

Arithmetic Unitarity II

The Light Cone of the Weil Form

Signature, Certified Windows, Damping and Ghost Birth

Thermodynamics of Arithmetic Action, Lemmas 14–29

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Abstract

Volume I (ai.viXra:2610.0045v2) showed that the Riemann Hypothesis (RH) is equivalent to unitarity, meaning the absence of negative-norm (“ghost”) directions, of the logarithmic scaling flow for the Weil form $\mathcal{Q}$. This volume measures that obstruction directly, in one geometric language: the Weil form is the transverse part of a *Lorentzian* form. For curves over $\mathbb{F}_q$, the Hodge index theorem makes $\mathrm{NS}(C \times C)_{\mathbb{R}}$ Lorentzian, the graph of Frobenius is the diagonal boosted by rapidity $\frac{1}{2} \log q$, and Weil’s bound is Cauchy–Schwarz on the transverse part (Ch. 14). For ζ , the pole plane minus $\mathcal{Q}$ is a form $\mathcal{L}$ that can be computed without zeros, and RH is equivalent to $\mathcal{L}$ having Lorentzian index ≤ 1 on every finite-dimensional subspace (Ch. 15, a restatement of Weil’s criterion). We then certify positivity of $\mathcal{Q}$ on every window of length ≤ 1.6 (Ch. 16, computer-assisted); give the arithmetic action a Lagrangian, Noether, Ostrogradsky and transmission-line reading (Ch. 17); make the damped (σ -shifted) window index exact, with $\Theta = \sup \mathrm{Re} \rho$ characterized by it (Ch. 18); separate trace bounds from Rosati positivity (Ch. 19) and read T_p as two boosts (Ch. 20); prove no-go theorems for zero-side certificates, prolate detectors, tropicalization and $\hbar$ -deformation (Chs. 21–23, 26); correct the explicit counting constants and certify zero-sampling inequalities on Platt’s zeros (Chs. 24, 24A, 24B); read each zero as a telegrapher line in a Krein space, with a passivity ladder from the functional equation to RH (Ch. 25); show that ghost birth is a two-stage event (Krein-space degeneration, then a Krein collision) (Ch. 27); and prove pole dominance for damped forms, with an exact closed-form threshold $A_{\sigma}^{\mathrm{pole}} = 2 \arcsin(2\sigma - 1)/\sqrt{\sigma(1 - \sigma)}$ (Chs. 28–29). Every statement carries a claim label (legend below). **RH is not proved.** The positivity source that geometry supplies for curves is not found for $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$; several natural candidates are proved not to supply it.

Status of this volume

This volume was assembled on 22 October 2026 from the standalone notes TAA Lemmas 14–29, using their latest versions. The note texts are reproduced verbatim except for (i) chapter-prefixed labels and canonical citation keys; (ii) two macro names renamed because they had different meanings in different notes (R_{cor} in Ch. 26 is printed with `\RcConvex`, S in Ch. 24B with `\SalignB`); (iii) “Questions for J.W.” renamed “Open questions”; and (iv) footnotes marked [Vol. 2 editorial note], which point to later corrections. All corrections are collected in Appendix A. Computations that were still running are *placeholders* (Appendix B; macro file `pending.tex`). No value has been filled in by estimate or extrapolation.

Statement of Intellectual Honesty

**The Riemann Hypothesis is NOT proved in this volume.
Nothing here implies it. No result here should be quoted as progress “towards a proof” beyond what is listed in Part 1.**

Numbers refer to this volume. Chapter k is the standalone note “TAA Lemma k ” (Chapter 23 is the note `tropical_note.tex`; Chapters 24A and 24B are the two supplements to Lemma 24). A statement numbered $k.s.n$ is statement $s.n$ of that note. References to Lemmas 1–13 are to Volume I.

Claim labels

The notes use several tag systems. Each tag belongs to exactly one of six volume classes.

Volume class	Meaning	Tags used in the chapters
Theorem	Proved: here, or a cited published theorem used as stated, or computer-assisted with rigorous (interval/ball) arithmetic. Sub-kinds are kept visible.	[THM], [THM], “rigorous”, “theorem”; cited: [CITED], [CITED]; computer-assisted: [THM,CA], [CA], [CERT]; inference from cited universal properties: [DEDUCTION]; definitions [DEF], [DEF] (not claims)
Restatement	Equivalent to a known statement (RH, GRH, $\Theta \leq \sigma$); no new content.	[RH-EQUIV], [$\equiv$ RH], [$\equiv \Theta \leq \sigma$, open for every $\sigma < 1$], [RESTATEMENT], [$\equiv$ GRH for the family], “reformulation”
Analogy	Structural parallel or heuristic argument; proves nothing.	[ANALOGY], [ANALOGY], [HEUR], Heuristic and Analogy environments, [CITED, SEMICLASSICAL]
Conjecture	A stated expectation, with evidence.	[CONJ], [CONJECTURAL]; items (S_n) in Part 2
Open	A question or gap explicitly left open.	[OPEN], Question environments, “Open questions” sections, Lemma 24.2.4
Numerical	Floating-point computation or cost estimate; not a proof.	[NUM], [NUM], [EST], “data”, “numerical check”; and the placeholders of Appendix B

A statement with two tags (e.g. [THM]+[NUM]) is a theorem whose numerical illustration is not part of the proof. “Certified” always means rigorous interval arithmetic (Arb). Certifying a *finite* window never implies RH.

Part 1. What we have proven

1a. Theorems with complete proofs in this volume.

- 14 Prop. [14.1.1](#): the constant $\frac{7}{8}$ in the smooth zero count, from $\arg \Gamma(\frac{1}{4} + \frac{iT}{2})$. Proof of Castelnuovo–Severi (Thm. [14.4.2](#)) from Hodge index.

- 15 Thm. 15.1.2 (the Lorentzian form $\mathcal{L} = \mathcal{P} - \mathcal{Q}$ is T_t -invariant; T_t is a boost on the pole plane); Thm. 15.1.3: $\sup_V n_+(\mathcal{L}|_V) = 1 + N_+$ [Theorem; the corollary “RH $\iff$ index ≤ 1 ” is a Restatement]; Prop. 15.2.1 (curves: $n_+ = 1$, rank $\leq 2 + 2g$).
- 17 Thms. 17.2.1, 17.3.1 (field and window actions; stationary values, κ_A non-decreasing); Props. 17.4.1, 17.4.2; Thm. 17.5.3 (the Weil action has no Ostrogradsky ghosts; its possible ghosts are complex-frequency pairs); Prop. 17.7.1; Thm. 17.9.8 (for curves, “distortionless” is Cauchy–Schwarz on the transverse part). Thms. 17.9.2, 17.9.9 are [Restatement].
- 18 Thm. 18.2.1 ($\mathcal{Q}_\sigma = \mathcal{W}^{(\sigma)} +$ point forms of the zeros right of σ); Cor. 18.2.2; Prop. 18.3.1 (monotonicity in A and σ); Thm. 18.3.3 (Θ -equivalence) [Restatement at each fixed σ]; Thm. 18.4.3 (explicit $\sigma_*(A) < 1$ for every window); Prop. 18.5.2 (invisibility exponent $\frac{1}{2}$, finitely many ghosts); Prop. 18.5.3.
- 19 Props. 19.1.2, 19.2.2; Thm. 19.2.3 (shadow vs. body: one trace bound is not the full statement for $g \geq 2$; integer “coin” polynomial $1 - 22T^2 + 49T^4$); Thm. 19.4.1 (J.W.’s volume/area sandwich: valid after one fix); Prop. 19.4.2. Thm. 19.3.2 is [Restatement].
- 20 Prop. 20.1.1; Thm. 20.2.1 ($T_p/\sqrt{p}$ as two boosts plus a transverse part, given Eichler–Shimura); Props. 20.2.3, 20.4.1, 20.4.2.
- 21 Thm. 21.2.1: no certificate from counting-only zero data exists for windows $A > A_{\text{crit}} = 2.113$; Props. 21.3.1, 21.5.1; Thm. 21.7.1 (three-band identity).
- 22 Props. 22.3.1, 22.4.1 (no subspace detects a ghost earlier than the full window), 22.4.2.
- 23 Props. 23.1.1, 23.1.2; Thm. 23.2.1; Prop. 23.2.2 (tropical positivity = single shadows).
- 24 Lemma 24.1.1 (corrected counting bound R_{cor}); Prop. 24.2.1 (sharp band edges collapse the frame bound).
- 24A Prop. 24A.1.1 (no penalty-free sampling bound for a finite set); Thm. 24A.3.1 (exact reduction to a finite model space).
- 24B Thm. 24B.2.3 (multi-block sampling bound with constants independent of the number of blocks).
- 25 Thm. 25.1.2, Props. 25.1.6, 25.1.7 (operator model of the zero space; $J_{\text{FE}}\beta J_{\text{FE}} = 1 - \beta$; three energies); Thm. 25.2.2 (telegrapher normal form; the line of ρ is distortionless iff $\text{Re } \rho = \frac{1}{2}$); Props. 25.2.3, 25.2.4, 25.2.6, 25.3.2, 25.3.3, 25.5.1, 25.6.3; Thm. 25.4.1 (last clause [Restatement]).
- 26 Lemma 26.1.2, Thm. 26.1.3 (inertia of the transported form is $\hbar$ -independent), Prop. 26.2.1, Thm. 26.3.2 (the Weil index is not first-order definable over $\mathbb{R}^{\text{convex}}$).
- 27 Prop. 27.1.1, Thm. 27.1.3 (an off-line pair is a boost composed with a rotation; RH $\iff [U_u, J_{\text{FE}}] = 0$, [Restatement]); Prop. 27.3.2; Lemma 27.4.1; Prop. 27.4.2 ($\kappa_A = n_{\mathbb{C}} + n_-$); Thm. 27.4.4 (two-stage mechanism, under its stated hypotheses); Thm. 27.9.3; Thm. 27.9.5.
- 28 Lemma 28.1.1, Prop. 28.2.2, Prop. 28.3.2 (density theorems bound the negativity mass of the damped symbol), Lemma 28.3.5, Thm. 28.3.6 (pole dominance).
- 29 Thm. 29.2.2; Prop. 29.2.4 (the inequality asked in Question 3.8 of Lemma 28 is false); Cor. 29.2.6; Thm. 29.2.9; Thm. 29.3.1 and Cor. 29.3.2 (exact A_σ^{pole} ; $c_\infty = \pi$; exact pair thresholds).

1b. Results resting on cited theorems (complete given the citations).

- 14 The chain prequantization $\rightarrow$ Lefschetz $(1, 1) \rightarrow$ Hodge index $\rightarrow$ Castelnuovo–Severi $\rightarrow$ Weil’s RH for curves (Thm. 14.2.1 and the cited theorems of §14.4).
- 19 Prop. 19.1.1; Thm. 19.2.1 (Rosati positivity is the full statement).
- 20 Cor. 20.2.2 (weight-2 Ramanujan, classical); Prop. 20.3.1.
- 25 Prop. 25.1.8 (passivity ladder; rungs from Wiener–Ikehara, de la Vallée Poussin, von Koch); Props. 25.3.1, 25.6.1 (classical).

- 26 Thm. 26.3.1 (Garg–Aujla monotonicity).
- 27 Prop. 27.2.1, Cor. 27.2.2; Thm. 27.4.3 (Krein–Gel’fand–Lidskiĭ definite-type stability).
- 28 Prop. 28.2.4 (Connes’ block decomposition); Cor. 28.3.3 (Ingham, Huxley, Guth–Maynard).
- 29 Cor. 29.2.12 (damped positivity at almost every height, given the cited density bounds).

1c. Computer-assisted, certified (ball arithmetic).

- 15 Inertia of $\mathcal{L}$ on 99 explicit subspaces of dimension up to 31 (certified LDL^T): signature $(1, n-1)$ whenever a timelike vector is present; a planted off-line quadruple gives $n_+ = 3$.
 - 16 Thm. 16.2.1: $\lambda^*(4) \geq 3.5368 \cdot 10^{-13}$ and $\lambda^*(e^{1.6}) \geq 9.2792 \cdot 10^{-18}$. Hence $\mathcal{Q}$ is positive definite on *every window of length* ≤ 1.6 . This is the largest certified window in this volume.
 - 18 Thm. 18.4.1 ($\inf_t w_1 = m_1 = 2 + \gamma - \log 4\pi$ exactly; $\lambda_\sigma(A) \geq m_\sigma$ for $\sigma \geq 1$ and all A); Thm. 18.4.2 (short windows, all $\sigma \geq \frac{1}{2}$).
 - 24 Certified rows of the Lemma 12 rerun with R_{cor} (Table 24.1; “pend.” cells are *not* certified, see Appendix B); the Lemma 18 tail constant $g(5000) = 0.07167$; Thm. 24.2.2 (padded frame bound on Platt’s zeros).
 - 24A Thms. 24A.4.1, 24A.5.1: per-block sampling inequality for all of PW_ω at $A = 2.4, \dots, 3.0$.
 - 24B Thm. 24B.4.1, Cor. 24B.5.1: every block of the Lemma 21 middle band certified at $A = 2.4, 2.6, 2.8, 3.0$. These are *sampling* inequalities, *not* window positivity (Appendix A, item B1.).
 - 29 [CERT] brackets of relative width $\leq 10^{-9}$ containing the closed form of Thm. 29.3.1.
- Certificates logged by the Lemma 24 Toeplitz/Levinson pipeline after the notes were written (window $A = 2$, and others) are *not* counted here; their write-up is pending (Appendix B).

Part 2. What we suspect to be true

Each item is labelled and its evidence stated. None is used in any proof of Part 1.

- (S1) **Damped detection length scales like $\varepsilon^{-1/2}$ at fixed abscissa** [Conjecture]. A ghost of displacement $\varepsilon = \beta - \sigma$ is first seen by $\kappa_{\sigma,A}$ at $A_\sigma^{\text{det}} \asymp \sqrt{b/\varepsilon}$, b the local background (Ch. 18, item (5)). *Evidence*: ghost-injection numerics (local exponent falling from 0.85 to 0.60); the rigorous lower bound with exponent $\frac{1}{2}$ (Prop. 18.5.2); the exact pair threshold $A^{\text{pair}}(\varepsilon, w) < \pi/\sqrt{\varepsilon w}$ (Cor. 29.3.2). This *replaces* Volume I’s guess $c/(\beta - \sigma)$ (Vol. I, S5).
- (S2) **Ghost birth is two-stage for ζ windows** [Numerical]. Birth ($\det \mathbf{G}_A = 0$) precedes the Krein collision ($\delta_b < \delta_c$, both decreasing in h), with the boost law $d_c(\delta) = 2.70 + 0.275 \log \frac{1}{\delta}$ against the birth law $d_b(\delta) = 2.40 + 0.172 \log \frac{1}{\delta}$ (parity 0; Ch. 27). *Evidence*: $\kappa_A = n_{\mathbb{C}} + n_-$ in all 1012 + 282 runs; the theorem is Thm. 27.4.4 under stated hypotheses, which are checked numerically, not proved, for ζ .
- (S3) **Window Grams are Lorentzian polynomials** [Numerical; on all windows a Restatement]. Lemma 15’s window Gram matrices with $h \geq 0.1$ are numerically Lorentzian in the sense of Brändén–Huh (Prop. 23.2.3).
- (S4) **Near-radical dimension** [Numerical]. $K(d) = 2d - \frac{7}{4} \pm 0.7$ for $2 \leq d \leq 13.5$, i.e. the Shannon number in $\log x$ minus $2N(2\pi d)$, matching Connes–Consani’s prolate count up to $\frac{1}{4}$ (Ch. 22).
- (S5) **$\hbar$ -threshold** [Numerical]. The true ζ Grams lose Lorentzian-ness for $\hbar < 1$, with $\hbar_* \rightarrow 1$ as the window grows (Ch. 26, certified inertia on 72 windows; the law $\hbar_* \rightarrow 1$ is a fit).
- (S6) **Hybrid certificates reach beyond $A = 1.6$** [Open]. Logged certificates at $A = 1.5, 1.55, 1.6$ and $A = 2$ (Appendix B) indicate that the Toeplitz/Levinson pipeline extends certified window positivity; the run at $A = 2.4$ is [**pending: P1**]. Until written up, nothing beyond $A = 1.6$ is claimed. Even a success at any finite A excludes no zero (Ch. 16, the corollary to Thm. 16.2.1; Volume I, Ch. 12).

- (S7) **Model-space approximation lemma** [Open]. Lemma 24.2.4; and a Kadec/Avdonin-type sampling theorem for the ordinates using local counting (Ch. 21, Open questions).
- (S8) **Arithmetic positivity source** [Open]. A Hodge-index/Rosati-type positivity for $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$. *Evidence*: only the proved curve case (Chs. 14, 17, 19, 20, 27). Caution, as in Volume I: such a structure exists iff RH, so its existence is not evidence; only a construction independent of the zeros would be. Chs. 22, 23 and 26 show that three natural candidates (prolate operators, tropicalization, $\hbar$ -deformation) do not supply it.
- (S9) **Analogies kept as analogies** [Analogy]. Bohr–Sommerfeld/Maslov for $\frac{7}{8}$ (Ch. 14; refined later, Appendix A); L_4/L_5 stability for ghost birth (Ch. 27, Addendum B); Heaviside’s distortionless line (Chs. 17, 25); the physical dictionary of the TAA vocabulary.

Part 3. Our final goal

The goal of the TAA programme is unchanged from Volume I: to *derive* RH as a unitarity (ghost-freedom) theorem. That means exhibiting a positivity principle on the “arithmetic surface” $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$, constructed *without* reference to the zeros, that does for ζ what the Hodge index theorem does on $C \times C$. In the language of this volume, the target is concrete. It is a reason why the zero-free Lorentzian form $\mathcal{L}$ has index ≤ 1 on every finite-dimensional subspace (Ch. 15). Equivalently, it is a Hilbert structure making the half-density scaling flow unitary (Ch. 19, Thm. 19.3.2), i.e. a flow that commutes with the functional-equation symmetry J_{FE} (Ch. 27, Thm. 27.1.3). It is the step from the passivity rung that is free (Ch. 25) to the rung $\delta = 0$.

Status. RH is **not** proven here. Every chapter does one of three things. It *restates* the obstruction in a sharper geometric form (Chs. 14, 15, 17–20, 25, 27). It *certifies* finitely many windows, which cannot imply RH (Chs. 16, 18, 24). Or it *proves that a candidate mechanism fails* (Chs. 21, 22, 23, 26, 28, 29). The unconditional positive results (pole dominance, the exact threshold A_σ^{pole} , the bound $\sigma_*(A)$) are data-blind. They hold for every “fake ζ ” with zeros in the closed strip, so they cannot give $\Theta < 1$, let alone $\Theta = \frac{1}{2}$.

Place in the series

The series is organized in volumes of about 10–20 lemmas, each titled by its overarching thread. Lemmas 14–47 are split into two volumes, with the break after Lemma 29:

Volume	Title	Contents and status
I	Arithmetic Unitarity I: Where the Riemann Hypothesis Lives in the Scaling Flow	Lemmas 1–13; ai.viXra:2610.0045v2
II	Arithmetic Unitarity II: The Light Cone of the Weil Form	Lemmas 14–29 (this volume): the obstruction measured on the Weil form itself (signature, certified windows, damping, Krein collisions, pole dominance)
III	Arithmetic Unitarity III: In Search of Positivity — G5, Polarization, and Hilbert versus Krein (proposed)	Lemmas 30–47 (in preparation): the axioms G1–G6, the G5 line, polarization, Hilbert vs. Krein, Verschiebung

The break after Lemma 29 is where the questions change. Up to Lemma 29 every note asks what the Weil form, its window restrictions and its damped versions can see. From Lemma 30 on, the notes ask which structure could *supply* positivity, in the form of the axioms G1–G6.

Contents

Abstract	i
Status of this volume	i
Statement of Intellectual Honesty	iii
Claim labels	iii
Part 1. What we have proven	iii
Part 2. What we suspect to be true	v
Part 3. Our final goal	vi
Place in the series	vi
Introduction	xi
Chapter 14. Lemma 14: Sommerfeld, integral classes and the Hodge index	1
14.1. (a) Bohr–Sommerfeld for $H = x\xi$ and the constant $\frac{7}{8}$	1
14.2. (b) Prequantization: Sommerfeld as integrality	2
14.3. (c) Lefschetz $(1, 1)$	3
14.4. (d) Weil’s proof via $C \times C$	3
14.5. Light-cone section	4
14.6. (g) Frobenius as a Lorentz boost	5
14.7. (h) Why $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ is not a surface	6
Status	6
Chapter 15. Lemma 15: The signature test	7
15.1. The Lorentzian Weil form	7
15.2. The $C \times C$ model	9
15.3. Certified computations for ζ	9
Status	10
Chapter 16. Lemma 16: Certified Weil positivity on windows of length $\log 4$ and 1.6	13
16.1. Setting and method	13
16.2. Certificates	14
16.3. Monotonicity of κ_A	14
16.4. Comparison with Zhu, and cost	15
Status	15
Chapter 17. Lemma 17: The arithmetic action as a Lagrangian	17
17.1. Three Lagrangian levels	18
17.2. L-I: field-theory form of the action	18
17.3. L-II: the variational principle on a window	19
17.4. Noether: scaling symmetry and its charge	20
17.5. Legendre transform, Ostrogradsky, and where ghosts come from	21
17.6. L-III: the point-particle level	23
17.7. The function-field model: a provably non-negative Lagrangian	23

17.8.	Scale-free remark: logarithmic kinetic operator; scale vs description	24
17.9.	Transmission-line form (Heaviside)	25
17.10.	Confluence of the three actions	28
17.11.	Dictionary: Lagrangian mechanics $\leftrightarrow$ arithmetic action	30
17.12.	Obstruction	31
17.13.	Open questions	31
	Status summary	32
Chapter 18.	Lemma 18: The σ -window index $\kappa_{\sigma,A}$:	33
18.1.	Setting	33
18.2.	Exact structure	34
18.3.	Monotonicity and the Θ -equivalence	35
18.4.	Unconditional results with explicit constants	36
18.5.	How the σ -detection length scales (ghost injection)	37
18.6.	Relation to κ_A , rigor status, obstruction	40
Chapter 19.	Lemma 19: “The $ a $ term is halfway; it needs a $\ a\ $.”	41
19.1.	Elliptic curves: the trace is an inner product, the degree is the norm	41
19.2.	Genus g : one trace bound is a shadow, the Rosati norm is the body	42
19.3.	Translation to ζ : one window is one section; all windows is the norm	45
19.4.	Volume vs area	47
	Obstruction	49
19.5.	Open questions	49
	Status	49
Chapter 20.	Lemma 20: T_p as two boosts	51
20.1.	Setting and the Hecke span in $\mathrm{NS}(X \times X)_{\mathbb{R}}$	51
20.2.	The two-boost theorem	52
20.3.	CM calibration	53
20.4.	The rapidity picture in $\mathbb{H}$ and the angle distributions	54
20.5.	What this gives, and what transfers to $\mathrm{Spec} \mathbb{Z}$	56
Chapter 21.	Lemma 21: Certifying Weil positivity from the zero side	57
21.1.	Setting	58
21.2.	The no-go theorem	58
21.3.	A sufficient condition in the model	59
21.4.	Comparison with Lemmas 12 and 18	59
21.5.	Restricted classes	60
21.6.	Prime-side costs: (b) Slepian basis, (c) Toeplitz/Levinson	60
21.7.	The three-band hybrid	60
21.8.	Obstruction, questions, status	62
Chapter 22.	Lemma 22: The prolate degeneration test	63
22.1.	What is proved and what is not	63
22.2.	The window dictionary	64
22.3.	Level 1: the operator W_λ is ghost-blind	66
22.4.	Levels 2 and 3: Weil-form prolate objects	66
22.5.	Erratum to Lemma 11’s ghost numerics	67
22.6.	Verdict	68
Chapter 23.	Lemma 23: Tropicalizing $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$	71
23.1.	Dequantization, done correctly; the zero side keeps only $\mathrm{Re} \rho$	71

23.2.	The decisive test on $C \times C/\mathbb{F}_q$	72
23.3.	Literature: what is proved, what is missing	74
23.4.	Conclusions, ranking, questions	75
Chapter 24.	Lemma 24: Lemma 21 follow-ups	77
24.1.	(1) Corrected zero-counting bound	77
24.2.	(2) Sampling the zeros: data, a certified padded frame bound, and what is missing	78
24.3.	(3) Hybrid certificate: projection and stop	80
24.4.	Status, obstruction, questions	80
Chapter 24A.	Lemma 24, supplement: Lemma 3.4 proved	83
24A.1.	Setting, and why the statement must carry a penalty	83
24A.2.	An explicit reproducing kernel	84
24A.3.	Exact reduction to a finite model space	84
24A.4.	The finite certificate on all of V	85
24A.5.	Main result	85
24A.6.	Side finding: the enclosure in Lemma 24, Thm. 3.2	86
24A.7.	What is proved, and what is still missing	86
Chapter 24B.	Lemma 24, supplement b: a multi-block sampling bound	89
24B.1.	Blocks and weights	89
24B.2.	The per-block inequality with a localised far field	90
24B.3.	Zero data above 446 000	91
24B.4.	Certification of every block	91
24B.5.	The final uniform inequality	92
24B.6.	What remains for the full hybrid	93
Chapter 25.	Lemma 25: The telegrapher components of the zero-waves x^ρ	95
25.1.	The operator reading	96
25.2.	One channel: the telegrapher normal form of a functional-equation pair	100
25.3.	The transfer-function reading	102
25.4.	Assembly: the explicit formula as a superposition of lines	104
25.5.	Curves over $\mathbb{F}_q$: the same lines, with $\delta = 0$ proved	105
25.6.	Energy positivity versus norm positivity	106
25.7.	What is missing	108
25.8.	Open questions	109
	Status summary	110
Chapter 26.	Lemma 26: Window Gram matrices under the Viro/Connes–Consani deformation $\theta_{1/\hbar}$	111
26.1.	The construction, from the Connes–Consani sources	111
26.2.	Where the limits land	112
26.3.	Monotonicity and the $\mathbb{R}^{\text{convex}}$ obstruction	113
26.4.	Numerics	113
26.5.	Verdict, rigor, obstruction, questions, ranking	114
Chapter 27.	Lemma 27: Ghost birth in the window Weil form and Krein collisions	117
0.	Verdict and status labels	117
27.1.	The hyperbolic plane of an off-line pair and the boost	118
27.2.	The curve case	119
27.3.	The frequency pencil	120
27.4.	The Krein-space theorems that transfer	121

27.5.	Numerics	122
27.6.	L_4 , Routh, MacKay: what is analogy	122
27.7.	Open questions	123
27.8.	Summary	123
27.9.	Addendum A (rev. 2): the two thresholds as the window grows	123
27.10.	Addendum B: a Lagrange-point dictionary	126
Chapter 28.	Lemma 28: (A) spatial modes for the zero telegrapher line;	129
28.1.	Setting	129
28.2.	Part A: spatial modes	130
28.3.	Part B: $\Theta < 1$	132
Chapter 29.	Lemma 29: (A) index versus negativity mass of the damped symbol;	137
29.1.	Setting	138
29.2.	Part A: index versus negativity mass (Question 3.8)	138
29.3.	Part B: the exact pole threshold (Question 3.9)	141
29.4.	Priority check (Theorem 3.6 of Lemma 28 and Theorem 29.3.1)	144
29.5.	Open questions	144
Appendix A.	Corrections	145
A.	Errata recorded in the chapters	145
B.	Later corrections to these chapters (Lemmas 30–47)	146
C.	Volume I items revisited	146
D.	Inconsistencies found while collating, and their handling	147
Appendix B.	Pending numerical results and logged certificates	149
	Placeholders	149
	Logged certificates, write-up pending	150
Appendix C.	Consolidated notation	151
	Conventions that differ between chapters	151
	Symbols by chapter	151
	Claim tags	153
Bibliography		155

Introduction

The thread of this volume. Volume I ended with one obstruction: positivity of the Weil form $\mathcal{Q}$ on windows of every length, or, structurally, a positive trace built without the zeros. For curves over $\mathbb{F}_q$ that positivity is the Hodge index theorem. This volume takes the Hodge index theorem literally and asks what it looks like on the ζ side. The answer organizing every chapter is a *light cone*. On $C \times C$ the span $\langle H, V \rangle$ is a hyperbolic plane, Frobenius is a boost of rapidity $\frac{1}{2} \log q$, and positivity lives on the spacelike complement. For ζ , the two poles of the completed zeta function, at $s = 0$ and $s = 1$, give the hyperbolic plane (the pole form $\mathcal{P}$, built from $\tilde{F}(\pm \frac{i}{2})$), the scaling flow is the boost, and the Weil form is minus the transverse part: $\mathcal{L} = \mathcal{P} - \mathcal{Q}$. An off-line zero is a second timelike direction, a ghost. The chapters measure that ghost: by signature, on certified windows, after damping to abscissa σ , through sampling of the actual zeros, as a lossy transmission line, and at the moment of its birth.

Chapter by chapter.

- Ch. 14 (light cone):** Bohr–Sommerfeld $\rightarrow$ prequantization $\rightarrow$ Lefschetz $(1, 1) \rightarrow$ Hodge index $\rightarrow$ Castelnuovo–Severi $\rightarrow$ Weil, every link a theorem for curves, and what is missing for $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$. Frobenius is a boost; the constant $\frac{7}{8}$ of the zero count is derived and checked. *Later errata:* the Maslov reading of $-\frac{1}{8}$ and the attribution of the $+1$ (Appendix A).
- Ch. 15 (signature test):** $\sup_V n_+(\mathcal{L}|_V) = 1 + N_+$: RH is the statement that the zero-free form $\mathcal{L}$ has Lorentzian index ≤ 1 . Certified inertia on 99 subspaces.
- Ch. 16 (certified windows):** Volume I’s certificate is extended from $\log 3$ to $\log 4$ and to 1.6, independently confirming Zhu’s window.
- Ch. 17 (Lagrangian):** The arithmetic action at three levels (field, window, particle), Noether charge, the Ostrogradsky analysis (ghosts can only be complex-frequency pairs), and Heaviside’s distortionless line, which for curves is Cauchy–Schwarz on the Hodge-negative part.
- Ch. 18 (σ -window index):** The damped form decomposes exactly into the Lagarias form plus point forms of the zeros right of σ . Θ is characterized by the damped index. Unconditional results for $\sigma \geq 1$ and short windows; a numerical scaling law for detection that refutes Volume I’s guess. *Later errata:* constants rechecked in Ch. 24.
- Ch. 19 (trace vs. Rosati):** One trace bound is Cauchy–Schwarz for one “shadow”; the full statement is Rosati positivity. An integer “coin” polynomial passes the first shadow and fails the second. J.W.’s volume/area sandwich is checked.
- Ch. 20 (T_p as two boosts):** Eichler–Shimura in Lorentzian form; CM calibration; Sato–Tate in the rapidity variable. Classical content, new presentation.
- Ch. 21 (zero-side no-go):** Counting-only data cannot certify windows beyond 2.113; a three-band hybrid (prime side, zero side, prime side) is the honest version of J.W.’s “lithography” idea. *Later errata:* the “sharp” cost column needs padding (Ch. 24).
- Ch. 22 (prolate test):** What Connes–Moscovici and Connes–Consani(–Moscovici) prove versus observe; the prolate operator does not detect ghosts earlier than window positivity. Contains the erratum to Volume I’s Lemma 11 ghost numerics (already applied in Volume I v2).

- Ch. 23 (tropicalization):** RH is tropical ($\Theta = \frac{1}{2}$); no known mechanism forcing it is. Plain tropicalization cannot yield a Hodge index theorem.
- Chs. 24, 24A, 24B (constants and sampling):** Corrected explicit counting constants (Trudgian 2014 $\rightarrow R_{\text{cor}}$), the Lemma 12 and Lemma 18 reruns, a certified padded frame bound on Platt’s zeros, its proof as a penalised per-block sampling inequality, and an n -independent multi-block version. *Pending:* uncertified cells of the Lemma 12 rerun; hybrid certificates (Appendix B).
- Ch. 25 (telegrapher):** Each functional-equation channel is a uniform telegrapher mode with Heaviside residual δ^2 . On the zero space the functional equation is one identity, $J_{\text{FE}}\beta J_{\text{FE}} = 1 - \beta$. A passivity ladder runs from the free rung $\beta \in [0, 1]$ to RH.
- Ch. 26 ($\hbar$ -deformation):** Neither reading of the Viro deformation adds anything to Ch. 15; it is not a detector.
- Ch. 27 (ghost birth):** An off-line pair is a boost composed with a rotation. Ghost birth is not a Krein collision but a degeneration of the Krein space, followed later by a genuine collision. *Later refinement:* Lemma 41 (Appendix A).
- Ch. 28 (spatial modes, $\Theta < 1$, pole dominance):** Hecke characters as the spatial variable; why density theorems cannot give $\Theta < 1$; pole dominance.
- Ch. 29 (index–mass link):** The inequality asked for is false and a corrected one is proved; the pole threshold is found in closed form. It records errata to Ch. 28.

What is and is not claimed. Nothing in this volume proves RH or any statement known to be equivalent to it. Equivalences with RH are labelled [Restatement]. Their value is that they place the obstruction in a specific geometric object: a second timelike direction of $\mathcal{L}$, a boost on the zero space, a nonzero Heaviside residual. Certified computations reach window length 1.6. By Volume I, Ch. 12, and Ch. 24 here, they exclude no zero: the explicit detection lengths are ≥ 9 .

Conventions and reading guide. Statements are numbered chapter.section.item. A cross-reference “Lemma k , Thm. $s.n$ ” in the text refers to statement $k.s.n$ here when $14 \leq k \leq 29$, and to Volume I when $k \leq 13$. “Lemma 24, supplement” and “L34” refer to Chapter 24A; “supplement b” and “L34b” to Chapter 24B. Lemmas 30–47 are cited by number; they form Volume III. Notation and the conventions that differ between chapters are collected in Appendix C. Files `num/...` named in the text are computation scripts and logs.

