of the approximate functional equation $\zeta(\sigma) = \sum_{n \leq A} n^{-\sigma} - \frac{A^{1-\sigma}}{1-\sigma} + \dots$ [ANALOGY].

This refines the review’s T5 (“the ζ -zeros are carried entirely by the analytically continued tail”). The tail is necessary, because without it nothing converges. But it is *universal*: the same counterterm would be subtracted for any weight sequence that is asymptotically constant. “Cusp” here means the *pole* at the parabolic exponent, not the arithmetic of large partial quotients.

6. THE POSITIVITY BARRIER AT $\delta_P = \frac{1}{2}$

Proposition 6.1 ([THEOREM] (a,b), [NUMERICAL] (c)). (a) For real $s > \frac{1}{2}$, $\mathcal{L}_s$ maps functions positive on $[0, 1]$ to functions positive on $[0, 1]$. All traces $\text{tr } \mathcal{L}_s^n$ are positive (Thm. 1.1(a): sums of $\prod |T'|^{-s}/(1 - (-1)^n \prod \dots)$ over periodic points [2, p. 210]). The PF eigenvalue $\lambda(s)$ is simple, positive, decreasing and log-convex, with $\lambda(1) = 1$. (b) $s = \frac{1}{2}$ is the critical exponent δ_P of the parabolic subgroup $\langle z \mapsto z + 1 \rangle$: $\sum_n |(\gamma^n)'|^s$ on the cusp orbit is $\sum n^{-2s}$. There $\mathcal{L}_s$ has the rank-one pole $N_0 f = \frac{1}{2} f(0)$ (Thm. 1.1(b)), evaluation at the cusp point. (c) For real $s < \frac{1}{2}$ the continued operator is not positive (Fig. 4, left): $(\mathcal{L}_s 1)(x) = \zeta(2s, 1+x) < 0$ on $[0, 1]$ (e.g. -10.4 to -11.0 at $s = 0.45$, -1.73 to -2.60 at $s = 0.1$); $\text{tr } \mathcal{L}_s < 0$; the dominant eigenvalue is negative (-9.18 at $s = 0.45$, -3.57 at 0.4) and becomes a complex pair for $s \leq 0.3$; $\text{tr } \mathcal{L}_s^2 = -0.338$ at $s = 0.3$ and -10.5 at $s = 0.1$. At the same time $\det(1 - \mathcal{L}_s)$ is ≈ 0 at $s = 1$, negative on $(\frac{1}{2}, 1)$ (-0.99 at 0.8) and positive below $\frac{1}{2}$ (18.3 at 0.45).

So the three exponents of the problem are ordered

$$\underbrace{\text{Re } s = \frac{1}{4}}_{\text{zeros } \rho/2} < \underbrace{\delta_P = \frac{1}{2}}_{\text{cusp pole; positivity ends}} < \underbrace{\delta = 1}_{\text{anomaly, } \lambda=1, \text{ Res } E=3/\pi}.$$

Every zero of $\zeta(2s)$ lies on the far side of the parabolic pole from the positive regime ([THEOREM], given Remark 1.3's location of the zeros in $0 < \text{Re } s < \frac{1}{2}$). The RH line is $\delta_P/2$, the centre of the functional equation $2s \leftrightarrow 1 - 2s$ of the parabolic sum $\zeta(2s)$ [THEOREM].

7. GHOST TEST: DOES ANYTHING POSITIVE NOTICE?

7.1. Design, and what an injection means. The reflex “put an off-line zero into the tail” needs care, because the tail is *universal* (Cor. 5.4): we cannot edit ζ inside it. A ghost must therefore be a deformation of the *operator*. The question is what it preserves. We use three families.

- **G0 (homotopy).** $\mathcal{L}_{s, \leq A} + \tau T_{s, > A}$ with $\tau \in (0, 1)$ (§5). Positive for real $s > \frac{1}{2}$. It is a genuine Ruelle operator (branch weights 1 and τ , independent of s), but it lowers δ below 1: the anomaly is lost.
- **G1 (anomaly-preserving dynamical ghost).** Branch weights $w_n = 1 + \eta$ for $n \leq A = 10$ and $w_n = \theta$ for $n > A$, with $\theta = \theta(\eta) > 0$ fixed by $\det(1 - \mathcal{L}_1^w) = 0$. This is a positive Ruelle operator for the Gauss map with a locally constant potential. Its PF eigenvalue at $s = 1$ is exactly 1, so $\delta = 1$ is kept and there is a positive eigenfunction. The zero is followed by continuation in η .
- **G2 (near-invisible analytic ghost).** Tail weight $1 + \epsilon(s)$ with $\epsilon(s) = (s - 1)(a + b(s - 1))e^{-s^2}$ and $a, b \in \mathbb{R}$ solved so that $\det(1 - \mathcal{L}_s^\epsilon) = 0$ *exactly* at $s_g = \frac{1}{4} + \delta_g + \frac{i}{2}\gamma_1$. ϵ is entire, real on $\mathbb{R}$, vanishes at $s = 1$, and is tiny on $[\frac{1}{2}, \infty)$, because $|e^{-s_g^2}| = e^{49.9}$ lets $a, b \sim 10^{-23}$ produce $\epsilon(s_g) = O(0.1-1)$. Being s -dependent, it is not a Ruelle operator. It is the most honest model of “a change that is invisible on the real axis”.

Meaning. None of these is the transfer operator of a Fuchsian group, and none is “ ζ with a moved zero”. They are positive (for real $s > \frac{1}{2}$) nuclear families with the same thermodynamic skeleton as Mayer's. The test asks whether that skeleton forces $\text{Re } s = \frac{1}{4}$.

7.2. Results.

Proposition 7.1 ([NUMERICAL]). (G0) For every $\tau \in (0, 1)$ tested, the $\rho_k/2$ -branch zero is off the line (Prop. 5.1). (G1) Results with $\theta > 0$ (Fig. 4, right):

η	$\theta(\eta)$	zero from $\rho_1/2$	zero from $\rho_2/2$	$\lambda_{\text{PF}}(1)$	$\lambda_{\text{PF}}(0.6)$
0	1	$0.25 + 7.0674i$	$0.25 + 10.5110i$	1	5.3271
0.02	0.8612	$0.2780 + 7.1147i$	$0.2107 + 10.4609i$	1	—
0.05	0.6549	$0.3313 + 7.1467i$	$0.2447 + 10.3596i$	1	—
0.10	0.3160	$0.3999 + 7.1638i$	$0.3256 + 10.3032i$	1	3.2731
-0.05	1.3517	$0.2778 + 6.9821i$	$0.3079 + 10.5441i$	1	—
-0.10	1.7107	$0.2994 + 6.9574i$	$0.3348 + 10.5575i$	1	7.3861

The PF eigenvalue at $s = 1$ equals 1 to 10^{-29} in every case. The PF eigenfunction stays positive on $[0, 1]$ (at $\eta = 0.1$: 2.088, 1.656, 1.365, 1.157, 1 at $x = 0, \frac{1}{4}, \dots, 1$, against $2/(1+x)$ for $\eta = 0$). $\text{tr } L_s^2 > 0$ at $s = 0.6, 1, 2$ (automatic for positive weights). ($\eta = 0.2$ needs $\theta < 0$ and is excluded.)

(G2) With $\delta_g = 0.05$ (0.15): $a = -3.79 \cdot 10^{-24}$, $b = -6.21 \cdot 10^{-25}$ ($a = 2.19 \cdot 10^{-23}$, $b = -3.41 \cdot 10^{-24}$). The zero is re-located at $0.30 + i\gamma_1/2$ ($0.40 + i\gamma_1/2$) to double precision. $\max |\epsilon(s)|$ over the grid $s \in \{0.5, 0.55, 0.75, 1, 1.5, 2, 3, 5\}$ is $1.4 \cdot 10^{-24}$ ($9.2 \cdot 10^{-24}$), while $\epsilon(s_g) = 0.014 + 0.176i$ ($0.075 + 1.006i$). At $s = 0.6, 1, 2$ the PF eigenvalue, $\text{tr } L$, $\det(1 - L)$ and $\det(1 + L)$ change by at most $5.0 \cdot 10^{-24}$ ($3.3 \cdot 10^{-23}$); at $s = 1$ by $< 10^{-135}$. There is no zero at the functional-equation partner $\frac{1}{4} - \delta_g + i\gamma_1/2$: $|\det| = 0.156$ (1.65). As a side effect, $\rho_2/2$ moves to $0.275 + 10.418i$, because e^{-s^2} grows with $\text{Im } s$.

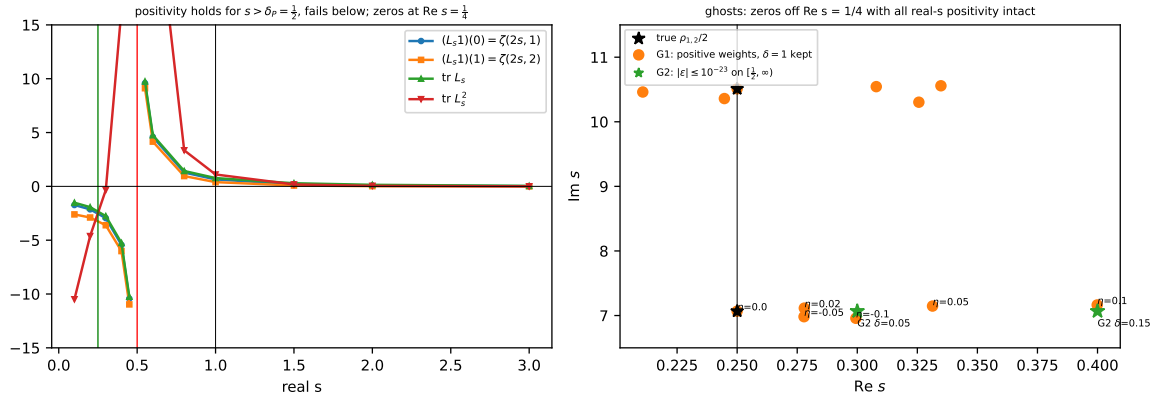


FIGURE 4. Left: positivity data of Mayer's $\mathcal{L}_s$ at real s . $(\mathcal{L}_s 1)(0)$, $(\mathcal{L}_s 1)(1)$, $\text{tr } \mathcal{L}_s$ and $\text{tr } \mathcal{L}_s^2$ are positive for $s > \delta_P = \frac{1}{2}$ (red line) and change sign through the parabolic pole. The zeros live at $\text{Re } s = \frac{1}{4}$ (green). Right: ghosts. G1 (positive weights, $\delta = 1$ kept; labels η) and G2 (invisible on $[\frac{1}{2}, \infty)$) put zeros off $\text{Re } s = \frac{1}{4}$.

7.3. No-go.

Proposition 7.2 (No-go for real- s positivity [THEOREM] (structure) + [NUMERICAL] (existence)). Let $\mathcal{P}$ be any property of a family $s \mapsto \mathcal{L}_s$ that depends only on the positive structure at real $s > \frac{1}{2}$: positivity on the cone of functions positive on $[0, 1]$; simplicity and positivity of the PF eigenvalue $\lambda(s)$; monotonicity and log-convexity of λ ; $\lambda(1) = 1$ with a positive eigenfunction ($\delta = 1$, “the anomaly”); positivity of $\text{tr } \mathcal{L}_s^n$; the sign pattern of $\det(1 \mp \mathcal{L}_s)$; or any inequality among these quantities that holds with slack. Then $\mathcal{P}$ does not imply that the eigenvalue-1 locus in $0 < \text{Re } s < \frac{1}{2}$ lies on $\text{Re } s = \frac{1}{4}$.

Proof sketch. G1 satisfies every property in the list exactly, except the numerical values of $\lambda(s)$ for $s \neq 1$. G2 changes all real- s data by $\leq 3.3 \cdot 10^{-23}$ while keeping $\epsilon(1) = 0$, so any inequality with slack $> 10^{-22}$ survives. Both have eigenvalue 1 off the line (Prop. 7.1). Existence is numerical (uncertified in N); the structural part (G1 and G2 preserve the listed properties) is elementary. For G2 it follows from $1 + \epsilon(s) > 0$ on $(\frac{1}{2}, \infty)$ and $\epsilon(1) = 0$. $\square$

Circularity flag. The real- s values of $\det(1 - \mathcal{L}_s)$ determine it everywhere by analytic continuation. So “real- s data detect RH” is trivially true *as information* and false *as positivity*. G2 shows the information sits at relative size $\sim e^{-\gamma^2/4}$ ($\sim 10^{-22}$ for γ_1), so it cannot be read by any stable positivity argument. The analogue in Lemma 73 is the Stirling factor hiding a high-zero ghost.

7.4. What the ghosts break: the cancellation input.

Proposition 7.3 ([THEOREM], elementary). (a) For branch weights w_n , identity (2) holds for all h iff w_n is constant in n . Indeed $(\mathcal{L}^w h)(z - 1) = w_1 z^{-2s} h(1/z) + \sum_{n \geq 1} w_{n+1} (z + n)^{-2s} h(\frac{1}{z+n})$. So the Lewis three-term equation, i.e. the bridge to period functions and Maass forms, is exactly the invariance of the weights under the parabolic shift $n \mapsto n + 1$. G0 and G1 break it at $n = A$. G2 breaks it for $n > A$ with an s -dependent factor. (b) The true zero set is symmetric under $s \mapsto \frac{1}{2} - \bar{s}$ (functional equation of $\zeta(2s)$). G2’s ghost has no partner (Prop. 7.1).

Verdict on J [HEURISTIC]. The outcome is a clean no-go, not a J candidate. Positivity of Mayer’s operator lives above $\delta_P = \frac{1}{2}$; the anomaly ($\delta = 1$, Gauss measure, $3/\pi$) is part of that positive structure; neither constrains the zeros. Every detector we found is an *identity*: the three-term equation (parabolic translation invariance), the functional equation, $\tau \equiv 1$. These hold exactly for the true operator and fail for the ghosts. They detect “this is not Mayer’s operator”, not “this zero is off the line”, so as RH tools they are *circular unless combined with a positivity that lives below* δ_P [OPEN]. The program’s requirement “ J must include a cancellation input” becomes concrete: the cancellation is the head-versus-pole-counterterm cancellation of Cor. 5.4, and the identity that encodes it is (2). A J would have to be a Hermitian form, positive *on period functions* (solutions of the three-term equation) for $0 < \operatorname{Re} s < \frac{1}{2}$, whose positivity forces $\operatorname{Re} s = \frac{1}{4}$. Lewis–Zagier’s pairing of period functions is the natural place to look (direction D3). Whether such a form can be non-circular is [OPEN]. A Davenport–Heilbronn-type warning applies [HEURISTIC]: a functional equation without an Euler product does not force zeros onto the line. The only arithmetic input in Mayer’s operator is that the cusp weights are all 1, and whether that suffices is exactly RH.

8. CONNECTIONS TO THE PROGRAM

- (a) *Where the cusp tail sits relative to the anomaly* ([THEOREM] for the exponents, [HEURISTIC] for the reading). There are three exponents: zeros at $\frac{1}{4}$, the parabolic pole at $\delta_P = \frac{1}{2}$ (Mayer’s rank-one residue $f \mapsto \frac{1}{2}f(0)$ at the cusp point), and the anomaly at $\delta = 1$ (PF eigenvalue 1, Gauss measure, $\operatorname{Res} E = 3/\pi = 1/\operatorname{vol}$). Lemma 69’s “anomaly positivity reaches only the edge” [TAA-L] becomes literal here. The positive structure ends at δ_P , half-way between the anomaly and the zeros. The review’s T3 (“positivity sees only the leading pole”) is confirmed for this operator, with a sharper statement: positivity sees everything above δ_P , and nothing at $\frac{1}{4}$ (Prop. 7.2).
- (b) *Lemma 72’s Theorem mb* [TAA-L]+[ANALOGY]. There, $S(X) = \Gamma(-\frac{1}{2}) \operatorname{Res} \varphi \cdot \sqrt{X} + \sum_{\rho} (\dots) X^{\rho/2-1/2}$: a smooth anomaly term plus zero fluctuations, with RH equivalent to the fluctuation decay rate. Here, $\det(1 - \mathcal{L}_s)$ near $\rho/2$ is “head – pole counterterm” (Cor. 5.4). Both say the zeros live in what remains after a universal pole term is subtracted. In Theorem mb the subtracted term is the $s = 1$ anomaly of φ . Here it is the δ_P pole of the cusp sum. These are different poles of the same parabolic structure, since $\varphi(s)$ contains $\zeta(2s - 1)/\zeta(2s)$ and the cusp sum is $\zeta(2s)$ [THEOREM].
- (c) *Lemma 7 and the review’s T6* [TAA-L]. RH is the resonance gap $\frac{1}{4}$ in the cusp channel. In Mayer’s picture the cusp channel is the even factor $\det(1 - \mathcal{L}_s)$, and the gap is the statement of Remark 1.3. Nothing here gives a gap, and Prop. 7.2 explains why thermodynamic (PF) methods cannot.
- (d) *The missing h^1 and the Deninger flow* ([t301u]–[t302u]) [ANALOGY]/[OPEN]. Mayer’s $\mathcal{L}_s$ addresses obstruction 2 of [t302u] (“it would have to be infinite-dimensional”). It is a nuclear

operator on an infinite-dimensional Banach space of holomorphic functions, and its Fredholm determinant carries all the zeros, so infinite dimension is no obstacle to a determinant formula [THEOREM]. But it fails obstruction 3 (“a flow, not a Frobenius”) in a precise way. Deninger’s conjectural shape is $\xi(s) \sim \det_\infty(\frac{s-\Theta}{2\pi} | h^1)$ for *one* operator Θ (the generator of a flow). Mayer gives $\det(1 - \mathcal{L}_s)$ for an *s-dependent family*, like a Ruelle zeta rather than a spectral determinant. Turning the family into a single generator is a version of the spectral-realization problem [OPEN]. It fails obstruction 5 (positivity) by Prop. 7.2: the only positivity available lives above δ_P . Finally, the analogue of “failure of the $\log n!/\mu$ cancellation” ([t301u]) is visible here as head versus counterterm [ANALOGY]. The smooth parts (x in $\psi(x)$; the pole counterterm here) cancel, and the zeros are what is left.

- (e) *Thin groups / Hensley* [NUMERICAL]+[HEURISTIC]. The $\delta_A \rightarrow 1$ deformation converges smoothly to the anomaly, and its expansion contains no zero information (§4). The zeros need the full parabolic sum plus its counterterm. This is why methods uniform in finite alphabets (BGS, Bourgain–Kontorovich) are RH-blind, as the review said.

9. CORRECTIONS

To the limit-set review (reviews/limitset_review.md, read-only):

- (C1) T2, “our OCR reading of Mayer 1991 suggests he conjectured they come from $\det(1 + L_s)$ ”: **refined**. The page image shows a *question* with that guess (§1). Efrat answered it. The correct statement (zeros in $\det(1 - \mathcal{L}_s)$) was, per Efrat, “conjectured by Mayer around the time we proved it”. Nothing contradicts anyone. The probe and Efrat agree.
- (C2) T2, “Efrat 1993: $Z_{\Gamma^+} = \det(1 - L_s)$; seen via Lewis–Zagier, not read in full”: **now read at source** (GDZ scan, pp. 207–210, 217–218). Confirmed, with the precise definition of $Z_\pm$ (Thm. 1.2).
- (C3) T2, “ L_s is positive for real s ”: **only for real $s > \frac{1}{2}$** . For real $s < \frac{1}{2}$ the continued operator maps 1 to a negative function, has negative or complex dominant eigenvalues, and $\text{tr } L_s^2 < 0$ at $s \leq 0.3$ (Prop. 6.1). This matters for any J idea built on “a positive operator whose spectrum carries the zeros”. At the zeros it is not positive.
- (C4) T2, “first even Maass form $R = 13.7798$ inconclusive at $N = 32$ ”: **settled**. It is a zero of $\det(1 - \mathcal{L}_s)$ (Table 2). The failure was the discretization, not the mathematics.
- (C5) T2, “a J built as a polarization of the Fredholm determinant is a concrete, non-heuristic candidate”: **not supported**. Every real- s positive structure is ghost-blind (Prop. 7.2).
- (C6) T5, “the ζ -zeros are carried entirely by the analytically continued tail”: **refined**. The tail is necessary but universal: an Euler–Maclaurin counterterm around the parabolic pole (Thm. 5.2, Prop. 5.3). The zeros come from the head-versus-counterterm cancellation (Cor. 5.4). “Cusp” means the parabolic pole, not the arithmetic of large partial quotients.
- (C7) T5, truncated zeros do not converge (distance 0.16–0.20): **confirmed** up to $A = 1280$ (distance 0.15–0.24).
- (C8) T5, the Hensley formula: **confirmed numerically** to $A = 10^5$, and the A^{-2} coefficient is now measured ($+1.02068/A^2$ in δ_A).
- (C9) T5, the probe’s δ_A values: **accurate only to $\sim 2 \cdot 10^{-7}$** (collocation $N = 20$). Examples: $\delta_{10} = 0.92573759115$ (review 0.9257377775), $\delta_{20} = 0.96539326353$ (review 0.9653935379). δ_2 was right to 7 decimals.
- (C10) T1, T4, T6: **not contradicted**. T1’s dictionary ($\text{Res } E = 3/\pi$ as the PS pole) matches Prop. 2.1 via $\zeta(2)$. T6’s Bourgain–Dyatlov quotes were not re-read here and remain as the review recorded them [CITED-sec].

To the chat [t301u]–[t302u] (numerics in num/175/chatcheck.json):

- (C11) “ $\psi(x) = \sum_k \mu(k) \log(\lfloor x/k \rfloor!)$; both sides 94.045311, 996.680912, 10013.396693”: **correct** (re-checked to 12 digits). $\psi(x) - x = -5.95, -3.32, +13.40$ and $M(x) = 1, 2, -23$: **correct**.

- (C12) “Replace $\log y!$ by $y \log y - y$; then the Möbius sum collapses to exactly x ”: **only formally**. The collapse uses the infinite sums $\sum \mu(k)/k = 0$ and $\sum \mu(k) \log k/k = -1$ (both equivalent to PNT). The actual finite sum $\sum_{k \leq x} \mu(k) (\frac{x}{k} \log \frac{x}{k} - \frac{x}{k})$ is 96.99, 995.99, 10021.10 at $x = 10^2, 10^3, 10^4$, not x . The smooth part cancels to x only asymptotically.
- (C13) “ $x \leftrightarrow \mathbb{L}$, the constant $\leftrightarrow \mathbb{K}$, $\sum x^\rho/\rho \leftrightarrow \text{trace on } h^1(\text{Spec } \mathbb{Z})$ ”: **correct as an analogy [ANALOGY]**. The pole at $s = 1$ gives x , the term at $s = 0$ gives $-\log 2\pi = -\zeta'(0)/\zeta(0)$, and the trivial zeros give $-\frac{1}{2} \log(1 - x^{-2})$.
- (C14) [t302u], obstructions 1–5: **correct as stated**, with one sharpening from this lemma. Obstruction 2 (infinite dimension) is not an obstruction to a *determinant formula*: Mayer’s nuclear $\mathcal{L}_s$ gives one. What remains are obstruction 3 (a family $\mathcal{L}_s$, not one generator Θ) and obstruction 5 (positivity below δ_P) (§8(d)).
- (C15) [t302u], “Deninger showed Γ can be written as a regularized determinant on an archimedean cohomology”: **correct as recalled**, but not re-read here [BIB] (Deninger, Invent. Math. 107 (1992)).

10. RANKED NEXT DIRECTIONS (SCORE = $0.6s + 0.4p$)

	Direction	s	p	score
D1	<i>Hecke triangle groups</i> G_q (Mayer–Mühlenbruch–Strömberg transfer operators): genuine Fuchsian groups with a cusp, $\delta = 1$, the same positive structure and anomaly $1/\text{vol}$, but non-arithmetic for $q \neq 3, 4, 6$. Locate their scattering poles: a <i>geometric</i> ghost. Do they leave $\text{Re } s = \frac{1}{4}$ -type lines, and which identity (three-term, Hecke operators) fails?	.75	.60	.69
D2	<i>Congruence subgroups and characters</i> (Chang–Mayer 2001 matrix transfer operators): Dirichlet L -functions in the cusp tail, parity per character, and a GRH ghost per channel (Lemma 73 analogue)	.60	.70	.64
D3	<i>A Hermitian form on period functions</i> : the Lewis–Zagier pairing of solutions of the three-term equation for $0 < \text{Re } s < \frac{1}{2}$. Is there a form whose positivity is equivalent to $\text{Re } s = \frac{1}{4}$, and is it circular?	.80	.35	.62
D4	<i>Certify the parity numerics</i> (Jenkinson–Pollicott Hardy-space bounds, Bandtlow-type approximation numbers) for $\rho_k/2$, $k \leq 12$, and the first Maass forms	.45	.85	.61
D5	<i>Mellin–Barnes for Mayer</i> : an exact “head + counterterm” formula for $\det(1 - \mathcal{L}_s)$ (the operator version of Lemma 72’s Theorem mb), with the counterterm’s residue structure explicit	.60	.50	.56
D6	<i>Closed form for Hensley’s A^{-2} coefficient</i> $c = 1.02068$ via second-order perturbation of the PF eigenvalue (RH-blind; completes T5)	.25	.70	.43

Still open from earlier: Lemma 73 F1 (.63), F2 (.62), F3 (.60); Lemma 72 F1 (.70); Lemma 71 F1 (.70).

11. QUESTIONS FOR J.W.

- Q1.** Do we adopt, as the official form of “ J must include a cancellation input”, the ordering *zeros at* $\frac{1}{4} < \text{parabolic pole } \delta_P = \frac{1}{2} < \text{anomaly } \delta = 1$? Positivity lives only above δ_P . The cancellation is head versus pole counterterm, encoded by the three-term (parabolic-translation) identity.
- Q2.** Do you accept the parity verdict (ζ zeros in $\det(1 - \mathcal{L}_s)$ with the even Maass forms) and the restatement “RH $\Leftrightarrow$ eigenvalue $+1$ of Mayer’s operator in $0 < \text{Re } s < \frac{1}{2}$ only on $\text{Re } s = \frac{1}{4}$ ” ([RESTATEMENT])?

- Q3.** Should the review’s “zeros carried by the cusp tail” be replaced by “zeros = finite parabolic head minus a universal pole counterterm” (Cor. 5.4)?
- Q4.** Should Prop. 7.2 (no real- s positivity of a transfer operator detects off-line zeros) be stated program-wide, alongside Lemma 72/73’s “unitarity is RH-blind”?
- Q5.** What comes next: D1 (Hecke triangle groups as geometric ghosts), D2 (congruence/character twists), D3 (a form on period functions), or an earlier direction (Lemma 73 F1–F3, Lemma 72 F1)? Any publication of Lemma 75 needs your approval.

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TAA LEMMA 76: RATIONALS IN $[0, 1)$. FAREY DISCREPANCY, FRANEL–LANDAU, AND THE FACTORIAL MOTIVE

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA. Nothing in this lemma is a step toward a proof that we can certify. Every RH-equivalent statement is labelled [RESTATEMENT]; every metaphor [ANALOGY] or [HEURISTIC]. Tags: [THEOREM] proved here (elementary) or classical with proof indicated; [CITED] checked at the source (`lit/lemma76/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, not opened; [TAA-L] earlier TAA lemma, not redone; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerical (floating point or multiprecision, not certified unless stated; exact integer arithmetic where stated).*

Question (J.W., [t306u], approved [t307u]). “It seems like this space from 0 to 1 and perhaps 0 to < 1 ought to connect to this problem of defining the factorial motive on $\mathbb{Q}$... any statement about the rational numbers and their relationship to integers ought to live in the space between 0 and 1?” We make this precise in seven parts: (A) $\mathbb{Q}/\mathbb{Z}$, roots of unity, $\widehat{\mathbb{Z}}$, Bost–Connes; (B) Franel–Landau, read at source, with numerics to $N = 10^8$; (C) Farey fractions as cusps, and exactly which part of Lemma 72/74’s cusp count the Farey discrepancy is; (D) Ramanujan sums as the Fourier side; (E) the Farey map and Lemma 75’s Gauss map; (F) a “factorial motive” on $[0, 1)$; (G) a ghost test.

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Date: October 23, 2026.

ANSWER IN BRIEF

- (1) $[0, 1) \cap \mathbb{Q}$ is a group, and its dual is $\widehat{\mathbb{Z}}$ (§2) [THEOREM]. $\mathbb{Q}/\mathbb{Z} \cong \mu_\infty \cong \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p$, with Pontryagin dual $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. Bost–Connes (read at source) build a C^* -dynamical system on $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times$ with partition function $\zeta(\beta)$, a phase transition at $\beta = 1$, and symmetry $\text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q}) \cong \widehat{\mathbb{Z}}^\times$ [CITED]. *New small identity* [THEOREM]+[NUMERICAL]: summing the BC state over the primitive points of level b gives $\sum_{d|b} \mu(b/d) d^{1-\beta}$, so the denominator zeta of the state is $\zeta(2s + \beta - 1)/\zeta(2s)$. As $\beta \rightarrow 0$ this is the Eisenstein scattering/Farey series $\zeta(2s - 1)/\zeta(2s)$. The zeros $\rho/2$ do not depend on β ; BC gives ζ as a partition function and gives *nothing* about RH.
- (2) *Franel–Landau, at source* (§3) [CITED]. Franel’s identity is $\sum_\nu \eta_\nu^2 = \frac{1}{\Phi}(I(N) - \frac{1}{12})$, where $I(N) = \int_0^1 (A_N(x) - \Phi x + \frac{1}{2})^2 dx = \frac{1}{12} \sum_{a,b \leq N} \frac{(a,b)^2}{ab} M(\frac{N}{a}) M(\frac{N}{b})$. $\text{RH} \Leftrightarrow I(N) = O(N^{1+\epsilon}) \Leftrightarrow \sum \eta_\nu^2 = O(N^{-1+\epsilon})$ (Franel) $\Leftrightarrow \sum |\eta_\nu| = O(N^{1/2+\epsilon})$ (Landau). Landau proves both directions through Littlewood’s $M(N) = O(N^{1/2+\epsilon})$, using $M(N) = \sum_\nu e^{2\pi i \rho_\nu}$ and $|M(N)| \leq 2\pi \sum |\eta_\nu|$. The sup-discrepancy is exactly $1/N$ (Dress), so it is RH-blind.
- (3) *Numerics* (§3) [NUMERICAL]. Exact Farey sequences to $N = 16000$ ($7.8 \cdot 10^7$ points). Your chat numbers are *correct*: Landau’s L^1 sum $\sum |\eta_\nu|$ goes $2.0437 \rightarrow 6.0642$ for $N = 100 \rightarrow 1600$, with slope 0.355 (0.356 to 16000). Franel’s identity holds to 10^{-9} , $N \cdot D_N = 1$ for every N , and $\sum_\nu e(\rho_\nu) = M(N)$. With the Gram identity, $I(N)$ is computed to $N = 10^8$: $I(N)/N \in [0.156, 0.212]$, with fitted exponent 1.004 on $[10^4, 10^8]$. The slope 0.355 is consistent with RH but is not evidence for it.
- (4) *Farey fractions are the cusps, and the Farey count is the constant term* (§4) [THEOREM]+[NUMERICAL]. $\Gamma_\infty \backslash \text{SL}_2(\mathbb{Z})/\Gamma_\infty \setminus \{\Gamma_\infty\} \cong \mathbb{Q}/\mathbb{Z}$, and c is the denominator. The x -Fourier modes of Lemma 74’s smoothed cusp count $\Theta_{x+iy}(T)$ are *exactly* smoothed Farey Weyl sums $\sum_c c_c(n) w(c)$: $n = 0$ gives the Farey count and $n = 1$ gives Mertens (checked to 10^{-38}). The scattering coefficient is $\phi_{\text{sc}}(s) = \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} \sum_c \varphi(c) c^{-2s}$, so Lemma 72’s Theorem mb is a transform of the Farey counting function. **Verdict on [t306]**: “the Farey discrepancy is the cusp-count fluctuation of Thm mb” is half right. Thm mb is the *count* (mode 0); the *discrepancy* is the modes $n \neq 0$. Both share $1/\zeta(2s)$ and hence the exponents. The dictionary is a Theorem; the RH content is a Restatement.
- (5) *Smoothed Farey count* (§4) [THEOREM]+[NUMERICAL]. $\sum_c \varphi(c) e^{-c^2/N^2} = \frac{3}{\pi^2} N^2 + \frac{1}{6} + \sum_\rho \frac{\Gamma(\rho/2) \zeta(\rho-1)}{2\zeta'(\rho)} N^\rho + O(N^{-2} \log N)$, verified to $5 \cdot 10^{-29}$. The anomaly $3/\pi^2$ is Lemma 74’s $3/\pi$ divided by π . The *sharp* Farey count is RH-blind: $\Phi(N) - 3N^2/\pi^2 \neq o(N)$ by an elementary argument, with $\text{RMS}/N = 0.335$ in every decade to 10^7 .
- (6) *Ramanujan sums are the Fourier side* (§5) [THEOREM]. $c_q(n)$ is the n -th Fourier coefficient of the level- q Farey measure; $\sum_{\rho \in \mathcal{F}_N} e(n\rho) = \sum_{d|n} dM(N/d)$. Franel’s identity is Parseval: $I(N) = \frac{1}{2\pi^2} \sum_{n \geq 1} W_N(n)^2/n^2$. Landau’s proof of Franel’s integral is the distribution relation $\sum_l \beta(x + l/m) = \beta(mx)$ on $\mathbb{Q}/\mathbb{Z}$.
- (7) *Gauss map* (§6) [THEOREM]+[NUMERICAL]. The Gauss map of Lemma 75 is the induced (jump) map of the Farey map, which is parabolic at the cusp 0. Even Lewis–Zagier period functions are fixed points of the Farey transfer operator $P_s f(x) = (1+x)^{-2s} [f(\frac{x}{1+x}) + f(\frac{1}{1+x})]$, and $1/x$ is the fixed point at $s = 1$ (checked). Farey order (by denominator) equidistributes to Lebesgue; Stern–Brocot order (by depth) equidistributes to d ? (Minkowski).
- (8) *Factorial motive* (§7) [THEOREM]/[HEURISTIC]/[OPEN]. Lerch’s formula says $\sqrt{2\pi}/\Gamma(a)$ is the zeta-regularized product $\prod_{n \geq 0} (n+a)$. Gauss multiplication, the Hurwitz distribution and the sawtooth are all *distribution relations on $\mathbb{Q}/\mathbb{Z}$* (checked to 10^{-40}). The identity $\psi(x) = \sum_k \mu(k) \log|x/k|!$ is exact. $\Gamma(k/n)$ -monomials are periods of CM abelian varieties / Fermat curves (Chowla–Selberg; lemniscate checked), and Gross–Koblitz is their p -adic analogue. **Verdict**: these are archimedean factors and CM periods (Hecke L -functions); they are *not* an h^1 over $\mathbb{Q}$ whose eigenvalues are the zeros of ζ [HEURISTIC]. An h^1 remains [OPEN].

- (9) *Ghost test* (§8) [NUMERICAL]+[THEOREM]. Surgery ghosts in M show up in Franel’s $I(N)$: $\delta = 0.2$ is clear from $N \approx 10^4$, $\delta = 0.1$ from $\sim 10^6$ (a marginal exponent shift), and $\delta = 0.05$ is masked to 10^8 . The same thresholds hold, in $T = N^2$, as Lemma 74’s sharp count. The Davenport–Heilbronn function (FE $\Lambda(s) = \Lambda(1-s)$ checked to 10^{-30} ; off-line zero $0.80851718+85.69934849i$) gives a Farey-type measure that is *positive* (weights ≥ 1 to $2 \cdot 10^6$), yet its Franel integral grows. **No-go:** no positivity in this chapter (Farey measure, Gram matrix, KMS states at real β) sees off-line zeros. Every RH-sensitive quantity here is a restatement of a cancellation bound on M . This agrees with Lemma 75: J needs a cancellation input.

1. CONVENTIONS AND SOURCES

Farey sequence. $\mathcal{F}_N$ is the set of reduced fractions $\rho \in (0, 1]$ with denominator $\leq N$, listed as $\rho_1 < \dots < \rho_\Phi = 1$, where $\Phi = \Phi(N) = \sum_{q \leq N} \varphi(q)$. This is Franel’s and Landau’s convention (positive reduced fractions ≤ 1), and it is the one your script `/tmp/farey.py` uses. $A_N(x) = \#\{\nu : \rho_\nu \leq x\}$; $\eta_\nu = \rho_\nu - \nu/\Phi$ (Landau’s notation); $\beta(y) = \{y\} - \frac{1}{2}$, so that $\lfloor y \rfloor = y - \frac{1}{2} - \beta(y)$ for all y . Franel’s $f(x) = [x] - x + \frac{1}{2} = -\beta(x)$. $M(x) = \sum_{n \leq x} \mu(n)$; $e(x) = e^{2\pi i x}$; $c_q(n) = \sum_{(a,q)=1} e(an/q)$. The Euler function is φ ; the Eisenstein scattering coefficient is $\phi_{\text{sc}}(s) = \xi(2s-1)/\xi(2s)$ (Lemma 72’s ϕ).

Sources read for this lemma [CITED]: Franel 1924 and Landau 1924 (Göttinger Nachrichten, pp. 198–206; page images from GDZ); Bost–Connes 1995 (IHES preprint M/95/38 = Selecta Math. 1); the abstract of Dress 1999 (JTNB 11); Lewis–Zagier 2001, Ch. IV (from Lemma 75’s folder). Everything else is [BIB] and listed as unverified in §12. Numerics are in `num/lemma76_*.py` with outputs in `num/176/`.

2. (A) $\mathbb{Q}/\mathbb{Z}$, ROOTS OF UNITY, $\widehat{\mathbb{Z}}$, AND BOST–CONNES

Proposition 2.1 (the group in $[0, 1)$ [THEOREM]). *$[0, 1) \cap \mathbb{Q}$ with addition mod 1 is the group $\mathbb{Q}/\mathbb{Z}$. Then: (i) $r \mapsto e(r)$ is an isomorphism $\mathbb{Q}/\mathbb{Z} \cong \mu_\infty$, the group of all roots of unity, and the level- q points $\{a/q : (a, q) = 1\}$ go to the primitive q -th roots. (ii) $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Q}_p/\mathbb{Z}_p$ by partial fractions. (iii) The Pontryagin dual of the discrete group $\mathbb{Q}/\mathbb{Z}$ is the profinite group $\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n = \prod_p \mathbb{Z}_p$, with pairing $\langle r, \hat{z} \rangle = e(r\hat{z})$. (iv) $\text{Aut}(\mathbb{Q}/\mathbb{Z}) = \widehat{\mathbb{Z}}^\times \cong \text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$ (Kronecker–Weber/cyclotomy).*

So J.W.’s “space between 0 and 1” has a precise meaning. As a *set* it is $[0, 1) \cap \mathbb{Q}$; as a *group* it is the torsion of $\mathbb{R}/\mathbb{Z}$; its *dual* $\widehat{\mathbb{Z}}$ is where the finite adeles $\mathbb{A}_f = \widehat{\mathbb{Z}} \otimes \mathbb{Q}$ live. The relation between rationals and integers is the exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$, and $[0, 1)$ is the fundamental domain of $\mathbb{Z}$ acting on $\mathbb{Q}$.

Theorem 2.2 (Bost–Connes 1995, Thm. 5 and Thm. 25 [CITED]). *There is a C^* -dynamical system $(\mathcal{C}_\mathbb{Q}, \sigma_t)$, the Hecke algebra of the almost-normal inclusion $P_\mathbb{Z}^+ \subset P_\mathbb{Q}^+$ of $ax + b$ groups, which contains $C^*(\mathbb{Q}/\mathbb{Z})$. It has a representation on $\ell^2(\mathbb{N}^*)$ with Hamiltonian $H\varepsilon_n = (\log n)\varepsilon_n$, so that $\text{Trace}(e^{-\beta H}) = \zeta(\beta)$. (a) For $0 < \beta < 1$ there is a unique KMS_β state (a type III_1 factor state). Its restriction to $C^*(\mathbb{Q}/\mathbb{Z})$ is a function ψ_β that is multiplicative in the denominator: $\psi_\beta(a/b) = \prod_{p^k \parallel b} p^{-k\beta \frac{1-p^{\beta-1}}{1-p^{-1}}}$. (b) For $\beta > 1$ the extreme KMS_β states $\varphi_{\beta, \alpha}(x) = \zeta(\beta)^{-1} \text{Tr}(\pi_\alpha(x)e^{-\beta H})$ are parametrized by the embeddings $\alpha : \mathbb{Q}^{\text{cycl}} \hookrightarrow \mathbb{C}$, i.e. by $\text{Gal}(\mathbb{Q}^{\text{cycl}}/\mathbb{Q})$. Their zero-temperature limits, evaluated on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$, are these embeddings, which is explicit class field theory for $\mathbb{Q}$.*

(The formula in (a) is partly lost in the OCR of the preprint. We checked it against our own derivation below.)

Proposition 2.3 (BC states on Farey levels; denominator zeta [THEOREM]+[NUMERICAL]). *For $\beta > 1$ and the standard embedding, $\varphi_\beta(e(\gamma)) = \zeta(\beta)^{-1} \sum_{n \geq 1} n^{-\beta} e(n\gamma)$. For every $b \geq 1$,*

$$(1) \quad \sum_{(a,b)=1} \varphi_\beta(e(\frac{a}{b})) = \zeta(\beta)^{-1} \sum_{n \geq 1} \frac{c_b(n)}{n^\beta} = \sum_{d|b} \mu(b/d) d^{1-\beta} = \varphi(b) \psi_\beta(1/b),$$

which therefore agrees with BC's ψ_β of Theorem 2.2(a) for all $\beta > 0$. Consequently

$$(2) \quad Z_\beta(s) := \sum_{b \geq 1} b^{-2s} \sum_{(a,b)=1} \psi_\beta(a/b) = \frac{\zeta(2s + \beta - 1)}{\zeta(2s)}.$$

As $\beta \rightarrow 0^+$, $\psi_\beta \rightarrow 1$ and $Z_\beta(s) \rightarrow \zeta(2s - 1)/\zeta(2s) = \sum_c \varphi(c) c^{-2s}$, the Farey zeta and (up to $\sqrt{\pi} \Gamma(s - \frac{1}{2})/\Gamma(s)$) the scattering coefficient $\phi_{\text{sc}}(s)$ (Prop. 4.2). At $\beta = 1$, $\psi_1 = \delta_0$ (the Haar state on $C^*(\mathbb{Q}/\mathbb{Z})$) and $Z_1 \equiv 1$.

Proof. $\sum_{(a,b)=1} e(an/b) = c_b(n)$, and Ramanujan's $\sum_n c_b(n) n^{-\beta} = \zeta(\beta) \sum_{d|b} \mu(b/d) d^{1-\beta}$ (which follows from $c_b(n) = \sum_{d|(b,n)} \mu(b/d) d$). For $b = p^k$ the right side is $p^{k(1-\beta)}(1 - p^{\beta-1}) = \varphi(p^k) \psi_\beta(1/p^k)$. Both sides are multiplicative in b . Then $\sum_b b^{-w} \sum_{d|b} \mu(b/d) d^{1-\beta} = \zeta(w + \beta - 1)/\zeta(w)$. Numerically (lemma76_bc.py): (1) and the product formula hold to $< 5 \cdot 10^{-31}$ for $b \in \{4, 9, 12, 30\}$, $\beta \in \{2, 3.5\}$, with the periodic zeta computed by Hurwitz sums. $\square$

Remark 2.4 (what BC gives and what it does not). *Gives* [CITED]: $\zeta(\beta)$ as an honest partition function; Lemma 74's Hagedorn point/anomaly at $\beta = 1$ as a phase transition (the same pole that gives $3/\pi$ there; the identification of the two poles is [THEOREM], the “thermodynamic” reading [ANALOGY]); the Galois group as a symmetry group; and $\mathbb{Q}/\mathbb{Z} \subset \Gamma_0 \backslash \Gamma/\Gamma_0$ as double cosets of a parabolic (unipotent) subgroup, the same structure as the Farey double cosets of §4. *Does not give* [THEOREM]: anything about the zeros. KMS states are defined for real β ; the trace $\zeta(\beta)$ converges only for $\beta > 1$; and (2) shows that the zeros $\rho/2$ of the denominator zeta are *independent* of β . Changing the state moves only the pole, from $s = 1$ ($\beta \rightarrow 0$) to $s = 1 - \beta/2$, and at $\beta = 1$ the numerator cancels them. This is the same pattern as Lemma 74's τ -independence and as the Farey wedge parameter x (Prop. 4.4): *the state changes amplitudes and the pole, never the zeros* [THEOREM]. That the pole slides exactly from the anomaly 1 to Lemma 75's parabolic exponent $\frac{1}{2}$ as β goes from 0 to 1 is a curiosity [HEURISTIC]. Connes' later approach (adele class space, trace formula; 1999) reduces RH to a positivity statement [BIB], which we have not opened.

3. (B) FRANEL–LANDAU, PRECISELY, AND NUMERICS TO $N = 10^8$

Theorem 3.1 (Franel 1924, pp. 198–201 [CITED]). *Put $\delta(x, N) = \sum_{a \leq N} f(ax) M[N/a]$, so that $A_N(x) = \delta(x, N) - \frac{1}{2} + \Phi x$, and $I(N) = \int_0^1 \delta(x, N)^2 dx$. Then*

$$(3) \quad I(N) = \frac{1}{12} \sum_{a,b \leq N} \frac{(a,b)^2}{ab} M\left[\frac{N}{a}\right] M\left[\frac{N}{b}\right] = \frac{1}{12} \sum_{r \leq N} \sigma(r) C\left[\frac{N}{r}\right]^2, \quad \sum_{\nu=1}^{\Phi} \eta_\nu^2 = \frac{1}{\Phi} \left(I(N) - \frac{1}{12} \right),$$

with $\sigma(r) = \prod_{p|r} (1 - p^{-2})$ and $C(n) = \sum_{a \leq n} \frac{1}{a} M[n/a]$, the summatory function of the coefficients of $\zeta(s+1)/\zeta(s)$. Hence $RH \Leftrightarrow I(N) = O(N^{1+\epsilon}) \Leftrightarrow \sum \eta_\nu^2 = O(N^{-1+\epsilon})$. The key integral is $\int_0^1 f(ax) f(bx) dx = (a,b)^2/(12ab)$.

Theorem 3.2 (Landau 1924, pp. 202–206 [CITED]). (i) $\mu(r)$ is the sum of the primitive r -th roots of unity, so $M(N) = \sum_{\nu \leq \Phi} e(\rho_\nu)$. With Littlewood (1912, [BIB]): $RH \Leftrightarrow \sum_\nu e(\rho_\nu) = O(N^{1/2+\epsilon})$. (ii) $RH \Leftrightarrow \sum_\nu |\eta_\nu| = O(N^{1/2+\epsilon})$ (Landau's (8)). (iii) $RH \Leftrightarrow \sum_\nu e(\nu/\Phi) \eta_\nu = O(N^{1/2+\epsilon})$. (iv) Under RH both sums satisfy the log log-sharpened bounds (10), (11). The proofs are two inequalities: $\sum |\eta_\nu| \leq N(\sum \eta_\nu^2)^{1/2}$ (Cauchy–Schwarz, (12)), and

$$(4) \quad |M(N)| = \left| \sum_\nu e(\nu/\Phi) (e(\eta_\nu) - 1) \right| \leq 2\pi \sum_\nu |\eta_\nu| \quad (N > 1).$$

Landau also gives an elementary proof of Franel's integral via $\sum_{l \bmod m} f(x + l/m) = f(mx)$, and he closes with “the number of true relations among (1), (2), (3), (8), (9), (10), (11) is an integer multiple of 7.”

Remark 3.3 (which “discrepancy”? [THEOREM]+[CITED]). Three different quantities are called the Farey discrepancy. Only the averaged ones carry RH. (a) Landau’s L^1 sum $\sum |\eta_\nu|$; RH $\Leftrightarrow O(N^{1/2+\epsilon})$. This is what `/tmp/farey.py` computes. (b) Franel’s L^2 sum $\sum \eta_\nu^2$; RH $\Leftrightarrow O(N^{-1+\epsilon})$; equivalently $I(N) = O(N^{1+\epsilon})$. (c) The sup (absolute) discrepancy $D_N = \sup_x |A_N(x)/\Phi - x|$, which equals *exactly* $1/N$ (Dress 1999, abstract [CITED]; order $1/N$ already in Niederreiter [BIB]). The sup is attained at the first Farey point $x = 1/N$ (Dress), which is a *lattice* effect, so (c) is RH-blind. [t306]’s “RH $\Leftrightarrow$ Farey discrepancy $O(N^{1/2+\epsilon})$ ” is correct for (a) only.

Numerics [NUMERICAL]. *Exact Farey* (`lemma76_farey.py`, `farey_exact.json`): all of $\mathcal{F}_N$, $N \leq 16000$ ($\Phi = 77816348$), sorted in float64 (gaps $\geq 1/N^2$), at 45 values of N . *Gram* (`lemma76_franel.py`, `franel_D2.json`): $I(N)$ from (3), with exact M from a Möbius sieve to 10^8 , grouped by $\lfloor N/d \rfloor$ via $(a, b)^2 = \sum_{d|(a,b)} J_2(d)$. Validated against the direct $O(N^2)$ Gram sum ($N = 10, 37, 100, 250$, agreement 10^{-15}) and against the exact Farey integral ($N = 1600, 6400, 16000$: 313.1405872, 1310.0328422, 3256.5913195, relative agreement $3 \cdot 10^{-15}$).

N	$\Phi(N)$	$\sum \eta_\nu $	$\sum \eta_\nu^2$	$I(N)$	$M(N)$	$ M /2\pi$	ND_N
100	3044	2.0437	$5.114 \cdot 10^{-3}$	15.650	1	0.159	1.000000
200	12232	3.5117	$3.183 \cdot 10^{-3}$	39.019	−8	1.273	1.000000
400	48678	3.9419	$1.554 \cdot 10^{-3}$	75.741	1	0.159	1.000000
800	194750	4.6635	$7.872 \cdot 10^{-4}$	153.385	1	0.159	1.000000
1600	778232	6.0642	$4.023 \cdot 10^{-4}$	313.141	−7	1.114	1.000000
6400	12451426	11.0180	$1.052 \cdot 10^{-4}$	1310.033	−25	3.979	1.000000
16000	77816348	14.4493	$4.185 \cdot 10^{-5}$	3256.591	−28	4.456	1.000000

TABLE 1. Exact Farey sequences. In all 45 runs: $|I - (\Phi \sum \eta^2 + \frac{1}{12})| \leq 1.2 \cdot 10^{-9}$; $\sum_\nu e(\rho_\nu) = M(N)$ to float precision; $\sum |\eta_\nu| \geq 2.37 \cdot |M(N)|/2\pi$.

- *Chat check.* Your numbers 2.04, 3.51, 3.94, 4.66, 6.06 and slope 0.35 reproduce exactly (slope 0.3548 on the five points). The definition used was Landau’s L^1 sum (a), over $(0, 1]$, with $\Phi(N)$ points. Fitted slopes: 0.356 on $[100, 16000]$, 0.365 on $[1600, 16000]$; the continuous $\int |A_N - \Phi x + \frac{1}{2}|$ gives 0.355 and 0.364.
- *What the slope means.* It is below $\frac{1}{2}$, which RH permits (RH gives only an upper bound), but it carries no RH information at this range. The L^1 sum is much smaller than $I(N)^{1/2}$ (14.4 against 57 at $N = 16000$) because $\delta(x, N)$ is spiky: its largest values sit next to fractions with small denominators (Fig. 1, left). Landau’s lower bound (4) holds with a margin of ≥ 2.37 .
- *Franel’s $I(N)$ to 10^8 .* $I(N)/N$: 0.1565 (10^2), 0.1944 (10^3), 0.2030 (10^4), 0.2032 (10^5), 0.2065 (10^6), 0.2117 (10^7), 0.2104 (10^8). Fitted exponents of I : 1.010 on $[10^2, 10^8]$, 1.004 on $[10^4, 10^8]$ and on $[10^6, 10^8]$. So $I(N) \asymp N$ with at most a slow (logarithmic-type) drift, in line with RH. The *size* $\approx 0.2N$ is set mainly by the deterministic small- y values of $M(y)$ (for $a > N/2$, $M(N/a) = 1$). The RH-sensitive part is the drift, i.e. the growth of $C(n)$.
- *Comparison with Mertens.* $M(N)/\sqrt{N}$ has RMS 0.189, 0.177, 0.173, 0.168, 0.169, 0.171 in the decades 10^2 – 10^8 (max 0.567 at 10^2 , 0.460 in $[10^7, 10^8]$). Franel’s and Landau’s statements are equivalent to Littlewood’s for M (Landau proves them via M). The Farey numerics are therefore a re-weighted look at the same M , not an independent test.

4. (C) FAREY FRACTIONS ARE CUSPS: WHAT THE FAREY DISCREPANCY IS IN LEMMA 72/74’S COUNT

Proposition 4.1 (Farey double cosets [THEOREM]). $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c, d \bmod c)$ is a bijection from $\Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z}) / \Gamma_\infty$ (with $c > 0$, modulo $\pm$) onto $\{(c, d \bmod c) : (c, d) = 1\}$. Since $d \mapsto -d/c \bmod 1$ is injective on $(\mathbb{Z}/c)^\times$, the nonidentity double cosets are in bijection with $\mathbb{Q}/\mathbb{Z}$, and c is the denominator.

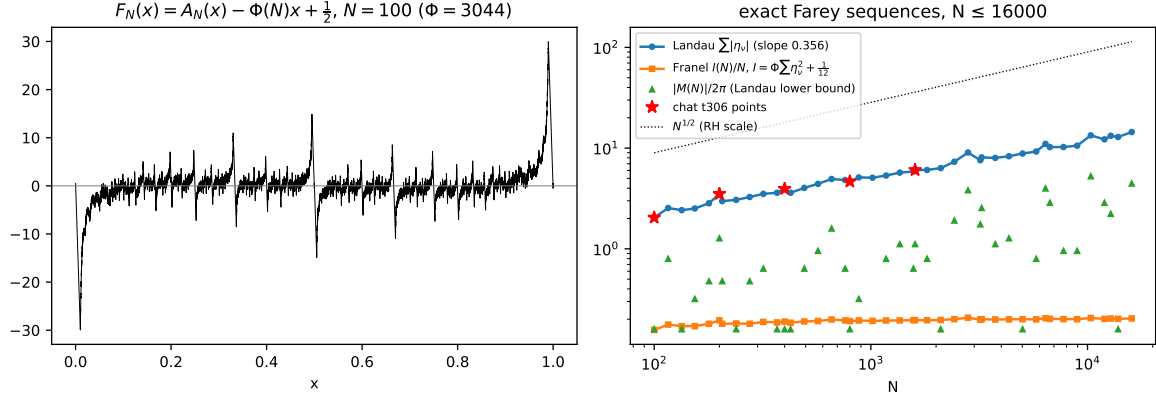


FIGURE 1. Left: Franel's $\delta(x, N) = A_N(x) - \Phi(N)x + \frac{1}{2}$ for $N = 100$. The spikes sit next to $0, \frac{1}{2}, \frac{1}{3}, \dots$. Right: exact Landau L^1 sum (with J.W.'s chat points, red stars), Franel's $I(N)/N$, Landau's lower bound $|M(N)|/2\pi$, and the RH scale $N^{1/2}$.

The cusp $\gamma^{-1}\infty = -d/c$ carries the Ford horoball of Euclidean diameter $1/c^2$, and $\mathcal{F}_N$ is the set of cusps in $(0, 1]$ whose Ford circle has diameter $\geq 1/N^2$. In Lemma 74's language, $\mathcal{F}_N$ consists of the cusps of height $\geq 1/N^2$.

Proposition 4.2 (scattering coefficient = Farey zeta [THEOREM]). $\phi_{\text{sc}}(s) = \frac{\xi(2s-1)}{\xi(2s)} = \sqrt{\pi} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{c \geq 1} \frac{\varphi(c)}{c^{2s}}$, because $\sum_c \varphi(c) c^{-w} = \zeta(w-1)/\zeta(w)$. Here $\varphi(c) = \#\{d \bmod c\}$ is the number of level- c cusps: the standard unfolding of the constant term of $E(z, s)$ over $\Gamma_\infty \backslash \Gamma/\Gamma_\infty$ (Iwaniec, [BIB]). Consequently, Lemma 72's $S(X) = \sum_m \frac{(-X)^m}{m!} \phi_{\text{sc}}(m + \frac{1}{2})$ (Thm. mb, [TAA-L]) is an integral transform of the Farey counting function $\Phi(N)$.

Theorem 4.3 (Fourier modes of the cusp count [THEOREM]+[NUMERICAL]). Let $\Theta_\tau(T) = \sum_{x \in \mathbb{P}^1(\mathbb{Q})} e^{-H_\tau(x)/T}$ be Lemma 74's smoothed height count, with $H_\tau(c:d) = |c\tau + d|^2/y$ and $\tau = x + iy$. For every $n \in \mathbb{Z}$,

$$(5) \quad \int_0^1 \Theta_{x+iy}(T) e(-nx) dx = \delta_{n0} e^{-1/(yT)} + \sqrt{\pi y T} \sum_{c \geq 1} \frac{c_c(n)}{c} \exp\left(-\frac{c^2 y}{T} - \frac{\pi^2 n^2 y T}{c^2}\right).$$

So the constant term ($n = 0$, $c_c(0) = \varphi(c)$) is a smoothed Farey count with $N^2 = T/y$, and the n -th mode is a smoothed Farey Weyl sum $\sum_{\rho \in \mathcal{F}} e(n\rho)$. For $n = 1$, $c_c(1) = \mu(c)$, so this mode is a smoothed Mertens function.

Proof. For $c \geq 1$ write $cx + d = c(x + d/c)$, split d into classes mod c , and unfold the x -integral over $[0, 1]$ into an integral over $\mathbb{R}$. Then use $\int_{\mathbb{R}} e^{-c^2 u^2/(yT)} e(-nu) du = \frac{\sqrt{\pi y T}}{c} e^{-\pi^2 n^2 y T/c^2}$ and $\sum_{(d,c)=1} e(nd/c) = c_c(n)$. The term $c = 0$ is $(0:1)$. Numerics (lemma76_cusp.py): $y = 0.7$, $T = 40$, a 128-point trapezoid in x (aliasing error $< e^{-900}$, since mode n decays like $e^{-\pi^2 n^2 y T/c^2}$ and $c \leq 68$). Left and right sides of (5) agree to $\leq 7 \cdot 10^{-38}$ for $n = 0, \dots, 4$. $\square$

Proposition 4.4 (the Farey wedge [THEOREM]). For $x \in (0, 1]$, $A_N(x) = \sum_{m \leq N} M(N/m) \lfloor mx \rfloor$ (Franel), so $\sum_c \#\{a \leq cx : (a, c) = 1\} c^{-w} = (x\zeta(w-1) - \frac{1}{2}\zeta(w) - Z_x(w))/\zeta(w)$ with $Z_x(w) = \sum_m \beta(mx) m^{-w}$. For rational $x = h/k$, $Z_x(w) = k^{-w} \sum_{r \leq k} \beta(rh/k) \zeta(w, r/k)$ is entire: the pole at $w = 1$ has residue $k^{-1} \sum_r \beta(rh/k) = 0$ by the distribution relation. So the discrepancy at x has the exponents $w = \rho$ (in N) with amplitude $\propto Z_x(\rho)/\zeta'(\rho)$. This is the exact analogue of Lemma 74's

$E(\tau, s) = Z_{Q_\tau}(s)/(2\zeta(2s))$: the exponents are universal, and the amplitudes depend on the region (x here, τ there).

Theorem 4.5 (smoothed Farey count and smoothed Mertens [THEOREM]+[NUMERICAL]). *Assume the zeros are simple, and take zero sums as symmetric limits. Then for $N > 0$,*

$$(6) \quad \sum_{c \geq 1} \varphi(c) e^{-c^2/N^2} = \frac{3}{\pi^2} N^2 + \frac{1}{6} + \sum_{\rho} \frac{\Gamma(\rho/2) \zeta(\rho-1)}{2\zeta'(\rho)} N^\rho + \sum_{n \geq 1} R_n^F(N),$$

$$(7) \quad \sum_{c \geq 1} \mu(c) e^{-c^2/N^2} = -2 + \sum_{\rho} \frac{\Gamma(\rho/2)}{2\zeta'(\rho)} N^\rho + \sum_{n \geq 1} R_n^M(N),$$

where $R_n = O(N^{-2n} \log N)$ are the residues at the double poles $w = -2n$ (Γ -pole times trivial zero).

Proof. Mellin: $e^{-u^2} = \frac{1}{2\pi i} \int \frac{1}{2} \Gamma(\frac{w}{2}) u^{-w} dw$, so the left sides are $\frac{1}{2\pi i} \int_{(3)} \frac{1}{2} \Gamma(\frac{w}{2}) N^w F(w) dw$ with $F = \zeta(w-1)/\zeta(w)$, resp. $1/\zeta(w)$. Shift left; Γ decays exponentially on vertical lines. The residues are: at $w = 2$, $\frac{1}{2} N^2/\zeta(2) = 3N^2/\pi^2$; at $w = 0$, $\zeta(-1)/\zeta(0) = \frac{1}{6}$, resp. $1/\zeta(0) = -2$; at $w = \rho$; and at $w = -2n$. *Numerics:* exact φ, μ up to $c \leq 11N$, 40 digits, 120 zero pairs, $R_1..R_3$ by contour integration. Residual LHS–RHS: $N = 100$: $2.8 \cdot 10^{-4}$ (no zeros) $\rightarrow 1.5 \cdot 10^{-6}$ (1 pair) $\rightarrow 2 \cdot 10^{-17}$ (all). $N = 1000$: $1.0 \cdot 10^{-3} \rightarrow 2.3 \cdot 10^{-6} \rightarrow 2.9 \cdot 10^{-25}$. $N = 3000$: $1.9 \cdot 10^{-3} \rightarrow 1.0 \cdot 10^{-5} \rightarrow 5.0 \cdot 10^{-29}$. For (7) at $N = 3000$ the final residual is $6.9 \cdot 10^{-27}$. The small- N floors (10^{-9} at $N = 10$) are the omitted $R_{n \geq 4}$. $\square$

Amplitudes [NUMERICAL]. The first zero has $|\Gamma(\rho_1/2) \zeta(\rho_1-1)/2\zeta'(\rho_1)| = 1.79 \cdot 10^{-5}$ in (6) and $1.46 \cdot 10^{-5}$ in (7). Compare $|1/(\rho_1 \zeta'(\rho_1))| = 0.0891$ in the sharp $M(x)$. The Gaussian smoothing suppresses the zeros by $\Gamma(\rho/2) \sim e^{-\pi\gamma/4}$, exactly as in Lemma 74's Theorem 6.1. The anomaly $3/\pi^2$ is Lemma 74's Peyre constant $3/\pi$ with the ellipse area πT replaced by the triangle area $N^2/2$: $\frac{1}{2} \cdot \frac{6}{\pi^2} = \frac{3}{\pi^2}$ [THEOREM].

Proposition 4.6 (the sharp Farey count is RH-blind [THEOREM]+[NUMERICAL]). *Let $E(N) = \Phi(N) - 3N^2/\pi^2$. For every prime p , $\max(|E(p)|, |E(p-1)|) \geq \frac{1}{2}((1 - \frac{6}{\pi^2})p - 1 + \frac{3}{\pi^2}) \approx 0.196p$. Hence $E(N) \neq o(N)$, so no exponent fit of the sharp count can see a ghost with $\delta < \frac{1}{2}$.*

Proof. $E(p) - E(p-1) = \varphi(p) - \frac{3}{\pi^2}(2p-1)$. Numerically [NUMERICAL]: the inequality holds for all 664 578 primes $p \leq 10^7$ (worst ratio 1.53). RMS of $E(N)/N$ is 0.3351 in each decade 10^2 – 10^7 , with range $[-0.092, 0.695]$ (Fig. 2, middle). $\square$

Remark 4.7 (verdict on the [t306]. claim [THEOREM]/[RESTATEMENT]/[ANALOGY]) “Farey fractions = cusps $-d/c \bmod 1$ ” is correct (Prop. 4.1) [THEOREM]. “The Farey discrepancy is the cusp-counting fluctuation of L72 Thm mb / L74's count” needs one correction. Thm mb and Lemma 74's count are the *counting* function, which is the constant mode $n = 0$ of (5) and the Dirichlet series $\zeta(2s-1)/\zeta(2s)$. The Farey *discrepancy* is the *non-constant* modes $n \neq 0$, i.e. the horizontal (x -) fluctuation of the cusp count, with Dirichlet series $\sigma_{1-2s}(n)/\zeta(2s)$. The two share $1/\zeta(2s)$ (primitivity, Lemma 72 §3.1) and therefore the exponent set $\{\rho/2\}$ in s , i.e. $T^{\rho/2} = N^\rho$. The dictionary is a [THEOREM]. The equivalence of each with RH is a [RESTATEMENT]. “The same fluctuation” without specifying the Fourier mode is an [ANALOGY]. Franel's $I(N)$ is the L^2 norm of the horizontal fluctuation (Parseval, Prop. 5.1).

5. (D) RAMANUJAN SUMS AND MÖBIUS: THE FOURIER SIDE ON $\mathbb{Q}/\mathbb{Z}$

Proposition 5.1 ([THEOREM]+[NUMERICAL]). (i) *The level- q Farey measure $\nu_q = \sum_{(a,q)=1} \delta_{a/q}$ on $\mathbb{R}/\mathbb{Z}$ has Fourier coefficients $\hat{\nu}_q(n) = c_q(n) = \sum_{d|(q,n)} \mu(q/d)d$; in particular $\hat{\nu}_q(1) = \mu(q)$. (ii) $\sum_{\rho \in \mathcal{F}_N} e(n\rho) = \sum_{q \leq N} c_q(n) = \sum_{d|n} d M(N/d) =: W_N(n)$. (iii) *Ramanujan:* $\sum_q c_q(n) q^{-s} =$*

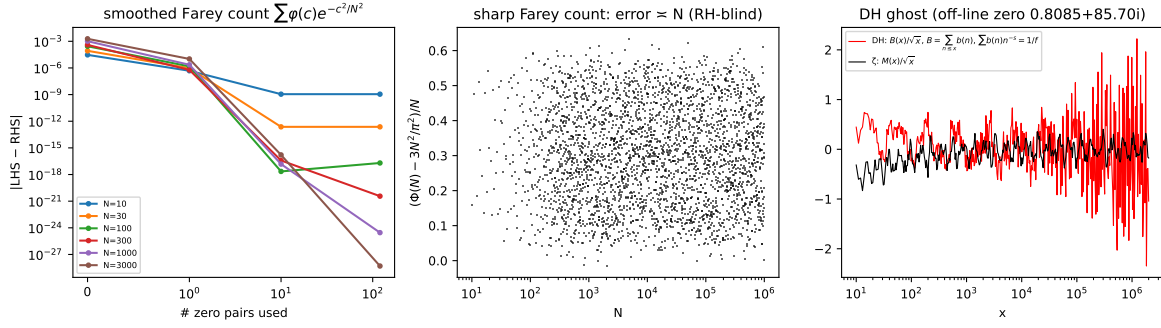


FIGURE 2. Left: smoothed Farey count (6), residual against the number of zero pairs used. Middle: the sharp Farey count error $E(N)/N$ (RH-blind, $\asymp N$). Right: the Davenport–Heilbronn ghost of §8: $B(x)/\sqrt{x}$ for $1/f_{\text{DH}}$ (red) against $M(x)/\sqrt{x}$ (black).

$\sigma_{1-s}(n)/\zeta(s)$. (iv) Since $A'_N = \sum_{\nu} \delta_{\rho_{\nu}} - \Phi$, the Fourier coefficients of $\delta(\cdot, N)$ are $W_N(n)/(2\pi i n)$ up to sign ($n \neq 0$), and Parseval gives

$$(8) \quad I(N) = \frac{1}{2\pi^2} \sum_{n \geq 1} \frac{W_N(n)^2}{n^2}.$$

This is the same as Franel’s (3), because $\sum_n \frac{1}{n^2} \sum_{d,e|n} d e M(\frac{N}{d}) M(\frac{N}{e}) = \zeta(2) \sum_{d,e} \frac{(d,e)^2}{de} M(\frac{N}{d}) M(\frac{N}{e})$.

Numerics [NUMERICAL]: (ii) holds exactly for $N = 500$, $n \leq 8$ ($W = -6, -8, -6, -12, -1, -32, -27, -20$). (iii) at $s = 2$, $n \leq 6$, truncated at $q \leq 2 \cdot 10^4$, agrees to $\leq 2 \cdot 10^{-7}$ (truncation). (8) truncated at $n \leq 4 \cdot 10^5$ gives 7.63429, 39.0176, 194.332 against the Gram values 7.63439, 39.0191, 194.372 ($N = 50, 200, 1000$); the slow tail is $\sum W^2/n^2$. **Interpretation** [THEOREM]: on $\mathbb{Q}/\mathbb{Z}$ the “position” side is the Farey points and the “momentum” side is $n \in \mathbb{Z} \subset \widehat{\mathbb{Z}}$. Möbius is the momentum-1 amplitude of each level. Franel’s theorem says that RH is the ℓ^2 size of the momentum profile $W_N(n)/n$, i.e. cancellation in μ weighted by divisors.