CHAPTER 14

Lemma 14: Sommerfeld, integral classes and the Hodge index

The light cone of $\mathrm{NS}(X)_{\mathbb{R}}$, Frobenius as a boost, and what is missing for $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$

Standalone note `lemma14.tex`, October 21, 2026.

Summary. We put one chain of ideas in order:

Bohr–Sommerfeld $\rightarrow$ prequantization (integrality) $\rightarrow$ Lefschetz $(1, 1) \rightarrow$ Hodge index
 $\rightarrow$ Castelnuovo–Severi $\rightarrow$ Weil’s RH for curves.

Every link is a theorem for curves over $\mathbb{F}_q$. We also record precisely what is absent for $\mathrm{Spec} \mathbb{Z} \times \mathrm{Spec} \mathbb{Z}$.

Hodge index makes $\mathrm{NS}(X)_{\mathbb{R}}$ Lorentzian. In that picture, Castelnuovo–Severi is a reversed Cauchy–Schwarz inequality, and the graph of Frobenius is, numerically, the diagonal boosted by rapidity $\frac{1}{2} \log q$ plus a transverse spacelike part. Weil’s bound is ordinary Cauchy–Schwarz on that transverse part.

The Berry–Keating count with Maslov $-\frac{1}{8}$ gives $\overline{N}(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8}$.¹ We derive the constant from $\arg \Gamma(\frac{1}{4} + \frac{iT}{2})$ and check it on the first 1000 zeros: the mean of $S(T)$ is $2 \cdot 10^{-4}$ with $\frac{7}{8}$, against -0.125 with $+1$.

Each statement is labelled *theorem*, *analogy* or *heuristic*. Nothing here bears on RH for ζ .

14.1. (a) Bohr–Sommerfeld for $H = x\xi$ and the constant $\frac{7}{8}$

PROPOSITION 14.1.1 (smooth zero count; theorem). *Let $\theta(T) = \arg \Gamma(\frac{1}{4} + \frac{iT}{2}) - \frac{T}{2} \log \pi$, with the continuous branch and $\theta(0) = 0$. Then*

$$\theta(T) = \frac{T}{2} \log \frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + \frac{1}{48T} + O(T^{-3}).$$

The Riemann–von Mangoldt formula $N(T) = \frac{1}{\pi}\theta(T) + 1 + S(T)$ (for T not an ordinate) therefore gives

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + S(T) + O(T^{-1}).$$

PROOF. Use Stirling’s formula $\log \Gamma(z) = (z - \frac{1}{2}) \log z - z + \frac{1}{2} \log 2\pi + \frac{1}{12z} + \dots$ at $z = \frac{1}{4} + \frac{iT}{2}$ and take imaginary parts. The term $-\frac{\pi}{8}$ is $\mathrm{Im}(-\frac{1}{4}) \log(iT/2) = -\frac{1}{4} \cdot \frac{\pi}{2}$.

So $\frac{7}{8} = 1 - \frac{1}{8}$: the $+1$ comes from the pole of ζ at $s = 1$ (argument principle on the rectangle), and the $-\frac{1}{8}$ comes from the Γ -factor.²

Numerically, at $T = 1000$ the difference $\theta/\pi + 1 - \overline{N}$ equals $6.631 \cdot 10^{-6} = 1/(48\pi T)$ to all printed digits (`num/lemma14_checks.out`). $\square$

¹[Vol. 2 editorial note] Later correction (Lemma 36, Vol. III): the $-\frac{1}{8}$ is a Stirling phase of $\Gamma(\frac{1}{4} + \frac{iT}{2})$, not an integer Maslov index in the standard convention. See Appendix A, item B4..

²[Vol. 2 editorial note] Refined by Lemma 36 (Vol. III): the $+1$ is $\frac{1}{\pi} \Delta \arg[s(s-1)]$, shared between the factors s and $s-1$ of ξ , asymptotically $\frac{1}{2}$ each. See Appendix A, item B3..

HEURISTIC 14.1.2 (Berry–Keating [7, 8]; semiclassical, not a theorem). Take the classical $H = x\xi$ on the region $x \geq l_x$, $\xi \geq l_\xi$, $H \leq E$. Its area is

$$A = E \log \frac{E}{l_x l_\xi} - E + l_x l_\xi$$

(checked symbolically). With $l_x l_\xi = 2\pi\hbar$, $E = \hbar T$ and a Maslov correction $-\frac{1}{8}$,

$$\frac{A}{2\pi\hbar} - \frac{1}{8} = \frac{T}{2\pi} \left(\log \frac{T}{2\pi} - 1 \right) + 1 - \frac{1}{8},$$

which is $\overline{N}(T)$ exactly, constant included [137].

What is *heuristic* here:

- the trajectories are not closed, so “Bohr–Sommerfeld” is an area count, not a quantization of periodic orbits;
- the cutoffs l_x, l_ξ are put in by hand, and the Planck-cell normalisation $l_x l_\xi = 2\pi\hbar$ is chosen precisely so that the +1 appears;
- the $-\frac{1}{8}$ is matched to the $-\pi/8$ of Proposition 14.1.1 rather than derived from an index computation. (For a closed orbit with two turning points the Maslov term would be $\frac{1}{2}$, as for the oscillator.)

What is a *theorem*: Proposition 14.1.1, i.e. that $\frac{7}{8} = 1 - \frac{1}{8}$, with $-\frac{1}{8}$ equal to $\frac{1}{\pi} \cdot (-\frac{\pi}{8})$ from $\arg \Gamma(\frac{1}{4} + \frac{iT}{2})$.

REMARK 14.1.3 (Connes’s count [35]; comparison). Connes cuts off by $|x| \leq \Lambda$, $|\xi| \leq \Lambda$ instead. In the semiclassical count the total is the area $\sim \frac{E}{2\pi} \cdot 2 \log \Lambda$ (up to normalisation), and the zeros enter with a *minus* sign: the smooth count is the area under the hyperbola removed from the box. They appear as an *absorption* spectrum.

Connes–Consani [31] show that this semiclassical computation and the Berry–Keating emission picture coincide up to that sign. Neither cutoff fixes the constant $\frac{7}{8}$ on its own; in both pictures it is a matching with the Γ -factor.

PROPOSITION 14.1.4 (numerical check; data). Write $\gamma_1 < \dots < \gamma_{1000}$ for the first 1000 ordinates, and let $\langle N - \overline{N}_c \rangle$ be the exact time average of $N(T) - (\frac{T}{2\pi} \log \frac{T}{2\pi e} + c)$ over $[\gamma_1, \gamma_{1000}]$. This average equals the mean of S plus $(\frac{7}{8} - c)$. We find

$$\langle N - \overline{N}_{7/8} \rangle = 2.17 \cdot 10^{-4}, \quad \langle N - \overline{N}_1 \rangle = -0.1248.$$

So the data select $\frac{7}{8}$ over the Maslov-free value 1. (That the mean of $S(T)$ tends to 0 is Littlewood’s theorem $\int_0^T S = O(\log T)$.)

14.2. (b) Prequantization: Sommerfeld as integrality

THEOREM 14.2.1 (Weil [156], Kostant [84], Souriau [141]). Let (M, ω) be symplectic. There is a Hermitian line bundle $L \rightarrow M$ with a connection ∇ of curvature $F_\nabla = -\frac{i}{\hbar} \omega$ if and only if $[\omega/2\pi\hbar] \in H^2(M, \mathbb{R})$ lies in the image of $H^2(M, \mathbb{Z})$. In that case $c_1(L) = [\omega/2\pi\hbar]$, and the choices of (L, ∇) form a torsor under $H^1(M, U(1))$.

REMARK 14.2.2 (Sommerfeld as holonomy; theorem). Let $\Lambda \subset M$ be Lagrangian. Then $\nabla|_\Lambda$ is flat, and its holonomy around a loop $\gamma = \partial\Sigma$ is $\exp(-\frac{i}{\hbar} \int_\Sigma \omega) = \exp(-\frac{i}{\hbar} \oint_\gamma p dq)$ when $\omega = dp \wedge dq$ near Σ . So “ $\nabla|_\Lambda$ has trivial holonomy” is *exactly* the old Bohr–Sommerfeld condition $\oint p dq \in 2\pi\hbar\mathbb{Z}$. The half-form (metaplectic) correction replaces it by $2\pi\hbar(\mathbb{Z} + \mu/4)$, where μ is the Maslov index ($\mu = 2$ for the oscillator).

Integrality of $[\omega/2\pi\hbar]$ is the global version of the same condition: every closed surface carries an integer number of Planck cells.

HEURISTIC 14.2.3 (one $\hbar$ per prime; labelled). The local factor $(1 - p^{-s})^{-1}$ has poles at $s = 2\pi i\mathbb{Z}/\log p$. These are the Bohr–Sommerfeld levels of a single periodic orbit of period $\log p$: the level spacing is $2\pi/\log p$, i.e. “the action quantum in the t variable is $\log p$ ”. Since the $\log p$ are $\mathbb{Q}$ -linearly independent (unique factorisation), no common $\hbar$ makes all primes integral at once. This is a theorem about the numbers $\log p$; reading it as an obstruction is heuristic.

14.3. (c) Lefschetz (1, 1)

THEOREM 14.3.1 (Lefschetz (1, 1) [60]). *Let X be compact Kähler. A class in $H^2(X, \mathbb{Z})$ whose image in $H^2(X, \mathbb{C})$ has type (1, 1) is c_1 of a holomorphic line bundle. Consequently $\text{NS}(X) = \text{im } H^2(X, \mathbb{Z}) \cap H^{1,1}$.*

PROOF. Use the exponential sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^* \rightarrow 0$: $c_1 : H^1(\mathcal{O}^*) \rightarrow H^2(\mathbb{Z})$ has image $\ker(H^2(\mathbb{Z}) \rightarrow H^2(\mathcal{O}) = H^{0,2})$. $\square$

COROLLARY 14.3.2 (prequantum $\Rightarrow$ ample; theorem). *Suppose ω is a Kähler form with $[\omega/2\pi\hbar]$ integral. By Theorem 14.2.1 and Lefschetz (1, 1), the prequantum bundle can be chosen holomorphic. By the $\partial\bar{\partial}$ -lemma it carries a metric with curvature form $\omega/\hbar$, so it is positive, hence ample (Kodaira), and X is projective.*

In words: a Kähler form satisfying Sommerfeld integrality is a polarisation, i.e. a timelike class in the sense of §14.5.

REMARK 14.3.3 (scope). Lefschetz (1, 1) is a statement over $\mathbb{C}$. Over $\mathbb{F}_q$, NS is defined directly by divisors modulo numerical equivalence; Weil’s argument uses that definition and needs no transcendental input. The chain “Sommerfeld $\rightarrow$ integral (1, 1)-class $\rightarrow$ divisor” is therefore literal over $\mathbb{C}$ and an *analogy* in characteristic p .

14.4. (d) Weil’s proof via $C \times C$

Let $C/\mathbb{F}_q$ be smooth, projective, geometrically connected, of genus g , and let $X = C \times C$. Put $l = C \times \text{pt}$ and $m = \text{pt} \times C$, so $l^2 = m^2 = 0$ and $l \cdot m = 1$.

Let Δ be the diagonal and $\Gamma = \{(x, F(x))\}$ the graph of the q -Frobenius. Then

$$\Delta^2 = 2 - 2g, \quad \Gamma^2 = q(2 - 2g), \quad \Gamma \cdot l = q, \quad \Gamma \cdot m = 1, \quad \Delta \cdot l = \Delta \cdot m = 1, \quad \Delta \cdot \Gamma = N_1 := \#C(\mathbb{F}_q)$$

[63, V Ex. 1.10].

THEOREM 14.4.1 (Hodge index [63], V Thm. 1.9). *On a projective surface, if $H^2 > 0$ and $D \cdot H = 0$, then $D^2 \leq 0$, with equality iff $D \equiv 0$. Equivalently, the intersection form on $\text{NS}(X)_{\mathbb{R}}$ has signature $(1, \rho - 1)$.*

THEOREM 14.4.2 (Castelnuovo–Severi [63], V Ex. 1.9). *For every divisor D on $C \times C$,*

$$D^2 \leq 2(D \cdot l)(D \cdot m).$$

PROOF. Let $E = D - (D \cdot m)l - (D \cdot l)m$. Then $E \cdot l = E \cdot m = 0$, so $E \cdot (l + m) = 0$ with $(l + m)^2 = 2 > 0$. Hodge index gives $0 \geq E^2 = D^2 - 2(D \cdot l)(D \cdot m)$. $\square$

THEOREM 14.4.3 (Weil [158]). $|N_1 - q - 1| \leq 2g\sqrt{q}$.

PROOF. Apply Theorem 14.4.2 to $D = r\Gamma + s\Delta$. Here $D \cdot l = rq + s$ and $D \cdot m = r + s$. Expanding gives

$$gqr^2 - (N_1 - q - 1)rs + gs^2 \geq 0 \quad \text{for all } r, s \in \mathbb{Z}.$$

A binary quadratic form that is nonnegative on $\mathbb{Z}^2$ is nonnegative on $\mathbb{R}^2$, so its discriminant is ≤ 0 : $(N_1 - q - 1)^2 \leq 4g^2q$. The same argument with F^m gives the bound for all N_m , hence $|\alpha_i| = \sqrt{q}$. $\square$

REMARK 14.4.4 (conventions; correction). With $H := l = C \times \text{pt}$ and $V := m = \text{pt} \times C$, the numerical class of Γ in $\langle H, V \rangle$ is $H + qV$, since $\Gamma \cdot H = q$ is the V -coefficient. The form “ $\Gamma_F = qH + V$ ” in the brief is correct only with the opposite naming $H := \text{pt} \times C$. The brief’s “ $D = a\Delta + b\Gamma$, $\Gamma \cdot H = q$, $\Gamma \cdot V = 1$ ” agrees with our convention.

PROPOSITION 14.4.5 (link to Lemma 3; data). *For the Lemma 3 curves, $E/\mathbb{F}_5$ with $P = 1 + 3T + 5T^2$ and $C_2/\mathbb{F}_7$ with $P = 1 + 3T + 4T^2 + 21T^3 + 49T^4$, the quantities $\delta \cdot \gamma = N_m - q^m - 1$ of §14.6 satisfy $|\delta \cdot \gamma| \leq 2gq^{m/2}$ for $m \leq 6$. For example:*

- E , $m = 4$: $N_4 = 675$, so $\delta \cdot \gamma = 49 \leq 50$;
- C_2 : $N_m = 11, 49, 398, 2377, \dots$, matching Lemma 3.

See `num/lemma14_checks.out`.

14.5. Light-cone section

(a) **Null coordinates.** With $u = t + x$ and $v = t - x$, we have $ds^2 = dt^2 - dx^2 = du dv$.

(b) **The hyperbolic plane $\langle H, V \rangle$.** Since $H^2 = V^2 = 0$ and $H \cdot V = 1$, the Gram matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. So $(aH + bV)^2 = 2ab$, which is $2 du dv$ under $(a, b) \leftrightarrow (u, v)$; the factor 2 is a normalisation.

H and V are null, and $H + V$ (ample) is timelike. By Hodge index, $\text{NS}(X)_{\mathbb{R}} = \langle H, V \rangle \oplus \langle H, V \rangle^{\perp}$ with the complement negative definite. So $\text{NS}(X)_{\mathbb{R}}$ is Lorentzian, and the ample cone lies inside the forward light cone. *Theorem.*

(c) **Castelnuovo–Severi as reverse Cauchy–Schwarz.** Write $D = aH + bV + D_0$ with $D_0 \perp \langle H, V \rangle$. Then $D \cdot V = a$, $D \cdot H = b$ and $D^2 = 2ab + D_0^2$. So Theorem 14.4.2 is equivalent to $D_0^2 \leq 0$, the negativity of the complement. For timelike x, y in a Lorentzian space one has $(x \cdot y)^2 \geq x^2 y^2$ (reverse Cauchy–Schwarz); Hodge index is exactly this statement for $y = H + V$.

HEURISTIC 14.5.1 (J.W.’s “ghosts”; labelled). A “ghost” is a putative extra timelike direction in $\langle H, V \rangle^{\perp}$. If one existed, $D_0^2 > 0$ would be possible, Castelnuovo–Severi would fail, and the discriminant argument in §14.4 would give no bound. “No ghosts” is Hodge index.

(d) **Binary quadratic forms as $\mathbb{R}^{2,1}$ (theorem, classical [37]).** For $f = ax^2 + bxy + cy^2$,

$$b^2 - 4ac = b^2 - (a + c)^2 + (a - c)^2,$$

which has signature $(2, 1)$. $\text{GL}_2(\mathbb{R})$ acts by substitution and multiplies the discriminant by $\det^2$, which gives $\text{PSL}_2(\mathbb{R}) \cong \text{SO}^+(2, 1)$.

Positive definite forms with fixed discriminant $D < 0$ form one sheet of a two-sheeted hyperboloid, identified with $\mathbb{H}$ via the root $\tau_f = (-b + \sqrt{D})/2a$. Degenerate forms $(nx - my)^2$ (with $D = 0$) are null rays, and the rational null rays are the cusps $m/n \in \mathbb{P}^1(\mathbb{Q})$. Examples:

- $x^2 + y^2$ has $D = -4$ and $\tau = i$;
- $x^2 + xy + y^2$ has $D = -3$ and $\tau = \rho = e^{2\pi i/3}$ (checked in sympy).

These are the two elliptic points; their stabilisers have orders 4 and 6 ($= w$).

(e) **Arithmetic surfaces: Faltings–Hriljac.**

THEOREM 14.5.2 (Faltings [52], Hriljac [70]). *Let $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$ be a regular arithmetic surface with generic fibre X_K , and use Arakelov intersection theory [3, 90]. For a divisor D of degree 0 on X_K , let $\Phi(D)$ be the vertical correction making $D + \Phi(D)$ orthogonal to every vertical component, including fibres at infinity. Then*

$$\langle D, D \rangle_{NT} = -(\bar{D} + \Phi(D))^2,$$

up to a normalising factor depending on the height convention. Consequently $(\bar{D} + \Phi(D))^2 \leq 0$, with equality iff $[D]$ is torsion in $\text{Jac}(X_K)$: the arithmetic Hodge index.

What is missing for $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$.

- Faltings–Hriljac needs a base $(\text{Spec } \mathcal{O}_K)$ over which $\mathcal{X}$ is a *curve*, and a Jacobian whose Néron–Tate height supplies the positivity. For ζ there is no base below $\text{Spec } \mathbb{Z}$ and no Jacobian.
- The proposed replacements are the scaling site [32] and its Riemann–Roch and Hodge-type programme [28, 22], and Deninger’s foliated dynamical systems [44]. Both are research programmes; neither has a proved Hodge index.

(f) Ribet’s theorem (theorem). Ramanujan’s congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ says $\Delta \equiv E_{12}$ modulo 691 on non-constant coefficients; here $691 \mid \text{num } B_{12}$ with $B_{12} = -691/2730$. We checked it for $n < 1200$.

Ribet [128] uses this cusp/Eisenstein congruence to construct an unramified $\mathbb{Z}/691$ -extension of $\mathbb{Q}(\mu_{691})$ on which Gal acts by ω^{-11} . That proves the converse of Herbrand’s theorem for this eigenspace.

Correction of emphasis: “ $691 \mid h(\mathbb{Q}(\mu_{691}))$ ” already follows from Kummer’s criterion [155], since $691 \mid B_{12}$ and $12 \leq 688$. Ribet’s contribution is the ω^{-11} -eigenspace, constructed from the congruence.

J.W.’s pasted heuristics, with corrections (labelled heuristic).

- For $T = \tan \theta$, $dT/d\theta = \sec^2 \theta = 2$ at 45° . It is finite; the blow-up is at 90° .
- $\Delta = (Q^3 - R^2)/1728$ with $Q = E_4$, $R = E_6$. We checked $= q \prod (1 - q^n)^{24}$ through q^{39} .
- Twistors [121]: a point of (complexified) space-time corresponds to one line $\mathbb{CP}^1 \subset \mathbb{PT}$, not to a point.
- None of these heuristics implies, or is claimed to imply, RH.

14.6. (g) Frobenius as a Lorentz boost

Write $\Delta \equiv (H + V) + \delta$ and $\Gamma \equiv (H + qV) + \gamma$ with $\delta, \gamma \in \langle H, V \rangle^\perp$. From the intersection numbers,

$$\delta^2 = -2g, \quad \gamma^2 = -2gq, \quad \Delta \cdot \Gamma = q + 1 + \delta \cdot \gamma = N_1.$$

The boost $B_\eta : (a, b) \mapsto (e^{-\eta}a, e^\eta b)$ preserves $2ab$. With $e^\eta = \sqrt{q}$,

$$B_\eta(H + V) = \frac{H + qV}{\sqrt{q}}.$$

So the hyperbolic part of $\Gamma/\sqrt{q}$ is the diagonal boosted by rapidity $\eta = \frac{1}{2} \log q$ (checked symbolically).

The transverse parts δ and $\gamma/\sqrt{q}$ both have square $-2g$. Since Hodge index makes them spacelike, Cauchy–Schwarz in the negative-definite complement gives $|\delta \cdot \gamma| \leq 2g\sqrt{q}$, which is Weil’s bound again.

Interpretation (equivalent to the theorem, not stronger): numerically, $\Gamma/\sqrt{q}$ is the diagonal boosted along $\langle H, V \rangle$, with a transverse part of the same length. The statement $|\alpha_i| = \sqrt{q}$ says that $F^*/\sqrt{q}$ acts on H^1 by a unitary “rotation”.

HEURISTIC 14.6.1 (comparison with the scaling flow; labelled). In the TAA scaling flow, the prime- p orbit has period $\log p$. Frobenius at q is a boost of rapidity $\frac{1}{2} \log q$. The factor $\frac{1}{2}$ is the $\sqrt{q}$ normalisation, the same factor as the “TAA time $t/2$ ” of Lemma 7.

Every first-quadrant ray $(1, k)$, $k > 0$, is timelike with rapidity $\frac{1}{2} \log k$. The slope $\sqrt{3}$ (60°) gives rapidity $\frac{1}{4} \log 3$, the hexagonal value of Lemma 7, i.e. “Frobenius at $q = \sqrt{3}$ ”. This is suggestive only: $\sqrt{3}$ is not a prime power, and no curve is attached.

14.7. (h) Why $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ is not a surface

- (1) *Theorem.* $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z}$, so $\text{Spec } \mathbb{Z} \times_{\mathbb{Z}} \text{Spec } \mathbb{Z} = \text{Spec } \mathbb{Z}$. A genuine product needs a base below $\mathbb{Z}$, the “field with one element”. Several $\mathbb{F}_1$ formalisms exist [95, 28], but none has an accepted intersection theory on $\text{Spec } \mathbb{Z} \times_{\mathbb{F}_1} \text{Spec } \mathbb{Z}$.
- (2) *Theorem (Arakelov).* On an arithmetic surface $\mathcal{X} \rightarrow \text{Spec } \mathcal{O}_K$, the fibre $F_{\mathfrak{p}}$ has $F_{\mathfrak{p}}^2 = 0$ and is numerically $\log N_{\mathfrak{p}}$ times a normalised fibre F (with $F \cdot D = \deg D_K$) [3, 52, 90]. Vertical fibres are null rays weighted by $\log N_{\mathfrak{p}}$.

Heuristic: this is the same $\log p$ as the period of the prime- p orbit and as the Bohr–Sommerfeld spacing $2\pi/\log p$ of the poles $2\pi i\mathbb{Z}/\log p$ (§2). The brief’s shorthand “ $\hbar\omega = \log p$ ” should read “orbit period $\log p$ ”, since the level spacing is $2\pi/\log p$.

- (3) *Heuristic.* For $C/\mathbb{F}_q$, the $2g$ zeros of $\zeta_C(\frac{1}{2} + it)$ fill one period window $0 \leq t < 2\pi/\log q$. Formally put $\log q = 1/d$ and count the zeros of ζ in $|t| < 2\pi d$. This gives an “effective genus” $g_{\text{eff}} = N(2\pi d) \rightarrow \infty$, so $\Delta^2 = 2 - 2g_{\text{eff}} \rightarrow -\infty$: the diagonal is infinitely spacelike-heavy, and no finite-dimensional $\text{NS}(X)_{\mathbb{R}}$ can hold it.
- (4) *Remark on a naive construction (neutral).* One might set $H \cdot V = 1$, take $D^2 = 2 - \sum_p \log p$ (one null fibre per prime), and invoke $\kappa = 0$ to close the argument. This fails for three reasons:
 - the weights are wrong: $\log p$ enters fibres, not self-intersections, and the sum diverges without a renormalisation that is itself the content of the explicit formula;
 - invoking $\kappa = 0$ is circular, since Lemma 1 gives only $\text{RH} \Leftrightarrow \kappa = 0$;
 - Faltings–Hriljac applies to curves over $\mathcal{O}_K$, not to this object.

Status

Item	Status	Evidence
$\frac{7}{8}$ from $\arg \Gamma$	theorem	Stirling; $1/(48T)$ check
BK count $= \bar{N}$	heuristic	cutoffs, Maslov by matching
mean $S \approx 0$ with $\frac{7}{8}$	data	$2.2 \cdot 10^{-4}$ vs -0.125
prequantization $\Leftrightarrow$ integrality	theorem	Kostant/Souriau/Weil
Lefschetz $(1, 1)$, prequantum $\Rightarrow$ ample	theorem	over $\mathbb{C}$; analogy in char p
Hodge index $\Rightarrow$ CS $\Rightarrow$ Weil	theorem	Hartshorne V; Lemma 3 data
Frobenius = boost $\frac{1}{2} \log q$ + transverse	theorem (reformulation)	symbolic
binary forms in $\mathbb{R}^{2,1}$	theorem	classical
Faltings–Hriljac	theorem	only over $\mathcal{O}_K$
Ribet 1976	theorem	congruence checked $n < 1200$
$\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ surface	does not exist (naively)	$\mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$
g_{eff} , ghosts, slope $\sqrt{3}$	heuristic	labelled

Obstruction. Each step of Weil’s chain uses a finite-dimensional Lorentzian $\text{NS}(X)_{\mathbb{R}}$ that contains Δ and Γ . For ζ :

- the ambient surface does not exist (by (h)(1));
- the diagonal’s transverse part would have infinite length (by (h)(3));
- the only proved arithmetic Hodge index (Faltings–Hriljac) lives one level up, over $\mathcal{O}_K$.

What would be needed is a positivity statement for the Weil distribution. That is equivalent to RH (Weil’s criterion), so it cannot be imported.

Open questions.

- (1) Should the “rapidity $\frac{1}{2} \log q$ vs. orbit period $\log p$ ” factor of 2 be identified with Lemma 7’s $t/2$ in the volume, or kept as a remark?
- (2) Is (h)(3)’s $g_{\text{eff}} = N(2\pi d)$ the normalisation you intended (window $|t| < 2\pi d$, $\log q = 1/d$)?

CHAPTER 15

Lemma 15: The signature test

The Weil form plus its pole plane is Lorentzian iff RH; certified inertia on diagonal, prime-fibre and boosted test families

Standalone note `lemma15.tex`, October 21, 2026.

Summary. Let $\mathcal{Q}$ be the Weil form of Lemma 1 (zero side), and let $\mathcal{P}$ be the rank-two pole form built from $\tilde{F}(\pm \frac{i}{2})$. Put $\mathcal{L} = \mathcal{P} - \mathcal{Q}$. By the explicit formula, $\mathcal{L}$ equals the prime side minus the archimedean term, so it can be computed *without zeros*. It is the analogue of the intersection form on the span of the Frobenius graphs $\Gamma_m \subset C \times C$.

We prove four things.

- (i) The scaling flow T_t preserves $\mathcal{L}$ and acts on the pole plane as a Lorentz boost of rapidity $t/2$.
- (ii) $\sup_V n_+(\mathcal{L}|_V) = 1 + N_+$, where V runs over finite-dimensional subspaces and $N_+ = \#\{\rho : \operatorname{Re} \rho > \frac{1}{2}\}$. Hence $RH \iff \mathcal{L}$ has Lorentzian index ≤ 1 on every finite-dimensional subspace. This is the exact analogue of the Hodge index theorem.
- (iii) For curves the analogous Gram matrices have $n_+ = 1$ exactly (Hodge index) and finite rank $\leq 2 + 2g$.
- (iv) For ζ , the inertia of $\mathcal{L}$ is certified in Arb (entry radii $\leq 2.4 \cdot 10^{-78}$, certified LDL^T) on 99 subspaces. These are built from cubic B-spline “diagonal”, “Frobenius-power” (boosted), “prime-fibre” and Hecke-symmetric families, of dimension up to 31. Whenever a timelike vector is present the signature is $(1, n - 1)$, and there is never a null direction.

A planted off-line quadruple is detected as $n_+ = 3 = 1 + N_+$.

Circularity: (ii) is a reformulation of Weil’s criterion, and finite certificates cannot imply RH. What is new is the precise Lorentzian dictionary and an unconditional, certified pipeline.

15.1. The Lorentzian Weil form

Notation is as in Lemma 1. $\tilde{F}(z) = \int F(u) e^{izu} du$, and $(T_t F)(u) = F(u - t)$, so $\widetilde{T_t F}(z) = e^{izt} \tilde{F}(z)$. $\mathcal{Q}(F, G) = \sum_\rho m_\rho \tilde{F}(\gamma_\rho) \overline{\tilde{G}(\bar{\gamma}_\rho)}$, with $\gamma_\rho = -i(\rho - \frac{1}{2})$.

DEFINITION 15.1.1. For $F \in C_c^\infty(\mathbb{R})$ let $a(F) = \tilde{F}(\frac{i}{2}) = \int F e^{-u/2}$ and $b(F) = \tilde{F}(-\frac{i}{2}) = \int F e^{u/2}$. Define

$$\mathcal{P}(F, G) = a(F) \overline{b(G)} + b(F) \overline{a(G)}, \quad \mathcal{L} := \mathcal{P} - \mathcal{Q}.$$

Write $H = F * G^*$. Then $\mathcal{P}(F, G) = \tilde{H}(\frac{i}{2}) + \tilde{H}(-\frac{i}{2})$ is exactly the pole term of the explicit formula. Lemma 1 eq. (EF) gives

$$(15.1.1) \quad \mathcal{L}(F, G) = H(0) \log \pi - \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{H}(r) \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{ir}{2} \right) dr + \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} (H(\log n) + H(-\log n)).$$

No zeros enter (15.1.1).

- THEOREM 15.1.2 (structure; unconditional). (a) $\mathcal{L}$ is Hermitian and T_t -invariant.
 (b) $a(T_t F) = e^{-t/2}a(F)$ and $b(T_t F) = e^{t/2}b(F)$. So on the pole plane (a, b) , where $\mathcal{P}$ is the hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of signature $(1, 1)$, the scaling flow acts as a Lorentz boost of rapidity $t/2$.
 (c) In particular, translation by $\log q$ is a boost of rapidity $\frac{1}{2} \log q$. This is the number-field form of “Frobenius = boost” (Lemma 14(g)).

PROOF. (a) holds for $\mathcal{P}$ by (b), and for $\mathcal{Q}$ by Lemma 1(i)–(ii). (b) and (c) follow from $\widetilde{T_t F}(\pm \frac{i}{2}) = e^{\mp t/2} \widetilde{F}(\pm \frac{i}{2})$. $\square$

THEOREM 15.1.3 (Lorentzian index; theorem). Let $n_+(\cdot)$ be the number of positive squares. Then

$$\sup_{V \subset C_c^\infty(\mathbb{R}), \dim V < \infty} n_+(\mathcal{L}|_V) = 1 + N_+ \in \{1, 2, \dots, \infty\}.$$

Consequently, RH holds if and only if $n_+(\mathcal{L}|_V) \leq 1$ for every finite-dimensional V . Moreover $\sup_V n_-(\mathcal{L}|_V) = \infty$ unconditionally.

PROOF. Upper bound. $\mathcal{P}$ has rank ≤ 2 and signature $(1, 1)$ on the image of (a, b) , so $n_+(\mathcal{P}|_V) \leq 1$. By Lemma 1(vi), $n_+(-\mathcal{Q}|_V) \leq \kappa(\mathcal{Q}) = N_+$. Since n_+ is subadditive (Weyl), $n_+(\mathcal{L}|_V) \leq 1 + N_+$.

Lower bound. Fix a finite $N \leq N_+$. Lemma 1(vi) gives $F_1, \dots, F_N \in C_c^\infty$ on which $\mathcal{Q}$ is negative definite. Fix a bump $\chi \geq 0$ with $a_0 = a(\chi) > 0$ and $b_0 = b(\chi) > 0$. For $\lambda > 0$ put

$$\chi^+ = T_{-\lambda} \chi / (e^{\lambda/2} a_0), \quad \chi^- = T_\lambda \chi / (e^{\lambda/2} b_0).$$

Then $a(\chi^+) = 1$, $b(\chi^+) = e^{-\lambda} b_0 / a_0 \rightarrow 0$, $b(\chi^-) = 1$ and $a(\chi^-) \rightarrow 0$.

For fixed G , $\mathcal{Q}(T_{\mp \lambda} \chi, G)$ is a sum over ρ of terms of modulus $\leq e^{\lambda |\operatorname{Re} \rho - \frac{1}{2}|} m_\rho |\tilde{\chi}(\gamma_\rho) \tilde{G}(\bar{\gamma}_\rho)|$. These are summable (by the bound lemma of Lemma 1), and $|\operatorname{Re} \rho - \frac{1}{2}| < \frac{1}{2}$. By dominated convergence:

- $\mathcal{Q}(\chi^\pm, G) \rightarrow 0$;
- $\mathcal{Q}(\chi^\pm, \chi^\pm) = \mathcal{Q}(\chi, \chi) e^{-\lambda} / a_0^2 \rightarrow 0$ (or with b_0);
- $\mathcal{Q}(\chi^+, \chi^-) = \mathcal{Q}(T_{-2\lambda} \chi, \chi) / (e^\lambda a_0 b_0) \rightarrow 0$, using $2|\operatorname{Re} \rho - \frac{1}{2}| < 1$.