6. (E) THE FAREY MAP, LEMMA 75’S GAUSS MAP, STERN–BROCOT AND LEWIS–ZAGIER

Proposition 6.1 (Farey map and Gauss map [THEOREM]). *The Farey map $F(x) = \frac{x}{1-x}$ on $[0, \frac{1}{2}]$, $F(x) = \frac{1-x}{x}$ on $[\frac{1}{2}, 1]$ has inverse branches $y \mapsto \frac{y}{1+y}$ (parabolic, fixing the cusp 0) and $y \mapsto \frac{1}{1+y}$. The jump transformation $x \mapsto F^{a_1(x)}(x)$, $a_1(x) = \lfloor 1/x \rfloor$, is exactly the Gauss map $x \mapsto \{1/x\}$ of Lemma 75. Indeed, each left-branch step lowers $1/x$ by 1, and after $a_1 - 1$ such steps one right-branch step returns $\{1/x\}$. So F runs through a continued-fraction digit a in a steps, and the Gauss map does it in one. F has the infinite invariant density $1/x$; the Gauss map has the finite one $1/((1+x) \log 2)$ (Lemma 75’s Perron–Frobenius eigenfunction at $s = 1$).*

Proposition 6.2 (the Farey transfer operator and Lewis–Zagier [THEOREM]+[NUMERICAL]). *Let $P_s f(x) = (1+x)^{-2s} [f(\frac{x}{1+x}) + f(\frac{1}{1+x})]$, the transfer operator of F with weight $|\text{Jac}|^s$. If ψ is even ($\psi(x) = x^{-2s} \psi(1/x)$), then ψ satisfies Lewis’ three-term equation $\psi(x) = \psi(x+1) + (x+1)^{-2s} \psi(\frac{x}{x+1})$ if and only if $P_s \psi = \psi$ on $(0, \infty)$. For odd ψ the second branch enters with a minus sign. In particular, the Lewis–Zagier function $\psi_s^+(z) = \sum_{m,n \geq 0}^* (mz+n)^{-2s}$ (LZ Ch. IV, (3.2) [CITED]) is a fixed point of P_s , and so is $\psi = 1/x$ at $s = 1$.*

Proof. Evenness gives $\psi(x+1) = (x+1)^{-2s} \psi(\frac{1}{x+1})$; substitute. **Numerics** (lemma76_maps.py): with $\psi_s^+ = \sum_{m \geq 1} [\zeta(2s, mz) - \frac{1}{2}(mz)^{-2s}] + \frac{1}{2}\zeta(2s)$, we get $|P_s \psi_s^+ - \psi_s^+|/|\psi_s^+| \leq 3 \cdot 10^{-26}$ at $x = 0.37, 0.8$ ($s = 1.5, 2$) and $\leq 3.5 \cdot 10^{-7}$ at $x = 0.1$ or $s = 1.25 + 3i$. The second figure is limited by the

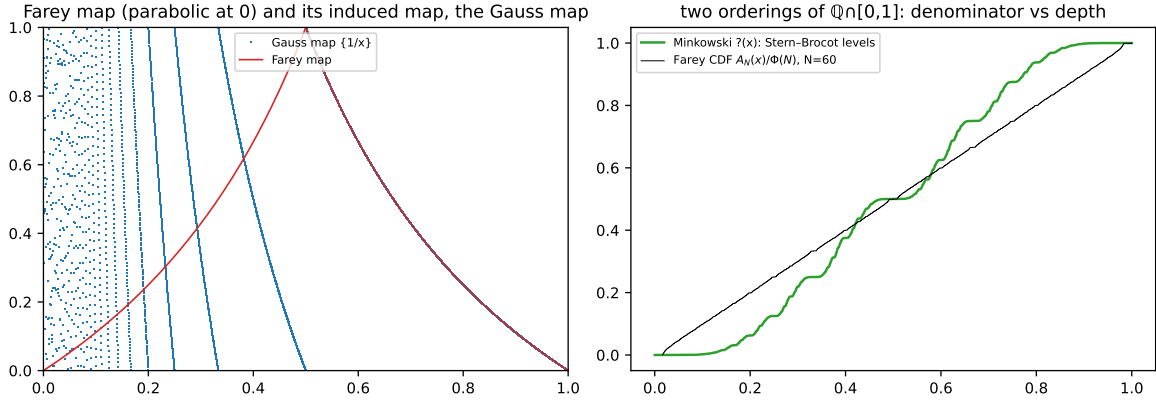


FIGURE 3. Left: the Farey map (parabolic at the cusp 0) and its induced map, the Gauss map. Right: two orderings of $\mathbb{Q} \cap [0, 1]$. The Farey CDF ($N = 60$) is close to x , while Minkowski's $?$ is the limit for Stern–Brocot depth.

convergence of the m -sum at argument $1/x$; the evenness defect shows the same floor. For $1/x$ at $s = 1$ the defect is $1.8 \cdot 10^{-15}$. $\square$

Link to Lemma 75 [THEOREM]+[TAA-L]. Lemma 75's parity verdict ($\rho/2$ and even Maass forms in $\det(1 - \mathcal{L}_s)$) is the statement that the scattering zeros carry *even* period functions, i.e. fixed points of P_s with the $+$ sign. P_s itself is not nuclear: its parabolic branch $y \mapsto y/(1+y)$ is the cusp, and that is why Lemma 75 had to induce to the Gauss operator and why the parabolic exponent $\delta_P = \frac{1}{2}$ appears there. Spectral theory of P_s (Isola, Bonanno–Graffi–Isola, Prellberg) is [BIB].

Two orderings of $\mathbb{Q} \cap [0, 1]$ [THEOREM] (classical, [BIB] for the measure-theoretic statements). Ordering by denominator gives Farey sequences, which equidistribute to Lebesgue measure; the rate is Franel–Landau. Ordering by depth in the Stern–Brocot tree (sum of continued-fraction digits) gives level sets that equidistribute to $d?$, where $?$ is Minkowski's question-mark function, singular with respect to Lebesgue. (Checked: $?(1/3) = \frac{1}{4}$, $?(2/5) = \frac{3}{8}$, $?((\sqrt{5}-1)/2) = \frac{2}{3}$, $?(\sqrt{2}-1) = \frac{2}{5}$.) The zeros of ζ enter only through the denominator order, i.e. through the height c^{-2s} (Lemma 74's $\text{Im } \gamma\tau$). The tree structure (mediants = Ford-circle tangencies = $\text{SL}_2(\mathbb{Z})$ combinatorics) is RH-blind. Fig. 3.

7. (F) A FACTORIAL MOTIVE ON $[0, 1]$?

Proposition 7.1 (Lerch as a regularized determinant [THEOREM]+[NUMERICAL]). *For $a \in (0, 1]$, $\log \Gamma(a) = \zeta'(0, a) - \zeta'(0)$ and $\zeta'(0) = -\frac{1}{2} \log 2\pi$. Equivalently, the zeta-regularized product of the spectrum $\{n + a\}_{n \geq 0}$ (number operator shifted by a) is $\prod_{n \geq 0}^{\text{reg}} (n + a) = \exp(-\zeta'(0, a)) = \sqrt{2\pi}/\Gamma(a)$. Checked at $a = \frac{1}{3}, \frac{1}{4}, \frac{2}{7}, 0.61$ to $2 \cdot 10^{-41}$ (`lemma76_motive.py`).*

Proposition 7.2 (one structure: distribution relations on $\mathbb{Q}/\mathbb{Z}$ [THEOREM]+[NUMERICAL]). *For $m \geq 1$ the following hold. (i) $\sum_{l \bmod m} \beta(x + \frac{l}{m}) = \beta(mx)$ (Landau's tool for Franel's integral). (ii) Hurwitz: $\sum_{a=1}^q \zeta(s, \frac{a}{q}) = q^s \zeta(s)$. (iii) Gauss multiplication: $\sum_{k=0}^{m-1} \log \Gamma(\frac{x+k}{m}) = \log \Gamma(x) + \frac{m-1}{2} \log 2\pi + (\frac{1}{2} - x) \log m$, which is (ii) differentiated at $s = 0$. Each says that a function on $\mathbb{Q}/\mathbb{Z}$ (or $\mathbb{R}/\mathbb{Z}$) is compatible with the multiplication-by- m maps, i.e. is a Kubert–Lang distribution ([BIB]). Checked to $\leq 3 \cdot 10^{-40}$ ($m \leq 7$; $q = 3, 5, 12$ at $s = \frac{1}{2} + 14.1347i$).*

Proposition 7.3 (the factorial identity of [t301]). *is exact [THEOREM]+[NUMERICAL]] $\log[x]! = \sum_{k \leq x} \psi(x/k)$ (Chebyshev), and Möbius inversion of $F(x) = \sum_k G(x/k)$ gives $\psi(x) = \sum_{k \leq x} \mu(k) \log[x/k]!$. Checked at $x = 10^2, 10^3, 10^4$ to 10^{-36} . Legendre: $\log[x]! = \sum_n \Lambda(n)[x/n]$,*

and $\lfloor x/n \rfloor = \frac{x}{n} - \frac{1}{2} - \beta(\frac{x}{n})$. So the arithmetic of the factorial is carried by the fractional parts $\{x/n\} \in [0, 1)$, the same sawtooth as Franel's.

Periods at rationals [CITED-sec]/[BIB]+[NUMERICAL]. Chowla–Selberg: periods of CM elliptic curves are monomials in $\Gamma(k/|d|)$. For $d = -4$: $2 \int_0^1 \frac{dt}{\sqrt{1-t^4}} = \frac{\Gamma(1/4)^2}{2\sqrt{2}\pi} = 2.62205755429211981\dots$ (checked to 10^{-22}), and the real period of $y^2 = x^3 - x$ is twice this (checked). Fermat curves $x^n + y^n = 1$ have periods $B(\frac{a}{n}, \frac{b}{n})$ (Rohrlich, [BIB]). Single values $\Gamma(k/n)$ are not periods in general; only certain monomials are (Deligne, Koblitz–Ogus, Anderson [BIB]). Gross–Koblitz is the p -adic analogue (Gauss sums as Morita Γ_p values) [BIB].

Remark 7.4 (assessment: does this give an h^1 over $\mathbb{Q}$? [HEURISTIC]/[OPEN]). What the $[0, 1)$ -factorial structure *is*: (a) the archimedean Euler factor. Γ is a regularized determinant with the *trivial* spectrum $\{-n - a\}$ (Prop. 7.1), Deninger's archimedean “ h^0 -type” local factor [BIB]. (b) A distribution on $\mathbb{Q}/\mathbb{Z}$ (Prop. 7.2), i.e. $\widehat{\mathbb{Z}}^\times$ -equivariant data. Its values are CM/abelian periods, whose motives are h^1 of CM abelian varieties over cyclotomic fields, with Hecke L -functions [BIB]. (c) The fractional-part sawtooth behind both Legendre's formula and Franel's Gram matrix. What it is *not*: none of (a)–(c) has the zeros of ζ as a spectrum. Γ has no zeros. The Kubert distribution is *universal* (it exists for every compatible system), and CM periods belong to other L -functions. [t302u]'s obstructions to an h^1 for $\text{Spec } \mathbb{Z}$ are untouched. The honest positive content is that the *gcd matrix* $\frac{(a,b)^2}{12ab} = \int \beta(ax)\beta(bx)dx$ is a Gram matrix of sawtooth functions in $L^2(0, 1)$, so the RH-sensitive object is a *vector* $(M(N/a))_a$ measured in a Hilbert space of functions on $[0, 1)$. That is the shape of the Nyman–Beurling–Báez-Duarte criterion ([BIB]; direction H1). Whether this can be made a cohomology is [OPEN]. The claim “the factorial motive lives on $[0, 1)$ ” is [HEURISTIC]: true for the archimedean and CM parts, unsupported for an h^1 .

8. (G) GHOST TEST: IS ANYTHING HERE MORE THAN A RESTATEMENT?

G1. Inventory. [THEOREM]/[RESTATEMENT]. *RH-equivalent* (restatements of Littlewood's $M(x) = O(x^{1/2+\epsilon})$): Landau's $\sum |\eta_\nu|$, Franel's $\sum \eta_\nu^2$ and $I(N)$, the Weyl sums $\sum_\nu e(\rho_\nu) = M(N)$ and $\sum e(\nu/\Phi)\eta_\nu$, the smoothed counts (6)–(7) read off to exponent $\frac{1}{2} + \epsilon$, and $I(N)$ in its Parseval form (8). Landau's own proofs, (4) and Cauchy–Schwarz, show that each is *equivalent to a bound on M* and adds no new input. *RH-blind*: the sup-discrepancy ($= 1/N$), the sharp Farey count ($E(N) \neq o(N)$), the Stern–Brocot/Minkowski structure, the leading anomaly $3/\pi^2$, and every positivity property listed in G4.

G2. Surgery ghosts in Franel's $I(N)$ (not circular for the masking question). [NUMERICAL]. Replace the first-zero term of M , $2 \text{Re}[x^{\rho_1}/(\rho_1 \zeta'(\rho_1))]$, by $\text{Re}[c_1(x^{\beta_+} + x^{\beta_-})]$ with $\beta_\pm = \frac{1}{2} \pm \delta + i\gamma_1$ and the same c_1 (this is continuous at $\delta = 0$, as in Lemma 74), and recompute $I(N)$ by (3) with the exact μ -part unchanged. *Reading*. $\delta \geq 0.2$ is unambiguous from $N \sim 10^4$ on (mean shift > 16

N	$M(N)$	true I/N	$\delta = 0.05$	0.1	0.2	0.3
10^4	−23	0.2030	0.2036	0.2059	0.2239	0.3213
10^5	−48	0.2032	0.2038	0.2063	0.2342	0.4719
10^6	212	0.2065	0.2074	0.2115	0.2606	0.7816
10^7	1037	0.2117	0.2148	0.2302	0.5636	7.704
10^8	1928	0.2104	0.2130	0.2271	0.6123	11.68
exponent $[10^6, 10^8]$		1.0044	1.0057	1.0154	1.211	1.612

TABLE 2. Franel's $I(N)/N$ with surgery ghosts (franel_D2.json, analysis.json). Exponents on $[10^4, 10^8]$: 1.0043 (true), 1.0054, 1.0125, 1.141, 1.493 ($\delta = 0.45$: 1.896).

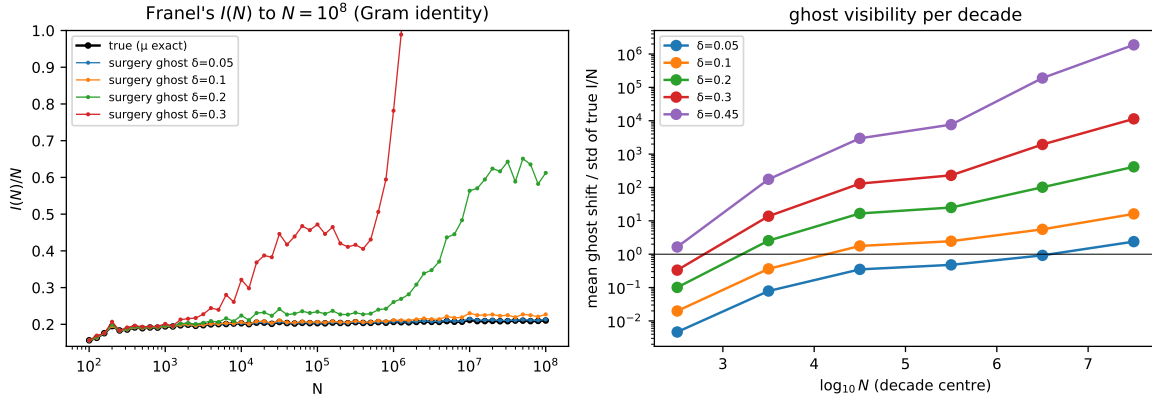


FIGURE 4. Left: Franel's $I(N)/N$ to $N = 10^8$ (exact μ), with surgery ghosts. Right: mean ghost shift per decade in units of the within-decade standard deviation of the true I/N . This criterion is optimistic; see the text for the exponent criterion.

within-decade standard deviations from 10^4). $\delta = 0.1$ shows up as an exponent excess of 0.008–0.011, comparable to the true curve's own slow drift (0.004–0.010 depending on the window), so it is marginal to 10^8 . $\delta = 0.05$ is masked to 10^8 : its shift (≈ 0.0026 at 10^8) is the size of the true decade-to-decade drift. With $T = N^2$, these thresholds are comparable to Lemma 74's sharp-count masking ($\delta \leq 0.1$ masked to $T = 10^{13}$, $N \approx 3 \cdot 10^6$; $\delta = 0.2$ visible past $T \approx 10^{9.5}$). This is not a coincidence: both measure $1/\zeta$ against lattice noise of the same order. Fig. 4.

G3. A Davenport–Heilbronn Farey measure. [NUMERICAL]+[THEOREM]. Franel's derivation uses only $[y] = y - \frac{1}{2} - \beta(y)$ and Möbius inversion. For *any* Dirichlet series $f = \sum a(n)n^{-s}$ with $a(1) = 1$, let $1/f = \sum b(n)n^{-s}$, $B = \sum_{n \leq x} b(n)$, $\zeta/f = \sum c(n)n^{-s}$ and $C = \sum_{n \leq x} c(n)$. Define the measure $\nu_N^f = \sum_{m \leq N} B(N/m) \sum_{b \leq m} \delta_{b/m}$. It gives a reduced fraction of denominator q the weight $C(N/q)$, and its Franel integral is $I_f(N) = \frac{1}{12} \sum \frac{(a,b)^2}{ab} B(\frac{N}{a}) B(\frac{N}{b})$ [THEOREM]. For $f = \zeta$, $C \equiv 1$ and ν_N^f is the counting measure on $\mathcal{F}_N$. Take the Davenport–Heilbronn function $f = 5^{-s}[\zeta(s, \frac{1}{5}) + \kappa \zeta(s, \frac{2}{5}) - \kappa \zeta(s, \frac{3}{5}) - \zeta(s, \frac{4}{5})]$, $\kappa = \frac{\sqrt{10-2\sqrt{5}-2}}{\sqrt{5}-1}$ (Titchmarsh §10.25, [BIB]; lemma76_dh.py):

- $\Lambda(s) = (5/\pi)^{s/2} \Gamma(\frac{s+1}{2}) f(s)$ satisfies $\Lambda(s) = \Lambda(1-s)$ to 10^{-30} at four test points. (We first tried a different form of the functional equation and it failed with residual $\sqrt{2}$; the form above is the one that holds.)
- Off-line zeros located by Newton to $|f| < 10^{-29}$: $0.80851718 + 85.69934849i$, $0.65083008 + 114.16334273i$, $0.57435605 + 166.47930591i$, $0.72425769 + 176.70246124i$.
- $B(x)/\sqrt{x}$ has RMS 0.35, 0.30, 0.33, 0.55, 1.04, 1.43 in the decades 10^1 – 10^6 . On $x > 10^5$, B correlates at 0.84 with the single term $2 \operatorname{Re}[x^\varrho/(\varrho f'(\varrho))]$, $\varrho = 0.8085 + 85.70i$ (predicted RMS scale 1.32 at 10^6 , observed 1.43).
- The measure ν_N^f is *positive*: $C(y) \geq 1$ for all $y \leq 2 \cdot 10^6$ (C grows like $y/f(1)$). Yet $I_f(N)/N$ goes from 0.30 (10^2) through 0.35–0.43 (10^3 – $6 \cdot 10^5$) to 0.70 and 0.93 ($8.8 \cdot 10^5$, $1.3 \cdot 10^6$), while $I_\zeta(N)/N$ stays at 0.20–0.21.

G4. Verdict: a no-go for positivity, and no new J . [THEOREM]+[NUMERICAL]+[HEURISTIC].

Proposition 8.1 (no positivity in this chapter sees off-line zeros). (i) *The Gram matrix $(\frac{(a,b)^2}{12ab}) = (\int_0^1 \beta(ax)\beta(bx)dx)$ is positive semidefinite, so $I \geq 0$ for every coefficient vector, ghost or not [THEOREM].* (ii) *Positivity of the Farey-type measure does not exclude off-line zeros: the Davenport–Heilbronn measure is positive (numerically, $y \leq 2 \cdot 10^6$) and its f has off-line zeros [NUMERICAL].* (iii) *KMS positivity is defined at real β , and the BC zeros are β -independent (Prop. 2.3) [THEOREM]. The RH-sensitive quantities are growth rates of cancellation sums (M, C, B) , as in Lemma 75’s verdict “ J needs a cancellation input”.*

The only structural feature here that separates ζ from Davenport–Heilbronn is $C \equiv 1$: the Farey measure is *exactly* the indicator of the reduced fractions (primitivity, $\mu * 1 = \delta$, i.e. the Euler product). That an Euler product plus a functional equation forces RH is the Selberg-class GRH, [CONJECTURE]. Using it would be circular. **J verdict:** no J candidate. Every functional here is either a restatement of the M -bound or RH-blind. This is a consistent no-go, and it adds a second independent family (Farey/BC) to Lemma 75’s (transfer operators).

9. PROGRAM: THREE “0 TO 1”S

(1) *Positions* $\mathbb{Q}/\mathbb{Z} = [0, 1) \cap \mathbb{Q}$ (Farey points, cusps). (2) *Momenta* $\hat{\mathbb{Z}}$ (characters, Ramanujan sums, Möbius as the momentum-1 amplitude). (3) *Exponents*, the critical strip $0 < \operatorname{Re} s < 1$ (Mellin dual of the scale group $\mathbb{R}_{>0}$). (1) and (2) are dual groups [THEOREM]. The functional equation intertwines them: Hurwitz’s formula $\zeta(1-s, \frac{h}{k}) = \frac{2\Gamma(s)}{(2\pi k)^s} \sum_{r \leq k} \cos(\frac{\pi s}{2} - \frac{2\pi r h}{k}) \zeta(s, \frac{r}{k})$ (checked to $3 \cdot 10^{-30}$, `lemma76_hurwitz.py`) exchanges a position h/k for characters while $s \leftrightarrow 1-s$ [THEOREM]. (3) is a different kind of object. “The strip is a second 0 to 1” is an [ANALOGY], made precise only through Mellin transforms like (6). In this picture RH says: the momentum profile of the Farey measures ($W_N(n)$, Möbius) has square-root cancellation, measured in the position space $L^2[0, 1)$. That is a [RESTATEMENT], but it is the cleanest form of J.W.’s “everything lives in $[0, 1)$ ” that we can state honestly.

10. CORRECTIONS TO MY CHAT CLAIMS

- C1. [t306] *Farey numbers.* Correct: 2.0437, 3.5117, 3.9419, 4.6635, 6.0642 for $N = 100 \cdots 1600$, slope 0.3548 (stated as 0.35). The definition used was Landau’s L^1 sum $\sum_{\nu \leq \Phi} |\rho_\nu - \nu/\Phi|$ over $(0, 1]$. Missing caveat: a slope below $\frac{1}{2}$ is *consistent* with RH (an upper bound) but is not evidence for it, and a $\delta \leq 0.1$ ghost would be invisible at that range (§8). Extended: 0.356 to $N = 16000$.
- C2. [t306] “*RH $\Leftrightarrow$ the Farey discrepancy is $O(N^{1/2+\epsilon})$.*” True for Landau’s L^1 sum. For Franel’s L^2 sum the exponent is $-1+\epsilon$ ($\sum \eta_\nu^2$), equivalently $I(N) = O(N^{1+\epsilon})$. *False* for the sup-discrepancy, which is exactly $1/N$ (Dress). Attribution: Franel proved the L^2 form; Landau added the L^1 form and two others; both rest on Littlewood 1912.
- C3. [t306] “*Farey fractions = cusps $-d/c \bmod 1$.*” Correct (Prop. 4.1).
- C4. [t306] “*so the discrepancy is the cusp-counting fluctuation of L72 Thm mb.*” Needs correcting. Thm mb and Lemma 74’s count are the *constant* Fourier mode (the Farey count). The discrepancy is the *non-constant* modes (Thm. 4.3). They share $1/\zeta(2s)$ and hence the exponents. Status: dictionary [THEOREM], RH content [RESTATEMENT], “same fluctuation” [ANALOGY].
- C5. [t306] $\mathbb{Q}/\mathbb{Z} = [0, 1) \cap \mathbb{Q} \cong \mu_\infty \cong \bigoplus \mathbb{Q}_p/\mathbb{Z}_p$, dual $\hat{\mathbb{Z}}$. Correct, with the caveat that $[0, 1)$ is a set of representatives and the group law is addition mod 1.
- C6. [t306] *Bost–Connes.* The partition function is $\zeta(\beta)$; that is correct. But the BC dynamics acts on the $\mathbb{N}^\times$ (scaling) part, and the $\mathbb{Q}/\mathbb{Z}$ part carries the Galois symmetry. The zeros appear nowhere in BC; the denominator zeta of its states has β -independent zeros (Prop. 2.3).
- C7. [t306] *Lerch’s $\log \Gamma(a) = \zeta'(0, a) - \zeta'(0)$.* Correct (Prop. 7.1).
- C8. [t306] “ *$T(k/n)$ are periods of Fermat/CM motives (Chowla–Selberg, Gross–Koblitz).*” Too loose. Only certain Γ -monomials are periods (e.g. $\Gamma(\frac{1}{4})^2/\sqrt{2\pi}$, Beta values). Chowla–Selberg is about

CM elliptic periods, Fermat curves have Beta periods, and Gross–Koblitz is a p -adic statement (Gauss sums via Morita’s Γ_p), not a statement about complex periods.

- C9. [t306] “The critical strip is a second 0 to 1.” [ANALOGY]. The strip is the exponent (Mellin) side, not $\mathbb{Q}/\mathbb{Z}$. The precise bridge is Hurwitz’s formula (§9).
- C10. [t301] $\psi(x) = \sum_k \mu(k) \log \lfloor x/k \rfloor!$. Exact (Prop. 7.3); the values 94.045311, 996.680912, 10013.396693 are correct.
- C11. [t301] “The smooth (Stirling) parts cancel to the main term x .” Only formally, as already corrected in Lemma 75 (finite sums 96.99, 995.99, 10021.10) [TAA-L]. Getting the main term x requires the prime number theorem (no zeros on $\text{Re } s = 1$), not termwise Stirling.
- C12. [t301] “The size of $\psi(x) - x$ is controlled by $M(x)$ (Littlewood).” Imprecise. Littlewood’s theorem is $\text{RH} \Leftrightarrow M(x) = O(x^{1/2+\epsilon})$. $\psi(x) - x$ and $M(x)$ are equivalent *as exponent statements*, but neither bounds the other pointwise (at $x = 10^4$: $\psi - x = 13.40$, $M = -23$).
- C13. [t302] *obstructions to an h^1* . Obstruction 2 was corrected in Lemma 75 [TAA-L]. The Γ /distribution structure of §7 removes none of the obstructions (Rem. 7.4).
- C14. [t299]. I have no record of the t299 claims in my files for this run, so nothing from t299 is corrected here. Please supply them if they need checking.

11. DIRECTIONS (RANKED BY $0.6s + 0.4p$)

Direction	s	p	score
H1 <i>Nyman–Beurling–Báez-Duarte as the $L^2[0, 1)$ form of this lemma.</i> Franel’s Gram matrix is $\langle \beta(ax), \beta(bx) \rangle$, while Báez-Duarte’s criterion uses the Gram matrix of $\{1/(kx)\}$ in $L^2(0, 1)$. Compare the two Gram matrices, their spectra, and the role of M (BIB, to be opened).	.80	.55	.70
H2 <i>Beurling-prime ghost.</i> Farey sets for generalized integers that have an Euler product and off-line zeros (Diamond–Montgomery–Vorhauer, BIB). This tests whether “ $C \equiv 1$ ” (primitivity) plus Farey structure forces anything.	.75	.60	.69
H3 <i>Franel–Landau over $\mathcal{O}_K$.</i> Cusps of $\text{PSL}_2(\mathcal{O}_K)$ and ζ_K zeros; ties to Lemma 73 and Lemma 74’s G2 (Peyre over $\mathcal{O}_K$).	.60	.70	.64
H4 <i>Optimal Farey statistic.</i> The weighting of the Gram form that maximizes ghost visibility at fixed N (cf. Lemma 74 G3).	.45	.80	.59
H5 <i>The non-nuclear Farey operator P_s against Mayer’s $\mathcal{L}_s$:</i> the spectral meaning of the three exponents $1, \frac{1}{2}, \frac{1}{4}$ (Isola et al., BIB).	.50	.65	.56
H6 <i>BC family $\zeta(2s + \beta - 1)/\zeta(2s)$.</i> Is there a modular object (an Eisenstein series twisted by KMS states) whose constant term it is?	.45	.65	.53

12. QUESTIONS FOR J.W.

- Q1. Shall we adopt as official: “the Farey *count* is the constant term of the cusp count (Thm mb, Lemma 74), and the Farey *discrepancy* is its horizontal fluctuation (modes $n \neq 0$, Ramanujan sums)”?
- Q2. Do you accept that Franel’s and Landau’s criteria (both L^2 and L^1) are restatements of Littlewood’s bound on M , as Landau’s own proof shows, so that they are not new handles on J ?
- Q3. Do you accept the factorial-motive verdict? Lerch, Gauss multiplication and Γ -periods give the archimedean factor, a distribution on $\mathbb{Q}/\mathbb{Z}$, and CM periods. They do not give an h^1 over $\mathbb{Q}$ (Heuristic/Open).
- Q4. Should the positivity no-go now be stated program-wide, covering Lemma 75 (transfer operators) and Lemma 76 (Farey measures, Gram matrices, KMS at real β), with “ J needs a cancellation input” as the standing conclusion?
- Q5. After Lemma 77 (codes, already queued), which next: H1 (Nyman–Beurling as the $L^2[0, 1)$ picture), H2 (Beurling-prime ghost), H3 (Franel–Landau over $\mathcal{O}_K$), or an older direction?

Nothing has been submitted or posted. Publication requires J.W.'s approval.

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TAA [TAA-L]: Lemmas 72 (Thm mb), 73, 74 (Thm 6.1, sharp count), 75 (parity, no-go); `STATE.md` §§81–84.

TAA LEMMA 77: CODES AS A FINITE LABORATORY FOR J . DUURSMA ZETA, TOEPLITZ–WEIL POSITIVITY, AND THE GOLAY/LEECH POINT

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA. The “Riemann hypothesis for codes” studied here (Duursma’s) is a statement about finitely many roots of an explicit polynomial; it is *not* the RH for $\zeta(s)$, and nothing proved about it transfers to ζ without a further argument that we do not have. Every transfer to ζ is labelled [ANALOGY] or [HEURISTIC]. Tags: [THEOREM] proved here or classical with proof indicated; [EXACT-COMP] a finite statement proved by exact rational arithmetic plus certified (Arb ball) root isolation, i.e. a computer-assisted theorem for the listed cases only; [CITED] checked at the source (`lit/lemma77/verified_refs.md`); [CITED-sec] via a named secondary source; [BIB] bibliographic record only, not opened; [TAA-L] earlier TAA lemma; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] floating/multiprecision numerics, not certified.*

Task (J.W., approved [t308u]–[t309u]). The Golay review proposed a lemma using self-dual codes as a finite setting in which a functional equation holds and RH can be true or false, so that the program’s question—*what input, if not positivity, forces the zeros onto the line?*—can be asked with everything computable. We do this in six parts: (1) Duursma zeta polynomials for Types I–IV; (2) the Toeplitz–Weil criterion as an exact finite form; (3) a ghost test; (4) transport to modular forms by Broué–Enguehard; (5) Duursma’s ultraspherical mechanism; (6) context (Leech, 691, Siegel–Weil, $\Pi_{25,1}$).

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Date: October 23, 2026.

1. ANSWER IN BRIEF

- (1) **The finite J is Krein, and κ counts ghosts.** [THEOREM] For any real polynomial P of degree 2γ satisfying Duursma’s functional equation, the Toeplitz matrix of zero power sums $T_N = [M_{i-j}]$ has inertia $(r - \kappa, \kappa, N - r)$ for every $N \geq r$. Here r is the number of distinct zeros and κ is the number of off-circle zero pairs (Theorem 4.1). So $\text{RH} \iff T_{2\gamma} \succeq 0$: one matrix size is enough, a sharper form than the review’s “for all N ”. This is a [RESTATEMENT] of the kind Li’s criterion is (L64). It adds no cancellation input, because its entries are functions of the zeros. What it does show is that the functional equation alone gives a *Pontryagin* form Π_κ . Hilbert positivity is the extra statement $\kappa = 0$.
- (2) **Extremal enumerators: no counterexample.** [EXACT-COMP] Every extremal (Mallows–Sloane) enumerator satisfies Duursma-RH, with $\kappa = 0$ certified, for Types I, III, IV with $n \leq 200$ and Type II with $n \leq 1000$. The Chinen formal enumerators (FE sign -1) do too, for $12 \leq n \leq 196$. These are *enumerators*, not codes. A negative coefficient proves that no code exists at 79 Type I lengths from $n = 32$, at 8 Type III lengths from $n = 72$, and at 42 Type IV lengths from $n = 102$ (Table 2). We found no negative coefficient for Type II up to $n = 1000$, and $[72, 36, 16]$ is open [BIB]. “Extremal $\Rightarrow$ RH” stays [OPEN] outside Duursma’s Type IV case $n \equiv 0 \pmod{6}$.
- (3) **The average is RH (Siegel–Weil for codes).** [THEOREM] The average weight enumerator of all self-dual codes of a given type and length (the mass average) has $P \propto (1 + u^{2\gamma+2})/(1 - \sqrt{2}u + u^2)$ for Type II and similar cyclotomic forms for I, III, IV, with $u = \sqrt{q}T$. Every zero lies at a root of unity on the critical circle (Theorem 6.1). At $n = 24$ the exact RH set on the Type II line is the window $A_4 \in [9.0452, 11.4644]$ around the average $A_4 = 10626/1025$, plus the isolated extremal point $A_4 = 0$ [EXACT-COMP]. At $n = 40$, a Type II code satisfies RH iff it is extremal [EXACT-COMP].
- (4) **Ghosts are Siegel-type and hug the pole.** [EXACT-COMP]/[NUMERICAL] In 209 of the 211 failing direct sums, the failure is one real off-circle pair. It sits at $s \in [0.71, 1)$, close to the pole $s = 1$ (as close as $1 - s = 4.8 \cdot 10^{-9}$). Two cases have complex off-line quadruples ($\mathbf{i}_2 \oplus e_8$ and $\mathbf{i}_2 \oplus e_8^2$). Coefficient positivity ($A_i \geq 0$, Z -coefficients > 0) cannot see these ghosts, and neither can point-count/Euler positivity ($N_m \geq 0$, $b_m \geq 0$) or the upper Weil bound. Only the Toeplitz/Li form sees them, and the real-axis sign change sees the real ones. Proposition 4.3 explains the prime-side blindness. This mirrors L75/L76.
- (5) **Broué–Enguehard transport: no-go.** [EXACT-COMP]/[NUMERICAL] At $n = 24$, map $W \mapsto f = E_{12} + \mu\Delta$. The RH set is $\{\mu = -32352/691\} \cup [97.90, 136.61]$. It is not determined by the sign of μ , by the Petersson size $|\mu|$, or by the 691 class (every integral W gives the same class, $691\mu \equiv 125$). The zero count of $L(f, s)$ on the line does not change at the RH boundaries. The Leech preimage ($A_4 = -3$) and the E_{12} preimage both fail RH. So “the Golay/Leech point” should read *the Golay point*.
- (6) **Duursma’s mechanism does not carry over as stated.** [EXACT-COMP] We confirm his ultraspherical identity (Jacobi($m + \frac{1}{2}, \frac{1}{2}$)) to 10^{-80} for Type IV with $n \equiv 0 \pmod{6}$. It *fails* (misfit $6 \cdot 10^{-5}$ –0.12) for Type IV with $n \not\equiv 0 \pmod{6}$ and for every extremal Type I–III family, even though RH holds there. In the proved case, the RH input is extremality: $d - 1$ forced vanishing conditions make P an orthogonal polynomial for a *positive weight on the zero side*. That is a cancellation input, not a positivity of the data. Chinen’s extremal enumerators have negative A_i and are still RH.

2. CONVENTIONS, SOURCES, AND WHAT WAS COMPUTED

Sources read at source [CITED]: Duursma, arXiv:math/0302172 (Defs. 1–2, $\deg P = n + 2 - d - d^\perp$, $P(1) = 1$, Thm. 2, §3 Mallows–Sloane bounds, §4); Chinen, arXiv:math/0510182 (Defs. 1.1–1.2, Thm. 1.4, Exs. 4.3–4.4, table §6); Chinen, arXiv:1709.03389 (abstract, §1). Also read: the

Golay review ([reviews/golay/](#), read-only), and STATE §73 (L64), §77 (L68), §79 (L70), §83–§85. Everything else is [BIB].

Computation. The scripts are `num/lemma77_*.py` and the outputs are `num/177/`. Weight enumerators, Gleason solves, $P(T)$ and power sums use exact `Fraction/fmpq` arithmetic. Roots come from FLINT/Arb `complex_roots` at 400 bits. A root reported with imaginary part *exactly* 0 is certified real. RH is decided on the exact polynomial $s(x^2) = R(x)R(-x)$ defined below: RH holds iff all roots of s are real and lie in $[0, 1]$. Certified isolation balls make this a proof for each listed case ([EXACT-COMP]). The codes G_{24} , QR_{32} , QR_{48} , e_8 , the tetracode, G_{12} , the ternary extended QR_{24} ($d = 9$) and the hexacode were built from generator matrices. Their full weight enumerators were enumerated by brute force, and self-orthogonality was checked (`codes.json`). Each extremal one agrees with the Gleason extremal polynomial.

3. PART 1: DUURSMAS ZETA POLYNOMIALS FOR TYPES I–IV

Definition 3.1 (Duursma; [CITED] math/0302172 Def. 1). For a linear $[n, k, d]_q$ code with weight enumerator $W = \sum A_i x^{n-i} y^i$, the zeta polynomial $P(T)$ is the unique polynomial of degree $\leq n - d$ such that $Z(T) = P(T)/((1 - T)(1 - qT))$ satisfies

$$[T^{n-d}] Z(T) (y + (x - y)T)^n = \frac{W - x^n}{q - 1}.$$

The genus is $\gamma = n + 1 - k - d$. For self-dual codes, $\deg P = 2\gamma$, $P(1) = 1$, and $P(T) = q^\gamma T^{2\gamma} P(1/(qT))$. Duursma-RH means: all zeros satisfy $|T| = q^{-1/2}$.

Lemma 3.2 (explicit coefficients; [THEOREM]). Write $Z = \sum z_j T^j$. Then $z_j = \sum_i A_i \binom{n-i}{n-d-j} / ((q - 1) \binom{n}{d+j})$, and $p_k = z_k - (1 + q)z_{k-1} + qz_{k-2}$.

Proof. The T^{n-d} coefficient of $Z(T)(y + (x - y)T)^n$ is $\sum_j z_j \binom{n}{d+j} y^{d+j} (x - y)^{n-d-j}$. On the other side, expand $x^{n-i} y^i = y^i ((x - y) + y)^{n-i}$ in the basis $y^w (x - y)^{n-w}$. The coefficient at $w = d + j$ is $\sum_{i \geq d} A_i \binom{n-i}{d+j-i}$, and $\binom{n-i}{d+j-i} = \binom{n-i}{n-d-j}$. The second formula is multiplication by $(1 - T)(1 - qT)$. $\square$

Definition 3.3 (exact RH test). Let $u = \sqrt{q}T$. The functional equation makes $u^{-\gamma}P$ a real combination of $u^j + u^{-j}$. So $u^{-\gamma}P = R(x) := p_\gamma + 2 \sum_{j \geq 1} p_{\gamma-j} q^{j/2} T_j(x)$, where $x = \frac{1}{2}(u + u^{-1})$ and T_j is the Chebyshev polynomial. Write $R = E + \sqrt{q}O$ with $E, O \in \mathbb{Q}[x]$ and set $s(x^2) := E^2 - qO^2 = R(x)R(-x)$. RH $\iff$ every root of R is real and lies in $[-1, 1]$ $\iff$ every root of s is real and lies in $[0, 1]$.

code	q	$[n, d], \gamma$	$P(T)$ (exact)	status
e_8	2	$[8, 4], 1$	$(2T^2 + 2T + 1)/5$	RH
G_{24}	2	$[24, 8], 5$	$(416T^{10} + 1248T^9 + \cdots + 78T + 13)/12597$	RH, $\kappa = 0$
QR_{32}	2	$[32, 8], 9$	$(200192T^{18} + \cdots + 391)/6633315$	RH
QR_{48}	2	$[48, 12], 13$	(in <code>codes.json</code>)	RH
G_{12}	3	$[12, 6], 1$	$(3T^2 + 3T + 1)/7$	RH
$QR_{24}^{(3)}$	3	$[24, 9], 4$	$(1053T^8 + 2106T^7 + \cdots + 78T + 13)/8398$	RH
tetracode, hexacode	3, 4	$\gamma = 0$	1	(MDS)
$G_{24} \oplus e_8$	2	$[32, 4], 13$	degree 26	fails , $\kappa = 1$

TABLE 1. Explicit codes (`codes.json`, `tables.json`). The e_8 entry disagrees with a secondary summary; see C6.

Type (q)	range	RH ($\kappa = 0$, certified)	negative coefficient $\Rightarrow$ no code
I (2)	$2 \leq n \leq 200$ even	all 100	$n = 32, 40, 42, 48, \dots$ (79 lengths); none at 30, 34, 36, 38, 44, 46, 54
II (2)	$8 \leq n \leq 1000, 8 \mid n$	all	none found
III (3)	$4 \leq n \leq 200, 4 \mid n$	all 50	$n =$ 72, 96, 120, 144, 168, 172, 192, 196
IV (4)	$2 \leq n \leq 200$ even	all 100	$n = 102, 108, 114, \dots$ (42 lengths)
formal (Chinen)	$12 \leq n \leq 196, n \equiv 4 \pmod{8}$	all	(formal: FE sign -1 , factor $2T^2 - 1$)

TABLE 2. Extremal (Mallows–Sloane) enumerators, $d = 2\lfloor n/8 \rfloor + 2, 4\lfloor n/24 \rfloor + 4, 3\lfloor n/12 \rfloor + 3, 2\lfloor n/6 \rfloor + 2$ ([CITED] math/0302172 §3). In every case d equals the bound. On-circle deviations are below $3.1 \cdot 10^{-120}$ (precision floor). Outputs: `extremal.json`, `typeII_far.json`.

Remark 3.4 (existence; [BIB]). Where a coefficient is negative, no code exists. Where every coefficient is non-negative, existence is a separate question that this lemma does not settle. Type I codes meeting $2\lfloor n/8 \rfloor + 2$ are known to exist only for a short list of small n (Ward; Conway–Sloane). Type II: the case $[72, 36, 16]$ is open, and Zhang (1999) proved that extremal Type II codes do not exist for all large n . Neither statement was opened here. We constructed extremal codes at $n = 8, 24, 32, 48$ ($e_8, G_{24}, QR_{32}, QR_{48}$).

Chinen cross-check [EXACT-COMP]. Chinen’s explicit $P_{12}, P_{20}, P_{28}, P_{36}$ (math/0510182 Ex. 4.3–4.4) agree exactly. His §6 table agrees at every printed n except $n = 60$. There the printed $A_{24} = -15622728$, while our value is -134295800 . The sum rule $\sum_i A_i = W(1, 1) = -2^{30}$ holds for ours. With the printed entry the sum would be -836395680 , so the printed entry is a misprint (`chinen60.json`).

4. PART 2: THE TOEPLITZ–WEIL CRITERION AS AN EXACT FINITE FORM

Let $\alpha_1, \dots, \alpha_{2\gamma}$ be the inverse roots, $P(T) = P(0) \prod (1 - \alpha_i T)$, and put $\beta_i = \alpha_i / \sqrt{q}$. RH means $|\beta_i| = 1$. Define the zero power sums $M_m = \sum_i \beta_i^m$ ($m \in \mathbb{Z}$) and $T_N = [M_{|i-j|}]_{0 \leq i, j < N}$. In terms of the code, $\sum \alpha_i^m = 1 + q^m - N_m$, where $Z(T) = \exp \sum_m N_m T^m / m$ defines the virtual point counts N_m (Newton’s identities, exact in $\mathbb{Q}$).

Theorem 4.1 (Krein inertia of the finite Weil form; [THEOREM]). *Let $P \in \mathbb{R}[T]$ with $P(0) \neq 0$, $\deg P = 2\gamma$, and $P(T) = q^\gamma T^{2\gamma} P(1/(qT))$. Let r be the number of distinct β_i , let c be the number of distinct β_i on $|\beta| = 1$, and let κ be the number of distinct β_i with $|\beta_i| > 1$. Then $r = c + 2\kappa$, and for every $N \geq r$*

$$\text{In}(T_N) = (c + \kappa, \kappa, N - r).$$

For $N < r$ the negative index is at most κ . Hence $\text{RH} \iff T_{2\gamma} \succeq 0 \iff T_N \succeq 0$ for all N , and κ , the Pontryagin index of the form, is the number of off-circle pairs.

Proof. The functional equation makes the multiset $\{\beta_i\}$ (with multiplicity) invariant under $\beta \mapsto 1/\beta$. Real coefficients make it invariant under $\beta \mapsto \bar{\beta}$. So it is invariant under $\beta \mapsto \beta^* := 1/\bar{\beta}$, with $m(\beta^*) = m(\beta)$. Moreover $\beta^* = \beta$ iff $|\beta| = 1$. Let $v(z) = (1, z, \dots, z^{N-1})^\top$. Then $(T_N)_{ij} = \sum_k m_k \beta_k^i \beta_k^{-j} = \sum_k m_k \beta_k^i \overline{(\beta_k^*)^j}$. Hence $T_N = V H V^*$, where $V = [v(\beta_k)]_{k \leq r}$ and H is block diagonal. H has an entry $m_k > 0$ for each on-circle β_k and a block $m_k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, of signature $(1, 1)$, for each pair $\{\beta_k, \beta_k^*\}$. For $N \geq r$, V is a Vandermonde matrix with distinct nodes and full column rank r , so Sylvester’s law of inertia gives $\text{In}(T_N) = \text{In}(H) \oplus (0, 0, N - r)$. For $N < r$, T_N is a principal submatrix of T_r , so by Cauchy interlacing its negative index is at most κ . Finally $r \leq 2\gamma$. $\square$

Corollary 4.2 (Li-type and Weil-bound forms; [THEOREM]). $RH \iff \lambda_m := 2\gamma - M_m \geq 0$ for all $m \geq 1 \iff N_m \geq q^m + 1 - 2\gamma q^{m/2}$ for all m . The upper bound $N_m \leq q^m + 1 + 2\gamma q^{m/2}$ (that is, $M_m \geq -2\gamma$) does not detect positive real ghosts.

Proof. $(\Rightarrow) |M_m| \leq 2\gamma$. $(\Leftarrow)$ Suppose $\rho = \max |\beta_i| > 1$, and let $\beta = \rho e^{i\varphi_j}$ be the zeros of maximal modulus. Then $M_m = \rho^m \sum_j m_j \cos(m\varphi_j) + O(\rho_2^m)$ with $\rho_2 < \rho$. By Dirichlet's simultaneous approximation theorem, $\cos m\varphi_j > \frac{1}{2}$ for all j at infinitely many m , so $M_m \rightarrow +\infty$ along these m . A real $\beta > 1$ adds $\beta^m > 0$ to M_m and never pushes it below -2γ . $\square$

Proposition 4.3 (pole shield; [THEOREM]). If $|\alpha_i| < q$ for all i , i.e. every zero has $\operatorname{Re} s < 1$ with $T = q^{-s}$, then $N_m = q^m(1 + o(1))$ and the Euler exponents $b_m = \frac{1}{m} \sum_{d|m} \mu(m/d) N_d = \frac{q^m}{m}(1 + o(1))$. Both are therefore positive for all large m , whatever ghosts lie in $\frac{1}{2} < \operatorname{Re} s < 1$.

The proof is immediate from $N_m = 1 + q^m - \sum \alpha_i^m$. Prime-side positivity can therefore only see ghosts through small- m accidents. Over all 224 direct sums with $\gamma > 0$ (`sums_alpha.json`), $\max_i |\alpha_i|/q < 1$ always holds. Yet its supremum approaches 1: $1 - 4.6 \cdot 10^{-9}$ for $\mathbf{i}_2^4 \oplus e_8^2 \oplus G_{24}^2$ [NUMERICAL]. Whether $|\alpha| < q$ holds for every self-dual code is [OPEN] (direction K4).

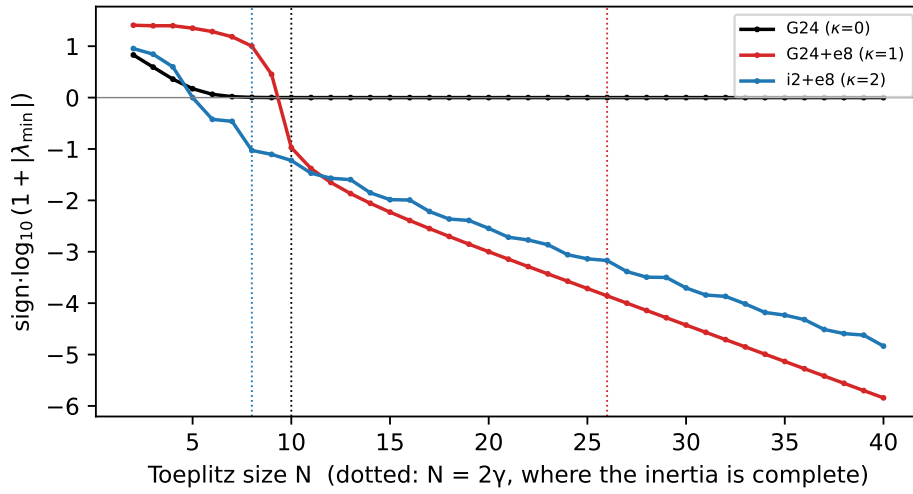


FIGURE 1. Minimal eigenvalue of T_N (signed log scale). G_{24} ($\kappa = 0$): $\lambda_{\min}(T_{10}) = 1.18 \cdot 10^{-4} > 0$, and the kernel has $|\lambda| < 10^{-59}$ for $N > 10$ (rank 10). $G_{24} \oplus e_8$ ($\kappa = 1$): $\lambda_{\min}(T_{11}) = -23.050$, $\lambda_{\min}(T_{41}) = -9.6487 \cdot 10^5$. $\mathbf{i}_2 \oplus e_8$ ($\kappa = 2$). The number of negative eigenvalues equals κ for $N \geq 2\gamma$ in every case (`ghost.json`).

Answer to (2). The Toeplitz–Weil criterion is a *restatement*, with no cancellation input. What it adds is structure. (a) The finite size 2γ suffices. (b) The functional equation, the analogue of MacWilliams duality, gives exactly a Pontryagin Π_κ form, so “unitarity is automatic, positivity is the extra theorem” is literally true in this finite model [THEOREM]. (c) There exist genuine self-dual codes with $\kappa = 1, 2, 3$ (Part 3). So no argument that uses only the functional equation, self-duality and $A_i \geq 0$ can prove RH for codes. The same no-go holds for ζ by analogy only [ANALOGY].

5. PART 3: THE GHOST TEST

A *ghost* is an off-circle pair $\{\beta, 1/\bar{\beta}\}$. We use $T = q^{-s}$, so the critical circle is $\operatorname{Re} s = \frac{1}{2}$ and the pole of Z is at $s = 1$. The test (`lemma77_ghost.py`, `lemma77_sums.py`) runs every positivity used in L64–L76 on genuine self-dual codes. Each code is built as a direct sum of the explicit codes of Table 1 together with $\mathbf{i}_2 = \{00, 11\}$ (and its $\mathbb{F}_4$ analogue $\mathbf{i}_2^{F_4}$ for Type IV).

criterion	ζ / TAA version	code version (this lemma)
Weil (L1)	explicit-formula form $W(g * \tilde{g}) \geq 0$ for all g	$c^* T_N c = \sum_i \sum_j c_j \beta_i^j ^2$ -type form, finite rank r
Li (L64)	$\lambda_n \geq 0$, infinitely many conditions	$\lambda_m = 2\gamma - M_m \geq 0$; equivalent to $T_{2\gamma} \succeq 0$
L64 Toeplitz / Bochner	truncations W_A ; positivity in the limit	exact at the single size $N = 2\gamma$ (Thm. 4.1)
FE alone	Krein/Pontryagin structure (Vol. 3 thesis)	$T_N \in \Pi_\kappa$ with κ = number of off-circle pairs: <i>literally</i> Krein
what proves it	unknown	curves: Hodge index / Castelnuovo positivity [BIB]; Type IV: Gegenbauer orthogonality (Part 5)

TABLE 3. Comparison. All criteria in rows 1–3 are [RESTATEMENT]: they are written in terms of the zeros. The entries of T_N are explicit rational functions of the A_i times powers of $q^{-1/2}$, but no combinatorial mechanism is known that makes the form positive.

Census [EXACT-COMP]. There are 228 direct sums: $q = 2$ up to $n = 72$, $q = 3$ up to 48, $q = 4$ up to 36. Only 17 satisfy RH: $\mathbf{i}_2, \mathbf{i}_2^2, \dots, \mathbf{i}_2^5, e_8, e_8^2, \mathbf{i}_2^2 \oplus e_8, G_{24}, QR_{32}$, the tetracode and its square, $G_{12}, QR_{24}^{(3)}, \mathbf{i}_2^{F_4}$ and its square, and the hexacode. Of the 211 failures, 209 have $\kappa = 1$ with a single *real* pair. The other two are $\mathbf{i}_2 \oplus e_8$ ($\kappa = 2$, one complex quadruple) and $\mathbf{i}_2 \oplus e_8^2$ ($\kappa = 3$: one real pair plus one complex quadruple).

code	(n, q, γ)	κ	ghost s	Li m_0	T_N	$N_m \geq 0$	$b_m \geq 0$	up. Weil
G_{24}	(24,2,5)	0	—	—	$\succeq 0$	yes	yes	yes
$G_{24} \oplus e_8$	(32,2,13)	1	0.97046119213	11	1 neg	yes	yes	yes
e_8^3	(24,2,9)	1	0.94703143429	10	1 neg	yes	yes	yes
G_{24}^2	(48,2,17)	1	0.98855988071	11	1 neg	yes	yes	yes
$\text{tet} \oplus G_{12}$	(16,3,6)	1	0.94313086205	6	1 neg	yes	yes	yes
hex^2	(12,4,3)	1	0.91675615731	3	1 neg	yes	yes	yes
$\text{tet}^2 \oplus G_{12}^3$	(44,3,20)	1	0.99999987357	7	1 neg	yes	yes	yes
$\mathbf{i}_2 \oplus e_8$	(10,2,4)	2	$0.86955 \pm 1.39341i$	7	2 neg	no ($m = 7$)	no ($m = 6, 7$)	no
$\mathbf{i}_2 \oplus e_8^2$	(18,2,8)	3	$0.84988; 0.68928 \pm 1.27812i$	13	3 neg	yes	no ($m = 6$)	yes

TABLE 4. Ghost test (`ghost.json`). The listed s is the ghost with $\text{Re } s > \frac{1}{2}$; its partner is $1 - \bar{s}$. “Li m_0 ” is the first m with $\lambda_m < 0$. N_m, b_m were checked for $m \leq 200$ and $m \leq 120$. For every code, $A_i \geq 0$ and the Z -coefficients (checked to T^{60}) are positive.

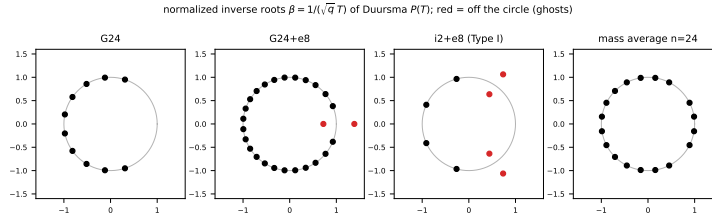


FIGURE 2. Normalized inverse roots β for G_{24} (on the circle), $G_{24} \oplus e_8$ (one real pair $\beta, 1/\beta$), $\mathbf{i}_2 \oplus e_8$ (complex quadruple) and the $n = 24$ mass average (24th roots of unity, Theorem 6.1).

What detects ghosts [THEOREM]+[EXACT-COMP]. The Toeplitz form detects every ghost, with exactly κ negative directions (Thm. 4.1). So do Li/the lower Weil bound (Cor. 4.2) and the exact $s(x^2)$ test. A sign change of P on $(0, 1)$ detects exactly the real ghosts. **What is blind** [EXACT-COMP]. (i) Data positivity: $A_i \geq 0$ and $z_j > 0$. Positivity of P 's coefficients is not even necessary, since e_8^2 is RH with mixed signs. (ii) Prime-side positivity, $N_m \geq 0$ and $b_m \geq 0$. This is blind for every real ghost in the census and for all large m by Proposition 4.3; it caught only the two complex cases, at $m = 6, 7$. (iii) The upper Weil bound. (iv) Integrality: N_m is non-integral already for G_{24} ($N_4 = 165/13$, $b_4 = 12/13$), so these zeta functions are not zeta functions of varieties. Integrality is therefore no test, and Euler-product positivity gives no information here.

Reading [ANALOGY]. The typical code ghost is a real Siegel-type pair close to the pole, Across the 211 failures the largest ghost has $s \in [0.713, 1 - 4.8 \cdot 10^{-9}]$ (`sums_alpha.json`). It is not a Davenport–Heilbronn-type complex zero. This is the L75/L76 picture: positivities on the data/prime side are blind to zeros in $\frac{1}{2} < \operatorname{Re} s < 1$ that the pole shields, and only the zero-side Krein form sees them.

6. PART 4: AVERAGES, THE RH REGION, AND BROUÉ–ENGUEHARD TRANSPORT

6.1. The mass average satisfies RH.

Theorem 6.1 (code Siegel–Weil average is cyclotomic; [THEOREM] for I, II). *Let $\bar{W}$ be the average weight enumerator over all binary self-dual codes of length n (Type I in the wide sense, n even). Let $\bar{W}^{\text{II}}$ be the average over all Type II codes ($8 \mid n$). Assume the classical constant-density fact [BIB] (MacWilliams–Sloane, Ch. 19): every admissible vector other than 0 and $\mathbf{1}$ lies in the same fraction c of the codes. Then, with $u = \sqrt{2}T$,*

$$P_{\text{I}} = c_{\text{I}}(1 + u^{2\gamma}), \quad P_{\text{II}} = c_{\text{II}} \frac{1 + u^{2\gamma+2}}{1 - \sqrt{2}u + u^2}, \quad c_{\text{I}} = \frac{1}{2^{n/2-1} + 1}, \quad c_{\text{II}} = \frac{1}{2^{n/2-2} + 1}.$$

In particular every zero is a root of unity in u , and Duursma-RH holds.

Proof. Type II: $d = 4$, $a_w = c$ for $4 \mid w$, $4 \leq w \leq n - 4$, and $a_n = 1$. Put $t = T/(1 - T)$. Then $a(t) = c \sum_{j \geq 0} t^{4j} - c \sum_{j \geq n/4-1} t^{4j} + t^{n-4} \equiv \frac{c(1-T)^4}{(1-T)^4 - T^4} + (1 - c)T^{n-4} \pmod{T^{n-3}}$. Since $(1 - T)^4 - T^4 = (1 - 2T)(1 - 2T + 2T^2)$, Definition 2 of Duursma (math/0302172) gives $P \equiv c/(1 - \sqrt{2}u + u^2) + (1 - c)T^{n-4} \equiv c \sum_k U_k(\cos \frac{\pi}{4})u^k + (1 - c)T^{n-4}$. Here $\deg P = 2\gamma = n - 6$. The u^{n-5} coefficient vanishes because $U_{n-5}(\cos \frac{\pi}{4}) = \sin((n-4)\frac{\pi}{4})/\sin \frac{\pi}{4} = 0$. The T^{n-4} coefficient vanishes iff $-c2^{(n-4)/2} + 1 - c = 0$, which is the stated c_{II} . This matches the cited count, and we did not assume it. The truncation identity $(1 - 2xu + u^2) \sum_{k \leq K} U_k u^k = 1 - U_{K+1}u^{K+1} + U_K u^{K+2}$ with $K = n - 6$, $U_{K+1} = 0$, $U_K = 1$ gives the closed form. Type I is the same computation with $d = 2$, $a_w = c$ on even w : $P(1 - T)^2/(1 - 2T) \equiv c(1 - T)^2/(1 - 2T) + (1 - c)T^{n-2} \pmod{T^{n-1}}$, so $P = c + (1 - c)T^{n-2}$. The functional equation then forces $c = 1/(2^\gamma + 1)$ with $\gamma = n/2 - 1$. $\square$

Types III, IV [EXACT-COMP]: the analogous closed forms $P_{\text{III}} \propto (1 + u^{2\gamma+2})/(1 + u^2)$ and $P_{\text{IV}} \propto (1 + u^{2\gamma+1})/(1 + u)$, with $c_{\text{III}} = 1/(3^{n/2-1} + 1)$ and $c_{\text{IV}} = 1/(2^{n-1} + 1)$, were verified exactly for $n \leq 120$ (`average_closed.json`). We did not write out a general- n proof. *Checks:* every average lies in the Gleason ring. Brute force at $n = 8$ gives 30 Type II codes and $\bar{A}_4 = 14 = \binom{8}{4}/5$ (`average.json`).

Remark 6.2 (reading; [ANALOGY]). In this finite model, the Eisenstein-like object (the average over the genus) has zeros at roots of unity: it is “supersingular”, with the zeros on the circle trivially. RH is then a statement about *how far an individual code may fluctuate from the average*. This is the code version of the Siegel–Weil/Eisenstein-plus-cusp split in L68 and L72.

6.2. The exact RH region.

Theorem 6.3 (RH sets on the Type II lines $n = 24, 32, 40$; [EXACT-COMP]). *The Type II enumerators of length $n \in \{24, 32, 40\}$ with $A_0 = 1$ form the line $W = \phi_8^{n/8} + a \phi_8^{n/8-3} \phi_{24}$. The event points*

(discriminant roots and boundary crossings of s) were found exactly and every gap was tested, which gives the following RH sets:

- $n = 24$: $A_4 \in \{0\} \cup [9.0451981788942352, 11.464413270750772]$. Integer points: $A_4 \in \{0, 10, 11\}$.
- $n = 32$: $A_4 \in \{0\} \cup [1.9892856145536041, 2.3078730314623818]$. Integer points: $\{0, 2\}$.
- $n = 40$: $A_4 \in \{0\} \cup [0.29375303675649533, 0.36070067949225861]$. No integer point with $A_4 \geq 1$.

Hence a Type II code of length 40 satisfies Duursma-RH iff it is extremal ($d = 8$).

Each window contains the mass average ($A_4 = 10626/1025 \approx 10.367$, $35960/16385 \approx 2.195$, $91390/262145 \approx 0.349$).

At $n = 24$, combine this with the Pless–Sloane list $A_4 \in \{0, 6, 12, 18, 24, 30, 42, 42, 66\}$ for the nine Type II codes [BIB]. Then G_{24} is the only Type II code of length 24 satisfying RH. This is conditional on the cited list. The isolated point $A_4 = 0$ is an extremality jump: as $A_4 \rightarrow 0$ the coefficient $p_0 \propto A_4$ vanishes, so d jumps from 4 to 8 and γ drops from 9 to 5. Four zero pairs leave through $T = 0$ and $T = \infty$, which is why $\kappa = 4$ next to $A_4 = 0$ (Fig. 3) [THEOREM].

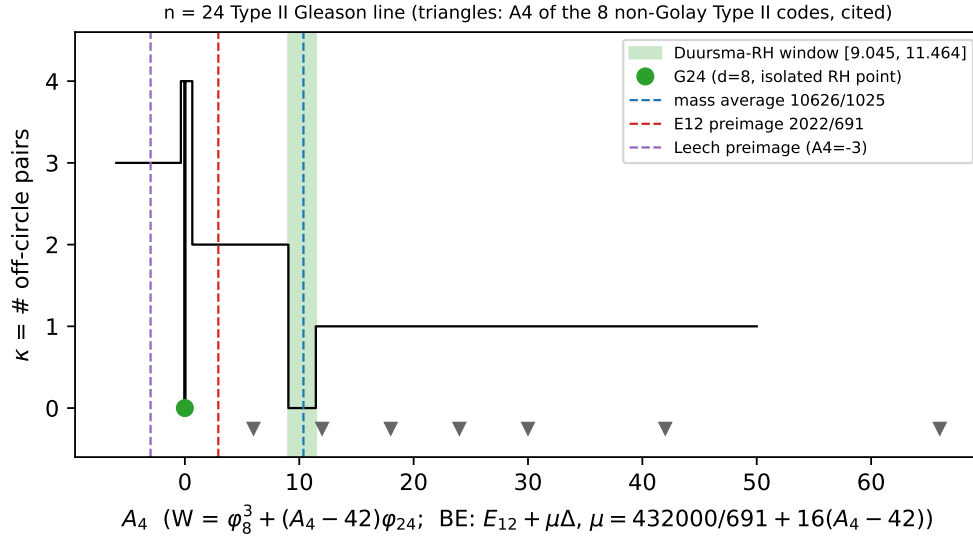


FIGURE 3. $n = 24$ Type II line: κ against A_4 (family.json). Green: the exact RH window. Marked: G_{24} ($A_4 = 0$, $\kappa = 0$), the mass average, the E_{12} preimage ($A_4 = 2022/691$, $\kappa = 2$), the Leech preimage ($A_4 = -3$, $\kappa = 3$), and the cited A_4 values of the eight non-Golay codes (triangles).

$n = 48$ (n48.json, n48zoom.json) [EXACT-COMP]/[NUMERICAL]. The plane is (A_4, A_8) . On the $d = 4$ sheet, the RH region found is a thin wedge around the mass average $(38916/838861, 377348994/4194305) \approx (0.04639, 89.967)$: 178 of 8281 points in the zoom box $[0.03, 0.065] \times [87, 93]$, and 8 of 14520 on the coarse grid. No integer point with $1 \leq A_4 \leq 5$ and $0 \leq A_8 \leq 400$ satisfies RH; each test is exact. On the $d = 8$ line $A_4 = 0$, RH holds for $A_8 \in [81.21571117144327, 99.75854797913836]$. The endpoints were bisected exactly, and the interval structure comes from a scan of step $1/8$ [NUMERICAL]. The extremal $d = 12$ point (QR_{48}) is RH. *Conjecture* [CONJECTURE]: no Type II code of length 48 with $d = 4$ satisfies RH.

6.3. Broué–Enguehard transport: no-go. Under $\phi_8 \mapsto E_4$, $\phi_{24} \mapsto 16\Delta$ [EXACT-COMP] (checked on q -expansions, context.json), $W = \phi_8^3 + a\phi_{24}$ maps to $f = E_4^3 + 16a\Delta = E_{12} + \mu\Delta$ with $\mu = 432000/691 + 16a$. This uses $E_4^3 - E_{12} = \frac{432000}{691}\Delta$, as in L68 [TAA-L].

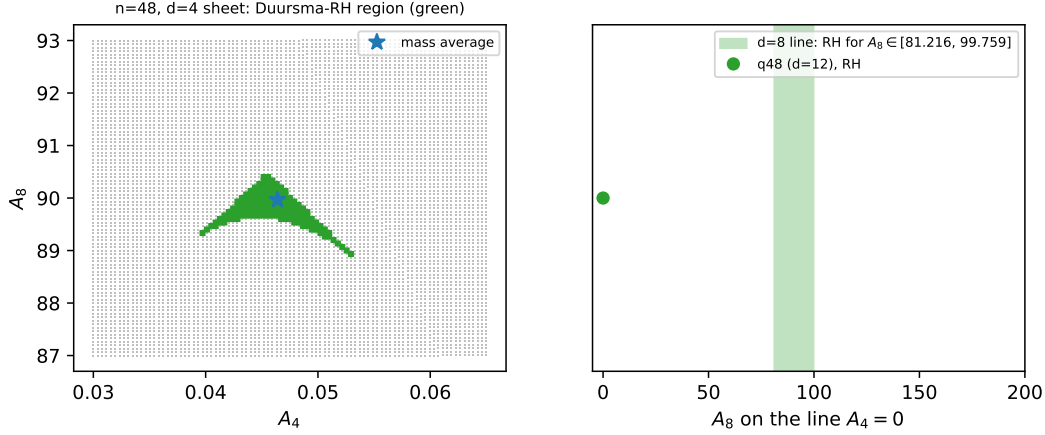


FIGURE 4. $n = 48$: the RH wedge around the mass average on the $d = 4$ sheet (left), and the $d = 8$ interval with the extremal QR_{48} point (right).

point	a	A_4	μ	RH	κ
G_{24} ($\Theta = E_4^3 - 672\Delta$)	-42	0	$-32352/691 \approx -46.82$	yes	0
Leech preimage ($E_4^3 - 720\Delta$)	-45	-3	$-65520/691 \approx -94.82$	no	3
E_{12} preimage	$-27000/691$	$2022/691$	0	no	2
mass average		$10626/1025$	≈ 119.05	yes	0
E_4^3	0	42	$432000/691 \approx 625.18$	no	1
RH window		$[9.045, 11.464]$	$[97.90, 136.61]$	yes	0

TABLE 5. The $n = 24$ points in the (E_4^3, Δ) plane (`family.json`).

Tests [EXACT-COMP] unless marked otherwise. (a) *Sign of the Δ -coefficient*: G_{24} has $\mu < 0$, the window has $\mu > 0$, and E_4^3 ($\mu > 0$) fails. So the sign does not decide RH. (b) *Petersson norm* of the cusp part, $\propto \mu^2$: the RH set $\{46.82\} \cup [97.9, 136.6]$ in $|\mu|$ excludes $\mu \in (-136.6, -97.9)$ and $\mu = +46.82$. So RH is not a function of $|\mu|$. (c) *691*: for every integral W ($a \in \mathbb{Z}$), $691\mu = 432000 + 16 \cdot 691a \equiv 432000 \equiv 125 \pmod{691}$. All codes therefore sit in the same nonzero class of the index-691 congruence module (L68), and the module cannot see RH. (d) *$L(f, s)$ zeros [NUMERICAL]*: we counted zeros in $[-1, 13] \times [0.5, 40]$ by the argument principle and on $\text{Re } s = 6$ by sign changes. The functional equation was checked to $1.3 \cdot 10^{-30}$, and $L(\Delta, s)$ against its Dirichlet series to $1.8 \cdot 10^{-14}$. E_{12} has 12 zeros, the zeros of $\zeta(s)$ and $\zeta(s-11)$, none on the line. The on-line count is 0 for $A_4 \leq 2$ and 1 for $A_4 \geq 3$ (on the integer grid), and nothing happens at the RH window or at G_{24} (Fig. 5). The reason is that on $\text{Re } s = 6$ the E_{12} part dominates, with $|\zeta(-5+it)| \sim t^{5.5}$. **Answer to (4): no-go.** The code-RH region is invisible to all four modular invariants tested. The modular image of the code average ($\mu \approx 119$) is also not E_{12} ($\mu = 0$), because Construction A does not carry the code mass to the lattice mass.