Put $\tilde{F}_k = F_k - a(F_k) \chi^+ - b(F_k) \chi^-$. Then $a(\tilde{F}_k), b(\tilde{F}_k) \rightarrow 0$ and $\mathcal{Q}(\tilde{F}_k, \tilde{F}_l) \rightarrow \mathcal{Q}(F_k, F_l)$, so $\mathcal{L}(\tilde{F}_k, \tilde{F}_l) \rightarrow -\mathcal{Q}(F_k, F_l)$, which is positive definite.

Let $e = \chi^+ + \chi^-$. Then $\mathcal{L}(e, e) \rightarrow 2$ and $\mathcal{L}(e, \tilde{F}_k) \rightarrow 0$. For λ large, the Gram matrix of $\mathcal{L}$ on $\operatorname{span}\{\tilde{F}_1, \dots, \tilde{F}_N, e\}$ is positive definite, so $n_+ \geq N + 1$. The case $N = 0$ gives $n_+ \geq 1$ unconditionally.

n_- is unbounded (sketch). Apply the interpolation argument of Lemma 1(vi) to the infinitely many zeros on the line (Hardy). This makes $\mathcal{Q}$ positive definite on subspaces of any finite dimension; making them primitive as above gives $\mathcal{L}$ negative definite there. $\square$

REMARK 15.1.4 (relation to the literature; circularity). Theorem 15.1.3 is equivalent to Weil’s criterion [157] together with the ghost count of Lemma 1(vi). The latter is in the spirit of Bombieri’s theorem [14]: for large truncations, the number of negative eigenvalues of Weil’s functional is half the number of off-line zeros. The only new content is the “+1”: the pole terms supply exactly one timelike direction, on which the scaling flow acts by boosts.

It therefore says nothing new about RH. Any proof of “ $n_+ \leq 1$ for all V ” is a proof of RH. Yoshida [163] shows that $\mathcal{Q}$ is positive definite on functions with sufficiently small support, so $n_+(\mathcal{L}|_V) \leq 1$ holds unconditionally for such V .

15.2. The $C \times C$ model

Let $C/\mathbb{F}_q$ have genus g and Frobenius eigenvalues α_j , and let Γ_m be the graph of F^m , with $\Gamma_0 = \Delta$. For $m \geq n$,

$$\Gamma_m \cdot \Gamma_n = q^n \Gamma_{m-n} \cdot \Delta = q^n N_{m-n}, \quad N_0 := \Delta^2 = 2 - 2g.$$

Normalising by $q^{(m+n)/2}$ and writing $e^\eta = \sqrt{q}$,

$$\tilde{G}_{mn} = 2 \cosh((m-n)\eta) - \sum_{j=1}^{2g} (\alpha_j / \sqrt{q})^{m-n}.$$

This is *literally* the Gram matrix of $\mathcal{L}$ on Frobenius translates, $\mathcal{P} \leftrightarrow 2 \cosh$ and $\mathcal{Q} \leftrightarrow \sum (\alpha_j / \sqrt{q})^k$, with the test functions replaced by Dirac masses on $\log q \cdot \mathbb{Z}$.

PROPOSITION 15.2.1 (theorem + exact computation). *(a) Hodge index gives $n_+(\tilde{G}) \leq 1$ for every n . Also $\text{rank } \tilde{G} \leq 2 + 2g$, since the Toeplitz part has $\text{rank} \leq \#\{\alpha_j\}$.*

(b) Exact rational LDL^T, for $n = 1, \dots, 25$ (`num/lemma15_ff.out`), gives:

- $E/\mathbb{F}_5$: $(n_+, n_-, n_0) = (0, 0, 1)$ for $n = 1$ (Δ null, $g = 1$), then $(1, n-1, 0)$ for $n = 2, 3, 4$, then $(1, 3, n-4)$.
- $C_2/\mathbb{F}_7$: $(0, 1, 0)$ for $n = 1$ ($\Delta^2 = -2$, spacelike), then $(1, n-1, 0)$ for $n \leq 6$, then $(1, 5, n-6)$.

The rank saturates at $2 + 2g$.

(c) Ghost model (not a curve): $q = 4$ with $\alpha/\sqrt{q} \in \{4, \frac{1}{4}\}$, which keeps the symmetry $z \mapsto 1/\bar{z}$. It has inertia $(2, 2, n-4)$: $n_+ = 2 = 1 + \#\text{ghosts}$.

15.3. Certified computations for ζ

Method.

- *Test functions.* $\phi_c(u) = M_4((u-c)/h+2)$, a cubic B-spline of half-width $2h$ centred at c . Then $H = \phi_c * \phi_{c'}^* = h M_8((u-d)/h+4)$ with $d = c - c'$, so $\mathcal{L}(\phi_c, \phi_{c'}) =: \ell(d)$.
- *Pole part.* Exactly, $a(\phi_0) = b(\phi_0) = h(\frac{\sinh(h/4)}{h/4})^4 =: a_0$ and $\mathcal{P} = 2a_0^2 \cosh(d/2)$.
- *Computing $\ell(d)$ from (15.1.1).* We use Arb (python-flint, 320 bits) with:
 - exact rational M_8 pieces;
 - a finite prime-power sum;
 - the archimedean term rewritten as $H_e(0)\psi(\frac{1}{4}) + 2 \int_0^\infty e^{-u/2} \frac{H_e(0) - H_e(u)}{1 - e^{-2u}} du$, integrated piecewise by certified Gauss–Legendre. On the first piece we use the entire form $-e^{u/2} r(u) / \sinh c(u)$; the tail is summed with a rigorous geometric bound.

All entry radii are $\leq 2.4 \cdot 10^{-78}$.

- *Cross-check.* $\mathcal{P} - \ell$ agrees with the zero sum $2 \sum_{\gamma > 0} \widetilde{\phi_0}(\gamma)^2 \cos(d\gamma)$ over the first 1000 zeros to 10^{-17} (for $h = \frac{1}{2}, 1$). This is a check of (15.1.1), not an input.
- *Inertia.* Symmetric pivoted elimination in ball arithmetic (with 2×2 pivots of certified negative determinant). The inertia is *certified* when no pivot ball contains 0 (Sylvester).

Results (all certified unless stated).

(A) *Diagonal-like.* For a single ϕ_0 , $\ell(0)/a_0^2 = 2 - 2g_{\text{eff}}(h)$:

h	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{1}{20}$	$\frac{1}{50}$	$\frac{1}{100}$
$\ell(0)/a_0^2$	2.000	2.000	1.982	0.215	-6.46	-38.5	-110.5
g_{eff}	10^{-8}	$6 \cdot 10^{-7}$	0.009	0.89	4.23	20.2	56.2

The diagonal turns from timelike to spacelike between $h = \frac{1}{10}$ and $\frac{1}{20}$, and $g_{\text{eff}} \rightarrow \infty$ as $h \rightarrow 0$. This is the certified form of Lemma 14(h)(3): $\Delta^2 = 2 - 2g_{\text{eff}} \rightarrow -\infty$. Under RH, $g_{\text{eff}}(h) = \sum_{\gamma > 0} (\widetilde{\phi}_0(\gamma)/a_0)^2$ is a smoothed zero count.

- (B) *Frobenius-power (boosted) family*. $V = \text{span}\{\phi_{ks} : 0 \leq k < n\}$ with $s = \log 2, \log 3$, $h \in \{\frac{1}{20}, \frac{1}{10}, \frac{1}{4}, \frac{1}{2}\}$ and $n \leq 12$ (59 cases).
- For $h \geq \frac{1}{10}$: $(1, n-1)$ for all $n \geq 2$.
 - For $h = \frac{1}{20}$, $s = \log 2$: $(0, 1), (0, 2)$, then $(1, n-1)$ from $n = 3$ on.
 - For $h = \frac{1}{20}$, $s = \log 3$: the Lorentzian onset is at $n = 2$.
- This is the same pattern as $C_2/\mathbb{F}_7$ (spacelike diagonal, then Lorentzian). The smallest |eigenvalue| ranges from $1.9 \cdot 10^{-3}$ (at $h = \frac{1}{20}$, $n = 12$) down to $6 \cdot 10^{-11}$ (at $h = \frac{1}{2}$, $n = 12$).
- (C) *Prime-fibre and Hecke families*. $V = \text{span}\{\phi_0, \phi_{\log p} : p \leq P\}$ and its Hecke-symmetric version $\{\phi_0, \phi_{\pm \log p}\}$, for $P \leq 47$ (dimension up to 31; 40 cases). All are $(1, n-1)$, except the single case $h = \frac{1}{20}$, $P = 2$, which is $(0, 2)$.
- (D) *Contrast with curves*. No certified null direction ever occurs ($n_0 = 0$ in all 99 cases), whereas curves have $n_0 = n - 2 - 2g$. This is the finite-rank versus infinite-rank dichotomy of Lemma 12. Soft near-degeneracy does appear: overlapping translates ($s < 4h$) leave 4–7 pivots uncertified at 320 bits (grids with $s = \frac{1}{20}, \frac{1}{10}$, $h = \frac{1}{20}$, run alongside (E)). These are reported as uncertified, not as zeros.
- (E) *Ghost injection (sanity check of Theorem 15.1.3)*. Add a hypothetical quadruple $\{\pm \gamma_0 \pm ib\}$ (i.e. $\text{Re } \rho = \frac{1}{2} + b$, $N_+ = 2$) to the zero side.
- $(\gamma_0, b) = (30, 0.25)$ and $(30, 0.4)$: inertia $(3, n-3)$, certified on the boosted grids with $s = \frac{3}{4}, 1$.
 - $(14.13, 0.1)$: inertia $(2, \cdot)$ or $(3, \cdot)$.
 - $(30, 0.1)$: not detected on these V (still $(1, n-1)$), consistent with the theorem being a supremum over V .

In every case $n_+ \leq 1 + N_+ = 3$.

Status

Statement	Status	Note
$T_t = \text{boost of rapidity } t/2 \text{ on pole plane}$	theorem	unconditional
$\sup_V n_+(\mathcal{L} _V) = 1 + N_+$; RH $\iff$ index ≤ 1	theorem	$\equiv$ Weil criterion + Lemma 1(vi)
$C \times C$: $n_+ = 1$, rank $\leq 2 + 2g$	theorem; exact data	Hodge index
ζ : $(1, n-1)$ on 99 listed V	certified (unconditional)	no zeros used; not evidence for RH
$g_{\text{eff}}(h) \rightarrow \infty$	certified values; limit under RH	Lemma 14(h)(3)
ghost $\Rightarrow n_+ = 3$	certified (hypothetical data)	sanity check

Obstruction. The certified V 's only see zeros with $|\gamma| \lesssim \text{a few}/h$, which are known to be on the line. Detecting a ghost at height T needs V adapted to T (Lemma 12), with the soft degeneracy getting worse. The signature test is a faithful reformulation, not a lever: “ $n_+ \leq 1$ for all V ” is RH.

Open questions.

- (1) Should the volume adopt $\mathcal{L} = \mathcal{P} - \mathcal{Q}$, the “intersection form”, as the primary object, with $\mathcal{Q}$ as its primitive part (the analogue of $\langle H, V \rangle^\perp$)?

- (2) Is the Hecke-symmetric family the “prime-fibre-like” class you intended, or did you mean $\log p$ -weighted null limits (Arakelov $F_p \equiv \log p \cdot F$)? The latter are boost limits ($\lambda \rightarrow \infty$) and sit on the null cone.

CHAPTER 16

Lemma 16: Certified Weil positivity on windows of length $\log 4$ and 1.6

(independent check of Zhu), with cost extrapolation

Standalone note `lemma16.tex`, October 21, 2026.

Summary. We extend Lemma 10’s computer-assisted certificate from $d = 3$ to $d = 4$ (primes 2, 3 active) and to window length 1.6 ($d = e^{1.6}$; $n = 2, 3, 4$ active). The method is unchanged (Lemma 10, Thm. 3.1): a frequency split at T , a Legendre block A_N in Arb ball arithmetic, and explicit super-exponential tail constants. We prove

$$\lambda^*(4) \geq 3.5368 \cdot 10^{-13}, \quad \lambda^*(e^{1.6}) \geq 9.2792 \cdot 10^{-18}.$$

By monotonicity, the Weil form is positive definite on every window of length ≤ 1.6 (the $d = 4$ bound is the sharper constant on the subrange $\leq \log 4$).

The value $9.2792 \cdot 10^{-18}$ is to be compared with Zhu’s claimed certified $8.9 \cdot 10^{-18} \leq \lambda^* \leq 2.27 \cdot 10^{-17}$ at support $[-0.8, 0.8]$ [165], and with our own uncertified Galerkin value $1.640 \cdot 10^{-17}$ (Lemma 11). We also prove that κ_A is non-decreasing in A (Remark 16.3.1). We extrapolate the cost: window 1.8 needs $\approx 400\times$ and window 2 needs $\approx 4 \cdot 10^4\times$ the $d = 4$ run, which took 15 minutes.

16.1. Setting and method

Notation is that of Lemma 10. $\lambda^*(d)$ is the infimum of $\mathcal{Q}(F, F)/\|F\|_2^2$ over F supported in an interval of length $L = \log d$, and $h = L/2$. Lemma 10, Thm. 3.1 gives, for any T with $\beta_T > 0$ and any N ,

$$\lambda^*(d) \geq \min(\lambda_{\min}(A_N), \beta_T - \varepsilon_D) - \varepsilon_B,$$

where:

- $\beta_T = \operatorname{Re} \psi(\frac{1}{4} + \frac{iT}{2}) - \log \pi - A_d$ and $A_d = \sum_{n < d} 2\Lambda(n)/\sqrt{n}$;
- A_N is the leading Legendre block of $\beta_T I + C + \mathcal{P}$;
- ε_B and ε_D are the explicit tail norms.

Implementation changes (`num/lemma16_certify.py`), all rigorous:

- (i) For $\operatorname{Re} z > \max(8, n + 2)$ we evaluate $j_n(z)$ by the exact upward recurrence $j_{k+1} = \frac{2k+1}{z} j_k - j_{k-1}$ from $j_0 = \frac{\sin z}{z}$ and $j_1 = \frac{\sin z}{z^2} - \frac{\cos z}{z}$, in ball arithmetic. This is stable for $k < |z|$. Otherwise we use Lemma 10’s entire ${}_0F_1$ form. Both expressions define the same entire function, so `acb.integral`’s analyticity requirement holds on every ball that is evaluated.
- (ii) Parity split: C and $\mathcal{P}$ preserve parity, so $A_N = A_N^{\text{even}} \oplus A_N^{\text{odd}}$ exactly, and $\lambda_{\min}$ is certified block by block (shifted Cholesky with residual bounded by Weyl’s inequality, as in Lemma 10).
- (iii) Entries are checkpointed to `num/lemma16_A_*.jsonl`, so the computation can resume after an interruption.

Regression. For $d = 3$, $T = 40$, 200 entries recomputed with the new code agree with Lemma 10’s stored entries to $2.0 \cdot 10^{-42}$, and all balls overlap.

16.2. Certificates

d	L	T	β_T	N	bits	$\lambda_{\min}(A_N) \geq$	ε_B	ε_D
3 (L10)	1.0986	40	0.8707	72	160	$1.4476 \cdot 10^{-8}$	$4.8 \cdot 10^{-26}$	$3.3 \cdot 10^{-56}$
4	1.3863	100	0.51846	160	200	$3.5368 \cdot 10^{-13}$	$2.0 \cdot 10^{-34}$	$1.4 \cdot 10^{-74}$
$e^{1.6}$	1.6	200	0.51846	280	240	$9.2792 \cdot 10^{-18}$	$1.0901 \cdot 10^{-27}$	$2.9357 \cdot 10^{-62}$

TABLE 16.1. Certified runs. The max entry radius is $2.5 \cdot 10^{-44}$ for $d = 4$ and $1.98 \cdot 10^{-35}$ for $e^{1.6}$. The Cholesky residuals are $\leq 10^{-43}$. For $d = 4$ the floating-point bottom eigenvalues are $3.93 \cdot 10^{-13}$ (even) and $2.07 \cdot 10^{-10}$ (odd). The run took 928s on 7 cores (`num/lemma16_d4_T100.log`, `lemma16_results.jsonl`).

THEOREM 16.2.1 (rigorous, computer-assisted). *For every $F \in \mathcal{D}$ supported in an interval of length $\leq \log 4$,*

$$\mathcal{Q}(F, F) \geq 3.5368 \cdot 10^{-13} \|F\|_2^2.$$

For length ≤ 1.6 the constant is $9.2792 \cdot 10^{-18}$.

PROOF. Apply Lemma 10, Thm. 3.1 with the certified numbers of Table 16.1. Use the monotonicity of λ^* in d and the reduction to real F , as in Lemma 10, Thm. 3.3. $\square$

REMARK 16.2.2 (relaxation loss). Compare with Lemma 11's Galerkin values of λ^* (uncertified, 120 modes/parity): $7.530 \cdot 10^{-13}$ (even) and $4.754 \cdot 10^{-10}$ (odd) at $d = 4$; $1.640 \cdot 10^{-17}$ (even) at $e^{1.6}$. The certified bound for $d = 4$ is therefore smaller by a factor 2.13 (even) and 2.29 (odd). This is the cost of discarding $\Psi_d - \beta_T > 0$ on $|t| > T$ (Lemma 10, Remark 3.2). At $e^{1.6}$ the factor is 1.77.

16.3. Monotonicity of κ_A

REMARK 16.3.1 (κ_A is non-decreasing; rigorous). Let $\mathcal{D}_A = \{F \in C_c^\infty(\mathbb{R}) : \text{supp } F \subset [-A/2, A/2]\}$. Let κ_A be the supremum of $\dim V$ over subspaces $V \subset \mathcal{D}_A$ on which $\mathcal{Q}$ is negative definite (Lemma 6). Then:

- (a) if $A \leq A'$ then $\kappa_A \leq \kappa_{A'}$;
- (b) κ_A does not depend on the position of the window;
- (c) if $\lambda^*(e^A) > 0$ then $\kappa_{A'} = 0$ for all $A' \leq A$.

PROOF. (a) $[-A/2, A/2] \subset [-A'/2, A'/2]$ gives $\mathcal{D}_A \subset \mathcal{D}_{A'}$. Every $V \subset \mathcal{D}_A$ on which $\mathcal{Q}$ is negative definite is also such a subspace of $\mathcal{D}_{A'}$, so the supremum over the larger family is at least as large.

(b) $\mathcal{Q}(T_t F, T_t G) = \mathcal{Q}(F, G)$ (Lemma 1(ii)), and T_t is a linear bijection from functions supported in I onto functions supported in $I + t$. So negative-definite subspaces correspond one-to-one, with the same dimension.

(c) If $\lambda^*(e^A) > 0$, then $\mathcal{Q}(F, F) \geq \lambda^* \|F\|^2 > 0$ for $0 \neq F \in \mathcal{D}_A$, so no nonzero negative vector exists and $\kappa_A = 0$. Then (a) gives $\kappa_{A'} \leq \kappa_A = 0$ for $A' \leq A$. $\square$

COROLLARY 16.3.2. $\kappa_A = 0$ for all $A \leq \log 4$ and for all $A \leq 1.6$. Combined with Lemma 12's certified ghost-detection lengths ($A_0 \geq 9.36$)¹, this still excludes nothing (Lemma 12, Cor. 5.1).

¹[Vol. 2 editorial note] With the corrected counting constant R_{cor} of Chapter 24 the smallest certified value is $A_0 = 9.14$ ($\gamma_0 = \gamma_1$, global, $\delta = 0.5$; Table 24.1). The conclusion is unchanged.

16.4. Comparison with Zhu, and cost

- **Zhu [165]** (unrefereed) claims $8.9 \cdot 10^{-18} \leq \lambda^* \leq 2.27 \cdot 10^{-17}$ for $\text{supp } f \subset [-0.8, 0.8]$. That is our window $L = 1.6$. We certify $\lambda^* \geq 9.2792 \cdot 10^{-18}$, which confirms positivity on Zhu's window independently. Our constant is above Zhu's $8.9 \cdot 10^{-18}$ and consistent with his upper bound $2.27 \cdot 10^{-17}$. Our implementation shares the frequency-split principle and the Legendre basis with Zhu's. It differs in the envelope (the exact digamma value at T), the software (Arb), the quadrature (`acb.integral`), the PSD test, and the Bessel evaluation.
- **Cost law.** We choose T so that $\beta_T \approx 0.518$, and N so that $x^N/(2N+1)!! < 10^{-35}$ with $x = Th$. Measured relative to the $d = 4$ run, the costs are as follows.

L	A_d	T	N	entries	cost (relative to $d = 4$)
$\log 4$	2.249	100	154	$5.9 \cdot 10^3$	1 (measured 928 s)
1.6	2.942	200	284	$2.0 \cdot 10^4$	≈ 6 (measured: 46 min (7 cores))
1.8	4.381	843	1102	$3.0 \cdot 10^5$	$\approx 4 \cdot 10^2$ (days)
2.0	5.852	3671	5062	$6.4 \cdot 10^6$	$\approx 4 \cdot 10^4$ (a year)
2.4	8.521	$5.3 \cdot 10^4$	$8.6 \cdot 10^4$	$1.9 \cdot 10^9$	$\approx 10^8$

So $T \approx 2\pi e^{A_d + \beta}$ with $A_d \sim 4e^{L/2}$ (Lemma 10, Remark 3.4): the cost is doubly exponential in L , and the required precision grows like $|\log \lambda^*|$ ($\lambda^*(e^{2.4}) \approx 7 \cdot 10^{-49}$, Lemma 11). Window 1.6 is the practical end of this method on this machine.

Status

Statement	Status	Note
$\lambda^*(4) \geq 3.5368 \cdot 10^{-13}$	certified	Arb, full tail
$\lambda^*(e^{1.6}) \geq 9.2792 \cdot 10^{-18}$	certified	independent check of Zhu
κ_A non-decreasing; $\kappa_A = 0$ for $A \leq \log 4$ (≤ 1.6)	theorem / certified	Remark 16.3.1
cost $\sim$ doubly exponential in L	extrapolation	measured to $L = 1.6$

Obstruction. This is the same as Lemma 10: pointwise control of the prime comb makes T grow like $e^{4e^{L/2}}$. Any fixed window is a finite fragment of RH, and certified windows (≤ 1.6) remain far below the detection lengths $A_0 \geq 9.36$ of Lemma 12.

Question for J.W. Should the volume cite Zhu's 1.6 as “independently confirmed” (with our weaker constant), or report only our certified range?

Lemma 17: The arithmetic action as a Lagrangian

Three levels of action, stationary values, Noether, Ostrogradsky, the transmission-line form, and the confluence of field, window and orbit

Standalone note lemma17.tex, October 21, 2026.

Summary. J.W. asked for a definition of “arithmetic action” that is fully consistent with Lagrange’s principle of stationary action. We give one, at three levels, and we say exactly where each level is a theorem and where it is only an analogy.

- **(L-I) Field action** on the whole log-scale line: $\mathcal{S}[F] = \mathcal{Q}(F, F) = \text{pole} + \langle F, K(D)F \rangle - \sum_{p,k} \frac{\log p}{p^{k/2}} 2 \operatorname{Re} \langle F, T_k \log_p F \rangle$. The kinetic symbol is $K = 2\theta' = \operatorname{Re} \psi(\frac{1}{4} + \frac{ir}{2}) - \log \pi$, i.e. the smooth zero density θ'/π . It is a logarithmic Laplacian: $K(r) = \log \frac{|r|}{2\pi} + O(r^{-2})$ (Thm. 17.2.1, Prop. 17.8.1).
- **(L-II) Window action** $\mathcal{S}_A$: $\mathcal{S}$ restricted to fields supported in an interval of length A . Hamilton’s principle with a norm constraint gives an Euler–Lagrange eigen-equation. The stationary values are the eigenvalues $\lambda_k(A)$ of a self-adjoint operator with compact resolvent; the lowest is λ^* ; the number of negative-action stationary points is κ_A ; κ_A is non-decreasing (Thm. 17.3.1). $\text{RH} \iff$ the trivial path is a minimum for every window, a Jacobi–Morse no-conjugate-point condition.
- **(L-III) Particle action**: closed orbits of period $m \log p$ are critical points of a Hamiltonian/Maupertuis action in Definition 5’s local model; their weights have Gutzwiller form.

We add a Noether theorem (charge = $\mathcal{Q}(\text{D}_{\text{sc}} F, F)$, spectrum = zero ordinates), an Ostrogradsky/Pais–Uhlenbeck analysis (ghosts of higher-derivative theories come from *negative residues at real frequencies*; the Weil action has all residues $m_\rho \geq 1$, so its only possible ghosts are *complex-frequency pairs*, exactly N_+ of them), the function-field model (finite-rank Toeplitz Lagrangian, genus as a mass term, $\mathcal{S}_X \geq 0$ by Weil/Rosati), a transmission-line (Heaviside) reformulation (for curves, “distortionless” is exactly Cauchy–Schwarz on the Hodge-index-negative transverse part of $\text{NS}(X)_{\mathbb{R}}$; for ζ , distortion $\iff$ an extra timelike direction of $\mathcal{L}$), and a section on how the three actions meet. **Nothing here proves RH.** Every equivalence with RH is a reformulation of Weil’s criterion; we flag each one.

Labels. [THM]: proved here or classical with proof indicated; [CITED]: a published theorem used as stated; [CA]: computer-assisted; [NUM]: numerical check, not a proof; [HEUR]: heuristic; [ANALOGY]: structural analogy, not a derivation. **Circularity flag** [RH-EQUIV]: the statement is equivalent to RH through Weil’s criterion (Lemma 1(iv)), so it is a reformulation, not progress.

Conventions (Lemma 1). $u = \log x$; $\tilde{F}(z) = \int F(u) e^{izu} du$; $\langle F, G \rangle = \int F \bar{G} = \frac{1}{2\pi} \int \tilde{F} \tilde{\bar{G}} dr$; $(T_t F)(u) = F(u - t)$; $H = F * G^*$, $G^*(u) = \overline{G(-u)}$, so $H(x) = \langle F, T_x G \rangle$ and $\tilde{H}(r) = \tilde{F}(r) \tilde{\bar{G}}(r)$ for real r . $\mathcal{Q}(F, G) = \sum_\rho m_\rho \tilde{F}(\gamma_\rho) \tilde{\bar{G}}(\bar{\gamma}_\rho)$ with $\rho = \frac{1}{2} + i\gamma_\rho$. $\text{D}_{\text{sc}} := i \frac{d}{du}$, so $\widetilde{\text{D}_{\text{sc}} F}(z) = z \tilde{F}(z)$ and $T_t = e^{it \text{D}_{\text{sc}}}$ (the sign is chosen so that Lemma 1(v)’s generator is multiplication by γ_ρ). $a(F) = \tilde{F}(\frac{i}{2})$, $b(F) = \tilde{F}(-\frac{i}{2})$, $\mathcal{P}(F, G) = a(F) \overline{b(G)} + b(F) \overline{a(G)}$ (Lemma 15).

17.1. Three Lagrangian levels

- DEFINITION 17.1.1 (the three actions). (L-I) *Global field action.* $\mathcal{S}[F] := \mathcal{Q}(F, F)$ for F in Lemma 1's class $\mathcal{A} \supset C_c^\infty(\mathbb{R})$, with no cutoff. Its “time” is $u \in \mathbb{R}$; its symmetry is T_t .
- (L-II) *Window action.* For an interval I of length A , $\mathcal{S}_A[F] := \mathcal{S}[F]$ on $\mathcal{D}_I = \{F \in \mathcal{D} : \text{supp } F \subset \bar{I}\}$, where $\mathcal{D} = \{F \in L^2 : \int |\tilde{F}|^2 \log(2 + |r|) dr < \infty\}$ is Lemma 10's form domain. By translation invariance only A matters. The constraint set is $\{\|F\| = 1\}$.
- (L-III) *Particle action.* On $M_p = T^*\mathbb{R}_+^*/\langle \phi_p \rangle$, $\phi_p(x, \xi) = (px, \xi/p)$, with $H_{\text{cl}} = x\xi$: $\mathcal{A}_T(\gamma) = \oint_\gamma \xi dx - \int_0^T H_{\text{cl}}(\gamma(t)) dt$ on loops $\mathbb{R}/T\mathbb{Z} \rightarrow M_p$ (Definition 5, Prop. 2.4), and the reduced (Maupertuis) action $S_0 = \oint \xi dx$ on an energy shell.

L-I and L-II are the same Lagrangian on different path spaces; L-III is a different kind of object (a classical action on loops). Section 17.10 explains where they meet.

17.2. L-I: field-theory form of the action

THEOREM 17.2.1 (kinetic + interaction + pole; [THM]). For $F \in C_c^\infty(\mathbb{R})$,

$$(17.2.1) \quad \mathcal{S}[F] = \underbrace{2 \operatorname{Re} (a(F) \overline{b(F)})}_{\mathcal{P}(F, F): \text{ pole}} + \underbrace{\langle F, \mathbf{K}(D)F \rangle}_{\text{kinetic}} - \underbrace{\sum_p \sum_{k \geq 1} \frac{\log p}{p^{k/2}} 2 \operatorname{Re} \langle F, T_{k \log p} F \rangle}_{\text{interaction}},$$

where $D = -i \frac{d}{du}$ and

$$\mathbf{K}(r) = \operatorname{Re} \psi\left(\frac{1}{4} + \frac{ir}{2}\right) - \log \pi = 2\theta'(r), \quad \theta(r) = \arg \Gamma\left(\frac{1}{4} + \frac{ir}{2}\right) - \frac{r}{2} \log \pi.$$

Equivalently $\langle F, \mathbf{K}(D)F \rangle = \int_{\mathbb{R}} |\tilde{F}(r)|^2 \frac{\theta'(r)}{\pi} dr$: the kinetic weight is the smooth zero density $\tilde{N}'(r) = \theta'(r)/\pi$. In position space,

(17.2.2)

$$\langle F, \mathbf{K}(D)F \rangle = - \int_{\mathbb{R}} (H(u) - H(0)) k_\infty(u) du - (\log \pi - \psi(\frac{1}{4})) H(0), \quad k_\infty(u) = \frac{e^{-|u|/2}}{1 - e^{-2|u|}},$$

with $H = F * F^*$ and $\log \pi - \psi(\frac{1}{4}) = \log 8\pi + \gamma_E + \frac{\pi}{2} = 5.3722$. Finally $\mathcal{L} = \mathcal{P} - \mathcal{S}$ (Lemma 15): $\mathcal{L}[F] = -\langle F, \mathbf{K}(D)F \rangle + \sum_{p,k} \frac{\log p}{p^{k/2}} 2 \operatorname{Re} \langle F, T_{k \log p} F \rangle$ contains no pole term and no zeros.

PROOF. Lemma 1 eq. (EF) with $G = F$. The pole term is $\tilde{H}(\frac{i}{2}) + \tilde{H}(-\frac{i}{2}) = \bar{a}b + b\bar{a}$. For real r , $\tilde{H}(r) = |\tilde{F}(r)|^2$, so $-H(0) \log \pi + \frac{1}{2\pi} \int \tilde{H} \operatorname{Re} \psi = \frac{1}{2\pi} \int |\tilde{F}|^2 \mathbf{K} = \langle F, \mathbf{K}(D)F \rangle$ by Plancherel ($\mathbf{K}$ is even, so the sign convention of D is immaterial). Since $H(-x) = \overline{H(x)}$ and $H(x) = \langle F, T_x F \rangle$, $H(\log n) + H(-\log n) = 2 \operatorname{Re} \langle F, T_{\log n} F \rangle$; write $n = p^k$. $\theta'(r) = \frac{1}{2} \operatorname{Re} \psi(\frac{1}{4} + \frac{ir}{2}) - \frac{1}{2} \log \pi$ by differentiating θ . For (17.2.2): Lemma 15 (§3, method) uses $\frac{1}{2\pi} \int \tilde{H} \operatorname{Re} \psi = H_e(0) \psi(\frac{1}{4}) + 2 \int_0^\infty e^{-u/2} \frac{H_e(0) - H_e(u)}{1 - e^{-2u}} du$, H_e the even part; this is the classical integral representation of ψ (Binet/Gauss form) applied termwise, and it is checked numerically below. Rewriting $2 \int_0^\infty (H_e(0) - H_e(u)) k_\infty = \int_{\mathbb{R}} (H(0) - H(u)) k_\infty$ gives (17.2.2). $\square$

REMARK 17.2.2 (signs; what “kinetic” does and does not mean). $\mathbf{K}$ is *not* positive: $\mathbf{K}(0) = -5.3722$, and $\mathbf{K}(r) < 0$ exactly for $|r| < r_0 = 6.28984$, the minimum of θ [NUM]. So the “kinetic” term is a high-frequency (ultraviolet) positive operator and an infrared negative one. The pole term is indefinite of signature (1, 1) (Lemma 10, §1), and the interaction is indefinite. Positivity of $\mathcal{S}$, when it holds, is a cancellation among three indefinite pieces. Table 17.1 shows a Gaussian whose action is ≈ 0 because pole 3.344, kinetic -2.238 and interaction -1.106 cancel to 16 digits.

REMARK 17.2.3 (frequency-space Lagrangian; [THM] under RH, distributional). Under RH, $\mathcal{S}[F] = \int_{\mathbb{R}} |\tilde{F}(r)|^2 dN_{\text{sym}}(r)$, where N_{sym} counts zeros at $\pm\gamma$. The Riemann–von Mangoldt formula

splits $dN = \frac{\theta'}{\pi} dr + dS$ (Weyl + oscillatory), and dS is formally $-\frac{1}{\pi} \sum_n \frac{\Lambda(n)}{\sqrt{n}} \cos(r \log n) dr$ plus the pole correction. Theorem 17.2.1 is the rigorous, smeared form of this split: *kinetic* = *Weyl term*, *interaction* = *oscillatory (Gutzwiller) term*. This is the content of the “Weyl + Gutzwiller” reading of Berry–Keating; the semiclassical interpretation is [ANALOGY].

TABLE 17.1. [NUM] Theorem 17.2.1 for Gaussian fields $F = e^{-(u-c)^2/2s^2} e^{-ig_0 u}$ (num/lemma17_checks.out). Prime side = pole + kinetic + interaction ($n < 4 \cdot 10^5$, closed-form H); zero side = 1000 zero pairs.

s	c	g_0	pole	kinetic	interaction	prime side	zero side
0.5	0	0	3.3442	-2.2379	-1.1064	$-3.5 \cdot 10^{-17}$	$6.4 \cdot 10^{-22}$
0.4	0.7	0	2.0927	-1.5823	-0.5104	$2.6486 \cdot 10^{-14}$	$2.6331 \cdot 10^{-14}$
0.6	0	γ_1	10^{-31}	0.8583	1.4037	2.2619467973	2.2619467973
0.6	0.3	30	10^{-141}	1.6617	0.5600	2.2216857570	2.2216857570

Rows 1–2: the action is tiny because $|\tilde{F}|$ is concentrated below γ_1 ; the residual agreement (10^{-16}) is at double-precision level. Rows 3–4: agreement to $6 \cdot 10^{-16}$. Also: $K = 2\theta'$ to 15 digits at $r = 1, 14.13, 100$; (17.2.2) equals the frequency form to 15 digits for $s = \frac{1}{2}$.

17.3. L-II: the variational principle on a window

THEOREM 17.3.1 (constrained stationary action; [THM]). *Fix a window I of length A .*

- (a) $\mathcal{S}_A$ is a closed Hermitian form on $L^2(I)$, bounded below, with form domain $\mathcal{D}_I$. By Kato’s first representation theorem [81, Thm. VI.2.1] it defines a self-adjoint operator S_A on $L^2(I)$, and S_A has compact resolvent. Its eigenvalues $\lambda_1(A) \leq \lambda_2(A) \leq \dots \rightarrow +\infty$.
- (b) (Euler–Lagrange = Lagrange multiplier.) $F \in \mathcal{D}_I$ with $\|F\| = 1$ is a stationary point of $\mathcal{S}_A$ on the sphere iff there is $\lambda \in \mathbb{R}$ with

$$(17.3.1) \quad \mathcal{Q}(F, G) = \lambda \langle F, G \rangle \quad \forall G \in \mathcal{D}_I, \quad \text{i.e.} \quad \mathbf{1}_I(\mathcal{W} * F) = \lambda F \quad \text{on } I,$$

where $\mathcal{W} * F = \mathcal{P}\text{-kernel} * F + K(D)F - \sum_{p,k} \frac{\log p}{p^{k/2}} (T_k \log p + T_{-k} \log p) F$. The multiplier equals the on-shell action: $\lambda = \mathcal{S}_A[F]$.

- (c) (Rayleigh–Ritz, Courant–Fischer [127, Thm. XIII.1].) $\lambda_k(A) = \min_{\dim V=k} \max_{F \in V, \|F\|=1} \mathcal{S}_A[F]$; in particular $\lambda_1(A) = \lambda^*(e^A)$ (Lemma 10).
- (d) (Ghost count.) The number of stationary points with negative action, counted as the dimension of their span, is $\#\{k : \lambda_k(A) < 0\} = \kappa_A$, the negative index of $\mathcal{Q}$ on $C_c^\infty(\mathbb{R})(I^\circ)$ (Lemma 6). It is finite for every A .
- (e) (Monotonicity.) $A \mapsto \lambda_k(A)$ is non-increasing, so $A \mapsto \kappa_A$ is non-decreasing; $\sup_A \kappa_A = N_+$.
- (f) [RH-EQUIV] $RH \iff \lambda_1(A) \geq 0$ for all $A \iff \kappa_A = 0$ for all A .

PROOF. (a) By Lemma 10, Lemma 1.1, $\mathcal{S}_A[F] = \mathcal{P}(F, F) + \frac{1}{2\pi} \int |\tilde{F}|^2 \Psi_d$, $d = e^A$, with $\Psi_d(r) = K(r) - \sum_{n < d} \frac{2\Lambda(n)}{\sqrt{n}} \cos(r \log n)$. Since $K(r) = \log \frac{|r|}{2\pi} + O(r^{-2})$ (Prop. 17.8.1) and the comb is bounded by A_d , there are $c, C > 0$ with $c \log(2 + |r|) - C \leq \Psi_d(r) \leq C \log(2 + |r|)$. The pole term satisfies $|\mathcal{P}(F, F)| \leq 2\|F\|^2 \int_I e^{|u|} du$ by Cauchy–Schwarz (window centred at 0), so it is a bounded rank-two perturbation. Hence $\mathcal{S}_A + C'\|\cdot\|^2$ is equivalent to the norm $\int |\tilde{F}|^2 \log(2 + |r|)$, which is closed on $\mathcal{D}_I$ (the support condition is preserved under L^2 limits). Compactness: the unit ball $\mathcal{B}$ of $\mathcal{D}_I$ satisfies $\frac{1}{2\pi} \int_{|r| > R} |\tilde{F}|^2 \leq 1/\log(2 + R)$, while $F \mapsto \mathbf{1}_{|r| < R} \tilde{F}$ is Hilbert–Schmidt on $L^2(I)$ (kernel $\mathbf{1}_I(u) \mathbf{1}_{|r| < R} e^{iru}$). So the embedding $\mathcal{D}_I \hookrightarrow L^2(I)$ is a norm limit of

compact maps, hence compact, and the resolvent is compact. (b) For $\|F\| = 1$ and $G \in \mathcal{D}_I$, $\frac{d}{d\epsilon}|_0(\mathcal{S}_A[F + \epsilon G] - \lambda\|F + \epsilon G\|^2) = 2\operatorname{Re}(\mathcal{Q}(F, G) - \lambda\langle F, G \rangle)$; replacing G by iG gives (17.3.1). The distributional form follows from $\mathcal{Q}(F, G) = \langle \mathcal{W} * F, G \rangle$ for $G \in C_c^\infty(\mathbb{R})(I^\circ)$ and the decomposition of Thm. 17.2.1. Taking $G = F$ gives $\lambda = \mathcal{S}_A[F]$. (c) is the min–max principle for $\mathcal{S}_A$. (d) The span of eigenvectors with $\lambda_k < 0$ is a maximal negative subspace, of dimension $\#\{\lambda_k < 0\}$; finiteness from (a). $C_c^\infty(\mathbb{R})(I^\circ)$ is a form core: dilate towards the centre (continuous in the form norm, since $\log(2 + |r/(1 - \epsilon)|) \leq \log(2 + |r|) + \log 2$ and dominated convergence) and then mollify; so the negative index on $C_c^\infty(\mathbb{R})(I^\circ)$ equals that on $\mathcal{D}_I$. (e) An interval of length $A' > A$ contains one of length A , and $\mathcal{S}$ is translation invariant; min–max over a larger space can only decrease. Any finite-dimensional negative subspace of $C_c^\infty(\mathbb{R})$ lies in some window, so $\sup \kappa_A = \kappa(\mathcal{Q}) = N_+$ (Lemma 1(vi)). (f) is (e) with Lemma 1(iv). $\square$

ANALOGY 17.3.2 (Jacobi–Morse: ghosts are conjugate points; [ANALOGY] with a precise half). In classical mechanics, the second variation of the action along a trajectory on $[0, T]$ with fixed ends is a Jacobi operator; its number of negative eigenvalues (Morse index) equals the number of conjugate points in $(0, T)$ and is non-decreasing in T (Morse index theorem, [104, Thm. 15.1]). For the oscillator $-\frac{d^2}{dt^2} - \omega^2$ with Dirichlet ends the index is $\lfloor \omega T / \pi \rfloor$. Theorem 17.3.1 has exactly this shape: $\mathcal{S}_A$ is a quadratic action on paths $u \mapsto F(u)$ with fixed (zero) data outside the window, its Morse index is κ_A , and it is non-decreasing in the “time length” A . Define the *conjugate lengths* $A_k^* = \inf\{A : \lambda_k(A) < 0\}$. Then [RH-EQUIV] $\text{RH} \iff A_1^* = \infty$, i.e. Jacobi’s necessary condition for a minimum holds for every window. Unconditionally $A_1^* > \log 3$ (Lemma 10), and under Lemma 12’s hypotheses an off-line zero forces $A_1^* \leq A_0$. The precise half is Thm. 17.3.1(c)–(e); the identification with conjugate points of a *classical* trajectory is an analogy, because $\mathcal{S}_A$ is non-local and there is no underlying point-particle Lagrangian. The finiteness of κ_A comes from $K \rightarrow +\infty$, the analogue of the strengthened Legendre condition.

TABLE 17.2. [NUM] Stationary values of $\mathcal{S}_A$ (Lemma 11’s exact geometric-side Galerkin, 240 Legendre modes; `num/lemma11_eigs240.json`). Lower bounds for $A \leq \log 3$ are certified in Lemma 10 [CA]. Right: first conjugate length A_1^* in Lemma 11’s ghost model (zero γ_k moved to $\frac{1}{2} + \delta + i\gamma_k$; `num/lemma11_ghost_table.json`).

A	λ_1 (even)	λ_2 (even)	λ_1 (odd)	ghost	A_1^* even	A_1^* odd
$\log 2$	$1.33 \cdot 10^{-3}$	0.661	$7.31 \cdot 10^{-2}$	$\gamma_1, \delta = 0.4$	0.80	1.00
1	$9.33 \cdot 10^{-7}$	$1.80 \cdot 10^{-2}$	$1.94 \cdot 10^{-4}$	$\gamma_1, \delta = 0.01$	1.20	1.30
$\log 3$	$5.53 \cdot 10^{-8}$	$1.46 \cdot 10^{-3}$	$1.49 \cdot 10^{-5}$	$\gamma_1, \delta = 10^{-8}$	1.80	1.80
1.6	$1.64 \cdot 10^{-17}$	$8.31 \cdot 10^{-12}$	$1.55 \cdot 10^{-14}$	$\gamma_4, \delta = 0.4$	1.30	1.30
2	$5.76 \cdot 10^{-30}$	$2.11 \cdot 10^{-23}$	$1.45 \cdot 10^{-26}$	$\gamma_4, \delta = 0.01$	1.50	1.60
2.6	$7.86 \cdot 10^{-62}$	$4.18 \cdot 10^{-54}$	$7.57 \cdot 10^{-58}$	$\gamma_4, \delta = 10^{-8}$	2.00	2.10

$\lambda_1(A)$ is non-increasing on the whole grid (20 windows). The ghost columns are grid resolution ($\Delta A \approx 0.1$); they are the “conjugate points” of a hypothetical zeta with one off-line pair, not statements about ζ .

17.4. Noether: scaling symmetry and its charge

PROPOSITION 17.4.1 (Noether charge; [THM]). (a) $\mathcal{S}[T_t F] = \mathcal{S}[F]$ (Lemma 1(ii)). Infinitesimally, $D_{\text{sc}} = i\frac{d}{du}$ is $\mathcal{Q}$ -symmetric: $\mathcal{Q}(D_{\text{sc}} F, G) = \mathcal{Q}(F, D_{\text{sc}} G)$ for $F, G \in C_c^\infty(\mathbb{R})$. (b) The Noether charge of the scaling symmetry is the quadratic form $\mathcal{E}[F] := \mathcal{Q}(D_{\text{sc}} F, F)$. It is real, conserved ($\mathcal{E}[T_t F] = \mathcal{E}[F]$), and has the spectral form

$\mathcal{E}[F] = \sum_{\rho} m_{\rho} \gamma_{\rho} \tilde{F}(\gamma_{\rho}) \overline{\tilde{F}(\bar{\gamma}_{\rho})}$. It is computable from primes alone via Thm. 17.2.1 with $(F, G) \rightarrow (D_{\text{sc}} F, F)$.

- (c) Under RH, $\mathcal{E}[F] = \sum_{\rho} m_{\rho} \gamma_{\rho} |\tilde{F}(\gamma_{\rho})|^2$, and the generator of the unitary realization is multiplication by γ_{ρ} with simple pure point spectrum $\{\gamma_{\rho}\}$ (Lemma 1(v)): the charge spectrum is the set of zero ordinates. For real F , $\mathcal{E}[F] = 0$ (the $\pm\gamma$ contributions cancel).

PROOF. (a) Differentiate $\mathcal{Q}(T_t F, T_t G) = \mathcal{Q}(F, G)$ at $t = 0$: $\mathcal{Q}(F', G) + \mathcal{Q}(F, G') = 0$; with $D_{\text{sc}} = i\partial$ and antilinearity in the second slot this is $\mathcal{Q}(D_{\text{sc}} F, G) = \mathcal{Q}(F, D_{\text{sc}} G)$. (b) Reality from (a) and Hermitian symmetry; conservation from $[D_{\text{sc}}, T_t] = 0$ and invariance; the spectral form from $\widetilde{D_{\text{sc}} F}(z) = z \tilde{F}(z)$. (c) Lemma 1(v). $\square$

PROPOSITION 17.4.2 (ghosts as complex charges; Krein/Pontryagin picture; [THM]). Assume $N_+ < \infty$. On the model space $\mathcal{K} = \ell^2(\mathcal{Z}_0, m) \oplus \bigoplus_{\rho \in \mathcal{Z}_+} \mathbb{C}^2$ with Krein form $[x, y] = \sum_{\mathcal{Z}_0} m x \bar{y} + \sum_{\mathcal{Z}_+} m_{\rho} (x_1 \bar{y}_2 + x_2 \bar{y}_1)$, the map $JF = ((\tilde{F}(\gamma))_{\mathcal{Z}_0}, (\tilde{F}(\gamma_{\rho}), \tilde{F}(\bar{\gamma}_{\rho}))_{\mathcal{Z}_+})$ satisfies $[JF, JG] = \mathcal{Q}(F, G)$ and $JD_{\text{sc}} = AJ$ with $A = \text{diag}(\gamma)$. A is $[\cdot, \cdot]$ -self-adjoint; its non-real eigenvalues are the pairs $\gamma_{\rho}, \bar{\gamma}_{\rho}$ ($\rho \in \mathcal{Z}_+$); each eigenvector of a non-real eigenvalue is neutral ($[e, e] = 0$); the root spaces of γ_{ρ} and $\bar{\gamma}_{\rho}$ are dually paired and span a hyperbolic plane. The number of non-real eigenvalues in the lower half-plane is $N_+ = \kappa(\mathcal{Q})$, which saturates Pontryagin's bound for self-adjoint operators in Π_{κ} [125, 74].

PROOF. $[JF, JG] = \mathcal{Q}(F, G)$ is Lemma 1 eq. (split). If $Ae = \mu e$ with $\mu \notin \mathbb{R}$, then $\mu[e, e] = [Ae, e] = [e, Ae] = \bar{\mu}[e, e]$, so $[e, e] = 0$. The rest is read off from the block form and Lemma 1(vi). $\square$

Reading. A ghost is not a negative-energy real mode; it is a pair of complex conjugate charges with zero norm, whose amplitudes grow and decay like $e^{\pm(\beta - \frac{1}{2})u}$ (Lemma 1(iii)). Numerics [NUM]: for $F = e^{-u^2/0.72} e^{-ig_0 u}$, prime side and 1000-zero side of $\mathcal{E}$ agree to 12 digits: $g_0 = \gamma_1$: $\mathcal{E} = 31.97199686$, $\mathcal{E}/S = 14.134725$ (the charge of a field localized at γ_1 is γ_1); $g_0 = 30$: 67.84840778 , ratio 30.54; $g_0 = 49.77$: 150.9692943 , ratio 49.41.

17.5. Legendre transform, Ostrogradsky, and where ghosts come from

S is non-local in u (the interaction couples u and $u \pm k \log p$, and $K(D)$ is a pseudodifferential operator of order 0^+). There is no finite-dimensional Legendre transform. We compare instead with the two classical sources of indefinite metrics.

THEOREM 17.5.1 (Ostrogradsky [116, 162]; [CITED], short proof). Let $L(q, \dot{q}, \dots, q^{(N)})$, $N \geq 2$, be nondegenerate ($\partial^2 L / \partial (q^{(N)})^2 \neq 0$). With $Q_k = q^{(k-1)}$ and $P_k = \sum_{j=k}^N (-\frac{d}{dt})^{j-k} \frac{\partial L}{\partial q^{(j)}}$, the Hamiltonian is $H = P_1 Q_2 + \dots + P_{N-1} Q_N + P_N \mathcal{A}(Q, P_N) - L(Q, \mathcal{A})$, where $q^{(N)} = \mathcal{A}(Q, P_N)$. It is linear in P_1 , hence unbounded below and above on phase space.

PROOF. Nondegeneracy lets one solve $P_N = \partial L / \partial q^{(N)}$ for $q^{(N)}$. P_1 appears only in $P_1 Q_2$. $\square$

PROPOSITION 17.5.2 (Krein weight of a mode = $K'(\omega)$; [THM]). Let K be a real polynomial and $S[q] = \int \bar{q} K(i\partial_t) q dt$ for complex q . The bilinear concomitant B defined by $\bar{q}_2 K(i\partial) q_1 - \overline{K(i\partial) q_2} q_1 = \frac{d}{dt} B(q_1, q_2)$ is conserved on solutions, and on exponentials $B(e^{-i\omega t}, e^{-i\mu t}) = i \frac{K(\omega) - K(\mu)}{\omega - \mu} e^{-i(\omega - \mu)t}$. Hence, if K has simple real roots ω_j , the conserved Hermitian form $-iB$ on solutions $q = \sum A_j e^{-i\omega_j t}$ is $\sum_j K'(\omega_j) |A_j|^2$, and the time-translation Hamiltonian is $\sum_j \omega_j K'(\omega_j) |A_j|^2$. In particular, for even K with positive roots $\omega_1 < \dots < \omega_N$ and positive leading coefficient, the signs of $K'(\omega_j)$ alternate, and exactly $\lfloor N/2 \rfloor$ positive-frequency modes carry negative energy (Pais–Uhlenbeck [118]; $N = 2$: one ghost).

PROOF. B is bilinear in $(q_1, \bar{q}_2)$ and polynomial in derivatives, so $B(e^{-i\omega t}, e^{-i\mu t}) = b(\omega, \mu)e^{-i(\omega-\mu)t}$ with b polynomial. The defining identity gives $(K(\omega) - K(\mu)) = -i(\omega - \mu)b(\omega, \mu)$ as polynomials, so b is i times the divided difference; on the diagonal $b = iK'(\omega)$, and off the diagonal at two roots $b = 0$. The Hamiltonian generating $q(t) \mapsto q(t+s)$ for the Krein structure $-iB$ is $[\Omega q, q]$ with $\Omega = \text{diag}(\omega_j)$ (the same structure as Prop. 17.4.1); that it coincides with Ostrogradsky's H is the standard identification of the Ostrogradsky symplectic form with the concomitant [162]. Sign alternation: K' changes sign between consecutive simple roots. $\square$

Check [NUM], sympy: for the real PU Lagrangian $\frac{1}{2}[\dot{q}^2 - (\omega_1^2 + \omega_2^2)q^2 + \omega_1^2\omega_2^2q^2]$ and $q = A \cos \omega_1 t + B \cos \omega_2 t$, Ostrogradsky's H equals $\frac{1}{2}(\omega_1^2 - \omega_2^2)(\omega_1^2 A^2 - \omega_2^2 B^2)$, which is exactly $\frac{1}{2} \sum_{\pm \omega_j} \omega K'(\omega) |a|^2$. So for $\omega_1 < \omega_2$ the slow mode is the ghost.

THEOREM 17.5.3 (the Weil action has no Ostrogradsky ghosts; its ghosts are complex pairs; [THM]). Put $\Xi(z) = \xi(\frac{1}{2} + iz)$, real on $\mathbb{R}$, with zeros γ_ρ .

- (a) (Weil weights are residues of Ξ'/Ξ .) The mode weights of $\mathcal{S}$ are $m_\rho = \text{Res}_{\gamma_\rho} \Xi'/\Xi \geq 1$, unconditionally. Thus no real-frequency mode has negative norm, and indefiniteness arises only from the non-real γ_ρ , as hyperbolic planes (Lemma 1(vi)): $\kappa(\mathcal{S}) = N_+$.
- (b) (The naive dispersion Lagrangian is ghost-ridden.) For the infinite-order “Lagrangian” $\bar{q} \Xi(i\partial_t) q$, whose on-shell modes are $e^{-i\gamma t}$, Prop. 17.5.2's weights are $\Xi'(\gamma_\rho)$. At consecutive simple real zeros with no real zero between them these have opposite signs (unconditional). So that Lagrangian has infinitely many Ostrogradsky ghosts; the explicit-formula action is its logarithmic derivative, which has none at real frequencies.
- (c) [RH-EQUIV] $RH \iff -\Xi'/\Xi$ is holomorphic in $\text{Im } z > 0 \iff -\Xi'/\Xi$ is a Herglotz (Nevanlinna) function there (maps $\text{Im } z > 0$ into $\text{Im } w \geq 0$). This is the arithmetic form of the positivity of a propagator's spectral measure; it is equivalent to Lagarias's criterion $\text{Re } \frac{\xi'}{\xi}(s) > 0$ on $\text{Re } s > \frac{1}{2}$ [87, Thm. 1.1].

PROOF. (a) Ξ'/Ξ has simple poles at the zeros with residues the multiplicities; Lemma 1's form puts weight m_ρ on the pair $(\gamma_\rho, \bar{\gamma}_\rho)$. (b) The computation of Prop. 17.5.2 is valid on finite sums of exponentials for any entire K , since $K(i\partial)e^{-i\omega t} = K(\omega)e^{-i\omega t}$. Ξ is real on $\mathbb{R}$ and changes sign at a simple real zero, so Ξ' alternates between consecutive ones. (c) By Hadamard, $\Xi(z) = \Xi(0) \prod_\gamma (1 - z^2/\gamma^2)$ (product over pairs $\pm\gamma$, absolutely convergent), so $\Xi'/\Xi(z) = \sum_\rho m_\rho/(z - \gamma_\rho)$ summed symmetrically. Under RH every γ is real and $\text{Im } \frac{1}{z-\gamma} = -\frac{\text{Im } z}{|z-\gamma|^2} < 0$ for $\text{Im } z > 0$, so $-\Xi'/\Xi$ is Herglotz. If RH fails, some $\bar{\gamma}_\rho$ lies in $\text{Im } z > 0$ and is a pole. With $s = \frac{1}{2} + iz$, $\text{Im } z > 0 \iff \text{Re } s < \frac{1}{2}$, and the functional equation turns this into Lagarias's half-plane. $\square$

TABLE 17.3. Two sources of indefinite metric.

	Higher-derivative / non-local field theory (PU [118], Stelle [143, 142], Ostrogradsky [116, 162])	Weil action $\mathcal{S}$
mode frequencies	real roots of K (stable case)	γ_ρ ; real iff RH
mode weights	$K'(\omega_j)$, alternating signs	$m_\rho \geq 1$, always positive [THM]
ghost type	real frequency, negative norm/energy	complex pair, zero-norm eigenvectors, hyperbolic plane
count	$\lfloor N/2 \rfloor$ (PU); massive spin-2 (Stelle)	N_+ [RH-EQUIV]
non-local limit	infinitely many ghosts unless $1/K$ entire-free [49]	$\Xi(i\partial)$: infinitely many; Ξ'/Ξ : none on $\mathbb{R}$

Numerics [NUM]: $\text{sign } \Xi'(\gamma_n) = (-1)^n$ for $n = 1, \dots, 20$.

17.6. L-III: the point-particle level

PROPOSITION 17.6.1 (recap of Definition 5; [THM]). (i) The primes are exactly the primitive closed orbits of the scaling flow on $\mathcal{Y} = \mathbb{A}_{\mathbb{Q}}/(\mathbb{Q}^{\times} \hat{\mathbb{Z}}^{\times})$, of length $\log p$, and orbit closure is the product formula (Definition 5, Lemma 1.3). (ii) In M_p the T -periodic Hamiltonian orbits are exactly the critical points of $\mathcal{A}_T$, and exist iff $T \in (\log p)\mathbb{N}$; on $H_{\text{cl}} = E$ the Maupertuis action is $S_0 = E m \log p$ and $\partial S_0 / \partial E = m \log p$ (Definition 5, Prop. 2.4). (iii) With monodromy $M_m = \text{diag}(p^m, p^{-m})$ in the covering chart, $\frac{\log p}{p^{m/2}} = \frac{\log p}{|\det(M_m - I)|^{1/2}} (1 - p^{-m})$: the explicit-formula weight has Gutzwiller form [62] up to the factor $(1 - p^{-m})$, and with the opposite sign (Definition 5, Prop. 2.5).

What is and is not a classical Lagrangian system. Is: each M_p (a symplectic manifold, a Hamiltonian, a Hamilton principle; [THM]); the disjoint union $\bigsqcup_p M_p$ has length spectrum $\{m \log p\}$. Is not: $\mathcal{Y}$ is not Hausdorff (Definition 5, Lemma 1.3(e)), so it carries no smooth structure and no classical action; in M_p the return map is the identity (the orbits form a degenerate family, not isolated hyperbolic orbits), so Gutzwiller's isolated-orbit formula does not apply to M_p itself; and no known classical system has a trace formula that *derives* the explicit formula. Connes's global trace formula would supply such a derivation and is equivalent to RH [35] [RH-EQUIV].

17.7. The function-field model: a provably non-negative Lagrangian

Let $X/\mathbb{F}_q$ have genus g , $\ell = \log q$, Frobenius eigenvalues $\alpha_i, \beta_i = \alpha_i/\sqrt{q}$, $c_n = \sum_i \beta_i^n$ ($c_{-n} = \bar{c}_n$ unconditionally, since $\{\beta_i\}$ is stable under $\beta \mapsto \bar{\beta}$ and $\beta \mapsto 1/\beta$), $N_n = \#X(\mathbb{F}_{q^n})$.

PROPOSITION 17.7.1 (Toeplitz Lagrangian; [THM]). For F supported in a window, with fibre vectors $v(x) = (F(x + j\ell))_j$, $x \in [0, \ell)$,

$$\begin{aligned} \mathcal{S}_X[F] &= \ell \int_0^\ell \sum_{j,k} \overline{v_k(x)} c_{j-k} v_j(x) dx \\ &= \underbrace{\ell \sum_{n \in \mathbb{Z}} 2 \cosh\left(\frac{n\ell}{2}\right) H(n\ell)}_{\text{pole}} \\ &\quad - \underbrace{\ell \sum_{n \neq 0} N_{|n|} q^{-|n|/2} H(n\ell)}_{\text{orbits}} + \underbrace{(2g - 2) \ell H(0)}_{\text{local} = -\chi(X) \ell \|F\|^2}. \end{aligned}$$

The orbit weights are $\sum_{\deg x | n} \deg(x) \ell \cdot q^{-n/2} = \sum \log N(x) N(x)^{-k/2}$ over closed points x with $k \deg x = n$: the exact analogue of $\Lambda(n) n^{-1/2}$. The genus enters as a constant mass term $-\chi \ell$, where ζ has the growing kinetic symbol $\mathbf{K}$ (Lemma 15: $\ell(0)/a_0^2 = 2 - 2g_{\text{eff}}$, $g_{\text{eff}} \rightarrow \infty$).

- (a) [CITED] By Weil's theorem ($|\beta_i| = 1$), c_n is a positive-definite sequence, every Toeplitz block is PSD, and $\mathcal{S}_X \geq 0$ on every window. The structural reason is the Hodge index theorem on $X \times X$ (Lemma 14(d)) or, equivalently, Rosati positivity: $\mathcal{S}_X[F] = \ell \int_0^\ell \text{Tr}(\xi_v \xi_v^\dagger) dv$ (Definition 5, Thm. 4.1).
- (b) [THM] The Toeplitz form has rank $\leq 2g$; the window margin is $\ell \lambda_{\min}(T_n)$ and is exactly 0 for $n > 2g$ (Lemma 12, Prop. 6.2). So the minimal action is ≥ 0 for all windows, with exact zero modes.

PROOF. Lemma 12's Poisson-summation identity $\mathcal{Q}_X(F) = \ell \sum_n c_n H(n\ell)$, and $H(n\ell) = \int_0^\ell \sum_k v_{k+n} \bar{v}_k$. Split $c_n = q^{|n|/2} + q^{-|n|/2} - N_{|n|} q^{-|n|/2}$ for $n \neq 0$ (from $N_n = q^n + 1 - \sum \alpha^n$) and $c_0 = 2g = 2 + (2g - 2)$. $\square$

Check [NUM]: $E/\mathbb{F}_5$ ($P(T) = 1 + 3T + 5T^2$), random complex fibre vectors, $n = 2, 3, 6$: Toeplitz value = pole + orbit + local to 12 digits; $\lambda_{\min}(T_2) = 0.658$, $\lambda_{\min}(T_n) = 0$ (to 10^{-15}) for $n \geq 3 = 2g + 1$.

17.8. Scale-free remark: logarithmic kinetic operator; scale vs description

- PROPOSITION 17.8.1 ([THM]). (a) $K(r) = \log \frac{|r|}{2\pi} - \frac{1}{24r^2} + O(r^{-4})$. Numerically $K(r) - \log \frac{r}{2\pi} = -4.17 \cdot 10^{-4}, -4.17 \cdot 10^{-6}, -4.17 \cdot 10^{-8}$ at $r = 10, 100, 1000$ [NUM]. Hence $K(D) = \frac{1}{2}L_\Delta - \log 2\pi + R$ with R bounded, where L_Δ is the logarithmic Laplacian of Chen–Weth [20] (symbol $2 \log |\zeta|$; in dimension one its off-diagonal kernel is $-1/|u - v|$). Consistently, $k_\infty(u) = \frac{1}{2|u|} + \frac{1}{4} + O(|u|)$ near 0, and k_∞ decays like $e^{-|u|/2}$.
- (b) (Scale covariance up to a constant.) Under the dilation of the log-scale variable $F_\lambda(u) = \lambda^{-1/2}F(u/\lambda)$ (that is $x \mapsto x^\lambda$), $\langle F_\lambda, \log |D| F_\lambda \rangle = \langle F, \log |D| F \rangle - \log \lambda \|F\|^2$. Among measurable even symbols k , “ $k(\lambda r) - k(r)$ independent of r for all $\lambda > 0$ ” holds iff $k = a \log |r| + b$ (Cauchy’s equation). So $\log |D|$ is the unique kinetic operator that is scale-covariant up to an additive mass term. The full $\mathcal{S}$ is not covariant under $u \mapsto \lambda u$: its covariance group in $\text{Diff}(\mathbb{R})$ is exactly $u \mapsto \pm u + b$ (Definition 5, Thm. 3.1), because the orbit lengths $\log p$ and the pole exponents $e^{\pm u/2}$ are rigid.
- (c) (Change of scale vs. change of description: one local formula.) For $v \leq \infty$ let $\mathcal{W}_v(H) = \int_{\mathbb{Q}_v^\times} H(\log |x|_v) \frac{|x|_v^{1/2}}{|1-x|_v} d^*x$, regularized at $x = 1$, with d^*x normalized by $\int_{1 \leq |x|_v \leq \Lambda} d^*x \sim \log \Lambda$. For $v = p$ the shells $|x|_p = p^{-k}$, $k \neq 0$, give exactly $\log p p^{-|k|/2} H(\pm k \log p)$: the interaction. For $v = \infty$, summing $x > 0$ and $x < 0$ gives the kernel $\frac{1}{2} \left(\frac{1}{2|\sinh(u/2)|} + \frac{1}{2\cosh(u/2)} \right) = k_\infty(u)$ of (17.2.2): the kinetic term. So $\mathcal{S} = \mathcal{P} - \sum_{v \leq \infty} \mathcal{W}_v^{\text{reg}}$; this is Weil’s local form of the explicit formula [157] as used by Connes [35] (the regularization constants at $x = 1$ are taken from there; our two kernel computations are elementary and checked numerically).

PROOF. (a) Stirling: $\psi(z) = \log z - \frac{1}{2z} - \frac{1}{12z^2} + O(z^{-4})$ with $z = \frac{1}{4} + \frac{ir}{2}$; take real parts. The difference symbol is $O(r^{-2})$, integrable, so R has a bounded continuous kernel. The expansion of k_∞ is a Taylor expansion. (b) $\tilde{F}_\lambda(r) = \lambda^{1/2} \tilde{F}(\lambda r)$; substitute. For the uniqueness, $k(e^s) =: h(s)$ satisfies $h(s+t) - h(s) = c(t)$, so c is additive and measurable, hence linear. (c) $|1-x|_p = 1$ for $|x|_p < 1$, $= |x|_p$ for $|x|_p > 1$, and $\text{vol}(\mathbb{Z}_p^\times) = \log p$ under the stated normalization; at ∞ , $x = \pm e^u$, $d^*x = du/2$ on each half-line, and $\frac{e^{u/2}}{|1 \mp e^u|}$ gives the two terms; $\frac{1}{4 \sinh a} + \frac{1}{4 \cosh a} = \frac{e^a}{2 \sinh 2a} = \frac{e^{-a}}{1 - e^{-4a}}$ with $a = u/2$. $\square$

Dictionary (precise). The archimedean place (*change of scale*) enters as a continuous, almost-local operator with singular support $\{0\}$; each prime (*change of description*) enters as translations by $\log p^k$, i.e. jumps supported on the value group $\log |\mathbb{Q}_p^\times| = \mathbb{Z} \log p$. The product formula $\log |q|_\infty = -\sum_p \log |q|_p$ for $q \in \mathbb{Q}_{>0}^\times$ says the translation $T_{\log q}$ seen archimedeanly equals the composite of p -adic shifts, which is why only $T_{\log q}$, $q \in \mathbb{Q}_{>0}$, appear in the interaction. Calling the archimedean term “kinetic” and the p -adic ones “interaction” is our naming; the mathematics is the identity $\mathcal{S} = \mathcal{P} - \sum_v \mathcal{W}_v$, in which all places enter by *one* formula. The RG reading (“unitary at every cutoff”) remains [ANALOGY].