7. PART 5: DOES DUURSMA'S ULTRASPHERICAL MECHANISM CARRY OVER?

The proved case [CITED] (math/0302172 Thm. 2, citing Duursma, *Discrete Math.*, “Extremal weight enumerators and ultraspherical polynomials” [BIB]). Take Type IV with $n = 3m+3$, m odd (so $n \equiv 0 \pmod{6}$), $d = m+3$, and $Q(T) = P(T)(1+2T)$. Then $Q(T^2/2) = \lambda_m C_m^{m+1}(\frac{1}{2}(T+T^{-1}))T^m$, where C_m^{m+1} has m real zeros in $[-1, 1]$. Hence RH. In our variables this says $R(X) \propto P_\gamma^{(m+\frac{1}{2}, \frac{1}{2})}(X)$ (Jacobi), using $C_{2\gamma+1}^\lambda(x) \propto x P_\gamma^{(\lambda-\frac{1}{2}, \frac{1}{2})}(2x^2-1)$.

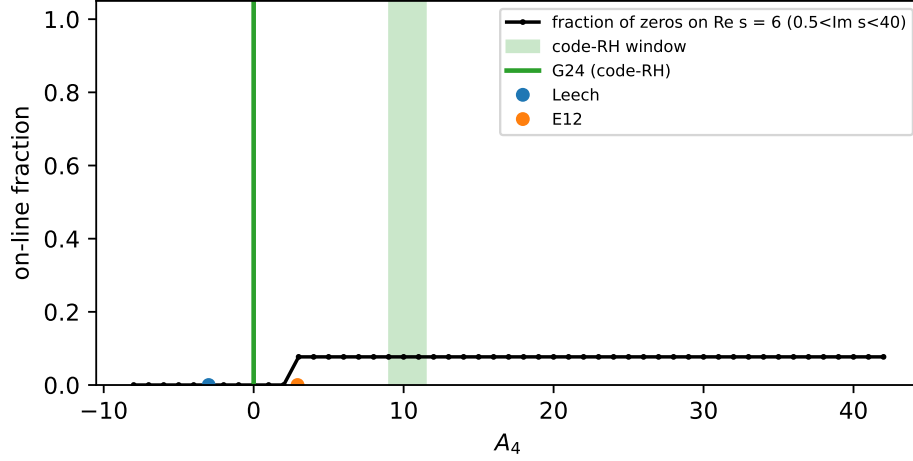


FIGURE 5. Fraction of zeros of $L(E_{12} + \mu\Delta, s)$ on $\text{Re } s = 6$ ($0.5 < \text{Im } s < 40$) against A_4 (lfun.json). The code-RH set (green) does not correlate with it.

Test (lemma77_jacobi.py) [EXACT-COMP]/[NUMERICAL]. Write R in $t = (1 - X)/2$. A Jacobi polynomial has hypergeometric coefficient ratios $c_{k+1}/c_k = (k - \gamma)(k + \gamma + \alpha)/((k + a + 1)(k + 1))$ with $\alpha = a + b + 1$. We fit (a, b) from $k = 0, 1$ and report the maximal relative misfit for $k \geq 2$.

- Type IV, $n \equiv 0 \pmod{6}$, $30 \leq n \leq 120$: $(a, b) = (m + \frac{1}{2}, \frac{1}{2})$ exactly, misfit $< 5 \cdot 10^{-80}$ (precision floor). **Duursma confirmed.**
- Type IV, $n \equiv 2, 4 \pmod{6}$: misfit $2.8 \cdot 10^{-3}$ – $5.8 \cdot 10^{-3}$, so not Jacobi. RH still holds (certified, Part 1).
- Type II: G_{24} misfit $8.8 \cdot 10^{-4}$ (best $a \approx 6.474$, $b \approx -0.348$). All others 0.05–0.12.
- Types I, III: misfits $6 \cdot 10^{-5}$ – $4.2 \cdot 10^{-2}$, never at the precision floor.
- Averages: I is exactly Jacobi($-\frac{1}{2}, -\frac{1}{2}$) (Chebyshev, consistent with $1 + u^{2\gamma}$); IV is exactly $(-\frac{1}{2}, \frac{1}{2})$; II is not Jacobi (misfit 0.049), being a ratio of cyclotomic factors.

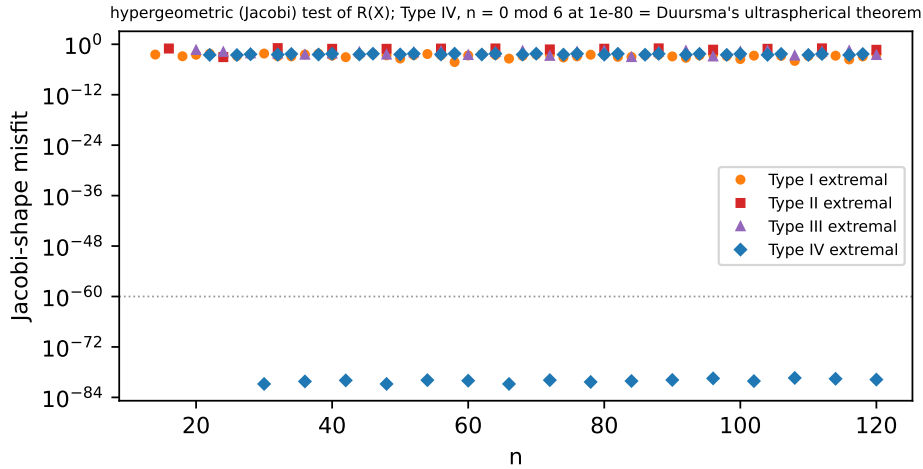


FIGURE 6. Jacobi misfit of extremal R by type and n (log scale). Only Type IV with $n \equiv 0 \pmod{6}$ sits at the precision floor.

Answer to (5). *No direct carry-over* [EXACT-COMP]: extremal Type II enumerators are not hypergeometric (not Jacobi) for any $n \leq 120$, so the identity cannot be transported as an identity. “Extremal Type II $\Rightarrow$ RH” is [OPEN]. The evidence is certified up to $n = 1000$ (Part 1) and there is no counterexample. Chinen (1709.03389, §1) reports that no extremal counterexample is known among actual codes. He also shows that for σ_q -invariant polynomials with $q = 6 + 2\sqrt{5}$, extremality is neither necessary nor sufficient for RH [CITED]. So a proof must use more than “maximal vanishing plus a MacWilliams-type invariance”. It has to use the specific group, i.e. the binary Type II Gleason ring.

What the input is, in the proved case [HEURISTIC] (reading), resting on [CITED]. The data A_i play no positive role. Chinen’s formal enumerators have negative A_i and are RH, while the codes of Part 3 have $A_i \geq 0$ and are not. The input is the $d - 1$ forced vanishings $A_1 = \dots = A_{d-1} = 0$, at the maximal d . These fix P uniquely, and in Duursma’s case they make P an orthogonal polynomial for a *positive weight on the zero side* (the Gegenbauer weight $(1 - x^2)^{m+1/2}$). Real-rootedness then follows from orthogonality. This is the program’s “cancellation, not positivity” in a finite model: positivity appears, but as a Hilbert structure *produced by* the maximal-cancellation condition, not as a positivity of the data. Direction K1 asks for the Type II analogue, i.e. a positive weight for which the extremal R is orthogonal, or a Krawtchouk/Christoffel–Darboux replacement.

8. PART 6: CONTEXT AND ERRATA

Leech and 691 [EXACT-COMP]. $\Theta_\Lambda = E_4^3 - 720\Delta = E_{12} - \frac{65520}{691}\Delta$, with q -expansion $1 + 196560q^2 + 16773120q^3 + 398034000q^4 + \dots$. $\Theta_{A(G_{24})} = E_4^3 - 672\Delta$, which has $48 = 24 \cdot 2$ roots (A_1^{24}). $E_4^3 - E_{12} = \frac{432000}{691}\Delta$, and $432000 \equiv 125, 65520 \equiv 566 \pmod{691}$ (`context.json`). The 691 congruence $\tau(n) \equiv \sigma_{11}(n)$ was checked for $n < 40$. The Leech theta is *not* the image of a code enumerator with $A_4 \geq 0$: its preimage has $A_4 = -3$.

Siegel–Weil [BIB]+[ANALOGY]. Siegel’s mass formula makes the mass-weighted average of the 24 Niemeier thetas equal to E_{12} (not recomputed; the review flags the same). The code analogue is Theorem 6.1. Under Broué–Enguehard the two averages do *not* correspond (Table 5: $\mu \approx 119$ against 0).

$\Pi_{25,1}$, **Weyl vector, horosphere** [BIB]+[ANALOGY]. $\sum_{k=0}^{24} k^2 = 4900 = 70^2$ [EXACT-COMP], so $\rho = (0, 1, \dots, 24 | 70)$ is null, and $\rho^\perp / \rho \cong \Lambda_{24}$ (Conway 1983, [BIB]). The Leech lattice is the translation lattice of the horosphere at the cusp ρ . This is the rank-24 analogue of the rank-1 cusp lattice of the L74 horocycle discussion [ANALOGY]. It is context only and is not used above.

Errata from the Golay review (re-checked here).

- **J12** (confirmed): “ $12+8 > 24+1$ ” is false. G_{24} satisfies Singleton with defect $n+1-k-d = 5 = \gamma$.
- **J13** (confirmed and sharpened): $\gamma = 5$ for G_{24} is Duursma’s *definition* $n+1-k-d$, with $\deg P = 2\gamma = 10$ (Table 1). It is not the genus of a curve. Its virtual point counts are not even integral ($N_4 = 165/13$, Part 3), so no variety over $\mathbb{F}_2$ has this zeta function.
- **J24** (confirmed): self-duality and large d do not force RH. $G_{24} \oplus e_8$ ($\kappa = 1$), e_8^3 , and 209 other direct sums fail. At $n = 40$, RH is equivalent to extremality (Theorem 6.3).

9. CORRECTIONS

These cover claims in the Golay review (`reviews/golay/golay_review.md`), the chat summaries of it, and my own intermediate statements.

- C1** *Toeplitz “for all N ”* (review §3): sharpened. The single size $N = 2\gamma$ suffices, and the negative index equals κ exactly (Thm. 4.1).
- C2** $G_{24} \oplus e_8$: -23.1 at size 11, $-9.6 \cdot 10^5$ at size 41: confirmed ($-23.050037, -964865.77$).
- C3** *Length-24 scan, integer $A_4 \in \{0, 10, 11\}$ over $[0, 120]$* : confirmed. It is now exact on the whole real line: RH set $\{0\} \cup [9.0452, 11.4644]$.

- C4** “ G_{24} is the only Type II code of length 24 with RH”: correct, but conditional on the Pless–Sloane A_4 list ([BIB]). The review labelled it Numerical; it is [EXACT-COMP] given that list.
- C5** Real pair at $\sigma \approx 0.97$, 0.03 : confirmed, $s = 0.970461192134$ and $1 - s$.
- C6** e_8 : the zbMATH review of Duursma’s DIMACS paper (secondary) says “[8, 4, 4] ... does not have the RH property”. This conflicts with our exact $P_{e_8} = (2T^2 + 2T + 1)/5$, whose zeros have $|T| = 2^{-1/2}$, so e_8 satisfies RH. The primary source was not opened, and the conflict is *flagged, not resolved* (a different code or normalization may be meant).
- C7** “Only extremal Type IV is proved”: Chinen (Thm. 1.4, [CITED]) states that all extremal Type IV codes satisfy RH, citing Duursma. Duursma’s own 2003 statement (Thm. 2, [CITED]) covers $n = 3m + 3$, m odd ($n \equiv 0 \pmod{6}$). The full paper was not opened. Our certified run covers all Type IV $n \leq 200$ regardless.
- C8** Chinen’s tables are formal enumerators (FE sign -1), not codes. The $n = 60$ entry A_{24} is misprinted (Part 1).
- C9** $n = 72$: [72, 36, 16] existence is open ([BIB]). Its extremal enumerator satisfies RH (certified), so RH gives no obstruction.
- C10** “Toeplitz positivity is a literal J ” (chat phrasing): it is a [RESTATEMENT]. What *is* literal is the Krein structure: the functional equation gives Π_κ .
- C11** Ghosts as Davenport–Heilbronn analogues (chat phrasing): in 209 of 211 failures the ghost is a real Siegel-type pair near the pole. Complex off-line zeros occur only in $\mathbf{i}_2 \oplus e_8$ and $\mathbf{i}_2 \oplus e_8^2$.
- C12** Genus-5 “variety” (J13): beyond J13, the virtual counts N_m of G_{24} are non-integral, so there is no variety at all.
- C13** “Golay/Leech point” (title and review): the Leech preimage ($A_4 = -3$) is not RH ($\kappa = 3$), and neither is the E_{12} preimage. The RH point is the *Golay point*. The title is kept as assigned.
- C14** Review G_3 , “averaging = Eisenstein”: on the code side this is now a theorem (Thm. 6.1). But Broué–Enguehard sends the code average to $\mu \approx 119$, not to E_{12} .
- C15** Review §5(iv), “ $n = 48, 72$ ”: $n = 48$ is done (wedge plus $d = 8$ interval). The three-parameter $n = 72$ map was *not* done.
- C16** My intermediate claim of a ghost with $s > 1$ at $n = 44$, $q = 3$ came from rounding T to six digits. The precise value is $s = 0.99999987357 < 1$. No ghost with $\operatorname{Re} s \geq 1$ has been found.
- C17** My intermediate Δ functional-equation check was symmetric by construction, so it tested nothing. The honest check is the Dirichlet-series comparison ($1.8 \cdot 10^{-14}$).

10. DIRECTIONS (RANKED BY $0.6s + 0.4p$)

direction	s	p	score
K2 Fluctuation radius. Measure the RH window around the mass average against n (relative widths 0.233, 0.145, 0.192 at $n = 24, 32, 40$) and compare it with the mass-distribution spread of actual A_4 . Target: a theorem “RH $\iff$ distance to average $< r_n$ ” plus extremal points, or a counterexample.	.70	.70	.70
K3 Transport the average theorem. Find a lattice/Eisenstein analogue in which the genus average is cyclotomic, and test Leech and Niemeier lattices inside it.	.70	.55	.64
K1 Orthogonality proof for extremal Type II. Find a positive weight (Krawtchouk/Christoffel–Darboux) making the extremal R orthogonal. This would be the cancellation-as-orthogonality mechanism of Part 5.	.75	.35	.59
K5 κ statistics over classified codes (e.g. Type II $n = 32, 40$), if classification data can be obtained.	.55	.62	.58
K4 Prove $ \alpha < q$ (the pole shield) for every self-dual code.	.55	.60	.57
K6 Duursma’s two-variable zeta (§4 at source): test whether its positivity sees ghosts.	.50	.60	.54

11. QUESTIONS FOR J.W.

- Q1** Adopt the reading: in codes, “Toeplitz–Weil positivity” is a [RESTATEMENT], and the functional equation gives exactly a Krein Π_κ form with κ = number of ghost pairs (Thm. 4.1)?
- Q2** Adopt “average is RH (cyclotomic); RH region = small fluctuation window around the average + isolated extremal points” (Thms. 6.1, 6.3) as the code-side model of Eisenstein + cusp?
- Q3** Accept the Broué–Enguehard no-go (sign, Petersson, 691, L -zeros) and rename “Golay/Leech point” to “Golay point”?
- Q4** Accept as the program-level lesson: prime-side and data-side positivities are blind to pole-hugging ghosts (Prop. 4.3), and the only proved RH mechanism (Duursma) uses maximal cancellation turned into orthogonality?
- Q5** Next: K2, K3, K1, or return to an older direction?

REFERENCES

Read at source (`lit/lemma77/verified_refs.md`).

- I. M. Duursma, *Results on zeta functions for codes*, arXiv:math/0302172 (2003).
- K. Chinen, *Zeta functions for formal weight enumerators and an analogue of the Mallows–Sloane bound*, arXiv:math/0510182 (2005).
- K. Chinen, *Extremal invariant polynomials not satisfying the Riemann hypothesis*, arXiv:1709.03389 (2017) (abstract and §1 only).

Secondary (opened, but not the primary): zbMATH/MaRDI review of Duursma’s DIMACS 2001 paper (the e_8 sentence, C6); MaRDI record of Duursma, *Discrete Math.* 268 (2003).

Not opened [BIB]: Duursma, *Weight distributions of geometric Goppa codes*, Trans. AMS (1999); *From weight enumerators to zeta functions*, Discrete Appl. Math. (2001); *A Riemann hypothesis analogue for self-dual codes*, DIMACS Ser. 56 (2001); *Extremal weight enumerators and ultraspherical polynomials*, Discrete Math. 268 (2003). Mallows–Sloane (1973); MacWilliams–Sloane, *The Theory of Error-Correcting Codes*, Ch. 19 (constant density, counts); Pless–Sloane (length-24 classification); Ward, Conway–Sloane (Type I existence); Zhang (1999, Type II nonexistence for large n); Rains–Sloane, *Self-dual codes* (survey); Broué–Enguehard (1972); Conway, J. Algebra 80 (1983) ($\Pi_{25,1}$); Siegel mass formula / Niemeier; Carathéodory–Toeplitz; Krein–Pontryagin; Weil/Bombieri for curves; Serre (point bounds).

TAA: L1, L64 (Li/Toeplitz), L68 (691), L72, L74 (horocycles), L75, L76 (STATE §73–§85); Golay review (`reviews/golay/`).

TAA LEMMA 78: CAUSALITY VERSUS LINEWIDTH. KRAMERS–KRONIG FOR THE MODULAR SCATTERING MATRIX, THE RESTRICTED TENSOR PRODUCT, CONNES’ COUNTERTERM, THE $\mathbb{F}_1$ “META-SPACE”, THE POINT AT INFINITY, AND THE LI DISK VERSUS THE ROSATI INVOLUTION

J.W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, here or anywhere in TAA. Nothing in this lemma proves it or brings a proof closer in a way we can measure. Several results below are *restatements* of RH and are labelled as such. Tags: [THEOREM] proved here (or classical with the proof indicated); [CITED] checked at the source (local copies listed in `lit/lemma78/verified_refs.md`); [CITED-sec] via a named secondary source we opened; [BIB] bibliographic record or recollection only, not opened; [TAA-L] earlier TAA lemma; [RESTATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] floating-point or multiprecision numerics (not certified).*

Task (J.W., approved [t321u]; additions [t322u]–[t328u]). Fold the chat thread [t311]–[t320] into one lemma: (A) Kramers–Kronig, causality and passivity for the modular scattering matrix, with a Davenport–Heilbronn (DH) ghost check and the “uniform linewidth $\frac{1}{4}$ ” form of RH; (B) the restricted tensor product; (C) Connes’ $\log \Lambda$ counterterm and the pole plane; (D) J.W.’s $\mathbb{F}_1$ “meta-space” (seeing $\text{Spec } \mathbb{Z}$ through $1 + 1 + 1 + \dots$); (E) the point at infinity; (F) the disk picture, plus the Li map versus the Rosati involution (t322u–t324u), the Γ -factor as archimedean Frobenius (t325u), and positivity from a quarter turn: the Weil operator, Fourier, orientation, and the Hilbert transform as the Kramers–Kronig operator (t326u–t328u). Only questions that bear on the polarization J were pursued. Every numerical run below was chosen to discriminate between alternatives.

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Date: October 23, 2026.

1. ANSWER IN BRIEF

- (1) **Causality, unitarity and passivity are automatic; RH is the linewidth (§2).** Put $\psi(s) = \xi(2s-1)/\xi(2s) = \frac{s-1}{s} \varphi(s)$, where $\varphi(s) = \Lambda(2s-1)/\Lambda(2s)$ is the scattering matrix of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$. Then [THEOREM]

$$\psi(s) = \prod_{\rho} \frac{s - a_{\rho}}{s + \bar{a}_{\rho} - 1}, \quad a_{\rho} = \frac{1+\rho}{2},$$

which is an infinite Blaschke product of exactly the Möbius maps of [t320]. The relation is checked to $2 \cdot 10^{-7}$ with 4520 zero pairs [NUMERICAL]. So ψ is inner in $\mathrm{Re} s > \frac{1}{2}$: it is unimodular on the line and bounded by 1 inside, and the group delay $-d \arg \psi / dr$ is a positive sum of Lorentzians of width $\mathrm{Re} \rho / 2$. None of this uses RH. $RH \iff \text{every line has width exactly } \frac{1}{4}$ [RESTATEMENT]. A DH ghost splits one line into a pair of widths $\frac{1}{4} \pm \frac{\delta}{2}$, e.g. 0.4043 and 0.0957 at $\gamma = 85.70$, while the delay stays positive (minimum 0.39 on $0 < r < 100$) and $|\psi_f| \leq 1$ on the grid [NUMERICAL]. Kramers–Kronig and passivity therefore encode “zeros lie in the strip” (the Euler-product, PNT-level content), not RH. The only passivity known to be *equivalent* to RH is that of a different transfer function, the exponential line $\exp(-2c\xi'/\xi)$ of Lemma 37 [TAA-L].

- (2) **Restricted tensor product (§3).** $\bigotimes_p' \ell^2(p^{\mathbb{N}}) \cong \ell^2(\mathbb{N})$ by unique factorization [THEOREM]. This is a tensor product of *local state spaces* (the Euler product as a partition function). It is not a fibre product of schemes. $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$ still makes $\mathrm{Spec} \mathbb{Z} \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{Z}$ the diagonal, and the Hilbert-space square $\ell^2(\mathbb{N})^{\otimes 2}$ has partition function $\zeta(s)\zeta(t)$, with no $h^1 \otimes h^1$ in it. The missing object is still an H^1 carrying a positive form.
- (3) **The pole plane is forced by the scaling flow; one-sided counterterms cannot produce it (§4).** On Lemma 10’s pole plane $P(F) = 2 \mathrm{Re}(a\bar{b})$, with $a = \int F e^{u/2}$ and $b = \int F e^{-u/2}$, the scaling flow acts as a Lorentz boost $(a, b) \mapsto (e^{c/2}a, e^{-c/2}b)$. The only flow-invariant quadratic forms on this plane are multiples of $\mathrm{Re}(a\bar{b})$ [THEOREM]. We reproduce Lemma 17’s Gaussian $(3.34421 - 2.23786 - 1.10635 = 9 \cdot 10^{-13})$ and its invariance under shifts [NUMERICAL]. Lemma 75’s cusp counterterm $A^{1-2s}/(2s-1) = \frac{1}{2s-1} - \log A + O(2s-1)$ carries one null line ($\mathbb{L}$) plus a log-cutoff term, which is the same shape as Connes’ $2h(1) \log \Lambda$ [THEOREM]+[ANALOGY]. By the invariance theorem it cannot reproduce P alone: the **1** line, i.e. the functional-equation partner, is required.
- (4) **$\mathbb{F}_1$ meta-space (§5).** J.W.’s “ $1+1+1+\dots$ ” picture is the Soulé/Kurokawa limit $q \rightarrow 1$, and for $N(q) = 1+q$ it gives exactly $1/(s(s-1))$. That is the singular part of Λ [THEOREM]/[CITED]. The Connes–Consani counting *distribution* $N(u) = u - \frac{d}{du} \sum_{\rho} \frac{u^{\rho+1}}{\rho+1} + 1$ exists and is positive on $(1, \infty)$ ([CITED]; we confirm their constant $\omega(1)$ to $5 \cdot 10^{-8}$ and the smoothed identity to 10^{-16} , [NUMERICAL]). But its positivity is manifest from $\Lambda(n) \geq 0$, so it holds whether or not RH does, and it already fails for $L(s, \chi_4)$, where GRH is expected [THEOREM]. So the counting positivity is RH-blind.
- (5) **Point at infinity (§6).** Adding ∞ gives the Γ -factor, the functional equation and the pole at 0. Manin (1995) proposes $h(\mathrm{Spec} \mathbb{Z}) = \mathbf{1} \oplus h^1 \oplus \mathbb{L}$ and says compactifying makes $\mathrm{Spec} \mathbb{Z}$ “similar to a projective curve” ([CITED]; a proposal, not a theorem). The pole at 0 is the $\Gamma(s/2)$ pole *surviving because* $\zeta(0) = -\frac{1}{2} \neq 0$. All other Γ poles are cancelled by the trivial zeros.
- (6) **Disks; Li versus Rosati (§7).** New [THEOREM]: the pole-centred Möbius map of the code β -plane, $w_q = (\beta - \sqrt{q})/(\sqrt{q}\beta - 1) = \sqrt{q}[s-1]_q/[s]_q$, tends to Li’s $1 - 1/s$ as $q \rightarrow 1$. This settles [t320]’s open question in the recentred form. The centre $\beta = 0$ of the plain β -disk is $\mathrm{Re} s = -\infty$, not $+\infty$ (correction). Li-type positivity seen from the pole catches pole-hugging (Siegel) ghosts at $m = 2$, against $m = 10\text{--}13$ from $\beta = 0$, but is slower on the complex $\mathbf{i}_2 \oplus e_8$ ghost ($m = 10$ against 7) [NUMERICAL]. *Li versus Rosati* (§7.3): the functional equation is the formal relation “adjoint = q -inverse” (Weil pairing, no positivity), while RH for curves comes from *positivity* of the Rosati involution (Milne Thm. 1.27/Cor. 1.29, [CITED]). Codes have the

first and not the second. For every ghost code, the Li-negativity index equals the first failure of the Weil (Castelnuovo) bound (7, 10, 11, 13), so $\lambda_n \geq 0$ is a restatement, not a source of positivity. *Gamma factor* (§6.1): $s = \pi$ plays no role. The reflection $s \mapsto 1 - \bar{s}$ is the archimedean Frobenius F_∞ , an involution with no positivity. *Quarter turn* (§7.4): over $\mathbb{C}$ a polarization *is* an oriented quarter turn (Hodge–Riemann). Fourier is a quarter turn that inverts scaling, so it is the FE again. The Hilbert transform in $\log x$, i.e. the Kramers–Kronig operator, is a quarter turn that commutes with scaling, but the Weil form on its $+$ -eigenspace (positive frequencies) is exactly half of Weil’s form, a restatement. Numerically it is negative on the DH ghost (down to -497) and positive for ζ [NUMERICAL].

Program verdict [HEURISTIC]. Every object in this lemma that is unconditionally positive is RH-blind: the inner function ψ , the group delay, the counting distribution $N(u)$, and Kramers–Kronig. The objects that are equivalent to RH are restatements: uniform linewidth, Li, Toeplitz, and the exponential line. The one place where positivity *produces* RH is the Rosati involution of a polarized Jacobian, and its $\mathbb{Q}$ -analogue is exactly the missing J . The quarter-turn analysis sharpens this. The involution (FE, F_∞ , Fourier) and an orientation (Hardy space, KK) are both available for $\mathbb{Q}$ and both RH-blind or tautological. What is missing is positivity on a space where scaling has multiplicity.

2. PART A: KRAMERS–KRONIG, CAUSALITY AND PASSIVITY FOR THE SCATTERING MATRIX

Objects. $\Lambda(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$ is the completed zeta, with simple poles at $s = 0, 1$ and $\Lambda(s) = \Lambda(1 - s)$. $\xi(s) = \frac{1}{2} s(s-1) \Lambda(s)$ is Riemann’s entire ξ . The constant term of the Eisenstein series of $\mathrm{SL}_2(\mathbb{Z})$ is $y^s + \varphi(s) y^{1-s}$ with $\varphi(s) = \Lambda(2s-1)/\Lambda(2s)$ (Lemma 73/76, [TAA-L]). The brief writes $\xi(2s-1)/\xi(2s)$ for φ . These differ by a rational factor:

$$\psi(s) := \frac{\xi(2s-1)}{\xi(2s)} = \frac{s-1}{s} \varphi(s) \quad [\text{THEOREM}] \text{ (direct; checked to } 10^{-31}, \text{ scatter.json).}$$

The factor $B(s) = (s-1)/s$ is itself a Blaschke factor ($|B| = 1$ on $\mathrm{Re} s = \frac{1}{2}$). It removes the pole of φ at $s = 1$, i.e. the $\mathbb{L}$ -pole.

Theorem 2.1 (all-pass factorization). *[THEOREM] Let ρ run over the nontrivial zeros of ζ , with multiplicity, in symmetric order $|\mathrm{Im} \rho| \leq T$, $T \rightarrow \infty$. For s not a pole,*

$$\psi(s) = \prod_{\rho} \frac{s - a_{\rho}}{s + \bar{a}_{\rho} - 1}, \quad a_{\rho} = \frac{1 + \rho}{2}.$$

Each factor is the Möbius map $w_a(s) = (s - a)/(s + \bar{a} - 1)$ of [t320] with centre $a = a_{\rho}$. Since $\mathrm{Re} a_{\rho} = \frac{1}{2} + \frac{\mathrm{Re} \rho}{2} > \frac{1}{2}$ for every ρ (no zeros on $\mathrm{Re} s = 0$), every factor has modulus 1 on $\mathrm{Re} s = \frac{1}{2}$ and modulus < 1 in $\mathrm{Re} s > \frac{1}{2}$. Hence $|\psi| = 1$ on the line and $|\psi| \leq 1$ on $\mathrm{Re} s > \frac{1}{2}$: ψ is inner (lossless, passive, causal), unconditionally.

Proof. $\xi(z) = \xi(0) \prod_{\rho} (1 - z/\rho)$ with zeros paired symmetrically (Hadamard; Davenport Ch. 12, [CITED-sec] via L25’s reference record). Hence $\psi(s) = \prod_{\rho} \frac{\rho - 2s + 1}{\rho - 2s} = \prod_{\rho} \frac{s - (1+\rho)/2}{s - \rho/2}$. The zero set is invariant under $\rho \mapsto 1 - \bar{\rho}$, which preserves $|\mathrm{Im} \rho|$, so the symmetric truncation is preserved. Re-indexing the denominator by $\rho' = 1 - \bar{\rho}$ gives $s - \rho'/2 = s - (1 - \bar{\rho})/2 = s + \bar{a}_{\rho} - 1$. For the modulus, $|s - a|^2 - |s + \bar{a} - 1|^2 = -(2 \mathrm{Re} s - 1)(2 \mathrm{Re} a - 1)$. Pointwise limits of products of factors of modulus ≤ 1 have modulus ≤ 1 . $\square$

Numerical check [NUMERICAL] (num/lemma78_scatter.py, exact Platt/LMFDB zeros up to $T = 5000$, 4520 pairs). At five points ($0.9 + 3i$, $2 + 10i$, $0.6 + 21i$, $0.3 + 7i$, $1.5 - 4i$), the truncated product has relative error $3 \cdot 10^{-3} - 2 \cdot 10^{-2}$ at $T = 5000$. Adding the smooth-density tail lowers this to $1.6 \cdot 10^{-7} - 1.1 \cdot 10^{-6}$ (Fig. 1, left). On a grid with $|\mathrm{Im} s| \leq 60$, $\max |\psi| = 0.99908, 0.99539, 0.97717, 0.95492, 0.91226, 0.80048$ at $\mathrm{Re} s = 0.51, 0.55, 0.75, 1, 1.5, 3$, and $||\psi| - 1| \leq 4 \cdot 10^{-31}$ on the line.

Corollary 2.2 (group delay = Lorentzian absorption spectrum; uniform linewidth). *[THEOREM]/[RESTATEMENT]* On $s = \frac{1}{2} + ir$ write $\psi = e^{i\Theta(r)}$. Then

$$\tau(r) := -\Theta'(r) = \sum_{\rho} \frac{2w_{\rho}}{w_{\rho}^2 + (r - \operatorname{Im} \rho/2)^2} > 0, \quad w_{\rho} = \frac{\operatorname{Re} \rho}{2},$$

and each zero contributes total delay 2π , so the winding of Θ counts zeros (“Weyl-law scattering phase”). Under RH every width is $\frac{1}{4}$. A zero $\rho = \frac{1}{2} + \delta + i\gamma$ and its partner $1 - \bar{\rho}$ give two lines at the same centre $r = \gamma/2$, with widths $\frac{1}{4} + \frac{\delta}{2}$ and $\frac{1}{4} - \frac{\delta}{2}$. Therefore

RH $\iff$ all lines of the scattering group delay have the same width $\frac{1}{4}$.

This is a *[RESTATEMENT]*. It adds no positivity: positivity of τ holds for any zero configuration in the open strip.

Proof. $\frac{d}{dr} \arg \frac{s-a}{s-b} = \operatorname{Re} \frac{1}{s-a} - \operatorname{Re} \frac{1}{s-b}$ with $s = \frac{1}{2} + ir$, $b = 1 - \bar{a}$, and $\operatorname{Re} a - \frac{1}{2} = \frac{1}{2} - \operatorname{Re} b = \frac{\operatorname{Re} \rho}{2}$. Numerically, τ computed from ζ directly agrees with the Lorentzian sum (4520 pairs plus tail) to $6 \cdot 10^{-8}$ at $r \in \{0.5, 3, 7.07, 10, 15, 20\}$ [NUMERICAL].

Kramers–Kronig: what it does and does not see [THEOREM]. The causal response is $\chi(s) = \sum_{\rho} \frac{1}{s - \rho/2}$, whose poles $\rho/2$ all lie in $\operatorname{Re} s < \frac{1}{2}$ whenever $0 < \operatorname{Re} \rho < 1$. So χ is analytic in $\operatorname{Re} s > \frac{1}{2}$, and on the line its real and imaginary parts are a Hilbert pair (KK). The real part is the Lorentzian absorption above; the imaginary part is the anomalous dispersion that changes sign across each line. This is J.W.’s prototype, and it holds exactly. But KK needs only that the poles lie on one side. That is “zeros in the strip”, which is the de la Vallée Poussin/PNT content supplied by the Euler product. KK is blind to *where* in the strip the lines sit. Note also that $\log \psi$ does not satisfy KK: ψ has zeros a_{ρ} in the causal half-plane, so it is maximally non-minimum-phase, and its whole phase is “excess phase” carried by the zeros (Bode’s gain–phase relation fails; cf. Lemma 48 on all-pass/Blaschke factors, [TAA-L]).

Causality vs passivity: which transfer function? [TAA-L]+[THEOREM]. “Passive $\iff$ RH” is true for the *exponential line* $\Theta_c = \exp(-2c\xi'/\xi)$ (Lemma 37, Thm. exp; Lagarias’ positive-real condition exponentiated). It is *false* for φ or ψ , which are passive unconditionally (Thm. 2.1). The difference is the shift by one: ψ compares ξ at $2s - 1$ and $2s$, whose critical strips land on opposite sides of $\operatorname{Re} s = \frac{1}{2}$ automatically. ξ'/ξ compares ξ with itself, so its positive-realness on $\operatorname{Re} s > \frac{1}{2}$ is equivalent to having no zeros there.

Ghost check: Davenport–Heilbronn. $f(s) = \sum a(n)n^{-s}$ is Lemma 76’s DH function, with $\Lambda_f(s) = (5/\pi)^{s/2} \Gamma(\frac{s+1}{2}) f(s) = \Lambda_f(1-s)$ (functional-equation residual 10^{-17}). Put $\psi_f(s) = \Lambda_f(2s - 1)/\Lambda_f(2s)$. Then [NUMERICAL]:

- $|\psi_f| - 1| \leq 1.4 \cdot 10^{-16}$ on the line. On $0 < \operatorname{Im} s < 100$, $\max |\psi_f| = 0.9961, 0.9616, 0.9071, 0.8248, 0.6926$ at $\operatorname{Re} s = 0.51, 0.6, 0.75, 1, 1.5$. The group delay is positive throughout $0 < r < 100$ (minimum 0.391; for ζ , 0.092).
- The four known off-line zeros give line pairs (widths $\frac{1}{4} \pm \frac{\delta}{2}$, matching the direct computation to all digits):

ρ	δ	width ρ	width $1 - \bar{\rho}$
$0.808517 + 85.699348i$	0.30852	0.40426	0.09574
$0.650830 + 114.163343i$	0.15083	0.32542	0.17458
$0.574356 + 166.479306i$	0.07436	0.28718	0.21282
$0.724258 + 176.702461i$	0.22426	0.36213	0.13787

- So passivity, unitarity, KK and positive delay all *miss* the DH ghosts. Only the linewidth sees them (Fig. 1, right: one narrow and one broad line at $r = 42.85$, where ζ has none).

- *Where passivity would fail.* If Λ_f has a zero with $\operatorname{Re} \rho > 1$, then $\rho/2$ is a pole of ψ_f in $\operatorname{Re} s > \frac{1}{2}$, and ψ_f is *active*. That DH-type functions have zeros in $\operatorname{Re} s > 1$ is a classical theorem we recall but did not open (Davenport–Heilbronn 1936 and successors, [BIB]). Our scan is reported in Remark 2.3.

Remark 2.3 (DH zeros with $\operatorname{Re} s > 1$: scope of our search). (a) [THEOREM]. Write $f_{\text{DH}}(s) = \sum a_n n^{-s}$ with $a_1 = 1$, $|a_n| \in \{0, \kappa, 1\}$, $\kappa = 0.28408$. Then $|f_{\text{DH}}(s)| \geq 1 - \sum_{n \geq 2} |a_n| n^{-\sigma}$, and the sum is < 1 for $\sigma \geq 1.5$ (it is 0.7398 at $\sigma = 1.5$, bounded above by the ζ tail beyond $n = 2 \cdot 10^5$). So there are *no* DH zeros with $\operatorname{Re} s \geq 1.5$, and ψ_f has no poles with $\operatorname{Re} s \geq \frac{3}{4}$. (b) [NUMERICAL], not certified. On the four lines $\sigma \in \{1.02, 1.08, 1.15, 1.30\}$, for $0 < t < 1500$ (step 0.05, `num/lemma78_dh_right_par.py`), $|f_{\text{DH}}|$ never drops below 0.2088 (at $\sigma = 1.02$), 0.279, 0.351, 0.480 respectively. No local minimum fell below 0.15, so Newton was never triggered and no zero was found. So in the range searched, ψ_f is passive, and the DH ghosts are visible *only* in the linewidth. The original serial run to $T = 1500$ was too slow and was replaced by this parallel run with the same range and a finer step. The strips $1 < \sigma < 1.02$ and $1.30 < \sigma < 1.5$ between our lines, and heights $t > 1500$, are not covered. The classical theorem ([BIB]) guarantees zeros in $\operatorname{Re} s > 1$ at *some* height, so passivity must fail eventually. Our search says only that it does not fail early.

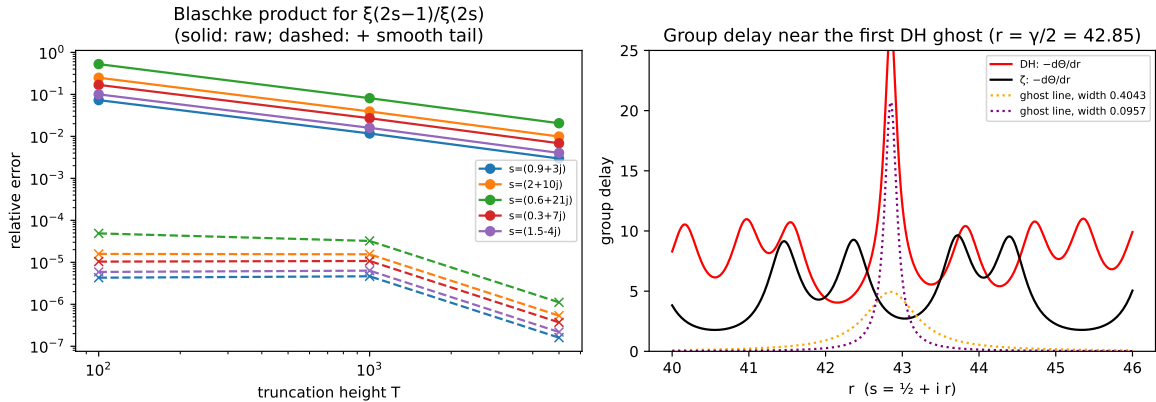


FIGURE 1. Left: relative error of the Blaschke product (Thm. 2.1) against truncation height (solid); with the smooth tail (dashed). Right: group delay $-\mathrm{d}\Theta/\mathrm{d}r$ for ζ (black) and for the DH function (red) near the first DH ghost. Dotted: the two Lorentzians of widths $\frac{1}{4} \pm \frac{\delta}{2}$ predicted by Cor. 2.2. The delay is positive in both cases; only the width distinguishes them.

Verdict on Part A [HEURISTIC]. J.W.’s anomalous-dispersion prototype fits *exactly*: the zeros are Lorentzian absorption lines of the modular scattering phase, and KK links absorption to dispersion. But the prototype’s built-in principles (causality, KK, passivity, unitarity) hold automatically here. RH is a further statement that all lines share one width, the analogue of a medium whose resonances share one damping constant. In Lemma 25’s language that is Heaviside’s distortionless condition ($R/L = G/C$: all modes attenuate equally, [TAA-L]). A mechanism forcing a common width would need a *conserved quantity tying each line’s damping to a single global constant*. That is a candidate description of what J must do (Direction L1).

3. PART B: THE RESTRICTED TENSOR PRODUCT AND WHY $\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$ IS THE OBSTRUCTION

What kind of objects. A *tensor product of state spaces* (Hilbert spaces) builds composite quantum systems. A *fibre product of schemes* $X \times_S Y$ builds a geometric product space. On affine

schemes it is the tensor product of *rings*, $\mathrm{Spec} A \times_{\mathrm{Spec} R} \mathrm{Spec} B = \mathrm{Spec}(A \otimes_R B)$. A *cohomology group* h^i is a linear invariant of a space, and for curves over $\mathbb{F}_q$ Frobenius acts on h^1 with eigenvalues giving the zeros. The “Hodge index theorem” is a sign statement about the intersection form on divisors of a surface: one positive direction, the rest negative.

Proposition 3.1. [THEOREM] (i) *Unique factorization gives a unitary isomorphism $\bigotimes_p' \ell^2(p^{\mathbb{N}}) \cong \ell^2(\mathbb{N})$, $\bigotimes_p e_{p^{k_p}} \mapsto e_{\prod p^{k_p}}$. The restricted tensor product is taken with respect to the vacuum e_{p^0} , so almost all factors are in the vacuum. The operator $n \mapsto n^{-s}$ factorizes, and $\mathrm{Tr} = \prod_p (1 - p^{-s})^{-1} = \zeta(s)$ for $\mathrm{Re} s > 1$. This is Julia’s/Bost–Connes’ “primon gas” (BC95, [CITED-sec] via CMR05 in Lemma 71). (ii) Likewise $L^2(\mathbb{A}_{\mathbb{Q}}) = \bigotimes_v' L^2(\mathbb{Q}_v)$ with respect to $\mathbf{1}_{\mathbb{Z}_p}$ (Tate’s thesis setting, [BIB]).*

Why this is not $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$ [THEOREM]. $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$, so the fibre product over $\mathrm{Spec} \mathbb{Z}$ is the diagonal $\mathrm{Spec} \mathbb{Z}$ again, a curve and not a surface. The Hilbert-space square $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ is perfectly good, but its partition function is $\zeta(s)\zeta(t)$: two independent gases. It contains no cross term that could play the role of $h^1 \otimes h^1 \subset h^2(C \times C)$ (Künneth), and $h^1 \otimes h^1$ is where, for curves, the graph of Frobenius and the Hodge index theorem live (Lemma 70’s Weil light cone, [TAA-L]). The tensor product in (i) is over the *primes*: it encodes the Euler product (independence of places), which is “point counting”. It is not over a base field $\mathbb{F}_1$. Manin’s question of what “absolute Descartes powers” $\mathrm{Spec} \mathbb{Z} \times \cdots \times \mathrm{Spec} \mathbb{Z}$ should be is stated as the central question of Manin (1995, §0, [CITED]). Connes–Consani’s best current answer is the square of the arithmetic site, whose reduced structure sheaf is the semiring $\mathrm{Conv}_{\geq}(\mathbb{N} \times \mathbb{N})$ of Newton polygons (CC14 abstract, [CITED]). No positivity theorem on it is known to us ([OPEN]).

Conclusion [HEURISTIC]. The restricted tensor product is the correct *state space* and gives the Euler product, i.e. the $\mathrm{Re} s > 1$ side. It does not give a space whose h^1 carries the zeros with a positive form. The obstruction is not that tensor products cannot be defined. It is that the one we can define has no h^1 .

4. PART C: CONNES’ COUNTERTERM, LEMMA 75’S CUSP COUNTERTERM, AND THE POLE PLANE

Connes 1999 at source [CITED] (`lit/lemma71/connes.txt`). The zeros appear “as an absorption spectrum, i.e. as missing spectral lines” (Introduction). Thm. 4 (semi-local): for $h \in \mathcal{S}(C_S)$ of compact support,

$$\mathrm{Trace}(R_{\Lambda} U(h)) = 2h(1) \log' \Lambda + \sum_{v \in S} \int_{k_v^*} \frac{h(u^{-1})}{|1 - u|} d^*u + o(1), \quad R_{\Lambda} = \hat{P}_{\Lambda} P_{\Lambda},$$

with P_{Λ} an infrared cutoff and $\hat{P}_{\Lambda}$ its Fourier conjugate (ultraviolet). The divergent term is proportional to $h(1)$, the identity component. For $h = g * g^*$ it is $\|g\|^2$: a *positive, infinite-rank, scale-blind* counterterm. Connes–Consani 2021, Thm. 1 ([CITED], used in Lemma 25): $W_{\infty}(g * g^*) \geq \mathrm{Tr}(\vartheta(g) \mathbf{S} \vartheta(g)^*) \geq 0$ only for $\mathrm{supp} g \subset [2^{-1/2}, 2^{1/2}]$ with $\hat{g}$ vanishing at $\frac{i}{2}$ and 0. Local archimedean positivity therefore holds only after the pole directions are removed.

The pole plane and the scaling flow. Lemma 10 ([TAA-L]): $\mathcal{P}(F) = 2 \mathrm{Re}(a\bar{b})$, $a = \int F(u) e^{u/2} du$, $b = \int F(u) e^{-u/2} du$, rank 2, signature (1,1). The null lines are $b = 0$ (only the $s = 1$ evaluation survives: $\mathbb{L}$) and $a = 0$ ($\mathbf{1}$).

Theorem 4.1 (scaling covariance forces the pole form). [THEOREM] *Let the scaling flow act by $(T_c F)(u) = F(u - c)$. Then $(a, b) \mapsto (e^{c/2} a, e^{-c/2} b)$, a Lorentz boost of rapidity $c/2$ on the pole plane, and $\mathcal{P}(T_c F) = \mathcal{P}(F)$. Conversely, every real quadratic form $Q(a, b)$ (Hermitian in the complex case) with $Q(e^{c/2} a, e^{-c/2} b) = Q(a, b)$ for all c is $\kappa \mathrm{Re}(a\bar{b})$ for some $\kappa \in \mathbb{R}$. In particular no one-sided form $|a|^2$, $|b|^2$, or any nonzero combination $\alpha|a|^2 + \beta|b|^2$ is flow-invariant.*

Proof. $\int F(u - c) e^{\pm u/2} du = e^{\pm c/2} \int F(u) e^{\pm u/2} du$. Writing $Q = \alpha|a|^2 + \beta|b|^2 + 2 \mathrm{Re}(\kappa' a\bar{b})$, invariance requires $\alpha e^c + \beta e^{-c}$ to be constant in c , so $\alpha = \beta = 0$. The cross term is invariant, and Hermitian symmetry plus the real structure (F real, $a, b \in \mathbb{R}$) leave the real multiple. $\square$

Numerics [NUMERICAL] (num/lemma78_pole.py). For Lemma 17's Gaussian ($\sigma = \frac{1}{2}$): pole $3.344208 = \pi e^{1/16}$, kinetic -2.237855 , primes -1.106353 , total $9.4 \cdot 10^{-13}$. This reproduces L17's Table. Shifted by $c = 1$ and $c = -2$: $(a, b) = (2.1320, 0.7843)$ and $(0.4757, 3.5150)$, while all three pieces are unchanged. A one-sided $|a|^2$ would move from 1.672 to 4.545 and 0.226. The odd Gaussian has $\mathcal{P} = -0.05225$, which is the negative direction (Fig. 2).

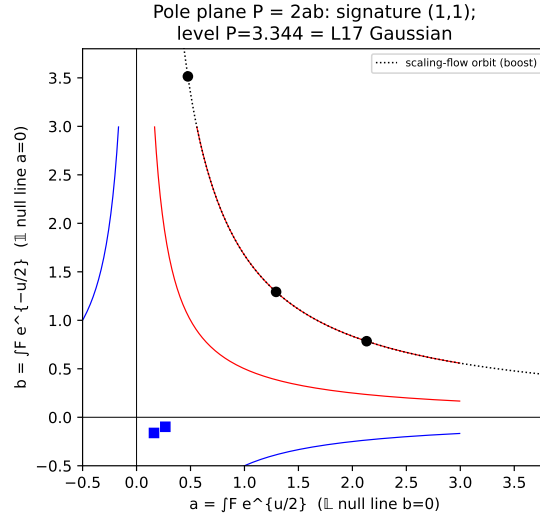


FIGURE 2. The pole plane in null coordinates (a, b) . Red/blue are level sets $\mathcal{P} = 2ab > 0 / < 0$. Black dots are the L17 Gaussian at shifts $c = 0, 1, -2$, lying on one boost orbit (dotted) with $\mathcal{P} = 3.344$. Blue squares are odd test functions, with $\mathcal{P} < 0$.

Test of the brief's hypothesis: does L75's cusp counterterm reproduce $\mathcal{P}$? Lemma 75 ([TAA-L]) found that the Mayer zeros $\rho/2$ equal the head of the transfer operator minus an Euler–Maclaurin counterterm for the parabolic pole at $s = \frac{1}{2}$ in Mayer's variable, i.e. at $2s = 1$, the pole of $\zeta(2s)$. Its leading part is $A^{1-2s}/(2s-1)$, and

$$\frac{A^{1-2s}}{2s-1} = \frac{1}{2s-1} - \log A + O((2s-1) \log^2 A) \quad \text{[THEOREM] (checked, pole.json).}$$

So the L75 counterterm is *one pole* (the $\mathbb{L}$ null line) *plus a logarithmic cutoff divergence*, with $\log A$ playing the role of Connes' $\log \Lambda$ [ANALOGY]. By Theorem 4.1, a counterterm with only the $\mathbb{L}$ direction cannot be scale-covariant and cannot reproduce $\mathcal{P}$. The **1** direction (pole at 0, from the archimedean place, Part E) must be supplied by the functional-equation partner. **Answer: no, not alone** [THEOREM] (for the quadratic-form question) + [HEURISTIC] (for the identification of L75's counterterm with the $\mathbb{L}$ line). Lemma 17's cancellation $3.344 - 2.238 - 1.106 \approx 0$ is the cancellation between the full invariant pole form and the two indefinite infinite-rank pieces. That cancellation is the “cancellation input” Lemma 75 asked for, and it is ζ -specific. For every entire L -function (Dirichlet $L(s, \chi)$, $\chi \neq 1$, where GRH is expected, and DH, where it fails) $\mathcal{P} \equiv 0$ [THEOREM]: no poles). So *the pole-plane cancellation cannot be the mechanism behind GRH in general*.

Same counterterm in Bombieri–Lagarias [CITED-sec] (Lagarias 2007, arXiv math/0404394, §9, pp. 38–39, opened). Li's coefficients are Weil's functional on test functions $g_n(x) = P_n(\log x) \mathbf{1}_{(0,1)}$, with P_n a Laguerre polynomial, whose Mellin transforms have poles at 0 and/or 1. The terms W_0, W_1 (the pole plane) diverge and must be cut off at $x = 1/T$, $x = T$, “offset by a corresponding divergence coming from the finite primes”. This is a third independent appearance of the same pole-versus-primes counterterm structure: L17, L75, and Connes.

5. PART D: THE $\mathbb{F}_1$ “META-SPACE” (J.W. [T315], [T317])

J.W.’s hunch, made precise. “At $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ one no longer counts with a prime q but always with 1, point-counted $1 + 1 + 1 + \dots$ ” is the $q \rightarrow 1$ limit of the $\mathbb{F}_1$ program.

Proposition 5.1 (Soulé’s limit; Kurokawa’s product). *[CITED]+[THEOREM] For a counting function $N(q)$, Soulé defines $\zeta_N(s) = \lim_{q \rightarrow 1} Z(q, q^{-s})(q-1)^{N(1)}$, where $Z(q, T) = \exp \sum_r N(q^r) T^r / r$ (CC10 = arXiv 1006.4810, eqs. (1)–(2), [CITED]). For a polynomial count $N(q) = \sum_k a_k q^k$ this gives $\zeta_N(s) = \prod_k (s-k)^{-a_k}$. (Proof: $Z = \prod_k (1 - q^{k-s})^{-a_k}$ and $(q-1)/(1 - q^{k-s}) \rightarrow 1/(s-k)$.) The product form is usually credited to Kurokawa ([BIB], via Manin 1995 [25]). For $N = 1 + q$ (the motive $\mathbf{1} \oplus \mathbb{L}$, i.e. $\mathbb{P}^1$) one gets $1/(s-1)$. CC10 states exactly this for the projective line ([CITED]), and we checked the limit numerically to 10^{-8} at $q-1 = 10^{-8}$ (cc.json). Moreover*

$$\Lambda(s) - \frac{1}{s(s-1)} \text{ is entire} \quad [\text{THEOREM}],$$

since $\text{Res}_{s=1} \Lambda = 1$ and $\text{Res}_{s=0} \Lambda = -1$. So “ $\mathbf{1} \oplus \mathbb{L}$ is the pole part of ζ ” is exact with Soulé’s normalization. Manin’s regularized-determinant normalization instead has $Z(T^{\otimes n}, s) = (s-n)/2\pi$ (Manin 1995 (1.30), [CITED]), and differs by 2π factors.

The linear part is $\mathbb{L}$, the remainder is h^1 [THEOREM]. $\psi(x) = \sum_{n \leq x} \Lambda(n) = x - \sum_{\rho} x^{\rho} / \rho - \log 2\pi - \frac{1}{2} \log(1 - x^{-2})$ (von Mangoldt; Davenport Ch. 17, [CITED-sec] via L25). The “knowable linear growth” x is the $\mathbb{L}$ term, and RH is a statement about the remainder. Connes–Consani make this an exact identity of distributions (CC10 Thm. 2.2, citing their Compositio 2010 paper, [CITED]):

$$N(u) = u - \frac{d}{du} \left(\sum_{\rho} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1} \right) + 1, \quad -\frac{\partial_s \zeta_{\mathbb{Q}}}{\zeta_{\mathbb{Q}}}(s) = \int_1^{\infty} N(u) u^{-s} d^*u,$$

and N is positive on $(1, \infty)$, while $N(1) = -\infty$ (“infinite genus”). Read term by term this is $\mathbb{L}(u)$, h^1 (the zeros), and $\mathbf{1}$ (the constant 1). It is the exact shape of $\#C(\mathbb{F}_q) = q - \sum \alpha + 1$ for a curve (their eq. (18)). This confirms what [t315] attributed to them “as I recall”.

Checks [NUMERICAL] (num/lemma78_cc.py). (i) Their constant: $\omega(1) = \sum_{\rho} \frac{1}{\rho+1} = \frac{1}{2} + \frac{\gamma}{2} + \frac{\log 4\pi}{2} - \frac{\zeta'(-1)}{\zeta(-1)} = 0.06906623$. Our 4520 pairs plus smooth tail give 0.06906628 (difference $4.6 \cdot 10^{-8}$, at the level of the tail estimate). (ii) With log-Gaussian test functions ($u_0 \in \{10, 30.5, 31, 100, 1000, 1009\}$, widths 0.002–0.05), the prime side $\sum \Lambda(n) f(n) + \int \kappa f d^*u$ equals the zero side $\int f(u+1) d^*u - \sum_{\rho} \tilde{f}(\rho)$ to relative $3 \cdot 10^{-16}$. Over a scan $u_0 \in [2.2, 60]$ the maximum absolute difference is $6 \cdot 10^{-14}$ (Fig. 3).

Does the counting positivity see ghosts? No [THEOREM]. The positivity of N on $(1, \infty)$ follows from $\Lambda(n) \geq 0$ and $\kappa(u) = u^2/(u^2 - 1) > 0$ (CC10 eqs. (8)–(9)). These are prime-side facts that hold whatever the zeros do. An off-line zero would change the h^1 term, but the identity would still hold with the same positive prime side, so positivity of N cannot exclude it. It is already false for $L(s, \chi_4)$, whose analogous coefficients $\chi_4(n)\Lambda(n)$ are negative at $n = 3, 7, 11, 19, 23, 27, 31, 43, 47, 59$ (cc.json), although GRH is expected there. Lemma 76’s DH “Farey measure” was positive while DH has ghosts, which is the same lesson ([TAA-L]). In codes (Lemma 77) the virtual counts N_m do go negative for the complex ghost $\mathbf{i}_2 \oplus e_8$ ($N_7 < 0$), but not for the 209 Siegel ghosts. So effectivity of counts catches some ghosts in finite models and none in the ζ case ([NUMERICAL] + [TAA-L]).

Scaling flow as Frobenius; orbits $\log p$ [CITED]+[ANALOGY]. Connes 1999 ([CITED]): for a curve over $\mathbb{F}_q$, the points over $\mathbb{F}_q$ form “a countable union of periodic orbits” of Frobenius, “parametrized by the set of places”, with periods “the analogues of the $\log p$ ”. In the characteristic- p case Frobenius is “not ... a flow but rather a single transformation” (p. 8). For $\mathbb{Q}$ the idele class group $C_{\mathbb{Q}} \supset \mathbb{R}_{+}^*$ acts on the adèle class space, and Thm. 4 above has one term per place v , including $v = \infty$, which contributes the archimedean principal-value term. So [t317]’s “one orbit for each prime, length $\log p$ ” is right for the finite places, with the archimedean place as an extra non-periodic contribution. The discrete clock m of $\mathbb{F}_{q^m}$ becomes the continuous clock $\log \lambda$.

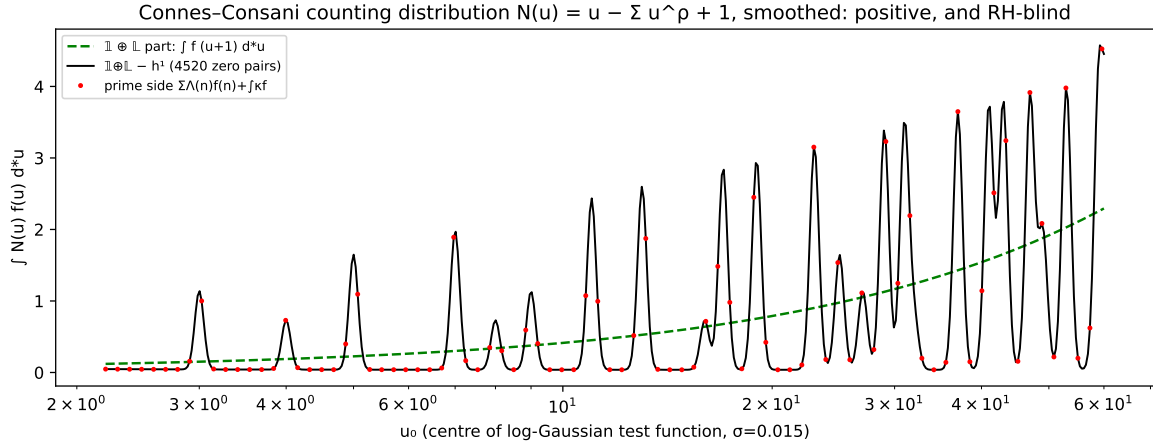


FIGURE 3. Smoothed Connes–Consani counting distribution. Green: the $\mathbf{1} \oplus \mathbb{L}$ part. Black: $\mathbf{1} \oplus \mathbb{L}$ minus the h^1 (zero) sum. Red: the prime side. Between prime powers the h^1 term cancels the smooth growth almost exactly; at prime powers it adds a spike. Positivity is manifest on the prime side.

The archimedean/non-archimedean swap; characteristic one [CITED]. Maslov dequantization $h \log(e^{a/h} + e^{b/h}) \rightarrow \max(a, b)$ (Litvinov, [CITED] via Lemma 23’s record) turns ordinary addition into idempotent addition, $\max(a, a) = a$. The ultrametric inequality $c \leq \max(a, b)$ is encoded by Viro’s *ultra-triangle hyperfield* $Y_\times$, the dequantization of the triangle hyperfield (Viro, arXiv 1006.3034, Intro, [CITED]). It is a multivalued sum, not the semifield max. The Connes–Consani arithmetic site (CC14, [CITED]) is the topos $\widehat{\mathbb{N}^\times}$ with structure sheaf the tropical semiring $\bar{\mathbb{N}} = (\mathbb{N} \cup \infty, \min, +)$ (min-plus, characteristic 1). Its points over $[0, 1]_{\max} \subset \mathbb{R}_{\max}^+$ form $\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}} / \widehat{\mathbb{Z}}^*$ (the adèle class space modulo $\widehat{\mathbb{Z}}^*$, not the full space). The Frobenius automorphisms Fr_λ , $\lambda \in \mathbb{R}_+^+$, of $\mathbb{R}_{\max}^+$ realize the scaling action. The Hasse–Weil zeta of the site is the complete ζ .

Does the $q \rightarrow 1$ limit see ghosts? [THEOREM] (negative) + [OPEN]. For finite polynomial counts, $q \rightarrow 1$ retains only the exponents k , the weights (Prop. 5.1). Every zero of a Kurokawa product $\prod (s - k)^{-a_k}$ sits at an integer, so there is no room for a ghost. For ζ the h^1 part survives only as a *distribution*, with $N(1) = -\infty$, and its positivity is RH-blind (above). Whether a refined $q \rightarrow 1$ object (e.g. CC’s scaling site with its tropical Riemann–Roch on periodic orbits, CC17/CC19, [CITED] via Lemma 23) carries a positivity that *does* see ghosts is open. CC19 itself calls the decisive step “the most problematic” (Lemma 23’s record, [CITED-sec]).

6. PART E: THE POINT AT INFINITY ([T318])

What kind of objects. The *Chow motive* $h(C)$ of a smooth projective curve is a formal object that splits as $\mathbf{1} \oplus h^1(C) \oplus \mathbb{L}$. Here $\mathbf{1} = h^0$ is the “point” piece (weight 0, the pole at $s = 0$), $\mathbb{L} = h^2$ is the “fundamental class” piece (weight 2, the pole at $s = 1$), and h^1 is where Frobenius acts with eigenvalues of weight 1 (the zeros). $\mathbb{P}^1$ is the case $h^1 = 0$ [THEOREM].

Arakelov’s compactification $\text{Spec } \overline{\mathbb{Z}} = \text{Spec } \mathbb{Z} \cup \{\infty\}$ adds the archimedean place, whose Euler factor is $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2)$. Only the completed Λ satisfies $\Lambda(s) = \Lambda(1 - s)$ [THEOREM]. *Precise origin of the pole at 0* [THEOREM]: $\Gamma(s/2)$ has poles at $0, -2, -4, \dots$. At $-2, -4, \dots$ they are cancelled by the trivial zeros of ζ , i.e. by the finite places. Only the pole at 0 survives, because $\zeta(0) = -\frac{1}{2} \neq 0$. So the pole at 0 is produced by ∞ and kept by the finite places. In the theta integral it is the $t \rightarrow \infty$ end (constant term 1 of θ), and the pole at 1 is the $t \rightarrow 0$ end (the term $t^{-1/2}$, i.e.

the $n = 0$ mode of the Poisson-dual theta). Concretely, from $\theta(t) = t^{-1/2}\theta(1/t)$,

$$\Lambda(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (t^{s/2} + t^{(1-s)/2}) \omega(t) \frac{dt}{t}, \quad \omega(t) = \sum_{n \geq 1} e^{-\pi n^2 t},$$

and both polar terms come from the $n = 0$ mode, one on each side of Poisson summation [THEOREM].

Manin’s proposal [CITED] (Manin 1995, §1.1 and §1.6, opened). Deninger’s formula (1.5) writes $Z(\overline{\text{Spec } \mathbb{Z}}, s)$ as a regularized determinant over a hypothetical H^1 , divided by the determinant of an “absolute motive $T^0 \oplus T^1$ ”. Manin adds: “Compactifying $\text{Spec } \mathbb{Z}$ to $\overline{\text{Spec } \mathbb{Z}}$ then makes it similar to a projective curve over $\mathbb{F}_q$ rather than to an affine one”, and $H(\overline{\text{Spec } \mathbb{Z}}) = T^0 \oplus H^1 \oplus T$. This is J.W.’s [t318] intuition, stated by Manin. It is a [CONJECTURE]/program, not a theorem: the H^1 is postulated.

Riemann–Roch for $\mathbb{Q}$ [CITED]/[CITED-sec]/[BIB]. Connes–Consani prove a Riemann–Roch theorem for divisors on $\overline{\text{Spec } \mathbb{Z}}$ (CC23, arXiv 2205.01391, abstract and Intro, [CITED]) with integer dimensions over the sphere spectrum. Their introduction credits the earlier theta-function approach of van der Geer–Schoof and Bost ([CITED-sec]). That Tate’s thesis contains a Riemann–Roch theorem (adelic Poisson summation) is our recollection only ([BIB]). All of these are Riemann–Roch on the curve $\overline{\text{Spec } \mathbb{Z}}$. None is on a surface.

What $(\overline{\text{Spec } \mathbb{Z}})^2$ would need for $\infty \times \infty$ to contribute to $\mathcal{P}$ [HEURISTIC]/[OPEN]. On $C \times C$ the plane spanned by the fibre classes $e_1 = \{pt\} \times C$ and $e_2 = C \times \{pt\}$ has intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: signature $(1, 1)$, with null lines e_1, e_2 (Lemma 70, [TAA-L]). Theorem 4.1 shows that $\mathcal{P}$ has exactly this shape, with the scaling flow as the boost preserving it. A surface theory would therefore need (i) two fibre classes, one through the finite places ($\mathbb{L}$ -type, pole at 1) and one through ∞ ($\mathbf{1}$ -type, pole at 0), with self-intersection 0 and mutual intersection 1; and (ii) a Frobenius-flow correspondence acting on their span by the boost. Item (ii) is free (Thm. 4.1). Item (i), $\infty \times \infty$ as an honest fibre with $e_1 \cdot e_2 = 1$, is exactly what no current theory supplies. Even if it were supplied, it would give only the $(1, 1)$ plane, which is the RH-blind part. The Hodge index content lives on its orthogonal complement (Lemma 70).

6.1. The archimedean place as a reflection (J.W. [t325]). J.W.’s hunch: the Γ -factor is the archimedean place because “ $s/2$ is a 90° rotation at $s = \pi$ ”, and conjugation is “a reflected 180° rotation”.

- (i) **Correction** [THEOREM]. The point $s = \pi$ plays no role. $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma(s/2) = \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s d^*x$ is the Mellin transform of the self-dual Gaussian $e^{-\pi x^2}$ (checked to $2 \cdot 10^{-31}$ at real/complex test points, $2 \cdot 10^{-7}$ at $s = \frac{1}{2} + 14.1i$ where the quadrature is oscillatory; `gamma.json`). The π comes from Fourier self-duality (Haar normalization $e^{-2\pi ixy}$), and the $/2$ comes from x^2 being a quadratic form. The setting is Tate’s thesis ([BIB]). A 90° rotation is multiplication by $e^{i\pi/2} = i$: the exponent $i\pi/2$, not the value $s = \pi$.
- (ii) **Rotation versus reflection** [THEOREM]. $s \mapsto 1 - s$ is the 180° rotation about $s = \frac{1}{2}$ (as a real-linear map, $\det = +1$). Composing with complex conjugation, which is legitimate because ζ is real on $\mathbb{R}$ and the zeros are closed under $\rho \mapsto \bar{\rho}$, gives $s \mapsto 1 - \bar{s}$, the reflection across the critical line ($\det = -1$). Under the Li map this reflection becomes the circle inversion $w \mapsto 1/\bar{w}$ (§7). So J.W.’s “reflected 180° rotation” is exactly the functional equation combined with conjugation.
- (iii) **The reflection is the archimedean Frobenius** ([BIB] for the general theory: Serre 1970 / Deligne 1979 on Γ -factors, recalled and not opened; [THEOREM] for the identities). At the real place the decomposition group is $\text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, F_\infty\}$, with F_∞ complex conjugation. The local factor of a motive at ∞ is $\Gamma_{\mathbb{R}}(s)$ on the $F_\infty = +1$ part and $\Gamma_{\mathbb{R}}(s+1)$ on the $F_\infty = -1$ part. For Dirichlet characters this is $\chi(-1) = +1$ (even) versus -1 (odd). The DH function used here (`LamDH`) has $\Gamma(\frac{s+1}{2})$ because its coefficients are odd. Legendre duplication gives $\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s)$ (checked to 10^{-31}), the factor of a complex place where F_∞ swaps two embeddings. So the reflection J.W. senses is, literally, the archimedean Frobenius F_∞ .

- (iv) **Positivity content: none by itself** [THEOREM]/[HEURISTIC]. F_∞ is an *involution*, the archimedean counterpart of the formal relation $\pi^\dagger = q\pi^{-1}$ in §7.3(2). It fixes the sign pattern (even/odd) but no inequality. For Hodge structures, positivity enters only through a *polarization* compatible with conjugation, i.e. the Hodge–Riemann bilinear relations with the Weil operator $C = i^{p-q}$ ([BIB], standard Hodge theory, not opened here). That is the archimedean analogue of Rosati positivity, and like it, it is an extra input beyond the involution. So [t325] correctly locates the *involution* at ∞ . The *polarization* J is still missing.

7. PART F: DISKS ([T319]–[T320]) AND THE LI MAP VERSUS THE ROSATI INVOLUTION ([T322]–[T324])

7.1. The Li disk and the Möbius family. [THEOREM] The map $w = 1 - 1/s$ sends $\operatorname{Re} s > \frac{1}{2}$ onto $|w| < 1$, since $|1 - 1/s| < 1 \iff |s - 1| < |s|$. It sends the line to the circle, $s = 1$ to 0, and $s = 0$ to ∞ , and $w(1 - \bar{s}) = 1/\overline{w(s)}$. Also $w(1 - s) = 1/w(s)$ identically. The general conformal map of $\{\operatorname{Re} s > \frac{1}{2}\}$ onto the unit disk is $w = e^{i\theta}(s - a)/(s + \bar{a} - 1)$, $\operatorname{Re} a > \frac{1}{2}$ (correction to [t320], which omitted $e^{i\theta}$ and the condition on a). Li (1997): $\text{RH} \iff \lambda_n = \sum_\rho [1 - (1 - 1/\rho)^n] \geq 0$ for all n , with λ_n from the derivatives of $s^{n-1} \log \xi(s)$ at $s = 1$ ([CITED-sec] zbMATH review, DOI checked). Bombieri–Lagarias (1999) extend this to general multisets with the $\rho \mapsto 1 - \bar{\rho}$ symmetry ([CITED-sec], zbMATH review and Lagarias 2007 §1).

7.2. The code β -disk and the q -Li map. For codes (Lemma 77), $T = q^{-s}$, and the normalized inverse roots are $\beta = 1/(\sqrt{q}T) = q^{s-1/2}$. The critical line is $|\beta| = 1$, the functional equation is $\beta \mapsto 1/\bar{\beta}$, and the poles are $s = 1 \mapsto \beta = \sqrt{q}$, $s = 0 \mapsto 1/\sqrt{q}$ [THEOREM]. **Correction:** the centre $\beta = 0$ corresponds to $\operatorname{Re} s \rightarrow -\infty$, not $+\infty$ as stated in [t320]. The Euler-product region $\operatorname{Re} s > 1$ is $|\beta| > \sqrt{q}$, *outside* the unit circle.

Theorem 7.1 (q -Li map). [THEOREM] Put $w_q(\beta) = \frac{\beta - \sqrt{q}}{\sqrt{q}\beta - 1}$. Then w_q is a Möbius map preserving $|\beta| = 1$. It sends the exterior $|\beta| > 1$ (i.e. $\operatorname{Re} s > \frac{1}{2}$) onto $|w| < 1$, $\sqrt{q} \mapsto 0$ (pole $s = 1$ to the centre), and $1/\sqrt{q} \mapsto \infty$, and it intertwines $\beta \mapsto 1/\bar{\beta}$ with $w \mapsto 1/\bar{w}$. In the variable s ,

$$w_q(q^{s-1/2}) = \frac{q^s - q}{\sqrt{q}(q^s - 1)} = \sqrt{q} \frac{[s-1]_q}{[s]_q}, \quad [x]_q = \frac{q^x - 1}{q - 1},$$

and $w_q(q^{s-1/2}) \rightarrow \frac{s-1}{s} = 1 - \frac{1}{s}$ as $q \rightarrow 1$, locally uniformly in $s \notin \{0\}$.

Proof. $|\sqrt{q}\beta - 1| = |\beta||\sqrt{q} - \bar{\beta}^{-1}| = |\sqrt{q} - \beta|$ on $|\beta| = 1$. Substituting $\beta = q^{s-1/2}$ gives the displayed identity (checked to 10^{-51}). Then $[x]_q \rightarrow x$. $\square$

Answer to [t320]’s open question ([THEOREM], in this form). The plain β -disk does *not* land on Li’s disk: as $q \rightarrow 1$ every bounded s -region collapses to $\beta = 1$, and only the blow-up $(\beta - 1)/\log q \rightarrow s - \frac{1}{2}$ survives, which is the half-plane. The pole-recentred disk w_q lands on Li’s disk exactly (Thm. 7.1; numerically $|w_q - (1 - 1/s)| \approx c(q - 1)^2$, e.g. $1.2 \cdot 10^{-32}$ at $q - 1 = 10^{-16}$; Fig. 4, right). [t320]’s phrase “the pole moving from the edge to the centre” should read “from outside the unit circle (on the $\operatorname{Re} s > \frac{1}{2}$ side) to the centre”.

[t316]’s ghost, checked [NUMERICAL] (disk.json). For $\mathbf{i}_2 \oplus e_8$ ($q = 2$, $g = 4$, $\kappa = 2$), $s = \frac{1}{2} \pm 0.3695528 \pm 1.3934054i$. The outer pair is $\beta = 0.73477 \pm 1.06266i$ ($|\beta| = 1.29195$), the inner pair $0.44021 \pm 0.63665i$ ($|\beta| = 0.77402$), and $|\beta_{\text{out}}|/|\beta_{\text{in}}| = 2^{2\delta} = 1.6692$. The real parts 0.73 and 0.44 do *not* straddle $\operatorname{Re} \beta = 0.5$ symmetrically. The symmetry is exact in s about $\operatorname{Re} s = \frac{1}{2}$ (functional equation), as said in [t316]. In the q -Li plane, $|w_q| = 0.83900$ and 1.19189 (reciprocal).

Which centre sees which ghost, and how fast [NUMERICAL] (first m with $\operatorname{Re} \sum (1 - z^m) < 0$, $m \leq 300$).

code	κ	ghost type	centre $\beta = 0$ (L77 Li)	q -Li (centre = pole)	$\max w_q $
G_{24}, e_8^2	0	none	never	never	1
$G_{24} \oplus e_8$	1	real, $s = 0.97046$	11	2	33.48
e_8^3	1	real	10	2	18.20
$i_2 \oplus e_8^2$	3	real + complex	13	2	5.74
$i_2 \oplus e_8$	2	complex	7	10	1.19

Pole-hugging (Siegel) ghosts sit next to the pole, i.e. next to the q -Li centre and its mirror at ∞ . So the pole-centred criterion catches them at once ($|w| = 33.5$ for $G_{24} \oplus e_8$). The complex ghost sits far from the pole and is caught faster from $\beta = 0$. *The truth value is centre-independent; the detection speed is not.* For DH ([HEURISTIC]: we estimate the index at which $|w|^n$ beats the smooth growth $\frac{n}{2} \log n$ of λ_n), Li's centre $a = 1$ gives $\log |w| = 4.2 \cdot 10^{-5}, 1.2 \cdot 10^{-5}, 2.7 \cdot 10^{-6}, 7.2 \cdot 10^{-6}$ for the four ghosts, so $n^* \approx 3.5 \cdot 10^5, 1.4 \cdot 10^6, 6.9 \cdot 10^6, 2.3 \cdot 10^6$. A centre $a = \frac{3}{4} + i\gamma$ next to a ghost gives $\log |w| = 0.6$ – 2.9 , i.e. immediate detection. Ghosts at height γ cost $\sim \gamma^2/\delta$ from $s = 1$ (shape [HEURISTIC]).

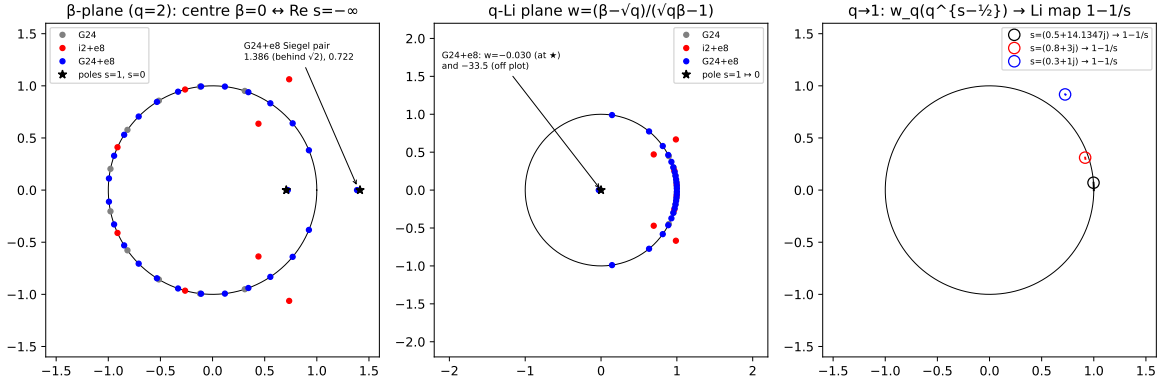


FIGURE 4. Left: zeros in the code β -plane ($q = 2$). Grey: G_{24} (RH). Red: $i_2 \oplus e_8$ (complex ghost quadruple, inverted pairs). Blue: $G_{24} \oplus e_8$ (Siegel pair near the pole $\sqrt{2}$). Middle: the same in the q -Li plane, with the pole at the centre. Right: $w_q(q^{s-1/2})$ as $q \rightarrow 1$ converging to Li's $1 - 1/s$ (circles) for three values of s .

2+1 gravity disk [ANALOGY]. The totally geodesic disk bounded by a primitive cycle in the Lemma 74 hyperbolic picture is a disk in a different space (2+1 dimensions: two space, one time). It has not been identified with the Li disk. Analogy only.

7.3. Li map versus the Rosati involution. (1) The inversion identity [THEOREM]. For every $z \neq 0$, $[1 - z^n] + [1 - z^{-n}] = (1 - z^n)(1 - z^{-n})$ (expand). If $|z| = 1$ then $z^{-n} = \bar{z}^n$ and the right side is $|1 - z^n|^2 \geq 0$. With $w(1 - s) = 1/w(s)$, the functional equation $\rho \mapsto 1 - \rho$ and real coefficients give $\lambda_{-n} = \overline{\lambda_n} = \lambda_n$, so

$$2\lambda_n = \sum_{\rho} (1 - w_{\rho}^n)(1 - w_{\rho}^{-n}) \stackrel{\text{RH}}{=} \sum_{\rho} |1 - w_{\rho}^n|^2 \geq 0.$$

This is the easy direction $\text{RH} \Rightarrow \lambda_n \geq 0$ [THEOREM]. It is also the Weil-form shape $\sum_{\rho} \hat{g}(\rho) \overline{\hat{g}(1 - \bar{\rho})}$ with $G_n(s) = 1 - (1 - 1/s)^n$. Bombieri–Lagarias showed that λ_n is Weil's functional for an explicit test function, $g_n(x) = P_n(\log x)$ on $(0, 1)$ with $P_n(x) = \sum_{j=1}^n \binom{n}{j} \frac{x^{j-1}}{(j-1)!}$ (a Laguerre polynomial L_n^1), $g_n(1) = \frac{n}{2}$, and 0 for $x > 1$, with the divergent pole terms regularized by cutoffs [[CITED-sec]: Lagarias 2007, math/0404394, §1 and §9, opened; the BL paper itself was behind a Cloudflare

challenge and was not opened]. On the codes, the identity holds to 10^{-38} , and $\sum |1 - \beta^n|^2 = 2\lambda_n$ holds to 10^{-43} for the RH codes but fails by 10^9 – 10^{11} for the ghost codes, as it must (`rosati.json`).

(2) Rosati: adjoint = q -inverse, with no positivity [CITED] (Milne, arXiv 1509.00797, §1, opened). For a polarization λ of an abelian variety A over $\mathbb{F}_q$, the Rosati involution is $\alpha^\dagger = \lambda^{-1} \alpha^\vee \lambda$, and the Weil pairing satisfies $e_\lambda(\alpha x, y) = e_\lambda(x, \alpha^\dagger y)$. Then $e_\lambda(x, \pi^\dagger \pi y) = e_\lambda(\pi x, \pi y) = q e_\lambda(x, y)$, so $\pi \pi^\dagger = q$ (proof of Milne’s Thm. 1.30). This uses only Galois-equivariance of the pairing, no positivity. In normalized form $(\pi/\sqrt{q})^\dagger = (\pi/\sqrt{q})^{-1}$: *the adjoint equals the inverse*. Since the adjoint has the same characteristic polynomial, the eigenvalues satisfy $\{\alpha\} = \{q/\alpha\} = \{q/\bar{\alpha}\}$. That is the functional equation, the same formal relation as Li’s $w \leftrightarrow 1/\bar{w}$ and as $\beta \leftrightarrow 1/\bar{\beta}$ [THEOREM]. With respect to an *indefinite* form this is the Krein-unitary situation of Lemma 77 (Π_κ , κ = number of ghost pairs, [TAA-L]).

(3) RH comes from positivity of $\dagger$, which does not refer to the zeros [CITED]. Milne Thm. 1.27: $\text{Tr}(\alpha \alpha^\dagger) > 0$ for $\alpha \neq 0$, proved from an ample divisor, $\text{Tr}(\alpha \alpha^\dagger) = \frac{2g}{(D^g)} (D^{g-1} \cdot \alpha^{-1} D) > 0$. Cor. 1.29: if $\alpha \alpha^\dagger = q$ then $|\rho(\alpha)| = \sqrt{q}$ for every embedding ρ . The positivity of $\text{Tr}(X \circ X')$ in Weil’s 1940s argument “expresses the positivity of the Rosati involution” (Milne §1 footnote, [CITED]). Connes 1999 tabulates “Riemann hypothesis $\leftrightarrow$ Castelnuovo positivity” ([CITED]). By contrast, $\lambda_n \geq 0$ for all n is a reformulation of the conclusion: [RESTATEMENT] (Lemma 77’s Krein inertia theorem gives the same for codes).

Codes: the involution without the positivity [NUMERICAL]+[THEOREM] (`num/lemma78_rosati.py`). For every code tested, the zero set is invariant under $\alpha \mapsto q/\bar{\alpha}$ to $2 \cdot 10^{-45}$ (the involution exists). But there is no Jacobian or polarization (Lemma 77: N_m non-integral for G_{24} ; $N_7 < 0$ for $\mathbf{i}_2 \oplus e_8$), and RH fails for the ghost codes. In each ghost case the first negative Li coefficient (centre $\beta = 0$) occurs at *exactly* the first failure of the Weil/Castelnuovo bound $|S_m| \leq 2gq^{m/2}$, $S_m = \sum \alpha^m$: $m = 7$ ($\mathbf{i}_2 \oplus e_8$), 10 (e_8^3), 11 ($G_{24} \oplus e_8$), 13 ($\mathbf{i}_2 \oplus e_8^2$). This is forced: $\lambda_m = 2g - q^{-m/2} S_m$, so $\lambda_m < 0 \iff S_m > 2gq^{m/2}$, the upper half of the Castelnuovo inequality. *For curves that inequality is a consequence of Rosati positivity; for codes it is simply false*. Li positivity and the Castelnuovo bound are the same numbers. What curves have and codes lack is the positive form that produces them.

(4) Squaring, conjugation, and $|w|^2$ (J.W. [t323]–[t324]) (`num/lemma78_square.py`, $\mathbf{i}_2 \oplus e_8$, centre $\beta = 0$).

- (a) [THEOREM]+[NUMERICAL]. Squaring commutes with inversion, $(1/\bar{w})^2 = 1/\bar{w^2}$, so $\{w^2\}$ again has the functional-equation symmetry, and its Li coefficients are λ_{2n} . First negative λ_n : $n = 7$, $\lambda_7 = -6.849242$. First negative even index: $n = 12$, $\lambda_{12} = -15.000000$ (confirmed). Squaring is a sub-sequence and adds nothing (it detects later).
- (b) [THEOREM]+[NUMERICAL]. The inversion $w \mapsto 1/w$ (which, composed with real-coefficient conjugation, is the FE) and the conjugation $w \mapsto \bar{w}$ agree on every zero iff $|w| = 1$ for every zero, i.e. iff RH ($\max |1/w - \bar{w}| = 0.518$ for $\mathbf{i}_2 \oplus e_8$; 10^{-45} for G_{24}). The conjugation form $\sum |1 - w^n|^2$ is positive by construction and RH-blind: its minimum over $n \leq 40$ is 16.537 (at $n = 1$) for $\mathbf{i}_2 \oplus e_8$.
- (c) [THEOREM]+[NUMERICAL]. “Complexified squaring” $w \mapsto w\bar{w} = |w|^2$: pairing w with $1/\bar{w}$ ($r = |w|$),

$$\sum_{\rho} (1 - |w_{\rho}|^{2n}) = - \sum_{\text{pairs}} (r^n - r^{-n})^2 \leq 0, \quad \text{with equality} \iff \text{RH.}$$

This is an exact criterion. It equals $-0.53650, -2.28993, -5.73064, \dots$ for $\mathbf{i}_2 \oplus e_8$ and is $\leq 10^{-44}$ in modulus for G_{24} . But $\rho \mapsto |w_{\rho}|^{2n}$ is *not holomorphic*. The explicit formula/trace formula evaluates only $\sum_{\rho} h(\rho)$ for h analytic in ρ (Mellin transforms of test functions), so this sum has no prime-side expression and cannot be computed without knowing the zeros [THEOREM] (that h must be analytic is the form of the explicit formula; [CITED-sec] Lagarias 2007 §9). It is a [RESTATEMENT] that is useless as a tool.

- (d) [THEOREM]. Weil’s Hermitian form already composes conjugation with inversion: $\langle f, g \rangle_W = W[f * \bar{g}]$, $\tilde{g}(x) = \overline{g(1/x)}$ (Lagarias 2007 (9.7), [CITED-sec]). Complexifying a real symmetric bilinear form to a Hermitian form preserves its signature (Sylvester’s law of inertia), so *complexification cannot create positivity*. The indefinite (Krein) structure for ghost configurations persists under any such step.

External text (pasted by J.W.) [HEURISTIC]. We treat it as non-authoritative. Its cited GitHub repository (UOR-Foundation/F1, not peer-reviewed) and a “Clay proceedings PDF” were *not opened* and are not cited. Two overclaims: (i) “ λ_n are algebraic power sums”: for ζ , λ_n is a conditionally convergent sum over infinitely many zeros. (Whether the zeros are transcendental is *unknown*, so “transcendental” should not be asserted either.) Only for codes and curves is λ_n a finite power sum of algebraic numbers. (ii) “Li artificially constructs the same algebraic outcome”: Li constructs an *equivalent statement*, not the positivity. The positivity for curves comes from Rosati/Castelnuovo, which Li does not supply.

7.4. Positivity from a quarter turn? The Weil operator, Fourier, and orientation (J.W. [t326]–[t327]). J.W.’s proposal [t326]: the geometric object behind positivity is a 90° rotation, a closed orbit of order 4. If we define one on $\text{Spec } \mathbb{Z}$ using the fourth roots of unity, the reflection, the conjugate and the positivity should all come out without reference to the zeros. [t327] adds: a polarization is the choice of counterclockwise versus clockwise, and the two rotations must be inequivalent. *Notation:* in this subsection C is the Weil operator and J_{cx} is a complex structure. Our polarization J (the missing positive form) is a different object.

- (i) **Over $\mathbb{C}$ the quarter turn really is the positivity** ([BIB], standard Hodge theory: Riemann’s bilinear relations, Hodge–Riemann relations and Riemann forms, not opened here; the algebra below is [THEOREM]). For a weight-1 Hodge structure $H^1 = H^{1,0} \oplus H^{0,1}$ the Weil operator $C = i^{p-q}$ acts by i on $H^{1,0}$ and by $-i$ on $H^{0,1}$. It is a real operator with $C^2 = -1$ (a quarter turn whose eigenvalues are fourth roots of unity). A polarization is a (-1) -symmetric form Q with $Q(Cx, Cy) = Q(x, y)$ and $Q(x, C\bar{x}) > 0$ on the relevant pieces. For an abelian variety this is a Riemann form E on the lattice with $E(J_{\text{cx}}x, J_{\text{cx}}y) = E(x, y)$ and $E(x, J_{\text{cx}}x) > 0$. The Rosati positivity $\text{Tr}(\phi\phi^\dagger) > 0$ of §7.3 is derived from this. So “positivity from a quarter turn” is *literally correct over $\mathbb{C}$* .
- (ii) **Orientation is the polarization’s choice** ([t327]; [THEOREM]). J_{cx} and $-J_{\text{cx}}$ both square to -1 . If $E(x, J_{\text{cx}}x) > 0$ then $E(x, -J_{\text{cx}}x) < 0$: the form is positive for one orientation and negative definite for the other. So, given the pairing, a polarization picks an *orientation* of the quarter turn, and the two orientations are inequivalent for that pairing, as J.W. says. Complex conjugation F_∞ (§6.1) swaps $i \leftrightarrow -i$, i.e. $H^{1,0} \leftrightarrow H^{0,1}$, and conjugates C to $C^{-1} = -C$. Polarizing therefore *breaks the archimedean-Frobenius symmetry by a choice*, and the pairing makes the chosen side positive.
- (iii) **Over $\mathbb{Q}$ /adeles the Fourier transform is a quarter turn** ([THEOREM]; diagnostic [NUMERICAL]). With $\mathcal{F}f(y) = \int f(x)e^{-2\pi ixy}dx$ and Hermite functions $h_n(x) = H_n(\sqrt{2\pi}x)e^{-\pi x^2}$, we have $\mathcal{F}h_n = (-i)^n h_n$, i.e. $\mathcal{F} = \exp(-i\pi N/2)$ with N the number (harmonic-oscillator) operator. Hence $\mathcal{F}^4 = 1$ with eigenvalues $\pm 1, \pm i$ (checked for $n \leq 7$ to $5 \cdot 10^{-26}$, `quarter.json`). Poisson summation, and so the functional equation, is Fourier invariance of the theta/adelic sum (Tate, [BIB]). Also $\mathcal{F}^{-1} = \mathcal{F}^3$ is the opposite orientation, the same $\pm$ pair as in (ii).
- (iv) **The obstruction: Fourier inverts the scaling flow** ([THEOREM]; [NUMERICAL] check). With $D_\lambda f(x) = \lambda^{1/2}f(\lambda x)$, $\mathcal{F}D_\lambda \mathcal{F}^{-1} = D_{1/\lambda}$ (checked to $3 \cdot 10^{-26}$). Equivalently, $\mathcal{F}$ anticommutes with the scaling generator S . Connes–Consani–Moscovici state this in the semilocal setting ($\mathcal{F}_S$ anticommutes with S and is the $\mathbb{Z}/2$ grading that becomes $s \mapsto -s$; arXiv:2310.18423, Prop. 4.2(ii), [CITED]). So $\mathcal{F}$ is again the *involution*: the functional equation, $s \mapsto 1 - s$, $w \leftrightarrow 1/\bar{w}$. It *reverses* the flow’s orientation. By contrast the Weil operator commutes with every endomorphism of the Hodge structure, Frobenius-type correspondences included. The

positivity that $\mathcal{F}$ does carry is the L^2 norm ($\mathcal{F}$ is unitary), and that is RH-blind. The *prolate* operator $W_\lambda = -\partial((\lambda^2 - x^2)\partial) + (2\pi\lambda x)^2$ commutes with $\mathcal{F}$ and with the cutoff projections $P_\lambda, \hat{P}_\lambda$ (Connes–Moscovici, arXiv:2112.05500, Thm. 2.6, [CITED]), but not with scaling. Symbolically $[x\partial, W_\lambda] = 2\lambda^2(\partial^2 + 4\pi^2 x^2) \neq 0$ (sympy, quarter.json). So it sits on the Fourier side of the dichotomy too.

- (v) **Physics: retarded versus advanced** ([THEOREM] for the KK statement, [ANALOGY] for the identification). The two orientations of i are the $\pm i\varepsilon$ prescriptions, i.e. retarded versus advanced Green’s functions. Kramers–Kronig needs a *choice* of upper-half-plane analyticity, and passivity is positivity relative to that choice. In Part A this choice is $\operatorname{Re} s > \frac{1}{2}$, and with it ψ is inner unconditionally (Thm. 2.1), so the orientation exists for free and is RH-blind. Choosing positive frequencies for the scaling flow gives Hardy space, i.e. the Lax–Phillips incoming/outgoing subspaces and de Branges spaces $\mathcal{H}(E)$ with $|E(\bar{z})| < |E(z)|$ in the upper half-plane. In that framework de Branges’s positivity conditions imply GRH. Conrey–Li (IMRN 2000; arXiv:math/9812166, opened, [CITED]) show they *fail* for the spaces attached to ζ and $L(s, \chi_4)$. Sarnak’s argument, sketched there, uses only the density of the values of $\log \zeta(s)$ in $\frac{1}{2} < \operatorname{Re} s < 1$ (Bohr), which is an Euler-product fact unrelated to RH. So the oriented Hilbert-space positivity tried so far is too strong, not too weak.
- (vi) **Over $\mathbb{F}_q$ there is no rotation** ([BIB]/[CITED]). Weil positivity for curves is the Hodge index theorem on $C \times C$ (algebraic, via an ample class; Milne Thm. 1.27/Cor. 1.29, [CITED], as in §7.3). ℓ -adic cohomology has no Weil operator and no complex structure. Positivity comes from the ample class (hard Lefschetz/Hodge index), not from a quarter turn. The quarter turn is the *archimedean* incarnation of polarization, not its definition.
- (vii) **Centrepiece: the Hilbert transform is a scaling-commuting quarter turn, and it only restates RH** (J.W. [t328], anomalous dispersion).
 - (a) [THEOREM]. In $u = \log x$ let H be the Hilbert transform, with multiplier $-i \operatorname{sgn} t$ on frequency t . Then $H^2 = -1$ on functions with no zero-frequency component. H commutes with translations in u , i.e. with the scaling flow D_λ , so it preserves the flow’s orientation, unlike $\mathcal{F}$, which sends $D_\lambda \mapsto D_{1/\lambda}$. Its eigenspaces are the Hardy spaces: $L^2 = H_+^2 \oplus H_-^2$, with $H = -i$ on the positive-frequency part H_+^2 and $H = +i$ on H_-^2 . So H is formally a Weil operator C for the frequency splitting. It is exactly the Kramers–Kronig operator: the real and imaginary parts of a causal (upper-half-plane analytic) response are Hilbert transforms of each other. Every bounded operator commuting with all translations of $L^2(\mathbb{R}, du)$ is a Fourier multiplier $m(D)$. So on $L^2(\mathbb{R}_+^\times, d^*x)$ the scaling-commuting quarter turns are exactly the multipliers with $m(t) \in \{\pm i\}$, and $\pm H$ are the ones that respect orientation.
 - (b) *The domain obstruction* [THEOREM]. Weil’s form is defined on test functions f whose transform $F(z) = \int f(u) e^{izu} du$ is analytic in the strip $|\operatorname{Im} z| \leq \frac{1}{2}$ (zeros $\rho = \frac{1}{2} + i\gamma_\rho$, γ_ρ complex off the line). If such an F vanished on $t < 0$ it would be 0. So H_+^2 contains *no* Weil test function, and “the Weil form on H_+^2 ” needs an extension of the projector $P_+ = \frac{1}{2}(1 + iH)$ off the real axis. The only extension analytic on each half-plane is $\mathbf{1}_{\operatorname{Re} z > 0}$, which gives $H \mapsto -i \operatorname{sgn} \operatorname{Re} z$. With it, the zero-side form $Q(f, g) = \sum_\rho F(\gamma_\rho) \overline{G(\bar{\gamma}_\rho)}$ satisfies, for real f : (1) $Q(P_+ f) = \frac{1}{2} Q(f)$, and (2) $Q(f, Hf) = 0$. For (1), the zero set is closed under $\gamma \mapsto -\bar{\gamma}$ and $F(-\bar{z}) = \overline{F(z)}$, so P_+ keeps exactly one of each pair. For (2), the terms at γ and $\bar{\gamma}$ (the functional-equation partner) cancel. Consequences: H is skew for Q ; $E(f, g) := Q(f, Hg)$ is alternating; and $E(f, Hf) = -Q(f, f)$. So the Hodge–Riemann shape $E(x, Cx) > 0$ holds with $C = -H$ *exactly when* Weil positivity holds. This is a [RESTATEMENT] of RH (via Weil’s criterion), neither stronger nor weaker.
 - (c) *TEST* [NUMERICAL] (num/lemma78_hilbert.py, 178/hilbert.json). The test uses the DH ghost $\rho = 0.808517 + 85.69935i$ together with the 11 on-line DH zeros in [76, 96] (found by sign changes of $\Lambda_{\text{DH}}(\frac{1}{2} + it)$, which is real to $5 \cdot 10^{-17}$; the gap 83.11–87.65 is where the ghost pair sits). The ζ zeros in the same window are used for comparison. Families:

Gaussians $F = e^{-a^2(z-c)^2/2}$ ($a = 2, 4, 6$), Hermite $H_k(a(z-c))e^{-a^2(z-c)^2/2}$ ($k = 1, 2$), and compactly supported $f = (1 - u^2/L^2)_+^4 e^{icu}$ ($L = 6, 10$, closed form via $J_{9/2}$). Each is taken in its real and its positive-frequency complex version, with the centre c scanned over $\gamma_0 \pm 1$. The minima over c of Q_+ for DH are -0.0889 (Gaussian, $a = 2$), -2.64 ($a = 4$), -32.8 ($a = 6$), -55.9 (Hermite $k = 1$), -497 (Hermite $k = 2$), -2.85 (compact $L = 6$), and -60.4 ($L = 10$). The real- f and complex- H_+^2 values agree to $2 \cdot 10^{-10}$, and $Q_+ = \frac{1}{2}Q$ throughout. For ζ the same test functions give $Q_+ > 0$ (e.g. 0.898, 23.2, 3.29). This is automatic, since every computed ζ zero is on the line. The twisted form satisfies $|Q(f, Hf)| \leq 6 \cdot 10^{-20}$ for both. Zeros outside the window contribute below 10^{-9} : Gaussian decay, and for compact f terms of size $|F|^2 \sim |\gamma - c|^{-10}$, which beat the $e^{2\delta L}$ growth of the far DH ghosts at 114–177. *Answer*: yes, the positive-frequency Weil form goes negative on the ghost and stays positive for ζ , but only because it is half of Weil’s form.

- (d) *Codes* [TAA-L]. On the code side, H_+^2 is the space of analytic polynomials in β , and the restricted form is Lemma 77’s one-sided Toeplitz form. Its negative index is κ , the number of ghost pairs (1 for $\mathbf{i}_2 \oplus e_8$). Again a restatement.
- (e) *Archimedean principal value* [BIB] (recalled: Weil 1952; Burnol; Yoshida 1992 on small support; not opened). Weil’s archimedean term is a principal-value/finite-part distribution. The Hilbert transform is already present there: the archimedean density $\frac{1}{2\pi} \operatorname{Re} \frac{\Gamma'_{\mathbb{R}}}{\Gamma_{\mathbb{R}}}(\frac{1}{2} + it)$ and the phase $\theta(t)$ are a Kramers–Kronig pair up to growth subtractions ([HEURISTIC]).
- (f) *Physics* ([THEOREM] for KK, [BIB] for Sommerfeld–Brillouin 1914). Near a resonance, dispersion is anomalous ($dn/d\omega < 0$). Phase and group velocity can then exceed c or be negative, but the front (signal) velocity never exceeds c , because the response is analytic in the upper half-plane. *Correction*: in anomalous dispersion it is mainly the *group* velocity that becomes superluminal or negative.
- (g) *Part A in this language* [THEOREM]+[NUMERICAL]. The poles of $\varphi = \Lambda(2s - 1)/\Lambda(2s)$ sit at $\rho/2$, and $\operatorname{Re} \rho/2 < \frac{1}{2}$ for every zero. So the causality and passivity of φ are RH-blind. The time delay on $\operatorname{Re} s = \frac{1}{2}$ is a sum of Breit–Wigner Lorentzians (verified to $6.1 \cdot 10^{-8}$ at $r = 0.5, \dots, 20$, across the first zeros, `scatter.json`). The anomalous-dispersion window of each line has full width $2 \cdot \frac{\operatorname{Re} \rho}{2}$: $\frac{1}{2}$ for every ζ zero, but 0.809 and 0.191 for the DH ghost pair at 85.70 (one broadened line, one sharpened). RH is the “distortionless medium” with uniform linewidth (Lemma 37, [TAA-L]).
- (viii) **Verdict** ([THEOREM] for the dichotomy and the multiplier theorem, [HEURISTIC] for the moral, [OPEN] for the question). Choosing an orientation is *necessary but not sufficient*. Fourier and its prolate commutant *reverse* the scaling flow: they are the functional-equation involution. The Hilbert transform is an orientation-preserving, scaling-commuting quarter turn compatible with the Weil pairing, exactly the shape asked for in [t326]. But it is a function of the scaling generator, so it carries no arithmetic, and the positivity it “produces” is Weil’s own: a restatement. On $L^2(\mathbb{R}_+^\times)$, where scaling has simple spectrum, every scaling-commuting quarter turn is such a multiplier. A non-tautological candidate therefore needs a space where the scaling flow has *multiplicity*, such as the even/odd sectors of $L^2(\mathbb{R})$ or the adèle class space of Connes–Consani, and a quarter turn that is not a function of the flow. The missing polarization J has to supply the positivity, not the rotation (Directions D1, D2).

8. CORRECTIONS TO THE CHAT CLAIMS [T311]–[T325]

- C1** *KK/causality versus passivity* ([t311]–[t312]). Refined. For the modular scattering matrix, causality (KK), unitarity *and* passivity all hold unconditionally (Thm. 2.1). “RH = passivity” is correct only for the exponential line $\exp(-2c\xi'/\xi)$, and that result is **Lemma 37** (Thm. exp), not Lemma 34 as the brief cites. The sharpened thesis (“unitarity and causality are automatic; RH is where the resonances sit”) is *confirmed* for φ , with passivity added to the automatic list.

- C2** *Uniform linewidth $\frac{1}{4}$; ghost widths $\frac{1}{4} \mp \frac{\delta}{2}$.* Confirmed (Cor. 2.2; DH table). Label [RESTATEMENT].
- C3** *Restricted tensor product as the missing product.* Corrected: it is the Euler-product state space, not $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$, and it has no h^1 (§3).
- C4** *Λ theta formula, both poles from the $n = 0$ mode.* Confirmed, with the precise split: pole at 1 from $t \rightarrow 0$ (dual $n = 0$ term $t^{-1/2}$), pole at 0 from $t \rightarrow \infty$ (constant term) (§6).
- C5** *“The pole at 0 literally is the point at infinity” ([t318]).* Sharpened: it is the $\Gamma(s/2)$ pole that survives because $\zeta(0) \neq 0$. The other Γ poles are cancelled by trivial zeros (finite places).
- C6** *Kurokawa $\zeta_{1 \oplus \mathbb{L}} = 1/(s(s-1))$.* Confirmed (Soulé limit, CC10 [CITED]). It equals the singular part of Λ exactly. Manin’s normalization differs by 2π factors. “Manin’s prediction”: Manin proposes $H(\overline{\text{Spec } \mathbb{Z}}) = T^0 \oplus H^1 \oplus T$ ([CITED], a proposal).
- C7** *$\psi(x) - x$ is the h^1 part.* Confirmed, with the lower-order terms $-\log 2\pi - \frac{1}{2} \log(1 - x^{-2})$, and made exact by CC’s $N(u) = u - \sum u^\rho + 1$.
- C8** *Scaling flow as Frobenius, one orbit per prime, length $\log p$.* Confirmed for finite places (Connes 1999 [CITED]). The archimedean place contributes a non-periodic term. In characteristic p , Frobenius is a single transformation, not a flow.
- C9** *“ $\max(a, a) = a$ is the ultrametric inequality turned into an algebraic rule”.* Corrected: idempotent addition is the max-plus *semifield*. The ultrametric inequality $c \leq \max(a, b)$ is encoded by Viro’s ultra-triangle *hyperfield* (multivalued sum). The arithmetic site uses $(\mathbb{N} \cup \infty, \min, +)$. Its points over $[0, 1]_{\max}$ are the adèle class space *modulo* $\widehat{\mathbb{Z}}^*$ (CC14).
- C10** *CC counting distribution “as I recall”.* Verified at source: $N(u) = u - \frac{d}{du} \sum \frac{u^{\rho+1}}{\rho+1} + 1$, positive on $(1, \infty)$, $N(1) = -\infty$. Their constant $\omega(1)$ is reproduced to $5 \cdot 10^{-8}$. Its positivity is RH-blind (§5).
- C11** *Arakelov gives $1 \oplus h^1 \oplus \mathbb{L}$, not $\mathbb{P}^1$.* Agreed as a proposal (Manin [CITED]); the h^1 is postulated ([CONJECTURE]).
- C12** *Tate / van der Geer–Schoof Riemann–Roch.* vdG–S [CITED-sec] via CC23; Tate’s thesis as Riemann–Roch [BIB] (recollection). CC23 proves a Riemann–Roch theorem for $\overline{\text{Spec } \mathbb{Z}}$ ([CITED]). All are on the curve, not the surface.
- C13** *Li map facts ([t319]).* All confirmed (§7). Refinement: [t319]’s mirror in the $\mathbf{i}_2 \oplus e_8$ plot is in the code β -plane, a different disk from Li’s, with the same inversion structure.
- C14** *General Möbius family $w = (s-a)/(s+\bar{a}-1)$ ([t320]).* Incomplete: multiply by $e^{i\theta}$ and require $\text{Re } a > \frac{1}{2}$. New: ψ is the Blaschke product of these maps over $a = a_\rho$ (Thm. 2.1).
- C15** *“The centre of the β -disk is $s = +\infty$ ” ([t320]).* **Wrong:** $\beta = q^{s-1/2} = 0$ at $\text{Re } s = -\infty$. The Euler-product region lies outside the circle.
- C16** *“Open whether $q \rightarrow 1$ of the β -disk lands on Li’s disk” ([t320]).* Settled in recentred form (Thm. 7.1). The plain β -disk does not land on it. “Pole from the edge to the centre” should read “from outside the circle to the centre”.
- C17** *[t316] numbers ($\delta = 0.3696$, $|\beta| = 1.292/0.774$, ratio $2^{2\delta}$, $s = \frac{1}{2} \pm 0.3696 \pm 1.3934i$, only two complex-ghost codes, $N_7 < 0$, $b_6, b_7 < 0$).* All confirmed.
- C18** *“Lemma 77’s Toeplitz/Krein result is the code version of Li” ([t319]).* Refined: L77’s Li form is centred at $\beta = 0$ ($s = -\infty$). The pole-centred analogue is the q -Li map, and the two detect different ghosts at different speeds (table in §7).
- C19** *2+1 gravity disk.* Remains [ANALOGY].
- C20** *t322 Rosati claims (1)–(4).* All confirmed at the labels given in §7.3. The Bombieri–Lagarias test-function statement is confirmed via Lagarias 2007 ([CITED-sec]). The external LLM text is non-authoritative and its two overclaims are flagged. Also flagged: “infinitely many transcendental zeros” (parent’s phrasing) asserts something unproven.
- C21** *t323–t324 squaring/conjugation/ $|w|^2$.* Values confirmed ($\lambda_7 = -6.849242$, $\lambda_{12} = -15$, conjugation minimum 16.537). The $|w|^2$ criterion is exact but non-holomorphic, so it is not computable by the explicit formula. Complexification preserves signature.

- C22** *t325 “ $s = \pi$ ”.* Corrected (§6.1). The reflection is the archimedean Frobenius F_∞ , an involution with no positivity of its own.
- C23** *t326 “positivity from a 90° rotation via 4th roots of unity, without the zeros”.* Correct over $\mathbb{C}$ (the Weil operator $C = i^{p-q}$ with Hodge–Riemann positivity), but not available over $\mathbb{Q}$ as stated. The arithmetic quarter turn $\mathcal{F}$ inverts scaling (it is the FE involution). Over $\mathbb{F}_q$ positivity is the algebraic Hodge index, not a rotation (§7.4).
- C24** *t327 “polarization = choosing counterclockwise vs clockwise”.* Confirmed: $\pm J_{\text{cx}}$ give forms of opposite sign, and choosing one breaks the F_∞ symmetry. Necessary, but not sufficient.
- C25** *t328 anomalous dispersion.* The Hilbert transform in $\log x$ is the orientation-preserving, scaling-commuting quarter turn and is the KK operator. But the Weil form on H_+^2 is exactly half of Weil’s form, so it is a restatement. Numerically it goes negative on the DH ghost (down to -497) and stays positive for ζ . Physics correction: in anomalous dispersion it is chiefly the group velocity that becomes superluminal or negative; the front velocity never exceeds c .

9. DIRECTIONS (RANKED BY $0.6s + 0.4p$)

s = significance for the polarization J , p = probability of a clean result within one lemma.

	direction	s	p	score
D1	Hilbert-transform polarization: the no-go (§7.4). Prove in general that every scaling-commuting quarter turn on $L^2(\mathbb{R}_+^\times)$ (a multiplier $m(D) \in \{\pm i\}$) produces a Hodge–Riemann form that only restates Weil’s (as shown here for $m = -i \text{sgn}$), including for weighted forms. Make the archimedean principal-value/KK pair rigorous.	.55	.80	.65
D2	Scaling-commuting quarter turn with multiplicity. Look for C with $C^2 = -1$, commuting with scaling but <i>not</i> a function of the scaling generator, on spaces where scaling has multiplicity: the even/odd sectors of $L^2(\mathbb{R})$, semilocal $S = \{2, \infty\}$ models (Connes–Consani–Moscovici), and the formal $2g$ -dimensional zero spaces of the Lemma 77 codes. Test whether the Weil form has the shape $E(x, Cx)$ for such a C ; for ghost codes the answer must be “no”.	.85	.30	.63
D3	Li index = Castelnuovo index for codes. Prove the observed coincidence (7, 10, 11, 13) of the first Li-negative n with the first failure of the Weil bound, or find a counterexample; then compare with the pole-centred q -Li speeds.	.50	.75	.60
D4	A computable form of the $w ^2$ criterion. The non-holomorphic criterion $\sum(1 - w_\rho ^{2n}) = 0$ becomes, via Jensen’s formula, a boundary integral of $\log \xi $, which <i>is</i> computable. Recalled: Balazard–Saias–Yor (1999), RH iff $\int \log \zeta(s) s ^{-2} ds = 0$ on the line ([BIB], open first). Map it to the code side and test the $\mathbf{i}_2 \oplus e_8$ ghost.	.60	.60	.60
D5	Orientation-relative passivity. Part A’s orientation $\text{Re } s > \frac{1}{2}$ is RH-blind. Look for a second transfer function besides Lemma 37’s exponential line whose passivity relative to the positive-frequency (Hardy) orientation of the scaling flow is RH-equivalent, and check it against the Conrey–Li obstruction.	.60	.55	.58
D6	(Spec $\mathbb{Z}$)² pole plane. Spell out what an Arakelov surface must supply for $\infty \times \infty$ to contribute to $\mathcal{P}$ (§6), as a list of axioms to test against Connes–Consani’s arithmetic site.	.80	.15	.54
D7	First active DH height (Rem. 2.3). Locate the first DH zero with $\text{Re } s > 1$ (certified argument-principle count on $1 < \sigma < 1.5$, beyond $t = 1500$ if needed), i.e. the first height where ψ_f stops being passive.	.30	.80	.50

10. QUESTIONS FOR J.W.

- Q1** Adopt the sharpened thesis: *for the modular scattering matrix, causality, unitarity and passivity are automatic, and RH is the statement that every resonance has linewidth exactly $\frac{1}{4}$* (Thm. 2.1, Cor. 2.2)?

- Q2** Accept that the “ $1 + 1 + 1 + \dots$ ” meta-space is the Soulé/Kurokawa $q \rightarrow 1$ limit, which recovers exactly the poles $\mathbf{1} \oplus \mathbb{L}$ of Λ , and that its counting positivity (Connes–Consani $N(u)$) is RH-blind (§5)?
- Q3** Accept “Li’s inversion = the functional-equation involution (Rosati’s *formal* relation), not Rosati *positivity*”, so $\lambda_n \geq 0$ is a restatement, and squaring, conjugation and complexification cannot add positivity (§7.3)?
- Q4** Accept the quarter-turn reformulation of the target: a polarization is an *oriented* quarter turn compatible with the pairing, and Fourier and the archimedean Frobenius supply only the involution (they reverse the scaling flow). The Hilbert transform (the Kramers–Kronig operator) is orientation-preserving and scaling-commuting, yet it only restates Weil’s form. The missing J must therefore supply positivity on a space where scaling has multiplicity, not merely a rotation (§§6.1, 7.4)?
- Q5** Next: D1 (the Hilbert-transform no-go), D2 (a quarter turn with multiplicity on finite models), D3 (Li = Castelnuovo index theorem for codes), D4 (Jensen/BSY form of the $|w|^2$ criterion), or something else?

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Not opened [BIB]: Davenport–Heilbronn (1936); Serre (1970) and Deligne (1979) on Γ -factors; Tate’s thesis (1950); Hodge–Riemann relations and Riemann forms (standard texts); Balazard–Saias–Yor (1999); Weil (1948); Castelnuovo–Severi. The GitHub document and “Clay proceedings PDF” cited in the external text pasted by J.W. were deliberately *not* opened or cited.

Code and data. `num/lemma78_*.py` (lib, scatter, disk, rosati, square, cc, pole, dh_right_par, gamma, quarter, figs); outputs `num/178/*.json`.

TAA LEMMA 79: THE HOLOMORPHIC HALF AND ITS CONJUGATE. RIEMANN–SIEGEL, THE Γ -PHASE AS GLUING, $n!$ TWICE, THE CONJUGATE STRUCTURE ON CONNES’ COKERNEL, AND WHAT CONJUGACY BUYS FOR DAVENPORT–HEILBRONN

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. Nothing in this lemma proves it or brings a proof closer in a way we can measure. Tags as in Lemma 78: [THEOREM] proved here (or classical with the proof indicated); [CITED] checked at the source (`lit/lemma79/verified_refs.md`); [CITED-sec] via a named secondary source we opened; [BIB] bibliographic record or recollection, not opened; [TAA-L] earlier TAA lemma; [RE-STATEMENT] restatement; [ANALOGY] analogy; [HEURISTIC] heuristic; [CONJECTURE] conjecture; [OPEN] open; [NUMERICAL] numerics (not certified).

Task (J.W., approved [t332u]; hunches [t330u]–[t331u]). J.W.’s hunch: positivity of the pairing of a holomorphic form with its antiholomorphic conjugate is the missing piece linking to the Hilbert transform/Kramers–Kronig thread of Lemma 78. In it, “ $1 + 1 + 1 + \dots$ ” is the holomorphic half, the path back from the point at infinity (perhaps via prime point counts) is the antiholomorphic half, and $n!$ should play a role. Five items: (1) holomorphic versus antiholomorphic halves; (2) Riemann–Siegel and the conjugacy defect off the line; (3) $n!$ as the archimedean factor and as Legendre’s bridge; (4) whether the conjugate structure survives on Connes’ cokernel (Direction D2 of Lemma 78, “multiplicity”); (5) what conjugacy alone buys, tested on Davenport–Heilbronn (DH). Every numerical run was chosen for its diagnostic value for the polarization J .

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1. ANSWER IN BRIEF

- (1) **Halves (§3).** A Dirichlet series is *causal* in the log variable $u = \log x$: its coefficients sit at $u = \log n \geq 0$, so its transform is analytic in a half-plane and obeys Kramers–Kronig. This is the line analogue of $H^{1,0}$ [THEOREM]/[ANALOGY]. The mirror half $\sum a_n n^{s-1}$ sits at $u = -\log n \leq 0$, i.e. at $x = 1/n \in (0, 1]$, Lemma 76’s range. The Γ -factor (the point at infinity) carries one half to the other. The pairing of a half with its conjugate on the line is a mean value,

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- $\frac{1}{T} \int_T^{2T} |A|^2 \approx \sum |a_n|^2/n$. It is positive for every Dirichlet polynomial, so it is RH-blind: the ratio to the Montgomery–Vaughan prediction is 1.024 for ζ and 1.005 for DH [NUMERICAL].
- (2) **Riemann–Siegel** (§4). On $\text{Re } s = \frac{1}{2}$ the mirror half is *exactly* the conjugate of the first, $e^{i\vartheta} B = \overline{e^{i\vartheta} A}$ [THEOREM](checked to $3 \cdot 10^{-19}$ for ζ and $2 \cdot 10^{-21}$ for DH). Off the line the defect is archimedean on average: $\log |B/A| \approx (\frac{1}{2} - \sigma) \log \frac{qt}{2\pi} + \frac{1}{2} \log M(\sigma)$, matched to 0.02–0.2 [NUMERICAL]. A zero off the line needs $A+B+R=0$. At the DH ghost $|B/A| = 0.805$, against the typical 0.374. But ζ reaches $|B/A| \geq 1$ at 3.7% of sampled heights on $\sigma = 0.8085$ and has no zeros there. So the modulus defect does not detect ghosts.
- (3) **$n!$ twice** (§5). (a) Γ is the zeta-regularized product of $a, a+1, a+2, \dots$ (Lerch), so $\vartheta(t) = \text{Im } \partial_z \zeta(0, \frac{1}{4} + \frac{it}{2}) - \frac{t}{2} \log \pi$. The “ $1+1+1$ ” counting gives the *archimedean* factor [THEOREM]. (b) Legendre/Chebyshev: $\log |x|! = \sum_k \psi(x/k)$ [THEOREM]. The two are linked formally only through Stirling’s constant, $\frac{1}{2} \log 2\pi = -\zeta'(0) = -\partial_z \zeta(0, 1)$ [THEOREM]. New [THEOREM]: $\log |x|!$ is *zero-blind*, since $\sum_k (x/k)^\rho = x^\rho \zeta(\rho) = 0$. Its Stirling error is $\leq 8.3 \cdot 10^{-5}$ for $n \geq 1000$, while $|\psi(x) - x|$ reaches 260 [NUMERICAL]. Both roles of $n!$ live in $\mathbf{1} \oplus \mathbb{L}$, never in h^1 (consistent with Lemma 76).
- (4) **Connes’ cokernel** (§6). The conjugate structure survives on the quotient, and that is the problem. The representation is unitary, so the cokernel’s spectrum is exactly the *critical* zeros, and non-critical zeros are “artificially eliminated” (Connes 1999, Thm. 1 and §VII, [CITED]). Positivity there is automatic and blind. Multiplicity does exist: the χ -sectors. On the $(\chi, \bar{\chi})$ plane mod 5, Galois conjugation gives a quarter turn C (i on $L(\chi)$, $-i$ on $L(\bar{\chi})$) that commutes with scaling and is not a function of the scaling generator [THEOREM]. Its real FE-even line is spanned by DH , and C maps DH to its FE-odd partner $DH^\perp$. The eigenlines $L(\chi), L(\bar{\chi})$ have Euler products, and their zeros are on the line in our range. The real combinations have ghosts [NUMERICAL].
- (5) **What conjugacy buys** (§7). DH has a functional equation with its own Γ -factor, so its halves are conjugate on the line, yet it has off-line zeros. Up to $T = 250$ the on-line fractions are 108/108 (ζ), 171/171 ($L(s, \chi) \bmod 5$), 161/171 = 0.942 (DH, 5 ghost pairs, including a new one at $0.8695 + 240.40i$) and 103/171 = 0.602 ($DH^\perp$, with zeros out to $\text{Re } s > 1.5$) [NUMERICAL]. Conjugacy buys a real Z and Hardy-type sign changes, i.e. “almost all” zeros on the line at best (recalled theorems, [BIB]). What separates 1 from 0.942 is the Euler product.

Program verdict [HEURISTIC]. Conjugacy (FE plus Γ -factor) is an *involution* and buys “most zeros on the line” at best: Selberg-type positive proportions, and for DH a proportion tending to 1 (recalled). It does not buy “all”. The holomorphic/antiholomorphic pairing is positive wherever it is defined on the line, and that positivity is blind. On a curve the same pairing is decisive because it lives on H^1 , the space on which Frobenius acts *with the zeros as eigenvalues*. The line pairing lives on Dirichlet polynomials, on which scaling acts with spectrum $\{\log n\}$. The target is unchanged: a positive pairing on a space whose spectrum is the zeros, i.e. on the cokernel *before* unitarity removes the ghosts, and it must separate the eigenlines $L(\chi)$ from their real combinations.

2. WHAT THE OBJECTS ARE

- h^1 is an *object of a category* of motives (Chow or numerical), not a space of functions. For a smooth projective curve C , $h(C) = h^0 \oplus h^1 \oplus h^2$ with $h^0 = \mathbf{1}$ (a point) and $h^2 = \mathbb{L}$ (the Lefschetz motive). Each cohomology theory realizes h^1 as a vector space of dimension $2g$ on which Frobenius acts, with eigenvalues the inverse zeros of $Z(C, T)$ [BIB]. “ h^1 over $\mathbb{Q}$ ” means a hypothetical such object for $\text{Spec } \mathbb{Z}$.
- **Cohomology** H^1 is a vector space attached functorially to a space. For a compact Riemann surface, $H^1(C, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$: holomorphic 1-forms and their complex conjugates, g each (Hodge decomposition). For a curve over $\mathbb{F}_q$, ℓ -adic H^1 is a $\mathbb{Q}_\ell$ -vector space of dimension $2g$ with no complex conjugation and no Hodge decomposition [BIB].

- **Hodge–Riemann relations** are an *inequality* on $H^{1,0}$: $i \int_C \omega \wedge \bar{\omega} > 0$ for $\omega \neq 0$ (the norm of a holomorphic form). Via the Riemann form this gives Rosati positivity over $\mathbb{C}$ (Lemma 78, subsection “Positivity from a quarter turn?”, [BIB]).
- **Hodge index theorem** is a statement about the *intersection form on divisors of a surface* S : on $\text{NS}(S) \otimes \mathbb{R}$ it has signature $(1, \rho - 1)$. Applied to $S = C \times C$ over $\mathbb{F}_q$ it yields Castelnuovo–Severi, hence Weil’s RH for curves. It is purely algebraic and uses an ample class, not a complex structure (Milne, [CITED] via Lemma 78).

3. HOLOMORPHIC AND ANTIHOLOMORPHIC HALVES (ITEM 1)

Proposition 3.1 (causality of the Dirichlet half [THEOREM]). *Let $D(s) = \sum_n a_n n^{-s}$ converge absolutely for $\text{Re } s > \sigma_a$. Then D is the Mellin/Laplace transform of the measure $\mu = \sum a_n \delta_{\log n}$ on $u = \log x$, which is supported on $u \geq 0$. Hence D is bounded and analytic in every half-plane $\text{Re } s \geq \sigma_0 > \sigma_a$, and on $\text{Re } s = \sigma_0$ its frequencies $\{\log n\}$ are one-sided. In particular, for $\sigma_0 > 1$, the real and imaginary parts of $\log \zeta(\sigma_0 + it)$ are conjugate functions in t : Hilbert transforms of each other, termwise on the one-sided almost-periodic series (Kramers–Kronig in the Euler-product region). The mirror half $\sum a_n n^{s-1}$ is the transform of $\sum a_n \delta_{-\log n}$, supported on $u \leq 0$, i.e. on $x = 1/n \in (0, 1]$. The functional equation*

$$\zeta(s) = \chi(s) \zeta(1-s), \quad \chi(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s),$$

and its DH analogue $f(s) = X(s)f(1-s)$, $X = \gamma(1-s)/\gamma(s)$, $\gamma(s) = (5/\pi)^{s/2} \Gamma(\frac{s+1}{2})$, multiply the anti-causal half by a pure Γ -ratio: the point at infinity does the gluing.

Proof. Immediate from absolute convergence. The KK statement holds because $\log \zeta(s) = \sum \Lambda(n) n^{-s} / \log n$ is bounded and analytic for $\text{Re } s \geq \sigma_0$ and tends to 0 as $\text{Re } s \rightarrow \infty$. $\square$

Relation to Lemma 78 [THEOREM]/[ANALOGY]. In the convention of Lemma 78, subsection “Positivity from a quarter turn?”, $u = \log x$ is the time variable and the Dirichlet half is *causal*: it lies in H_+^2 wherever it is square-integrable. The Riemann–Siegel splitting (§4) shows that on the critical line $Z(t) = 2 \text{Re}(e^{i\vartheta} A)$ is twice the real part of the rotated causal half, the analytic-signal shape. The anti-causal half brings *no new information* there: it is the complex conjugate.

The pairing is a mean value, hence blind [THEOREM]+[NUMERICAL] (num/lemma79_mv.py). For any Dirichlet polynomial $A(t) = \sum_{n \leq N} a_n n^{-1/2-it}$, Montgomery–Vaughan gives $\int_T^{2T} |A|^2 dt = \sum |a_n|^2 n^{-1} (T + O(n))$ ([BIB], recalled). This is the line analogue of $i \int \omega \wedge \bar{\omega} = \|\omega\|^2 > 0$. Numerically, on $[100, 200]$ the ratio to the diagonal is 1.0236 for ζ ($N = 5$) and 1.0051 for DH ($N = 12$). Also $\int Z^2 = 2 \int |A|^2 (1 + c)$ with $c = 0.113$ and 0.079 (the oscillatory cross term). The positivity is the same for the ghost-bearing DH: blind.

Why the curve pairing is decisive and the line pairing is not [HEURISTIC]. On a curve, $i \int \omega \wedge \bar{\omega}$ is a form on $H^{1,0} \subset H^1$, and Frobenius (over $\mathbb{F}_q$, through the algebraic Hodge index) or the Riemann form (over $\mathbb{C}$) acts on that space with the zeros as eigenvalues. On the line, the half A lives in the space of Dirichlet polynomials. There the scaling flow has spectrum $\{\log n\}$, the prime side, and the zeros appear only as a *cokernel* (§6). Positivity on the space where the primes act is automatic. The missing positivity is on the space where the zeros act.

4. RIEMANN–SIEGEL: CONJUGATE ON THE LINE, A QUANTIFIED DEFECT OFF IT (ITEM 2)

Write $s = \sigma + it$, $N = \lfloor \sqrt{qt/2\pi} \rfloor$ ($q = 1$ for ζ , $q = 5$ for DH),

$$A(s) = \sum_{n \leq N} a_n n^{-s}, \quad B(s) = \chi(s) \sum_{n \leq N} a_n n^{s-1}, \quad \zeta \text{ (or } f_{\text{DH}}) = A + B + R,$$

with χ replaced by X for DH. On the line this is the Riemann–Siegel formula $Z(t) = 2 \sum_{n \leq N} n^{-1/2} \cos(\vartheta(t) - t \log n) + R(t)$ with $e^{-2i\vartheta} = \chi(\frac{1}{2} + it)$.

Proposition 4.1 (conjugacy on the line, defect off it [THEOREM]). (i) On $\sigma = \frac{1}{2}$: $e^{i\vartheta} B = \overline{e^{i\vartheta} A}$ exactly, for every real-coefficient Dirichlet polynomial with a functional equation of sign +1. (ii) Off the line, $|\chi(\sigma + it)| = (t/2\pi)^{1/2-\sigma}(1 + O(1/t))$, and likewise $|X(\sigma + it)| = (5t/2\pi)^{1/2-\sigma}(1 + O(1/t))$ (Stirling). Hence $\log |B/A| = (\frac{1}{2} - \sigma) \log \frac{qt}{2\pi} + \log \left| \frac{\sum a_n n^{s-1}}{\sum a_n n^{-s}} \right| + O(1/t)$, and the second term has mean square governed by $M(\sigma) = \sum |a_n|^2 n^{2\sigma-2} / \sum |a_n|^2 n^{-2\sigma}$.

Proof. (i) On the line $n^{s-1} = n^{-1/2+it} = \overline{n^{-s}}$, and $|\chi| = 1$ there. (ii) Stirling for $\Gamma(1-s)/\Gamma$ -ratios. $\square$

Numerics [NUMERICAL] (num/lemma79_rs.py, 179/rs.json; 200 heights on the line, 300 heights $t \in [60, 200]$ per σ).

	σ	mean $\log B/A $	prediction	max $\log B/A $	frac. $ B/A \geq 1$	min $\frac{ A+B+R }{ A + B }$
ζ	0.6	-0.183	-0.197	2.12	5.3%	0.087
	0.7	-0.392	-0.394	2.81	4.0%	0.152
	0.8085	-0.622	-0.606	4.23	3.7%	0.218
	0.9	-0.836	-0.784	1.22	2.3%	0.274
DH	0.6	-0.377	-0.336	0.73	2.7%	0.103
	0.7	-0.759	-0.669	1.16	2.0%	0.144
	0.8085	-1.165	-1.024	2.18	1.7%	0.197
	0.9	-1.510	-1.317	0.81	1.0%	0.296

On the line, conjugacy holds to $2.6 \cdot 10^{-19}$ (ζ) and $2.1 \cdot 10^{-21}$ (DH) relative to $|A|$. The Stirling form of $|\chi|$, $|X|$ holds to $2.2 \cdot 10^{-6}$ in log for $t \geq 60$. The remainder satisfies $|R|(t/2\pi)^{1/4} \leq 0.91$ (ζ) and 0.65 (DH). At the DH ghost $\rho = 0.808517 + 85.699348i$ ($N = 8$): $|B/A| = 0.805$, against the typical $\exp(\text{prediction}) = 0.374$, with $|R|/|A| = 0.263$. So the ghost lies on an excursion where the halves are far better balanced than typical (Fig. 1), and the remainder closes the gap.

Diagnostic verdict [HEURISTIC]. The conjugacy defect off the line is archimedean (Stirling) on average, the same for ζ and DH. Balance of moduli $|B/A| \geq 1$ is not rare: ζ attains it at 1.7–5.3% of sampled heights with no zero nearby, because a zero needs phase cancellation too. The defect is therefore *not* a ghost detector, and conjugacy on the line, being an identity, carries no RH information. Quantitatively, an off-line zero at σ requires a deviation of size $(\sigma - \frac{1}{2}) \log \frac{qt}{2\pi}$ in $\log |\sum a_n n^{s-1}| / |\sum a_n n^{-s}|$. This is the mechanism behind zero-density estimates ([BIB]), which make ghosts *rare*, not absent.

5. $n!$ TWICE: THE ARCHIMEDEAN FACTOR AND LEGENDRE’S BRIDGE (ITEM 3)

Proposition 5.1 (two roles, one constant [THEOREM]). (a) (Lerch; Lemma 76 Prop. “lerch”, [TAA-L]) $\partial_z \zeta(z, a)|_{z=0} = \log \Gamma(a) - \frac{1}{2} \log 2\pi$, so $\Gamma(a) = \sqrt{2\pi} / \prod_{n \geq 0}^{\text{reg}} (n + a)$. The Γ -factor is the zeta-regularized “factorial” of the shifted integers, i.e. “ $1 + 1 + 1 + \dots$ ” counted from a . Consequently

$$\vartheta(t) = \arg \Gamma\left(\frac{1}{4} + \frac{it}{2}\right) - \frac{t}{2} \log \pi = \text{Im } \partial_z \zeta\left(0, \frac{1}{4} + \frac{it}{2}\right) - \frac{t}{2} \log \pi \pmod{2\pi}.$$

(b) (Legendre/Chebyshev; Lemma 76 Prop. “fact”, [TAA-L]) $\log[x]! = \sum_n \Lambda(n) \lfloor x/n \rfloor = \sum_{k \leq x} \psi(x/k)$. (c) Formal link. The Stirling constant is the same number in both: $\log n! = n \log n - n + \frac{1}{2} \log n + \frac{1}{2} \log 2\pi + O(1/n)$, with $\frac{1}{2} \log 2\pi = -\zeta'(0) = -\partial_z \zeta(0, 1)$. In the Perron integral of $-\zeta'(s)x^s/s$ (the Dirichlet series of $\log n = (1 * \Lambda)(n)$), the constant comes from the residue at $s = 0$, and $\zeta'(0)$ is computed from the functional equation, i.e. from the Γ -factor. (d) Zero-blindness of Legendre’s bridge. $-\zeta'(s) = \zeta(s) \cdot (-\zeta'/\zeta)(s)$ has no poles at the zeros of ζ : the factor ζ (the “ $1*$ ”, counting) cancels the poles of ζ'/ζ . Equivalently, the zero terms of $\sum_k \psi(x/k)$ combine to $-\sum_\rho \frac{x^\rho}{\rho} \sum_k k^{-\rho} = -\sum_\rho \frac{x^\rho}{\rho} \zeta(\rho) = 0$ (formally, with $\sum_k k^{-\rho}$ Cesàro/Euler–Maclaurin regularized). So $\log[x]!$ carries no zero oscillation.

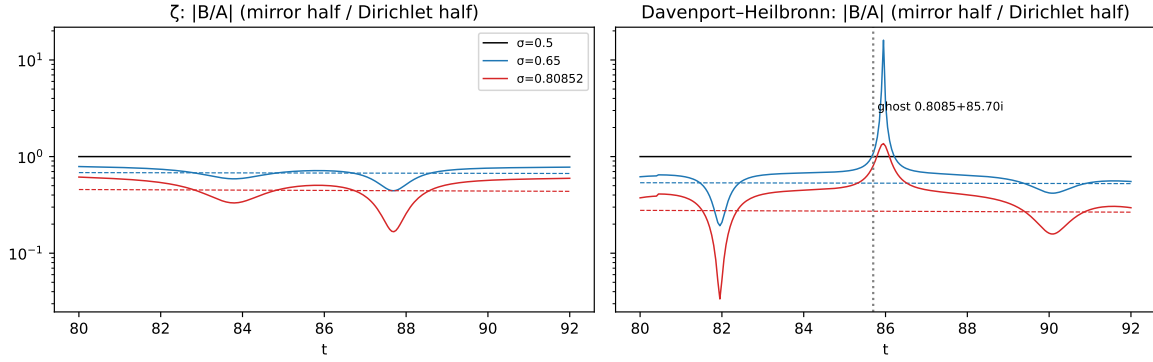


FIGURE 1. $|B/A|$ for ζ (left) and DH (right) on $t \in [80, 92]$ at $\sigma = \frac{1}{2}$ (identically 1), 0.65 and 0.8085. The dashed curves are the archimedean predictions $(qt/2\pi)^{1/2-\sigma}$. The DH ghost at $t = 85.70$ is marked: it sits on an excursion of $|B/A|$ above 1 (peak ≈ 1.4 at $\sigma = 0.8085$). In this window ζ stays below 1, but over $t \in [60, 200]$ it exceeds 1 at 3.7% of sampled heights without any off-line zero (table). An excursion is necessary for a ghost but not sufficient.

Proof. (a), (b): classical (Lemma 76). (c) Stirling plus Lerch at $a = 1$. (d) ζ' is holomorphic on $\mathbb{C} \setminus \{1\}$. $\square$

Numerics [NUMERICAL] (num/lemma79_factorial.py). Lerch holds to $5 \cdot 10^{-32}$ at $a = 1, \frac{1}{4}, \frac{1}{4} + 7.07i, \frac{1}{4} + 50i$. $-\zeta'(0) = \frac{1}{2} \log 2\pi$ exactly at 30 digits. ϑ from Lerch agrees with **siegeltheta** to 10^{-28} (mod 2π). Legendre's identity holds to $4 \cdot 10^{-16}$ (double precision) at $x = 10, 97.5, 10^3, 12345.6, 199999$. Zero-blindness: for $1000 \leq n \leq 2 \cdot 10^5$, $|\log n! - \text{Stirling}| \leq 8.33 \cdot 10^{-5}$ ($= 1/12n$), while $|\psi(x) - x|$ reaches 260 and has $\text{rms}(\psi(x) - x)/\sqrt{x} = 0.208$, the zero oscillation. The regularized $\sum_{k \leq K} k^{-\rho_1} - \frac{K^{1-\rho_1}}{1-\rho_1} - \frac{K^{-\rho_1}}{2}$ tends to $\zeta(\rho_1) = 0$: $3.7 \cdot 10^{-5}, 1.2 \cdot 10^{-6}, 3.7 \cdot 10^{-8}$ at $K = 10^3, 10^4, 10^5$.

Verdict [THEOREM]/[HEURISTIC]. The two roles of $n!$ are formally linked (one constant, $-\zeta'(0)$, and Stirling for both $\log[x]!$ and ϑ), but both sit in the smooth part $\mathbf{1} \oplus \mathbb{L}$. $\vartheta/\pi + 1$ is the smooth zero count $N_0(T)$, and $\log[x]!$ is zero-blind. Neither touches h^1 , in agreement with Lemma 76's factorial-motive verdict. The positivity Legendre offers, $v_p(n!) \geq 0$, gives only Chebyshev-strength bounds for ψ ([BIB]).

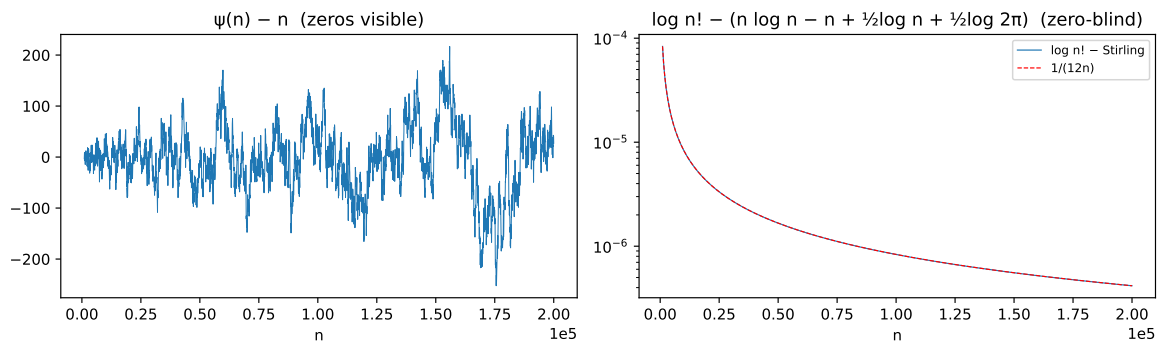


FIGURE 2. Left: $\psi(n) - n$, which carries the zeros. Right: $\log n! - (n \log n - n + \frac{1}{2} \log n + \frac{1}{2} \log 2\pi)$, zero-blind and equal to $1/12n + \dots$. Same range $10^3 \leq n \leq 2 \cdot 10^5$.

6. DOES THE CONJUGATE STRUCTURE SURVIVE ON CONNES' COKERNEL? (ITEM 4; LEMMA 78 D2)

Theorem 6.1 (Connes 1999, Thm. 1 and the remarks after Cor. 2, [CITED]). *For $\delta > 1$, the scaling generator D_χ on the cokernel $\mathcal{H}_\chi = L_\delta^2(C_k)_\chi / E(L_\delta^2(X)_0)$ has discrete spectrum. It is the set of imaginary parts of the zeros of $L(\tilde{\chi}, s)$ with real part $\frac{1}{2}$, with multiplicity the largest $n < \frac{1+\delta}{2}$ not exceeding the multiplicity of the zero. Connes notes that the space L_δ^2 “artificially eliminates the non critical zeros”. With the cutoff ($\delta = 0$) “the critical zeros will appear per se, the non critical ones will appear as resonances”. Positivity of the Weil distribution is then equivalent to RH.*

Proposition 6.2 (the unitary cokernel sees exactly the critical zeros: a line model [THEOREM]). *Let f be a Dirichlet series with a functional equation, so that $\Lambda_f(\frac{1}{2} + it)$ is real-analytic. Let $\mathcal{R} = \{\Lambda_f(\frac{1}{2} + it)G(t)\}$, with G ranging over a Paley–Wiener class. A compactly supported distribution h on $\mathbb{R}_t$ annihilates $\mathcal{R}$ iff $\Lambda_f(\frac{1}{2} + it)h = 0$, i.e. iff h is supported on the real zeros $\{t : \Lambda_f(\frac{1}{2} + it) = 0\}$, with derivatives up to the multiplicity minus one. Scaling acts unitarily on $L^2(\mathbb{R}_+^\times)$, and the involution $g \mapsto \tilde{g}$ is compatible with it. Both descend to the quotient, where the induced positive form is the restriction of the L^2 norm. Hence the conjugate (unitary) structure survives on the quotient, the positive form is automatically compatible with the flow, and the spectrum is $\{\gamma \in \mathbb{R}\}$. Off-line zeros have complex γ and never appear.*

Proof. $\langle \Lambda G, h \rangle = 0$ for all G means $\Lambda h = 0$ as a distribution. Division by a real-analytic function gives the support statement. Unitarity is preserved by compressing to the orthogonal complement of a closed invariant subspace. $\square$

Consequence [RESTATEMENT]. On the cokernel, “compatibility of the flow with the positive form after the quotient” is automatic. The RH content is *completeness*: $N_{\text{on}}(T) = N(T)$ for all T , where N_{on} counts the cokernel spectrum and N all zeros. This is exactly what §7 measures. For ζ , $N_{\text{on}} = N$ to $T = 250$ (all 108 zeros with $0 < \gamma \leq 250$ in the Platt/LMFDB table used since Lemma 78; that these are all the zeros is the published verification, [BIB]).