17.9. Transmission-line form (Heaviside)

PROPOSITION 17.9.1 (telegrapher’s equation and Heaviside’s condition; [THM]). *Let $v_{xx} = LCv_{tt} + (RC + LG)v_t + RGV$, $L, C > 0$. With $\mu = \frac{RC+LG}{2LC}$ and $v = e^{-\mu t}w$,*

$$w_{xx} = LC w_{tt} - \frac{(RC - LG)^2}{4LC} w.$$

Hence the line is distortionless iff $RC = LG$ (Heaviside 1887, The Electrician, reprinted in [66, Vol. II, Arts. 35 and 42]): then $v = e^{-\mu t}f(x - ct)$, $c = (LC)^{-1/2}$, for arbitrary f ; the dispersion relation is $k(\omega) = \omega/c + iR\sqrt{C/L}$, so every frequency has the same attenuation and the same speed. If $RC \neq LG$ the term $\frac{(RC-LG)^2}{4LC}w$ (a Klein–Gordon term of negative mass squared in $LCw_{tt} = w_{xx} + \kappa w$) makes attenuation and speed frequency-dependent.

PROOF. Direct substitution; verified symbolically (num/lemma17_heaviside.out): the reduced equation, the travelling-wave solution at $RC = LG$, and $k(\omega)$. $\square$

THEOREM 17.9.2 (arithmetic distortionless condition; [THM], reformulation, [RH-EQUIV]). *Let $E(u) = e^{-u/2}(\psi(e^u) - e^u)$. In $u = \log x$, each zero contributes $-x^\rho/\rho = -e^{\beta u}e^{i\gamma u}/\rho$: a wave of frequency γ and amplitude growth $e^{\beta u}$. The following are equivalent:*

- (i) RH: every spectral component has the same exponent $\beta = \frac{1}{2}$ (“no distortion”);
- (ii) $\psi(x) = x + O(x^{1/2+\varepsilon})$ for every $\varepsilon > 0$;
- (iii) $E(u) = O_\varepsilon(e^{\varepsilon u})$ for every $\varepsilon > 0$: the normalized signal has no exponentially growing component.

Under RH, $E(u) = O(u^2)$ (von Koch [154]: $\psi(x) = x + O(\sqrt{x} \log^2 x)$), and E is almost periodic in u in a generalized (Besicovitch B^2) sense, with Fourier series $-\sum_\gamma e^{i\gamma u}/\rho$ and a limiting distribution (Wintner [161], who shows this property is in fact equivalent to RH; the general framework with logarithmic densities is [131], §2). An off-line zero is a frequency with a different attenuation: it adds $e^{(\beta-\frac{1}{2})u}$ growth to E , which is exactly “distortion”.

PROOF. (i) $\Rightarrow$ (ii): von Koch. (ii) $\Rightarrow$ (i): $-\zeta'(s) = \frac{s}{s-1} + s \int_1^\infty (\psi(x) - x)x^{-s-1}dx$ for $\operatorname{Re} s > 1$, and under (ii) the integral converges and is holomorphic on $\operatorname{Re} s > \frac{1}{2} + \varepsilon$, so ζ has no zeros there. (ii) $\iff$ (iii) is the substitution $x = e^u$. $\square$

PROPOSITION 17.9.3 (the cusp is a distortionless line; [THM]). *In Lax–Phillips’s wave equation $u_{tt} = Lu$, $L = y^2\Delta + \frac{1}{4}$ on $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ ([91]; Lemma 7, Thm. 3.1, read in [92]), restrict to x -independent data in the cusp and put $r = \log y$, $u = e^{r/2}W$. Then*

$$e^{-r/2}\left(y^2\partial_y^2 + \frac{1}{4}\right)u = W_{rr}, \quad \text{whereas} \quad e^{-r/2}y^2\partial_y^2u = W_{rr} - \frac{1}{4}W.$$

So the half-density substitution produces a mass term $-\frac{1}{4}W$ of the same (negative-mass-squared) sign and form as Heaviside’s $\frac{(RC-LG)^2}{4LC}w$, and the shift $+\frac{1}{4}$ in L cancels it exactly, leaving the free wave equation $W_{tt} = W_{rr}$ in the cusp. This is an exact algebraic correspondence: $\frac{(RC-LG)^2}{4LC} \leftrightarrow \frac{1}{4}$, $e^{-\mu t} \leftrightarrow e^{r/2}$.

PROOF. $y^2\partial_y^2 = \partial_r^2 - \partial_r$ and $(\partial_r^2 - \partial_r)(e^{r/2}W) = e^{r/2}(W_{rr} - \frac{1}{4}W)$; checked symbolically. $\square$

REMARK 17.9.4 (the “bounce from infinity”; [CITED] with an [ANALOGY] label on the Heaviside reading). An incoming wave y^{1-s} travels up the cusp and returns as $\varphi(s)y^s$, with $\varphi(s) = \zeta^*(2s-1)/\zeta^*(2s) = \frac{s}{s-1} \frac{\xi(2s-1)}{\xi(2s)}$ (Lemma 7 normalization; the naive ratio $\xi(2s-1)/\xi(2s)$ is off by $\frac{s-1}{s}$). φ is unimodular on $\operatorname{Re} s = \frac{1}{2}$ unconditionally (a lossless reflection). The zeros of ζ appear as resonances: poles of φ at $\rho/2$, $\sigma(B'') = \{(\rho-1)/2\}$, and Pavlov–Faddeev’s criterion

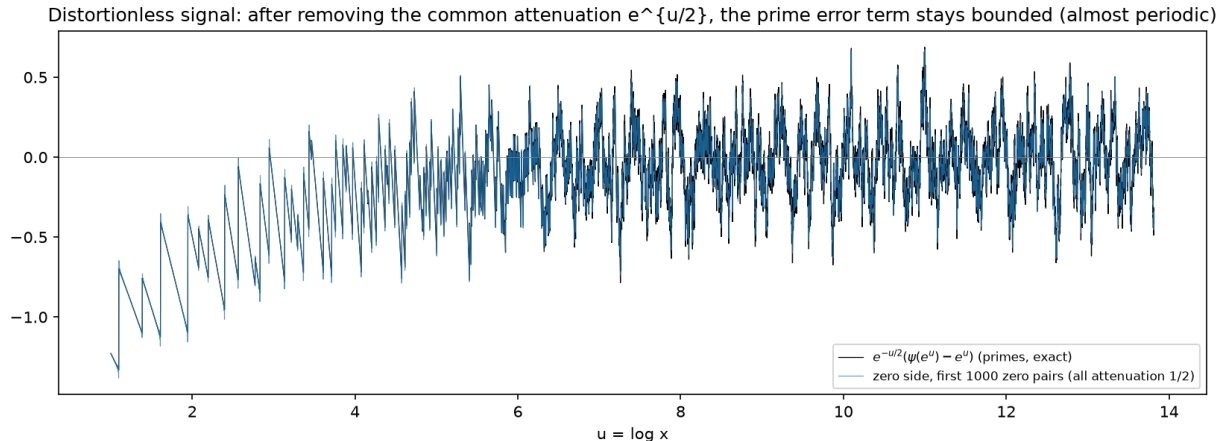


FIGURE 17.1. [NUM] $E(u)$ from primes ($x \leq 10^6$, black) and from the first 1000 zero pairs (blue, plus the trivial-zero and constant terms). Away from prime powers the rms difference is 0.031; $\max_{u>8} |E| = 0.69$. A hypothetical zero $0.6 + 1000i$ would add a component of size $8 \cdot 10^{-3}$ at $u = 13.8$, growing to 44 at $u = 100$ (num/lemma17_heaviside.py).

RH $\iff \lim t^{-1} \ln \|Z_t(B+i)^{-1}\| = -\frac{1}{4}$ [120] [RH-EQUIV]. In Heaviside’s language: the cusp line itself is distortionless (Prop. 17.9.3), and RH says every echo decays at the common rate $\frac{1}{4}$ (Lemma 7, Thm. 4.1(i): $s(B'') = -\frac{1}{4}$ iff RH). Unconditionally $\|Z_t\| = 1$ for all $t \geq 0$ (Volume 1, Thm. 7.4.1(ii)): the norm sees no attenuation at all, which is why norm estimates cannot reach RH (Lemma 7). The identification “resonance width = attenuation constant” is [ANALOGY]; the mass-term cancellation is Prop. 17.9.3.

PROPOSITION 17.9.5 (Caldirola–Kanai Lagrangian of the line; [THM]). *The Euler–Lagrange equation of $\mathcal{L}_{\text{tel}} = \frac{1}{2}e^{2\mu t}(LCv_t^2 - v_x^2 - RGV^2)$ is the telegrapher’s equation (checked symbolically); this is the field version of the Caldirola–Kanai damped-oscillator Lagrangian [18, 80]; Bateman’s doubling [6] is the alternative variational treatment. With $v = e^{-\mu t}w$, $\mathcal{L}_{\text{tel}}$ becomes $\frac{1}{2}(LCw_t^2 - w_x^2 + \kappa w^2)$ plus a total derivative, $\kappa = \frac{(RC-LG)^2}{4LC}$.*

ANALOGY 17.9.6 (the half-density as Caldirola–Kanai factor; [ANALOGY]). In TAA, $f(x) = x^{-1/2}F(\log x)$ (Lemma 1, §1) is the substitution that turns dilation into a unitary translation, exactly as $e^{\mu t}$ turns the lossy line into a conservative one and $e^{r/2}$ does in the cusp. After it, L-I’s kinetic operator $K(D) \approx \log \frac{|D|}{2\pi}$ is the “free line”; the half-density shift $\frac{1}{2}$ is the common attenuation; RH is “all components share it”. The precise content is Thm. 17.9.2 and Prop. 17.9.3; the transmission-line reading adds no positivity.

17.9.1. Distortionless = Hodge index. Here the transmission-line picture is a theorem rather than an analogy: for curves, “no distortion” is proved, and the proof is Cauchy–Schwarz on the transverse part of a Lorentzian space.

PROPOSITION 17.9.7 (curves: distortionless $\iff$ RH; [THM]). *Let $X/\mathbb{F}_q$ have genus g , $N_n = \#X(\mathbb{F}_{q^n}) = q^n + 1 - \sum_{i=1}^{2g} \alpha_i^n$. In the log-scale variable $u = n \log q$ put $\beta_i = \log |\alpha_i| / \log q$. Then $\alpha_i^n = e^{\beta_i u} e^{i\varphi_i n}$ with $\varphi_i = \arg \alpha_i$: the point count is a carrier $q^n + 1 = e^u + 1$ plus $2g$ waves of attenuation exponents β_i . The normalized signal $e^{-u/2}(N_n - q^n - 1) = -\sum_i (\alpha_i / \sqrt{q})^n$ is bounded (almost periodic in n) iff all $\beta_i \leq \frac{1}{2}$, and the functional equation ($\alpha \mapsto q/\alpha$ permutes the α_i)*

makes this equivalent to: all $\beta_i = \frac{1}{2}$. So RH for $X \iff$ the line is distortionless, in the sense of Thm. 17.9.2.

PROOF. If some $|\alpha_i| > \sqrt{q}$, then $\sum_n (\sum_i \alpha_i^n) T^n = \sum_i \frac{\alpha_i T}{1 - \alpha_i T}$ has radius of convergence $1/\max |\alpha_i| < q^{-1/2}$, so $|\sum_i \alpha_i^n| q^{-n/2}$ is unbounded. The converse is clear. The lower bound follows from $\{\alpha_i\} = \{q/\alpha_i\}$. $\square$

THEOREM 17.9.8 (the distortionless bound is Cauchy–Schwarz on the transverse part; [THM], classical). On $X \times X$ let $H = X \times \text{pt}$, $V = \text{pt} \times X$ ($H^2 = V^2 = 0$, $H \cdot V = 1$), let Δ be the diagonal and Γ_n the graph of F^n (Lemma 14, §§(d), (g); Lemma 15, $C \times C$ model). Then, numerically,

$$\begin{aligned} \Delta &= (H + V) + \delta, & \Gamma_n &= (H + q^n V) + \gamma_n, & \delta, \gamma_n &\perp \langle H, V \rangle, \\ \delta^2 &= -2g, & \gamma_n^2 &= -2g q^n, & N_n = \Gamma_n \cdot \Delta &= \underbrace{q^n + 1}_{\text{carrier}} + \underbrace{\delta \cdot \gamma_n}_{\text{waves}}, & \gamma_m \cdot \gamma_n &= -q^{\min(m,n)} \sum_i \alpha_i^{|m-n|} \text{ (real part)}. \end{aligned}$$

- (a) Carrier. $(H + q^n V)/q^{n/2} = e^{\eta_n} H + e^{-\eta_n} V$ up to the swap $H \leftrightarrow V$; that is, the hyperbolic part of $\Gamma_n/q^{n/2}$ is the diagonal's hyperbolic part $H + V$ boosted by rapidity $\eta_n = \frac{n}{2} \log q$ (Lemma 14(g)). The carrier $q^n + 1 = (H + q^n V) \cdot (H + V)$ is the timelike inner product of the boosted and unboosted carrier vectors; it equals $2q^{n/2} \cosh \eta_n$.
- (b) Hodge index $\Rightarrow$ transverse spacelike. By Hodge index, $\langle H, V \rangle^\perp \subset \text{NS}(X)_\mathbb{R}$ is negative definite [63, Thm. V.1.9] (Lemma 14, light-cone section). So δ, γ_n are spacelike.
- (c) Cauchy–Schwarz $\Rightarrow$ attenuation $\leq \frac{1}{2}$. On a negative-definite space, $|\delta \cdot \gamma_n| \leq \sqrt{\delta^2 \gamma_n^2} = \sqrt{2g \cdot 2g q^n} = 2g q^{n/2}$. So $|\sum_i \alpha_i^n| \leq 2g q^{n/2}$ for all n , whence $\max \beta_i \leq \frac{1}{2}$ by Prop. 17.9.7.
- (d) Duality $\Rightarrow$ attenuation $\geq \frac{1}{2}$. Poincaré duality on H^1 (equivalently the functional equation of Z_X) gives $\alpha \mapsto q/\alpha$, so $\min \beta_i = 1 - \max \beta_i \geq \frac{1}{2}$. Hence all $\beta_i = \frac{1}{2}$.

In words: the carrier lives in the hyperbolic plane $\langle H, V \rangle$ and is boosted, the waves live in its orthogonal complement, and the Cauchy–Schwarz inequality there is exactly the distortionless bound.

PROOF. The intersection numbers $\Gamma_n \cdot H = q^n$, $\Gamma_n \cdot V = 1$, $\Delta \cdot H = \Delta \cdot V = 1$, $\Delta^2 = 2 - 2g$, $\Gamma_n^2 = q^n(2 - 2g)$, $\Gamma_m \cdot \Gamma_n = q^{\min} N_{|m-n|}$ are in Lemma 14, §(d), and Lemma 15, $C \times C$ model. For $D = aH + bV + D_0$ one has $D \cdot V = a$, $D \cdot H = b$, which gives the decompositions; then $\delta^2 = 2 - 2g - 2$, $\gamma_n^2 = q^n(2 - 2g) - 2q^n$, and $\Gamma_n \cdot \Delta = (H + q^n V) \cdot (H + V) + \delta \cdot \gamma_n$, $\gamma_m \cdot \gamma_n = q^{\min} N_{|m-n|} - q^m - q^n$. (a) is a computation in the hyperbolic plane. (b)–(d) are as stated. This is Weil's proof via Castelnuovo–Severi, in the form of Lemma 14. $\square$

THEOREM 17.9.9 (ζ : distortion $\iff$ extra timelike direction; [THM], reformulation [RH-EQUIV]). In $\mathcal{L} = \mathcal{P} - \mathcal{S}$ (Lemma 15), the pole plane $\mathcal{P}$ is the carrier: it is the smeared form of the main term x of $\psi(x)$ (the $\tilde{H}(\pm \frac{i}{2})$ terms of the explicit formula), it has signature (1,1), and the scaling flow acts on it by boosts of rapidity $t/2$ (Lemma 15, Theorem “structure”). The zero terms are the waves. The following are equivalent:

- (i) a distorted wave exists: some zero has $\beta > \frac{1}{2}$, i.e. $E(u)$ of Thm. 17.9.2 has an exponentially growing component;
- (ii) $N_+ \geq 1$;
- (iii) the “transverse part” ($-\mathcal{S}$ on finite-dimensional spaces) has a timelike direction: $\sup_V n_+(\mathcal{L}|_V) = 1 + N_+ \geq 2$.

More precisely, the number of extra timelike directions is N_+ , the number of distorted waves (Lemma 15, Theorem “Lorentzian index”; Lemma 1(vi)).

PROOF. (i) $\iff$ (ii) by definition of N_+ and the symmetry $\rho \mapsto 1 - \bar{\rho}$. (ii) $\iff$ (iii) is Lemma 15, Theorem “Lorentzian index”. (i) $\iff$ growth of E is Thm. 17.9.2. $\square$

TABLE 17.4. [NUM]/[CA] Carrier/wave split for the Lemma 3 curves (num/lemma17_hodge.out; exact integer arithmetic for N_n , and N_n agrees with Lemma 3 for $n \leq 4$). CS ratio = $|\delta \cdot \gamma_n|/(2gq^{n/2}) \leq 1$. The last columns give the inertia of the Gram matrix of $\{H, V, \Delta, \Gamma_1, \dots, \Gamma_8\}$ and the largest eigenvalue of the normalized transverse Gram $(\gamma_m \cdot \gamma_n)/q^{(m+n)/2}$.

	n	N_n	carrier $q^n + 1$	wave $\delta \cdot \gamma_n$	CS ratio
$E/\mathbb{F}_5, g = 1$ ($P = 1 + 3T + 5T^2$)	1	9	6	3	0.671
	3	108	126	-18	0.805
	4	675	626	49	0.980
	10	9768627	9765626	3001	0.480
$C_2/\mathbb{F}_7, g = 2$ ($P = 1 + 3T + 4T^2 + 21T^3 + 49T^4$)	1	11	8	3	0.283
	3	398	344	54	0.729
	5	16541	16808	-267	0.515
	10	282483289	282475250	8039	0.120
model					
		all β_i	inertia (n_+, n_-, n_0)		max eig. transverse
$E/\mathbb{F}_5$		$\frac{1}{2}$	$(1, 3, 7)$		$2.5 \cdot 10^{-15}$ ($= 0$)
$C_2/\mathbb{F}_7$		$\frac{1}{2}$	$(1, 5, 5)$		$7.7 \cdot 10^{-16}$ ($= 0$)
ghost model $q = 4, \alpha/\sqrt{q} \in \{4, \frac{1}{4}\}$ (Lemma 15; not a curve)		$\frac{3}{2}, -\frac{1}{2}$	$(2, 2, 7)$		$+7.0 \cdot 10^4$

For the ghost model the CS ratio grows like 2^{2n-1} (524288 at $n = 10$) and $(1/n) \log_q |\delta \cdot \gamma_n| \rightarrow \frac{3}{2}$: a distorted wave goes together with a timelike transverse direction (inertia $n_+ = 2$).

REMARK 17.9.10 (no new leverage; the missing ingredient). Theorem 17.9.9 is a *reformulation*: each equivalence is Weil’s criterion in another coordinate system, and it gives no new handle on RH. Comparing it with Thm. 17.9.8 shows exactly what is missing. For curves, “transverse is spacelike” is *proved geometrically* by the Hodge index theorem on the surface $X \times X$: it comes from an ample class, a finite-dimensional $\text{NS}(X)_{\mathbb{R}}$ containing Δ and every Γ_n , and Riemann–Roch. For ζ there is a carrier plane (the poles) and a Lorentzian form $\mathcal{L}$, but there is no surface “ $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ ” whose intersection form restricts to $-\mathcal{S}$ on the orthogonal complement of the pole plane. With such a surface, negativity would follow from an ampleness/Riemann–Roch argument, and Cauchy–Schwarz would then *be* the distortionless bound $\beta \leq \frac{1}{2}$. The missing ingredient is a *geometric reason for transverse spacelikeness*. It is the same gap as Lemma 14’s Obstruction and the Hodge-index question of Definition 5; the transmission-line language names it but does not supply it.

17.10. Confluence of the three actions

- (a) **Trace-formula identity** [THM]. The explicit formula (Lemma 1(i)) equates L-I’s spectral sum $\sum_{\rho} m_{\rho} \tilde{H}(\gamma_{\rho})$ with pole + archimedean – L-III’s orbit sum $\sum_{(p,m)} \ell^{\#} e^{-\ell/2} (H(\ell) + H(-\ell))$, $\ell = m \log p$. It has the form of a Gutzwiller trace formula (periodic orbits $\leftrightarrow$ spectrum). *Not available*: a derivation of it from a classical (or quantum) system whose periodic orbits are L-III’s orbits. Connes’s framework would provide one, and its global positivity is equivalent to RH.
- (b) **L-II is where both are evaluated** [THM]. For $\text{supp } F$ in a window of length A , H is supported in $[-A, A]$, so exactly the orbits with $k \log p < A$ enter (Lemma 10, Lemma 1.1), while the zeros enter through $\tilde{F}(\gamma)$, which can resolve ordinates up to $T_{\text{res}} \approx 2\pi e^A$ (Lemma 11). $\lambda_1(A)$ is the stationary value at which the finitely many orbits and the smeared zeros balance.

- (c) **Interference, literally** [NUM]. (Fig. 17.2.) *Zero-waves* $\rightarrow$ *prime staircase*: $\psi_0(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2})$ (Riemann–von Mangoldt; [38, Ch. 17]); the waves x^{ρ}/ρ add constructively into jumps of size $\log p$ at $x = p^k$ and cancel elsewhere. Max error on $[1.5, 50]$ away from integers: 1.44 (100 zero pairs), 0.36 (1000). *Prime-waves* $\rightarrow$ *zero peaks*: $-\sum_{n \leq X} \frac{\Lambda(n)}{\sqrt{n}} (1 - \frac{\log n}{\log X}) \cos(t \log n)$ peaks at the zero ordinates. For $X = 10^3$ and $t \in [5, 60]$ there are exactly 13 peaks for the 13 zeros, each within 0.018. For $X = 10^5$ every zero has a peak within 0.006 (with a few low side-lobes as well). This is the dual reading of the same identity (Remark 17.2.3).
- (d) **Where they do not overlap.** (1) The archimedean term has no orbit: it is a continuous “change of scale” term, not a closed orbit of L-III (Prop. 17.8.1(c)). (2) The pole plane $\mathcal{P}$ (Lemma 15) is a boundary term at $u \rightarrow \pm\infty$ with no orbit and no zero. (3) Ghosts: stationary points of L-II with negative action (Thm. 17.3.1(d)) have no L-III counterpart; in L-I they are complex-charge pairs (Prop. 17.4.2), invisible to any single orbit. (4) $\mathcal{Y}$ is not a manifold, so L-III exists only prime by prime (M_p), never globally.

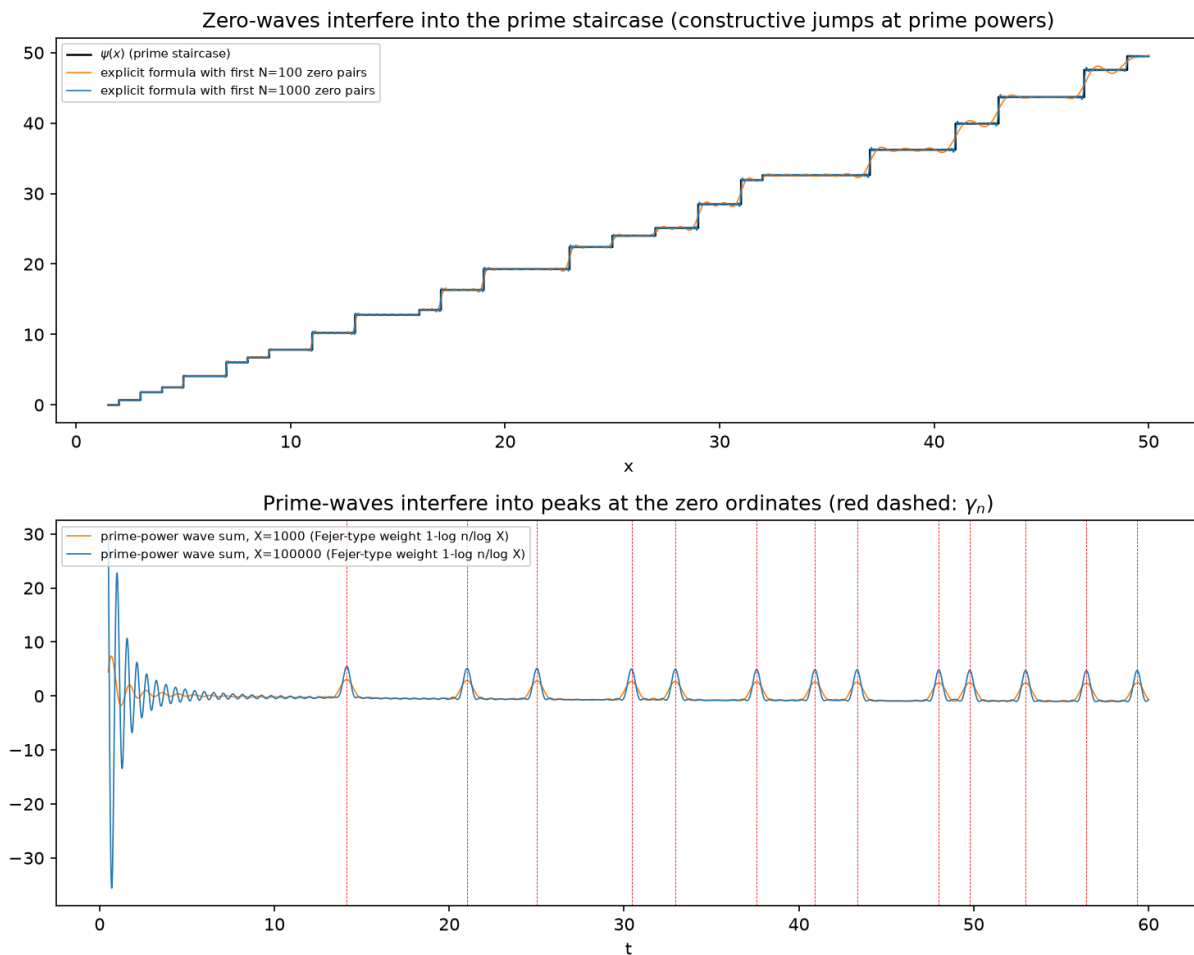


FIGURE 17.2. [NUM] Top: $\psi(x)$ and the explicit formula with 100 and 1000 zero pairs. Bottom: smoothed prime-power wave sums for $X = 10^3, 10^5$; red dashed lines are γ_n . The large oscillation at small t is the pole term (num/lemma17_interference.py).

TABLE 17.5. Objects across the three levels.

object	L-I (field, all u)	L-II (window A)	L-III (orbits)	shared?
prime power p^k	interaction $T_{\pm k \log p}$	iff $k \log p < A$	closed orbit, period $k \log p$	yes [THM]
zero ρ	mode $\tilde{F}(\gamma_\rho)$, charge γ_ρ	resolved if $ \gamma \lesssim T_{\text{res}}$	none (spectral side)	L-I/L-II
archimedean place	kinetic $K(D)$	same, windowed	no orbit	L-I/L-II
pole plane $\mathcal{P}$	rank 2, $(1, 1)$, boost	rank 2	none	no
weights $\frac{\log p}{p^{k/2}}$	coupling constants	same	Gutzwiller amplitude $\times (1 - p^{-k})$	yes (sign flipped)
ghost	complex pair, N_+	$\lambda_k(A) < 0$, κ_A	none	no
symmetry T_t	exact; charge $\mathcal{E}$	broken by window	flow on M_p , $\mathcal{Y}$	yes
Frobenius	boost of rapidity $\frac{1}{2} \log q$ (L15)	–	closing element (Def. 5)	yes
manifold?	field on $\mathbb{R}$	field on I	M_p yes; $\mathcal{Y}$ no	–

17.11. Dictionary: Lagrangian mechanics $\leftrightarrow$ arithmetic action

Lagrangian mechanics	arithmetic action	status
time t	log-scale $u = \log x$	definition
path $q(t)$	field $F(u)$ (half-density in x)	definition
action $S[q]$	$\mathcal{S}[F] = \mathcal{Q}(F, F)$	[THM] (well-defined)
kinetic term	$\langle F, K(D)F \rangle$, $K = 2\theta' \approx \log \frac{ D }{2\pi}$	[THM]
potential / interaction	$-\sum \frac{\log p}{p^{k/2}} 2 \operatorname{Re} \langle F, T_{k \log p} F \rangle$	[THM]
boundary term	pole plane $\mathcal{P}$	[THM]
fixed endpoints, time interval $[0, T]$	support in a window of length A	[THM]
Hamilton's principle	(17.3.1), multiplier = on-shell action	[THM]
Jacobi field / Morse index	negative-action eigenvectors / κ_A	[THM]; Morse [ANALOGY]
conjugate point	A_k^* ; RH $\iff A_1^* = \infty$	[RH-EQUIV]
Legendre condition	$K \rightarrow +\infty$ (finite index)	[THM]
Noether charge of t -translation	$\mathcal{E}[F] = \mathcal{Q}(D_{\text{sc}} F, F)$; spectrum $\{\gamma_\rho\}$	[THM] (spectrum under RH)
Ostrogradsky ghost	absent: residues $m_\rho \geq 1$	[THM]
complex-frequency ghost pair	off-line zero pair; count N_+	[THM], [RH-EQUIV]
propagator positivity	$-\Xi'/\Xi$ Herglotz	[RH-EQUIV]
periodic orbit, Maupertuis action	prime orbit in M_p , $S_0 = Em \log p$	[THM]
Gutzwiller trace formula	explicit formula (form only, sign flipped)	[THM] (form); derivation not available
distortionless line ($RC = LG$)	all $\beta_\rho = \frac{1}{2}$	[RH-EQUIV]; reading [ANALOGY]
distortionless bound (curves)	Cauchy-Schwarz on $\langle H, V \rangle^\perp$ (Hodge index)	[THM]
distorted wave (ζ)	extra timelike direction of $\mathcal{L}$	[RH-EQUIV]

Caldirola–Kanai $e^{\mu t}$	half-density $x^{-1/2}$; cusp $e^{r/2}$	[THM] (cusp), [ANALOGY]
curve over $\mathbb{F}_q$ (model case)	Toeplitz Lagrangian, mass $-\chi\ell$, $\mathcal{S}_X \geq 0$	[CITED] (Weil)

17.12. Obstruction

Every level is now a genuine variational object, and L-II is a genuine Hamilton principle with a Morse index. But positivity, i.e. “the trivial path is a minimum for every window”, is not produced by any of the three structures. The kinetic term is positive only in the ultraviolet ($|r| > 6.29$); the pole and interaction terms are indefinite; L-III supplies the orbit sum with the Gutzwiller *form* and the wrong sign, but no classical system whose quantization would make $\mathcal{S}$ a norm. In the function-field case that missing ingredient is the Hodge index theorem/Rosati positivity, which makes the Toeplitz Lagrangian a sum of squares; for ζ the corresponding statement is Weil positivity, which is RH. The windowed margin $\lambda_1(A)$ decays like $e^{-2\pi^2 e^A}$ (Lemma 11), so no finite computation reaches all A . The transmission-line picture moves the problem to “every echo has width $\frac{1}{4}$ ” and does not remove it.

Outlook [HEUR]. J.W.’s view is that the confluence of the three actions (field, window, orbit), with constructive interference at prime powers and at zero ordinates, is the underlying structure of TAA. What this note establishes is narrower: the three actions are tied together by one identity (the explicit formula), they agree on the prime-orbit data, and they disagree exactly on the archimedean term, the pole plane and the ghosts. Whether a single structure realizes all three *with* positivity is the open Hodge-index question of the Hodge-index question of Definition 5.

17.13. Open questions

- (1) Should the volume adopt L-II (window action + norm constraint) as *the* definition of “arithmetic action”, with L-I as its $A \rightarrow \infty$ limit and L-III as its orbit content? It is the only level where Hamilton’s principle, stationary values and the Morse index all hold literally.
- (2) The naming kinetic = archimedean, interaction = p -adic is forced by the signs but is not unique (one could absorb the pole plane into the kinetic term). Which split do you want in the text?
- (3) Should Theorem 17.5.3 (“no Ostrogradsky ghosts; only complex pairs”) replace the Nash/Stelle analogy of Lemma 1, Remark 1.4? It is sharper: Stelle’s ghost is a real-frequency negative-norm mode, which the Weil action provably never has.