Multiplicity exists: the χ -sectors [THEOREM]. Theorem 6.1 is stated sector by sector, so $L^2(X)$ decomposes over the characters χ of $C_{\mathbb{Q},1} \cong \hat{\mathbb{Z}}^\times$, and scaling preserves every sector. This is the multiplicity asked for in Lemma 78 D2. Take $\chi \bmod 5$ with $\chi(2) = i$ (odd, order 4). On the two-dimensional multiplicity space $\langle e_\chi, e_{\bar{\chi}} \rangle$ define

$$C = i \text{ on } e_\chi, \quad C = -i \text{ on } e_{\bar{\chi}}.$$

Then $C^2 = -1$, and C commutes with scaling but is *not* a function of the scaling generator, since it acts differently on two sectors with the same scaling spectrum. The inversion $u \mapsto u^{-1}$ on $\hat{\mathbb{Z}}^\times$, which sends $\chi \mapsto \bar{\chi}$, conjugates C to $-C$, just as F_∞ conjugates the Weil operator to its inverse (Lemma 78). The real structure (real Dirichlet coefficients) is $F_c = cL(s, \chi) + \bar{c}L(s, \bar{\chi})$, and C acts on it by $c \mapsto ic$, a rotation by 90° . With $\Lambda(s, \chi) = \varepsilon\Lambda(1-s, \bar{\chi})$, F_c is FE-even iff $c/\bar{c} = \bar{\varepsilon}$ and FE-odd iff $c/\bar{c} = -\bar{\varepsilon}$. So C exchanges the FE-even and FE-odd real lines.

- FE-even real line: $c = \frac{1-i\kappa}{2}$, coefficients $(1, \kappa, -\kappa, -1)$. This is **exactly the Davenport–Heilbronn function** [THEOREM]; numerically $\Lambda/\Lambda(1-s) = 1 \pm 3 \cdot 10^{-17}$.
- FE-odd real line: $C \cdot \text{DH} = \text{DH}^\perp$, coefficients $(\kappa, -1, 1, -\kappa)$, with $\Lambda/\Lambda(1-s) = -1 \pm 8 \cdot 10^{-17}$ [NUMERICAL].
- C -eigenlines: $L(s, \chi)$ and $L(s, \bar{\chi})$, which have *Euler products*. Here $\varepsilon = 0.850651 + 0.525731i$.

Diagnostic outcome (§7, [NUMERICAL]). In our range the zeros of the C -eigenlines are all on the line, while both real lines carry ghosts. $\text{DH}^\perp$ even has zeros with $\text{Re } s > 1$ (its leading coefficient $\kappa = 0.284$ is smaller than $2^{-\sigma}$ for $\sigma < 1.81$). So a scaling-commuting quarter turn with multiplicity *exists* but does not supply positivity. It rotates the good, Euler-product eigenlines’ real combinations into one another, and those real combinations are where the ghosts are. In Hodge theory, positivity $iQ(\omega, \bar{\omega}) > 0$ is likewise evaluated on *eigenvectors* of C paired with their conjugates, not on the

real lattice. [HEURISTIC]: the polarization J must be compatible with the Hecke/Euler-product structure, i.e. with Frobenius at every finite prime, and not only with scaling and a quarter turn.

7. WHAT CONJUGACY ALONE BUYS: THE DAVENPORT–HEILBRONN TEST (ITEM 5)

DH has a functional equation of sign $+1$ with its own Γ -factor $\gamma(s) = (5/\pi)^{s/2} \Gamma(\frac{s+1}{2})$. By Proposition 4.1 its two halves are exactly conjugate on the line. It has no Euler product. We compare four functions with the same conductor and Γ -factor (and ζ) by counting $N(T)$, all zeros with $0.3 < \text{Im } s \leq T$ (argument principle on the FE-symmetric contour $\frac{1}{2} + 0.3i \rightarrow \sigma_R + 0.3i \rightarrow \sigma_R + iT \rightarrow \frac{1}{2} + iT$, doubled), against $N_{\text{on}}(T)$, the sign changes of the real function $Z(t)$ on the line (step 0.02, refined). By Proposition 6.2, N_{on} is the spectral count of the unitary cokernel, and $N - N_{\text{on}}$ counts the zeros it cannot see. Scripts `num/lemma79_count.py`, `lemma79_count3.py`.

function	structure	FE sign	σ_R	$N(250)$	$N_{\text{on}}(250)$	on-line fraction
ζ	Euler product	$+1$	$-$	108	108	1
$L(s, \chi) \bmod 5$	Euler product, C -eigenline	ε	1.5	171	171	1
DH	FE-even real line	$+1$	1.5	171	161	0.942
$\text{DH}^\perp = C \cdot \text{DH}$	FE-odd real line	-1	3.2	171	103	0.602

[NUMERICAL]. The contour values are integers by construction (both endpoints lie on the line, where Λ is real or of fixed phase). Their reliability rests on the step size (0.02, refined whenever an increment exceeds 0.8 rad) and is cross-checked by the smooth count $\frac{T}{2\pi} \log \frac{5T}{2\pi e} = 170.8$ at $T = 250$. For DH and $L(s, \chi)$ the abscissa $\sigma_R = 1.5$ is certified zero-free (Lemma 78, Rem. “dhright”, and the Euler product). For $\text{DH}^\perp$ the leading coefficient is only $\kappa = 0.284$. It is zero-free for $\sigma \geq 3.2$ by the same argument ($\sum_{n \geq 2} |a_n| n^{-3.2} = 0.1478 < \kappa$), but on the narrow contour $\sigma_R = 1.5$ we count only 123 zeros, so $\text{DH}^\perp$ has zeros with $\text{Re } s > 1.5$. The DH off-line zeros with $\text{Re } s > \frac{1}{2}$ found by grid search and Newton are $0.808517 + 85.6993i$, $0.650830 + 114.1633i$, $0.574356 + 166.4793i$, $0.724258 + 176.7025i$ and (new) $0.869531 + 240.4047i$. That is 5 pairs, exactly $N - N_{\text{on}} = 10$, an independent consistency check. For $\text{DH}^\perp$, the $171 - 103 = 68$ invisible zeros split into 20 off-line zeros in $-0.5 < \sigma < 1.5$ and 48 with $\sigma > 1.5$ or $\sigma < -0.5$ (FE partners). Each comes with its partner $1 - \bar{\rho}$.

What conjugacy buys [THEOREM]/[BIB]/[HEURISTIC].

- *An involution, hence a real $Z(t)$, hence sign changes.* Every sign change is a zero on the line. This is Hardy’s method, and it gives infinitely many zeros on the line for any such function ([BIB]).
- *Proportions.* For ζ : Selberg (a positive proportion), Levinson ($> \frac{1}{3}$), Conrey ($> \frac{2}{5}$), all [BIB]. These use mollifiers, i.e. the Dirichlet structure plus the FE. For DH-type combinations of L -functions, almost all zeros are on the line (Selberg; Bombieri–Hejhal, conditional, [BIB]), and there are $\gg T$ off-line zeros (Davenport–Heilbronn; Voronin, [BIB]). Our 0.942 at $T = 250$ is consistent with an on-line fraction $1 - O(1/\log T)$. Conjugacy thus buys “almost all”, never “all”.
- *What decides “all”.* Among four functions with identical Γ -factor and conductor, the ghosts occur exactly on the real lines without an Euler product. The difference between 0.942 and 1 is the multiplicative structure, which conjugacy does not see. This is the same lesson as §6: J must see the Euler product.

8. CORRECTIONS TO THE CHAT CLAIMS [T330]–[T331] AND TO THE BRIEF

J.W.’s claims are paraphrased from the brief we were given; the chat text itself was not available to us.

- C1** “*The positivity of pairing a holomorphic form with its antiholomorphic conjugate is the missing piece*”. Correct on a curve over $\mathbb{C}$: $i \int \omega \wedge \bar{\omega} > 0$ (Hodge–Riemann) is the archimedean form of the polarization. On the critical line, however, the analogous pairing is a mean value, positive for every Dirichlet polynomial, DH included, and therefore blind (§3). The missing piece is this pairing on a space whose spectrum is the zeros.

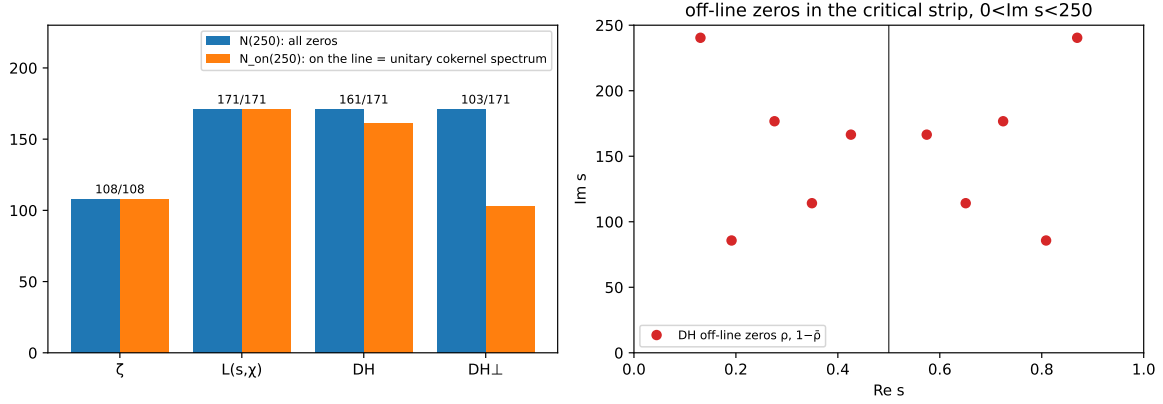


FIGURE 3. Left: $N(250)$ (all zeros) against $N_{\text{on}}(250)$ (sign changes on the line, which is the spectrum of the unitary cokernel) for ζ , $L(s, \chi) \bmod 5$, DH and $DH^\perp$. Right: the DH off-line zeros ρ and $1 - \bar{\rho}$ in the critical strip. ζ and $L(s, \chi)$ have none in this range.

- C2** “ $1+1+1+\dots$ is the holomorphic half”. Corrected. Counting the integers $a, a+1, a+2, \dots$ and regularizing gives $\Gamma(a)$ (Lerch), i.e. the *archimedean factor*, the gluing at the point at infinity. The holomorphic half is $\sum a_n n^{-s}$, whose structure (Euler product) is prime-by-prime.
- C3** “The path back from the point at infinity, perhaps via prime point counts, is the antiholomorphic half”. Half right. The mirror half is $\chi(s) \sum a_n n^{s-1}$: the Γ -ratio (the point at infinity) times the *same* coefficients. Prime counts enter both halves identically, and the point at infinity alone supplies the gluing.
- C4** Link to the Hilbert transform/Kramers–Kronig. Correct and made precise: the Dirichlet half is causal in $\log x$ (Prop. 3.1), and KK holds in the Euler-product region. Lemma 78 already showed that the Hilbert-transform polarization is a restatement.
- C5** “ $n!$ should play a role”. Confirmed twice: Lerch/Stirling (the Γ -phase ϑ) and Legendre/Chebyshev. The two are formally linked by one constant, $-\zeta'(0) = \frac{1}{2} \log 2\pi$. Both are in $\mathbf{1} \oplus \mathbb{L}$, and Legendre’s bridge is zero-blind (Prop. 5.1).
- C6** Brief, item 1: “ $\sum n^{-1/2-it}$ has one-sided spectrum”. Sharpened: the series diverges on the line, so one-sidedness is rigorous for $\text{Re } s > 1$ and for the finite Riemann–Siegel halves. With $n^{-it} = e^{-it \log n}$, the frequencies are $-\log n$ in the $e^{i\omega t}$ convention; the half is one-sided either way. “Line analogue of $H^{1,0}$ ” is [ANALOGY].
- C7** Brief, item 2. Confirmed: exact conjugacy on the line (Prop. 4.1). Off the line the defect is quantified and is archimedean on average. It does not detect ghosts.
- C8** Brief, item 4: “this is $D2$ multiplicity”. Refined. The multiplicity is the χ -sector decomposition of Connes’ space, and a scaling-commuting quarter turn C exists there. Its real FE-even line is DH itself, so C supplies no positivity (§6). “Compatibility of the flow with the positive form after the quotient” is automatic; the RH content is completeness of the spectrum.
- C9** Brief, item 5: zero proportions for ζ . Hardy (infinitely many on the line), Selberg (a positive proportion), Levinson ($> \frac{1}{3}$), Conrey ($> \frac{2}{5}$) are all [BIB], recalled and not opened, as is the statement that for DH-type combinations almost all zeros lie on the line (Selberg; Bombieri–Hejhal, conditional).

9. DIRECTIONS (RANKED BY $0.6s + 0.4p$)

s = significance for the polarization J , p = probability of a clean result within one lemma.

	direction	<i>s</i>	<i>p</i>	score
E1	Hilbert-transform no-go (Lemma 78 D1, carried over). Every scaling-commuting multiplier quarter turn only restates Weil’s form.	.55	.80	.65
E2	The $(\chi, \bar{\chi})$ Hodge–Riemann form. Write $iQ(x, \bar{x})$ on the C -eigenline H_χ with Weil’s Hermitian form for $L(\chi)$. Decide whether it is a restatement of $\text{GRH}(\chi)$ (expected) or carries extra structure from the pairing $\chi \leftrightarrow \bar{\chi}$. Test on DH and $\text{DH}^\perp$, which have no Euler product, so their Weil form exists only on the zero side.	.60	.70	.64
E3	The cokernel before unitarity. In Connes’ cutoff picture ($\delta = 0$), non-critical zeros are resonances. Build a finite cutoff model for DH and test whether a positivity defect localizes at the ghosts, i.e. at the gap $N - N_{\text{on}}$.	.80	.35	.62
E4	Excursions versus zero density. Relate the frequency of $ B/A \geq 1$ at abscissa σ to zero-density counts $N(\sigma, T)$, using DH (where off-line zeros are plentiful) against ζ .	.45	.70	.55
E5	$\text{DH}^\perp$ as an active example. Its zeros with $\text{Re } s > 1$ make $\psi_{\text{DH}^\perp}$ non-passive (Lemma 78, Part A). Quantify.	.30	.90	.54
E6	The circle of real combinations F_c, $c = 1$. Only two points have a functional equation. Track zeros as c rotates and test whether the on-line fraction peaks at the FE points.	.50	.60	.54

10. QUESTIONS FOR J.W.

- Q1** Accept that on the line the holomorphic/antiholomorphic pairing is a mean value, positive and blind, and that the curve pairing is decisive only because it lives on H^1 , where the zeros are eigenvalues?
- Q2** Accept the correction that “ $1 + 1 + 1$ ” regularizes to Γ (the gluing at infinity), while primes enter both halves identically?
- Q3** Accept the $n!$ verdict: two roles, one constant $-\zeta'(0)$, both in $\mathbf{1} \oplus \mathbb{L}$, with Legendre’s bridge zero-blind?
- Q4** Accept the multiplicity finding: a scaling-commuting quarter turn exists on the $(\chi, \bar{\chi})$ plane, its FE-even real line *is* DH, and so J must be compatible with the Euler-product (Hecke) structure, with positivity evaluated on eigenlines?
- Q5** Next: E1, E2, E3, or something else?

REFERENCES

Read at source (`lit/lemma79/verified_refs.md`).

- A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, arXiv:math/9811068 (1999): Thm. 1, Cor. 2 and the discussion following it.

Earlier TAA lemmas [TAA-L]: Lemma 76 (Lerch, Legendre/Chebyshev, the factorial-motive verdict); Lemma 78 (Blaschke/linewidth, the quarter-turn subsection, the Hilbert transform, the Davenport–Heilbronn data and `lemma78_lib.py`); Lemma 77 (codes); Lemma 37 (exponential line).

Not opened [BIB]: Riemann–Siegel (Siegel 1932; Edwards, *Riemann’s Zeta Function*); Montgomery–Vaughan mean value theorem (1974); Hardy (1914), Selberg (1942), Levinson (1974), Conrey (1989) on zeros on the line; Selberg and Bombieri–Hejhal (1995) on linear combinations; Davenport–Heilbronn (1936); zero-density estimates; Chebyshev (1852); Hodge–Riemann relations and the Hodge index theorem (standard texts); Platt’s verification of zeros (data used via Lemma 78).

Code and data. `num/lemma79_*.py` (lib, rs, mv, factorial, count, count3, figs; count2 was abandoned as too slow and replaced by count3); outputs `num/179/*.json`.

TAA LEMMA 80: A NO-GO FOR MULTIPLIER POLARIZATIONS. POSITIVITY FROM SCALING-COMMUTING OPERATORS ON A SINGLE COPY IS BLIND, A RESTATEMENT OF WEIL'S CRITERION, USELESS, OR ZERO-DEPENDENT

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA. This lemma is a *no-go* result: it closes a route and opens none. Tags as in Lemmas 78–79: [THEOREM] proved here (or classical with proof indicated); [CITED] checked at source; [CITED-sec] secondary; [BIB] recalled, not opened; [TAA-L] earlier TAA lemma; [RESTATEMENT] re-statement; [ANALOGY] analogy; [HEURISTIC] heuristic; [OPEN] open; [NUMERICAL] numerics (not certified).

Task (J.W., approved [t336u], Direction E1 of Lemma 79 / D1 of Lemma 78). Prove in general that any positivity built from a scaling-commuting operator on a single copy (a Fourier multiplier) is either blind or a restatement of Weil's criterion. State the hypotheses precisely and test on Davenport–Heilbronn (DH).

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1. ANSWER IN BRIEF

On $L^2(\mathbb{R}_+^\times, d^*x)$, every bounded operator commuting with scaling is a multiplier $m(D)$ (Prop. 3.1). From such operators one can build two kinds of quadratic forms. Hilbert-space forms $\langle Af, Bf \rangle$ are positive or not *independently of the zeros* (blind). Weil-twisted forms reduce to a single weighted zero sum $Q_w(f) = \sum_\rho w(\gamma_\rho) F(\gamma_\rho) \overline{F(\bar{\gamma}_\rho)}$ with w entire (Thm. 3.2). For Q_w there are exactly four cases:

- (i) $w \geq 0$ on $\mathbb{R}$ and w nonzero at the zeros: positivity $\iff$ RH [RESTATEMENT]. If $w = nn^*$, then literally $Q_w(f) = Q(Nf)$.
- (ii) $w < 0$ at some known zero on the line: fails unconditionally (useless).
- (iii) w vanishes at a ghost: hides it, but then w encodes the zeros (zero-dependent).
- (iv) Mixtures with Hilbert-space forms change nothing: case (i) again.

On DH [NUMERICAL], the weights 1 , $1 + z^2/100$, $\cos(z/3) + 1.5$ and $e^{-z^2/400}$ all go negative on the ghost at 85.70 (minima -994 , $-7.4 \cdot 10^4$, -540 and $-1.1 \cdot 10^{-5}$). The size of each minimum tracks $|w(\text{ghost})|$. The same weights are positive on ζ . The sign-changing $\cos(z/2)$ fails for ζ too (-13.4).

Date: October 23, 2026.

The ghost-vanishing weight is positive on DH (≥ 0 at our precision): it hides the ghost. The identity $Q_{nn^*} = Q(N\cdot)$ holds to $1.4 \cdot 10^{-21}$. *Escape routes* (outside the hypotheses): operators that do not commute with scaling (Fourier and prolate, the involution, Lemma 78), spaces with multiplicity (Lemma 79), and non-quadratic constructions.

2. SETTING AND HYPOTHESES

Single copy. $\mathcal{H} = L^2(\mathbb{R}_+^\times, d^*x) \cong L^2(\mathbb{R}, du)$, $u = \log x$. Scaling $U_\lambda f(u) = f(u - \log \lambda)$ is translation. Fourier: $F(z) = \int f(u) e^{izu} du$; on the line the frequency is t , with $s = \frac{1}{2} + it$.

Test class $\mathcal{T} = C_c^\infty(\mathbb{R})$. Then F is entire of exponential type and decays rapidly on horizontal lines. This is the class in which Weil's criterion is stated (Weil 1952; Bombieri 2000, [BIB]).

Zero-side Weil form. For an L -function with a functional equation $\Lambda(s) = \Lambda(1-s)$ (here ζ or DH), write the zeros as $\rho = \frac{1}{2} + i\gamma_\rho$, with $\gamma_\rho \in \mathbb{C}$ for off-line zeros, and set

$$Q(f) = \sum_{\rho} F(\gamma_\rho) \overline{F(\bar{\gamma}_\rho)}, \quad f \in \mathcal{T}.$$

For ζ this is Weil's functional on $g = f * \tilde{f}$. By the explicit formula it equals pole terms minus prime and archimedean terms. The zero set is closed under $\gamma \mapsto \bar{\gamma}$ (functional equation) and $\gamma \mapsto -\bar{\gamma}$ (real coefficients), so $Q(f) \in \mathbb{R}$.

Theorem 2.1 (Weil's criterion, [BIB] for ζ ; same proof for DH). *All zeros lie on the line $\iff Q(f) \geq 0$ for all $f \in \mathcal{T}$.*

For DH the direction " $\Leftarrow$ " is the localization argument. A Gaussian F centred at $c = \gamma_0 + \tau$ with $\tau = \pi/(2a^2\delta)$ makes a ghost pair $\gamma_0 \mp i\delta$ contribute $\approx -2e^{a^2(\delta^2 - \tau^2)}$, which beats the on-line terms, each bounded by $\sup |F|^2$ (sketched here; this is the mechanism behind the DH tests of Lemma 78, [TAA-L]).

3. THE NO-GO THEOREM

Proposition 3.1 (commutant of scaling [THEOREM]). *A bounded operator M on $\mathcal{H}$ with $MU_\lambda = U_\lambda M$ for all λ is a multiplier, $\widehat{M}f = m\hat{f}$ with $m \in L^\infty(\mathbb{R})$. M maps $\mathcal{T}$ into $\mathcal{T}$ only if m is the restriction of an entire function of exponential type, polynomially bounded on $\mathbb{R}$. Examples: polynomials in D (differential operators), e^{iaz} (translations themselves), and combinations of these.*

Proof. Translation-invariant bounded operators on $L^2(\mathbb{R})$ are Fourier multipliers (classical). If $m\hat{f}$ is entire of exponential type for every $f \in C_c^\infty$, then m is the quotient of two such functions and has no poles (vary f). Paley–Wiener–Schwartz then bounds its type and growth. $\square$

Theorem 3.2 (multiplier no-go [THEOREM]). *Assume (H1) single copy: operators are taken from the commutant of scaling on $\mathcal{H}$ (multiplicity one). Assume (H2) the positivity is a Hermitian form on $\mathcal{T}$ of one of the following types:*

- (a) Hilbert type: $B(f) = \langle Af, Bf \rangle_{\mathcal{H}} = \frac{1}{2\pi} \int a(t) \overline{b(t)} |F(t)|^2 dt$, with $A = a(D)$, $B = b(D)$ bounded;
- (b) Weil-twisted type: $Q_{A,B}(f) = \sum_{\rho} (aF)(\gamma_\rho) \overline{(bF)(\bar{\gamma}_\rho)}$, with A, B preserving $\mathcal{T}$ (Prop. 3.1);
- (c) mixtures: $Q_{A,B} + B'$, with B' of type (a).

Assume (H3) the multipliers are chosen without knowledge of the zeros. Then:

- (1) Type (a) is positive iff $\operatorname{Re}(ab) \geq 0$ a.e. This is independent of the zeros: blind.
- (2) Type (b) equals Q_w with $w(z) = a(z)b(\bar{z})$ entire. It is Hermitian iff $w(\bar{z}) = \overline{w(z)}$, i.e. w is real on $\mathbb{R}$. Then:
 - (2i) if $w \geq 0$ on $\mathbb{R}$ and $w(\gamma) \neq 0$ at every zero, $Q_w \geq 0$ on $\mathcal{T} \iff RH$. If in addition $w = nn^*$ with n entire of exponential type ($n^*(z) = \overline{n(\bar{z})}$; this factorization exists for $w \geq 0$ on $\mathbb{R}$ of exponential type, Akhiezer–Krein, [BIB]), then $Q_w(f) = Q(Nf)$ exactly: a restatement of Weil's criterion on the subclass NT ;

- (2ii) if $w(\gamma_0) < 0$ at some zero $\gamma_0 \in \mathbb{R}$, then Q_w takes negative values whether or not RH holds: useless;
- (2iii) Q_w can be ≥ 0 in the presence of a ghost γ_g only if $w(\gamma_g) = 0$ (more precisely, only if w vanishes on every ghost), which violates (H3): zero-dependent.
- (3) Type (c): $Q_w + B' \geq 0$ is again $\iff$ RH under (2i). Concentrated test functions make the ghost term grow like $e^{a^2\delta^2}|w(\gamma_g)|$, while $|B'(f)| \leq \|a'b'\|_\infty \|f\|^2$ stays bounded. So adding Hilbert-type forms neither helps nor hurts.

Consequently, every positivity built from scaling-commuting operators on a single copy is blind, a restatement, useless, or zero-dependent.

Proof. (1) Plancherel. (2) $(aF)(\gamma)\overline{(bF)(\bar{\gamma})} = a(\gamma)\overline{b(\bar{\gamma})}F(\gamma)\overline{F(\bar{\gamma})}$. The terms at γ and $\bar{\gamma}$ are conjugate iff $w(\bar{\gamma}) = \overline{w(\gamma)}$. (2i) Under RH, $Q_w = \sum w(\gamma)|F(\gamma)|^2 \geq 0$. Conversely, the localization argument of Thm. 2.1 applies with the ghost term multiplied by $w(\gamma_g) \neq 0$; its phase is absorbed into the choice of τ . For $w = nn^*$: $n(\gamma)\overline{n(\bar{\gamma})} = n(\gamma)n^*(\gamma) = w(\gamma)$. (2ii) Concentrate F at γ_0 , a known on-line zero. The term $w(\gamma_0)|F(\gamma_0)|^2 < 0$ dominates the others, which are Gaussian-small, and the ghosts, if any, are suppressed by $e^{-a^2(\gamma_g-\gamma_0)^2}$ against $e^{a^2\delta^2}$ (choose γ_0 with $|\gamma_g - \gamma_0| > \delta$ for the finitely many ghosts in a window; on-line zeros are dense enough, Hardy). (2iii) is the contrapositive of the localization. (3) as stated. $\square$

Scope of the hypotheses [HEURISTIC]. (H1) excludes spaces with multiplicity, where the comutant is matrix-valued (Lemma 79's χ -sectors). (H2) excludes non-sesquilinear tests (determinants, counting, Toeplitz/Krein *index* arguments, which are restatements by Lemma 77). Lemma 78's Hilbert transform is the boundary case: $m = -i \operatorname{sgn}$ is not entire, so it does not preserve $\mathcal{T}$. With the only half-plane-analytic extension it gives $Q(P_+f) = \frac{1}{2}Q(f)$ and $Q(f, Hf) = 0$ (Lemma 78, [TAA-L]), which is type (2i) with $w = \frac{1}{2}$, or degenerate. (H3) is the honest content of “without presupposing the zeros”.

4. TEST ON DAVENPORT–HEILBRONN

What DH is. $f = \frac{1-i\kappa}{2}L(\chi) + \frac{1+i\kappa}{2}L(\bar{\chi}) \bmod 5$. It has a functional equation and no Euler product, and it has off-line zeros (“ghosts”). In $[0, 250]$ there are 5 ghost pairs, the first at $0.80852 + 85.6993i$ (Lemma 79, [TAA-L]). In the variable γ of Thm. 2.1 the first ghost is $\gamma_g = 85.6993 - 0.30852i$.

Run (num/lemma80_nogo.py $\rightarrow$ num/180/nogo.json, [NUMERICAL]).

- Zeros: all 171 DH zeros up to height 250 (ghosts included) and the ζ zeros up to 250.
- Test functions: real Gaussian/Hermite, $F(z) = G(z) + \overline{G(-\bar{z})}$ with $G(z) = H_k(a(z - c))e^{-a^2(z-c)^2/2}$ (Hermite polynomial H_k), for $(a, k) \in \{(4, 0), (6, 0), (4, 2)\}$.
- Centres: for DH, $c \in [\gamma_0 - 1, \gamma_0 + 1]$; for ζ , $c = 14 + 0.7j$ with $j < 300$.
- Zeros are summed within ± 12 of c ; the neglected terms are $< e^{-200}$.

weight w	$ w(\gamma_g) $	min Q_w on DH	min Q_w on ζ	verdict (Thm. 3.2)
1 (Weil)	1	−993.9	$+1.7 \cdot 10^{-177}$	(2i) restatement
$1 + z^2/100 = nn^*$, $n = 1 - iz/10$	74.4	$-7.41 \cdot 10^4$	$+5 \cdot 10^{-177}$	(2i), $= Q(Nf)$
$e^{-z^2/400}$	$1.06 \cdot 10^{-8}$	$-1.08 \cdot 10^{-5}$	$+1 \cdot 10^{-177}$	(2i), damped not hidden
$\cos(z/3) + 1.5$	0.538	−540	$+2.6 \cdot 10^{-177}$	(2i)
$\cos(z/2)$	0.452	−455.8	−13.35 ($c = 169.4$)	(2ii) useless
$\frac{(z-\gamma_g)(z-\bar{\gamma}_g)(z+\gamma_g)(z+\bar{\gamma}_g)}{ \gamma_g ^4}$	0	$+4.8 \cdot 10^{-83}$	$+1.6 \cdot 10^{-177}$	(2iii) zero-dependent

Diagnostic value for J . Across the (2i) weights, the DH minimum is $\approx -994|w(\gamma_g)|$ (ratios 74.6, $1.09 \cdot 10^{-8}$ and 0.543 against $|w|$ of 74.4, $1.06 \cdot 10^{-8}$ and 0.538). The multiplier only rescales the ghost's contribution; it cannot change its sign. This is the content of (2i): a positivity built from a multiplier *is* Weil's form seen through w . The identity $Q_{nn^*}(f) = Q(Nf)$ was checked separately, to $1.4 \cdot 10^{-21}$.

The positive ζ minima are tiny (10^{-177}) because centres between zeros make $|F(\gamma)|^2$ Gaussian-small. They are signs, not margins. The zero-dependent weight hides the targeted ghost only: its positivity on DH holds in the window around 85.70, and the other four ghosts would require four more factors in w .

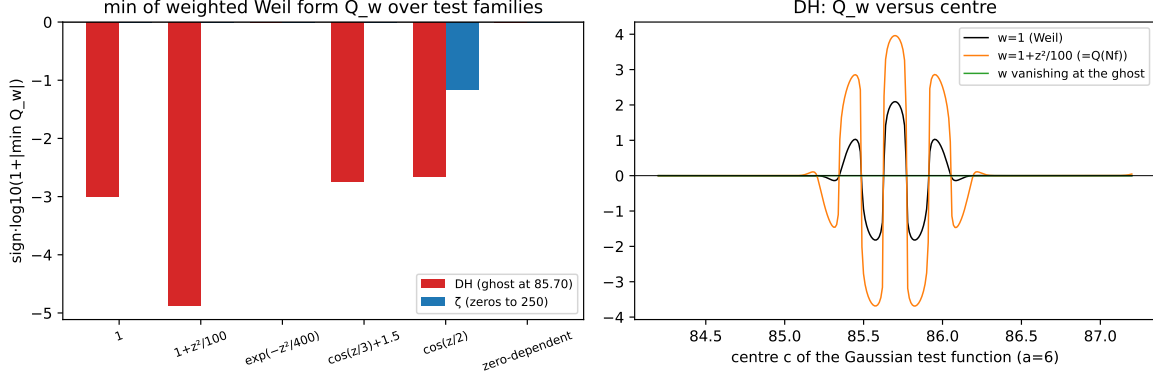


FIGURE 1. Left: signed $\log_{10}(1 + |\min Q_w|)$ for DH (red) and ζ (blue). The $e^{-z^2/400}$ bar is $-1.1 \cdot 10^{-5}$, invisible at this scale, but negative. Right: DH, Q_w versus the centre of a Gaussian ($a = 6$). Weil's form and $Q(Nf)$ oscillate in sign near the ghost. The weight that vanishes at the ghost is flat and nonnegative.

5. CORRECTIONS

- C1** *Lemma 78 D1 / Lemma 79 E1 as stated* (“every scaling-commuting multiplier quarter turn only restates Weil’s form”). This is *almost* right, and it is now a theorem with two additions. Sign-changing multipliers are *useless*, not restatements (2ii): they fail even for ζ , here -13.35 . Multipliers that vanish at ghosts can hide them (2iii), so the statement needs (H3).
- C2** “*Blind*” was used loosely in Lemma 78 for both Hilbert-space positivity and Weil-type positivity. Sharpened: only type (a) is blind. Type (b) sees the zeros exactly as much as Weil’s form does, no more and no less.
- C3** *Damping weights*. A decaying weight such as $e^{-z^2/400}$ seems to “suppress” high ghosts. It does not: the sign survives at size 10^{-5} . Positivity is a sign condition, and weights that never vanish cannot remove signs.
- C4** *The Hilbert transform* of Lemma 78 is the boundary case of (H2): $-i \operatorname{sgn}$ is not entire. Its positivity statements were (2i) with $w = \frac{1}{2}$, or degenerate, consistent with Thm. 3.2.

6. DIRECTIONS (RANKED BY $0.6s + 0.4p$)

	direction	s	p	score
F1	Multiplicity version (Lemma 81). Replace the scalar w by a matrix-valued multiplier on the χ -sectors. Find the analogue of (2i)–(2iii).	.70	.75	.72
F2	Non-commuting operators (Fourier, prolate; Lemma 82). The only single-copy escape. Test whether finite-cutoff positivity sees ghosts.	.80	.45	.66
F3	Non-quadratic tests : determinants and Hankel/Toeplitz minors built from multipliers. Prove they are still restatements (via Lemma 77 Krein inertia).	.45	.70	.55
F4	Optimal ghost amplification : $\sup_w Q_w(\text{ghost}) /\ w\ $. This gives a quantitative ghost detector, but is still a restatement.	.30	.85	.52
F5	Weights from the arithmetic (w built from Λ or μ), which might evade (H3) legitimately.	.55	.30	.45

7. QUESTIONS FOR J.W.

- Q1** Accept Thm. 3.2 as the formal close of the “Hilbert transform / multiplier polarization” line (D1/E1)?
- Q2** Accept (H3) as the precise meaning of “without presupposing the zeros”? Is there a weight class you have in mind that is legitimately arithmetic (F5)?
- Q3** Agree that the only single-copy escape is an operator that does *not* commute with scaling (Fourier/prolate, the involution), and that this is where Connes’s cutoff lives (Lemma 82)?
- Q4** Agree that the useful lesson for J is that the multiplier only *rescales* each zero’s contribution, so a polarization must come from structure that *changes which vectors are compared* (multiplicity, Lemma 81)?
- Q5** Proceed to Lemmas 81–83 as queued (already in progress)?

REFERENCES

Not opened [BIB]: A. Weil, *Sur les “formules explicites”* (1952); E. Bombieri, *Remarks on Weil’s quadratic functional* (2000); N. I. Akhiezer, *Lectures on Integral Transforms* (factorization of nonnegative entire functions of exponential type); Paley–Wiener–Schwartz; Davenport–Heilbronn (1936); Hardy (1914).

Earlier TAA lemmas [TAA-L]: Lemma 77 (Krein inertia), Lemma 78 (Hilbert transform, quarter turn, DH localization), Lemma 79 (DH ghosts to $T = 250$, χ -sectors).

Code and data. num/lemma80_nogo.py, num/lemma80_figs.py $\rightarrow$ num/l80/nogo.json, nogo.log; figure build80/figs/lemma80_nogo.pdf.

**TAA LEMMA 81: THE $(\chi, \bar{\chi})$ QUARTER TURN AND ITS HODGE–RIEMANN
FORM.
RIEMANN’S FIRST RELATION HOLDS, THE SECOND IS GRH, AND THE
 $E \times E$ TEMPLATE**

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven, and neither is GRH for any $L(\chi)$. Tags as in Lemma 80.*

Task (J.W., approved [t336u], Direction E2 of Lemma 79). For the quarter turn C (i on the $L(\chi)$ sector, $-i$ on the $L(\bar{\chi})$ sector, $\chi \bmod 5$ with $\chi(2) = i$), build its Hodge–Riemann-type form $Q(x, C\bar{x})$ and compare it with Weil’s Hermitian form on the two-dimensional sector. Compare positivity on eigenlines with positivity on the real axis, where DH lives. Determine what Euler-product input is needed. Verify J.W.’s $E \times E$ Gram computation and use it as the template. Added [t342u]: Möbius strip, branched double covers and Verschiebung (§6). Added [t346u]: Cassini ovals, the saddle at $\frac{1}{2}$ and the de Bruijn–Newman flow (§7).

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1. ANSWER IN BRIEF

- (1) **The Hodge–Riemann form is Weil’s form [THEOREM]/[RESTATEMENT].** On the sector $V = H_\chi \oplus H_{\bar{\chi}}$, Weil’s *bilinear* form B pairs χ with $\bar{\chi}$. With $E(x, y) = B(x, Cy)$, E is alternating and $H_\chi, H_{\bar{\chi}}$ are isotropic. This is the exact analogue of *Riemann’s first bilinear relation*, and it holds unconditionally (character orthogonality). The Hodge–Riemann form $h(x) = E(Cx, \bar{x})$ equals Weil’s Hermitian form $Q_\chi(f_1) + Q_{\bar{\chi}}(f_2)$. So *Riemann’s second relation* ($h > 0$) is $\text{GRH}(\chi)$: a restatement. Numerically the identities hold to $3.5 \cdot 10^{-30}$.
- (2) **Eigenlines versus real axis [THEOREM].** The two give the same thing: h on the eigenline H_χ is Q_χ , and B on the real axis $\{(f, \bar{f})\}$ is $2Q_\chi$. Neither is closer to a proof.
- (3) **DH is not on this real axis [THEOREM].** DH is a real point of C acting on *coefficient* vectors (Lemma 79). The Weil distribution is additive over *products* of L -functions, not over sums of Dirichlet series. So no vector or subspace of V carries DH’s zero-side form. Its log-derivative

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coefficients are non-local: $b(n) \neq 0$ at 5502 non-prime-powers $n \leq 2 \cdot 10^4$, against 0 for $L(\chi)$, where $b = \chi\Lambda$ to $7 \cdot 10^{-16}$.

- (4) **Euler-product input** [THEOREM]/[HEURISTIC]. The Euler product supplies (i) locality (b supported on prime powers, $|b(n)| \leq \log n$), which makes the prime side a sum of local traces, and (ii) the sector splitting, i.e. the first relation. It does *not* supply (iii) the signature. On $E \times E$, (iii) comes from the Hodge index theorem.
- (5) **$E \times E$ template** [THEOREM] (J.W.'s run confirmed). $\det G = a^2 - 4q$. The complement block $\begin{pmatrix} -2 & -a \\ -a & -2q \end{pmatrix}$ is negative definite iff $a^2 < 4q$. For $y^2 = x^3 + x + 1/\mathbb{F}_{101}$: $N = 105$, $a = -3$, signature $(1, 3)$. A fake $a = 21$ gives $(2, 2)$, and the Hodge index theorem catches the ghost. The zero-side analogue, a Weil Gram matrix on 25 Gaussians near height 85.7, has exactly one negative eigenvalue for DH (its ghost pair) and none for ζ , $L(\chi)$ or $L(\chi)L(\bar{\chi})$ [NUMERICAL]. The difference is that on $E \times E$ the signature is known *a priori*; here it is the conjecture.
- (6) **Branched double covers** (§6, [t342u]) [THEOREM]/[ANALOGY]. Ghosts are the points that have passed a branch point: the Hasse cover $a = q^s + q^{1-s}$ and the archimedean $\lambda = s(1-s)$, both branched at $s = \frac{1}{2}$. Lemma 77's real code ghosts and the fake $a = 21$ land in $(0, \frac{1}{4})$; DH's ghosts are non-real in λ . A bridge table marks what transfers to $\text{Spec } \mathbb{Z}$.
- (7) **Cassini ovals and the heat flow** (§7, [t346u]) [THEOREM]/[CITED]/[RESTATEMENT]. Cassini ovals with foci $\rho, 1 - \rho$ are pulled-back circles under λ . Every pair shares one saddle, $s = \frac{1}{2}$. $\text{RH} \iff$ the zeros of $\Xi(\sqrt{z})$ are positive. Under the de Bruijn–Newman flow, collisions are *moving* branch points, distinct from the fixed one. Causality is RH-blind; irreversibility of the forward flow is the honest version, and “no collision for $t \geq 0$ ” is a restatement. DH's five ghosts below 250 land by $t = 0.2306$.

2. WHAT THE OBJECTS ARE

- A *Hodge structure of weight 1* is a real vector space $V_{\mathbb{R}}$ with a complex structure C ($C^2 = -1$, the Weil operator, i on $H^{1,0}$ and $-i$ on $H^{0,1}$). A *polarization* is an alternating form E with (first relation) $E(Cx, Cy) = E(x, y)$, equivalently $H^{1,0}$ isotropic, and (second relation) $E(x, Cx) > 0$ for real $x \neq 0$. Example: $H^1(X, \mathbb{R})$ of a curve, a vector space, with E the cup product.
- *Cohomology* H^1 is a vector space attached to a space; h^1 is the motive, an object of a category whose realizations are such vector spaces.
- The *Hodge index theorem* is a statement about the intersection form, a symmetric integer bilinear form on the Néron–Severi group $\text{NS}(S)$ of divisor classes on a surface S : it has signature $(1, \rho - 1)$.
- The *sector* H_{χ} is the χ -isotypic part of Connes's space under $\hat{\mathbb{Z}}^{\times}$ (Lemma 79). Concretely, its Weil form is that of $L(\chi)$.

3. THE SECTOR MODEL AND THE HODGE–RIEMANN FORM

Let $Z(\chi)$ be the zeros of $L(\chi)$, written $\rho = \frac{1}{2} + i\gamma$. The functional equation $\Lambda(s, \chi) = \varepsilon\Lambda(1-s, \bar{\chi})$ together with $\overline{L(\bar{s}, \chi)} = L(s, \bar{\chi})$ gives $Z(\chi) = \overline{Z(\chi)}$ (in γ) and $Z(\bar{\chi}) = -Z(\chi)$. For test functions $f \in C_c^{\infty}(\mathbb{R})$ (complex-valued), with $F(z) = \int f e^{izu} du$, define

$$W_{\chi}(f, g) = \sum_{\gamma \in Z(\chi)} F(\gamma)G(-\gamma) \quad (\text{bilinear}), \quad Q_{\chi}(f) = \sum_{\gamma \in Z(\chi)} F(\gamma)\overline{F(\bar{\gamma})} \quad (\text{Hermitian}).$$

Weil's criterion for $L(\chi)$ [BIB] (Weil 1952; the version with characters) is: $\text{GRH}(\chi) \iff Q_{\chi} \geq 0$. Complex conjugation $f \mapsto \bar{f}$ acts by $F \mapsto F^{\dagger}$, $F^{\dagger}(z) = \overline{F(-\bar{z})}$, so $Q_{\chi}(f) = W_{\chi}(f, \bar{f})$.

On $V = H_{\chi} \oplus H_{\bar{\chi}} \ni x = (f_1, f_2)$ put

$$B(x, y) = W_{\chi}(f_1, g_2) + W_{\bar{\chi}}(f_2, g_1), \quad \sigma(f_1, f_2) = (\bar{f}_2, \bar{f}_1), \quad C = (i, -i), \quad E(x, y) = B(x, Cy).$$

Theorem 3.1 (the sector polarization [THEOREM]). (a) B is symmetric and C -invariant, and H_{χ} and $H_{\bar{\chi}}$ are B -isotropic. σ is an antilinear involution commuting with C .

- (b) E is alternating, $E(Cx, Cy) = E(x, y)$, and the eigenlines are E -isotropic: Riemann's first bilinear relation, unconditionally.
- (c) $h(x) := E(Cx, \sigma x) = Q_\chi(f_1) + Q_{\bar{\chi}}(f_2)$ for every $x \in V$. On the eigenline H_χ : $h = Q_\chi$. On the real axis $V^\sigma = \{(f, \bar{f})\}$: $B(x, x) = -E(x, Cx) = 2Q_\chi(f)$.
- (d) Hence “Riemann's second relation” for (V, C, E) , i.e. $h > 0$, or equivalently $-E(x, Cx) > 0$ on V^σ , holds iff $\text{GRH}(\chi)$. This is [RESTATEMENT] of Weil's criterion.

Proof. (a) $W_\chi(f, g) = W_{\bar{\chi}}(g, f)$ by $Z(\bar{\chi}) = -Z(\chi)$, which gives symmetry. $B(Cx, Cy) = i \cdot (-i)B(x, y)$, since each term pairs a χ - with a $\bar{\chi}$ -component. Isotropy holds because B only pairs across sectors. $\sigma C = C\sigma$ by direct check. (b) $E(y, x) = B(y, Cx) = B(Cx, y) = B(C^2x, Cy) = -E(x, y)$. (c) $E(Cx, \sigma x) = B(Cx, C\sigma x) = B(x, \sigma x) = W_\chi(f_1, f_1) + W_{\bar{\chi}}(f_2, \bar{f}_2)$. On V^σ : $B(x, x) = W_\chi(f, f) + W_{\bar{\chi}}(\bar{f}, \bar{f}) = 2Q_\chi(f)$. (d) Combine (c) with Weil's criterion. $\square$

Numerics (`num/lemma81_sector.py` $\rightarrow$ `num/181/sector.json`, [NUMERICAL]). We used random complex Gaussians and the zeros of $L(\chi)$ and $L(\bar{\chi})$ in [74, 98]: 16 and 17 zeros; the $L(\bar{\chi})$ zeros were recomputed by `findroot`. The defects were 0 for symmetry, alternation, the eigenline identity and the real-axis identity, and $3.5 \cdot 10^{-30}$ for $h = Q_\chi + Q_{\bar{\chi}}$. These are identities, so these numbers only check the code. Their diagnostic content for J is that the HR construction adds no information beyond Weil's form.

Reading [ANALOGY]. Riemann's first relation ($H^{1,0}$ isotropic) is *free* here: it is character orthogonality, i.e. the sector structure coming from the Euler product. The second relation is the whole problem. On a curve, both come from geometry, the second from the positivity of $i \int \omega \wedge \bar{\omega}$.

4. WHERE DH LIVES, AND WHAT THE EULER PRODUCT SUPPLIES

Proposition 4.1 (DH is not a vector of V [THEOREM]). *The Weil distribution is additive over products: $W_{L_1 L_2} = W_{L_1} + W_{L_2}$, so V carries $W_{L(\chi)L(\bar{\chi})}$. $DH = \frac{1-i\kappa}{2}L(\chi) + \frac{1+i\kappa}{2}L(\bar{\chi})$ is a sum of Dirichlet series. Its zero set is disjoint from $Z(\chi) \cup Z(\bar{\chi})$, and no subspace of V , or restriction of B , gives its zero-side form. DH 's real line is the real axis of C acting on the coefficient plane $\mathbb{C}L(\chi) \oplus \mathbb{C}L(\bar{\chi})$ (Lemma 79), a different space from V .*

Proof. The log-derivative is additive under products. A common zero ρ of DH and $L(\chi)$ would force $L(\bar{\chi})(\rho) = 0$ as well. At heights ≤ 250 there are none [NUMERICAL]. The closest approach is $6.006 \cdot 10^{-5}$, at height 48.4778: the DH zero is at 48.47778658 and the $L(\chi)$ zero at 48.47784664, both refined to 30 digits, with $|L(\bar{\chi})| = 4.5 \cdot 10^{-4}$ there. In general a common zero is not excluded, but it is not generic. The last statement is Lemma 79. $\square$

Locality (`num/lemma81_gram.py`, $n \leq 2 \cdot 10^4$, [NUMERICAL]). Write $-f'/f = \sum b(n)n^{-s}$. For $L(\chi)$, $b = \chi\Lambda$ to $6.7 \cdot 10^{-16}$: supported on prime powers, with $\max |b|$ growing like $\log n$ (fitted exponent 0.11). For DH , $b(n) \neq 0$ at 5502 non-prime-powers (the first are 6, 12, 14, 18, 21, 24, 26, 28), and $\max |b|$ grows like $n^{0.61}$. So DH has no prime-side Weil form: the “geometric side” of its explicit formula is not a sum over primes. This is what the Euler product buys.

input	$E \times E/\mathbb{F}_q$	sector V of $L(\chi)$
(i) locality	$N = \Delta \cdot \Gamma_F = 1 - a + q$ is a point count	$b = \chi\Lambda$ on prime powers (Euler product)
(ii) splitting / first relation	$\text{NS} = \langle H, V \rangle \oplus$ complement	$H_\chi \perp_B H_{\bar{\chi}}$ (character orthogonality)
(iii) signature	Hodge index (ampleness) [BIB]	missing = $\text{GRH}(\chi)$ [OPEN]
Frobenius as a vector	graph $\Gamma_F \in \text{NS}$	none: scaling is an operator, not a class

5. THE $E \times E$ TEMPLATE: THE HODGE INDEX THEOREM CATCHES A GHOST

Let $E/\mathbb{F}_q$ be an elliptic curve and $S = E \times E$. Take the classes $H = E \times \{0\}$, $V = \{0\} \times E$, the diagonal Δ and the Frobenius graph Γ_F . Their intersection numbers are $H^2 = V^2 = \Delta^2 = \Gamma_F^2 = 0$

(genus 1), $H \cdot V = H \cdot \Delta = V \cdot \Delta = V \cdot \Gamma_F = 1$, $H \cdot \Gamma_F = q = \deg F$, and $\Delta \cdot \Gamma_F = N = \#E(\mathbb{F}_q)$. So

$$G = \begin{pmatrix} 0 & 1 & 1 & q \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & N \\ q & 1 & N & 0 \end{pmatrix}.$$

Proposition 5.1 ([THEOREM], num/lemma81_gram.py, exact in sympy). $\det G = (N - q - 1)^2 - 4q = a^2 - 4q$, where $a = q + 1 - N$. In the basis $H, V, D' = \Delta - H - V, F' = \Gamma_F - H - qV$, G becomes $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} -2 & -a \\ -a & -2q \end{pmatrix}$. The complement has trace < 0 and determinant $4q - a^2$. Hence: signature $(1, 3)$ iff $a^2 < 4q$; $(1, 2, 1)$ (one null direction) iff $a^2 = 4q$; $(2, 2)$ iff $a^2 > 4q$. The Hodge index theorem (signature $(1, \rho - 1)$ on $\text{NS}(S)$, with $H + V$ ample) allows at most one positive eigenvalue on any subspace, hence $|a| \leq 2\sqrt{q}$ (Hasse; Weil's proof via Castelnuovo–Severi, [BIB]).

J.W.'s run confirmed [NUMERICAL]. $y^2 = x^3 + x + 1$ over $\mathbb{F}_{101}$ has $N = 105$ projective points, $a = -3$, $2\sqrt{q} = 20.0998$. The eigenvalues are $(-145.19, -1.000, -0.0186, 146.21)$, signature $(1, 3)$. A fake N with $a = 21$ gives $(-128.98, -1.000, +0.0022, 129.98)$, signature $(2, 2)$. The ghost shows up as one extra positive eigenvalue. At $a = 2\sqrt{q}$ we get $(1, 2, 1)$. The scan over $a \in [-30, 30]$ shows the middle eigenvalue changing sign exactly at $|a| = 2\sqrt{q}$ (Fig. 1, left).

The zero-side analogue [NUMERICAL]. Take the Weil Gram matrix $M_{jk} = \sum_{\gamma} F_j(\gamma) \overline{F_k(\gamma)}$ on 25 Gaussians $e^{-a^2(z-c_j)^2/2}$ with $a = 1.5$, $c_j = 80, 80.5, \dots, 92$ (real-symmetrized for the self-dual cases).

zero set	zeros in $[74, 98]$	negative eigenvalues	$\lambda_{\min}$
ζ	10	0	≥ 0
$L(\chi)$ (sector form Q_χ)	16	0	≥ 0
$L(\chi)L(\bar{\chi})$ (all of V)	33	0	≥ 0
DH	17 (incl. one ghost pair)	1	-1.078

A ghost pair $\gamma_g, \bar{\gamma}_g$ contributes $uv^* + vu^*$, with $u_j = F_j(\gamma_g)$ and $v_j = F_j(\bar{\gamma}_g)$: a hyperbolic plane, signature $(1, 1)$. An on-line zero contributes $uu^* \geq 0$. So the number of negative eigenvalues counts ghost pairs: this is the Krein/Pontryagin index κ of Lemma 77 [TAA-L]. It is the zero-side analogue of the extra positive eigenvalue on $E \times E$. The sign is flipped only by convention: the complement there is negative definite, and Weil's form here is positive.

Target shape for E2 [HEURISTIC]. A proof along the $E \times E$ template would need three things in the sector:

- (i) a lattice-like space containing a “Frobenius graph” vector whose pairings are point counts, i.e. the prime-local $b(n)$;
- (ii) a hyperbolic plane split off. For ζ this is the pole plane $s = 0, 1$ (Lemma 78 §C). For nonprincipal χ there is no pole, so the whole sector should be definite;
- (iii) an a priori reason (ampleness) for the signature.

Theorem 3.1 gives (ii) and, with the Euler product, the locality half of (i). The vector in (i) and the reason in (iii) are missing [OPEN].

6. MÖBIUS STRIP, BRANCHED DOUBLE COVERS AND VERSCHIEBUNG

Added at J.W.'s request [t342u]. Each item is labelled. The common thread: a *double cover branched at the centre* ($s = \frac{1}{2}$, $|a| = 2\sqrt{q}$) and an *involution* ($s \mapsto 1 - s$, $F \mapsto V$). Ghosts are the points that have passed through the branch point. On objects: a *line bundle* on S^1 is a family of lines over the circle, and a *section* chooses a vector in each line, continuously.

Theorem 6.1 (Möbius bundle [THEOREM]). *Let $\mathcal{M} = [0, 2\pi] \times \mathbb{R}/(0, v) \sim (2\pi, -v)$ be the Möbius line bundle on S^1 . Its sections are the antiperiodic functions $f(\theta + 2\pi) = -f(\theta)$, i.e. functions of*

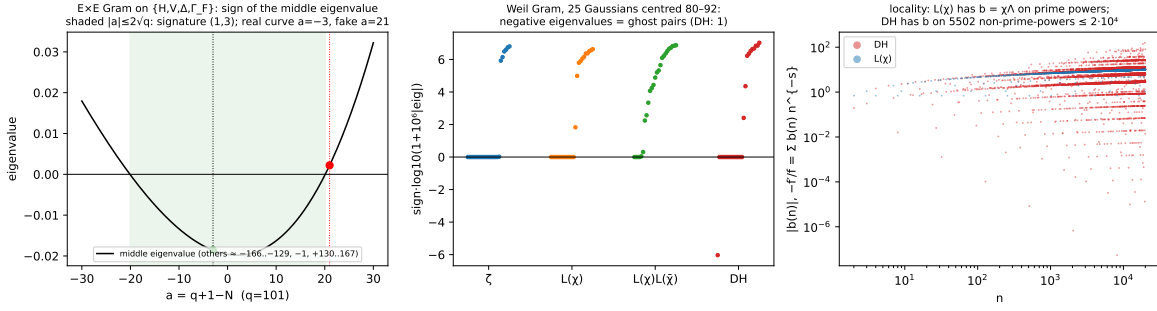


FIGURE 1. Left: $E \times E$ Gram, the middle eigenvalue against a (the other three are about $-166 \dots -129$, -1 and $+130 \dots +167$). The sign change is at $|a| = 2\sqrt{101}$. Red dot: fake $a = 21$. Middle: Weil Gram eigenvalues on 25 Gaussians near height 85.7; only DH has a negative one (its ghost). Right: $|b(n)|$ of $-f'/f$. $L(\chi)$ (blue) is supported on prime powers; DH (red) is spread over all n .

period 4π that are odd under the half-shift. One example is the “width” $\cos(\theta/2)$: this is the spinor, or antiperiodic, boundary condition. Every continuous section vanishes somewhere (intermediate value theorem on $[0, 2\pi]$, since $f(2\pi) = -f(0)$): a forced zero. Moreover $\mathcal{M} \otimes \mathcal{M} \cong \mathbf{1}$, and $\mathcal{M}$ is the unique nontrivial real line bundle ($w_1 \neq 0$).

Theorem 6.2 (root number forces a central zero [THEOREM]). *On the Li disk $w = 1 - 1/s$ the critical line maps to $|w| = 1$, and $s \mapsto 1 - s$ becomes $w \mapsto 1/w$, with fixed points $w = 1$ ($s = \infty$) and $w = -1$ ($s = \frac{1}{2}$). If $\Lambda(s) = \varepsilon \Lambda(1 - s)$ with $\varepsilon = -1$, then $\Lambda(\frac{1}{2}) = 0$. More precisely, $\text{ord}_{s=1/2} \Lambda \equiv (1 - \varepsilon)/2 \pmod{2}$. With real coefficients, $t \mapsto \Lambda(\frac{1}{2} + it)$ is then odd. It is a “section of the Möbius bundle” along the line, folded at $w = -1$ (this last phrase is [ANALOGY]). For an elliptic curve $E/\mathbb{Q}$, BSD and the parity conjecture identify this with odd rank, $\varepsilon = (-1)^{\text{rank}}$ ([BIB]; parity for Selmer groups is Dokchitser–Dokchitser).*

Proof. $|1 - 1/s| = |s - 1|/|s| = 1 \iff \text{Re } s = \frac{1}{2}$. Also $1 - \frac{1}{1-s} = \frac{-s}{1-s} = (1 - \frac{1}{s})^{-1}$. Put $s = \frac{1}{2}$: $\Lambda(\frac{1}{2}) = -\Lambda(\frac{1}{2})$. For the order, expand both sides of $\Lambda(\frac{1}{2} + z) = \varepsilon \Lambda(\frac{1}{2} - z)$ in z . $\square$

Theorem 6.3 (Hasse as a branched double cover [THEOREM]). *For $E/\mathbb{F}_q$, the Frobenius eigenvalues satisfy $\alpha + \bar{\alpha} = a$ and $\alpha \bar{\alpha} = q$; they are the roots of $x^2 - ax + q$. The circle $\alpha = \sqrt{q}e^{i\theta}$ maps 2 : 1 onto the Hasse interval $a = 2\sqrt{q} \cos \theta \in [-2\sqrt{q}, 2\sqrt{q}]$, branched at the endpoints $\theta = 0, \pi$. Past a branch point ($a^2 > 4q$) the roots are real, $(r, q/r)$, with $r \neq \sqrt{q}$: a ghost. The discriminant of $x^2 - ax + q$ equals $a^2 - 4q = -\det$ of the complement block of Prop. 5.1. So the $E \times E$ Gram flip $(1, 3) \rightarrow (2, 2)$ happens exactly at the branch point. For $q = 101$: $a = -3$ gives $\alpha = -1.5 \pm 9.937i$ on the circle; $a = 21$ (> 20.0998) gives $\alpha = 13.541, 7.459$, real, with product 101 (num/l81/branch.json, [NUMERICAL]).*

Theorem 6.4 ($\lambda = s(1 - s)$, the archimedean branched cover [THEOREM]). *$\lambda = s(1 - s)$ is the quotient by $s \leftrightarrow 1 - s$, a double cover branched only at $s = \frac{1}{2}$ ($\lambda = \frac{1}{4}$). Writing $s = \beta + i\gamma$: $\lambda = \beta(1 - \beta) + \gamma^2 + i\gamma(1 - 2\beta)$. Hence:*

- the critical line maps onto $[\frac{1}{4}, \infty)$, real;
- the real segment $(0, 1)$ maps onto $(0, \frac{1}{4}]$, real. A Siegel-type real ghost reaches it by passing the branch point along the axis, exactly as a Hasse root pair passes $a = \pm 2\sqrt{q}$;
- off-line non-real zeros map to non-real λ . A quartet $\{s, 1 - s, \bar{s}, 1 - \bar{s}\}$ maps to one pair $\lambda, \bar{\lambda}$.

On $\mathbb{H}^2$, λ is the Laplace eigenvalue of $E(z, s)$. The critical line is the continuous spectrum, $(0, \frac{1}{4})$ is Selberg’s exceptional range, and non-real λ is forbidden for eigenvalues, but not for resonances (item 6).

Where the ghosts land (`num/lemma81_branch.py` $\rightarrow$ `num/181/branch.json`, Fig. 2, [NUMERICAL]).

- DH off-line zeros: $0.8085 + 85.70i \mapsto \lambda = 7344.53 - 52.88i$; $0.6508 + 114.16i \mapsto 13033.50 - 34.44i$; $0.5744 + 166.48i \mapsto 27715.60 - 24.76i$; $0.7243 + 176.70i \mapsto 31223.96 - 79.25i$; $0.8695 + 240.40i \mapsto 57794.52 - 177.67i$, together with the conjugates. All are non-real, at distance $|\gamma(1 - 2\beta)|$ from the axis.
- Lemma 77 real code ghosts (Siegel-type) all land in $(0, \frac{1}{4})$, on the far side of the branch point: G24+e8 0.02867, e8³ 0.05016, G24+G24 0.01131, i2+e8+e8 0.12759, tet+G12 0.05364, hex+hex 0.07631, and tet²+G12³ $1.26 \cdot 10^{-7}$ (almost at the pole $s = 1$).
- The complex code ghosts i2+e8 ($2.0550 \mp 1.0299i$) and i2+e8+e8 ($1.8478 \mp 0.4838i$) land off the axis.
- The fake Hasse pair $a = 21$ lands at $\lambda = 0.24583 < \frac{1}{4}$, just past the branch point, like a Siegel ghost. The real pair $a = -3$ lands at $0.38900 > \frac{1}{4}$.

Caveat [THEOREM]. For function-field (and code) zetas, s lives modulo $2\pi i / \log q$, and $s \leftrightarrow 1 - s$ has a second fixed point, $s = \frac{1}{2} + i\pi / \log q$. The fake pair $a = -21$ sits there and lands off-axis under λ ($0.7092 \mp 0.0880i$). The correct function-field cover is $a = q^s + q^{1-s} = 2\sqrt{q} \cosh((s - \frac{1}{2}) \log q)$, which is exactly Thm. 6.3, branched at both points. So $\lambda = s(1 - s)$ is the archimedean shadow of the Hasse cover.

Theorem 6.5 (Verschiebung, Rosati, pushforward [THEOREM]; reading [ANALOGY]). *For $E/\mathbb{F}_q$ the dual isogeny $V = \hat{F}$ satisfies $FV = VF = [q]$ and $F + V = [a]$, since $F^2 - aF + q = 0$ and $V = qF^{-1}$. V is the Rosati adjoint of F for the principal polarization: $F^\dagger = V = qF^{-1}$ (Mumford, Abelian Varieties §21, [BIB]). For the squaring map $p : S^1 \rightarrow S^1$, $z \mapsto z^2$: $p_*\mathbf{1} \cong \mathbf{1} \oplus \mathcal{M}$. The two summands are even and odd functions under the deck transformation $z \mapsto -z$; the odd ones are the antiperiodic sections of Thm. 6.1. **Flagged analogy** [ANALOGY]: “ $F \leftrightarrow V$ is the deck involution, the trivial summand is the $\varepsilon = +1$ (even) part and $\mathcal{M}$ the $\varepsilon = -1$ (odd) part”. This matches the parity of $t \mapsto \Lambda(\frac{1}{2} + it)$ (Thm. 6.2). It provides no positivity: the deck involution exists for DH and for every self-dual ghost example alike.*

Bridge table for Spec $\mathbb{Z}$.

curve over $\mathbb{F}_q$	$\text{Spec } \mathbb{Z}$	status
<i>Bridged</i>		
B1 $FV = q$, Rosati $F^\dagger = qF^{-1}$	functional equation $s \leftrightarrow 1-s$ (formal Rosati relation, Lemma 78)	bridged [THEOREM]
B2 Hasse cover $a = q^s + q^{1-s}$	$\lambda = s(1-s)$, branched at $\frac{1}{2}$	bridged [THEOREM]
B3 root number of h^1	$\varepsilon = +1$ for ζ ; $\varepsilon = -1$ forces a central zero	bridged [THEOREM]
<i>Candidate bridges</i>		
C1 Frobenius acting on H^1	Connes’s scaling flow: it exists, and its trace formula holds distributionally (Lemma 79, [CITED])	open
C2 zeros = eigenvalues of F	$\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$: ζ zeros are only scattering <i>resonances</i> of $\varphi = \xi(2s-1)/\xi(2s)$, not eigenvalues of Δ , so self-adjointness does not constrain them. Connes’s cokernel makes them spectral, but only the on-line ones (Lemma 79)	cannot bridge as stated
<i>Unbridged</i>		
U1 H^1 finite-dimensional, cup product into $H^2 = \mathbb{L}$	a trace-class H^1 with a pairing into $\mathbb{L}$	open [OPEN]
U2 $C \times C$ with ample class, Hodge index	a surface “ $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ ”. Connes–Consani’s arithmetic and scaling sites exist ([BIB]), but positivity is unproven	open [OPEN]
U3 polarization from geometry	single-copy multiplier polarization: excluded by the no-go (Lemma 80; Lemma 78)	cannot bridge as stated

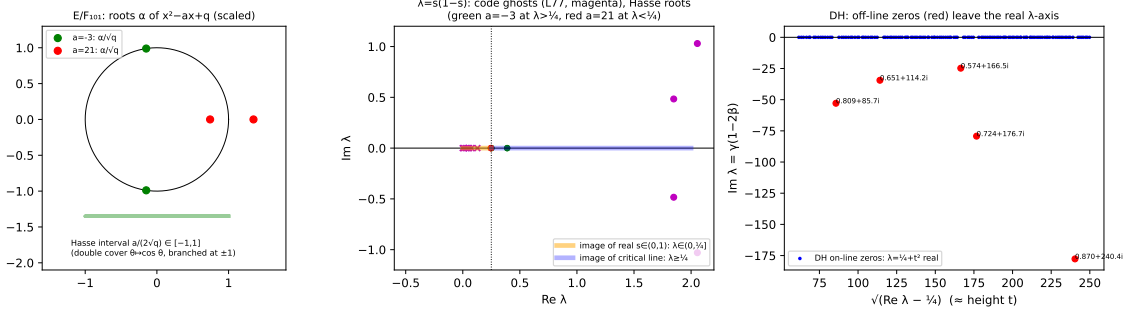


FIGURE 2. Left: roots $\alpha/\sqrt{q}$ of $x^2 - ax + q$, $q = 101$. The real curve ($a = -3$, green) lies on the circle; the fake $a = 21$ (red) gives a real pair past the branch point. Centre: the $\lambda = s(1-s)$ plane near the branch point $\frac{1}{4}$. Lemma 77’s real code ghosts ($\times$) and the fake Hasse pair lie in $(0, \frac{1}{4})$; complex code ghosts ($\circ$) lie off the axis. Right: DH, with the on-line zeros on the real λ -axis and the five off-line zeros at $\text{Im } \lambda = \gamma(1-2\beta)$ (one of each conjugate pair shown).

7. CASSINI OVALS, THE SADDLE AT $\frac{1}{2}$, AND THE DE BRUIJN–NEWMAN HEAT FLOW

Added at J.W.’s request [t346u]. On objects: a *Cassini oval* with foci a, b is a level set $|s-a||s-b| = c$; a *lemniscate* is the one through the midpoint; the *heat flow* deforms a function by $\partial_t H = -\partial_{zz} H$.

Theorem 7.1 (Cassini ovals are pulled-back circles [THEOREM]). $s(1-s) - \rho(1-\rho) = -(s-\rho)(s-(1-\rho))$. Hence, with $\lambda_0 = \rho(1-\rho)$, the Cassini ovals $|s-\rho||s-(1-\rho)| = c$ are exactly the pullbacks under $\lambda = s(1-s)$ of the circles $|\lambda - \lambda_0| = c$. Since $d\lambda/ds = 1-2s$, $s = \frac{1}{2}$ is the only critical point (value $\frac{1}{4}$): the common saddle of every pair. The separatrix is the pullback of the circle through $\lambda = \frac{1}{4}$, of radius $|\frac{1}{4} - \lambda_0| = |\rho - \frac{1}{2}|^2$. This is the lemniscate through $\frac{1}{2}$ (a figure eight around the two foci). Smaller circles pull back to two ovals, and larger ones to a single oval.

Proof. Expand: $s - s^2 - \rho + \rho^2 = -(s-\rho)(s+\rho-1)$. Then $|\lambda - \lambda_0| = |s-\rho||s-(1-\rho)|$, and the rest is the Morse picture of $|\lambda - \lambda_0|$. $\square$

Reconciliation with Lemmas 55–56 ([TAA-L]). Lemma 55, Prop. 2.2: “by Hadamard, $|\xi|$ is a Cassinian function with infinitely many foci”. This is consistent with the present theorem. Grouping the Hadamard factors in pairs $\{\rho, 1-\rho\}$ gives $\xi(s) = G(\lambda) = G(0) \prod_{\text{pairs}} (1 - \lambda/\lambda_\rho)$, so $|\xi|$ is the product of the two-focus Cassini functions above. The saddles differ, however. Lemma 55’s saddles and separatrices are critical points of $|\xi|$ (zeros of ξ' , one per gap between line zeros under RH, with the longitudinal/transverse orientation of Lemma 55 Thm. 4.2 and Lemma 56). Here the saddle is the critical point of λ itself, a single one common to all pairs. The two coincide only at $s = \frac{1}{2}$, which is always a critical point of ξ : $\xi'(\frac{1}{2}) = G'(\frac{1}{4})\lambda'(\frac{1}{2}) = 0$ ([THEOREM]). Lemma 56 Thm. 6.1 (the 90° rotation of the lemniscate at a zero collision) is the moving-branch-point phenomenon of Thm. 7.3(c) below.

Theorem 7.2 (one cover, not a stack; $\Xi(\sqrt{z})$ [THEOREM]). All pairs $\{\rho, 1-\rho\}$ share the single branch point $s = \frac{1}{2}$ of the single double cover $\lambda : \mathbb{C} \rightarrow \mathbb{C}$. $\Xi(t) = \xi(\frac{1}{2}+it)$ is even, so $\Xi(t) = G(\frac{1}{4}+t^2)$, and $\Xi(\sqrt{z}) = G(\frac{1}{4}+z)$ is entire of order $\frac{1}{2}$ in z . A zero γ of Ξ gives $z = \gamma^2$: $z > 0$ iff γ is real (on-line); $z < 0$ iff γ is purely imaginary (a real Siegel-type zero $s \in (0, 1)$); $z \notin \mathbb{R}$ otherwise. Hence RH $\iff$ all zeros of $\Xi(\sqrt{z})$ are real and positive [RESTATEMENT]. ($\gamma = 0$ is excluded since $\xi(\frac{1}{2}) \neq 0$.)

Theorem 7.3 (de Bruijn–Newman [CITED]/[THEOREM]). (a) [CITED] (Polymath 15, arXiv:1904.12438, (1.1)–(1.2); Rodgers–Tao, arXiv:1801.05914 §1; both read in Lemma 56). $H_t(z) = \int_0^\infty e^{tu^2} \Phi(u) \cos(zu) du$ with $\Phi(u) = \sum_n (2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u}) e^{-\pi n^2 e^{4u}}$, $H_0(z) = \frac{1}{8} \xi(\frac{1}{2} + \frac{iz}{2})$, and $\partial_t H_t = -\partial_{zz} H_t$. Newman: there is Λ such that H_t has only real zeros iff $t \geq \Lambda$. RH $\iff \Lambda \leq 0$. Rodgers–Tao 2018/2020: $\Lambda \geq 0$. Polymath 15: $\Lambda \leq 0.22$. Platt–Trudgian 2021, Cor. 2: $\Lambda \leq 0.2$ (correction to the brief: 0.2 is Platt–Trudgian, Polymath gives 0.22). (b) [THEOREM] (formal; [BIB] for the rigorous version, Pol19 Prop. 3.1). Simple zeros move by $\dot{z}_k = 2 \sum_{j \neq k} (z_k - z_j)^{-1}$. Forward in t , real zeros repel, and a conjugate pair $x \pm iy$ alone obeys $\dot{y} = -1/y$, i.e. $y^2 = y_0^2 - 2t$, so it lands at $t_c = y_0^2/2$. Backward, real zeros attract, collide and leave the line as complex pairs. (c) [THEOREM]. Near a collision (t_c, z_c) , $H_t(z) \approx a[(z - z_c)^2 - 2(t - t_c)]$, which solves the equation in (a). So $z = z_c \pm \sqrt{2(t - t_c)}$: a square-root branch point in the parameter t , which moves with (t_c, z_c) . It is distinct from the fixed branch point $z = 0$ ($s = \frac{1}{2}$): H_t is even in z for every t , so the cover of Thm. 7.2 persists along the flow. In the computed examples, Λ is the last collision time.

Correction: causality is not the mechanism ([THEOREM] via Lemma 78). Half-plane analyticity and Kramers–Kronig constrain the poles of a response function, not its zeros. Lemma 78 showed that the scattering function is causal unconditionally, and DH is equally causal. The honest version of “an arrow of time selects the line” is the irreversibility of the forward heat flow: it is smoothing, real zeros repel, and complex pairs land and never leave. “No collision for $t \geq 0$ ” (every zero of H_0 already real) is RH: [RESTATEMENT].

Numerics ([NUMERICAL]; num/lemma81_heat.py reuses Lemma 56’s contour engine lemma56_heatcore, validated there to $6 \cdot 10^{-12}$; num/181/heat.json).

- ζ , real zeros of H_t for $z < 135$ ($z = 2\gamma$), $t \in [-1.4, 0.3]$ (191 steps). There are 14 zeros for $t \leq -1.29$ and 16 for $t \geq -1.285$. One pair is born at $z \approx 121$, matching Lemma 56’s collision $t_c = -1.28527$ ($z = 121.0$). At $t = 0$ the zeros agree with 2γ to $2.3 \cdot 10^{-6}$ (linear interpolation

on a 0.004 grid). The minimal gap among the first 20 zeros is 1.485, 2.391, 2.969, 3.241 at $t = -1, -0.5, 0, 0.3$: repulsion forward, attraction backward.

- *DH* (even completion H_t^{DH} with Φ_{DH} even; data from Lemma 56 v2, `num/156/dhflow.json`, [TAA-L]). The five ghosts below 250 land at $t_c = 0.17651$ (85.70, $\delta = 0.3085$), 0.04453 (114.16, 0.1508), 0.01103 (166.48, 0.0744), 0.09466 (176.70, 0.2243) and 0.23061 (240.40, 0.3695). The single-pair prediction $t_c = y_0^2/2 = 2\delta^2$ gives ratios 0.927, 0.979, 0.998, 0.941, 0.844: neighbours speed up the landing. Lemma 56 v3 certified $0.230607 \leq \Lambda_{\text{DH}} \leq 1.6025$.

Limits. The ζ backward run stops at $t = -1.4$ and $z < 135$, and the high collisions of Lemma 56 are not recomputed. DH was not run backward here: its ghosts only move farther off the line, and none is born. Nothing here bears on $\Lambda \leq 0$.

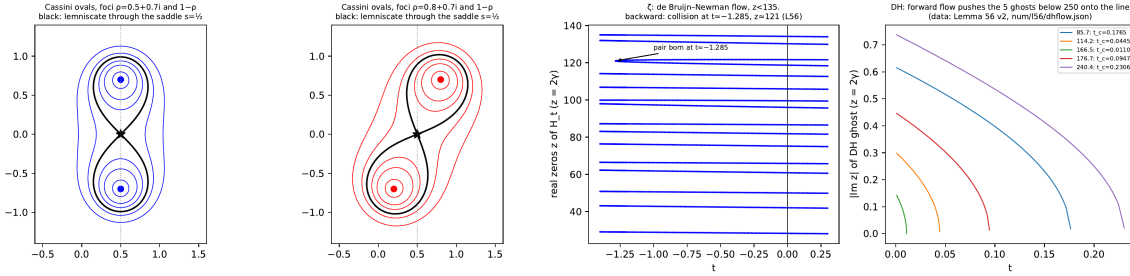


FIGURE 3. Left two panels: Cassini ovals for an on-line pair (foci $\frac{1}{2} \pm 0.7i$) and an off-line pair (foci $0.8 + 0.7i$ and $0.2 - 0.7i$). The lemniscate (black) passes through the common saddle $s = \frac{1}{2}$ (star). Third: real zeros of H_t for ζ ; a pair is born at $t \approx -1.285$, $z \approx 121$. Right: imaginary parts of DH's ghosts below height 250 under the forward flow (Lemma 56 data), all landing by $t = 0.2306$.

8. CORRECTIONS

- C1** *Lemma 79, $Q_4/C8$* (“positivity evaluated on eigenlines” as the target). Sharpened: on the sector, eigenline positivity and real-axis positivity are the *same* number, Q_χ (Thm. 3.1(c)). Choosing eigenlines buys nothing.
- C2** *Lemma 79, “the FE-even real line of C is DH”*. True on the *coefficient* plane. On the test-function sector V , the real axis of C carries $2Q_\chi$, not DH (Prop. 4.1). The two “real axes” are different objects, and the phrase “where DH lives” in the brief has to be read on the coefficient side.
- C3** *Analogy $C \leftrightarrow$ Weil operator*. It is now partly a theorem: the first Riemann relation holds exactly. The second is GRH, so the analogy is complete only modulo the conjecture. This is a restatement, not progress.
- C4** *$J.W.$ ’s $E \times E$ run*. Confirmed in every number ($N = 105$, $(1, 3)$; fake $|a| = 21$ gives $(2, 2)$). One refinement: $a = \pm 2\sqrt{q}$ is not attainable for $q = 101$ prime (it is not an integer), and at the boundary the form is degenerate, $(1, 2, 1)$, not indefinite.
- C5** *“Hodge index catches a ghost”*. The precise zero-side counterpart is the Krein index κ of Weil’s Gram matrix (Lemma 77), and it is a count of ghost pairs, not a proof device. On $E \times E$ the Hodge index is a *theorem* that forbids ghosts. On the zero side it is the *definition* of a ghost.

9. DIRECTIONS (RANKED BY $0.6s + 0.4p$)

	direction	s	p	score
G1	Function-field sector check: run Thm. 3.1 for a Dirichlet character over $\mathbb{F}_q(T)$, where the sector is H^1 of a curve and the second relation is a theorem. Identify which input proves it.	.65	.75	.69
G2	A Frobenius-graph vector for the sector. Find a vector whose B -pairings are the local terms $\chi(p) \log p$ (the analogue of Γ_F); test on function-field $L(\chi)$ first, where it exists.	.80	.45	.66
G3	Krein index statistics for DH on long windows: $\kappa(T) =$ ghost pairs, versus Lemma 79's $N - N_{\text{on}}$.	.35	.90	.57
G4	Near-coincidences of DH and $L(\chi)$ zeros where $ L(\bar{\chi}) $ is small. Quantify, as a diagnostic of how DH zeros move off the line.	.30	.80	.50
G5	Ample class: is there an “ample” vector (positive self-pairing, forced by geometry) in Connes’s space? This is the missing (iii).	.85	.15	.57

10. QUESTIONS FOR J.W.

- Q1** Accept Thm. 3.1: the $(\chi, \bar{\chi})$ Hodge–Riemann form *is* Weil’s Hermitian form, the first relation is free, and the second is GRH [RESTATEMENT]?
- Q2** Accept that “the real axis where DH lives” is on the coefficient plane, and that the Weil distribution is additive only over products, so DH has no place in the sector?
- Q3** Accept the three-input table (locality, splitting, signature) as the precise meaning of “what the Euler product buys”: it gives the first two, and the third is missing?
- Q4** Accept the $E \times E$ template (one hyperbolic plane, a definite complement containing a Frobenius-graph vector, and an a priori signature) as the target shape for J ?
- Q5** Next: G1 (the function-field sector, where the second relation is a theorem), or G2?

REFERENCES

Not opened [BIB]: A. Weil, *Sur les courbes algébriques*. . . (1948) and *Sur les “formules explicites”* (1952); Hartshorne, *Algebraic Geometry* V.1 (Hodge index) and Ex. V.1.10 (Hasse via $E \times E$); Griffiths–Harris, Ch. 0.6–0.7 and 2.6 (Riemann bilinear relations); Davenport–Heilbronn (1936).

Read at source (in Lemma 56, `lit/lemma56/verified_refs.md`) [CITED]: B. Rodgers, T. Tao, *The de Bruijn–Newman constant is non-negative*, Forum Math. Pi 8 (2020) e6, arXiv:1801.05914 (§1); D.H.J. Polymath, *Effective approximation of heat flow evolution of the Riemann ξ function, and a new upper bound for the de Bruijn–Newman constant*, Res. Math. Sci. 6 (2019) 31, arXiv:1904.12438 (§1, §3); D. Platt, T. Trudgian, *The Riemann hypothesis is true up to $3 \cdot 10^{12}$* , Bull. LMS 53 (2021) 792–797 (Cor. 2). Records only [BIB]: de Bruijn 1950; Newman 1976; Mumford, *Abelian Varieties*; Dokchitser–Dokchitser (parity).

Earlier TAA lemmas [TAA-L]: Lemmas 55–56 (Cassini $|\xi|$, saddles, heat flow, Λ_{DH}), Lemma 77 (Krein inertia), Lemma 78 (pole plane), Lemma 79 (quarter turn C , DH ghosts), Lemma 80 (multiplier no-go).

Code and data. `num/lemma81_{gram,sector,figs,branch,heat,cassini}.py` $\rightarrow$ `num/l81/{gram,sector,branch,heat}.json`; `figures/build81/figs/lemma81_{sector,branch,cassini}.pdf`.

**TAA LEMMA 82: CONNES'S CUTOFF FOR DAVENPORT–HEILBRONN
VERSUS ζ .
THE FINITE-SUPPORT WEIL FORM SEES THE GHOST AT FINITE Λ ,
BEFORE ANY UNITARITY OR COMPLETENESS STEP**

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven.* Nothing here bounds the cutoff uniformly. Tags as in Lemma 80.

Task (J.W., approved [t336u], Direction E3 of Lemma 79). Apply Connes's cutoff model (finite Λ ; the Sonin and prolate picture) to DH and to ζ . Does the finite-cutoff object see DH's off-line zeros *before* the unitarity/completeness step? Which positivity appears? Numerics where feasible.

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1. ANSWER IN BRIEF

- (1) **Yes, the finite cutoff sees the ghost** [THEOREM]+[NUMERICAL]. The Weil quadratic form restricted to support $[\Lambda^{-1}, \Lambda]$ is computed exactly from the geometric side: primes and DH coefficients up to Λ^2 , plus the archimedean integral, validated against the zero side to $3\text{--}4 \cdot 10^{-6}$. For DH it acquires two negative eigenvalues (one ghost quartet) at $L^* = 2 \log \Lambda^* \approx 3.71\text{--}3.73$, i.e. $\Lambda^* \approx 6.4$, using $n \leq 41$. The minimizer sits at the ghost frequency 85.7. For ζ , $\lambda_{\min}$ stays at the numerical floor ($\geq -10^{-9}$) for all $L \leq 12$.
- (2) **Which positivity** [THEOREM]/[RESTATEMENT]. The archimedean part alone is positive for both ζ and DH, hence blind. Small-support positivity also holds for DH ($L < L^*$), hence blind. Positivity for all L is RH.
- (3) **Before unitarity** [THEOREM]. Ghosts are visible at finite Λ as negative directions of the form. Connes's unitarity/completeness step is not where they first appear; it is where they would be discarded.
- (4) **For J** [HEURISTIC]. ζ 's positivity is marginal (near-null directions, the ζ -cycles of Connes–Consani). A polarization must explain the absence of sign flips in these directions.

Date: October 23, 2026.

2. WHAT THE OBJECTS ARE

- **Cutoff.** Fix $\Lambda > 1$ and restrict test functions on $\mathbb{R}_+^\times$ to support in $[\Lambda^{-1}, \Lambda]$. In $u = \log x$ this is $[-L/2, L/2]$ with $L = 2 \log \Lambda$. Then $g = f * \tilde{f}$ is supported in $[-L, L]$, and the prime side of the explicit formula is a *finite* sum over $n \leq e^L = \Lambda^2$.
- **Weil quadratic form at cutoff L :** $Q_L(f) = \sum_\rho F(\gamma_\rho) \overline{F(\bar{\gamma}_\rho)}$ for f supported in $[-L/2, L/2]$, a Hermitian form on a Hilbert space. By the explicit formula it equals (poles) + (archimedean integral) − (finite prime sum). Weil positivity for all L is RH (Weil, [BIB]). Connes–Consani–Moscovici state the same thing as a property $P(n)$, “involving only the Euler product”, on support $[1/n, n]$ (CCM 2023, arXiv:2310.18423, intro, [CITED]).
- **Connes 1999** (arXiv:math/9811068, [CITED]). Thm. 1 ($\delta > 1$, L_δ^2) realizes only the *critical* zeros as spectrum. §VIII: with the cutoff on L^2 ($\delta = 0$), “the critical zeros will appear per se, the non critical ones will appear as resonances ... by proving positivity of the Weil distribution ... equivalent to RH”.
- **Sonin space and prolate operator** (CCM 2023 abstract and intro, [CITED]). The prolate wave operator W_λ commutes with the cutoff projections. Its positive spectrum produces “extremely small eigenvalues of the Weil quadratic form restricted to functions with fixed compact support” (Connes–Consani, *Spectral triples and ζ -cycles*, cited there as [11]), whose eigenfunctions reconstruct low zeros. Its negative spectrum is identified with the Sonin space (cited as [12]).

3. THE COMPUTATION

For $\Lambda(s) = \gamma(s)L(s)$ self-dual with real coefficients, $-L'/L = \sum b(n)n^{-s}$, and $h(t) = \int g(u)e^{itu}du$:

$$\sum_\rho h(\gamma_\rho) = [h(\tfrac{i}{2}) + h(-\tfrac{i}{2})]_\zeta + \frac{1}{2\pi} \int_{\mathbb{R}} h(t) 2 \operatorname{Re} \frac{\gamma'}{\gamma}(\tfrac{1}{2} + it) dt - \sum_{n \geq 2} \frac{b(n)}{\sqrt{n}} (g(\log n) + g(-\log n))$$

([BIB], standard; the bracket is present only for ζ). For ζ : $\gamma = \pi^{-s/2} \Gamma(\frac{s}{2})$ and $b = \Lambda$. For DH: $\gamma = (5/\pi)^{s/2} \Gamma(\frac{s+1}{2})$, and b comes from the Dirichlet convolution $\sum_{d|n} b(d)a(n/d) = a(n) \log n$. It is *not* supported on prime powers (Lemma 81).

Basis. Sine modes $f_j(u) = \sin(\omega_j(u + L/2))$ on $[-L/2, L/2]$ with $\omega_j = j\pi/L$. Their transforms are entire, with the closed form $F_j(t) = \omega_j(e^{-itL/2} - (-1)^j e^{itL/2})/(\omega_j^2 - t^2)$. The cross-correlations g_{jk} are computed in closed form. Frequency bands are $\omega_j \in [70, 102]$ (around DH’s ghost $\gamma_g = 85.70$) and $[40, 130]$. The archimedean integral uses Simpson’s rule on $[-4000, 4000]$ with step 0.004. Code: `num/lemma82_{lib,validate,cutoff,lstar,figs}.py`.

Validation ([NUMERICAL], `num/182/validate.json`). At $L = 6$ with modes $j \in \{1, 2, 5, 9, 14\}$, the geometric-side matrix agrees with the zero-side matrix $\sum_\rho F_j(\gamma) F_k(-\gamma)$ (zeros to height 250, ghosts included) to $2.9 \cdot 10^{-6}$ for ζ and $4.0 \cdot 10^{-6}$ for DH, against entries of size up to 0.02 and 10.5. The residual is the zero tail beyond 250. So the normalization is right, and the geometric side is *exactly* what Connes’s cutoff sees: primes up to Λ^2 plus the archimedean place.

4. RESULTS

Scan over the cutoff (`num/182/cutoff.json`, band $\omega \in [70, 102]$, N modes, [NUMERICAL]).

L	N	DH $\lambda_{\min}$	DH #neg	$\zeta \lambda_{\min}$	arch. only, ζ	arch. only, DH
2	20	2.988	0	1.21	2.40	4.01
3	31	2.238	0	$2.3 \cdot 10^{-2}$	3.59	6.01
4	40	-1.340	2	$3.2 \cdot 10^{-9}$	4.81	8.03
5	51	-4.512	2	≈ 0	6.00	10.03
6	61	-10.83	2	≈ 0	7.20	12.03
8	81	-42.34	2	≈ 0	9.61	16.05
10	102	-123.2	2	≈ 0	12.00	20.05
12	122	-314.6	2	≈ 0	14.41	24.07

Here “ ≈ 0 ” means $|\lambda_{\min}| \leq 10^{-9}$ (between $-9.7 \cdot 10^{-10}$ and $-1.5 \cdot 10^{-13}$). This is the numerical floor, consistent with $\lambda_{\min} \geq 0$. These are the “extremely small eigenvalues” of Connes–Consani’s ζ -cycles: on band-limited functions of compact support the prime sum almost exactly cancels the archimedean part. The DH minimizer peaks at frequency 85.2–86.7 for $L \geq 5$, i.e. at the ghost.

The threshold (num/182/1star.json, step 0.1, linear interpolation). DH turns negative at $L^* = 3.733$ (band [70, 102]) and $L^* = 3.709$ (band [40, 130]). The wider band can only lower L^* , since the min is over a larger space. So $\Lambda^* = e^{L^*/2} \approx 6.4$, and the prime-side sum runs over $n \leq e^{L^*} \approx 41$. ζ stays $\geq -2.6 \cdot 10^{-12}$ at every $L \in [3, 4]$ in both bands.

Robustness ([NUMERICAL]). At $L = 6$, doubling T and halving the step changes matrix entries by $\leq 2.5 \cdot 10^{-7}$, DH’s $\lambda_{\min}$ by $2.6 \cdot 10^{-7}$ ($-10.833849 \rightarrow -10.833849$), and ζ ’s $\lambda_{\min}$ from $-5.6 \cdot 10^{-13}$ to $-1.8 \cdot 10^{-12}$, both at the floor.

Count of negative directions ([THEOREM]). A ghost quartet $\{\gamma_g, \bar{\gamma}_g, -\gamma_g, -\bar{\gamma}_g\}$ contributes $4 \operatorname{Re}[F(\gamma_g)\overline{F(\bar{\gamma}_g)}]$ on real test functions. As a real quadratic form this has rank 4 and signature $(2, 2)$, so we expect two negative directions per quartet in the band, and we observe two. In Lemma 81 there was one, because that basis used a single real-symmetrized Gaussian per centre.

Growth [HEURISTIC]. The normalized depth $|\lambda_{\min}|/\|f\|^2$ (with $\|f_j\|^2 = L/2$) grows with local slope 0.39 over $L \in [9, 12]$, against the single-ghost Paley–Wiener scale $e^{\delta L}$, $\delta = 0.3085$. It stays below $4 \sinh(\delta L)/\delta$ (Fig. 1, right). This is not a rigorous bound: every ghost contributes, including high ones with larger δ .

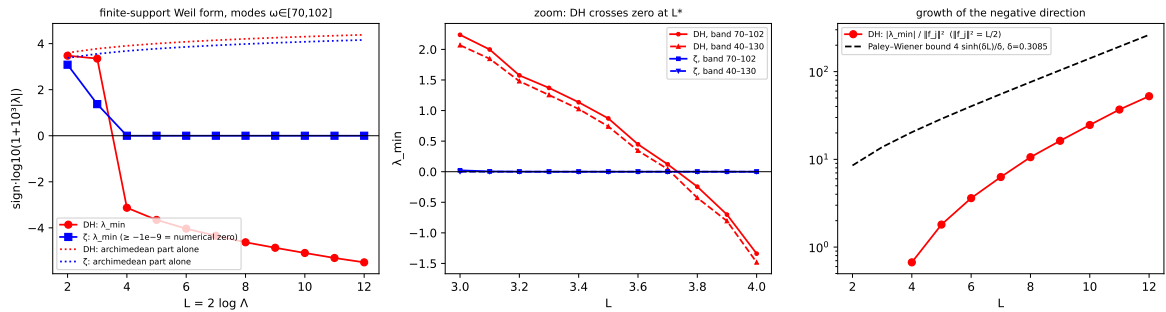


FIGURE 1. Left: signed $\log_{10}(1 + 10^3|\lambda|)$ of the minimal eigenvalue of the finite-support Weil form, for DH (red) and ζ (blue), with the archimedean part alone (dotted, positive for both). Middle: DH crosses zero at $L^* \approx 3.71$ – 3.73 ; ζ stays at the numerical floor. Right: DH depth per unit norm against the single-ghost scale.

5. WHAT THE CUTOFF SEES, AND WHAT POSITIVITY APPEARS

Theorem 5.1 (finite cutoff detects every ghost [THEOREM]). *Let Λ be self-dual, entire, with a zero $\gamma_g = x_g - i\delta$, $\delta > 0$. Then there is $L^* < \infty$ such that Q_L is not positive semidefinite for all $L > L^*$.*

Consequently, *Weil positivity at cutoff L is a finite- L statement that fails for DH at finite L , before any limit $L \rightarrow \infty$ and before any unitarity or completeness step.*

Proof. By Paley–Wiener, $\{F : f \in C_c^\infty(-L/2, L/2)\}$ is dense in the Paley–Wiener space of type $L/2$, and $\bigcup_L$ is the whole test class $\mathcal{T}$. Weil’s criterion (Lemma 80, Thm. 2.1) gives some $f \in \mathcal{T}$ with $Q(f) < 0$. That f has compact support, contained in some $[-L^*/2, L^*/2]$. $\square$

Which positivity ([THEOREM]/[CITED]/[RESTATEMENT]).

- (i) *The archimedean part alone* is positive definite on every band we used, for both ζ and DH. It is blind: positive for both. This matches Connes–Consani’s archimedean Weil positivity (CC 2021, Selecta 27; via Lemma 25, [TAA-L]).
- (ii) *Small-support positivity*: $Q_L \geq 0$ for $L < L^*$ holds for DH too ($L \leq 3.7$). Positivity on small support is therefore not an RH discriminator; for ζ it is a theorem only for small L (CC 2021 / Yoshida, [BIB]).
- (iii) *Full positivity* $Q_L \geq 0$ for all L is RH (Weil; CCM’s $P(n)$): [RESTATEMENT].

Before unitarity, answered ([THEOREM]). Yes. At finite Λ the Weil form sees DH’s off-line zeros through a *negative direction*, at $\Lambda^* \approx 6.4$, using primes up to 41 and the non-local DH coefficients $b(n)$, $n \leq 41$. In Connes’s 1999 formulation the same zeros appear as resonances of the cutoff (§VIII, [CITED]). Unitarity would discard them. The finite- Λ quadratic form keeps them as negative eigenvalues. So the cokernel step is not where ghosts are first visible. They are visible as soon as Λ^2 exceeds roughly the inverse ghost distance scale ($e^{L^*} \approx 41$ here).

Diagnostic value for J [HEURISTIC]. A polarization J must make $Q_L \geq 0$ manifest for all L . For ζ , the numerics show the margin collapses to the floor (10^{-9} – 10^{-13}) already at $L \geq 4$ on band-limited functions: positivity is *marginal*, saturated by near-radical directions (the ζ -cycles). Any J must explain these near-null directions, and their absence of a sign flip, rather than a uniform gap.

6. CORRECTIONS

- C1** *Lemma 79, E3 premise* (“test whether a positivity defect localizes at the ghosts, i.e. at the gap $N - N_{\text{on}}$ ”). Answered: yes, the defect localizes at the ghost (eigenvector frequency 85–87). The count of negative directions is 2 per quartet on real functions, not 1.
- C2** *“Positivity on small support supports RH”*. It does not by itself: DH satisfies it for $L \leq 3.7$.
- C3** *Connes 1999 §VIII* is about the trace formula with cutoff, where non-critical zeros enter as resonances. Our finite- L Weil form is the quadratic-form version, as in CCM 2023, not Connes’s operator. The two agree on what they see, but they are distinct objects.
- C4** *Sonin/prolate*. We did not build the prolate operator. Our modes are sine modes, so the Sonin-space statements are [CITED] from CCM and are not tested here.

7. DIRECTIONS (RANKED BY $0.6s + 0.4p$)

	direction	s	p	score
I1	L^* versus ghost geometry : measure L^* for each DH ghost (and for Epstein ghosts, Lemma 83), and test $L^* \sim c/\delta$ or $c \log(\cdot)/\delta$.	.55	.80	.65
I2	Near-null directions of ζ : compute them in higher precision (mpmath) and compare with Connes–Consani’s ζ -cycles. Do they reconstruct the low zeros?	.70	.55	.64
I3	Prolate basis : redo the scan in the prolate eigenbasis and test the Sonin-space claims on DH.	.60	.50	.56
I4	Prime-side localization : which $n \leq 41$ terms drive DH’s negativity (non-local $b(n)$ versus prime powers)?	.45	.75	.57
I5	Uniform-in-L margin for ζ on bands: is $\lambda_{\min}$ exactly 0 in the limit, i.e. a radical?	.65	.30	.51

8. QUESTIONS FOR J.W.

- Q1** Accept Thm. 5.1: the finite cutoff sees ghosts as negative directions, before unitarity, and for DH at $\Lambda^* \approx 6.4$?
- Q2** Accept that archimedean positivity and small-support positivity are both blind (both hold for DH)?
- Q3** Is the marginality of ζ 's positivity (floor-level $\lambda_{\min}$) the right target for J , i.e. explaining the near-radical directions?
- Q4** Should we build the prolate basis (I3), or first the cheaper I1/I4?
- Q5** Combine with Lemma 83: Epstein ghosts at finite cutoff?

REFERENCES

Read at source [CITED]: A. Connes, arXiv:math/9811068 (1999), Thm. 1 and §VIII (local copy `lit/lemma30`); A. Connes, C. Consani, H. Moscovici, *Zeta zeros and prolate wave operators*, arXiv:2310.18423 (abstract and §1), which cites Connes–Consani, *Spectral triples and ζ -cycles* (Enseign. Math. 2023) and Connes–Moscovici (PNAS 2022) as [11], [12].

Earlier TAA lemmas [TAA-L]: Lemma 25 (CC 2021 archimedean positivity), Lemma 78 (causality), Lemma 79 (DH ghosts), Lemma 80 (Weil criterion form), Lemma 81 (non-local $b(n)$, Gram count).

Not opened [BIB]: Weil 1952; Yoshida 1992; Bombieri 2000.

Code and data. `num/lemma82_{lib,validate,cutoff,lstar,figs}.py` → `num/182/{validate,cutoff,lstar}.json, logs; figure build82/figs/lemma82_cutoff.pdf.`

**TAA LEMMA 83: FROM LAPLACE–RUNGE–LENZ TO $\mathbb{H}^3$ AND THE
BIANCHI GROUPS.
COULOMB PHASES ARE THE ARCHIMEDEAN PLACE; CLOSED ORBITS
SEE ONLY EULER-PRODUCT CHANNELS**

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven.* Tags as in Lemma 80. Fock 1935, Bander–Itzykson 1966, Bertrand 1873, Elstrodt–Grunewald–Mennicke (EGM) and Sarnak 1983 are recalled, not opened ([BIB]).

Task (J.W., approved [t336u]: “the LRL/coulomb $\rightarrow H^3$ bianchi lemma”). The chain $\text{LRL/Coulomb} \rightarrow SO(4)/SO(3,1)$ (Fock, Bander–Itzykson) $\rightarrow \mathbb{H}^3 \rightarrow$ Bianchi groups $\text{PSL}_2(\mathcal{O}_K)$ and Dedekind zetas. Compare the Coulomb S -matrix Gamma ratio with the modular $\sqrt{\pi}\Gamma(s - \frac{1}{2})/\Gamma(s)$ and with the Riemann–Siegel ϑ . Compare Bertrand (closed orbits, rationality) with Selberg (closed geodesics, Lefschetz fixed points). Decide whether closed-orbit counting on Bianchi quotients sees a DH-type ghost, such as the Epstein zeta of a non-principal genus. Correct the external text pasted at [t334]. Added [t345u]: elliptic point versus cusp (§6).

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1. ANSWER IN BRIEF

- (1) **Coulomb = archimedean place [THEOREM].** The archimedean Gamma ratio of hyperbolic scattering *is* an s -wave Coulomb S -matrix. On $\mathbb{H}^2$, $\varphi_{\text{mod}}(\frac{1}{2} + ir) = \pi^{2ir} S_{-1/2}(r) \zeta(1 - 2ir)/\zeta(1 + 2ir)$, i.e. $d = 2$. On $\mathbb{H}^3$ (Masri’s normalization) the ratio is $\Gamma(1 - s)/\Gamma(1 + s) = S_0$, i.e. $d = 3$. Formally, $\vartheta(t) = \sigma_{-3/4}(t/2) - \frac{t}{2} \log \pi$. The checks hold to 10^{-29} . The bare $\sqrt{\pi}\Gamma(s - \frac{1}{2})/\Gamma(s)$ is not unimodular (3.77 at $r = 0.3$). The Bohr ladder of Coulomb poles is cancelled by trivial zeros, except $s = 1$ (the residual spectrum).
- (2) **Bertrand versus Selberg [BIB]/[ANALOGY].** Kepler has all orbits closed with one length (S^3 great circles), which gives maximal degeneracy and no zeta. Bianchi quotients have isolated closed geodesics, $\ell = 2 \log |\lambda|$ with $\text{tr} \in \mathcal{O}_K$, as Lefschetz fixed points of the flow. “Rationality” there means algebraic traces and Hecke symmetry, not rational period ratios.

Date: October 23, 2026.

- (3) **No ghost seen by closed orbits** [THEOREM]+[NUMERICAL]. For $d = -20$ ($h = 2$), the Epstein zeta of $x^2 + 5y^2$ equals $\zeta L(\chi_{-20}) + L(\chi_{-4})L(\chi_5)$ (checked to $4 \cdot 10^{-19}$). It has many off-line zeros: $N(200) = 253$, $N_{\text{on}}(200) = 183$, so 70 off-line zeros. The first, at $0.93297 + 15.66825i$, sits where the two channels are equal and opposite. The trace formula sees only $\det \Phi = \pm \prod \varphi_\chi$, which has winding number 0 around every ghost tested; the partial zeta has winding number 1. The Bianchi geometry diagonalizes into Euler-product channels, and DH-type combinations are invisible to closed-orbit counting.
- (4) **Elliptic point versus cusp** (§6, [t345u]) [THEOREM]. $\tau = i$ is a quarter turn ($S^2 = -I$, $j - 1728$ has a double zero there, Rosati $J^\dagger = J^{-1}$), and the cusp gives $\varphi(\frac{1}{2}) = -1$, $E(z, \frac{1}{2}) \equiv 0$. In the trace formula the elliptic terms are zero-independent and exponentially small at large r : the *transport gap*. A ghost inserted in φ moves the scattering term by 0.715 and leaves the elliptic terms unchanged. The Bianchi link is open.
- (5) **Corrections to [t334]**. LRL is not a Lefschetz operator; $[K_i, K_j]$ is a Lie bracket, not a cup product; Casimir positivity is the energy sign and is RH-blind; $\text{Im } G > 0$ is Herglotz passivity (Lemma 78), not Hodge–Riemann.

2. WHAT THE OBJECTS ARE

- **Laplace–Runge–Lenz (LRL) vector** $\mathbf{A} = \mathbf{p} \times \mathbf{L} - \hat{\mathbf{r}}$ (units $m = e = \hbar = 1$). It is a conserved *vector operator* for $H = \frac{1}{2}p^2 - 1/r$. On an energy shell $E < 0$ put $\mathbf{K} = \mathbf{A}/\sqrt{-2E}$. Then $[L_i, L_j] = i\varepsilon_{ijk}L_k$, $[L_i, K_j] = i\varepsilon_{ijk}K_k$, $[K_i, K_j] = i\varepsilon_{ijk}L_k$, so $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ via $\mathbf{J}_\pm = \frac{1}{2}(\mathbf{L} \pm \mathbf{K})$. For $E > 0$, $\mathbf{K} = \mathbf{A}/\sqrt{2E}$ gives $[K_i, K_j] = -i\varepsilon_{ijk}L_k$, i.e. $\mathfrak{so}(3, 1)$ (Pauli 1926; Fock 1935; Bargmann 1936; Bander–Itzykson 1966, [BIB]).
- **Fock’s projection** (1935, [BIB]). In momentum space, stereographic projection $\mathbf{p} \mapsto S^3$ (with $p_0^2 = -2E$) turns the bound-state Schrödinger equation into the Laplacian on S^3 . The bound states are the harmonics of degree $n - 1$, with degeneracy n^2 and $E_n = -1/(2n^2)$. For $E > 0$ the same construction lands on the hyperboloid $\mathbb{H}^3$ (Bander–Itzykson II, [BIB]). There the scattering states are eigenfunctions of $\Delta_{\mathbb{H}^3}$ of the free hyperbolic space, with no lattice.
- **Bianchi groups** $\text{PSL}_2(\mathcal{O}_K)$, $K = \mathbb{Q}(\sqrt{d})$ imaginary quadratic, act on $\mathbb{H}^3$ with finite covolume and h_K cusps (Elstrodt–Grunewald–Mennicke, *Groups Acting on Hyperbolic Space*, [BIB]). Their Eisenstein series have scattering matrix $\Phi(s)$ with channels $\varphi_\chi = \Lambda(s, \chi)/\Lambda(1 + s, \chi)$ indexed by class-group characters, and $\det \Phi = \pm \prod_\chi \varphi_\chi$ (Lemma 73, [TAA-L]; Efrat–Sarnak 1985, [CITED] there).
- **Closed geodesics** of $\Gamma \backslash \mathbb{H}^3$ correspond to loxodromic conjugacy classes $[\gamma]$. If λ is the eigenvalue with $|\lambda| > 1$, then $\lambda + \lambda^{-1} = \text{tr } \gamma \in \mathcal{O}_K$, and the complex length is $2 \log \lambda$, with real length $\ell = 2 \log |\lambda|$. They are the “periodic orbits” (the fixed points of the geodesic flow, in the Lefschetz sense) on the geometric side of Selberg’s trace formula (EGM Ch. 6, [BIB]).

3. COULOMB S -MATRIX VERSUS THE ARITHMETIC SCATTERING PHASES

The Coulomb partial-wave S -matrix in three dimensions is $S_l(\eta) = \Gamma(l + 1 - i\eta)/\Gamma(l + 1 + i\eta)$, with Sommerfeld parameter η . The Coulomb phase is $\sigma_l(\eta) = \arg \Gamma(l + 1 + i\eta)$. In d dimensions, $l + 1$ is replaced by $l + \frac{d-1}{2}$ ([BIB]).

Proposition 3.1 (Gamma factors are s -wave Coulomb [THEOREM]). *Let $\varphi_{\text{mod}}(s) = \sqrt{\pi} \Gamma(s - \frac{1}{2}) \zeta(2s - 1) / (\Gamma(s) \zeta(2s))$ be the scattering function of $\text{PSL}_2(\mathbb{Z})$.*

- (a) ($\mathbb{H}^2$, $d = 2$) On $s = \frac{1}{2} + ir$: $\varphi_{\text{mod}} = \pi^{2ir} S_{-1/2}(r) \frac{\zeta(1 - 2ir)}{\zeta(1 + 2ir)}$. The bare ratio $\sqrt{\pi} \Gamma(s - \frac{1}{2}) / \Gamma(s)$ is not unimodular; for example $|\cdot| = 3.771$ at $r = 0.3$. Only after the functional equation moves $\Gamma_{\mathbb{R}}$ across does the archimedean part become the $d = 2$ s -wave Coulomb matrix ($l + \frac{d-1}{2} = \frac{1}{2}$).

- (b) ($\mathbb{H}^3$, $d = 3$) In Masri's normalization (Lemma 73, Masri's theorem, [TAA-L]), $\Phi(s) = (2\pi/\sqrt{|d|})^{2s} \frac{\Gamma(1-s)}{\Gamma(1+s)} Z(1+s)^{-1} PZ(1-s)$. At $s = ir$ the Gamma ratio is exactly $S_0(r)$, the 3D s -wave Coulomb matrix.
- (c) (Riemann–Siegel) $\vartheta(t) = \sigma_{-3/4}(t/2) - \frac{t}{2} \log \pi$. This is formal ($l = -\frac{3}{4}$, “ $d = \frac{3}{2}$ ”): ϑ is the phase of $\Gamma_{\mathbb{R}}(\frac{1}{2} + it)$, the degree-one archimedean factor.
- (d) (Bohr ladder) S_l has poles at $l + 1 + i\eta = -k$, i.e. the hydrogen bound states $\eta = -in$, $E_n = -1/(2n^2)$. In Φ the poles of $\Gamma(1-s)$ at $s = n \geq 2$ are cancelled by the trivial zeros $\zeta_C(1-n) = 0$. Only $s = 1$ survives, the residual spectrum (the constant function), with residue $\sqrt{|d|}/(w \text{ vol}) \cdot J$ (Lemma 73).

Proof. (a) $\varphi_{\text{mod}}(s) = \Lambda(2s-1)/\Lambda(2s)$, and $\Lambda(2ir) = \Lambda(1-2ir)$. The powers of π combine to π^{2ir} . (b) is immediate. (c) $\vartheta(t) = \arg \Gamma(\frac{1}{4} + \frac{it}{2}) - \frac{t}{2} \log \pi$. (d) $\Lambda_C(s) = (\sqrt{|d|}/2\pi)^s \Gamma(s) \zeta_C(s)$ is regular at $s = 1 - n$, $n \geq 2$, while Γ has a pole there. $\square$

Numerics (num/lemma83_coulomb.py, 30 digits, [NUMERICAL]). $|S_l| = 1$ to 10^{-31} for $l \in \{0, 1, 2, -\frac{1}{2}, -\frac{3}{4}\}$. The identity (c) holds to $2.5 \cdot 10^{-29}$, and (a) to $4.1 \cdot 10^{-30}$, with $|\varphi_{\text{mod}}| - 1$ at $2 \cdot 10^{-31}$. In (b) the two sides are identical. For (d), $|\zeta_K(1-n)| = 0$ for $n = 2, \dots, 6$ and $d = -4, -3, -20$, while $\zeta_K(0) = -h/w (\frac{1}{4}, \frac{1}{6}, 1)$.

Diagnostic value for J [THEOREM]. Coulomb is *only the archimedean place*. Its S -matrix is unimodular identically, and so is every Gamma ratio, for every η . Everything arithmetic sits in the extra factor $\zeta(1-2ir)/\zeta(1+2ir)$, respectively $Z(1+s)^{-1} PZ(1-s)$. By Lemma 73's Thm. “unitarity is RH-blind”, unitarity of that factor holds with ghosts as well. Coulomb physics therefore supplies no positivity that could see a zero.

4. BERTRAND, CLOSED ORBITS, RATIONALITY VERSUS SELBERG

Bertrand (1873, [BIB]). Among central potentials, only $-1/r$ and r^2 make all bounded orbits closed. For Kepler this is the conservation of $\mathbf{A}$ (the perihelion does not precess). In Fock's picture all bounded orbits are great circles of S^3 : *one* length, 2π . The “trace formula” of S^3 is then trivial, and its spectrum, Bohr's n^2 -fold degenerate levels, is as degenerate as possible.

Selberg on $\Gamma \backslash \mathbb{H}^3$ ([BIB], EGM Ch. 6). Closed geodesics have lengths $\ell = 2 \log |\lambda|$ with $\lambda + \lambda^{-1} \in \mathcal{O}_K$, logarithms of algebraic units of degree ≤ 4 . These lengths are not rationally related, but they come with *exponentially large multiplicities* (class numbers of quadratic orders over $\mathcal{O}_K$; Sarnak 1983, [BIB]). This is the “arithmetic chaos” of Bogomolny–Georgot–Giannoni–Schmit (1992, [BIB]): Poisson level statistics despite chaotic dynamics.

Rationality: the correct dictionary [ANALOGY].

	Kepler / S^3 (Fock)	$\text{PSL}_2(\mathcal{O}_K) \backslash \mathbb{H}^3$
symmetry	$SO(4)$ ($E < 0$), $SO(3, 1)$ ($E > 0$)	lattice $\Gamma \subset SO(3, 1)^0 \cong \text{PSL}_2(\mathbb{C})$
closed orbits	all, one length (Bertrand)	countably many, $\ell = 2 \log \lambda $, $\text{tr} \in \mathcal{O}_K$
“rationality”	period ratios rational (all = 1)	traces algebraic integers; Hecke operators commute with Δ
fixed points	none isolated (degenerate)	isolated, weights $\ell_0/ \lambda - \lambda^{-1} ^2$
spectrum	$E_n = -1/(2n^2)$, multiplicity n^2	Maass + continuous ($\lambda = 1 + r^2$) + scattering Φ
zeta-type function	none (archimedean Γ only)	Selberg Z_Γ ; $\det \Phi = \pm \prod \varphi_\chi$

The Lefschetz reading of periodic orbits is correct: in the trace formula each closed geodesic is a fixed point of the flow, with weight $\ell_0/|\lambda - \lambda^{-1}|^2$, which up to normalization is the Lefschetz factor $|\det(1 - P_\gamma)|^{-1/2}$ of the Poincaré map ([BIB]). It is a statement about the geodesic flow, not about $\mathbf{A}$.

5. DOES CLOSED-ORBIT COUNTING SEE A DH-TYPE GHOST?

Setting (Lemma 73, [TAA-L]). For $K = \mathbb{Q}(\sqrt{-5})$, $d = -20$, $h = 2$, the classes are $C_1 = [x^2 + 5y^2]$ and $C_2 = [2x^2 + 2xy + 3y^2]$, and $w = 2$. The partial zetas satisfy $\zeta_{C_1} \pm \zeta_{C_2} = L_K(s, \chi)$, with $L_K(s, \chi_0) = \zeta L(\chi_{-20})$ and $L_K(s, \chi_{\text{gen}}) = L(\chi_{-4})L(\chi_5)$. So the Epstein zeta is

$$Z_{x^2+5y^2}(s) = w \zeta_{C_1}(s) = \zeta(s)L(s, \chi_{-20}) + L(s, \chi_{-4})L(s, \chi_5),$$

a DH-type sum of two Euler products sharing a functional equation, with completion $(\sqrt{20}/2\pi)^s \Gamma(s)Z$. The decomposition was checked against Lemma 73's Chowla–Selberg routine at $s = 2, 0.7 + 5i$ and $0.5 + 14.2i$: relative error $\leq 3.6 \cdot 10^{-19}$ for both forms.

Zeros (`num/lemma83_epstein.py`, [NUMERICAL]). On the line, the completed function is real to $2.6 \cdot 10^{-14}$ (relative), and we counted sign changes at step 0.04. $N(T)$ comes from the argument principle; it is an integer at every T to 10^{-14} , which checks the constant.

	T	20	40	60	80	100	120	140	160	180	200
$N(T)$	12	31	53	79	105	133	162	192	221	253	
$N_{\text{on}}(T)$	10	25	39	57	79	101	120	136	161	183	
off-line	2	6	14	22	26	32	42	56	60	70	

The search (`lemma83_ghosts.py`, $t < 60$) found exactly the 7 zeros with $\beta > \frac{1}{2}$ predicted by $N - N_{\text{on}} = 14$: $0.93297 + 15.66825i$, $0.93767 + 29.98340i$, $0.69693 + 36.37406i$, $0.82319 + 44.00011i$, $0.63491 + 46.75841i$, $0.59744 + 52.74327i$, $0.81117 + 58.50774i$. At each, $|\zeta L(\chi_{-20})| = |L(\chi_{-4})L(\chi_5)| \neq 0$ (between 0.29 and 1.15): the two channels cancel and neither vanishes.

Theorem 5.1 (closed orbits see channels only [THEOREM]; numerics [NUMERICAL]). *The Selberg trace formula for $\text{PSL}_2(\mathcal{O}_K) \backslash \mathbb{H}^3$ (EGM Ch. 6, [BIB]) equates the closed-geodesic side with discrete spectrum, plus a continuous-spectrum term involving $(\det \Phi)' / \det \Phi$ and $\text{tr } \Phi(0)$. By Lemma 73, $\det \Phi = \pm \prod_{\chi} \Lambda(s, \chi) / \Lambda(1 + s, \chi)$, and each $\Lambda(s, \chi)$ is an Euler product (Hecke L -function of a class-group character). Hence a zero of a partial zeta ζ_C that is not a zero of any $L_K(s, \chi)$, for instance every Epstein ghost above, contributes nothing to the closed-orbit counting function.*

Numerics (`lemma83_channel.py`, radius 0.02). Around the first four ghosts the winding numbers were: $Z_{x^2+5y^2}$: 1, 1, 1, 1; trivial channel: 0, 0, 0, 0; genus channel: 0, 0, 0, 0; $\det \Phi$: 0, 0, 0, 0 ($|\det \Phi| = 7.0 \cdot 10^{-3}$, $1.4 \cdot 10^{-3}$, $8.6 \cdot 10^{-4}$, $6.6 \cdot 10^{-4}$); $\Phi_{11} = \frac{1}{2}(\varphi_0 + \varphi_{\text{gen}})$: 0, 0, 0, 0. Here $P = I$ and both characters are real.

Diagnostic value for J [HEURISTIC]. This is the geometric version of Lemma 81's finding. The geometry, through Hecke symmetry, the class-group characters and the cusp structure, *diagonalizes* into Euler-product channels. The ghost lives in a non-diagonal combination, which the geometry never uses as a spectral object. Bianchi quotients are thus *automatically DH-proof*, but for a structural reason that says nothing about the channels' own zeros. Whether ζ_K 's zeros are on the line is again the open question.

6. ELLIPTIC POINT VERSUS CUSP: TRANSPORTING J 'S POSITIVITY

Added at J.W.'s request [t345u]. The base case is $\tau = i$ on $\text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$; the Bianchi analogues follow. Numerics: `num/lemma83_elliptic.py` $\rightarrow$ `num/183/elliptic.json` (25 digits). On objects: an *elliptic point* is a point with nontrivial finite stabilizer, and a *cusp* is a fixed point at infinity of a parabolic subgroup.

Theorem 6.1 ($\tau = i$ is a quarter turn [THEOREM]). $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ acts by $\tau \mapsto -1/\tau$ and fixes i , with $dS/d\tau|_i = 1/\tau^2|_i = -1$ (a rotation by π of the tangent space), and $S^2 = -I$. So S is a quarter turn J in $\text{SL}_2(\mathbb{Z})$, of order 2 in $\text{PSL}_2(\mathbb{Z})$. Hence $j - 1728$ has a double zero at i : $j(i) = 1728$, $|j'(i)| < 10^{-39}$, $j''(i) = -49655.130\dots$ [NUMERICAL]. On $E_i = \mathbb{C}/\mathbb{Z}[i]$ (CM, $j = 1728$), S is multiplication by $[i]$. For completeness, $\rho = e^{2\pi i/3}$ is elliptic of order 3 (stabilizer $\langle ST \rangle$), j has a triple zero there ($|j'|, |j''| < 10^{-20}$, $|j'''(\rho)| = 2.74 \cdot 10^5$), and $E_\rho = \mathbb{C}/\mathbb{Z}[\rho]$ has automorphism $[\rho]$.

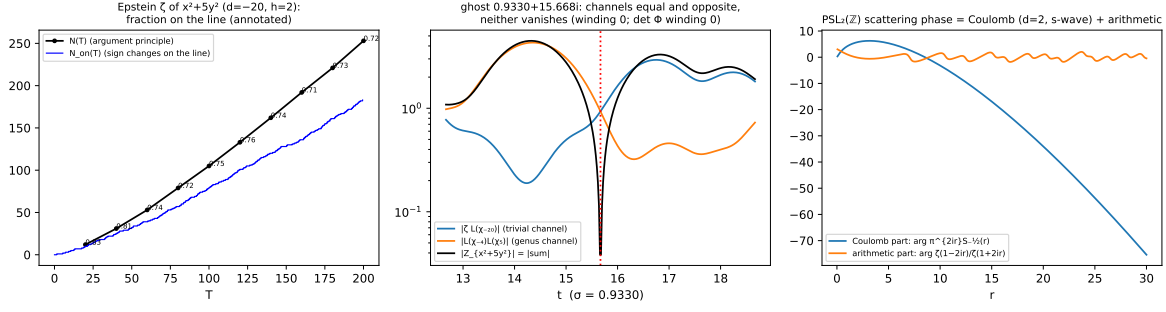


FIGURE 1. Left: Epstein zeta of $x^2 + 5y^2$, $N(T)$ against $N_{\text{on}}(T)$, with the on-line fraction annotated (0.72 at $T = 200$). Centre: along $\sigma = 0.9330$, the two Euler-product channels have equal modulus at the ghost 15.668, and their sum vanishes. Right: the $\text{PSL}_2(\mathbb{Z})$ scattering phase splits into a Coulomb part, $\arg \pi^{2ir} S_{-1/2}(r)$, and an arithmetic part, $\arg \zeta(1 - 2ir)/\zeta(1 + 2ir)$.

Theorem 6.2 (Rosati at E_i ; archimedean conjugation [THEOREM]). $\text{End}(E_i) = \mathbb{Z}[i]$. The Rosati involution of the principal polarization is the dual isogeny, i.e. complex conjugation, so $J^\dagger = [-i] = -J = J^{-1}$, and $\text{Tr}(\varphi\varphi^\dagger) = 2|\varphi|^2 > 0$ (Rosati positivity, [BIB] Mumford §21). Choosing the polarization amounts to choosing J versus $-J$, i.e. an orientation. On moduli, $\tau \mapsto -\bar{\tau}$ preserves the standard fundamental domain, fixes i , and satisfies $j(-\bar{\tau}) = j(\tau)$ (deviation 0 at $\tau = 0.31 + 1.07i$, [NUMERICAL]). Its fixed locus in $X(1)$ is the image of the boundary of the fundamental domain together with the imaginary axis, i.e. the real locus $j \in \mathbb{R}$ (curves defined over $\mathbb{R}$).

Theorem 6.3 (the cusp: Tate curve and $E(z, \frac{1}{2}) \equiv 0$ [THEOREM]). At $\tau \rightarrow i\infty$ the curve degenerates to the Tate curve ($q = e^{2\pi i\tau} \rightarrow 0$, multiplicative reduction). The Eisenstein series has constant term $y^s + \varphi(s)y^{1-s}$, with $\varphi(s) = \xi(2s-1)/\xi(2s)$, where $\xi(w) = \pi^{-w/2}\Gamma(w/2)\zeta(w)$ is the completed zeta (poles at 0, 1; not Riemann's entire ξ , for which the ratio at $\frac{1}{2}$ would be +1). Then $\varphi(\frac{1}{2}) = -1$ and $E(z, \frac{1}{2}) \equiv 0$. The zeros ρ of ζ give poles of φ at $s = \rho/2$, which are scattering resonances with $\text{Re } s = \frac{1}{4}$ under RH, not eigenvalues.

Proof (residue argument). Near $s = \frac{1}{2}$: $\xi(2s-1) \sim -1/(2s-1)$ and $\xi(2s) \sim +1/(2s-1)$ (residues ∓ 1 at 0, 1), so $\varphi \rightarrow -1$. The nonconstant Fourier terms carry the factor $2/\xi(2s)$, which vanishes at $s = \frac{1}{2}$. The constant term is $y^{1/2}(1 + \varphi(\frac{1}{2})) = 0$. $\square$

Numerics [NUMERICAL]. $\varphi(\frac{1}{2} + \epsilon) = -1 - 3.9076\epsilon$ for $\epsilon = 10^{-4}, 10^{-8}, 10^{-12}$. $E(0.3 + 1.1i, \frac{1}{2} + \epsilon) = -3.8996\epsilon$ for $\epsilon = 10^{-3}, 10^{-6}, 10^{-9}$ (Fourier expansion, $n \leq 40$). At $s = \rho_1/2 + 10^{-8}$, $|\varphi| = 5.1 \cdot 10^7$ (the pole). *Sign remark.* $\varphi(\frac{1}{2}) = -1$ is a property of the scattering matrix at its centre. ζ 's root number is $\varepsilon = +1$, which is a different sign attached to a different object (Lemma 81, Thm. “root number”).

The trace formula ([BIB]; Hejhal, Iwaniec *Spectral Methods* Thm. 10.2, normalizations not re-derived here). For even h with transform g , the identity reads

$$\sum_j h(t_j) + \frac{1}{4\pi} \int h(r) \left(-\frac{\varphi'}{\varphi} \right) \left(\frac{1}{2} + ir \right) dr = I(h) + H(h) + \mathcal{E}(h) + P(h),$$

- $I(h) = \frac{\text{vol}}{4\pi} \int r h(r) \tanh(\pi r) dr$, the identity term;
- $H(h)$, the sum over closed geodesics (hyperbolic classes) with $\log N(P_0) g(\log NP)/(NP^{1/2} - NP^{-1/2})$;
- $\mathcal{E}(h) = \sum_{m \in \{2,3\}} \sum_{k=1}^{m-1} \frac{1}{2m \sin(\pi k/m)} \int h(r) \frac{\cosh(\pi(1-2k/m)r)}{\cosh \pi r} dr$, the elliptic terms from i and ρ ;
- $P(h)$, the parabolic terms ($g(0) \log 2$, $\int h \psi(1 + ir)$, and $h(0)$ times a constant involving $\varphi(\frac{1}{2})$).

The elliptic and parabolic terms appear in one identity, but $\mathcal{E}(h)$ depends only on m and k .

Theorem 6.4 (the transport gap [THEOREM]). (a) $\mathcal{E}(h)$ is independent of the zeros of ζ , and $|\mathcal{E}(h)| \leq C \int |h(r)| e^{-2\pi r/m_{\max}} dr$ with $m_{\max} = 3$. The elliptic terms are exponentially insensitive to spectral parameters $r \gg 1$. (b) Hence no inequality on $\mathcal{E}$, positivity included, constrains the scattering poles. A change of the poles of φ must be balanced by $\sum_j h(t_j)$ or $H(h)$, never by $\mathcal{E}$. (c) Positivity at the elliptic point ($J^\dagger = J^{-1}$, $\text{Tr } \varphi \varphi^\dagger > 0$) is positivity on $\text{End}(E_i)$, a rank-2 lattice. The cusp's data are ζ 's zeros. No map in the trace formula carries one to the other. That is the gap.

Proof. (a) The integrand is $h(r) \cosh(\pi(1-2k/m)r) / \cosh(\pi r) \leq 2|h(r)| e^{-2\pi k r/m}$. (b) follows from (a). (c) is a description of the terms. $\square$

Diagnostic: insert a ghost [NUMERICAL]. We replaced the first zero pair $\frac{1}{2} \pm i\gamma_1$ of ξ by the quartet $0.8 \pm i\gamma_1$, $0.2 \pm i\gamma_1$ (Lemma 73's Q -factor). The resulting poles of φ sit at $\text{Re } s = 0.4$ and 0.1 , still in $\text{Re } s < \frac{1}{2}$, so unitarity holds (Lemma 73). Take h a Gaussian at $r_0 = \gamma_1/2$ of width 0.6 .

- The scattering term moves from 0.68093 to 1.39617 .
- The elliptic terms are identical in both models: $5.9 \cdot 10^{-10}$ (order 2) and $4.5 \cdot 10^{-7}$ (order 3).
- $\min_{r \leq 40} (-\varphi'/\varphi)(\frac{1}{2} + ir)$ is -3.754 (true) against -3.734 (ghost): no qualitative change.

The ghosted formula is inconsistent only as an *identity*: the $\Delta = 0.715$ would have to be absorbed by the discrete spectrum or the geodesics, which are fixed by the group. No positivity statement on the elliptic or scattering side flags it.

Bianchi analogues [BIB]/[OPEN]. Elliptic elements of $\text{PSL}_2(\mathcal{O}_K)$ have finite stabilizers: cyclic of order 2 or 3, dihedral, or A_4 (EGM Ch. 7, [BIB]). An example is $\text{diag}(i, -i)$ for $\mathbb{Z}[i]$, a quarter turn about a geodesic. In $\mathbb{H}^3$ elliptic elements fix whole geodesics, so their trace-formula terms involve the axis lengths (units) and are less “local” than in $\mathbb{H}^2$. The cusps carry ζ_K through Φ (Lemma 73; §5). We know of *no* structure linking elliptic-point positivity (CM by orders in K or quaternion orders) to the cusp scattering poles, beyond the trace identity itself. Status: **open** [OPEN]. Thm. 6.4(a,b) carries over verbatim to the purely elliptic (finite-order rotation) contributions [HEURISTIC].

Remark 6.5 (the branch point is local; $X(1)$ is orientable [THEOREM]). Near i , the quotient map is $z \mapsto z^2$ in a local coordinate: a branched double cover with cone angle π , *not* a Möbius band. $\text{PSL}_2(\mathbb{Z})$ acts by orientation-preserving maps, so $X(1) = \text{PSL}_2(\mathbb{Z}) \backslash \mathbb{H} \cup \{\infty\}$ is an orientable orbifold, topologically a sphere, with cone points of orders 2 and 3 and one cusp. A Möbius-type (non-orientable) structure arises only after quotienting by the orientation-reversing $\tau \mapsto -\bar{\tau}$ ($\text{PGL}_2(\mathbb{Z})$). That gives a disk with reflector boundary, the real locus of Thm. 6.2. The Lemma 81 Möbius/branch picture therefore applies to the local covers and to that real structure, not to $X(1)$ itself.

7. CORRECTIONS

The external LLM text pasted by J.W. at [t334] was not available to us verbatim. We correct the four claims as relayed in the brief.

- C1** “*LRL is a Lefschetz operator.*” No. The Lefschetz operator is cup product with a Kähler class, $H^k \rightarrow H^{k+2}$; with its adjoint and the degree it forms an $\mathfrak{sl}_2$, and hard Lefschetz plus Hodge–Riemann give positivity. **A** is a vector operator that commutes with H . With **L** it generates $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ on a shell, and it shifts l by ± 1 inside a fixed n -shell. The resemblance (an $\mathfrak{sl}_2$ acting on a graded space) is at most [ANALOGY], and it carries no positivity.
- C2** “[K_i, K_j] is a cup product.” No. $[K_i, K_j] = \pm i \varepsilon_{ijk} L_k$ is a Lie bracket: antisymmetric, satisfying the Jacobi identity, degree-preserving. A cup product is associative and graded-commutative, $H^p \times H^q \rightarrow H^{p+q}$.
- C3** “*Casimir positivity is the polarization.*” No. $\mathbf{J}_\pm^2 = j(j+1) \geq 0$ and $\mathbf{L}^2 + \mathbf{K}^2 = n^2 - 1$ come from unitarity of a compact-group representation, i.e. from $E < 0$. For $E > 0$ ($SO(3, 1)$, principal series) the Casimir corresponds to $\lambda = 1 + r^2$ on $\mathbb{H}^3$, positive by $-\Delta \geq 0$. This is RH-blind:

the zeros of ζ_K are *scattering resonances* (zeros and poles of Φ), not eigenvalues (Lemma 81, bridge C2).

- C4** “ $\text{Im } G > 0$ is *Hodge–Riemann*.” No. Positivity of Im of a resolvent or Green’s function in the upper half-plane is the Herglotz (Nevanlinna) property, i.e. passivity and causality. Lemma 78 showed it is automatic and blind, holding for DH-type examples as well.
- C5** *Rationality* (brief). “Closed orbits $\leftrightarrow$ rationality” is right for Kepler (rational frequency ratio $1 : 1$). For Bianchi quotients the correct notion is integrality of traces ($\text{tr } \gamma \in \mathcal{O}_K$, lengths $= 2 \log |\lambda|$ of algebraic units), with exponentially large multiplicities (Sarnak 1983, [BIB]).

8. DIRECTIONS (RANKED BY $0.6s + 0.4p$)

	direction	s	p	score
H1	Trace formula with a non-diagonal test: Hecke-twisted traces on $\text{PSL}_2(\mathcal{O}_K)$ that weight cusps unequally. Can any geometric quantity isolate ζ_C rather than the channels? Expected no (Hecke equivariance).	.60	.65	.62
H2	Epstein ghost statistics for several $h = 2, 3$ forms: the off-line fraction against T , compared with Bombieri–Hejhal.	.35	.85	.55
H3	Fock map for the scattering states made explicit: Coulomb $E > 0$ states as $\mathbb{H}^3$ eigenfunctions, and their behaviour under $\text{PSL}_2(\mathcal{O}_K)$ (automorphic Coulomb).	.45	.55	.49
H4	Dimension formula: $\mathbb{H}^{n+1}$ Gamma ratio = the $(n+1)$ -dimensional s -wave Coulomb matrix, for all n (Hilbert modular and $SO(n, 1)$ cases).	.25	.85	.49
H5	Arithmetic chaos multiplicities as a positivity source: whether the class-number multiplicities on the geometric side constrain the spectral side.	.55	.25	.43

9. QUESTIONS FOR J.W.

- Q1** Accept “Coulomb = the archimedean place”: every Gamma ratio is unimodular for free, and the arithmetic lives in $\zeta(1 - 2ir)/\zeta(1 + 2ir)$, respectively $Z(1 + s)^{-1}PZ(1 - s)$?
- Q2** Accept the four [t334] corrections (Lefschetz, cup product, Casimir, Herglotz)? Could you paste the exact [t334] text, so that we can check the corrections against it verbatim?
- Q3** Accept Thm. 5.1: Bianchi closed-orbit counting is DH-proof because the geometry diagonalizes into Euler-product channels, and this says nothing about the channels’ own zeros?
- Q4** Is the Bertrand/Selberg dictionary (one length versus isolated Lefschetz fixed points with algebraic traces) the comparison you intended?
- Q5** Accept the transport gap (Thm. 6.4): elliptic-point positivity (Rosati at E_i) cannot reach the cusp’s scattering poles through the trace formula? Next: H1, or a Bianchi elliptic/cusp study?

REFERENCES

Earlier TAA lemmas [TAA-L]: Lemma 73 (Bianchi scattering, Masri’s matrix, Efrat–Sarnak, unitarity blindness), Lemma 78 (Herglotz/passivity), Lemma 79 (Connes’s cokernel), Lemma 81 (sectors, bridge table).

Not opened [BIB]: W. Pauli (1926); V. Fock, Z. Phys. 98 (1935) 145; V. Bargmann (1936); M. Bander, C. Itzykson, Rev. Mod. Phys. 38 (1966) 330, 346; J. Bertrand, C. R. Acad. Sci. 77 (1873); J. Elstrodt, F. Grunewald, J. Mennicke, *Groups Acting on Hyperbolic Space* (Springer 1998); P. Sarnak, Acta Math. 151 (1983); D. Hejhal, *The Selberg Trace Formula for $\text{PSL}(2, R)$* , vol. 2; H. Iwaniec, *Spectral Methods of Automorphic Forms*; D. Mumford, *Abelian Varieties*; E. Bogomolny, B. Georgeot, M.-J. Giannoni, C. Schmit, Phys. Rep. 291 (1997) and PRL 69 (1992); Davenport–Heilbronn (1936); Potter–Titchmarsh (1935); Bombieri–Hejhal (1995).

Code and data. num/lemma83_{lib,coulomb,epstein,ghosts,channel,elliptic,figs}.py
 $\rightarrow$ num/183/{coulomb,epstein,ghosts,channel,elliptic}.json; figure
 build83/figs/lemma83_epstein.pdf.

TAA LEMMA 84: ALBERT’S CLASSIFICATION, THE $\mathbb{Q}$ -STRUCTURE AND THE AMPLE CLASS. ROSATI POSITIVITY IS WEIL’S PROOF; ONE NEGATIVE DIRECTION PER GHOST PAIR

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. Statement of intellectual honesty. *The Riemann hypothesis (RH) is not proven,* here or anywhere in TAA. Tags as in Lemma 80: [THEOREM] proved here (or standard with proof sketched), [CITED]/[CITED-sec] cited from a checked source / a checked secondary source, [BIB] recalled bibliography not opened, [TAA-L] earlier TAA lemma, [RESTATEMENT] equivalent to RH, [ANALOGY] analogy, [NUMERICAL] computation, [OPEN] open, [HEURISTIC] heuristic.

Task (J.W., approved [t348u]: “Albert’s classification and the $\mathbb{Q}$ -rational structure and ample class”). (1) Positive involutions and Albert’s classification; Rosati positivity from ampleness. (2) The algebra $\mathbb{Q}[F, V]$ and its correct Möbius split; Weil’s proof via Rosati; a genus-2 test. (3) Corrections to the Möbius text pasted at [t348] and to a second text pasted at [t349] (relayed by the parent, together with the parent’s own quick check). (4) Candidate $\mathbb{Q}$ -structures for $\text{Spec } \mathbb{Z}$. (5) The ample class: Arakelov theory, Faltings–Hriljac, Connes–Consani. (6) Directions and questions.

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GLOSSARY (RESTATED EACH TIME IT IS USED)

- *Cohomology* H^1 of a curve C of genus g : a vector space of dimension $2g$ (ℓ -adic over $\mathbb{Q}_\ell$, or Betti over $\mathbb{Q}$ when C is complex), on which Frobenius F acts with characteristic polynomial the Weil polynomial $P(T)$. $h^1(C)$ is the *motive*: the object in a category of which H^1 is a realization. Over $\mathbb{C}$, $H^1 = H^{1,0} \oplus H^{0,1}$.
- *Hodge index theorem*: on the Néron–Severi group $\text{NS}(S) \otimes \mathbb{R}$ of a surface S the intersection form has signature $(1, \rho - 1)$; equivalently it is negative definite on the orthogonal complement of an ample class. For $S = C \times C$ this is Weil’s route to RH for C (Lemma 81).

Date: October 23, 2026.

- *Polarization / ample class*: an ample line bundle L on an abelian variety A gives $\lambda = \varphi_L : A \rightarrow A^\vee$; the *Rosati involution* is $\varphi^\dagger = \lambda^{-1} \varphi^\vee \lambda$ on $\text{End}^0(A) = \text{End}(A) \otimes \mathbb{Q}$.

1. ANSWER IN BRIEF

- (1) **Albert [THEOREM]/[CITED-sec]**. A positive involution on a number field is either the identity on a totally real field or complex conjugation on a CM field (proved in §2); the full classification has types I–IV. Rosati positivity $\text{Tr}(\varphi\varphi^\dagger) > 0$ is a consequence of ampleness (Mumford, *Abelian Varieties*, §21 “Positivity of the Rosati involution”).
- (2) **The correct Möbius split [THEOREM]**. In $A = \mathbb{Q}[F]$ with $F^\dagger = V = qF^{-1}$: in genus 1, $A = \mathbb{Q} \cdot 1 \oplus \mathbb{Q} \cdot (F - V)$, $\dagger = +1$ on the even line, -1 on the odd line, and $(F - V)(F - V)^\dagger = 4q - a^2$. The odd element squares to a negative rational iff A is CM iff $|a| < 2\sqrt{q}$. In genus g the even part is $K_0 = \mathbb{Q}[F + V]$ and the odd part $K_0 \cdot (F - V)$; $\text{Tr}(xx^\dagger)$ is positive definite iff all $|\alpha| = \sqrt{q}$, and its number of negative directions equals the number κ of off-circle reciprocal pairs $\{\alpha, q/\alpha\}$.
- (3) **Genus-2 test [NUMERICAL]**. $y^2 = x^5 + x^4 + x$ over $\mathbb{F}_7$: $N_1 = 4$, $N_2 = 54$, $P = T^4 - 4T^3 + 10T^2 - 28T + 49$, signature $(4, 0)$. Fakes give $(3, 1)$ (one ghost pair) or $(2, 2)$ (two real pairs, or a complex quartet). Exhaustive exact scan of 13 135 squarefree Weil-shaped quartics for $q \in \{2, 3, 5, 7, 11\}$: zero mismatches between “positive definite” and “RH”, and $\#\text{negatives} = \kappa$ in every case. In genus 1, $\text{Tr}(xx^\dagger)$ on $\mathbb{Q}[F]$ is exactly minus the NS($E \times E$) complement Gram (checked at $q = 101$).
- (4) **[t348] text [THEOREM]**. Setup and $\star^2 = -1$ are right; $\star$ is a complex structure, not an involution; all signs come from the form $\langle x, -i \star y \rangle$, which is indefinite by construction and vanishes on real functions; $\star$ is a rotation-invariant multiplier, so the Lemma 80 no-go applies; no zero data enter, so nothing about RH follows. Numerically the $\star$ -form Gram is *identical* for ζ and DH, while Weil’s form separates them.
- (5) **$\mathbb{Q}$ -structure for Spec $\mathbb{Z}$ [OPEN]**. Hecke algebras give type I for free (self-adjointness), but the type IV piece (Ramanujan) needs Deligne. A finite-dimensional $\mathbb{Q}[F]$ for ζ is impossible; an infinite-dimensional substitute would need a positive state, which is Weil positivity again.
- (6) **Ample class [THEOREM]/[OPEN]**. Faltings–Hriljac (arithmetic Hodge index = Néron–Tate positivity) holds, and we check it on 37a1 and 389a1, but it is height positivity for curves *over* Spec $\mathbb{Z}$, not RH. What “ample on Spec $\mathbb{Z} \times \text{Spec } \mathbb{Z}$ ” means is open.