Status summary

Statement	Status	Note
Thm 17.2.1 kinetic+interaction+pole; $K = 2\theta'$; local kernel	[THM], [NUM]	EF of Lemma 1; agreement 10^{-16}
Thm 17.3.1(a)–(e) window variational principle	[THM]	Kato, min-max
Thm 17.3.1(f), $A_1^* = \infty$	[RH-EQUIV]	Weil criterion
Analogy 17.3.2 Morse index	[ANALOGY] (pre- cise half [THM])	
Prop 17.4.1 Noether charge	[THM]	spectrum $\{\gamma\}$ under RH
Prop 17.4.2 complex charges, neutral vectors	[THM]	$N_+ < \infty$
Thm 17.5.1 Ostrogradsky	[CITED]	Woodard 2015
Prop 17.5.2 Krein weight $K'(\omega)$, PU	[THM], [NUM]	sympy
Thm 17.5.3(a),(b)	[THM]	(c) [RH-EQUIV]
Prop 17.6.1 L-III recap	[THM]	Definition 5
Prop 17.7.1 Toeplitz Lagrangian, mass $-\chi\ell$	[THM]; positivity [CITED]	Weil, Rosati
Prop 17.8.1 log-Laplacian, scale covariance, lo- cal formula	[THM]; PV con- stants [CITED]	
Prop 17.9.1, 17.9.3, 17.9.5	[THM]	sympy
Thm 17.9.2	[THM], [RH- EQUIV]	von Koch, Wintner
Prop 17.9.7, Thm 17.9.8 (curves: distortionless = Hodge index + CS)	[THM] (classical), [NUM]	Weil; Lemma 14
Thm 17.9.9 (ζ : distortion $\iff$ extra timelike)	[THM], [RH- EQUIV]	no new leverage
Heaviside/resonance reading; half-density as CK factor	[ANALOGY]	
Confluence (a),(b),(d)	[THM]; deriva- tion from a classical system not available	
Confluence (c), Figs.	[NUM]	

Files. num/lemma17_checks.py/.out/.json, num/lemma17_interference.py/.out,
num/lemma17_heaviside.py/.out, num/lemma17_hodge.py/.out,
img/lemma17_interference.png, img/lemma17_normalized_signal.png. Reused:
num/zeros1000.txt, num/lemma11_eigs240.json, num/lemma11_ghost_table.json.

CHAPTER 18

Lemma 18: The σ -window index $\kappa_{\sigma,A}$:

exact decomposition, the Θ -equivalence, unconditional windows, and how σ -detection length scales

Standalone note `lemma18.tex`, October 21, 2026.

Summary. $\kappa_{\sigma,A}$ is the negative index of Lemma 9's damped Weil form $\mathcal{Q}_\sigma(F) = \mathcal{Q}(e^{-(\sigma - \frac{1}{2})|u|} H_F)$ on functions supported in a window of length A ; $\kappa_{1/2,A} = \kappa_A$. (1) *Exact structure (rigorous)*. For $\sigma \notin \{\text{Re } \rho\}$, $\mathcal{Q}_\sigma = \mathcal{W}^{(\sigma)} + \sum_{\text{Re } \rho > \sigma} m_\rho \pi_\rho^\sigma$: the Lagarias form plus one rank-2 point form of signature $\leq (1, 1)$ for each zero to the right of σ , evaluated at $z_\rho^\sigma = -i(\rho - \sigma)$. Every zero with $\text{Re } \rho \leq \sigma$ contributes a positive form. (2) *Monotonicity (rigorous)*. The window margin $\lambda_\sigma(A)$ is non-increasing in A and *non-decreasing in σ* (a Cauchy average of modulations). Hence $\kappa_{\sigma,A} = 0 \Rightarrow \kappa_{\sigma',A'} = 0$ for $\sigma' \geq \sigma$, $A' \leq A$, and the detection length A_σ^{det} is non-decreasing in σ . The *count* $\kappa_{\sigma,A}$ is not monotone in σ : in a ghost model it is ≥ 6 at $\sigma = 0.52$ but ≤ 2 at $\sigma = \frac{1}{2}$. (3) *The Θ -equivalence (rigorous)*. $\kappa_{\sigma,A} = 0$ for all A iff no zero has $\text{Re } \rho > \sigma$. Thus $\Theta = \min\{\sigma : \kappa_{\sigma,A} = 0 \ \forall A\}$, which is Lemma 9's “RG in σ ” made exact. At $\sigma = \frac{1}{2}$ this is RH; for $\frac{1}{2} < \sigma < 1$ it is the quasi-Riemann hypothesis $\Theta \leq \sigma$, open for every such σ . (4) *Unconditional, explicit*. (a) For $\sigma \geq 1$ and every A : $\lambda_\sigma(A) \geq m_1 = 2 + \gamma - \log 4\pi = 0.0461914\dots$, and m_1 is *exactly* $\inf_t w_1(t)$ (convexity on $|t| \leq 13.5$, Arb on $[13.5, 5000]$, explicit $N(T)$ beyond). (b) For every $\sigma \geq \frac{1}{2}$ and $A \leq 1.6$: $\lambda_\sigma(A) \geq 9.2792 \cdot 10^{-18}$ (from Lemma 16 by (2)). (c) For every A there is an explicit $\sigma_*(A) < 1$ with $\kappa_{\sigma,A} = 0$ on $[\sigma_*(A), 1]$, and $1 - \sigma_*(A) \approx 0.023/A^2$ (e.g. $\sigma_*(10) = 0.99977$). No zero-free region or zero-density estimate can give “ $\kappa_{\sigma,A} = 0$ for all A ” with $\sigma < 1$, because that statement is equivalent to $\Theta \leq \sigma$. (5) *How σ -detection length scales (numerical, ghost injection)*. At fixed abscissa the detection length of a ghost of displacement $\varepsilon = \beta - \sigma$ grows with local exponent $p = -d \log A_\sigma^{\text{det}} / d \log \varepsilon$ falling from 0.85 to 0.60 as ε goes from 0.2 to 0.009 ($\sigma = 0.7$). It obeys $A_\sigma^{\text{det}} |S| / 2\pi \in [0.24, 0.87]$, where $|S|$ is the measure of $\{w_\sigma < 0\}$, and the counts follow the Landau–Pollak law $\kappa_{\sigma,A} \approx A|S|/2\pi$. The guess $A_\sigma^{\text{det}} \approx c/(\beta - \sigma)$ of Lemma 11 (and of the ranking item) is *not* what happens at fixed σ . The data point to $A_\sigma^{\text{det}} \asymp \sqrt{b/\varepsilon}$, with b the local background; a rigorous lower bound with exponent $\frac{1}{2}$ holds.

18.1. Setting

Notation follows Lemmas 1, 9–12. For $F, G \in C_c^\infty(\mathbb{R})$ put $H_{F,G}(u) = \int F(y + u) \overline{G(y)} dy$, $H_F = H_{F,F}$, $\widehat{F}(z) = \int F(u) e^{izu} du$, $\|F\|^2 = \int |F|^2 = \frac{1}{2\pi} \int |\widehat{F}|^2$, and $\gamma_\rho = -i(\rho - \frac{1}{2})$. Zeros are counted with multiplicity m_ρ , $\Theta = \sup \text{Re } \rho$, and $c = c_\sigma = \sigma - \frac{1}{2} \geq 0$.

DEFINITION 18.1.1. For $\sigma \geq \frac{1}{2}$ let $G_\sigma = e^{-c|u|} H_{F,G}$ and

$$\mathcal{Q}_\sigma(F, G) := \sum_\rho m_\rho \widehat{G}_\sigma(\gamma_\rho) \quad (\text{Lemma 9; } \mathcal{Q}_{1/2} = \mathcal{Q}).$$

Let $\mathcal{D}_A$ be the real $F \in C_c^\infty$ supported in an interval of length A . Then $\kappa_{\sigma,A} := \sup\{\dim V : V \subset \mathcal{D}_A, \mathcal{Q}_\sigma < 0 \text{ on } V \setminus 0\}$, $\lambda_\sigma(A) := \inf_{0 \neq F \in \mathcal{D}_A} \mathcal{Q}_\sigma(F)/\|F\|^2$ and $A_\sigma^{\det} := \inf\{A : \kappa_{\sigma,A} \geq 1\}$. The Lagarias form is $\mathcal{W}^{(\sigma)}(F) = \frac{1}{2\pi} \int |\widehat{F}|^2 w_\sigma$ with $w_\sigma(t) = 2 \operatorname{Re} \frac{\xi'}{\xi}(\sigma + it) = \sum_\rho m_\rho 2 \operatorname{Re} \frac{1}{\sigma + it - \rho}$ [39, Ch. 12], [87]. Finally $N_+(\sigma) = \#\{\rho : \operatorname{Re} \rho > \sigma\}$.

$\kappa_{1/2,A} = \kappa_A$, and $\lambda_{1/2}(A) = \lambda^*(e^A)$ is Lemma 10's margin. G_σ is Lipschitz, compactly supported, and smooth off $u = 0$, so two integrations by parts on each half-line give $|\widehat{G}_\sigma(z)| \leq C(1 + |z|)^{-2}$ for $|\operatorname{Im} z| \leq \frac{1}{2}$. The series therefore converges absolutely, as in Lemma 1, Def. 1.3. $\mathcal{Q}_\sigma$ is Hermitian and T_t -invariant ($H_{T_t F} = H_F$). For complex F , $\mathcal{Q}_\sigma(F) = \mathcal{Q}_\sigma(\operatorname{Re} F) + \mathcal{Q}_\sigma(\operatorname{Im} F)$: the cross term of H_F is i times an odd function, $e^{-c|u|}$ is even, and the zero set is symmetric under $\gamma \mapsto -\gamma$. So indices and margins may be computed on complex functions; this is the complexification, which preserves inertia.

18.2. Exact structure

THEOREM 18.2.1 (σ -decomposition; rigorous). *Let $\sigma > \frac{1}{2}$ with $\sigma \neq \operatorname{Re} \rho$ for all ρ , and $F \in C_c^\infty$. Then, with absolutely convergent sums,*

$$\mathcal{Q}_\sigma(F) = \mathcal{W}^{(\sigma)}(F) + \sum_{\operatorname{Re} \rho > \sigma} m_\rho \pi_\rho^\sigma(F), \quad \pi_\rho^\sigma(F) = \widehat{F}(z) \overline{\widehat{F}(\bar{z})} + \widehat{F}(\bar{z}) \overline{\widehat{F}(z)}, \quad z = z_\rho^\sigma := -i(\rho - \sigma).$$

Each π_ρ^σ is a Hermitian form of rank ≤ 2 and signature $\leq (1, 1)$.

PROOF. Group each zero $\rho = \frac{1}{2} + \delta + i\gamma_0$ ($\delta \geq 0$) with $\rho^* = 1 - \bar{\rho}$ (once if $\delta = 0$). The pair's kernel is $e^{-c|u|}(e^{i\gamma_0 u} + e^{i\gamma_0^* u}) = e^{i\gamma_0 u} 2 \cosh(\delta u) e^{-c|u|} = e^{i\gamma_0 u} (e^{(\delta-c)|u|} + e^{-(\delta+c)|u|})$. For $b > 0$, $\int e^{-b|u|} e^{i\gamma_0 u} H_F(u) du = \frac{1}{2\pi} \int |\widehat{F}(t)|^2 \frac{2b}{b^2 + (t - \gamma_0)^2} dt$. If $\delta < c$ (that is, $\operatorname{Re} \rho < \sigma$), both exponents are negative, and the two Lorentzians are $2 \operatorname{Re} \frac{1}{\sigma + it - \rho}$ and $2 \operatorname{Re} \frac{1}{\sigma + it - \rho^*}$. If $\delta > c$, write $e^{\varepsilon|u|} = 2 \cosh(\varepsilon u) - e^{-\varepsilon|u|}$ with $\varepsilon = \delta - c = \operatorname{Re} \rho - \sigma$. The term $-e^{-\varepsilon|u|}$ gives the Lorentzian $-\frac{2\varepsilon}{\varepsilon^2 + (t - \gamma_0)^2} = 2 \operatorname{Re} \frac{1}{\sigma + it - \rho}$. The term $2 \cosh(\varepsilon u) e^{i\gamma_0 u}$ gives $\widehat{H}_F(\gamma_0 - i\varepsilon) + \widehat{H}_F(\gamma_0 + i\varepsilon)$, and $\widehat{H}_F(z) = \widehat{F}(z) \widehat{F}(\bar{z})$ (Lemma 1, proof of (i)); note $\gamma_0 - i\varepsilon = z_\rho^\sigma$. Summing all Lorentzians gives w_σ . Absolute convergence: Poisson smoothing of $|\widehat{F}|^2$ at γ is $O((1 + \gamma^2)^{-1})$ uniformly for widths in $(0, 1]$, and $|\widehat{H}_F(z)| = O((1 + |z|)^{-2})$ in the strip; both are summable against $\sum m_\rho (1 + \gamma^2)^{-1} < \infty$. Signature: $2 \operatorname{Re}(ab) = \frac{1}{2}(|a + b|^2 - |a - b|^2)$. $\square$

COROLLARY 18.2.2 (index bookkeeping; rigorous). (a) *If $N_+(\sigma) < \infty$, then $|\kappa_{\sigma,A} - \kappa_{\sigma,A}^W| \leq N_+(\sigma)$ for every A .* (b) *If $\Theta \leq \sigma$, then $\mathcal{Q}_\sigma = \mathcal{W}^{(\sigma)} \geq 0$ (Lemma 9, Prop. 4.1).* (c) *If $N_+(\sigma) < \infty$ and $w_\sigma < 0$ on a set of positive measure, then $\sup_A \kappa_{\sigma,A} = \infty$.* (d) *If ρ_0 is a zero with $\beta_0 = \operatorname{Re} \rho_0 > \frac{1}{2}$, then $w_\sigma(\gamma_0) \rightarrow -\infty$ as $\sigma \uparrow \beta_0$. So every off-line zero produces a range $(\beta_0 - \eta, \beta_0)$ of abscissae where (c) applies, provided $N_+(\sigma) < \infty$ there.*

PROOF. (a) Each $m_\rho \pi_\rho^\sigma$ adds or removes at most one negative square on any finite-dimensional V . (c) Pick k disjoint intervals I_j on which $w_\sigma \leq -\eta < 0$, and $F_j = e^{-it_j x} \varphi(x/R)/\sqrt{R}$ with $t_j \in I_j$ and $\varphi \in C_c^\infty$ fixed. As $R \rightarrow \infty$, $|\widehat{F}_j|^2$ concentrates in I_j . Since $\widehat{\varphi}$ is Schwartz and $w_\sigma^\pm = O(\log(2 + |t|))$, the $\mathcal{W}^{(\sigma)}$ -Gram matrix tends to a diagonal matrix with entries $\leq -\eta$. Then $\kappa_{\sigma,A}^W \geq k$ for $A \geq$ the support length, and (a) gives $\kappa_{\sigma,A} \geq k - N_+(\sigma)$. (d) ξ'/ξ has a simple pole with residue $m_{\rho_0} > 0$ at ρ_0 , and $\operatorname{Re} \frac{m}{\sigma - \beta_0} \rightarrow -\infty$; the remaining terms are bounded near ρ_0 . $\square$

PROPOSITION 18.2.3 (prime side; rigorous). For $\frac{1}{2} \leq \sigma \leq 1$, real $F \in \mathcal{D}_A$ and $a = 1 - \sigma$,

$$\begin{aligned} \mathcal{Q}_\sigma(F) = & \int (e^{a|u|} + e^{-\sigma|u|}) H_F(u) du - \|F\|^2 \log \pi \\ & + \frac{1}{2\pi} \int |\widehat{F}(t)|^2 \operatorname{Re} \psi\left(\frac{\sigma}{2} + \frac{it}{2}\right) dt - \sum_{n < e^A} 2\Lambda(n) n^{-\sigma} H_F(\log n). \end{aligned}$$

For $\sigma < 1$ the first term equals $2c_+c_- + \frac{1}{2\pi} \int |\widehat{F}|^2 2 \operatorname{Re} \left(\frac{1}{\sigma+it} + \frac{1}{\sigma-1+it} \right) dt$, with $c_\pm = \int F e^{\pm au}$. At $\sigma = \frac{1}{2}$ this is Lemma 10's frequency form, with Lemma 10's Ψ_d and pole term.

PROOF. Apply the explicit formula (Lemma 1, Thm. 2.1(i)) to G_σ . It holds for this continuous, piecewise-smooth, compactly supported even function by mollification: both sides are continuous under C^0 -convergence with uniformly bounded $\int |G''|$ away from 0, since the zero side is absolutely convergent and the prime side finite. In the pole term, $\int G_\sigma 2 \cosh \frac{u}{2} = \int H_F(e^{(1-\sigma)|u|} + e^{-\sigma|u|})$. In the primes, $G_\sigma(\log n) = n^{-c} H_F(\log n)$. In the archimedean term, $\widehat{G}_\sigma = P_c * |\widehat{F}|^2$ with the Cauchy kernel P_c . Since $\operatorname{Re} \psi(\frac{1}{4} + \frac{z}{2})$ is harmonic in $\operatorname{Re} z > -\frac{1}{2}$ and $O(\log |z|)$, its Poisson extension from $\operatorname{Re} z = 0$ to $\operatorname{Re} z = c$ is $\operatorname{Re} \psi(\frac{\sigma}{2} + \frac{it}{2})$. The last sentence uses $e^{a|u|} = 2 \cosh(au) - e^{-a|u|}$. $\square$

Checks (`num/lemma18_checks.py/json`). (i) For an injected quadruple $\beta_0 = 0.75$, $\gamma_0 = 27.7$ at $\sigma = 0.62$ and a complex window function, the four-term zero sum equals the Lorentzian part plus π_ρ^σ to $1.4 \cdot 10^{-14}$; the parts are -0.4606 and $+6.7372$. (ii) Prop. 18.2.3 against $\mathcal{W}^{(\sigma)}$ computed from ξ'/ξ (`mpmath`), for a non-even F on a window of length 1.5: the differences are $1.0 \cdot 10^{-9}$, $4.5 \cdot 10^{-10}$, $1.7 \cdot 10^{-10}$ at $\sigma = 0.6, 0.75, 0.9$. Equality is expected by Thm. 18.2.1, since no zeta zero below the verified height has $\beta > \sigma$, and for a smooth window function the influence of zeros beyond $3 \cdot 10^{12}$ is negligible.

18.3. Monotonicity and the Θ -equivalence

PROPOSITION 18.3.1 (rigorous). (a) $\kappa_{\sigma,A}$ is non-decreasing in A , and $\lambda_\sigma(A)$ is non-increasing in A . (b) For $\sigma' > \sigma \geq \frac{1}{2}$ and $F \in C_c^\infty$,

$$\mathcal{Q}_{\sigma'}(F) = \int_{\mathbb{R}} \mathcal{Q}_\sigma(M_s F) d\mu(s), \quad M_s F(x) = e^{isx} F(x), \quad d\mu = \frac{(\sigma' - \sigma)/\pi}{(\sigma' - \sigma)^2 + s^2} ds.$$

Hence $\lambda_{\sigma'}(A) \geq \lambda_\sigma(A)$. In particular $\kappa_{\sigma,A} = 0 \Rightarrow \kappa_{\sigma',A'} = 0$ for all $\sigma' \geq \sigma$, $A' \leq A$; $A_\sigma^{\det}$ is non-decreasing in σ ; and $\lambda_\sigma(A) \geq \lambda^*(e^A)$ (Lemma 10's margin). (c) $\sigma \mapsto \kappa_{\sigma,A}$ is lower semicontinuous; in particular $\kappa_A \leq \liminf_{\sigma \downarrow \frac{1}{2}} \kappa_{\sigma,A}$.

PROOF. (a) $\mathcal{D}_A \subset \mathcal{D}_{A'}$, together with translation invariance (Lemma 16, Rem. 3.1). (b) $H_{M_s F}(u) = e^{isu} H_F(u)$, so $\widehat{G_\sigma^{M_s F}}(z) = \widehat{G_\sigma^F}(z + s)$. Also $e^{-(\sigma' - \sigma)|u|} = \int e^{isu} d\mu(s)$, so $\widehat{G_{\sigma'}^F}(z) = \int \widehat{G_\sigma^F}(z + s) d\mu(s)$ by Fubini. Sum over ρ and interchange: $\sum_\rho m_\rho \int |\widehat{G_\sigma}(\gamma_\rho + s)| d\mu(s) \leq C \sum_\rho m_\rho (1 + \gamma_\rho^2)^{-1} < \infty$, because the Cauchy average of $(1 + |x|)^{-2}$ is $O((1 + x^2)^{-1})$. Since $\|M_s F\| = \|F\|$ and $M_s F$ has the same support, $\mathcal{Q}_{\sigma'}(F) \geq \lambda_\sigma(A) \|F\|^2$ (complex F allowed, §1). (c) $\mathcal{Q}_\sigma(F)$ is continuous in σ for fixed F (dominated convergence on the zero side). A negative definite V of finite dimension stays negative under small perturbation, because its unit sphere is compact. $\square$

REMARK 18.3.2 (counts are not monotone in σ). Averaging (b) preserves the margin, but a Cauchy average of forms with $\leq k$ negative squares can have more. Cor. 18.2.2(c,d) predicts that $\sup_A \kappa_{\sigma,A} = \infty$ just below every $\beta > \frac{1}{2}$ (when $N_+(\sigma) < \infty$), while $\sup_A \kappa_A = N_+$ (Lemma 1, Thm. 2.1(vi)). Table 18.3 shows this in the ghost model: $\kappa_{0.52,80} \geq 6$, whereas $\kappa_{80} \leq N_+ = 2$.

THEOREM 18.3.3 (Θ -equivalence; rigorous). For $\sigma \geq \frac{1}{2}$ the following are equivalent: (a) no zero has $\operatorname{Re} \rho > \sigma$; (b) $\kappa_{\sigma,A} = 0$ for all A ; (c) $\mathcal{Q}_\sigma(F) \geq 0$ for all $F \in C_c^\infty$. Consequently

$$\Theta = \min\{\sigma \geq \tfrac{1}{2} : \kappa_{\sigma,A} = 0 \text{ for all } A\},$$

and the set in braces is the interval $[\Theta, \infty)$. At $\sigma = \frac{1}{2}$, (a) is RH $[\equiv \text{RH}]$ (Weil's criterion [157, 14]). For $\frac{1}{2} < \sigma < 1$, (a) is the quasi-Riemann hypothesis $\Theta \leq \sigma$ $[\equiv \Theta \leq \sigma$, open for every $\sigma < 1$]. For $\sigma \geq 1$, (a) holds (classically, $\beta < 1$ for every zero).

PROOF. (b) $\Leftrightarrow$ (c): every F lies in some $\mathcal{D}_A$. (a) $\Rightarrow$ (c): Lemma 9, Prop. 4.1, which needs only $\sigma \geq \beta$ for all ρ . (c) $\Rightarrow$ (a), the argument of Lemma 1, Thm. 2.1(iii): let $F, G \in C_c^\infty$ with $\operatorname{supp} H_{F,G} \subset [-R, R]$, and let $t > R$. Then $H_{T_t F, G}(u) = H_{F, G}(u - t)$ is supported in $u > 0$, where $e^{-c|u|} = e^{-cu}$. Hence

$$\mathcal{Q}_\sigma(T_t F, G) = e^{-ct} L(t), \quad L(t) = \sum_{\rho} a_{\rho} e^{i\gamma_{\rho} t}, \quad a_{\rho} = m_{\rho} \widehat{F}(\gamma_{\rho} + ic) \overline{\widehat{G}(\bar{\gamma}_{\rho} - ic)}, \quad \sum |a_{\rho}| < \infty.$$

If $\mathcal{Q}_\sigma \geq 0$, Cauchy–Schwarz and T_t -invariance give $|\mathcal{Q}_\sigma(T_t F, G)|^2 \leq \mathcal{Q}_\sigma(F) \mathcal{Q}_\sigma(G)$. Thus $|L(t)| \leq C e^{ct}$, and Lemma 1, Lemma 3.2 (the Laplace lemma) gives $a_{\rho} = 0$ whenever $\operatorname{Re} \rho - \frac{1}{2} > c$, i.e. $\operatorname{Re} \rho > \sigma$. Given a zero ρ_0 with $\operatorname{Re} \rho_0 > \sigma$, choose F, G with $\widehat{F}(\gamma_{\rho_0} + ic) \neq 0 \neq \widehat{G}(\bar{\gamma}_{\rho_0} - ic)$ (Lemma 1, Lemma 3.1). This is a contradiction. The minimum is attained by (a) $\Rightarrow$ (b) at $\sigma = \Theta$, and the set is an up-interval by Prop. 18.3.1(b). $\square$

Reading. Lemma 9 asked for an “RG in σ ” in which Θ is the first scale, coming down from $\sigma = 1$, at which a negative square appears. Thm. 18.3.3 is that statement, exactly and unconditionally as an *identity*. Evaluating it is RH-hard: deciding whether $\Theta < 1$ is already open.

18.4. Unconditional results with explicit constants

THEOREM 18.4.1 ($\sigma \geq 1$; rigorous, computer-assisted).

$\inf_{t \in \mathbb{R}} w_1(t) = w_1(0) = 2 + \gamma - \log 4\pi = 0.046191417932 \dots =: m_1$. For every $\sigma \geq 1$ and every A ,

$$\lambda_{\sigma}(A) \geq m_{\sigma} := \inf w_{\sigma} \geq m_1, \quad \text{so } \kappa_{\sigma,A} = 0,$$

and $\lim_{A \rightarrow \infty} \lambda_1(A) = m_1$.

PROOF. For $\sigma \geq 1 \geq \Theta$, Thm. 18.2.1 (or Lemma 9) gives $\mathcal{Q}_\sigma = \mathcal{W}^{(\sigma)} \geq m_{\sigma} \|F\|^2$. Each Lorentzian in w_{σ} has width $\sigma - \operatorname{Re} \rho > 0$, so $w_{\sigma'} = P_{\sigma' - \sigma} * w_{\sigma}$ (Poisson semigroup) and $m_{\sigma'} \geq m_{\sigma}$ for $\sigma' \geq \sigma \geq 1$. *Value of m_1 .* (i) $|t| \leq 13.5$. Each term $2b/(b^2 + (t - \gamma)^2)$ of w_1 has $b \in (0, 1]$, and it is convex where $|t - \gamma| > b/\sqrt{3}$. No zero has $|\gamma| < 14.13$ [124, 114], so every term is convex on $|t| \leq 13.5$. w_1 is even, hence minimal at 0, where $w_1(0) = 2\frac{\xi'}{\xi}(1) = 2 + \gamma - \log 4\pi$. (ii) $13.5 \leq t \leq 5000$ (num/lemma18_margin.py, Arb [78], 128 bits). We use $w_1(t) \geq w_1(t_c) - r \sup_{|t - t_c| \leq r} |w_1'|$, with w_1' enclosed from the order-2 Taylor series of $\log \xi$ evaluated on balls, and adaptive bisection. There were 498,650 evaluations and no failure, and the minimum certified lower bound is 0.2242. (iii) $t \geq 5000$. Every zero contributes $\geq \frac{1}{2(1+(t-\gamma)^2)}$, from $\frac{2\beta}{\beta^2+y^2} \geq \frac{1}{1+y^2}$ for the member of its pair with $\beta \geq \frac{1}{2}$. Hence $w_1(t) \geq \frac{N(t+H)-N(t-H)}{2(1+H^2)}$, and Trudgian's explicit bound [151] (as in Lemma 12) gives at $H = 10$ the value $g(t) = \frac{M(t+H)-M(t-H)-R(t+H)-R(t-H)}{2(1+H^2)}$, with $g(5000) \geq 0.05639$.¹ Also $g' > 0$ on $[5000, \infty)$, since $\frac{H}{\pi t} > 2 \frac{0.112+0.278/\log(t-H)}{t-H}$ there. All three ranges give $\geq 0.0462 > m_1$ outside $t = 0$. The limit follows by concentrating $\widehat{F}$ at $t = 0$ (Lemma 11, Prop. 3.2). $\square$

¹[Vol. 2 editorial note] Trudgian's 2014 bound is reported to be erroneous (Hasanalizade–Shen–Wong). Chapter 24 redoes this step with the corrected R_{cor} : $g(5000) = 0.07167$, and Theorem 18.4.1 stands. See Appendix A.

THEOREM 18.4.2 (short windows, all σ ; rigorous, computer-assisted). *For every $\sigma \geq \frac{1}{2}$: $\lambda_\sigma(A) \geq 1.447 \cdot 10^{-8}$ for $A \leq \log 3$, $\geq 3.5368 \cdot 10^{-13}$ for $A \leq \log 4$, and $\geq 9.2792 \cdot 10^{-18}$ for $A \leq 1.6$. Hence $\kappa_{\sigma,A} = 0$ for all $\sigma \geq \frac{1}{2}$ and $A \leq 1.6$.*

PROOF. Prop. 18.3.1(b) with Lemma 10, Thm. 3.1 and Lemma 16, Thm. 2.1. $\square$

THEOREM 18.4.3 (every window: an explicit σ -neighbourhood of 1; rigorous). *For $A > 0$ and $\frac{1}{2} \leq \sigma \leq 1$, $a = 1 - \sigma$, put*

$$E(\sigma, A) = \min\left(2\left[\frac{e^{aA}-1}{a} - A\right], A(e^{aA} - 1)\right) + \frac{2(1-\sigma)}{\sigma} + \frac{1-\sigma}{2} \psi'\left(\frac{\sigma}{2}\right) + \sum_{n < e^A} 2\Lambda(n)(n^{-\sigma} - n^{-1}).$$

Then $\lambda_\sigma(A) \geq m_1 - E(\sigma, A)$. E is decreasing in σ , and $E(1, A) = 0$. Hence $\kappa_{\sigma,A} = 0$ for $\sigma \in [\sigma_(A), 1]$, where $E(\sigma_*(A), A) < m_1$. Certified values (Arb, `num/lemma18_sigstar.json`):*

A	1	1.6	2	3	5	10	15
$\sigma_*(A)$	0.99230	0.99477	0.99627	0.99794	0.99914	0.999774	0.999898
$A^2(1 - \sigma_*)$	0.0077	0.0134	0.0149	0.0185	0.0214	0.0226	0.0229

PROOF. Subtract Prop. 18.2.3 at 1 from the same formula at σ , and use $\mathcal{Q}_1 \geq m_1 \|F\|^2$ (Thm. 18.4.1). The $-\log \pi$ terms cancel. Then use $|H_F| \leq \|F\|^2$, $\text{supp } H_F \subset [-A, A]$ and $\int |H_F| \leq A \|F\|^2$ for the pole kernels $e^{a|u|} - 1$ and $e^{-\sigma|u|} - e^{-|u|}$. For the archimedean symbol, $|\text{Re } \psi(\frac{\sigma}{2} + \frac{it}{2}) - \text{Re } \psi(\frac{1}{2} + \frac{it}{2})| \leq \frac{1-\sigma}{2} \psi'(\frac{\sigma}{2})$, since $|\psi'(x + iy)| \leq \psi'(x)$; then apply Parseval. For the finitely many primes, use $|H_F(\log n)| \leq \|F\|^2$. Each term is decreasing in σ . $\square$

So the unconditional region where $\kappa_{\sigma,A} = 0$ is $\{A \leq 1.6\} \cup \{\sigma \geq \sigma_*(A)\}$, with $1 - \sigma_*(A) \sim 0.023A^{-2}$; the constant is $\approx m_1/2$, set by the pole and prime terms.

PROPOSITION 18.4.4 (what zero-free regions and zero density do and do not give). (a) *For any $\sigma < 1$ and every A , the statement “ $\kappa_{\sigma',A} = 0$ for all A and all $\sigma' \geq \sigma$ ” is equivalent to $\Theta \leq \sigma$ (Thm. 18.3.3). It is not implied by any known result. The Mossinghoff–Trudgian–Yang region $\beta < 1 - 1/(5.558691 \log |\gamma|)$ ($|\gamma| \geq 2$) [108] allows $\beta \rightarrow 1$. Zero-density bounds, e.g. Ingham’s [73] or $N_+(\sigma) \cap \{|\gamma| \leq T\} \leq T^{30(1-\sigma)/13+o(1)}$ (Guth–Maynard [61]), bound counts, while a single zero with $\text{Re } \rho > \sigma$ already gives $\sup_A \kappa_{\sigma,A} \geq 1$. (b) What [108, 124] do give: every zero with $\text{Re } \rho > \sigma$ has $|\gamma| \geq \max(e^{1/(5.558691(1-\sigma))}, 3 \cdot 10^{12})$. Thm. 18.4.3 is consistent with this. A zero with $\beta > \sigma_*(A)$ has displacement $\varepsilon \leq 1 - \sigma_*(A) \approx 0.023/A^2$, and its point form changes $\mathcal{Q}_\sigma$ on $\mathcal{D}_A$ by $O(\varepsilon A^2) \|F\|^2$. So Thm. 18.4.3 excludes nothing that is not already excluded. This is the σ -analogue of Lemma 12’s “vacuous today”.*

18.5. How the σ -detection length scales (ghost injection)

Model (as in Lemmas 11, 15): the true zeta zeros plus one quadruple $\beta_0 \pm i\gamma_0$, $1 - \beta_0 \pm i\gamma_0$, with $\gamma_0 \in \{27.72, 100.07\}$ mid-gap. By Thm. 18.2.1 the model form is $\mathcal{W}^{(\sigma)}$ (from ξ'/ξ , `mpmath`) plus the ghost’s two Lorentzians plus its point forms. *Basis*: $F_n(x) = e^{-i\gamma_0 x} (1 - x^2/h^2)^2 P_n(x/h)$, $|x| \leq h$, $A = 2h$, $n < N = \max(14, 0.7h + 12)$; transforms are exact via spherical Bessel functions. *Output*: the number of negative generalized eigenvalues, which is a lower bound for $\kappa_{\sigma,A}$ in the model, and A_σ^{det} by geometric search and bisection. The computation is float64 and *not certified* (`num/lemma18_core.py`, `lemma18_ghost.py`, `lemma18_ghost2.py`, `lemma18_analyze.py`). Without the ghost no negative eigenvalue occurs in any run. *Findings*. (1) *Landau–Pollak count*. Away from detection threshold, $\kappa_{\sigma,A}$ tracks $A|S|/2\pi$ to within ± 1 , where $S = \{w_\sigma < 0\}$ near γ_0 (Table 18.3; e.g. 7 vs 6.1, 4 vs 3.7, 1 vs 1.2). This is the quantitative form of Cor. 18.2.2(c) [89]. (2) *Threshold*. $A_\sigma^{\text{det}}|S|/2\pi$ lies in $[0.24, 0.87]$ in all 59 detection runs, and increases slowly as $\varepsilon \downarrow 0$ (Tables 18.1, 18.2). (3) *Exponent*. At fixed $\sigma = 0.7$ the local exponent p falls from 0.85 to 0.60 ($\gamma_0 = 27.72$) and from 0.66 to 0.57 ($\gamma_0 = 100.07$). At

TABLE 18.1. σ -detection length A_σ^{det} of one injected ghost quadruple $(\beta_0 \pm i\gamma_0, 1 - \beta_0 \pm i\gamma_0)$ at fixed abscissa σ (fixed background), as $\varepsilon = \beta_0 - \sigma \downarrow 0$. $|S|$ = measure of $\{t : w_\sigma(t) < 0\}$ near γ_0 (ghost Lorentzians included); $b = w_\sigma^\zeta(\gamma_0) + 2/(\sigma + \beta_0 - 1)$ is the local positive background; p = local slope $-d \log A_\sigma^{\text{det}} / d \log \varepsilon$. Numerical (float64, modulated window basis, $\kappa \geq 1$ found by bisection to $\pm 1\%$); source `num/lemma18_ghost2.jsonl`.