2. POSITIVE INVOLUTIONS AND ALBERT’S CLASSIFICATION

Definition 2.1. Let D be a finite-dimensional semisimple $\mathbb{Q}$ -algebra. An *involution* is a $\mathbb{Q}$ -linear anti-automorphism $x \mapsto x^\dagger$ with $x^{\dagger\dagger} = x$. It is *positive* if $\text{Tr}_{D/\mathbb{Q}}(xx^\dagger) > 0$ for all $x \neq 0$ (reduced or regular trace; the conditions are equivalent).

Theorem 2.2 (commutative case [THEOREM]). *Let K be a number field with a positive involution $\dagger$. Then either (i) $\dagger = \text{id}$ and K is totally real, or (ii) $\dagger$ has order 2, its fixed field K_0 is totally real, K is a totally imaginary quadratic extension of K_0 (a CM field), and $\sigma(x^\dagger) = \overline{\sigma(x)}$ for every embedding $\sigma : K \rightarrow \mathbb{C}$. Conversely these involutions are positive.*

Proof. $\dagger$ is a field automorphism of order ≤ 2 . Write $\text{Tr}_{K/\mathbb{Q}}(y) = \sum_\sigma \sigma(y)$. (i) If $\dagger = \text{id}$, $\text{Tr}(x^2) = \sum_{\text{real}} \sigma(x)^2 + \sum_{\text{complex pairs}} 2 \text{Re } \tau(x)^2$. A complex place contributes $2 \text{Re } z^2$, an indefinite form on $\mathbb{C} \cong \mathbb{R}^2$; since K is dense in $K \otimes \mathbb{R}$, positivity forces no complex places. (ii) If $\dagger$ has order 2, $\text{Tr}_K(x^2) = 2 \text{Tr}_{K_0}(x^2)$ on K_0 , so K_0 is totally real by (i). Write $K = K_0(\sqrt{d})$ with $d \in K_0$; $\sqrt{d}^\dagger = -\sqrt{d}$. For $x = u\sqrt{d}$, $\text{Tr}_K(xx^\dagger) = -2 \text{Tr}_{K_0}(u^2 d)$, positive for all u iff $-d$ is totally positive, i.e. K is totally imaginary over K_0 ; then $\dagger$ induces complex conjugation in every embedding. The converse: $\text{Tr}(x\bar{x}) = \sum_\sigma |\sigma(x)|^2$. $\square$ $\square$

Theorem 2.3 (Albert’s classification [CITED-sec]). *Let D be a division algebra of finite dimension over $\mathbb{Q}$ with positive involution $\dagger$, centre K , and $K_0 = \{x \in K : x^\dagger = x\}$. Then K_0 is totally real and D is of one of four types:*

- I** $D = K = K_0$ totally real, $\dagger = \text{id}$;
- II** D a quaternion algebra over totally real $K = K_0$, split at every real place ($D \otimes_{K,\sigma} \mathbb{R} \cong M_2(\mathbb{R})$); $x^\dagger = a^{-1}\bar{x}a$ with $a^2 \in K$ totally negative, $\bar{x}$ the canonical involution;
- III** D a quaternion algebra over totally real $K = K_0$, definite at every real place ($\cong \mathbb{H}$); $x^\dagger = \bar{x}$;
- IV** K a CM field over K_0 , D a division algebra with centre K , $\dagger|_K$ complex conjugation.

For $D = \text{End}^0(A)$, A simple, all types occur, with numerical restrictions tabulated in Mumford §21, p. 202 ([CITED-sec]: table and page via secondary sources, see refs).

Theorem 2.4 (Rosati positivity from ampleness [CITED-sec]). *Let A be an abelian variety over a field k and L an ample line bundle, $\lambda = \varphi_L$. The Rosati involution $\varphi \mapsto \varphi^\dagger = \lambda^{-1}\varphi^\vee\lambda$ on $\text{End}^0(A)$ is positive: $\text{Tr}(\varphi\varphi^\dagger) > 0$ for $\varphi \neq 0$ (Mumford, Abelian Varieties, Ch. IV, §21 “Positivity of the Rosati involution”; the proof uses Riemann–Roch $\chi(L)^2 = \deg \varphi_L$ and an intersection-number formula for $\text{Tr}(\varphi\varphi^\dagger)$ that is positive by ampleness; formula recalled, not re-checked). Hence $\text{End}^0(A)$ for simple A is a division algebra of Albert type I–IV.*

Remark. Ampleness enters only through a positive intersection number. Without an ample class one still has the involution (duality, $\varphi \mapsto \varphi^\vee$ up to a chosen isogeny), but not its positivity. This is the template for §7: *involution* $\leftrightarrow$ functional equation (available for ζ), *positivity* $\leftrightarrow$ ample class (missing for ζ).

3. THE ALGEBRA $\mathbb{Q}[F, V]$, ITS MÖBIUS SPLIT, AND WEIL’S PROOF

Theorem 3.1 (genus 1: even and odd lines [THEOREM]). *Let $E/\mathbb{F}_q$ be an elliptic curve, $a = q + 1 - \#E(\mathbb{F}_q)$, F Frobenius, $V = \hat{F}$ Verschiebung. Then $FV = q$, $F + V = a$, and the Rosati involution of the principal polarization is $F^\dagger = V$. Let $A = \mathbb{Q}[F] \cong \mathbb{Q}[T]/(T^2 - aT + q)$ and assume $a^2 \neq 4q$. Then*

$$A = \mathbb{Q} \cdot 1 \oplus \mathbb{Q} \cdot w, \quad w = F - V, \quad 1^\dagger = 1, \quad w^\dagger = -w, \quad w^2 = a^2 - 4q, \quad ww^\dagger = 4q - a^2.$$

For $x = u + vw$: $\text{Tr}(xx^\dagger) = 2u^2 + 2(4q - a^2)v^2$. Hence $\dagger$ is positive $\iff a^2 < 4q \iff w^2 \in \mathbb{Q}_{<0} \iff A$ is imaginary quadratic (CM) with $\dagger =$ complex conjugation. In the basis $(1, F)$ the trace form is $\begin{pmatrix} 2 & a \\ a & 2q \end{pmatrix}$, which is exactly -1 times the Gram matrix $\begin{pmatrix} -2 & -a \\ -a & -2q \end{pmatrix}$ of the complement of $\langle H, V \rangle$ in $\text{NS}(E \times E) = U \oplus \text{End}(E)(-1)$, of determinant $4q - a^2$.

Proof. $w = 2F - a$, so $w^2 = 4F^2 - 4aF + a^2 = a^2 - 4q$ by $F^2 = aF - q$; $w^\dagger = 2V - a = -w$. $\text{Tr}(1) = 2$, $\text{Tr}(w) = 0$, $\text{Tr}(w^2) = 2(a^2 - 4q)$. The NS statement: $\deg(m + nF) = m^2 + amn + qn^2$ and $\text{End}(E)(-1)$ carries $-2\deg$ as quadratic form (Lemma 81, Thm. $E \times E$). $\square$ $\square$

This is the correct *algebraic* Möbius split: $\dagger$ is a genuine involution ($\dagger^2 = 1$), the odd line is the “twisted” one, and its sign is the arithmetic content $4q - a^2 > 0$, i.e. RH for E . Nothing in the split is automatic: for the fake $a = 21$, $q = 101$, $ww^\dagger = -37 < 0$ [NUMERICAL].

Theorem 3.2 (genus g ; Weil’s proof via Rosati [THEOREM], [CITED-sec]). *Let $P(T) = \prod_{i=1}^{2g} (T - \alpha_i) \in \mathbb{Z}[T]$ be squarefree with root set stable under $\alpha \mapsto q/\alpha$ (the functional equation). On $A = \mathbb{Q}[T]/(P)$, $F^\dagger = qF^{-1}$ is a ring involution, and*

$$\text{Tr}(xx^\dagger) = \sum_i x(\alpha_i) x(q/\alpha_i).$$

(a) $\text{Tr}(xx^\dagger)$ is positive definite $\iff$ all $|\alpha_i| = \sqrt{q}$. (b) Its number of negative directions equals $\kappa =$ the number of reciprocal pairs $\{\alpha, q/\alpha\}$ with $|\alpha| \neq \sqrt{q}$ (a complex off-circle quartet $\{\alpha, \bar{\alpha}, q/\alpha, q/\bar{\alpha}\}$ counts as two pairs). (c) Writing $K_0 = \mathbb{Q}[F + V]$ (degree g) and $w = F - V$: $A = K_0 \oplus K_0 w$, $\dagger = \pm 1$ on the two summands, the trace form is block diagonal, $2\text{Tr}_{K_0}(u^2)$ on K_0 and $-2\text{Tr}_{K_0}(u^2 w^2)$ on $K_0 w$. Positivity $\iff K_0$ totally real and $w^2 = (F + V)^2 - 4q$ totally negative $\iff A$ is a product of CM fields with $\dagger =$ complex conjugation. (d) (Weil, as in Mumford §21.) If P is the Weil polynomial

of a curve $C/\mathbb{F}_q$, then $\mathbb{Q}[F] \subset \text{End}^0(J_C)$ and $\dagger$ is the restriction of the Rosati involution of the theta polarization, which is positive (Thm. 2.4). Hence all $|\alpha_i| = \sqrt{q}$: RH for C .

Proof. $A \otimes \mathbb{R} = \prod_v A_v$ over the real roots and the conjugate pairs of complex roots. Since $\dagger$ permutes the roots by $\alpha \mapsto q/\alpha$, the form splits into orbit blocks. On-circle complex pair ($q/\alpha = \bar{\alpha}$): $2|z|^2$, signature $(2, 0)$. Real root $\alpha = \pm\sqrt{q}$: $x(\alpha)^2$, $(1, 0)$. Real off-circle pair $\alpha \neq q/\alpha$ real: $2uv$ on $\mathbb{R}^2$, $(1, 1)$. Complex off-circle quartet: $2\text{Re}(zw')$ on $\mathbb{C}^2$, $(2, 2)$. Rationality: a rational quadratic form is positive definite iff it is so over $\mathbb{R}$. (c) follows from $\dagger(F \pm V) = \pm(F \pm V)$ and $\text{Tr}_A = 2\text{Tr}_{K_0}$ on K_0 , $\text{Tr}_A(K_0 w) = 0$. (d) is the cited argument. $\square$ $\square$

3.1. Numerics: a real genus-2 curve and fakes [NUMERICAL]. Script `num/lemma84_albert.py` (exact integer arithmetic, signatures by Descartes' rule on the characteristic polynomial of the integer Gram matrix $G_{ij} = \text{Tr}(F^i V^j) = q^{\min(i,j)} s_{|i-j|}$, s_k the power sums). The point counts over $\mathbb{F}_7$ and $\mathbb{F}_{49}$ are done directly (with $\mathbb{F}_{49} = \mathbb{F}_7[u]/(u^2 - 3)$) and N_1 is re-checked by brute force; $\#J(\mathbb{F}_7) = P(1) = 28$.

case	(c_1, c_2)	roots a_i of $x^2 + c_1 x + c_2 - 2q$	κ	sig $\text{Tr}(xx^\dagger)$	even odd
$y^2 = x^5 + x^4 + x/\mathbb{F}_7$	$(-4, 10)$	$2 \pm 2\sqrt{2}$	0	$(4, 0)$	$(2, 0) (2, 0)$
both $ a_i > 2\sqrt{7}$	$(0, -20)$	± 5.83	2	$(2, 2)$	$(2, 0) (0, 2)$
two real pairs	$(1, -28)$	6, -7	2	$(2, 2)$	$(2, 0) (0, 2)$
one real pair	$(-8, 26)$	6, 2	1	$(3, 1)$	$(2, 0) (1, 1)$
complex quartet	$(1, 30)$	$-\frac{1}{2} \pm 3.97i$	2	$(2, 2)$	$(1, 1) (1, 1)$
boundary (not squarefree)	$(0, -14)$	$\pm 2\sqrt{7}$	–	$(2, 0, 2)$	$(2, 0) (0, 0, 2)$

Here $P = T^4 + c_1 T^3 + c_2 T^2 + 7c_1 T + 49$. The block decomposition of (c) agrees with the root-side computation to $2 \cdot 10^{-12}$. These reproduce the parent's floating-point check `/tmp/g2.py` $((4, 0), (3, 1), (2, 2), (2, 2))$ with exact integer polynomials and a real curve.

Exhaustive scan. For $q \in \{2, 3, 5, 7, 11\}$, all integer (c_1, c_2) with $|c_1| \leq 4\lfloor\sqrt{q}\rfloor + 6$, $|c_2| \leq 6q + 10$ and P squarefree: 13 135 polynomials, of which 785 satisfy RH. Mismatches between “ $\text{Tr}(xx^\dagger)$ positive definite” and “all $|\alpha| = \sqrt{q}$ ”: 0. The count of negative directions equals κ in all 13 135 cases; signatures occurring: $(4, 0)$ $[\kappa = 0]$ 785, $(3, 1)$ $[\kappa = 1]$ 8518, $(2, 2)$ $[\kappa = 2]$ 3832. The closed form of the RH region, $|c_1| \leq 4\sqrt{q}$ and $2\sqrt{q}|c_1| - 2q \leq c_2 \leq c_1^2/4 + 2q$, also matches with 0 mismatches (`lemma84_t349.py`).

Diagnostic for J . This is the finite-dimensional prototype of the Krein index κ of Lemmas 77/81: a positivity whose failure is *counted* by ghost pairs, one negative direction per reciprocal pair. A candidate J for ζ should reproduce exactly this count on DH (Lemma 82: two negative directions per DH quartet on real test functions, i.e. one per reciprocal pair in each of the two conjugate halves).

4. CORRECTIONS TO THE MÖBIUS TEXT PASTED AT [T348]

Summary of the text. Möbius bundle on S^1 ; fibre-linear functions $\psi(\theta)y$ force $\psi(\theta + 2\pi) = -\psi(\theta)$, so $\psi = \sum c_n e^{i(n+\frac{1}{2})\theta}$; the “involution” $\psi^*(\theta) = \psi(\theta + \pi)$ satisfies $\star^2 = -1$ and $e^{i(n+\frac{1}{2})\theta} \mapsto (-1)^n e^{i(n+\frac{1}{2})\theta}$; claims “ n even positive-definite, n odd negative-definite”; a table base circle $\rightarrow \mathbf{1} \oplus \mathbb{L}$, fibres $\rightarrow H^1 \otimes H^1$, antipodal map $\rightarrow$ Rosati, even/odd modes $\rightarrow$ positive/negative eigenspaces; claims this forces the complement negative-definite and hence RH; offers an S -matrix bounded in $[0, 1]$. Numerics: `num/lemma84_mobius.py` $\rightarrow$ `num/184/mobius.json` ($M = 16$, i.e. 32 modes).

M1 *Setup, antiperiodicity*, $\star^2 = -1$, *eigenvalues* $(-1)^n i$: correct [THEOREM]. (Defects $7 \cdot 10^{-15}$ [NUMERICAL].)

M2 $\star$ is a complex structure, not an involution. An involution has order 2; $\star$ has order 4 ($\star^2 = -1$). On real antiperiodic functions, in the basis $\cos(n + \frac{1}{2})\theta, \sin(n + \frac{1}{2})\theta$, $\star$ is orthogonal and antisymmetric (defects $2 \cdot 10^{-15}, 4 \cdot 10^{-15}$), i.e. a rotation by $\frac{\pi}{2}$ in each 2-plane. Compare

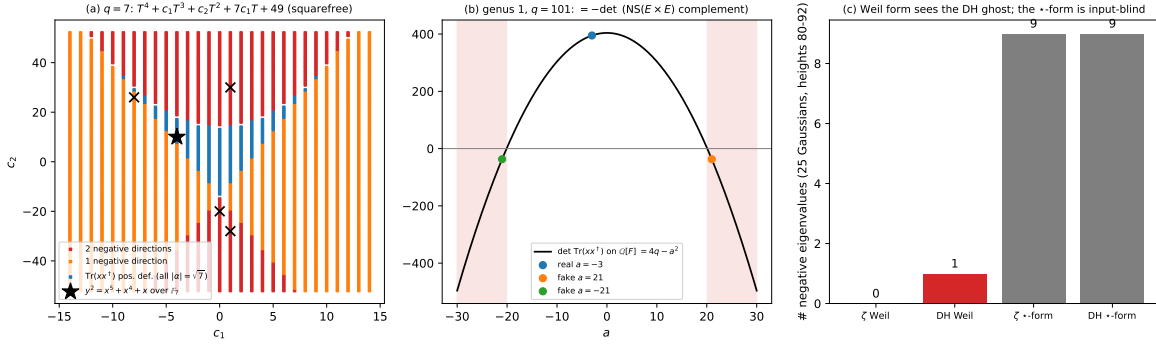


FIGURE 1. (a) $q = 7$: sign pattern of $\text{Tr}(xx^\dagger)$ on $\mathbb{Q}[F]$ over the (c_1, c_2) plane; blue = positive definite = RH region; star: the real curve; crosses: the fakes of the table. (b) Genus 1, $q = 101$: $\det \text{Tr}(xx^\dagger) = 4q - a^2 = -\det$ of the $\text{NS}(E \times E)$ complement; the real curve $a = -3$ and the fakes $a = \pm 21$. (c) Diagnostic of §4: on 25 Gaussian test functions (heights 80–92) Weil’s form has 0 negative eigenvalues for ζ and 1 for DH, while the $\star$ -form of [t348] has 9 negative eigenvalues for both (identical matrices).

Thm. 3.1: the *algebraic* Möbius split uses a genuine involution $\dagger$, and its odd line carries an arithmetic sign.

- M3** L^2 is positive on all modes [THEOREM]. The Gram matrix is $2\pi I$ (eigenvalues 6.2832 to 10^{-14}). No mode is “negative-definite” for the global inner product.
- M4** Where the signs actually come from [THEOREM]. They appear only in $h(x, y) = \langle x, -i \star y \rangle$: $-i \star$ is self-adjoint with eigenvalues $(-1)^n$, so h has signature (M, M) , indefinite *by construction* ((16, 16) [NUMERICAL]). The conjugate of mode n is mode $-n - 1$, of opposite parity, so the “even” sector is not defined over $\mathbb{R}$ (let alone $\mathbb{Q}$): on real x , $\langle x, \star x \rangle = 0$ ($9 \cdot 10^{-15}$) and h vanishes. The honest structure is a compatible triple $(g, \star, \omega)$, $g = \langle \cdot, \cdot \rangle > 0$, $\omega(x, y) = \langle x, \star y \rangle$ alternating, $g(x, y) = -\omega(x, \star y)$ (defect 10^{-14}): a weight-one “Hodge structure” whose polarization is *positive because g was chosen positive*. The $\pm i$ eigenspaces play the role of $H^{1,0}, H^{0,1}$ and the sign alternation is the i^{p-q} of Hodge–Riemann, automatically satisfied. No arithmetic is involved.
- M5** Lemma 80 applies. $\star$ commutes with every rotation ($\|[\star, R_\phi]\| = 0$): it is the Fourier multiplier $e^{i\pi\xi}$ on a single copy. By Lemma 80’s no-go, a multiplier with no zero input yields only blind or restated positivity.
- M6** Internal contradiction. The text says half of the modes are positive and half negative, then that “the entire $H^1 \otimes H^1$ complement” is negative-definite. Both cannot hold, and neither concerns NS.
- M7** The table is metaphor [ANALOGY]. The base circle is not $\mathbf{1} \oplus \mathbb{L}$; the pole plane $\langle H, V \rangle$ of $C \times C$ is a hyperbolic plane U of signature $(1, 1)$; the antipodal map is not a Rosati involution ($\dagger^2 = 1$, $\star^2 = -1$); the fibres are real lines, not $H^1 \otimes H^1$.
- M8** Cannot force RH. No zeros, primes or L -function enter the construction. Diagnostic [NUMERICAL]: on the 25 Gaussian test functions of Lemma 81 (centres 80, 80.5, $\dots$, 92), Weil’s zero-side Gram has 0 negative eigenvalues for ζ and 1 (-1.0775) for DH; the $\star$ -form Gram on the same functions sampled at half-integer frequencies has signature $(9, 9)$ for both, and the two matrices are identical (difference 0). Fig. 1(c).
- M9** $|S| \leq 1$ is unitarity (Lemmas 78, 83): it holds for DH-type scattering as well, so it is RH-blind. “Bounded in $[0, 1]$ ” is a statement about $|S|^2$ or transmission probabilities, again unitarity.

5. CORRECTIONS TO THE SECOND TEXT PASTED AT [T349]

Relayed by the parent agent (verbatim text not seen by us); each item checked.

- N1** “*Roots on $|t| = q$.*” No: $|\alpha| = \sqrt{q}$ for H^1 (weight 1); $|\alpha| = q$ is weight 2.
- N2** “*Ampleness makes the form positive on the base scaffold.*” No: the pole plane $\langle H, V \rangle \subset \text{NS}(C \times C)$ has signature $(1, 1)$. Ampleness gives $(H + V)^2 = 2 > 0$ and, by Hodge index, *negativity* on the complement of $\langle H, V \rangle$.
- N3** “*A genus-2 fake always has signature $(+, +, -, -)$ ” / “*a single negative eigenvalue*”. Both wrong as universal claims. The number of negatives is κ , one per off-circle reciprocal pair: $(3, 1)$ for one real pair (8518 of the 12350 fakes in our scan), $(2, 2)$ for two real pairs or a complex quartet (3832).*
- N4** “*Fake iff $a_1^2 - 4a_2 + 8q < 0$.*” This is $\text{disc}(x^2 + a_1x + a_2 - 2q) < 0$, i.e. the roots a_i of the real Weil polynomial are non-real: the complex-quartet case only. It is sufficient for failure of RH (2614 of 2614 cases) but not necessary: 9736 fakes have $\text{disc} \geq 0$ (real a_i with some $|a_i| > 2\sqrt{q}$). The correct RH region is $|a_1| \leq 4\sqrt{q}$, $2\sqrt{q}|a_1| - 2q \leq a_2 \leq a_1^2/4 + 2q$ (0 mismatches, [NUMERICAL]).
- N5** “*Atkin–Lehner is the $\dagger$ for Hecke.*” At level 1 there is no Atkin–Lehner operator; the Petersson adjoint makes every T_n self-adjoint, so $\dagger = \text{id}$ and the Hecke algebra is of type I (§6). At level N , W_N is an involution of $S_k(\Gamma_0(N))$ and conjugates T_p^* to T_p for $p \mid N$. In any case the zeros of ζ are not Hecke eigenvalues.
- N6** “*Weil’s involution is the Fourier transform.*” No: Weil’s involution is $f \mapsto f^*(x) = \overline{f(x^{-1})}$ on the multiplicative group (with f^* in the convolution algebra); Fourier transform enters via the functional equation, not as the involution.
- N7** “*Schwartz functions give the $\mathbb{Q}$ -form.*” No: Schwartz space has no natural $\mathbb{Q}$ -structure compatible with Weil’s form; finding one is the $\mathbb{Q}$ -structure problem (§6).
- N8** “*Arakelov intersection lives at the smooth/archimedean places.*” No: it is the sum of the finite-fibre intersection numbers (weighted by $\log \#$ residue fields) plus archimedean contributions from Green’s functions.
- N9** “*Faltings–Hriljac on $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$.*” No: it is a theorem for arithmetic surfaces, i.e. curves over $\text{Spec } \mathbb{Z}$ (or $\mathcal{O}_K$). $\text{Spec } \mathbb{Z} \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z} = \text{Spec } \mathbb{Z}$; the needed product is over a hypothetical base below $\mathbb{Z}$ (§7).

6. THE $\mathbb{Q}$ -STRUCTURE FOR $\text{Spec } \mathbb{Z}$: CANDIDATES

For a curve, Weil’s proof needs three things: a $\mathbb{Q}$ -algebra ($\mathbb{Q}[F] \subset \text{End}^0(J)$), an involution ($\dagger$, from duality), and positivity (from an ample class). Each candidate below is labelled.

- (a) **Hecke algebras with Petersson adjointness** [THEOREM] (standard; [BIB] Diamond–Shurman, Iwaniec). On $S_k(\text{SL}_2(\mathbb{Z}))$ the T_n commute and are self-adjoint for the Petersson product, and they preserve the lattice of forms with integral q -expansion. So $\mathbf{T} \otimes \mathbb{Q}$ is a product of totally real fields with $\dagger = \text{id}$: *Albert type I, for free*. The Ramanujan bound is a *type IV* statement about a different algebra: for each p , $\mathbb{Q}[X]/(X^2 - a_pX + p^{k-1})$ with $X^\dagger = p^{k-1}/X$ is CM with positive $\dagger$ iff $|a_p| < 2p^{(k-1)/2}$ (Thm. 3.1 with $q = p^{k-1}$). That is Deligne’s theorem (1974, via the Weil conjectures for Kuga–Sato varieties, [BIB]): the positivity comes from geometry, not from Petersson.
 [NUMERICAL] (num/lemma84_hecke.py). S_{24} , basis $\Delta E_4^3, \Delta^2$: $T_2 = (\frac{696}{20736000} \frac{1}{384})$, $\chi(T_2) = x^2 - 1080x - 20468736$, $\text{disc} = 2^6 3^2 144169$: Hecke field $\mathbb{Q}(\sqrt{144169})$, totally real. Trace form on $\mathbb{Q}[T_2]$: $(\frac{2}{1080} \frac{1080}{42103872})$, $\det = 83041344 > 0$ (type I positive). All T_p , $p \leq 97$, commute with T_2 and have real eigenvalues; $\max_p |a_p|/2p^{23/2} = 0.880$ (at $p = 2$). Diagnostic for J : Hecke compatibility gives total reality for free but not the Ramanujan/RH-type sign. A J built from Hecke self-adjointness alone would be blind in exactly this way.
- (b) **Bost–Connes** [BIB] (Bost–Connes 1995; Connes–Marcolli). The C^* -dynamical system $(C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times, \sigma_t)$ has partition function $\zeta(\beta)$, a phase transition at $\beta = 1$, and an arithmetic

$\mathbb{Q}$ -subalgebra whose values on extremal KMS_∞ states generate $\mathbb{Q}^{\text{ab}}$, with $\text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\times$ acting by symmetries. It provides a $\mathbb{Q}$ -structure with a Galois action, but ζ appears as a partition function (poles and the Euler product), not through its zeros, and KMS positivity ($\varphi(x^*x) \geq 0$) holds unconditionally: blind.

- (c) **Weil’s form on rational test functions** [RESTATEMENT]/[OPEN]. Weil positivity $W(f * f^*) \geq 0$ with $f^*(x) = \overline{f(x^{-1})}$ is equivalent to RH ([RESTATEMENT]). Restricting to a $\mathbb{Q}$ -span (e.g. $\mathbb{Q}$ -valued step functions on $\mathbb{R}_+^\times$ with rational break points, or functions with rational Mellin data) is possible, but it gives no finite-dimensional algebra, no trace, and no reason for positivity. It reformulates; it does not supply a polarization.
- (d) **Why no finite-dimensional $\mathbb{Q}[F]$** [THEOREM]/[OPEN]. If ζ ’s zeros were the eigenvalues of F on a finite-dimensional $\mathbb{Q}$ -space, ζ would have finitely many zeros ([THEOREM]: it has infinitely many). Even for an infinite-dimensional $\mathbb{Q}$ -structure on which a “Frobenius” acts with spectrum $\{\rho\}$, one would need algebraicity properties of the γ ’s; whether any γ is algebraic, or the γ ’s are $\mathbb{Q}$ -linearly independent, is unknown [OPEN].
- (e) **An infinite-dimensional Albert substitute** [OPEN]. One would need a $*$ -algebra $\mathcal{A}$ over $\mathbb{Q}$ (or a C^* -algebra with a $\mathbb{Q}$ -form), an affiliated scaling/Frobenius, and a *trace or state* τ with $\tau(xx^*) \geq 0$, such that τ restricted to the algebra generated by test functions is Weil’s distribution. Positivity of τ is RH. The substitute is non-circular only if τ arises from a geometric positivity (an ample class) and not by definition. Connes’s trace formula supplies τ as a distribution, not as a state.

7. THE AMPLE CLASS

Theorem 7.1 (Faltings–Hriljac: arithmetic Hodge index [CITED-sec]). *Let $\mathcal{X} \rightarrow \text{Spec } \mathbb{Z}$ be a regular arithmetic surface with generic fibre a curve $X/\mathbb{Q}$ of genus ≥ 1 . Arakelov intersection $(\cdot, \cdot)_{\text{Ar}}$ is the sum of finite-fibre intersection numbers (weighted by $\log \#\kappa(p)$) plus archimedean terms from Green’s functions on $X(\mathbb{C})$ (Arakelov 1974). For a degree-0 divisor D on X , let $\hat{D}$ be its extension, corrected by vertical components to be orthogonal to all fibre components and with harmonic archimedean part. Then $(\hat{D}, \hat{D})_{\text{Ar}} = -2\hat{h}_{NT}([D])$ (Faltings 1984; Hriljac 1985; normalization of $\hat{h}$ as in those papers, constant not re-checked). Hence $(\cdot, \cdot)_{\text{Ar}}$ is negative semi-definite on this complement, and negative definite modulo torsion: an arithmetic Hodge index theorem.*

What it gives. Positivity of the Néron–Tate height on $J(\mathbb{Q}) \otimes \mathbb{R}$ (heights of rational points), the arithmetic counterpart of the Hodge index theorem on a surface *fibred over* $\text{Spec } \mathbb{Z}$. It does not give RH: the surface is a curve over $\mathbb{Z}$, not a square of $\text{Spec } \mathbb{Z}$, and its H^1 is that of X , not of ζ ([THEOREM]).

[NUMERICAL] (num/lemma84_height.py, Tate’s limit $\hat{h} = \lim 4^{-n}h(x(2^n P))$, exact rationals to $n = 9$). 37a1 $y^2 + y = x^3 - x$, $P = (0, 0)$: $\hat{h} = 0.0511114082$ (LMFDB regulator 0.0511114082, recalled; agreement 10^{-9}). 389a1 $y^2 + y = x^3 + x^2 - 2x$, $P = (0, 0)$, $Q = (1, 0)$: Gram $\begin{pmatrix} 0.326999 & 0.058523 \\ 0.058523 & 0.476712 \end{pmatrix}$, positive definite, $\det = 0.1524594$ (LMFDB regulator 0.1524602, agreement $8 \cdot 10^{-7}$, limited by $n = 9$). Diagnostic for J : this is the only proved *arithmetic* Hodge index, and its positivity is unconditional, so it cannot distinguish ζ from DH. It tells J what an ample-type positivity over $\mathbb{Z}$ looks like: finite places and archimedean Green’s functions both enter.

Connes–Consani [CITED]. *Riemann–Roch for $\overline{\text{Spec } \mathbb{Z}}$* , Bull. Sci. Math. 187 (2023) 103293 (arXiv:2205.01391): for Arakelov divisors D , $\dim_{\mathbb{S}[\pm 1]} H^0(D) - \dim_{\mathbb{S}[\pm 1]} H^1(D)$ equals a modified $\deg D + \log 2$ (their Thm. 1.1), with integer-valued dimensions over the sphere spectrum and Serre duality. *The scaling site*, C. R. Math. 354 (2016) 1–6, and *Geometry of the scaling site*, Selecta Math. 23 (2017) 1803–1850: the topos $[0, \infty) \times \mathbb{N}^\times$ with a tropical structure sheaf, obtained from the arithmetic site by extension of scalars to $\mathbb{R}_+^{\text{max}}$. These provide Riemann–Roch-type statements (the H^0/H^1 half of Weil’s proof), but no surface $\overline{\text{Spec } \mathbb{Z}} \times \overline{\text{Spec } \mathbb{Z}}$ with an intersection pairing and an ample class has been constructed ([OPEN]).

What “ample on $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ ” would have to mean [OPEN]. A “surface” S over a base below $\mathbb{Z}$ with: classes H, V (fibres) and diagonal Δ with $(H \cdot V) = 1$; correspondences Γ_f attached to test functions f (graphs of scaling); an ample class η with $\eta^2 > 0$; and a Hodge index theorem on $\langle H, V \rangle^\perp$. Then $-(\Gamma_f^\perp \cdot \Gamma_f^\perp) \geq 0$ would be Weil positivity, i.e. RH ([RESTATEMENT] for the implication; [OPEN] for existence).

ingredient of Weil’s proof	status for ζ	label
$\mathbb{Q}$ -algebra containing F	none finite-dimensional; Hecke (type I), Bost–Connes ($\mathbb{Q}^{\text{ab}}$) partial	[OPEN]
involution $\dagger$	functional equation $s \leftrightarrow 1 - s$, Weil’s f^*	exists [THEOREM]
Riemann–Roch	Connes–Consani 2023	exists [CITED]
intersection pairing on a square ample-type positivity	none	[OPEN]
	Arakelov/Faltings–Hriljac, but over curves over $\mathbb{Z}$	exists, wrong object [CITED-sec]
Hodge index on complement of $\langle H, V \rangle$	$\equiv$ Weil positivity	[RESTATEMENT]
Krein count κ of failures	works for DH (Lemmas 77, 82) and fakes here	[NUMERICAL]/[THEOREM]

8. CORRECTIONS

- C1** *Brief, (2): “odd line $F - V$ ” in genus g .* The even/odd split $\mathbb{Q} \cdot 1 \oplus \mathbb{Q}(F - V)$ holds in genus 1. For $g \geq 2$ the even part is $K_0 = \mathbb{Q}[F + V]$ (dimension g) and the odd part $K_0(F - V)$; the odd element squares to a *totally negative element* of K_0 , not a rational, iff CM (Thm. 3.2(c)).
- C2** *“The trace form goes indefinite exactly when RH fails”:* confirmed for squarefree P (theorem plus 13 135 exact cases). At the boundary (repeated roots such as $a_i = \pm 2\sqrt{q}$) $\mathbb{Q}[T]/(P)$ is not reduced and the form is only semidefinite: (2, 0, 2) in the table.
- C3** *Rosati positivity “from ampleness”:* correct, but the involution needs the polarization. $F^\dagger = V$ uses the principal (theta) polarization; another polarization changes $\dagger$ by conjugation, not its positivity.
- C4** *Counting:* the correct invariant is $\kappa = \text{number of off-circle reciprocal pairs}$, not “one negative eigenvalue per fake” (§5, N3).
- C5** *Hecke algebra as the $\mathbb{Q}$ -structure:* type I positivity is free and therefore cannot carry RH-type information; the RH-type (Ramanujan) sign lives on the local type IV algebras and required Deligne.

9. DIRECTIONS (RANKED BY $0.6s + 0.4p$)

direction	s	p	score
K3 Function-field ample check: for genus-2 curves and fakes, build $\text{NS}(C \times C)$ -type lattices explicitly $(H, V, \Delta, \Gamma_{F^k})$ and show that a fake polynomial is exactly a Hodge-index violation with κ negative directions.	.50	.85	.64
K2 Hecke-compatible J: test whether a J split into type I (Hecke) and type IV (local Satake) parts can be defined on the Lemma 81 sectors; numerics on Satake algebras.	.55	.75	.63
K1 Infinite-dimensional Albert substitute: a $\mathbb{Q}$ -form of the scaling/Weil algebra with a geometric positive state.	.90	.20	.62
K4 A square of the scaling site: does Connes–Consani’s framework admit a product with an intersection pairing and an ample class?	.85	.15	.57
K5 Bost–Connes $\mathbb{Q}$-structure and Weil’s form: restrict Weil’s distribution to the arithmetic subalgebra and test signatures on DH-type twists.	.60	.30	.48

10. QUESTIONS FOR J.W.

- Q1** Accept Thm. 3.2: positivity of $\mathrm{Tr}(xx^\dagger)$ on $\mathbb{Q}[F]$ is RH for P , and failure is counted by κ (one negative direction per ghost pair). Is this the algebraic form of the $E \times E$ Gram check you wanted?
- Q2** Accept the [t348] corrections ($\star$ is a complex structure; the sign split comes from an indefinite form built by hand; Lemma 80 applies; nothing about zeros enters) and the [t349] corrections?
- Q3** Which $\mathbb{Q}$ -structure should we pursue: Hecke (type I free, type IV = Deligne), Bost–Connes, or Weil’s form on rational test functions?
- Q4** Accept that Faltings–Hriljac is height positivity for curves over $\mathrm{Spec} \mathbb{Z}$ and not RH, and that “ample on $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$ ” is the open target?
- Q5** Next: K3 (function-field ample check, most likely to succeed) or K2 (Hecke-compatible J)?

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Code and data. num/lemma84_{albert,t349,mobius,hecke,height,figs}.py $\rightarrow$ num/l84/{albert,t349,mobius,hecke,height}.json;
figure build84/figs/lemma84_albert.pdf. The $q = 101$ reference script /tmp/mob.py is copied to num/l84/mob_reference_copy.py.

TAA LEMMA 85: MORDELL–WEIL LATTICES, THE AMPLE CLASS, AND BSD. WHERE THE RATIONAL LATTICE EXISTS

J. W. MCGREEVY; FORMALIZATION BY GROK BOT

ABSTRACT. **Statement of intellectual honesty.** *The Riemann hypothesis (RH) is not proven*, here or anywhere in TAA; neither is BSD. Tags as in Lemmas 80–84: [THEOREM] proved here or standard with proof sketched, [CITED]/[CITED-sec] checked source / checked secondary source, [BIB] recalled, not opened, [TAA-L] earlier TAA lemma, [RESTATEMENT] equivalent to RH, [ANALOGY] analogy, [CONJECTURE] conjecture, [OPEN] open, [NUMERICAL] computation, [HEURISTIC] heuristic. Every citation is listed in `lit/lemma85/verified_refs.md`. Every L -value is computed here (PARI 2.17.2); none is recalled.

Task (J.W., approved [t352u]; hunch [t351u]). The $\mathbb{Q}$ -algebra object around $h^1 \otimes h^1$, the ample class and the Möbius line bundle are related to BSD, to the EDSAC $\prod N_p/p$ data, and to Mordell’s finite generation (how many generators of infinite order). (A) Shioda–Tate and Artin–Tate. (B) An explicit elliptic surface over $\mathbb{F}_q$: Frobenius eigenvalues, count of eigenvalue q versus Shioda–Tate, Mordell–Weil Gram matrix, ghost tests linked to κ . (C) EDSAC plots for ranks 0–3 with a ghost variant. (D) Unit theorem and class number formula as BSD for $\mathbb{G}_m$. (E) Möbius and parity. (F) Synthesis. (G) Chord–tangent on the Möbius circle (claims [t353s1478]). (H) Euler products and spinor characters (claims [t354–t355]). (I) Modularity and the divergent region (claims [t357s1491]). Corrections to the parent’s chat claims [t351s1471–1473].

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GLOSSARY (RESTATED WHERE USED)

- *Cohomology* H^i of a variety over $\bar{\mathbb{F}}_q$: ℓ -adic vector spaces with a Frobenius action; for a surface S , $H^2(\bar{S}, \mathbb{Q}_\ell)$ has dimension b_2 . $h^1(C)$ is the motive of a curve, a categorical object whose realizations are H^1 's; $h^1 \otimes h^1 \subset h^2(C \times C)$.
- *Néron–Severi group* $\text{NS}(S)$: divisors modulo algebraic equivalence, a finitely generated abelian group with the integral, symmetric intersection form. Its rank ρ is the Picard number.
- *Hodge index theorem*: on $\text{NS}(S) \otimes \mathbb{R}$ the intersection form has signature $(1, \rho - 1)$; it is negative definite on the orthogonal complement of any class of positive square (e.g. an ample class).
- *Elliptic surface* $\pi : S \rightarrow C$ over k with zero section O ; generic fibre E/K , $K = k(C)$; m_v = number of components of the fibre F_v ; *trivial lattice* $T = \langle O, F, \text{non-identity fibre components} \rangle \subset \text{NS}(S)$.

1. ANSWER IN BRIEF

- (1) **Where the rational lattice exists** [THEOREM]/[CITED-sec]. Over a function field $K = \mathbb{F}_q(C)$ the Mordell–Weil group is a lattice inside $\text{NS}(S)$ (Shioda), with $\mathbb{Q}$ -valued heights, positive by Hodge index; its rank is the multiplicity of the Frobenius eigenvalue q (Tate’s conjecture, equivalent to BSD for E/K).
- (2) **Explicit surface** [NUMERICAL]. $S : y^2 = x^3 - x + (t^3 + 2t^2 - t + 1)^2$ over $\mathbb{F}_5$ (12 fibres I_1). From point counts over $\mathbb{F}_{5^k}$, $k \leq 8$: $L(E/K, T) = (1 - 5T)^4(1 + 5T)^2(1 + 25T^2)$, satisfying the functional equation exactly. Eigenvalue q has multiplicity 4 on H^1 and 6 on H^2 . 14 integral $\mathbb{F}_5$ -sections span a Mordell–Weil lattice of rank 4 with Gram matrix $A_3 \oplus A_1$ (det 8); the NS Gram on $O, F, P_1, \dots, P_4$ has signature $(1, 5)$. Shioda–Tate: $6 = 2 + 0 + 4$, so the eigenvalue count matches the rank. K3 family $y^2 = x^3 + \alpha t^4 x + t^5(t^2 + ct + 1)$ over $\mathbb{F}_7$ (33 surfaces): $\varepsilon = -1$ in 18 cases, each with rank $E(\mathbb{F}_7(t)) = 1$.
- (3) **Ghosts** [NUMERICAL]. Fake Weil-shaped L ’s break different counters. A ghost at q violates Tate’s inequality against the explicit lattice; a ghost at $q\zeta$ breaks $\bar{\rho} = b_2$; an off-circle pair raises κ (Lemma 84). An on-circle non-root-of-unity fake is invisible to κ and caught only by the algebraic count.
- (4) **EDSAC** [NUMERICAL]. $\prod_{p \leq x} N_p/p$ to $x = 10^8$ for 11a1, 37a1, 389a1, 5077a1. The fitted exponents are 0.004, 0.990, 2.020, 2.958 (fits over $[10^4, 10^8]$). The ratio to Goldfeld–Conrad’s $\sqrt{2}e^{\gamma}r!/L^{(r)}(E, 1)(\log x)^r$ at 10^8 is 0.972, 0.999, 1.032, 0.953. A ghost quartet ($\delta = 0.25$) produces residual oscillations growing like $x^\delta/\log x$, to ± 1.6 versus ± 0.15 – 0.35 ; the same picture holds for $\psi(x) - x$.
- (5) $\mathbb{G}_m$ [THEOREM]. $\zeta(0) = -\frac{1}{2} = -hR/w$ is the class number formula for $\mathbb{Q}$ (rank 0, $w = 2$); for $\mathbb{Q}(\sqrt{5})$, $\zeta'_K(0) = -0.2406059125 = -\frac{1}{2} \log \varphi$. $\mathbb{Q}$ ’s own lattice is zero. Over $\mathbb{Q}$ the heights carry archimedean Green’s-function terms, and no $\mathbb{Q}$ -structure is known.
- (6) **Parity** [THEOREM]/[ANALOGY]. $\varepsilon = -1$ forces the eigenvalue q (function fields, a theorem) and an odd order at the centre (number fields). The Möbius forced zero (Lemma 81) is the same mechanism, an odd section under an involution, at the level of the L -function; the link to an actual rational point (Heegner, Tate) is a theorem only where the Tate conjecture or Gross–Zagier–Kolyvagin applies.
- (7) **Synthesis** [THEOREM]/[OPEN]. In Weil’s proof BSD-type counting (algebraic classes) and RH (the sign on H^1) are tied by one fact: *Frobenius is algebraic*, i.e. its graph lies in $\text{NS}(C \times C)$. For $\text{Spec } \mathbb{Z}$ the count side is trivial ($\zeta(0) = -\frac{1}{2}$) and the sign side has no algebraic Frobenius. Nothing here supplies one.

- (8) **Circle** [THEOREM]/[NUMERICAL]. The chord law on $x^2 + y^2 = 1$ is multiplication on the norm-one torus $\{z\bar{z} = 1\}$ (circular points $(1 : \pm i : 0)$). Tangent = doubling with kernel ± 1 , whose sign bundle is the Möbius bundle (antiperiodic under 2π). $T(\mathbb{Q}) \cong \mu_4 \oplus \mathbb{Z}^{(\infty)}$ (generators $\pi_p/\bar{\pi}_p$): at height ≤ 600 the rank is already 51, while Mordell holds for elliptic curves. 2-descent on $y^2 = x^3 - 36x$: the real condition $E(\mathbb{R})/2E(\mathbb{R}) = \pi_0$ cuts 64 classes to 32; the image is 8. For $y^2 = x^3 - x$, $F^2/p = \pi/\bar{\pi}$, so Frobenius gives the circle’s generators.
- (9) **Spinor characters** [THEOREM]/[NUMERICAL]. $a_{p^k}/p^{k/2} = U_k(\cos \theta_p)$ (1088 exact checks, 0 failures). Sato–Tate holds to 10^{-3} at $p \leq 10^7$ for 11a1 and 37a1. Off-circle ghosts break Hasse locally. But local unitarity plus Sato–Tate (a “PNT”) gives no rate, and RH is a rate: no route from local to global positivity.
- (10) **Modularity** [THEOREM]/[NUMERICAL]. $\Lambda(E, s) = \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy$ is entire. The Fricke split formula gives $L'(1)$ for 37a1 and $L''(1)$ for 389a1 to 16 digits, where the Euler product diverges. $f|W_N = wf$, $\varepsilon = -w$ (checked to 10^{-29}), and $\varepsilon = -1$ forces $f(i/\sqrt{N}) = 0$. Split-invariance detects fakes (wrong w , changed a_n inside an $e^{-2\pi ny_1}$ window) but is RH-blind: Davenport–Heilbronn passes to 10^{-41} with an off-line zero at $0.8085 + 85.6993i$, whereas the L82 cutoff detector rejects it.

2. (A) SHIODA–TATE, MORDELL–WEIL LATTICES, TATE AND ARTIN–TATE

Let $\pi : S \rightarrow C$ be a relatively minimal elliptic surface with zero section O over a field k , not a product, generic fibre E/K , $K = k(C)$.

Theorem 2.1 (Shioda–Tate [CITED-sec]). (*Schütt–Shioda*, Elliptic surfaces, *Thm. 6.3 and Cor. 6.13.*) *The map $P \mapsto \bar{P} \bmod T$ is an isomorphism $E(K) \cong \text{NS}(S)/T$, and for k algebraically closed*

$$\rho(S) = \text{rank } T + \text{rank } E(K) = 2 + \sum_v (m_v - 1) + \text{rank } E(K).$$

Over $k = \mathbb{F}_q$, with $\text{NS}(S) := \text{NS}(\bar{S})^{\text{Gal}}$ (equal to the group of $\mathbb{F}_q$ -rational classes; $\text{Br}(\mathbb{F}_q) = 0$), the same isomorphism gives $\text{rank NS}(S) = 2 + \text{rank } T(\mathbb{F}_q) + \text{rank } E(\mathbb{F}_q(C))$, where $T(\mathbb{F}_q)$ counts Frobenius orbits of non-identity fibre components [THEOREM].

Theorem 2.2 (Mordell–Weil lattice [CITED-sec]). (*Shioda 1990; Schütt–Shioda §11, Thm. 11.5.*) *Let $\varphi : E(K) \rightarrow \text{NS}(S) \otimes \mathbb{Q}$ be the splitting map sending P to $\bar{P}$ projected orthogonally to T . Then $\langle P, Q \rangle = -\varphi(P) \cdot \varphi(Q)$ is symmetric and bilinear, positive definite on $E(K)/E(K)_{\text{tor}}$ (because $T^\perp$ is negative definite by Hodge index: T contains $O + 2\chi F$ of positive square for $\chi \geq 1$), and $\mathbb{Q}$ -valued (in $\frac{1}{m}\mathbb{Z}$). Explicitly*

$$\langle P, Q \rangle = \chi + \bar{P} \cdot \bar{O} + \bar{Q} \cdot \bar{O} - \bar{P} \cdot \bar{Q} - \sum_v \text{contr}_v(P, Q), \quad \langle P, P \rangle = 2\chi + 2\bar{P} \cdot \bar{O} - \sum_v \text{contr}_v(P),$$

with $\chi = \chi(\mathcal{O}_S)$ (1 rational, 2 K3) and $\text{contr}_v \geq 0$ from the inverse Cartan matrix of F_v . For $k = \mathbb{F}_q$, the Néron–Tate height is this pairing times $\log q$ (up to the normalization convention, a factor 1 or 2) [BIB].

Conjecture 2.3 (Tate; Tate’s inequality is a theorem [CITED-sec]). *For a smooth projective surface $S/\mathbb{F}_q$, $\text{rank NS}(S) \leq \text{multiplicity of the eigenvalue } q \text{ of Frobenius on } H^2(\bar{S}, \mathbb{Q}_\ell)$ (theorem, via the cycle class map), with equality conjectured (Tate). Known: rational surfaces (all of H^2 is algebraic, [THEOREM]), products of curves and abelian surfaces (Tate 1966), elliptic K3 surfaces with section (Artin–Swinerton-Dyer 1973), Fermat surfaces (Schütt–Shioda §14.6).*

Theorem 2.4 (BSD for $E/K \Leftrightarrow$ Tate for S [CITED-sec]). (*Tate, Bourbaki 306 (1966).*) *$L(E/K, T) = \det(1 - \text{Frob } T \mid H^1(\bar{C}, j_* R^1 \pi_* \mathbb{Q}_\ell))$ is a polynomial, $H^2(\bar{S})$ is the direct sum of the trivial-lattice part (eigenvalues q -roots of unity) and this H^1 twisted, and*

$$\text{rank } E(K) \leq \text{ord}_{T=1/q} L(E/K, T) = \text{mult. of } q \text{ on } H^1(\bar{C}, j_* R^1 \pi_*),$$

with equality $\Leftrightarrow$ Tate's conjecture for S . The eigenvalue q is a zero of $L(E, s)$ at the centre $s = 1$ ($T = q^{-s}$) and, in $\zeta(S, s) = \frac{P_1 P_3}{P_0 P_2 P_4}$, part of the pole at $s = 1$ (from P_2). The leading terms correspond: the Artin–Tate formula for S ($\# \text{Br}(S)$, $\det \text{NS}$) is equivalent to the BSD formula for E/K ($\# \text{Sha}$, regulator), Gordon 1979; Milne 1975 proved that Tate's conjecture implies the Artin–Tate formula ([CITED-sec]; Grothendieck's $\text{Br}(S) \leftrightarrow \text{Sha}(E/K)$ [BIB]).

Theorem 2.5 ($E \times E$ [THEOREM], Tate 1966 [CITED]). $\text{NS}(E \times E) = \mathbb{Z}H \oplus \mathbb{Z}V \oplus \text{End}_{\mathbb{F}_q}(E)$, so $\text{rank NS} = 2 + \text{rank End}_{\mathbb{F}_q}(E)$. On $H^2 = H^2 \otimes H^0 \oplus H^1 \otimes H^1 \oplus H^0 \otimes H^2$ the eigenvalues are q, q and $\alpha_i \alpha_j$. By Tate's isogeny theorem $\text{rank End}_{\mathbb{F}_q}(E) = \#\{(i, j) : \alpha_i \alpha_j = q\}$, which is 2 if $\alpha \neq \bar{\alpha}$ and 4 if $\alpha = \bar{\alpha} = \pm\sqrt{q}$. The complement Gram has $\det = 4q - a^2$ (Lemmas 81, 84): Hodge index on NS is RH for E .

3. (B) AN EXPLICIT ELLIPTIC SURFACE OVER $\mathbb{F}_5$, A K3 FAMILY OVER $\mathbb{F}_7$, AND GHOSTS

3.1. The rational elliptic surface. [NUMERICAL] (num/lemma85_surface.py, PARI ellcard over $\mathbb{F}_{5^k}$; library num/lemma85_lib.gp.) The surface was built to contain the sections $(0, y_0)$ and $(1, y_0)$:

$$S : y^2 = x^3 - x + B(t), \quad B = y_0^2 = t^6 - t^5 + 2t^4 - 2t^3 - 2t + 1, \quad y_0 = t^3 + 2t^2 - t + 1 \quad \text{over } \mathbb{F}_5.$$

Δ has degree 12 and is squarefree, and the fibre at ∞ is smooth, so there are 12 fibres of type I_1 (all irreducible): $T = \langle O, F \rangle$, $\rho(\bar{S}) = 10$, $E(\mathbb{F}_5(t)) \cong E_8$ (Schütt–Shioda Cor. 8.4).

$A_k = \sum_{t \in \mathbb{P}^1(\mathbb{F}_{5^k})} (5^k + 1 - \#E_t(\mathbb{F}_{5^k}))$ for $k = 1, \dots, 8$:
 $-10, -100, -250, -5000, -6250, -62500, -156250, -3125000$. With $L(T) = \exp(\sum A_k T^k / k)$:

$$L(E/K, T) = 1 - 10T + 250T^3 - 1250T^4 + 6250T^5 - 156250T^7 + 390625T^8 = (1 - 5T)^4 (1 + 5T)^2 (1 + 25T^2).$$

The functional equation $c_{8-j} = \varepsilon q^{8-2j} c_j$ holds exactly with $\varepsilon = +1$; it was *not* used in the computation.

- *Eigenvalues*: $q \cdot \{1, 1, 1, 1, -1, -1, i, -i\}$, all q times roots of unity, as Tate's theorem for rational surfaces requires ($\bar{\rho} = b_2 = 10$).
- *Count*: the multiplicity of q is 4 on $H^1(j_* R^1)$ and $2 + 4 = 6$ on H^2 .
- *Sections*: an exhaustive search of integral sections ($\deg x \leq 2$, $\deg y \leq 3$) over $\mathbb{F}_5$ finds 14. Their height Gram ($\langle P, P \rangle = 2$, $\langle P, Q \rangle = 1 - P \cdot Q$, with $P \cdot Q$ computed over $\bar{\mathbb{F}}_5$ including $t = \infty$) has rank 4 and no negative eigenvalue. A basis $P_1 = (0, y_0)$, $P_2 = (2t^2 + 2t, 2t^3 + 2t^2 - 2t + 1)$, $P_3 = (1, y_0)$, $P_4 = (2t - 2, t^3 + 2t^2 - t)$ has Gram $\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \cong A_3 \oplus A_1$, $\det = 8$.
- *Shioda–Tate = Tate*: $\text{rank NS}(S) = 2 + 0 + 4 = 6 =$ multiplicity of q on H^2 . BSD for $E/\mathbb{F}_5(t)$ (rank part) is confirmed numerically, and it is a theorem here since Tate holds for rational surfaces.
- *Hodge index*: the NS Gram on $O, F, P_1, \dots, P_4$ ($O^2 = P_i^2 = -1$, $F^2 = 0$, $O \cdot F = P_i \cdot F = 1$, $P_i \cdot O = 0$) has signature $(1, 5)$ and $\det = -8$.
- *Beyond $\mathbb{F}_5$* : the eigenvalues $-q$ (multiplicity 2) and $\pm iq$ become q^2 resp. q^4 over $\mathbb{F}_{25}$ resp. $\mathbb{F}_{625}$. By Tate, the rank is 6 over $\mathbb{F}_{25}(t)$ and 8 over $\mathbb{F}_{625}(t)$ ([THEOREM], not separately counted). The $-q$ classes are *antiperiodic* under Frobenius, and the $\pm iq$ pair is a quarter turn ($\text{Frob}^2 = -q^2$ there), cf. the $\star$ of Lemma 84. The Möbius reading is [ANALOGY]; the computation is [THEOREM].

3.2. An elliptic K3 family over $\mathbb{F}_7$. [NUMERICAL] (num/lemma85_k3.py.) $y^2 = x^3 + \alpha t^4 x + t^5(t^2 + ct + 1)$ has II^* at $t = 0, \infty$ and four I_1 (we keep the 33 pairs $(\alpha, c) \in \mathbb{F}_7^\times \times \mathbb{F}_7$ for which this holds). So $\text{rank } T = 2 + 8 + 8 = 18$, $\deg L = 22 - 2 - 16 = 4$, and L is determined by $A_1, \dots, A_4$; the functional equation holds in all 33 cases. Results:

ε	$\text{rank } E(\mathbb{F}_7(t))$	$\bar{\rho}$	# surfaces	example L
-1	1	20	18	$-(7T-1)(7T+1)(49T^2-3T+1)$
+1	0	18	9	$2401T^4 + 392T^3 + 42T^2 + 8T + 1$
+1	0	20	3	$(7T+1)^2(49T^2-10T+1)$
+1	2	20	3	$(7T-1)^2(49T^2+10T+1)$

All $|\gamma| = q$ (Deligne). Factors like $49T^2 - 3T + 1$ have γ/q not a root of unity: *transcendental* eigenvalues, where RH is a genuine constraint not supplied by NS. Rank parity equals $(1 - \varepsilon)/2$ in all 33 cases (§6). Here rank means the eigenvalue count, equal to the Mordell–Weil rank by Artin–Swinnerton-Dyer’s theorem.

3.3. Ghost tests [NUMERICAL]. `num/lemma85_ghosts.py`: the fakes keep integrality and the functional equation, and change one factor of the true L . Counters: the multiplicity of q on H^2 (Tate: $= \text{rank NS}(S)$, and ≥ 6 because the explicit lattice exists); $2 + \#\{\gamma = q\zeta\}$ (Tate: $= \bar{\rho} = b_2 = 10$); $\kappa = \#$ off-circle reciprocal pairs; and the Lemma 84 trace form $\text{Tr}(xx^\dagger)$ on $\mathbb{Q}[F]$, $F^\dagger = q^2 F^{-1}$.

change	mult. q	$2 + \#\{q\zeta\}$	κ	sig $\text{Tr}(xx^\dagger)$
true –	6	10	0	(4, 0)
G1 $(1 - 5T)^2 \rightarrow 1 - 11T + 25T^2$	4	8	1	(5, 1)
G2 $1 + 25T^2 \rightarrow 1 - 12T + 25T^2$	6	8	1	(3, 1)
G3 $1 + 25T^2 \rightarrow 1 + 2T + 25T^2$	6	8	0	(4, 0)
G4 $(1 + 5T)^2(1 + 25T^2) \rightarrow (1 - 3T + 5T^2)(1 - 15T + 125T^2)$	6	6	2	(3, 2)

Reading [THEOREM] (given the numbers). G1 puts a ghost *at* q : the multiplicity drops to 4 below the rank 6 of an explicitly constructed NS sublattice, which contradicts Tate’s (proved) inequality. The rational lattice detects ghosts at the BSD eigenvalue. G2 and G4 are off-circle: κ and the trace form detect them (one negative direction per reciprocal pair, Lemma 84), and $\bar{\rho}$ drops below b_2 . G3 is on the circle but not a root of unity: it satisfies the Weil bounds, so $\kappa = 0$ and the trace form is positive. It is detected only by the algebraic count $\bar{\rho} = b_2$, which holds for rational surfaces. *Diagnostic for J* : a positivity counter (κ) and an algebraicity counter (NS rank) catch different ghosts. For ζ only the first kind is conceivable, since there is no NS.

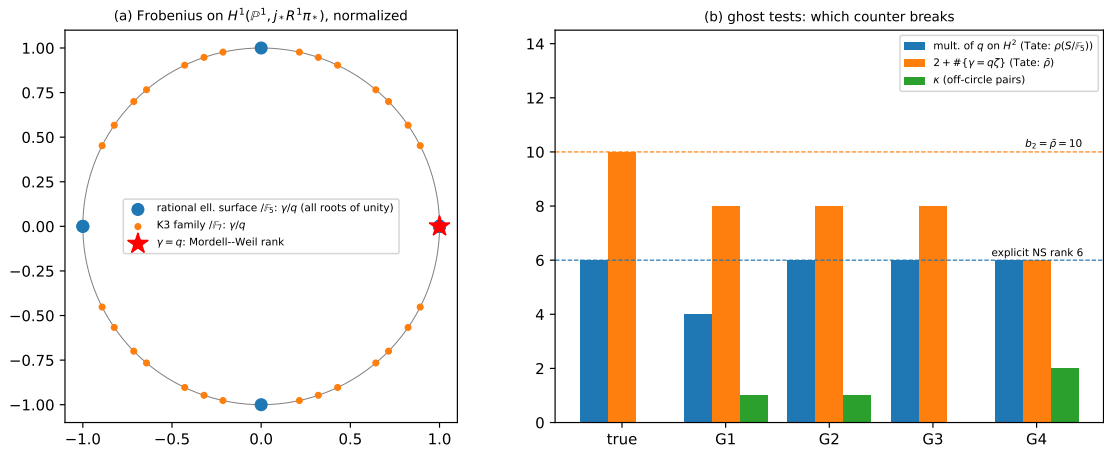


FIGURE 1. (a) Normalized Frobenius eigenvalues γ/q on $H^1(\mathbb{P}^1, j_* R^1 \pi_*)$. Blue: the rational surface over $\mathbb{F}_5$ ($\pm 1, \pm i$, all roots of unity). Orange: the 33 K3 surfaces over $\mathbb{F}_7$ (transcendental eigenvalues on the circle). Star: $\gamma = q$, the Mordell–Weil rank. (b) Ghost tests G1–G4: which counter breaks (dashed: the explicit NS rank 6 and $b_2 = 10$).

4. (C) EDSAC: $\prod_{p \leq x} N_p/p$ FOR RANKS 0–3, $\psi(x) - x$, AND A GHOST

Theorem 4.1 (Goldfeld 1982, in Conrad’s form [CITED]). (*Conrad, Canad. J. Math.* 57 (2005), Thm. 1.1 and (1.1)–(1.2).) Let $N_p = \#E_{\text{ns}}(\mathbb{F}_p)$, i.e. $p + 1 - a_p$ for $p \nmid N$ and $p - a_p$ for $p \mid N$. If $\prod_{p \leq x} N_p/p \sim A(\log x)^r$ with $A > 0$, then $L(E, s)$ satisfies RH, $r = \text{ord}_{s=1} L(E, s)$, and

$$A = \frac{\sqrt{2} e^{r\gamma} r!}{L^{(r)}(E, 1)}.$$

The hypothesis is strictly stronger than RH. Conrad calls it “much deeper”; it is the “Deep Riemann Hypothesis” of Kimura–Koyama–Kurokawa (*Lett. Math. Phys.* 104 (2014)). It is [CONJECTURE] for every $E/\mathbb{Q}$.

[NUMERICAL] (num/lemma85_edsac.{py,gp}: ellap for all $p \leq 10^8$; r and $L^{(r)}(E, 1)$ from ellanalyticrank and cross-checked against lfund derivatives, identical.)

curve	r	$L^{(r)}(E, 1)/r!$	A_{pred}	slope fit $[10^4, 10^8]$	ratio at 10^8	residual range
11a1	0	0.2538418609	5.571	0.004	0.972	$[-0.149, 0.119]$
37a1	1	0.3059997738	8.231	0.990	0.999	$[-0.145, 0.135]$
389a1	2	0.7593165003	5.908	2.020	1.032	$[-0.207, 0.214]$
5077a1	3	1.7318499001	4.614	2.958	0.953	$[-0.304, 0.353]$

Ratio = $\prod_{p \leq x} (N_p/p) / (A_{\text{pred}}(\log x)^r)$. Residual = $\log \prod N_p/p - r \log \log x - \log A_{\text{pred}}$ over $x \in [10^4, 10^8]$. Fits over $[10^6, 10^8]$: $-0.025, 0.972, 2.069, 3.038$. The data are consistent with Goldfeld–Conrad, including the $\sqrt{2}$, but at 10^8 they cannot *prove* anything ([HEURISTIC]: the approach is $O(1/\log x)$ with oscillations).

Ghost variant [THEOREM] (derivation) +[NUMERICAL]. Write $\lambda_p = a_p/\sqrt{p}$ and $\vartheta_E(t) = \sum_{p \leq t} \lambda_p \log p$. The explicit formula gives $\vartheta_E(t) \approx -\sum_{\rho} t^{\rho}/\rho$ over the zeros $\rho = \frac{1}{2} + \delta + i\gamma$ of $L(E, s + \frac{1}{2})$. Partial summation of $\log \prod N_p/p = -\sum \lambda_p/\sqrt{p} + \dots$ then shows that a zero contributes $+2 \text{Re } x^{\delta+i\gamma}/((\delta+i\gamma) \log x)$. On the line ($\delta = 0$) this is a bounded, decaying oscillation. An off-line quartet ($\pm\delta$) replacing an on-line pair at height γ adds a term growing like $x^{\delta}/\log x$. With $\delta = 0.25$, $\gamma = 6$ the residual of 37a1 grows to $[-1.45, 1.66]$ (all four curves give $[-1.5, 1.7]$), versus ≤ 0.35 for the true data (Fig. 2(b)). For ζ : moving the first zero pair to $\beta = 0.75, 0.25$ makes $\max |\psi(x) - x|/\sqrt{x}$ on $[10^4, 10^8]$ equal to 13.7, versus 0.677 for the true ψ (sieve to 10^8 ; Fig. 2(c)).

Diagnostic for J. The EDSAC product reads the *central* zero (its order r is the exponent) and the *off-line* zeros (growing oscillation) in one plot. It is RH-sensitive the way $\psi(x) - x$ is, but it carries no positivity: it is a criterion, not a polarization. A ghost is visible at finite x only when δ is not small; Lemma 82’s cutoff detection is the analogous finite statement for Weil’s form.

5. (D) Spec $\mathbb{Z}$ ’s BSD: THE UNIT THEOREM AND THE CLASS NUMBER FORMULA

Theorem 5.1 (class number formula at $s = 0$ [THEOREM], standard). For a number field K with r_1 real, r_2 complex places: $\text{ord}_{s=0} \zeta_K = r_1 + r_2 - 1 = \text{rank } \mathcal{O}_K^\times$ (Dirichlet), and $\zeta_K(s) \sim -\frac{hR}{w} s^{r_1+r_2-1}$.

[NUMERICAL] (num/lemma85_psi.py). $K = \mathbb{Q}$: $\zeta(0) = -0.5 = -hR/w$ with $h = R = 1, w = 2$, unit rank 0. $K = \mathbb{Q}(\sqrt{5})$: $\zeta_K = \zeta L(\cdot, \chi_5)$, $L(0, \chi_5) = 0$ ($< 10^{-31}$), $\zeta'_K(0) = -0.240605912529802 = -\frac{1}{2} \log \varphi$ ($h = 1, R = \log \varphi, w = 2$), and $\text{Res}_{s=1} \zeta_K = L(1, \chi_5) = 0.430408940964004 = 2^{r_1} hR/(w\sqrt{5})$.

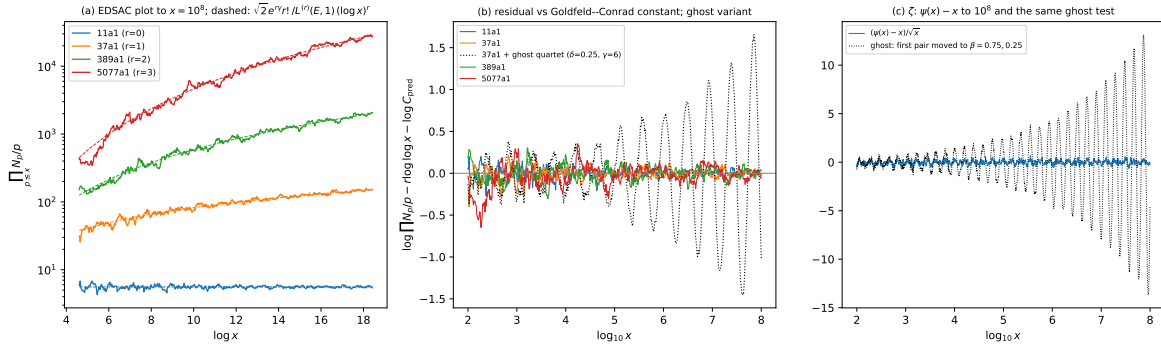


FIGURE 2. (a) $\prod_{p \leq x} N_p/p$ against $\log x$ for 11a1, 37a1, 389a1, 5077a1 to $x = 10^8$ (log scale); dashed: $A_{\text{pred}}(\log x)^r$ with Goldfeld–Conrad’s constant. (b) Residual after removing $r \log \log x + \log A_{\text{pred}}$; dotted: 37a1 with a ghost quartet ($\delta = 0.25$, $\gamma = 6$). (c) $(\psi(x) - x)/\sqrt{x}$ to 10^8 and the same with the first zero pair moved off the line.

E/K (BSD)	$\mathbb{G}_m/K$ (class number formula)	status
$E(K)$, Mordell–Weil (finitely generated)	$\mathcal{O}_K^\times$ (finitely generated, Dirichlet)	[THEOREM] both
$\text{rank} = \text{ord}_{s=1} L(E, s)$	unit rank $= \text{ord}_{s=0} \zeta_K(s)$	[CONJECTURE] / [THEOREM]
torsion $E(K)_{\text{tor}}$	roots of unity $\mu(K)$, w	[ANALOGY]
$\text{Sha}(E/K)$	class group, h	[ANALOGY] (both measure failure of a local–global principle)
regulator $\det \langle P_i, P_j \rangle_{NT}$	regulator $\det(\log \sigma_i(u_j))$	[ANALOGY]
periods, Tamagawa c_p	$2^{r_1} (2\pi)^{r_2} \sqrt{ D }$	[ANALOGY]

(The common frame is the Tamagawa number conjecture, [BIB].) $\mathbb{Q}$ ’s own lattice is trivial [THEOREM]: $\mathbb{Z}^\times = \{\pm 1\}$ has rank 0, the regulator is the empty determinant 1, and $\zeta(0) = -\frac{1}{2}$ is the whole “BSD formula”. The count side for $\text{Spec } \mathbb{Z}$ is empty. The zeros of ζ are not seen by it: they lie in H^1 , the lattice lies in H^0/H^2 (pole and unit data).