σ	γ_0	ε	A_σ^{det}	$ S $	$A_\sigma^{\text{det}} S /2\pi$	p	b	$A_\sigma^{\text{det}}\sqrt{\varepsilon/b}$
0.7	27.72	0.2	4.8	0.652	0.50		3.49	1.15
0.7	27.72	0.14	6.5	0.526	0.55	0.85	3.86	1.24
0.7	27.72	0.1	8.6	0.432	0.59	0.83	4.16	1.34
0.7	27.72	0.07	11.4	0.353	0.64	0.78	4.41	1.43
0.7	27.72	0.05	14.6	0.293	0.68	0.74	4.60	1.52
0.7	27.72	0.035	18.8	0.242	0.72	0.71	4.75	1.62
0.7	27.72	0.025	23.6	0.202	0.76	0.68	4.86	1.70
0.7	27.72	0.018	29.3	0.170	0.79	0.65	4.94	1.77
0.7	27.72	0.0125	36.8	0.141	0.83	0.63	5.00	1.84
0.7	27.72	0.009	44.9	0.119	0.85	0.60	5.05	1.90
0.7	100.07	0.2	6.7	0.549	0.58		3.97	1.50
0.7	100.07	0.14	8.4	0.464	0.62	0.64	4.34	1.51
0.7	100.07	0.1	10.5	0.391	0.65	0.66	4.64	1.54
0.7	100.07	0.07	13.3	0.325	0.69	0.67	4.89	1.59
0.7	100.07	0.05	16.8	0.273	0.73	0.68	5.08	1.66
0.7	100.07	0.035	21.1	0.227	0.76	0.65	5.23	1.73
0.7	100.07	0.025	26.1	0.191	0.79	0.63	5.34	1.79
0.7	100.07	0.018	32.0	0.162	0.82	0.62	5.42	1.85
0.7	100.07	0.0125	39.9	0.134	0.85	0.60	5.49	1.90
0.7	100.07	0.009	48.2	0.114	0.87	0.57	5.53	1.94
0.55	27.72	0.2	3.2	0.478	0.24		6.71	0.55
0.55	27.72	0.14	4.6	0.362	0.27	1.07	8.37	0.60
0.55	27.72	0.1	6.8	0.280	0.30	1.12	10.04	0.68
0.55	27.72	0.07	10.1	0.217	0.35	1.12	11.80	0.78
0.55	27.72	0.05	14.3	0.173	0.39	1.04	13.37	0.88
0.55	27.72	0.035	20.6	0.137	0.45	1.01	14.85	1.00
0.55	27.72	0.025	28.5	0.112	0.51	0.97	16.04	1.13
0.55	27.72	0.018	38.2	0.092	0.56	0.89	16.99	1.24
0.55	27.72	0.0125	51.9	0.075	0.62	0.84	17.82	1.37
0.55	27.72	0.009	67.1	0.063	0.67	0.78	18.39	1.48

TABLE 18.2. A_σ^{det} for fixed ghosts as $\sigma \uparrow \beta_0$ (the background changes with σ). Entries $A_\sigma^{\text{det}} [A_\sigma^{\text{det}}|S|/2\pi]$. Source `num/lemma18_ghost.jsonl`.

ghost $(\beta_0, \gamma_0) \setminus \varepsilon$	0.2	0.14	0.1	0.07	0.05	0.035	0.025	0.018
(0.6, 27.72)	—	—	—	8.5 [0.26]	14.3 [0.39]	20.9 [0.50]	28.0 [0.59]	35.9 [0.65]
(0.75, 27.72)	3.2 [0.24]	6.0 [0.42]	8.5 [0.53]	11.4 [0.62]	14.6 [0.68]	18.5 [0.74]	23.0 [0.78]	28.2 [0.82]
(0.9, 27.72)	4.8 [0.50]	6.6 [0.60]	8.5 [0.68]	10.9 [0.74]	13.4 [0.78]	16.7 [0.82]	20.3 [0.85]	24.5 [0.87]
(0.75, 100.07)	5.5 [0.39]	7.8 [0.50]	10.3 [0.59]	13.3 [0.67]	16.8 [0.73]	21.0 [0.78]	25.7 [0.81]	31.1 [0.84]

$\sigma = 0.55$, where the partner Lorentzian of width $\sigma + \beta_0 - 1 \leq 0.3$ is itself narrow, $p \approx 1$ for $\varepsilon \geq 0.035$, falling to 0.78 at $\varepsilon = 0.009$. (4) *Height*. At $\sigma = 0.7$ and equal ε , A_σ^{det} is 7–40% larger at $\gamma_0 = 100.07$ than at 27.72, consistent with a background growing like $\log \frac{\gamma_0}{2\pi}$.

TABLE 18.3. Negative index on the local (complex, modulated) basis for the ghost $(\beta_0, \gamma_0) = (0.75, 27.72)$, a lower bound for $\kappa_{\sigma,A}$ in the model, against the Landau–Pollak count $A|S|/2\pi$. Without the ghost every entry is 0. At $\sigma = \frac{1}{2}$ the model has $\kappa_A \leq N_+ = 2$ for all A (Lemma 1, Thm. 2.1(vi)). Source `num/lemma18_ghost.jsonl`.

σ	0.52	0.55	0.6	0.65	0.68	0.7	0.72	0.73	0.74	0.745
$\kappa, A = 20$	2	2	2	1	1	1	0	0	0	0
$A S /2\pi, A = 20$	1.5	1.5	1.4	1.2	1.1	0.9	0.7	0.6	0.4	0.3
no ghost, $A = 20$	0	0	0	0	0	0	0	0	0	0
$\kappa, A = 80$	6	7	6	5	4	4	3	2	1	1
$A S /2\pi, A = 80$	6.2	6.1	5.7	5.0	4.3	3.7	3.0	2.4	1.7	1.2
no ghost, $A = 80$	0	0	0	0	0	0	0	0	0	0

HEURISTIC 18.5.1 (two regimes; not proved). Let b be the local positive background at γ_0 (zeta part plus the partner’s $2/(\sigma + \beta - 1)$) and $\varepsilon = \beta - \sigma$. The ghost Lorentzian $-2\varepsilon/(\varepsilon^2 + \tau^2)$ beats b on $|\tau| < \sqrt{2\varepsilon/b - \varepsilon^2}$, so $|S| \approx 2\sqrt{2\varepsilon/b}$ for $\varepsilon \ll 2/b$. With (2) this gives

$$(\text{dip regime, } \varepsilon \lesssim 2/b) : \quad A_\sigma^{\text{det}} \approx \frac{2\pi c}{|S|} \approx \pi c \sqrt{\frac{b}{2\varepsilon}}, \quad c \approx 0.25\text{--}0.9.$$

For $\varepsilon \gtrsim 2/b$ there is no dip, and detection must come from the point form, at most one square per ghost (Cor. 18.2.2(a)). In that regime Lemma 12’s two-bump argument transfers verbatim with $\delta \rightarrow \varepsilon$, which gives the c/ε law with $c \approx \log(K \log \gamma_0)$ (Prop. 18.5.3). For an actual zero above $3 \cdot 10^{12}$, $b \gtrsim \log \frac{\gamma_0}{2\pi} \geq 26$, so the dip regime is only $\beta - \sigma \lesssim 0.07$. The “ $c/(\beta - \sigma)$ as $\sigma \uparrow \beta$ ” guess of Lemma 11, §5 conflated the two regimes: as $\sigma \uparrow \beta$ one enters the *dip* regime, where the growth is $\asymp \varepsilon^{-1/2}$, not ε^{-1} .

PROPOSITION 18.5.2 (invisibility, exponent $\frac{1}{2}$; rigorous for finitely many ghosts). *Suppose the zeros with $\text{Re } \rho > \sigma$ form finitely many quadruples q (displacements ε_q). Let w_σ^{refl} be w_σ with each ghost Lorentzian $-2\varepsilon_q/(\varepsilon_q^2 + (t \mp \gamma_q)^2)$ replaced by its negative (reflection of the ghost across $\text{Re } s = \sigma$). Then for $F \in \mathcal{D}_A$*

$$\mathcal{Q}_\sigma(F) \geq \left(\inf w_\sigma^{\text{refl}} - \sum_q \frac{8(\cosh \varepsilon_q A - 1)}{\varepsilon_q} \right) \|F\|^2.$$

In particular, with $m^{\text{refl}} := \inf w_\sigma^{\text{refl}}$, a single ghost is invisible while

$$A < \varepsilon^{-1} \text{arccosh}(1 + \varepsilon m^{\text{refl}}/8) \approx \sqrt{m^{\text{refl}}/(4\varepsilon)}.$$

PROOF. In the proof of Thm. 18.2.1, $e^{\varepsilon|u|} = e^{-\varepsilon|u|} + 2\sinh(\varepsilon|u|)$. The first term produces $\mathcal{W}^{(\sigma)}$ with the reflected weight, which is a sum of positive Lorentzians. For the second, since $|H_F| \leq \|F\|^2$,

$$\left| \int 4\cos(\gamma_q u) \sinh(\varepsilon_q |u|) H_F(u) du \right| \leq 4\|F\|^2 \int_{-A}^A \sinh(\varepsilon_q |u|) du = \frac{8\|F\|^2 (\cosh \varepsilon_q A - 1)}{\varepsilon_q}.$$

□

Prop. 18.5.2 has the exponent $\frac{1}{2}$ of Heuristic 18.5.1, but its constant is far off. It uses the global infimum of w_σ^{refl} ($\approx 0.09(\sigma - \frac{1}{2})$ at $t \approx 0$) and $|H_F| \leq \|F\|^2$. For $\sigma = 0.7$, $\varepsilon = 0.05$ it gives $A_\sigma^{\text{det}} \geq 0.3$, against 14.6 observed. A localized version, with b in place of the global infimum, would make the $\sqrt{b/\varepsilon}$ lower bound rigorous. We did not do this.

PROPOSITION 18.5.3 (Lemma 12's two-bump identity at abscissa σ ; exact). *With Lemma 12's $\varphi, \Phi = \varphi \star \varphi, I_y = \int \varphi \cosh(yu)du, F = e^{-i\gamma_0 u}(\varphi(u - c_0) - \varphi(u + c_0))$ and $c_0 > r$: the pair $\rho_0, 1 - \bar{\rho}_0$ contributes exactly*

$$-\left(2e^{2c_0\varepsilon}I_\varepsilon^2 + 2e^{-2c_0w}I_w^2 - 2J_\sigma\right), \quad J_\sigma = \int (e^{\varepsilon|v|} + e^{-w|v|})\Phi(v)dv, \quad \varepsilon = \beta_0 - \sigma, \quad w = \sigma + \beta_0 - 1,$$

to $\mathcal{Q}_\sigma(F)$, which is $-4I_\delta^2(\cosh 2c_0\delta - 1)$ at $\sigma = \frac{1}{2}$ (Lemma 12, Lemma 2.1). Every zero with $\operatorname{Re} \rho \leq \sigma$ contributes a value in $[0, (P_{\sigma-\beta} + P_{\sigma+\beta-1}) * 4\hat{\Phi}]$.

PROOF. The pair kernel is $K = e^{\varepsilon|u|} + e^{-w|u|}$ (proof of Thm. 18.2.1). $H_F = e^{-i\gamma_0 u}(2\Phi - \Phi(\cdot - 2c_0) - \Phi(\cdot + 2c_0))$, and on $\operatorname{supp} \Phi(\cdot - 2c_0) \subset (0, \infty)$ one has $K(v + 2c_0) = e^{2c_0\varepsilon}e^{\varepsilon v} + e^{-2c_0w}e^{-wv}$, with $\int e^{\pm yv}\Phi = I_y^2$. For $\operatorname{Re} \rho \leq \sigma$ the kernel is a sum of Poisson kernels, and $\widehat{H}_f = \widehat{\Phi}(2 - 2\cos 2c_0\tau) \in [0, 4\hat{\Phi}]$. $\square$

A certified σ -version of Lemma 12's table additionally needs decay majorants for the Poisson-smoothed $\widehat{\Phi}$ (decay $O(\tau^{-2})$ instead of $O(\tau^{-2m})$). This was not done.

18.6. Relation to κ_A , rigor status, obstruction

Relation to κ_A . $\kappa_{1/2,A} = \kappa_A$, $\lambda_\sigma(A) \geq \lambda^*(e^A)$, $A_\sigma^{\det} \geq A_{1/2}^{\det}$, and $\kappa_A \leq \liminf_{\sigma \downarrow \frac{1}{2}} \kappa_{\sigma,A}$ (Prop. 18.3.1). At $\sigma = \frac{1}{2}$ the Lorentzians cancel ($w_{1/2} = 0$ off the ordinates, since $\xi(\frac{1}{2} + it)$ is real), only point forms remain, and $\sup_A \kappa_A = N_+$. For $\sigma > \frac{1}{2}$ the dip appears, and the index near a ghost becomes Landau–Pollak-extensive. The σ -direction trades the hole problem of $\sigma = \frac{1}{2}$ (Lemma 11: λ^* collapses like $e^{-2\pi^2 d}$) for a *hole-free* margin. Thm. 18.4.1 makes this exact at $\sigma = 1$ (m_1 for all A). The price is that zeros are seen only to the right of σ .

Rigor status. Rigorous: Thm. 18.2.1, Cor. 18.2.2, Prop. 18.2.3, Prop. 18.3.1, Thm. 18.3.3, Thm. 18.4.3, Prop. 18.5.2 (finitely many ghosts), Prop. 18.5.3. Rigorous and computer-assisted: Thm. 18.4.1 (Arb), Thm. 18.4.2 (inherits Lemmas 10, 16), and the table in Thm. 18.4.3.

Numerical: §5 and the checks. Heuristic: Heuristic 18.5.1. RH-equivalent: Thm. 18.3.3 at $\sigma = \frac{1}{2}$. Equivalent to the open quasi-RH $\Theta \leq \sigma$: Thm. 18.3.3 for $\frac{1}{2} < \sigma < 1$, and Prop. 18.4.4(a).

Obstruction. The σ -family gives a clean rigorous *identity* for Θ (Thm. 18.3.3) and clean *unconditional* positivity in the corner $\{\sigma \geq \sigma_*(A)\} \cup \{A \leq 1.6\}$. The corner is thin because every unconditional input is PNT-level: m_1 at $\sigma = 1$, perturbed by $O((1 - \sigma)A^2)$. Moving it requires positivity of $\mathcal{Q}_\sigma$ on long windows at fixed $\sigma < 1$. By Thm. 18.3.3 that is a zero-free half-plane statement in disguise, and Lemma 10's method pays

$\log T_1 \approx \sum_{n < e^A} 2\Lambda(n)n^{-\sigma} \approx \frac{2}{1-\sigma}e^{(1-\sigma)A}$, doubly exponential in A for every $\sigma < 1$ (polynomial only at $\sigma = 1$). The detection side is also unfavourable: ghosts above $3 \cdot 10^{12}$ sit in a background $b \geq 26$, so they are invisible until $A \gtrsim \sqrt{b/\varepsilon}$ in the dip regime or $\log(Kb)/\varepsilon$ in the point regime.

QUESTION 18.6.1. (1) Is $\liminf_{\varepsilon \rightarrow 0} A_\sigma^{\det} \sqrt{\varepsilon/b}$ positive and finite (exponent exactly $\frac{1}{2}$), and does the threshold constant $A_\sigma^{\det}|S|/2\pi$ tend to 1? A localized Prop. 18.5.2 would settle the lower half. (2) Does J.W. want the σ -version of Lemma 12's certified table (point regime, $\delta \rightarrow \varepsilon$)? Cost: decay majorants for Poisson-smoothed $\widehat{\Phi}$, about a day. (3) TAA reading: should the “ σ -RG” be read with σ as an energy scale (index counts states below a scale), or as a damping (Lax–Phillips e^{-ct})? Thm. 18.3.3 supports either reading. Remark 18.3.2 says the index count is not an RG-monotone (“ c -theorem”) quantity in σ ; only the margin is.

CHAPTER 19

Lemma 19: “The $|a|$ term is halfway; it needs a $\|a\|$.”

Trace bounds, Rosati norms, and shadow tomography

Standalone note `lemma19.tex`, October 21, 2026.

Summary. J. W.’s slogan is made precise. For an elliptic curve the trace $a = \text{Tr } F$ enters Hasse’s bound only through an inner product $a = 2\langle 1, F \rangle$ for the positive-definite degree form, with norm $\|F\|^2 = \deg F = q$. The bound $|a| \leq 2\sqrt{q}$ is the Cauchy–Schwarz inequality, and the full statement is that $F/\sqrt{q}$ is an isometry. For $g = 1$ one trace bound is equivalent to the full statement. For $g \geq 2$ it is not: we give an *integer* “coin” polynomial $1 - 22T^2 + 49T^4$ over $\mathbb{F}_7$ that passes the $n = 1$ Weil bound and fails at $n = 2$, and a family that passes every $n \leq N$ and fails later. The bound for *all* n is equivalent to $|\alpha_i| = \sqrt{q}$ (radius of convergence and duality); so does positivity of the Rosati trace form, which geometry supplies all at once. On the ζ side, one window A is one finite Toeplitz-type section, all windows together are Weil positivity, and the needed object is a positive form making the half-density-normalized scaling flow unitary. Its existence is equivalent to RH. §19.4 checks J. W.’s volume/area sandwich (AM–GM on $\prod \alpha_i = q^g$ against Schur on $\text{Tr } \pi\pi^\dagger = 2gq$). It is a valid proof with a fixable gap: Schur must run on $\mathbb{R}[\pi]$ with the Rosati form, not on V_ℓ . Once fixed, the volume is redundant. **Nothing here bears on RH beyond reformulation.**

Labels (as in Lemma 17): [THM] proved here or classical with proof; [CITED]; [NUM]; [ANALOGY]; [HEUR]; [RH-EQUIV]: equivalent to RH by Weil’s criterion, a reformulation. Numerics: `num/lemma19_checks.py/.out/.json`, `num/lemma19_volarea.py/.out`, `num/lemma19_tachyon.py/.out`. References verified: `lit/lemma19/verified_refs.md`.

19.1. Elliptic curves: the trace is an inner product, the degree is the norm

Let $E/\mathbb{F}_q$ be an elliptic curve with q -Frobenius F , dual isogeny $\hat{\phi}$ and $\phi\hat{\phi} = [\deg \phi]$. The Verschiebung is $V = \hat{F}$, so $FV = VF = [q]$, and $F + \hat{F} = [a]$ with $a = q + 1 - \#E(\mathbb{F}_q)$.

- PROPOSITION 19.1.1 (Rosati = dual isogeny; [CITED]+[THM]). (a) $\deg : \text{End}(E) \rightarrow \mathbb{Z}$ is a positive-definite quadratic form [138, Cor. III.6.3]. Its polarization $\langle \phi, \psi \rangle = \frac{1}{2}(\phi\hat{\psi} + \psi\hat{\phi})$ makes $\text{End}(E) \otimes \mathbb{R}$ a Euclidean space with $\|\phi\|^2 = \deg \phi$. The involution $\phi \mapsto \hat{\phi}$ is the Rosati involution for the principal polarization.
- (b) $\|1\|^2 = 1$, $\|F\|^2 = q$, $\langle 1, F \rangle = \frac{a}{2}$, and $\deg(m - nF) = m^2 - amn + qn^2$. In particular $\deg(1 - F^n) = \#E(\mathbb{F}_{q^n})$.
- (c) (Hasse [65], [138, Thm. V.1.1].) Positivity of $\deg$ on $\mathbb{Z}^2$ gives positivity on $\mathbb{R}^2$, hence discriminant $a^2 - 4q \leq 0$. Equivalently, Cauchy–Schwarz: $|a| = 2|\langle 1, F \rangle| \leq 2\|1\|\|F\| = 2\sqrt{q}$.
- (d) With $\alpha, \bar{\alpha}$ the roots of $T^2 - aT + q$, $(\alpha + \bar{\alpha})^2 - 4\alpha\bar{\alpha} = (\alpha - \bar{\alpha})^2$; so $a^2 - 4q \leq 0$ iff $\alpha \notin \mathbb{R}$ or $\alpha = \bar{\alpha}$, iff $|\alpha| = \sqrt{q}$. On $\mathbb{Z}[F] \otimes \mathbb{R} \cong \mathbb{C}$, $F/\sqrt{q}$ is multiplication by $e^{i\theta}$, $\alpha = \sqrt{q}e^{i\theta}$: an isometry.

PROOF. (b) $\deg(m - nF) = (m - nF)(m - n\hat{F}) = m^2 - mn(F + \hat{F}) + n^2F\hat{F}$. $1 - F$ is separable with kernel $E(\mathbb{F}_q)$, and likewise for F^n . (c) A nonnegative binary form on $\mathbb{Z}^2$ is nonnegative on $\mathbb{Q}^2$ by homogeneity and on $\mathbb{R}^2$ by continuity. (d) Algebra. $\square$

Reading of J. W.’s slogan [THM]. $|a|$ is a *linear* quantity, the inner product $2\langle 1, F \rangle$; its bound is “halfway”. The statement it comes from is *quadratic*: $\|F\|^2 = \deg F = q$ together with $F\hat{F} = q$. That makes $F/\sqrt{q}$ orthogonal, and Cauchy–Schwarz turns the norm into the trace bound. For $g = 1$ the two are equivalent: the multiset $\{\alpha, q/\alpha\}$ is closed under conjugation, and if it is real then $|\alpha + q/\alpha| \geq 2\sqrt{q}$ with equality iff $|\alpha| = \sqrt{q}$. So *one* shadow suffices in genus one. This fails from $g = 2$ on (§19.2).

PROPOSITION 19.1.2 (the discriminant cone of Lemma 14; [THM]). *Identify the degree form with the point $f_E = (A, B, C) = (1, -a, q)$ in the space of real binary quadratic forms, with discriminant $D = B^2 - 4AC = B^2 - (A + C)^2 + (A - C)^2$ of signature $(2, 1)$ (Lemma 14, light-cone section, (d)). Then $D(f_E) = a^2 - 4q$ and the root $\tau_{f_E} = (-B + \sqrt{D})/2A$ is the Frobenius eigenvalue α , with $|\tau_{f_E}|^2 = C/A = q$. The three cases are:*

position of f_E	D	Frobenius/ $\sqrt{q}$ on $\mathbb{Z}[F] \otimes \mathbb{R}$	TAA name (Lemma 17)
inside the cone (definite form)	< 0	rotation by θ , $\alpha = \sqrt{q}e^{i\theta}$	elliptic = distortionless
on the cone (null ray)	$= 0$	$\alpha = \pm\sqrt{q}$ (supersingular, $F = \pm\sqrt{q}$)	parabolic, boundary
outside the cone (indefinite)	> 0	boost, $\alpha/\sqrt{q} = \pm e^{\pm\eta}$, $\cosh \eta = \frac{ a }{2\sqrt{q}}$	hyperbolic = ghost

For a genuine curve the third row is excluded by Prop. 19.1.1(a), and the “4” in $a^2 - 4q$ is the $4AC$ of the cone. Lemma 15’s ghost model ($q = 4$, $\alpha/\sqrt{q} \in \{4, \frac{1}{4}\}$, $a = 8.5$, $D = 56.25$) sits outside the cone.

PROOF. Direct substitution, together with Lemma 14’s identification of definite forms with points of $\mathbb{H}$ via τ_f . In the hyperbolic case $\alpha, q/\alpha$ are real and $\alpha/\sqrt{q} = \pm e^{\pm\eta}$ with $2 \cosh \eta = |a|/\sqrt{q}$. $\square$

REMARK 19.1.3 (squaring map, Levi-Civita/Kustaanheimo–Stiefel; precise identity + [ANALOGY]). Write $\alpha = z^2$ with $|z|^2 = \sqrt{q}$ (Lemma 13’s Levi-Civita map $x = z^2$, $r = |x| = |z|^2$; KS [86] in dimension four). Then $a = 2 \operatorname{Re} z^2$, $a^2 - 4q = (z^2 - \bar{z}^2)^2 = -4(\operatorname{Im} z^2)^2$, and Hasse reads $|\operatorname{Re} z^2| \leq |z|^2$. This identity is precise but trivial. It restates the norm condition $|\alpha| = \sqrt{q}$ as “ α is the square of a point on the circle of radius $q^{1/4}$ ”; it does not prove it. The other “4” ($|dx/dz|^2 = 4r$ in Lemma 13) and the 4 of the discriminant both come from a degree-two map ($z \mapsto z^2$, and the polarization identity $(x + y)^2 - (x - y)^2 = 4xy$). Identifying the two is an *analogy only*: no statement about Frobenius follows from KS regularization.

Numerics [NUM]. $E : y^2 = x^3 + x + 1$ over $\mathbb{F}_5$: by brute force $N_1 = 9$, $N_2 = 27$ (as in Lemma 3); $a = -3$, $D = -11$, $\tau_{f_E} = \alpha = -\frac{3}{2} + \frac{\sqrt{11}}{2}i$, $|\alpha|^2 = 5$ to 10^{-12} ; $\deg(1 - F^n) = N_n = 9, 27, 108, 675, 3069, 15552$ ($n \leq 6$). $\min_{0 < \max(|m|, |n|) \leq 30} \deg(m - nF) = 1$. The Cauchy–Schwarz ratios $|a_n|/2q^{n/2}$ are 0.671, 0.100, 0.805, 0.980, 0.510, 0.296.

19.2. Genus g : one trace bound is a shadow, the Rosati norm is the body

Let $X/\mathbb{F}_q$ have genus g , Frobenius eigenvalues $\alpha_1, \dots, \alpha_{2g}$ (a multiset closed under $\alpha \mapsto \bar{\alpha}$ and, by duality, $\alpha \mapsto q/\alpha$), and traces $s_n = \sum_i \alpha_i^n = q^n + 1 - N_n$.

THEOREM 19.2.1 (Rosati positivity is the full statement; [CITED]). *Let J be the Jacobian with its principal polarization and Rosati involution $\dagger$, and π the Frobenius endomorphism. Then*

$\mathcal{B}(\phi, \psi) = \text{Tr}(\phi\psi^\dagger | V_\ell J)$ is positive definite on $\text{End}(J) \otimes \mathbb{R}$ [109, Ch. IV], [82, Thm. 6.0.14]; $\pi\pi^\dagger = [q]$; and consequently every $|\alpha_i| = \sqrt{q}$ [82, Thm. 6.0.15], [158, 159]. On $\mathbb{R}[\pi]$,

$$\mathcal{B}(\phi, \psi) = \sum_{i=1}^{2g} \phi(\alpha_i) \psi(q/\alpha_i), \quad \mathcal{B}(1, 1) = 2g, \quad \mathcal{B}(\pi^n, \pi^n) = 2gq^n, \quad \mathcal{B}(1, \pi^n) = s_n.$$

PROOF OF THE DISPLAYED FORMULA. $\pi^\dagger = q\pi^{-1}$, and Tr on V_ℓ is the sum over the eigenvalues of π ; $\sum_i q^n/\alpha_i^n = \sum_i \alpha_i^n$ by duality. $\square$

PROPOSITION 19.2.2 (trace bound = a 2×2 minor; [THM]). *The Weil bound at level n , $|s_n| \leq 2gq^{n/2}$, is exactly positivity of the 2×2 Gram matrix of $\mathcal{B}$ on $\{1, \pi^n\}$, i.e. the Cauchy–Schwarz inequality $|\mathcal{B}(1, \pi^n)|^2 \leq \mathcal{B}(1, 1)\mathcal{B}(\pi^n, \pi^n)$.*

THEOREM 19.2.3 (shadow vs. body; [THM]). *Let $M = \{\alpha_1, \dots, \alpha_{2g}\} \subset \mathbb{C}^\times$ be any multiset closed under $\alpha \mapsto \bar{\alpha}$ and $\alpha \mapsto q/\alpha$, $s_n = \sum \alpha_i^n$, and define $\mathcal{B}_M$ on $\mathbb{R}[T]$ by the formula of Thm. 19.2.1.*

- (a) (All shadows $\Rightarrow$ body.) *If $|s_n| \leq Cq^{n/2}$ for all $n \geq 1$ (any C), then every $|\alpha_i| \leq \sqrt{q}$, and by duality every $|\alpha_i| = \sqrt{q}$.*
- (b) (The body is a single positivity.) *$\mathcal{B}_M$ is positive semidefinite iff every $|\alpha_i| = \sqrt{q}$.*
- (c) (One shadow is not enough, $g \geq 2$.) *For every $N \geq 1$ there is such an M with $g = 2$ satisfying $|s_n| \leq 4q^{n/2}$ for $1 \leq n \leq N$ and not unitary. Explicitly $M = \sqrt{q}\{e^\eta, e^{-\eta}, e^{i\theta}, e^{-i\theta}\}$ with $\theta = \frac{\pi}{N+1}$ and $0 < \eta \leq \frac{1}{N} \text{arccosh}(2 - \cos \theta)$. For $N = 1$ there is even an integer example (§19.2.1).*

PROOF. (a) $\sum_{n \geq 1} s_n T^n = \sum_i \frac{\alpha_i T}{1 - \alpha_i T}$. By the bound the left side converges for $|T| < q^{-1/2}$. The right side is rational with poles exactly at the $1/\alpha_i$ (the residues $-1/\alpha_i \cdot m_i$ do not cancel, since distinct poles stay distinct). So $|1/\alpha_i| \geq q^{-1/2}$. Duality maps $\max |\alpha|$ to $q/\max |\alpha|$, so $\min |\alpha_i| \geq \sqrt{q}$. (b) If all $|\alpha_i| = \sqrt{q}$, then $q/\alpha_i = \bar{\alpha}_i$ and $\mathcal{B}_M(\phi, \phi) = \sum |\phi(\alpha_i)|^2 \geq 0$. Conversely let $|\alpha| \neq \sqrt{q}$. The conjugation-stable sets $S = \{\alpha, \bar{\alpha}\}$ and $S' = \{q/\alpha, q/\bar{\alpha}\}$ are disjoint ($q/\alpha = \alpha$ or $q/\alpha = \bar{\alpha}$ would force $|\alpha| = \sqrt{q}$). Take $\phi \in \mathbb{R}[T]$ with $\phi = 1$ on S , $\phi = -1$ on S' and $\phi = 0$ on $M \setminus (S \cup S')$ (real interpolation, possible since the data are conjugation-invariant). Every term of $\mathcal{B}_M(\phi, \phi) = \sum_i \phi(\alpha_i)\phi(q/\alpha_i)$ is either 0 or $(1)(-1)$, and at least one is the latter, so $\mathcal{B}_M(\phi, \phi) < 0$. (c) M is closed under both involutions. $s_n/q^{n/2} = 2 \cosh n\eta + 2 \cos n\theta$. For $1 \leq n \leq N$, $n\theta \in [\theta, \pi - \theta]$, so $\cos n\theta \leq \cos \theta$ and $2 \cosh n\eta + 2 \cos n\theta \leq 2(2 - \cos \theta) + 2 \cos \theta = 4$; the lower bound $\geq 0 > -4$ is clear. M is not unitary since $\eta > 0$, and by (a) the bound fails for some $n > N$. $\square$

19.2.1. The integer “coin”. Take $q = 7$, $g = 2$ and $P_{\text{coin}}(T) = (1 - 6T + 7T^2)(1 + 6T + 7T^2) = 1 - 22T^2 + 49T^4$. It satisfies the functional equation ($T^4 P(1/qT)q^2 = P(T)$) and has integer coefficients. Its reciprocal roots are $\pm(3 \pm \sqrt{2})$, real, with $|\alpha|/\sqrt{7} \in \{0.599, 1.668\}$: two hyperbolic pairs, i.e. two “ghosts” in the sense of Prop. 19.1.2. [NUM]/[THM]:

n	s_n	bound $4 \cdot 7^{n/2}$	pass	“ N_n ” = $7^n + 1 - s_n$
1	0	10.58	yes	8
2	44	28	no	6
3	0	74.08	yes	344
4	772	196	no	1630

The first failing level is $n = 2$. Gram matrices of $\mathcal{B}$: on $\{1, \pi\}$, $(\begin{smallmatrix} 4 & 0 \\ 0 & 28 \end{smallmatrix})$, positive (the $n = 1$ shadow is round); on $\{1, \pi^2\}$, $(\begin{smallmatrix} 4 & 44 \\ 44 & 196 \end{smallmatrix})$, determinant -1152 ; on $\{1, \pi, \pi^2, \pi^3\}$ the eigenvalues are $-39.22, -5.60, 205.6, 1439.2$ (indefinite, as Thm. 19.2.3(b) requires). It is not the L -polynomial of any curve: “ N_2 ” = $6 < 8$ = “ N_1 ”, but $X(\mathbb{F}_7) \subset X(\mathbb{F}_{49})$. The trace test at $n = 1$ cannot see this.