The archimedean obstruction (precise). (i) Over $K = \mathbb{F}_q(C)$ the Néron–Tate pairing is $\log q$ times $\mathbb{Q}$ -valued intersection numbers on S (Thm. 2.2), so a *rational lattice* exists [THEOREM]. (ii) Over $\mathbb{Q}$, $\hat{h} = \sum_v \lambda_v$ with λ_∞ given by archimedean Green’s functions on $E(\mathbb{C})$, and λ_p rational multiples of $\log p$. Regulators are real numbers (389a1: 0.15246, Lemma 84), and no $\mathbb{Q}$ -structure on them is known ([OPEN]; rationality of any normalization is not expected, [HEURISTIC]). (iii) Positivity survives: Faltings–Hriljac identifies $\hat{h}$ with minus the Arakelov self-intersection on the arithmetic surface (Lemma 84, [CITED-sec]). So over $\mathbb{Q}$ one has a *real* positive lattice without a known $\mathbb{Q}$ -form. The archimedean place is exactly where rationality is lost.

6. (E) MÖBIUS, PARITY, HEEGNER POINTS

Theorem 6.1 ($\varepsilon = -1$ forces the central eigenvalue [THEOREM]). (a) *Function fields*: if $L(T) = \varepsilon(qT)^d L(1/(q^2T))$ with $\varepsilon = -1$, then $L(1/q) = -L(1/q) = 0$, so q is an eigenvalue on $H^1(j_* R^1)$ with odd multiplicity. Where Tate’s conjecture is known (rational, elliptic K3), rank $E(K)$ is odd, hence ≥ 1 : a section is forced. Checked: 18 of 18 K3 surfaces with $\varepsilon = -1$ have rank 1; all 33 have rank $\equiv (1 - \varepsilon)/2$. (b) $E/\mathbb{Q}$: $\Lambda(s) = \varepsilon \Lambda(2 - s)$ gives $(-1)^{\text{ord}_{s=1} L} = \varepsilon$ (analytic rank parity). (c) Lemma 81: an antiperiodic section of the Möbius bundle has a forced zero, and $\Lambda(\frac{1}{2}) = 0$ when $\varepsilon = -1$, by the same mechanism: an odd function under an involution vanishes at a fixed point.

What is known beyond (b) [BIB] (bibliographic data checked): p^∞ -Selmer parity $\equiv$ root number for all $E/\mathbb{Q}$ and all p (Dokchitser–Dokchitser 2010; earlier cases and totally real fields by

Nekovář). If $\text{ord}_{s=1} L = 1$ then $\text{rank } E(\mathbb{Q}) = 1$ and Sha is finite: Gross–Zagier (1986) build the Heegner point, Kolyvagin (1990) bounds the Selmer group, modularity supplies L ([BIB] for the auxiliary non-vanishing theorems). Parity of $\text{rank } E(\mathbb{Q})$ itself, and any statement for $\text{ord} \geq 3$, is open ([OPEN]; 5077a1 has three explicit generators, but no theorem forces them).

Analogy levels. [THEOREM]: (a), (b), (c) as stated. [THEOREM]: in function fields, “forced zero $\Rightarrow$ forced section” holds where Tate is known. [ANALOGY]: “Möbius forced zero $\leftrightarrow$ Heegner point”. The Heegner point is a construction (CM points on $X_0(N)$), not a topological consequence of a sign, and it is known to be non-torsion only when the zero is simple. [ANALOGY]: the rational surface’s $-q$ eigenspace as an “antiperiodic” (Möbius) local system, rational after the quadratic extension $\mathbb{F}_{25}$ (§3). These are mechanisms, not proofs of anything about ζ .

7. (F) SYNTHESIS: ONE EIGENVALUE COUNTED, THE OTHERS CONSTRAINED

Theorem 7.1 (how Weil’s proof links the two [THEOREM]/[CITED-sec]). *For a curve $C/\mathbb{F}_q$ and $S = C \times C$: (i) the multiplicity of q on $H^2(\bar{S})$ counts $\text{NS}(S)$ (Tate, known for products of curves), the BSD-type count; (ii) RH for C is a sign condition on $H^1(C)$; (iii) they are linked because Frobenius is algebraic: its graph Γ_F is a divisor, so Γ_F and its powers lie in $\text{NS}(S)$, and Hodge index on $\langle H, V \rangle^\perp \cap \text{NS}$ forces $|\alpha| = \sqrt{q}$ (Lemma 81; for $E \times E$, $\det = 4q - a^2 > 0$, Lemmas 81, 84). Without (iii), (i) says nothing about the eigenvalues that are not q .*

For elliptic surfaces (§3) the same split is visible. The BSD eigenvalue q is counted by the Mordell–Weil lattice. The transcendental eigenvalues (K3 family) obey $|\gamma| = q$ by Deligne, not by NS. Ghost G3 shows that the algebraic count and the positivity count are independent.

The missing lattice for $\text{Spec } \mathbb{Z}$: the [t350] list. J.W.’s [t350] reply named three obstructions. We keep the working list O1–O6 of Lemmas 81 and 84 and map it onto them.

[t350] obstruction	O-items and evidence	effect of Lemma 85
(1) No rational lattice for Weil’s form; no finite $\mathbb{Q}[F]$ (Albert cannot apply as stated)	O1 (no finite-dimensional $\mathbb{Q}$ -algebra with a Frobenius for ζ , L84); O3 (no algebraic Frobenius: the scaling flow is not a correspondence); O6 (the count side for ζ is trivial: unit rank 0, $\zeta(0) = -\frac{1}{2}$, §5)	Sharpened: the rational lattices of §§2–3 exist <i>because</i> Frobenius is algebraic (Thm. 7.1(iii)); the norm-one torus F^2/q (§8) is that $\mathbb{Q}$ -structure in rank 2
(2) No surface $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ with an ample class (Faltings–Hriljac: height positivity, not RH; Connes–Consani: Riemann–Roch, no Hodge index on the square)	O2 (no surface with intersection pairing, L81, L84); O4 (archimedean place: real heights with no $\mathbb{Q}$ -form; positivity survives through Faltings–Hriljac, §5). Connes–Consani, Bull. Sci. Math. 187 (2023): Riemann–Roch for $\text{Spec } \mathbb{Z}$ [CITED-sec] (L84 refs)	Located: the archimedean term is exactly where rationality is lost; the real local condition (Thm. 8.3) is its 2-descent shadow
(3) ζ ’s positivity is critical with zero margin	O5 (state positivity = RH, circular, L84). L82: $\lambda_{\min}$ of the cutoff Weil form for ζ is $\geq -9.7 \times 10^{-10}$, a numerical floor (CCM’s “minuscule eigenvalues”), versus finite negative eigenvalues for Davenport–Heilbronn. Rodgers–Tao: $\Lambda \geq 0$, so RH $\Leftrightarrow \Lambda = 0$ [CITED]	New angle (§9): the local positivity (Hasse/Ramanujan) has full margin at every prime. The zero margin is purely global (rate of cancellation)

Verdict [THEOREM]/[OPEN]. Rational lattices with Hodge-index positivity exist exactly where the base is a curve over a finite field (function-field Mordell–Weil lattices, $\text{NS}(C \times C)$). Over $\mathbb{Q}$, Mordell–Weil lattices exist but are real-valued (O4), and they belong to the H^1 of an elliptic curve, not of ζ . Nothing here moves the missing lattice. What it does is sharpen [t350](1) to the absence of

an algebraic Frobenius (O3) and [t350](2) to the archimedean loss of rationality (O4). [t350](3) is untouched: §9 shows that full local positivity at every prime plus Sato–Tate does not produce any global margin. Mordell’s finite generation and BSD live entirely on the count side, and §8 shows that even finite generation fails for the genus-0 circle group, whose rational points are generated by Frobenius-type elements $\pi/\bar{\pi}$.

8. (G) CHORD AND TANGENT ON THE MÖBIUS CIRCLE

Objects. $T = \{x^2 + y^2 = 1\}$ is a conic over $\mathbb{Q}$. Its points at infinity are the *circular points* $(1 : \pm i : 0)$. With $z = x + iy$, T minus these two points is the *norm-one torus* $T = R_{\mathbb{Q}(i)/\mathbb{Q}}^1 \mathbb{G}_m = \{z\bar{z} = 1\}$, an algebraic group of dimension 1. It is *not* an elliptic curve (genus 0), and that difference is why Mordell fails for it.

Theorem 8.1 (chord law = angle addition [THEOREM]). *Fix $O = (1, 0)$. For $P, Q \in T$, the line through O parallel to PQ meets T again at $P \oplus Q$, and $\oplus$ is multiplication $z_P z_Q$, i.e. addition of angles (inscribed-angle theorem). For $P = Q$ the chord becomes the tangent, giving the doubling map $z \mapsto z^2$ with kernel $T[2] = \{\pm 1\}$. Parallel lines meet at the same point at infinity, so the construction depends only on the conic and its points at infinity. When these are a conjugate pair (circle) one gets the non-split torus $\{z\bar{z} = 1\}$; when they are real (hyperbola $x^2 - y^2 = 1$) one gets the split torus $\mathbb{G}_m$.*

Link to the Rosati conjugation [THEOREM] (for elliptic curves), otherwise [ANALOGY]. For $E/\mathbb{F}_q$ with $\text{End}^0(E) \supseteq K = \mathbb{Q}(F)$ imaginary quadratic, the Rosati involution restricts to complex conjugation on K , and $FF^\dagger = q$ (Lemma 84). So $u = F/F^\dagger = F^2/q$ is a $\mathbb{Q}$ -point of the norm-one torus $R_{K/\mathbb{Q}}^1 \mathbb{G}_m$, i.e. of $\{u\bar{u} = 1\}$. The eigenvalues of $h^1 \otimes h^1$ are $\alpha^2, \alpha\bar{\alpha}, \bar{\alpha}\alpha, \bar{\alpha}^2$. Normalized by q they are $z^2, 1, 1, \bar{z}^2$ with $z = \alpha/\sqrt{q}$, and the two 1’s are the norm-one equation $z\bar{z} = 1$: they give the eigenvalue q that counts $\text{NS}(E \times E)/\langle E \times 0, 0 \times E \rangle$ (Thm. 2.5). Multiplying on the circle adds Frobenius angles, $\alpha_i \alpha_j \leftrightarrow \theta_i + \theta_j$. The pair of circular points corresponds to the pair of conjugate embeddings $(\chi, \bar{\chi})$ of K [ANALOGY]. Example [NUMERICAL]: for $y^2 = x^3 - x$ (32a1) and all 145 primes $p \equiv 1 \pmod{4}$ among the first 300 primes, $|a_p| = 2A$ where $p = A^2 + B^2$ with A odd. So $F = \pm(A \pm Bi)$ is a Gaussian prime, and $F^2/p = \pi/\bar{\pi}$ are exactly the generators of $T(\mathbb{Q})$ in Thm. 8.4.

Theorem 8.2 (Möbius bundle [THEOREM]). (a) *The doubling map $T(\mathbb{R}) \rightarrow T(\mathbb{R})$, $w \mapsto w^2$, is a principal $\{\pm 1\}$ -bundle. The Möbius line bundle M is the bundle associated with the sign character of $\{\pm 1\}$. Its sections are the functions $f(w)$ with $f(-w) = -f(w)$, i.e. in the base angle $\theta = 2 \arg w$, $f(\theta + 2\pi) = -f(\theta)$: antiperiodic under 2π , periodic under 4π . (b) $M \otimes M$ is trivial ($\text{sign}^2 = 1$; $w_1(M) \in H^1(S^1, \mathbb{Z}/2) = \mathbb{Z}/2$ is the generator and $2w_1 = 0$). (c) $T(\mathbb{R})/\{\pm 1\} = \mathbb{R}P^1$, and M is the tautological line bundle of $\mathbb{R}P^1$, which is the real locus of $\mathcal{O}(-1)$ on $\mathbb{P}^1$. This is the same bundle as in (a), because the quotient map $S^1 \rightarrow S^1/\pm 1 = \mathbb{R}P^1$ is the doubling map.*

Theorem 8.3 (the real place in 2-descent [THEOREM]; Silverman Ch. X [BIB]). *For $E/\mathbb{R}$: $E(\mathbb{R})^0 \cong \mathbb{R}/\mathbb{Z}$ is divisible, so $E(\mathbb{R})/2E(\mathbb{R}) \cong \pi_0(E(\mathbb{R}))$, of order 2 if $\Delta > 0$ (an “egg”) and 1 if $\Delta < 0$. In 2-descent, $\delta : E(\mathbb{Q})/2E(\mathbb{Q}) \hookrightarrow H^1(\mathbb{Q}, E[2])$ must land at ∞ in $\delta_{\mathbb{R}}(E(\mathbb{R})/2E(\mathbb{R}))$: this is the real local condition of the 2-Selmer group. For the circle, $T(\mathbb{R})$ is connected and $T(\mathbb{R})/2 = 0$, so there is no real condition.*

Explicit 2-descent [NUMERICAL] (num/lemma85_circle.py). $E : y^2 = x(x-6)(x+6)$ ($\Delta = 2^{12}3^6 > 0$, $E(\mathbb{Q})_{\text{tors}} = (\mathbb{Z}/2)^2$). PARI’s `ellrank` (2-descent) gives $\text{rank} = 1$, generator $P = (-3, 9)$. The descent map is $\delta(x, y) = (x, x-6) \in (\mathbb{Q}^\times/\mathbb{Q}^{\times 2})^2$, with the usual replacements at 2-torsion. The 8 classes of $E(\mathbb{Q})/2E(\mathbb{Q})$ are

point	O	(6, 0)	(12, 36)	(18, -72)	(0, 0)	(-6, 0)	(-3, 9)	(-2, -8)
δ	(1, 1)	(6, 2)	(3, 6)	(2, 3)	(-1, -6)	(-6, -3)	(-3, -1)	(-2, -2)
component	id	id	id	id	egg	egg	egg	egg

The real condition: on the identity component $x \geq 6$, so the signs are $(+, +)$; on the egg $-6 \leq x \leq 0$, so they are $(-, -)$. Hence $\delta_{\mathbb{R}}(E(\mathbb{R})/2E(\mathbb{R})) = \{(+, +), (-, -)\} \cong \pi_0$. Of the 64 classes (d_1, d_2) , $d_i \in \{-1, 2, 3\}$, exactly 32 pass the real condition. A search finds rational points in exactly the 8 above. The 2- and 3-adic conditions cut the 32 down to $2^{1+2} = 8$, as **ellrank**'s upper bound 1 requires. The generator P lies on the egg: it is the point whose real image is the nontrivial ‘‘Möbius’’ class. For 37a1 ($\Delta = 37 > 0$): $P = (0, 0)$ is on the egg ($x \in [-1.107, 0.270]$), and nP alternates egg/identity for $n = 1, \dots, 8$ (numerically).

Arithmetic Möbius covers [ANALOGY]. $[2] : E \rightarrow E$ is an $E[2]$ -torsor, and its twists (2-coverings) are classified by $H^1(\mathbb{Q}, E[2])$, just as $w \mapsto w^2$ on T is a μ_2 -torsor with twists $H^1(\cdot, \mu_2) = \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ (Kummer). The real piece is a theorem (Thm. 8.3). Calling the global 2-coverings ‘‘arithmetic Möbius bundles’’ is an analogy: $E[2]$ has rank 2, not 1, and the cocycle lives in Galois cohomology, not in $H^1(S^1, \mathbb{Z}/2)$.

Theorem 8.4 (the rational circle group is not finitely generated [THEOREM]). $T(\mathbb{Q}) = \{z \in \mathbb{Q}(i) : z\bar{z} = 1\} = \{w/\bar{w} : w \in \mathbb{Q}(i)^\times\}$ (Hilbert 90). By unique factorization in $\mathbb{Z}[i]$,

$$T(\mathbb{Q}) \cong \mu_4 \oplus \bigoplus_{p \equiv 1 \pmod{4}} \mathbb{Z} \cdot \frac{\pi_p}{\bar{\pi}_p}, \quad p = \pi_p \bar{\pi}_p,$$

which is free of infinite rank modulo $\mu_4 = \{\pm 1, \pm i\}$ (note $i = (1 + i)/(1 - i)$). In particular $T(\mathbb{Q})/2T(\mathbb{Q})$ is infinite: the ‘‘weak Mordell theorem’’ fails for the circle. The reason is that T has genus 0 and is affine, and nothing bounds the ramification of the 2-coverings $w \mapsto w^2$ (every $p \equiv 1 \pmod{4}$ contributes).

[NUMERICAL]: for points $(a/c, b/c)$ with $c \leq H$, the subgroup they generate has rank exactly $\#\{p \leq H : p \equiv 1 \pmod{4}\}$ (exponent vectors in $\mathbb{Z}[i]$, computed): $H = 100, 300, 600$ give 132, 380, 764 points and ranks 11, 29, 51. For 37a1 the rank is 1 at every height ($\hat{h}(P) = 0.0511$, $\log H(x(nP))$ grows like n^2). See Fig. 3.

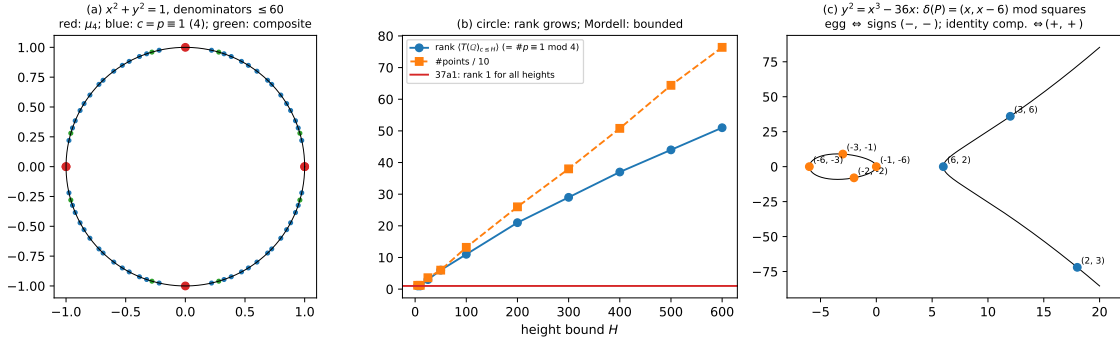


FIGURE 3. (a) Rational points on $x^2 + y^2 = 1$ with denominator ≤ 60 ; red: μ_4 . (b) Rank of the subgroup generated by points of height $\leq H$ (equal to the number of primes $p \equiv 1 \pmod{4}$ up to H) versus Mordell–Weil rank 1 for 37a1. (c) Real locus of $y^2 = x^3 - 36x$ and representatives of $E(\mathbb{Q})/2E(\mathbb{Q})$ labelled by $\delta(P) \bmod$ squares: the sign of δ is the component, which is the real local condition of 2-descent.

9. (H) EULER PRODUCTS AND SPINOR CHARACTERS

Objects. For $p \nmid N$, $L_p(E, s)^{-1} = (1 - \alpha_p p^{-s})(1 - \bar{\alpha}_p p^{-s})$ with $\alpha_p \bar{\alpha}_p = p$, $\alpha_p + \bar{\alpha}_p = a_p$. Hasse’s theorem gives $\alpha_p = \sqrt{p} e^{i\theta_p}$, $\theta_p \in [0, \pi]$. So $\text{diag}(e^{i\theta_p}, e^{-i\theta_p})$ is a conjugacy class in $SU(2)$, the *Satake parameter*. $\zeta_p(s)^{-1} = 1 - p^{-s}$ has degree 1 (one eigenvalue, $= 1$). J.W.’s picture of ζ (each prime

contributes its geometric series; unique factorization selects one term from each, e.g. $\frac{1}{4^s} \cdot \frac{1}{3^s} \cdot 1 \cdots = \frac{1}{12^s}$) is $\zeta(s) = \prod_p \sum_k p^{-ks}$ [THEOREM], valid for $\text{Re } s > 1$.

Theorem 9.1 (spin- $k/2$ characters [THEOREM]). *For $p \nmid N$: $a_{p^k}/p^{k/2} = \sin((k+1)\theta_p)/\sin \theta_p = U_k(\cos \theta_p)$, the character of Sym^k (spin $k/2$) of $SU(2)$. The proof is the recursion $a_{p^{k+1}} = a_p a_{p^k} - p a_{p^{k-1}}$ against $U_{k+1} = 2xU_k - U_{k-1}$. Hence $|a_{p^k}| \leq (k+1)p^{k/2}$. For $p \mid N$ instead $a_{p^k} = a_p^k$ with $a_p \in \{0, \pm 1\}$, not a U_k . Also $U_k(-x) = (-1)^k U_k(x)$: $\theta \mapsto \pi - \theta$ (equivalently $\theta \mapsto \theta + \pi$ up to the Weyl symmetry, i.e. multiplication by $-I \in SU(2)$) acts by $(-1)^k$. Arithmetically this is the quadratic twist by a character with $\chi(p) = -1$, $a_p \mapsto -a_p$.*

Spin and Möbius [THEOREM]. On the maximal torus, $SU(2) \rightarrow SO(3)$ is $e^{i\theta} \mapsto e^{2i\theta}$, the doubling map of Thm. 8.1 with kernel $\{\pm I\}$. The odd- k characters (half-integer spin) are odd under $-I$, so they are *sections of the Möbius bundle* of Thm. 8.2 over the $SO(3)$ -torus. The even- k characters are functions on it. Calling μ “fermionic” is Spector’s (Comm. Math. Phys. 127 (1990)): $1/\zeta(s) = \prod_p (1 - p^{-s}) = \sum \mu(n) n^{-s}$, each prime occupied at most once, with sign $(-1)^\#$ [THEOREM] for $\text{Re } s > 1$. The identification $\mu = (-1)^F$ is [ANALOGY] (Spector’s interpretation).

Theorem 9.2 (Sato–Tate [CITED-sec]). *For non-CM $E/\mathbb{Q}$, $\{\theta_p\}$ is equidistributed for $\frac{2}{\pi} \sin^2 \theta d\theta$, the pushforward of Haar measure on $SU(2)$. Equivalently, by Serre’s argument, $\frac{1}{\pi(x)} \sum_{p \leq x} U_k(\cos \theta_p) \rightarrow 0$ for every $k \geq 1$, which is equivalent to $L(\text{Sym}^k E, s)$ being holomorphic and non-vanishing on $\text{Re } s = 1$. Proved for E with multiplicative reduction somewhere by Harris–Shepherd-Barron–Taylor (Ann. of Math. 171 (2010)) with Clozel–Harris–Taylor and Taylor [BIB]. For all non-CM newforms of weight ≥ 2 , hence all non-CM $E/\mathbb{Q}$, by Barnet-Lamb–Geraghty–Harris–Taylor (Publ. RIMS 47 (2011)). For CM curves the measure is $\frac{1}{2} \delta_{\pi/2} + \frac{1}{2\pi} d\theta$ [BIB].*

Numerics [NUMERICAL] (num/lemma85_spin.py).

- (i) The U_k identity, exact integer arithmetic, 37a1: 1088 tests (first 200 primes $\neq 37$, $k = 0..12$, $p^k < 10^{12}$), **0 failures**. $\max |a_{p^k}|/((k+1)p^{k/2}) = 1$, attained at $k = 0$.
- (ii) Sato–Tate, $p \leq 10^7$ (664 578 primes). Mean of $U_k(\cos \theta_p)$ for $k = 1..8$: $|\cdot| \leq 0.0009$ (11a1), ≤ 0.0013 (37a1). Moments $E[(2 \cos \theta)^{2m}]$: 1, 1.000, 1.999, 4.998, 13.99 (11a1), against the Catalan numbers 1, 1, 2, 5, 14. CM 32a1: $E[U_4] = 0.9995$, $E[U_8] = 0.9988$, moments 1, 1, 3.00, 9.99, 35.0, as the CM measure predicts ($\frac{1}{2}(1 + (-1)^{k/2})$ for even k). Fig. 4(a,b).
- (iii) Orthogonality: the Gram matrix $G_{jk} = \overline{U_j} U_k$ ($j, k \leq 6$) deviates from I by at most 0.0024 (11a1), 0.0026 (37a1), with $\lambda_{\min} = 0.995, 0.991$. For CM it deviates by 2.99. G is automatically positive semi-definite (a Gram matrix of a positive measure), so this checks *equidistribution*, not positivity.
- (iv) Rate: $S_k(x) = \sum_{p \leq x} U_k(\cos \theta_p) \log p$ at $x = 10^7$ gives $|S_k|/\sqrt{x} = 1.16, 1.26, 1.01$ (11a1, $k = 1, 2, 3$) and 0.44, 2.20, 2.58 (37a1), consistent with square-root cancellation. Sato–Tate only gives $S_k = o(x)$.

Ghost tests [NUMERICAL] (37a1, $p \leq 10^6$). A “Weil-shaped” factor with real a_p and $\alpha\beta = p$ but $|\alpha| = p^{1/2+\delta}$ forces α, β real, i.e. $\cos \theta_p = \pm \cosh(\delta \log p)$, $|\cos \theta_p| > 1$, $\theta_p \in i\mathbb{R}$.

ghost	frac. $ \cos \theta > 1$	$\max U_6 /7$	$E[U_2]$	$E[(2 \cos \theta)^8]$ (Catalan 14)	$\max G - I $
true 37a1	0	≤ 1	−0.0007	13.99	0.003
5% of p , $\delta = 0.1$	0.050	607	0.70	2927	3.9×10^5
all p , $\delta = 0.05$	1	12.0	4.87	1208	4.5×10^3
0.1% of p , $\delta = 0.25$	0.0009	1.4×10^8	0.59	3.1×10^8	2.4×10^{14}
uniform θ (on circle)	0	≤ 1	1.00	69.9	5.98

What breaks. Off-circle ghosts break Hasse (locally, prime by prime): characters leave $[-(k+1), k+1]$ exponentially in k and the moments explode, even for a 0.1% sparse set. The on-circle “uniform” ghost keeps Hasse and breaks only Sato–Tate (as the genuine CM curve 32a1 does, whose RH status is the same as 37a1’s: open).

Local positivity $\rightarrow$ global positivity? No, and here is precisely what is missing ([THEOREM] for the equivalences, [OPEN] for the gap).

- (a) *Local unitarity is known.* Hasse ($|\alpha_p| = \sqrt{p}$, i.e. θ_p real, Ramanujan for E) is a theorem at every p . So is $|U_k| \leq k + 1$.
- (b) *Sato–Tate is a “prime number theorem”.* It is equivalent to non-vanishing of all $L(\text{Sym}^k E, s)$ on the edge $\text{Re } s = 1$ (normalized) and controls $S_k(x) = o(x)$.
- (c) *RH is a rate.* RH for $L(E, s)$ is equivalent to $\sum_{p \leq x} 2 \cos \theta_p \log p = O(x^{1/2+\varepsilon})$ (explicit formula; the prime-square term $\sum_{p \leq \sqrt{x}} (U_2 - 1) \log p \sim -\sqrt{x}$ is of the same order and is where Conrad’s $\sqrt{2}$ comes from, see below). Neither (a) nor (b) yields a rate. The zeros are a global object, while the Satake parameters are a family of independent local data plus one global limit law.
- (d) *The random model satisfies RH-type cancellation.* With independent random θ_p (Rademacher or Haar), square-root cancellation holds almost surely (Wintner 1944 for random multiplicative ± 1 [BIB]). What is missing for a given E is control of the *correlations* of θ_p across primes at scale $\sqrt{x}$. That is a global statement, equivalent to RH itself ([RESTATEMENT]), not supplied by $SU(2)$ structure.
- (e) *Where Sato–Tate does enter the critical line.* Conrad’s Thm. 1.2 [CITED]: the $\sqrt{2}$ in Goldfeld’s constant is $\sqrt{2}^{R_0}$, where R_0 is the order at $s = 1$ of the second-moment product $\prod (1 - \alpha_p^2 p^{-s})^{-1} (1 - \bar{\alpha}_p^2 p^{-s})^{-1}$ (normalized $|\alpha_p| = 1$). This product is $L(\text{Sym}^2 E, s)/\zeta(s)$ up to finitely many factors, so $R_0 = 1$, which is the $k = 2$ Sato–Tate moment (this instance known since Gelbart–Jacquet [BIB]) $E[2 \cos 2\theta] = E[U_2] - 1 = -1$. The Euler product identity holds only for $\text{Re } s > 1$ (normalized; conditionally on $\text{Re } s = 1$ for ζ by PNT). At the centre, convergence of the partial products *is* the Deep RH (§4), strictly stronger than RH.

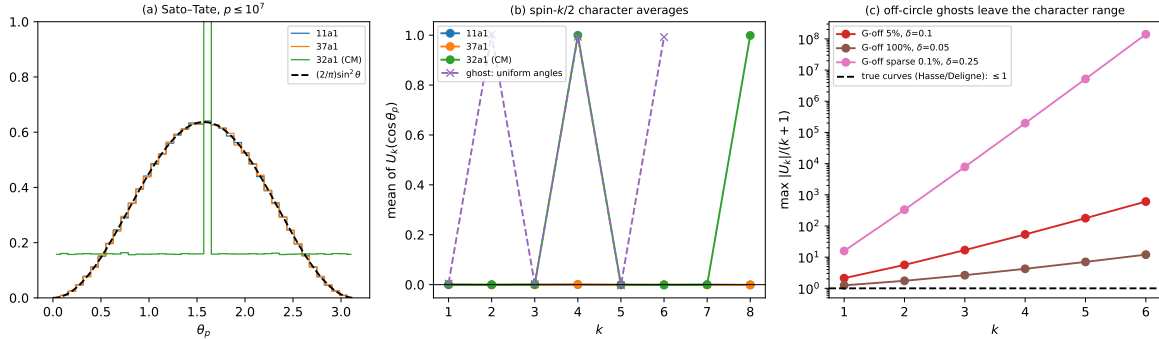


FIGURE 4. (a) Histograms of θ_p , $p \leq 10^7$, for 11a1, 37a1 (non-CM) and 32a1 (CM; spike at $\pi/2$) against $\frac{2}{\pi} \sin^2 \theta$. (b) Means of the spin- $k/2$ characters $U_k(\cos \theta_p)$: zero for non-CM (Sato–Tate), $\frac{1}{2}(1 + (-1)^{k/2})$ for CM, 1 for even k in the uniform ghost. (c) Off-circle ghosts: $\max |U_k|/(k+1)$ exceeds the Hasse/Deligne bound 1 exponentially in k , even when only 0.1% of primes are altered.

10. (I) MODULARITY AND THE DIVERGENT REGION

Objects. $S_2(\Gamma_0(N))$ is the space of weight-2 cusp forms on $\Gamma_0(N)$. A newform is $f(\tau) = \sum a_n q^n$, $q = e^{2\pi i \tau}$, $a_1 = 1$, a Hecke eigenform. $X_0(N)$ is the compactified modular curve over $\mathbb{Q}$ and $J_0(N)$ its Jacobian. The Fricke involution is $W_N : \tau \mapsto -1/(N\tau)$, with $(f|_2 W_N)(\tau) = N^{-1} \tau^{-2} f(-1/(N\tau))$.

Theorem 10.1 (modularity [CITED-sec]; Mellin integral [THEOREM]). (*Wiles 1995 and Taylor–Wiles 1995 for semistable E ; Breuil–Conrad–Diamond–Taylor 2001 for all $E/\mathbb{Q}$.*) There is a newform $f \in S_2(\Gamma_0(N))$, $N = \text{conductor}$, with $a_p(f) = a_p(E)$ for all p . Then

$$\Lambda(E, s) := N^{s/2} (2\pi)^{-s} \Gamma(s) L(E, s) = N^{s/2} \int_0^\infty f(iy) y^{s-1} dy = \int_0^\infty f\left(\frac{iy}{\sqrt{N}}\right) y^{s-1} dy,$$

since $\int_0^\infty e^{-2\pi ny} y^{s-1} dy = \Gamma(s) (2\pi n)^{-s}$. The integral converges for all $s \in \mathbb{C}$: $f(iy)$ decays like $e^{-2\pi y}$ as $y \rightarrow \infty$, and as $y \rightarrow 0$ because f is a cusp form also at the cusp 0 (by W_N below). So Λ is entire. The path is the geodesic from the cusp 0 to the cusp $i\infty$.

Theorem 10.2 (Fricke symmetry, forced zero [THEOREM]). $f|W_N = wf$ with $w = \pm 1$, and $\Lambda(E, s) = \varepsilon \Lambda(E, 2-s)$ with $\varepsilon = -w$ (the factor $i^k = -1$ for $k = 2$). On the imaginary axis this reads $f(i/(Ny)) = \varepsilon Ny^2 f(iy)$, so $g(y) = y f(iy)$ satisfies $g(1/(Ny)) = \varepsilon g(y)$: the map $s \leftrightarrow 2-s$ is $y \leftrightarrow 1/(Ny)$, with fixed point $y_0 = 1/\sqrt{N}$ ($\tau = i/\sqrt{N}$). If $\varepsilon = -1$, then g is odd under an involution and vanishes at its fixed point: $f(i/\sqrt{N}) = 0$, as well as $\Lambda(1) = 0$. Conjugating by $\tau \mapsto \sqrt{N}\tau$ turns W_N into $S : \tau \mapsto -1/\tau$ with fixed point $\tau = i$ (Lemma 83’s $\tau = i$) [THEOREM].

The identification with the Möbius pinch point of Lemma 81 has two levels. The mechanism (odd section, fixed point, forced zero) is [THEOREM] and is the same as in Thm. 6.1(c) and §8. Calling the Fricke fixed point “the” pinch point is [ANALOGY].

Split formula [THEOREM]. Splitting the integral at any $y_1 > 0$ and applying Fricke to $\int_0^{y_1}$ gives

$$\Lambda(s) = N^{s/2} \sum_{n \geq 1} a_n \left[(2\pi n)^{-s} \Gamma(s, 2\pi n y_1) + \varepsilon N^{1-s} (2\pi n)^{s-2} \Gamma(2-s, \frac{2\pi n}{N y_1}) \right],$$

which converges exponentially for every s . At $y_1 = y_0$, $s = 1$ it reduces to the classical $L(E, 1) = 2 \sum \frac{a_n}{n} e^{-2\pi n/\sqrt{N}}$ ($\varepsilon = +1$) and $L'(E, 1) = 2 \sum \frac{a_n}{n} E_1(2\pi n/\sqrt{N})$ ($\varepsilon = -1$). The result is independent of y_1 if and only if the functional equation holds with these a_n and this ε .

Numerics [NUMERICAL] (num/lemma85_modular.py, mpmath 30 digits; lemma85_dh.py).

- *Inside the divergent region.* At $s = 1$ the Euler product does not converge absolutely (it needs $\text{Re } s > 3/2$; its partial products behave like $(\log x)^{-r}$ only under Deep RH, §4). The split formula gives $L'(E, 1) = 0.30599977383405230182$ for 37a1 (400 terms) and $L''(E, 1) = 1.5186330005768535405$ for 389a1 (1500 terms). These agree with PARI’s e11L1 (0.3059997738340523, 1.5186330005768536) to all 16 printed digits.
- *Fricke eigenvalue.* $f(i/(Ny))/(Ny^2 f(iy))$ at $y/y_0 \in \{0.5, 0.8, 1.3, 2\}$ gives $\varepsilon = -1.000$ (37a1) and $+1.000$ (389a1), errors $< 10^{-29}$, so $w = +1, -1$, matching PARI’s root numbers. Forced zero: $f(i/\sqrt{37}) = -5 \times 10^{-32}$ (37a1); $f(i/\sqrt{389}) = -0.918$ (389a1, $\varepsilon = +1$, no forced zero). Fig. 5(a).
- *Split-invariance.* $|\Lambda(1+i; y_1) - \Lambda(1+i; y_0)|$ for $y_1/y_0 \in \{0.5, 0.7, 1.5, 2\}$ is $\leq 1.3 \times 10^{-31}$ (37a1) and $\leq 3.3 \times 10^{-30}$ (389a1).

Ghost detector from modularity [NUMERICAL].

fake (same a_n otherwise)	37a1: max spread over y_1	389a1: max spread
true	1.3×10^{-31}	3.3×10^{-30}
wrong w (ε flipped)	0.159	1.06
$a_{53} \rightarrow a_{53} + 2$ (multiplicatively propagated)	4.7×10^{-14}	2.5×10^{-5}
$a_{211} \rightarrow -a_{211}$, $y_1 \geq y_0/2$	1.8×10^{-31} (missed)	5.3×10^{-16}
$a_{211} \rightarrow -a_{211}$, $y_1 = 0.1y_0$ / $0.07y_0$ (3200 terms)	4.1×10^{-11} / 2.8×10^{-8}	–

A coefficient change at n enters with weight $\approx e^{-2\pi n \min(y_1, 1/(Ny_1))}$, so with D working digits its detection window is $n \lesssim D \ln 10 / (2\pi \min(y_1, 1/(Ny_1)))$. Pushing y_1 away from y_0 widens the window at the cost of more terms (Fig. 5(b,c)). In principle every finite change is detected [THEOREM]: a nonzero finite Dirichlet polynomial $D(s)$ cannot satisfy Γ -factor functional equations (Stirling growth in $|t|$ versus boundedness of D on vertical lines).

Is it RH-blind? Yes [THEOREM]/[NUMERICAL]. The detector tests the functional equation and nothing else. Davenport–Heilbronn: $f_{\text{DH}} = \frac{1-i\kappa}{2}L(s, \chi) + \frac{1+i\kappa}{2}L(s, \bar{\chi})$, $\chi \bmod 5$, $\chi(2) = i$, $\kappa = (\sqrt{10 - 2\sqrt{5}} - 2)/(\sqrt{5} - 1) = 0.2840790438$, coefficients $(1, \kappa, -\kappa, -1, 0)$ periodic mod 5. Its theta series $\theta(t) = \sum a_n n e^{-\pi n^2 t/5}$ satisfies $\theta(1/t) = t^{3/2}\theta(t)$ (ratio 1.000 at $t = 0.5, 0.8, 1.3, 2$), and the split formula for $(\pi/5)^{-(s+1)/2}\Gamma(\frac{s+1}{2})f_{\text{DH}}(s)$ is t_1 -invariant to 1.2×10^{-41} at $s = 0.7 + 3i$. Yet f_{DH} has the off-line zero

$$\rho = 0.808517182456637 + 85.699348485377592i \quad (|f_{\text{DH}}(\rho)| < 10^{-38}, \text{ mirror } 1 - \bar{\rho} \text{ also a zero}),$$

found here by Newton iteration from Hurwitz-zeta values (Davenport–Heilbronn 1936 proved such zeros exist). So DH passes the modularity detector, and the L82 cutoff detector rejects it (negative eigenvalues of the cutoff Weil form for $L > L^* \approx 3.7$, Lemma 82). *Modularity sees the functional equation; Weil positivity sees the zeros.* ζ also passes (Riemann’s split at $t = 1$).

Eichler–Shimura and local ampleness [CITED-sec]. For $p \nmid N$, $X_0(N)$ has good reduction and $T_p = \text{Frob} + \text{Ver}$ on $J_0(N)_{\mathbb{F}_p}$ (Eichler 1954, Shimura 1958). So $a_p(f)$ is the trace of Frobenius on the f -isotypic 2-dimensional piece of $H^1(X_0(N)_{\mathbb{F}_p})$ (for E : on $T_\ell E$). It is not the trace on all of $J_0(N)$, which is the sum over a basis of $S_2(\Gamma_0(N))$. Hasse’s bound for f (Ramanujan–Petersson in weight 2) is then Weil’s RH for the curve $X_0(N)/\mathbb{F}_p$, whose proof uses the ample class and Hodge index on $X_0(N) \times X_0(N)$ over $\mathbb{F}_p$ (Lemma 81). For E itself Hasse (1933) suffices, since E is a curve over $\mathbb{F}_p$.

What modularity gives, and does not ([THEOREM], [OPEN]).

- It gives continuation, functional equation, root number as a Fricke eigenvalue, Heegner points (CM points on $X_0(N)$ pushed to E by the parametrization $X_0(N) \rightarrow E$; Gross–Zagier, Kolyvagin), and symmetric-power automorphy, hence Sato–Tate (§9).
- It does not give positivity. ζ has had “modularity” since Riemann’s second proof: the Mellin transform of θ with Jacobi’s $\theta(1/y) = \sqrt{y}\theta(y)$, which is the Fricke involution W_4 for $\theta(2\tau) = \sum q^{n^2}$ of weight $\frac{1}{2}$ on $\Gamma_0(4)$ [THEOREM]. RH is still open. Zero-free regions for $L(E, s)$ are of classical (de la Vallée Poussin) type [BIB].
- Map to [t350]: (1) the Hecke algebra $\mathbb{T} \otimes \mathbb{Q}$ is a finite $\mathbb{Q}$ -algebra with $T_p = \text{Frob} + \text{Ver} \bmod p$. That is a rational lattice for each p , not for the zeros of $L(E, s)$ (and none for ζ). (2) The ample class is local: $X_0(N) \times X_0(N)$ over each $\mathbb{F}_p$. $X_0(N)$ over $\mathbb{Z}$ is an arithmetic surface, but its square carries no Hodge index controlling the zeros (O2, O4). (3) Margin: untouched. The modularity detector is RH-blind (DH), so it adds no global margin.

11. CORRECTIONS TO THE CHAT CLAIMS [T351S1471–1473]

- (1) “ $\rho = 2 + \sum(m_v - 1) + \text{rank}$.” Correct over $\bar{k}$, for a relatively minimal elliptic surface with a section that is not birationally a product. Over $\mathbb{F}_q$, replace $m_v - 1$ by the number of Frobenius orbits of non-identity components, summed over closed points (Thm. 2.1).
- (2) “Néron–Tate height = Shioda pairing.” Correct up to normalization: $\hat{h} = \langle, \rangle \cdot \log q$ (times deg of the base field, and times 1 or 2 depending on the convention for $\hat{h}$). The values lie in $\frac{1}{m}\mathbb{Z}$ for an m fixed by the bad fibres.
- (3) “Rank = multiplicity of q ; leading terms match.” In general $\text{rank} \leq \text{multiplicity}$ (Tate’s inequality). Equality is equivalent to Tate/BSD for E/K , known in specific classes (rational, K3, dominated by products of curves). The leading-term equivalence Artin–Tate $\Leftrightarrow$ BSD is due to Gordon (1979); Milne (1975) proved that Tate’s conjecture (for ℓ) implies the Artin–Tate formula. The $E \times E$ statement was correct.
- (4) “ $\varepsilon = (-1)^{\text{rank}}$.” Correct for the *analytic* rank. For the algebraic rank it needs Sha finite. Known: p^∞ -Selmer parity for all $E/\mathbb{Q}$ and all p (Dokchitser–Dokchitser 2010; Nekovář for earlier cases). Gross–Zagier–Kolyvagin gives $\text{rank} = \text{analytic rank} \leq 1$, using modularity and auxiliary non-vanishing theorems.

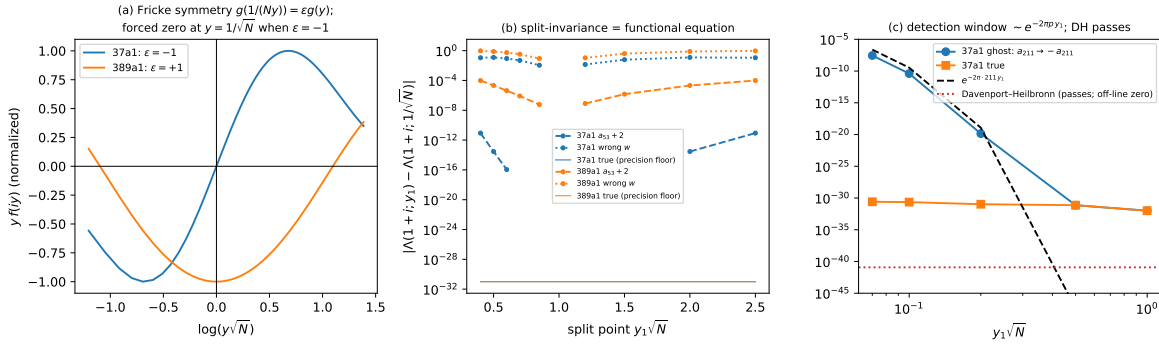


FIGURE 5. (a) $g(y) = yf(iy)$ against $\log(y\sqrt{N})$: odd about the Fricke fixed point for 37a1 ($\varepsilon = -1$, forced zero), even for 389a1. (b) Split-invariance of the Mellin integral at $s = 1 + i$: true forms sit at the precision floor; a wrong w or $a_{53} + 2$ breaks it. (c) Detection window: the ghost $a_{211} \rightarrow -a_{211}$ in 37a1 is visible only for split points $y_1 \lesssim 0.2/\sqrt{37}$, tracking $e^{-2\pi \cdot 211 y_1}$. Davenport–Heilbronn (dotted) passes at 10^{-41} despite its off-line zero.

- (5) “ $\prod N_p/p \sim C(\log x)^r$.” Correct as a conditional theorem, with exact constant $C = \sqrt{2}e^{\gamma}r!/L^{(r)}(E, 1)$ (Goldfeld 1982, under modularity at the time; Conrad 2005, *Canad. J. Math.* 57). The asymptotic is strictly stronger than RH (“Deep RH”, Kimura–Koyama–Kurokawa 2014). It is a conjecture for every E .
- (6) “ $\zeta(0) = -\frac{1}{2}$ is BSD for $\mathbb{Q}$; $\zeta(\frac{1}{2}) \dots$ ” The first part is correct: $-\frac{1}{2} = -hR/w$. But $\zeta(\frac{1}{2}) = -1.4603545088$ plays no role: the BSD point for $\mathbb{G}_m$ is $s = 0$ (equivalently $s = 1$), not the centre.
- (7) *Rank records/averages*. Elkies (2006): rank ≥ 28 , exact rank unknown. Elkies–Klagsbrun (announced 29 Aug 2024): rank ≥ 29 , exactly 29 under GRH. Average rank ≤ 1.5 is Bhargava–Shankar, *Ann. of Math.* 181 (2015); later improvements were not checked here.

Claims on the Möbius circle [t353s1478].

- (G1) *Chord law = angle addition; norm-one torus; circular points; link to Rosati; $\alpha_i \alpha_j$* . Correct (Thm. 8.1). The Rosati link is a theorem for elliptic curves ($F/F^\dagger = F^2/q$ lies on the norm-one torus of $\mathbb{Q}(F)$), and the circular points $\leftrightarrow (\chi, \bar{\chi})$ is an analogy. Precision: the normalized $h^1 \otimes h^1$ eigenvalues are $z^2, 1, 1, \bar{z}^2$, and the two 1’s are the norm-one equation itself.
- (G2) *Tangent = doubling; Möbius = sign bundle; “antiperiodic with period 4π ”; $M \otimes M$ trivial; tautological = $\mathcal{O}(-1)$* . Correct, with one wording fix: sections are antiperiodic under $\theta \mapsto \theta + 2\pi$, hence periodic with period 4π . The two descriptions (sign bundle of the doubling cover; tautological bundle of $\mathbb{RP}^1$) are the same bundle, not separate facts, since $S^1 \rightarrow \mathbb{RP}^1$ is the doubling map.
- (G3) *$E(\mathbb{R})/2E(\mathbb{R}) = \pi_0$ is the real local condition (Theorem); 2-coverings as arithmetic Möbius covers (Analogy)*. Correct as labelled. Verified on $y^2 = x^3 - 36x$ (sign rule, $64 \rightarrow 32 \rightarrow 8$).
- (G4) *$T(\mathbb{Q}) \cong \mathbb{Z}/4 \oplus$ free of infinite rank, generators $\pi/\bar{\pi}$, Hilbert 90*. Correct. Precision: Hilbert 90 gives $T(\mathbb{Q}) = \{w/\bar{w}\}$, and the free basis comes from unique factorization in $\mathbb{Z}[i]$; $\mathbb{Z}/4 = \mu_4$ includes $i = (1+i)/(1-i)$ (the ramified prime). Also $T(\mathbb{Q})/2T(\mathbb{Q})$ is infinite, so weak Mordell fails.

Euler product and spinor claims [t354–t355].

- (H1) *Degree-1 ζ factor; degree-2 E factor*. Correct for $p \nmid N$; at $p \mid N$ the E factor has degree ≤ 1 ($1 - a_p p^{-s}$, $a_p \in \{0, \pm 1\}$).

- (H2) $a_{p^k}/p^{k/2} = U_k(\cos \theta_p)$; $\theta \rightarrow \theta + \pi$ gives $(-1)^k$. Correct for $p \nmid N$ (fails at $p \mid N$, where $a_{p^k} = a_p^k$). The sign is the central element $-I$, and its Möbius reading is a theorem on the torus (odd spin = sections of the Möbius bundle over the $SO(3)$ torus).
- (H3) *Sato–Tate, BLGHT 2011*, $\frac{2}{\pi} \sin^2 \theta d\theta$. Correct. Precision: BLGHT (Publ. RIMS 47) completes all non-CM $E/\mathbb{Q}$; curves with multiplicative reduction somewhere (including 11a1 and 37a1) were already covered by Harris–Shepherd–Barron–Taylor (Ann. of Math. 171, 2010) with Clozel–Harris–Taylor and Taylor.
- (H4) $1/\zeta$ *fermionic rows give μ (Spector); factorization only for $\operatorname{Re} s > 1$; RH and BSD live where it diverges; link to Deep RH*. Correct, with precision. The Euler product for ζ still converges conditionally on $\operatorname{Re} s = 1$, $s \neq 1$ (PNT). For $L(E, s)$ the BSD point $s = 1$ is the *centre* of the critical strip (absolute convergence needs $\operatorname{Re} s > 3/2$), and convergence of the partial products there is the Deep RH, and its $\sqrt{2}$ is a Sato–Tate (Sym^2) effect (§9, item (e)).
- (H5) *Local positivity to global?* Agreed: no. The precise gap is a rate ($o(x)$ vs. $O(x^{1/2+\varepsilon})$), i.e. cross-prime correlations (§9).

Modularity claims [t357s1491].

- (I1) *Modularity; Mellin integral converges for all s ; before modularity $L(E, 1)$ undefined except CM; geodesic $0 \rightarrow i\infty$* . Correct with three fixes. (a) Normalization: $\int_0^\infty f(iy)y^{s-1}dy = (2\pi)^{-s}\Gamma(s)L(E, s)$, and the completed Λ needs the extra $N^{s/2}$, equivalently $\Lambda = \int_0^\infty f(iy/\sqrt{N})y^{s-1}dy$. (b) Convergence at $y \rightarrow 0$ uses that f is cuspidal at 0 (via W_N). (c) “Undefined” should be “not known to continue”: before 1995 continuation was known for CM curves (Hecke, Deuring; as BCDT note) and for curves known to be modular (e.g. quotients of $J_0(N)$ such as 11a1, 37a1, by Eichler–Shimura). Wiles 1995, Taylor–Wiles 1995 (semistable), BCDT 2001 (all).
- (I2) *W_N swaps the cusps and fixes $i/\sqrt{N}$; $f|W_N = wf$, $\varepsilon = -w$; $s \leftrightarrow 2 - s$ is $y \leftrightarrow 1/(Ny)$; pinch = Fricke fixed point*. Correct, with the convention $(f|_2 W_N)(\tau) = N^{-1}\tau^{-2}f(-1/(N\tau))$ (checked numerically: $w = +1$ for 37a1, -1 for 389a1). Upgrade: for $\varepsilon = -1$, $f(i/\sqrt{N}) = 0$ is a theorem. The pinch-point identification is [ANALOGY].
- (I3) *$a_p = \text{trace of Frobenius on } J_0(N) \bmod p$; Hasse = Weil RH for $X_0(N)/\mathbb{F}_p$; ample class behind it*. Needs correction: a_p is the trace on the f -isotypic 2-dimensional part (the trace on all of $J_0(N)$ sums over a basis of $S_2(\Gamma_0(N))$), and only for $p \nmid N$. Hasse for E needs only $E/\mathbb{F}_p$ (Hasse 1933). Weil RH for $X_0(N)/\mathbb{F}_p$ gives Ramanujan–Petersson for all weight-2 forms of level N , with the ample class on $X_0(N)_{\mathbb{F}_p}^2$ behind it (correct).
- (I4) *Symmetric-power automorphy (BLGHT; Newton–Thorne 2021) and edge non-vanishing give Sato–Tate*. Correct, with precision. BLGHT gives *potential* automorphy of all Sym^n , which already suffices for Sato–Tate. Newton–Thorne (Publ. IHÉS 134 (2021), two papers) prove actual automorphy of $\operatorname{Sym}^n f$ for all non-CM f , after part I covered level 1 and a class including semistable elliptic curves.
- (I5) *Heegner points need $X_0(N) \rightarrow E$* . Correct (Gross–Zagier; Kolyvagin uses the Heegner system), plus the auxiliary non-vanishing results noted in (4).
- (I6) *Limits: ζ has “modularity” (Riemann’s second proof; level-4 Fricke), RH open; continuation and FE but no positivity; no zero-free region beyond classical; ample class local*. Correct. Demonstrated: the modularity detector is RH-blind (DH passes with an off-line zero). The [t350] map is in §10.

[t350] list (Q4 answer). Aligned in §7. One precision on (3): the L82 value is $\lambda_{\min} \geq -9.7 \times 10^{-10}$, a numerical floor, *not* a computed negative eigenvalue. Positivity of ζ ’s Weil form is equivalent to RH and is not contradicted.

12. RANKED DIRECTIONS

Score = 0.6·significance+0.4·probability (subjective, [HEURISTIC]). Updated after (G), (H), (I).

direction	sig	prob	score
L3 NS($C \times C$) for a genus-2 curve: Frobenius-graph lattice, Hodge index $\leftrightarrow \kappa$	.55	.80	.65
L1 function-field Goldfeld–Conrad: $\prod N_v/q^{\deg v}$ for the surfaces of §3 (exact, provable)	.50	.85	.64
L8 two detectors on one family: run the L82 cutoff Weil form on $L(E, s)$ (37a1, 389a1) next to the split/FE test of §10; quantify what each sees (FE vs zeros)	.50	.80	.62
L6 function-field Sato–Tate vs. RH: over $\mathbb{F}_q(t)$ both are theorems (Deligne); exhibit where algebraic Frobenius converts local data into a rate	.55	.70	.61
L4 forced sections from $\varepsilon = -1$: explicit generators in the 18 K3 cases (Heegner analogue)	.45	.70	.55
L2 unit lattice as “Mordell–Weil lattice of $\mathbb{G}_m$ ”; a Hodge-index test for regulators	.60	.45	.54
L7 norm-one tori F^2/q as a $\mathbb{Q}$ -structure across CM curves (§8)	.40	.75	.54
L5 search for a $\mathbb{Q}$ -structure on archimedean height regulators	.70	.10	.46

13. QUESTIONS FOR J.W.

- Q1. Do you accept the summary “the rational lattice exists over function fields; the BSD count is the multiplicity of the eigenvalue q ; RH constrains the remaining eigenvalues; Frobenius algebraicity links them”?
- Q2. Are corrections (1)–(7), (G1)–(G4), (H1)–(H5), (I1)–(I6) acceptable, in particular (3) (Gordon vs. Milne), (H2) (the U_k identity fails at $p \mid N$), (I1) (the $N^{s/2}$ normalization; “not known” rather than “undefined”) and (I3) (a_p is the trace on the f -part, not on $J_0(N)$)?
- Q3. Should the archimedean obstruction O4 (real heights; positivity survives via Faltings–Hriljac, rationality is lost) become the focus, or O3 (algebraic Frobenius)? Do you accept the §10 slogan “modularity sees the functional equation, Weil positivity sees the zeros” (DH passes the first and fails the second)?
- Q4. (Replaces the [t350] request, now answered.) Do you accept the §9 verdict that local unitarity plus Sato–Tate gives no route to RH, the gap being a cross-prime rate? Should [t350](3) (“zero margin”) be phrased as “full local margin, zero global margin”?
- Q5. Next lemma: L3 (genus-2 NS($C \times C$), .65), L1 (function-field EDSAC, .64), L8 (both detectors on $L(E, s)$, .62), or L6 (.61)?

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All entries are checked in `lit/lemma85/verified_refs.md` (“seen” = text opened; others with bibliographic data confirmed).

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CODE AND DATA

`num/lemma85_lib.gp`, `lemma85_surface.py`, `lemma85_ghosts.py`, `lemma85_k3.py`,
`lemma85_edsac.{gp,py}`, `lemma85_psi.py`, `lemma85_figs.py`, `lemma85_circle.py`,
`lemma85_spin.py`, `lemma85_figsGH.py`, `lemma85_modular.py`, `lemma85_dh.py`, `lemma85_figI.py`;
 outputs in `num/185/*`.json. Nothing was posted or submitted.

APPENDIX A

Consolidated corrections

Every note ends with a corrections section that records which claims made in the working conversation were wrong, and how they were fixed. Those claims came from pasted texts, from J.W.’s hunches stated as claims, or from the assistant’s own chat replies. Chapter texts were not edited for this volume. The table points to each section and gives its gist; the full text is in the chapter. “Note p.” is the page inside the note; “vol. p.” is the volume page of this part. Only the chapters of this part (Chapters 66–85) are listed; the others are in the same appendix of Volume IVa.

Ch.	Section	note p.	vol. p.	Gist
66	8. Corrections to Grok Bot’s chat claims	11	11	K1–K2 refined (strict Ramanujan inequality; mod-691 degeneration precise); the bridge is where positivity is lost.
67	11. Corrections to Grok Bot’s chat claims	12	25	C1–C10: poles at $\rho/2$; “hence Weil’s prime sum” partly wrong; saddle angle 51.2° .
68	8. Corrections to Grok Bot’s chat claims	15	44	It is E_{12} , not Δ , whose zeros miss $\operatorname{Re} s = 6$; graded-commutativity claim corrected.
69	11. Corrections to the chat claims	15	61	C1–C13.
70	12. Corrections to the chat claims	11	74	C1–C12: $p \mapsto 1/p$ swap only normalized; Casimir sign; Tate curve local.
71	10. Corrections to the chat claims	13	89	C1–C14: closure list = supersingular; precession $\neq$ Kepler; Hamiltonian $\frac{1}{2}g_p$ exists but circular.
72	7. Corrections to the chat claims	12	103	C1–C8: 2D “Ric = λg is EFE with Λ ” wrong; Λ -term = curvature is analogy.
73	11. Corrections to the chat claims	13	118	C1–C12: primitive (c, d) over $\mathbb{Z}$ is a real null direction; cusps $\leftrightarrow \operatorname{Cl}(K)$ correct and refined.
74	8. Corrections to the chat claims and the brief	13	134	C1–C16: Jensen RH-blind; Busemann up to $2h_{\text{FS}}$; a $\oplus$ conflation.
75	9. Corrections	13	149	C1–C14: review items T1–T6 checked; Fredholm-polarization J unsupported.
76	10. Corrections to my chat claims	12	163	C1–C14: “discrepancy $\iff$ RH” only for L^1/L^2 , not sup; Bost–Connes has no zeros.
77	9. Corrections	11	176	C1–C17: matrix size 2γ suffices; a zb-MATH conflict flagged; a misprint in Chinen’s table.
78	8. Corrections to the chat claims [t311]–[t325]	16	194	C1–C25 (claims from t311–t328).
79	8. Corrections to the chat claims [t330]–[t331] and to the brief	7	204	C1–C9: the line pairing is blind, the curve pairing decisive; $1 + 1 + 1 = \Gamma$ via Lerch.
80	5. Corrections	4	210	C1–C4: the D1 statement and its useless/zero-dependent cases; “blind” only for Hilbert type.

Ch.	Section	note p.	vol. p.	Gist
81	8. Corrections	9	220	C1–C8, incl. the addenda (correct function-field cover; dBN bound 0.2 attributed to Platt–Trudgian).
82	6. Corrections	4	225	C1–C4: count is 2 per quartet; small-support positivity not discriminating; operator vs. quadratic form.
83	7. Corrections	6	232	C1–C5 to a pasted text (paraphrased; verbatim text unavailable): LRL is not Lefschetz; Casimir sign RH-blind.
84	4. Corrections to the Möbius text pasted at [t348]	4	237	M1–M9, N1–N9, C1–C5: $ \alpha = \sqrt{q}$ not q ; fake signatures $(3, 1)/(2, 2)$; a sufficient-not-necessary discriminant criterion corrected.
	5. Corrections to the second text pasted at [t349]	6	239	
	8. Corrections	8	241	
85	11. Corrections to the chat claims [t351s1471–1473]	14	256	(1)–(7), (G1)–(G4), (H1)–(H5), (I1)–(I6); e.g. the BSD point is the centre of the strip; the U_k identity fails at $p \mid N$.

Corrections across chapters. Some later chapters correct or sharpen earlier ones; the later chapter states this in its “Prior TAA work” or “Corrections” section. Example: Ch. 81’s second addendum reconciles its Cassini saddle with Ch. 55, Prop. 2.2 (Volume IVa).

APPENDIX B

Open questions

Each note ends with “Questions for J.W.” (usually Q1–Q5). They are reproduced in full in the chapters; this appendix lists their topics and locations for the chapters of this part (Chapters 66–85); the others are in the same appendix of Volume IVa. **Status.** Unless marked, no answer from J.W. is recorded in the notes, so the questions are open. Two kinds of exception: answers recorded in a later note (marked *Answered*); questions of the form “which direction next”, which were in effect settled by the following lemma requests. No volume-level computations are pending: unlike Volume III, this volume contains no placeholders.

Ch.	§	note p.	vol. p.	Topics
66	10	12	12	Accept the corrected reading of the bridge; adopt the E_{12} theorem and ablation row; H1 or H2 next; H3 for L67; Vol. 4 inclusion and L60 erratum.
67	13	15	28	Adopt the t255u answer and t256u verdict with the cone theorem; θ -ladder counter; Ramanujan–Serre naming; Vol. 4 inclusion.
68	10	16	45	Adopt the t258u/t259u answers; the parity section; the t263u section; next P1 or P2; Vol. 4 inclusion.
69	13	16	62	Anomaly = $1/\text{area}$ and reflected half; denting as Poisson width; reach verdict; ghosts respect reflection; next D1/D3/D8.
70	14	12	75	Modulus as numerator of Euler-depleted $\zeta(1-k)$ at both cusps; weight 2 as pole plane; level- N reflection (E1); congruences RH-neutral; next.
71	12	14	90	Adopt Thm. NS; accept the $\frac{1}{2}g_p/\text{circular-orbit}$ verdict; closure $\iff$ supersingular; principalization; next.
72	9	13	104	Accept “negativity RH-blind, RH = fluctuation size”; adopt Thm. mb; extend to Selberg’s $\frac{1}{4}$; primitivity $\leftrightarrow$ principalization; next.
73	13	15	120	Class group as obstruction to integral spinor square roots; scattering unitarity automatic [Restatement]; fluctuation thesis by channel; 3+1 route; next.
74	10	14	135	E as anticanonical height zeta (Peyre constant); $\oplus$ correction; disk metaphor; close the polarization line (open: can $Z_{Q_r}(\rho/2) = 0?$); next.
75	11	14	150	Official ordering “zeros $\frac{1}{4} < \delta_P = \frac{1}{2} < \text{anomaly } 1$ ”; parity verdict; “head – pole counterterm”; program-wide no-go; next.
76	12	13	164	Count = mode 0, discrepancy = modes $n \neq 0$; Franel/Landau as restatements of Littlewood; factorial-motive verdict; program-wide no-go; next.
77	11	13	178	Krein/ κ reading; average and window model; Broué–Enguehard no-go; prime-side blindness lesson; next.
78	10	18	196	Thesis “causality automatic, RH = linewidth $\frac{1}{4}$ ”; meta-space = Soulé limit; Li = involution, not Rosati positivity; quarter turn needs multiplicity; next.

Ch.	§	note p.	vol. p.	Topics
79	10	9	206	Line pairing blind, curve pairing decisive; $1 + 1 + 1 = \Gamma$; $n!$ verdict; multiplicity and Hecke compatibility; next.
80	7	5	211	Accept the no-go as closing D1/E1; meaning of (H3) and arithmetic weights; the Connes-cutoff escape; multiplicity; proceed to L81–83.
81	10	10	221	Accept Hodge–Riemann = Weil; DH on the coefficient side; three-input table; $E \times E$ target shape; next.
82	8	5	226	Accept the detection theorem; blind positivities; marginality as a J target; prolate basis vs. geometry; Epstein ghosts at finite cutoff.
83	9	7	233	Coulomb = the archimedean place; the [t334] corrections (Q2 asks for the verbatim text); Thm. 5.1 (closed orbits DH-proof); the Bertrand/Selberg dictionary; the transport gap (Thm. 6.4); next.
84	10	9	242	Accept Thm. 3.2 (κ counts failures of $\text{Tr}(xx^\dagger) > 0$); the [t348]/[t349] corrections; which $\mathbb{Q}$ -structure (Hecke, Bost–Connes, Weil on rational test functions); Faltings–Hriljac is not RH, the ample class on $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ is the open target; next K3 or K2.
85	13	17	259	Accept the function-field summary; accept corrections (1)–(7), (G1)–(I6); focus O4 vs. O3 and the §10 slogan; Q4 (replacing the [t350] request) the §9 verdict (local $\nRightarrow$ global); next lemma.

APPENDIX C

Consolidated notation

Each chapter defines its notation at the start of the note, and glosses objects such as h^1 , H^1 , the Hodge–Riemann relations and the Hodge index theorem where it uses them. This appendix lists the global conventions and the symbols that different chapters use differently. It covers the whole of Volume IV, so it also names chapters of Volume IVa; chapter numbers are the same in both parts.

Global conventions

- $s = \sigma + it$; nontrivial zeros $\rho = \beta + i\gamma$, $\delta_\rho = \beta - \frac{1}{2}$; mirror $\rho^* = 1 - \bar{\rho}$; $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$; $\Xi(t) = \xi(\frac{1}{2} + it)$. Some chapters use $s = \frac{1}{2} - iz$ (Lemma 33/34/37 coordinates; Ch. 48) or $\lambda = s(1-s)$ (Chs. 81, 85).
- *Ghost*. An off-line zero. Most chapters count both members of a pair $\{\rho, \rho^*\}$ together. Ch. 48 calls a zero with $\beta > \frac{1}{2}$ a ghost, so each pair contributes one ghost.
- *Davenport–Heilbronn*. f_{DH} is the Davenport–Heilbronn combination of the two L -functions mod 5, with completed form $\Lambda(s) = (5/\pi)^{s/2}\Gamma(\frac{s+1}{2})f_{\text{DH}}(s)$, self-dual. It has a functional equation and no Euler product. Its first off-line zeros are $0.80851718 + 85.69934849i$ and $0.65083008 + 114.16334273i$ (Ch. 76; the same values recur in Chs. 78–85).
- *Over $\mathbb{F}_q$* . F is Frobenius on H^1 (an ℓ -adic vector space; h^1 is the motive, a categorical object). $V = qF^{-1}$ is the Verschiebung, written $\tilde{V}$ in Ch. 54. $F^\dagger = qF^{-1}$ is the Rosati-type involution on $\mathbb{Q}[F]$ (Ch. 84). On H^2 of a surface, Ch. 85 uses $F^\dagger = q^2F^{-1}$.
- *Rosati, Albert, Hodge index*. The Rosati involution is the adjoint for a polarization. Albert’s classification lists the division algebras with a positive involution (types I–IV). The Hodge index theorem states that the intersection form on $\text{NS}(S) \otimes \mathbb{R}$ of a surface has signature $(1, \rho - 1)$ (restated in Chs. 79, 81, 84, 85).

Symbols used differently in different chapters

Symbol	Meanings
κ	<i>Main meaning</i> (Chs. 77, 81, 82, 84, 85): the number of off-line (off-circle) reciprocal pairs, i.e. the negative index of the Weil-type form; a complex off-circle quartet counts as 2. Ch. 82 counts negative directions of Q_L on real functions, 2 per quartet. <i>Ch. 56 (and Chs. 59, 63)</i> : the official saddle counter $\kappa = \tau + S^+$ (Kim’s window identity), which equals the number of ghosts in the cases stated there; Ch. 59 adds tower counters $\kappa_n = G_n$, with $\kappa = \kappa_0$. <i>Chs. 50–51</i> : κ_λ is a Krein sign and $\hat{\kappa}$ a normalized Krein norm.
k	Winding number and ghost charge ($\text{ch} = -k$), Chs. 48, 53, 54; weight of a modular form, Chs. 66–70, 85; spin index of U_k , Ch. 85.
Q, W	The Weil form. Truncations: W_A (support $[-A/2, A/2]$; Chs. 57, 64), Q_L (finite support, Connes cutoff $\Lambda = e^{L/2}$; Ch. 82), Q_w (weighted, Ch. 80), Q_χ (sector form, Ch. 81), T_N (Toeplitz form of zero power sums, Ch. 77).
J	A compatible complex structure (the sought polarization) throughout. J_F is Frobenius’s own (Ch. 54); J_ε (Ch. 50); J_p or $J_{\mathbb{Q}}$ local models (Chs. 64–66).

Symbol	Meanings
Λ	Completed L -function (most chapters); de Bruijn–Newman constant (Λ , Λ_{DH} ; Chs. 56, 81); Connes’s cutoff (Ch. 82).
E	An elliptic curve; the Eisenstein series $E(\tau, s)$ or $E(z, s)$ (Chs. 67–75); E_k holomorphic Eisenstein series; $E(\cdot, \cdot)$ an alternating form (Ch. 81).
ϕ, φ	Scattering coefficient of the Eisenstein series (normalizations stated per chapter: $E \sim y^s + \phi(s)y^{1-s}$ on $\mathbb{H}^2$; $E \sim y^{1+s} + \phi(s)y^{1-s}$ on $\mathbb{H}^3$, Ch. 73); Euler’s totient in Farey counts (Ch. 76).
ε, w	Root number; for a newform, $f W_N = wf$ and $\varepsilon = -w$ (Ch. 85).
$\rho, \bar{\rho}$	Zero of an L -function; Picard number $\rho(S)$ and geometric Picard number $\bar{\rho}$ (Ch. 85).

Claim tags

The tags are mapped to volume classes in the table after the Statement of Intellectual Honesty.
Local variants:

- Ch. 77 has [EXACT-COMP] (exact arithmetic plus certified root isolation).
- Ch. 57 has “[THM, computer-assisted]” and [CROSS-CHECK, uses zeros].
- Ch. 58 has [METAPHOR].
- Chs. 84–85 have [CITED-sec].
- Chs. 48–56 use the short forms [THM], [NUM], [CONJ].