Coin family [NUM] (Thm. 19.2.3(c), q arbitrary, $\eta = 0.99 \times$ the maximum): passes all $n \leq N$, and first fails at

	N	1	2	5	10	20
first failing n		2	4	10	20	41
first non-PSD Toeplitz section $T_k = (c_{i-j})_{i,j < k}$, $c_n = s_n/q^{n/2}$		$k = 3$	4	4	4	5

The last row is worth noting [NUM]. A single *joint* Toeplitz section of size $k \leq 5$, built only from $c_0, \dots, c_{k-1}$ (each of which passes its own trace test), already detects the coin. The individual tests are the 2×2 minors $\{0, n\}$ of these sections. For $E/\mathbb{F}_5$ and $C_2/\mathbb{F}_7$ every section T_k , $k \leq 40$, is PSD, and $\mathcal{B}$ on $\{1, \dots, \pi^{2g-1}\}$ has eigenvalues $\{1, 11\}$ and $\{1.31, 25.8, 180.7, 1392\}$. For C_2 : brute force gives $N_1 = 11$, $N_2 = 49$, matching Lemma 3.

REMARK 19.2.4 (shadow tomography: the eclipse; [ANALOGY] for the picture, [THM] for the step). *The picture* (J.W.; cf. the “cosmic distance ladder” presentation of Tao with Sanderson [145], used only as motivation). Aristotle inferred that the Earth is a sphere because *every* lunar-eclipse shadow is a circular arc; one round shadow could have come from a coin seen face-on. *The dictionary*. The “body” is the multiset M , the “shadow in direction n ” is the level- n trace test $|c_n| \leq c_0$ (the $\{1, \pi^n\}$ minor of $\mathcal{B}$), and “round body” means $|\beta_i| = 1$ for all i , $\beta_i = \alpha_i/\sqrt{q}$. *The rigorous content* [THM]:

- (i) *One shadow is a coin*: P_{coin} passes $n = 1$ and first fails at $n = 2$; for every N the coin family passes $n \leq N$ and fails later (Thm. 19.2.3(c)). In genus 1 one shadow suffices (§1).
- (ii) *All shadows give the sphere*: $|c_n| \leq C$ for all $n \Rightarrow \max |\beta_i| \leq 1$ by the radius of convergence of $\sum_n s_n T^n$, and duality gives $\min |\beta_i| \geq 1$ (Thm. 19.2.3(a)).
- (iii) *Tomography, Carathéodory–Toeplitz*: the sequence $(c_n)_{n \in \mathbb{Z}}$ ($c_{-n} = \bar{c}_n$ by the two involutions) has every finite Toeplitz section T_k PSD iff $c_n = \int z^n d\mu$ for a positive measure μ on the unit circle (Carathéodory, Toeplitz, Herglotz; [59, Ch. 4]). For our finite atomic $c_n = \sum_i \beta_i^n$, such a representation gives $|c_n| \leq \mu(\mathbb{T}) = c_0$ for all n , hence all $|\beta_i| = 1$ by (ii). So “all sections PSD” is again the round body. For a curve, Weil’s theorem gives this (Lemma 12, Prop. 6.2; Lemma 17, Prop. 7.1).
- (iv) *Geometry* [CITED]: in $\mathbb{R}^3$, a convex body all of whose orthogonal projections onto planes are discs is a ball. All the discs have the same radius (the width $w(u)$ equals $2r(P)$ for every plane $P \ni u$, and any two directions lie in a common plane), so the projections are congruent, and a convex body in $\mathbb{R}^{n+1}$, $n \geq 2$, with congruent hyperplane projections is a ball ([105, Thm. 5.2]; see also [54]). In $\mathbb{R}^2$ this fails (constant-width curves). Aristotle’s eclipse shadows are central projections in a restricted family of directions, so the historical inference is heuristic.

19.3. Translation to ζ : one window is one section; all windows is the norm

PROPOSITION 19.3.1 (dictionary; [THM] for the function-field column).

	curve $X/\mathbb{F}_q$	ζ (TAA)
one shadow	$ s_n \leq 2gq^{n/2}$, one n	—
one finite section	$T_k \succeq 0 \iff \mathcal{Q}_X \geq 0$ on a window of length kl (Lemma 12, Prop. 6.2)	$\mathcal{Q} \geq 0$ on a window of length A ; only $\Lambda(n)$ with $n < e^A$ enter (Lemma 10, Lemma 1.1)
all sections/windows	Carathéodory–Toeplitz $\iff \beta_i = 1$	Weil positivity [RH-EQUIV]
the norm (“body”)	Rosati $\mathcal{B}$, positive by geometry (ampleness)	missing: Thm. 19.3.2
$\sqrt{-}$ -unitarity	$\pi\pi^\dagger = q$, $\pi/\sqrt{q}$ isometry	$x^{-1/2}$ half-density: $\lambda^{-1/2}f(x/\lambda)$ unitary, i.e. T_t

Status of the window family: $\mathcal{Q} \geq 0$ is certified for windows $A \leq 1.6$ (Lemma 10 for $A \leq \log 3$; Lemma 16 for $A \leq 1.6$, where only $n = 2, 3, 4$ are active). These are finitely many shadows, i.e. finite sections. The certification cost grows doubly exponentially in A (Lemma 16), and Lemma 12’s detection lengths for an off-line zero are $A_0 \geq 9.36$, so no finite stock of windows excludes anything.

THEOREM 19.3.2 (the inner product the ζ side needs; [THM], [RH-EQUIV]). The following are equivalent.

- (i) RH.
- (ii) $\mathcal{Q}(F, F) \geq 0$ for all $F \in C_c^\infty(\mathbb{R})(\mathbb{R})$ (Weil [157]; Lemma 1(iv)).
- (iii) There are a Hilbert space $\mathcal{H}$, a linear map $J : C_c^\infty(\mathbb{R})(\mathbb{R}) \rightarrow \mathcal{H}$ with dense range, and a strongly continuous unitary group U_t on $\mathcal{H}$ such that $\langle JF, JG \rangle_{\mathcal{H}} = \mathcal{Q}(F, G)$ and $JT_t = U_t J$. In x -variables, $f(x) = x^{-1/2}F(\log x)$ and T_t is $f \mapsto e^{-t/2}f(e^{-t}x)$: the scaling flow is $\sqrt{-}$ -unitary for a positive form equal to the explicit-formula form, the exact analogue of $\pi\pi^\dagger = q$ with $\mathcal{B} \geq 0$.

PROOF. (i) $\iff$ (ii) is Weil’s criterion. (ii) $\Rightarrow$ (iii): complete $C_c^\infty(\mathbb{R})$ modulo the null space of $\mathcal{Q}$ (Cauchy–Schwarz makes it a subspace); T_t preserves $\mathcal{Q}$ (Lemma 1(ii)) and so descends to isometries with dense range, hence unitaries. Strong continuity follows from continuity of $t \mapsto \mathcal{Q}(T_t F, G)$ (Lemma 1, absolutely convergent zero sum) and density. (iii) $\Rightarrow$ (ii): $\mathcal{Q}(F, F) = \|JF\|^2 \geq 0$. □

REMARK 19.3.3 (what (iii) must *not* be confused with). Positive forms making T_t unitary are plentiful: $L^2(du)$ is one. The content of (iii) is the identity $\langle JF, JG \rangle = \mathcal{Q}(F, G)$, i.e. that the *prime side* of the explicit formula (pole + archimedean $-\sum \Lambda(n)n^{-1/2}(\dots)$) is the Gram form of some positive structure. For curves this is supplied in the opposite order: $\mathcal{B}$ is defined from the polarization and is positive because an ample class is ample [82, Thm. 6.0.14], and the trace formula then identifies $\mathcal{Q}_X$ with it (Definition 5, Thm. 4.1). The ζ analogue would be an a-priori positive “polarization” whose trace formula is the explicit formula. This is Connes’s programme [35], with positivity still open. Lemma 17, §9.1 named the same gap “a geometric reason for transverse spacelikeness”.

19.3.1. Shadows, ghosts, tachyons.

REMARK 19.3.4 (dictionary [ANALOGY]; items (c), (d) are [THM]/known results). (a) *Ghost* = *negative direction* = *failed shadow* [THM]. For ζ , the window- A test fails iff some F supported in a window of length A has $\mathcal{Q}(F, F) < 0$, iff $\lambda_1(A) < 0$, iff $\kappa_A \geq 1$ (Lemma 17, Thm. 3.1). The ghosts are exactly these negative directions, and $\sup_A \kappa_A = N_+$

(Lemma 1(vi)). For curve-type multisets, a failed level- n shadow is a negative vector of $\mathcal{B}$ in $\text{span}\{1, \pi^n\}$. The negative index of $\mathcal{B}$ counts the off-circle pairs $\{\alpha, q/\alpha\}$ [NUM]: the integer coin has 2 pairs and 2 negative eigenvalues $(-39.2, -5.60)$. The coin family has 1 pair and 1 negative eigenvalue for $N = 1, 5, 20$ $(-618, -7.5 \cdot 10^{-4}, -2.2 \cdot 10^{-8})$; the last needs 60-digit arithmetic, `num/lemma19_tachyon.out`). This is the finite form of Lemma 1(vi).

- (b) *Off-line zero = complex frequency = growing mode.* With $\rho = \frac{1}{2} + i\gamma_\rho$ (Lemma 1), a zero $\rho = \frac{1}{2} + \delta + i\gamma$ has $\gamma_\rho = \gamma - i\delta$. Under the scaling flow its mode is $e^{i\gamma_\rho t} = e^{i\gamma t} e^{\delta t}$, growing for $t \rightarrow +\infty$, and the partner $\bar{\gamma}_\rho$ decays (Lemma 17, Prop. 4.1) [THM]. *Tachyon reading* [ANALOGY]: a Klein–Gordon mode with $\omega^2 = k^2 + m^2$, $m^2 < 0$, has imaginary ω for $k^2 < -m^2$ and grows exponentially. The match is literal only for purely imaginary γ_ρ ($\gamma = 0$, i.e. a real zero in $(\frac{1}{2}, 1)$), where $\gamma_\rho^2 = -\delta^2 < 0$. For ζ that case does not occur: $\zeta(\sigma) < 0$ on $(0, 1)$, since $(1 - 2^{1-\sigma})\zeta(\sigma) = \sum_{n \geq 1} (-1)^{n-1} n^{-\sigma} > 0$ [THM] ([NUM]: $\zeta(0.1), \dots, \zeta(0.9) < 0$). A genuine off-line zero would have $\gamma \neq 0$ and complex $\gamma_\rho^2 = \gamma^2 - \delta^2 - 2i\gamma\delta$: an oscillating *and* growing mode. That is a complex-conjugate pair rather than a tachyon proper.
- (c) *Negative norm and complex frequency coincide* [THM]/[CITED]. Let A be self-adjoint in a Krein space $(\mathcal{K}, [\cdot, \cdot])$.

(i) If $Ax = \mu x$ with $\mu \notin \mathbb{R}$, then $[x, x] = 0$. If $Ax = \mu x$, $Ay = \nu y$ and $\mu \neq \bar{\nu}$, then $[x, y] = 0$. (Proof: $\mu[x, y] = [Ax, y] = [x, Ay] = \bar{\nu}[x, y]$.)

(ii) Hence the root spaces of μ and $\bar{\mu}$ are neutral, and orthogonal to all other root spaces. When the root space of $\{\mu, \bar{\mu}\}$ is non-degenerate (always the case in Π_κ for isolated eigenvalues [74, 12]), they are dually paired, and for simple μ the eigenvectors x, y with $[x, y] = 1$ span a plane with Gram $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, of signature $(1, 1)$. Its negative vector $x - y$ is the ghost.

(iii) In a Pontryagin space Π_κ the number of eigenvalues of A in $\text{Im } \mu > 0$, counted with algebraic multiplicity, is at most κ (Pontryagin [125]; [74, Ch. II]; [12]).

So *complex frequency* $\Rightarrow$ *negative norm*, always. The converse needs the absence of negative-weight real modes. That fails for Pais–Uhlenbeck (real-frequency ghosts with weight $K'(\omega) < 0$, Lemma 17, Prop. 5.2), but holds for the Weil form, whose real modes have weight $m_\rho \geq 1$ (Lemma 17, Thm. 5.3(a): “ghosts only as complex pairs”). For $\mathcal{Q}$, therefore, negative norm $\iff$ complex frequency, and the count saturates (iii): $\kappa(\mathcal{Q}) = N_+$ (Lemma 17, Prop. 4.2). [NUM]: in the 2×2 model $A = \text{diag}(\mu, \bar{\mu})$, $\mu = 14.134725 - 0.3i$, $[e_1, e_1] = [e_2, e_2] = 0$, $[e_1, e_2] = 1$, $[e_1 - e_2, e_1 - e_2] = -1$, preserved by e^{itA} , while $|e^{i\mu t}| = e^{0.3t}$.

- (d) *Selberg’s $\frac{1}{4}$: tachyons in the Lax–Phillips line* [THM]/[CITED]. For a Maass cusp form, $-y^2 \Delta \phi = \lambda \phi$ with $\lambda = s(1-s) = \frac{1}{4} + t^2$, and Lemma 7’s wave equation $u_{tt} = Lu$, $L = y^2 \Delta + \frac{1}{4}$, gives $L\phi = (\frac{1}{4} - \lambda)\phi$. So $u = e^{i\omega\tau} \phi$ requires $\omega^2 = \lambda - \frac{1}{4} = t^2$. An exceptional eigenvalue $\lambda < \frac{1}{4}$ means $t \in i\mathbb{R}$ and $\omega^2 < 0$: a *literal* tachyonic, exponentially growing mode of the same equation that Lemma 17, Prop. 9.3 showed is mass-free in the cusp. Known results:
- Selberg conjectured $\lambda_1 \geq \frac{1}{4}$ for congruence subgroups and proved $\lambda_1 \geq \frac{3}{16}$ [133].
 - Kim–Sarnak proved $\lambda_1 \geq \frac{1}{4} - (\frac{7}{64})^2 = \frac{975}{4096} \approx 0.23804$ [83] (Appendix 2 to Kim’s paper, from the bound $\theta = \frac{7}{64}$ towards Ramanujan). So any tachyon has $|\omega| \leq \frac{7}{64}$ ($\omega^2 \geq -0.01196$).
 - For $\text{SL}_2(\mathbb{Z})$ the conjecture is known (Selberg, Roelcke; see [51]). The first cusp form has spectral parameter $r_1 = 9.5336952613\dots$, $\lambda_1 = \frac{1}{4} + r_1^2 = 91.14134533\dots$ (LMFDB [147], rigorous record 1.0.1.1.1): no tachyons.

Relation to ζ [ANALOGY]. In Lax–Phillips (Lemma 7), the zeros of ζ do *not* appear as Maass eigenvalues. They appear as scattering resonances at $(\rho - 1)/2$. An off-line zero would be a resonance damped more slowly than the common rate $\frac{1}{4}$, not a growing mode, and $\|Z_t\| = 1$ holds unconditionally. So Selberg’s conjecture (no tachyons among cusp forms) and RH (no

distortion among resonances) are different statements, and neither implies the other in this framework. The tachyon picture fits RH in the Weil-form setting of (b)–(c), where an off-line zero literally produces a growing mode, but not in the scattering setting.

19.4. Volume vs area

J. W.’s proposal: the *volume* of Frobenius is its determinant on the $2g$ -dimensional Tate module, $\deg \pi = \det(\pi | V_\ell J) = \prod_i \alpha_i = q^g$, which is unconditional; the *area* is the Rosati trace $\mathrm{Tr}(\pi\pi^\dagger) = \mathrm{Tr}(\pi V) = \mathrm{Tr}[q] = 2gq$. The proposed sandwich is

$$2gq \stackrel{\text{AM-GM}}{\leq} \sum_{i=1}^{2g} |\alpha_i|^2 \stackrel{\text{Schur}}{\leq} \mathrm{Tr}(\pi\pi^\dagger) = 2gq,$$

so both are equalities, and AM–GM equality gives $|\alpha_i|^2 = q$ for all i . The left inequality is correct: the geometric mean of the $|\alpha_i|^2$ is $(q^{2g})^{1/2g} = q$. Here $\deg \pi = q^g$ [102, §II.1, proof of Thm. 1.1], and $\deg \phi = \det(\phi | V_\ell)$ [82, Prop. 6.0.10]. Schur’s inequality is $\sum |\lambda_k|^2 \leq \mathrm{Tr}(AA^*)$ for an operator A on a finite-dimensional complex Hilbert space, with equality iff A is normal [146], [9, Thm. II.3.6] (Weyl’s majorant theorem, of which it is a case).

THEOREM 19.4.1 (verdict on the sandwich; [THM], inputs [CITED]). *(a) The gap. As stated on $V_\ell J$, the Schur step has no Hilbert space to stand on. The only structure on V_ℓ for which $\dagger$ is the adjoint is the Weil pairing e^λ , with $e^\lambda(\phi x, y) = e^\lambda(x, \phi^\dagger y)$. It is alternating (skew-symmetric) and $\mathbb{Q}_\ell(1)$ -valued [102, §§I.13–14], so it is not positive and is not Hermitian. Moreover, a positive-definite Hermitian form on $V_\ell \otimes \mathbb{C}$ with $\pi^* = q\pi^{-1}$ exists iff π is semisimple with all $|\alpha_i| = \sqrt{q}$. ($\Leftarrow$: declare an eigenbasis orthonormal. $\Rightarrow$: then $\pi/\sqrt{q}$ is unitary.) Assuming such a form is assuming the conclusion.*

- (b) The fix. Run Schur instead on $H = \mathbb{R}[\pi] = \mathbb{Q}[\pi] \otimes \mathbb{R}$ with the Rosati trace form $\langle x, y \rangle = \mathcal{B}(x, y) = \mathrm{Tr}(xy^\dagger | V_\ell)$. This form is positive definite by Rosati positivity [102, Thm. I.14.3], [82, Thm. 6.0.14], and H is $\dagger$ -stable since $\pi^\dagger = q\pi^{-1}$ [102, Lemma II.1.2]. Let m be the minimal polynomial of π over $\mathbb{Q}$, $d = \deg m$, and L_π multiplication by π on H . Then:
- (i) $L_\pi^* = L_{\pi^\dagger}$, because $\langle \pi x, y \rangle = \mathrm{Tr}(\pi xy^\dagger) = \mathrm{Tr}(xy^\dagger \pi) = \mathrm{Tr}(x(\pi^\dagger y)^\dagger)$.
 - (ii) Area: $\mathrm{Tr}_H(L_\pi L_\pi^*) = \mathrm{Tr}_H(L_q) = qd$.
 - (iii) Volume: the eigenvalues of L_π are the roots $\mu_1, \dots, \mu_d$ of m (its characteristic polynomial is m). Since $\dagger$ is a ring automorphism of the commutative algebra $\mathbb{Q}[\pi]$ with $\pi \mapsto q/\pi$, the root multiset of m is stable under $\mu \mapsto q/\mu$, so $\prod_k |\mu_k|^2 = q^d$. AM–GM and Schur now give $qd \leq \sum_k |\mu_k|^2 \leq qd$, so $|\mu_k| = \sqrt{q}$. Every eigenvalue α_i of π on V_ℓ is a root of m , because $m(\pi) = 0$ in $\mathrm{End}^0(J)$, which acts on V_ℓ . Hence all $|\alpha_i| = \sqrt{q}$. So the sandwich is a valid proof with a fixable gap. The fix moves the Hilbert space from V_ℓ to $\mathbb{R}[\pi]$. Multiplicities change from $2g$ to d , but the volume survives because of the q/μ symmetry. Semisimplicity of $\mathbb{Q}[\pi]$ is not needed.
- (c) Once fixed, the volume is redundant. $\mathbb{R}[\pi]$ is commutative, so L_π commutes with $L_\pi^* = L_{\pi^\dagger}$ and is normal. Then $L_\pi^* L_\pi = q$ says that $L_\pi/\sqrt{q}$ is unitary, and $|\mu_k| = \sqrt{q}$ follows with no AM–GM. This is the CM-type argument in operator language. Milne’s Lemma II.1.3 and Kedlaya’s Thm. 6.0.15 [102, 82] show that $\mathbb{Q}[\pi] \otimes \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$, with $\dagger$ positive and hence the identity on each $\mathbb{R}$ and complex conjugation on each $\mathbb{C}$, and then $\bar{\mu} = q/\mu$. All the content is in the area side, i.e. positivity of $\mathcal{B}$. The volume $\prod |\alpha_i| = q^g$ carries nothing beyond the functional equation.

PROPOSITION 19.4.2 (why genus 1 is “volume = area”; [THM]). *Let M be any multiset as in Thm. 19.2.3, and call M volume-positive if $\prod_i f(\alpha_i) \geq 0$ for all $f \in \mathbb{R}[T]$ (for a curve this is $\deg f(\pi) \geq 0$).*

- (a) *M is volume-positive iff every real element of M has even multiplicity.*
- (b) *For $g = 1$: $\deg \phi = \phi\phi^\dagger$ is a quadratic form, with $\mathcal{B}(\phi, \phi) = 2 \deg \phi$ ($\phi^\dagger$ is the adjugate $\text{Tr } \phi - \phi$ of the 2×2 matrix ϕ , and $\phi\phi^\dagger = \det \phi$ by Cayley–Hamilton). Volume and area are the same form up to the factor 2. For $\deg F$ this reads $\alpha\bar{\alpha} = |\alpha|^2 = q$ and $\text{Tr } FF^\dagger = 2|\alpha|^2 = 2q$. Volume-positivity alone gives RH. A non-real pair has $|\alpha|^2 = \alpha\bar{\alpha} = \deg F = q$ outright. A real non-unitary pair $\{\alpha, q/\alpha\}$ makes $(x - \alpha)(x - q/\alpha)$ change sign, which (a) forbids. This is Hasse’s proof (Prop. 19.1.1).*
- (c) *For $g \geq 2$, volume-positivity does not imply unitarity. A non-real quadruple $\{\alpha, \bar{\alpha}, q/\alpha, q/\bar{\alpha}\}$ with $|\alpha| \neq \sqrt{q}$ is volume-positive and has volume q^2 , but it is not unitary. Here $\deg$ is a form of degree $2g \geq 4$ whose positivity is not a Cauchy–Schwarz positivity.*

PROOF. (a) $\prod_i f(\alpha_i) = \prod_{\alpha \notin \mathbb{R}} |f(\alpha)|^2 \cdot \prod_{\alpha \in \mathbb{R}} f(\alpha)$, since the non-real part pairs by conjugation. If a real α has odd multiplicity, take $f = T - \alpha \pm \varepsilon$ to change the sign. (b) and (c) follow from (a), Thm. 19.2.3(b), and $\prod = q^g$. $\square$

Numerics [NUM] (num/lemma19_volarea.py, .out). $H = \mathbb{R}[T]/(m)$ with Gram matrix $\mathcal{B}_M(T^i, T^j)$ and $L =$ companion matrix. In every case $L^\sharp = G^{-1}L^\top G$ equals multiplication by q/T , and $\text{Tr}(LL^\sharp) = qd = 2gq$:

	volume $\prod \alpha_i$	q^g	$\sum \alpha_i ^2$	area $2gq$	$\min_{x \in \mathbb{R}} \chi_\pi(x)$	$\mathcal{B}$ inertia
$E/\mathbb{F}_5$	5	5	10	10	2.75	(2, 0)
$C_2/\mathbb{F}_7$	49	49	28	28	13.30	(4, 0)
coin $1 - 22T^2 + 49T^4$	49	49	44	28	−72	(2, 2)
coin family, $N = 1$	49	49	109.5	28	−970.7	(3, 1)
non-real quadruple, $ \alpha /\sqrt{7} = e^{\pm 0.3}$	49	49	33.19	28	26.17	(2, 2)

For $C_2/\mathbb{F}_7$ the sandwich is tight on both sides ($28 = 28 = 28$). *The coin violates the Schur (area) inequality: $\sum |\alpha_i|^2 = 44 > 28$.* Since its roots are real, $\sum |\alpha_i|^2 = s_2$, so this is exactly its failed $n = 2$ shadow of §19.2.1. It satisfies the volume identity $\prod \alpha_i = 49$ and, automatically, AM–GM ($44 \geq 28$). Schur cannot apply because $\mathcal{B}_{\text{coin}}$ is indefinite, and $\text{Tr}(LL^\sharp) = 28$ is then just a number. The coin *also* fails volume-positivity: $\deg(3 - \pi)$ would be $\chi(3) = 81 - 198 + 49 = -68 < 0$. This gives a one-line proof that it is no curve. The non-real quadruple passes both volume tests and fails only the area test, so in genus ≥ 2 the area is the inequality that does the work.

REMARK 19.4.3 (ζ -side translation; [RH-EQUIV] where stated, otherwise [ANALOGY]). *Volume* corresponds to the unconditional symmetry data: the functional equation $\xi(s) = \xi(1 - s)$, i.e. zeros closed under $\rho \mapsto 1 - \bar{\rho}$ (the analogue of $\alpha \mapsto q/\bar{\alpha}$, so “ $\prod |\alpha_i| = q^g$ ”), and the fact that $\Xi(t) = \xi(\frac{1}{2} + it)$ is real on $\mathbb{R}$. The analogue of volume-positivity is “ Ξ changes sign only at zeros of odd order”. Sign changes of Ξ locate zeros *on* the line, but by Prop. 19.4.2(c) an off-line quadruple $\{\rho, \bar{\rho}, 1 - \rho, 1 - \bar{\rho}\}$ is invisible to every volume-type test. *Area* corresponds to a positive form for which the half-density-normalized scaling flow is unitary, equal to the explicit-formula form. That is Thm. 19.3.2(iii), equivalent to RH. There is no finite sandwich: ζ has infinitely many atoms, $\sum |\cdot|^2$ and Tr diverge, and AM–GM has no direct analogue. The only part of the sandwich that survives is (c) of Thm. 19.4.1: “positive form + $\pi\pi^\dagger = q \Rightarrow$ unitary”. Its ζ form is (iii) $\Rightarrow$ (i) of Thm. 19.3.2. **No new leverage.**

Obstruction

Every finite family of trace or window tests is a shadow, and Thm. 19.2.3(c) shows that finite families of trace tests cannot imply unitarity even in genus 2. For curves, the body (Rosati positivity) is supplied by geometry, the shadows follow from it by Cauchy–Schwarz, and the all- n radius-of-convergence argument recovers the body from the shadows only because all n are available at once *from* the body. For ζ we have certified finitely many shadows (windows ≤ 1.6), at a doubly exponential cost per window, and no body. Producing the body is Theorem 19.3.2(iii), which is equivalent to RH. The “volume (determinant, i.e. the functional equation) is unconditional on both sides. By §19.4 it cannot see non-real off-line quadruples, so it adds nothing to the body.

19.5. Open questions

- (1) Is your $\|a\|$ the Rosati norm $\mathcal{B}(\pi, \pi)^{1/2} = \sqrt{2gq}$ (as used here), or the degree? For $g = 1$ they agree up to the factor $2g = 2$: $\mathcal{B}(\phi, \phi) = 2 \deg \phi$, and $\deg(1 - \pi^n) = N_n$. For $g \geq 2$, $\deg \phi = \det(\phi | V_\ell)$ is a form of degree $2g$, not quadratic; the quadratic norm is $\mathcal{B}$.
- (2) Should the volume state the “one shadow suffices in genus one” phenomenon as the reason the p -adic/elliptic toy models look easier than they are?
- (3) (§19.4) Is “volume meant as $\deg = \det$ on V_ℓ (degree $2g$), as here? Then in genus ≥ 2 the volume never constrains a non-real off-circle quadruple (Prop. 19.4.2(c)), and all the leverage is in the area. Do you have a use for volume-positivity (“ Ξ changes sign only at odd zeros) beyond detecting real ghosts like the coin?
- (4) The joint Toeplitz section detected every coin at size ≤ 5 (§19.2.1). Do you want a lemma bounding the section size needed for $2g$ atoms? It would be a finite-dimensional statement with no ζ analogue, since ζ has infinitely many atoms.

Status

Prop. 19.1.1 (deg positive definite; Hasse = Cauchy–Schwarz)	[CITED]+[THM]	Silverman, Hasse
Prop. 19.1.2 (discriminant cone; $\tau_{f_E} = \alpha$)	[THM]	Lemma 14
LC/KS reading	identity [THM]; link [ANALOGY]	
Thm. 19.2.1 (Rosati positivity $\Rightarrow$ RH for curves)	[CITED]	Mumford, Kedlaya, Weil
Prop. 19.2.2, Thm. 19.2.3(a)–(c)	[THM]	elementary
Integer coin, coin family, Toeplitz sections	[NUM]	lemma19_checks.out
Rem. 19.2.4 picture / steps (i)–(iii) / (iv)	[ANALOGY] / [THM] / [CITED]	GS58, Montejano
Prop. 19.3.1 (window = section)	[THM] (curves); ζ column descrip- tive	Lemmas 10, 12, 16
Thm. 19.3.2	[THM], [RH- EQUIV]	no new leverage
Rem. 19.3.4(a),(c) ghosts = negative directions = complex pairs	[THM]/[CITED], [NUM]	IKL, Bognár, Pontryagin; Lemma 17
Thm. 19.4.1 (sandwich: gap on V_ℓ , fix on $\mathbb{R}[\pi]$, volume redundant)	[THM]; inputs [CITED]	Milne AV, Kedlaya, Schur/Bhatia
Prop. 19.4.2, volume/area table	[THM], [NUM]	lemma19_volarea.out
Rem. 19.4.3	[ANALOGY]; area side [RH-EQUIV]	Thm. 19.3.2
Rem. 19.3.4(b),(d) tachyon reading / Selberg, Kim–Sarnak, $\lambda_1(\mathrm{SL}_2\mathbb{Z})$	[ANALOGY] / [CITED]	Selberg, Kim–Sarnak, LMFDB

CHAPTER 20

Lemma 20: T_p as two boosts

The Eichler–Shimura class on $X_0(N)^2$, CM calibration, and the rapidity picture

Standalone note `lemma20.tex`, October 21, 2026.

Summary. On $X \times X$ with $X = X_0(N)_{\mathbb{F}_p}$ ($p \nmid N$) we make the two-boost reading of the Eichler–Shimura congruence $T_p \equiv \Gamma_F + \Gamma_F^t$ precise in $\mathrm{NS}(X \times X)_{\mathbb{R}}$. With $H = X \times \mathrm{pt}$, $V = \mathrm{pt} \times X$ and $\eta = \frac{1}{2} \log p$:

$$\frac{\Gamma_F}{\sqrt{p}} = (B_{\eta} \oplus R_p) \Delta, \quad \frac{\Gamma_F^t}{\sqrt{p}} = (B_{-\eta} \oplus R'_p) \Delta, \quad \frac{T_p}{\sqrt{p}} \equiv 2 \cosh \eta (H + V) + \tau_p,$$

where $B_{\pm\eta}$ are boosts of $\langle H, V \rangle$ and R_p, R'_p are rotations of the negative-definite complement W . Further, $\tau_p^2 = -\frac{2}{p} \sum_j a_j(p)^2$ (sum over a Hecke eigenbasis of $S_2(\Gamma_0(N))$), and for genus one $\tau_p = 2 \cos \theta_p \delta$ exactly. Hence T_p is numerically a *pure* pair of boosts if and only if T_p kills $S_2(\Gamma_0(N))$. This happens for every inert prime on the CM curves $X_0(27), X_0(32), X_0(36), X_0(49)$ and on the genus-3 curve $X_0(64) \cong \{x^4 + y^4 = z^4\}$, whose cusp forms all have CM by $\mathbb{Q}(i)$. The inert primes are “null-trace”: $\gamma_F \perp \delta$, a right-angle rotation; the vector γ_F is not null. In the binary-form model of Lemma 14 the same data are a right triangle in $\mathbb{H}$ with legs η (Eisenstein) and $s_p = \operatorname{artanh}(a_p/2\sqrt{p})$ (cuspidal): $\cosh d(i, \alpha_p) = \cosh \eta \cosh s_p$. Sato–Tate becomes the density $\frac{2}{\pi} \operatorname{sech}^3 s$, and CM becomes $\frac{1}{2} \delta_0 + \frac{1}{2\pi} \operatorname{sech} s$; both are checked for $p < 10^6$. **None of this goes beyond Eichler–Shimura plus Weil.** It is the classical proof of weight-2 Ramanujan, rewritten in Lorentzian form. The only piece that transfers to $\operatorname{Spec} \mathbb{Z}$ is the boost (Eisenstein/pole) part, which is the ζ pole kernel $2 \cosh(u/2)$ evaluated at $u = \log p$. The rotation part has no GL_1 content, and its global analogue is Weil positivity [RH-EQUIV].

Labels (as in Lemmas 17, 19): [THM] proved here, or classical with proof; [CITED]; [NUM]; [ANALOGY]; [HEUR]; [RH-EQUIV], equivalent to RH by Weil’s criterion. Numerics: `num/lemma20_checks.py/.out/.json`, `num/lemma20_fig.py/.pdf`. References and LMFDB labels: `lit/lemma20/verified_refs.md`.

20.1. Setting and the Hecke span in $\mathrm{NS}(X \times X)_{\mathbb{R}}$

Let $N \geq 1$, let $p \nmid N$ be prime, and let $X = X_0(N)_{\mathbb{F}_p}$, the good reduction of $X_0(N)$ [71]. It is a smooth projective geometrically connected curve of genus g . Write $F : X \rightarrow X$ for its p -Frobenius (degree p), $N_m = \#X(\mathbb{F}_{p^m})$, $s_m = p^m + 1 - N_m = \sum_{i=1}^{2g} \alpha_i^m$, and $a_j(p)$ ($1 \leq j \leq g$) for the T_p -eigenvalues on a Hecke eigenbasis of $S_2(\Gamma_0(N))$ (oldforms with multiplicity). By Eichler–Shimura [47, 135], [46, Ch. 8], the characteristic polynomial of F on H^1 is $\prod_j (x^2 - a_j(p)x + p)$ [CITED].

Conventions are those of Lemma 14, §5 and Lemma 19, Prop. 1.2: $H = X \times \mathrm{pt}$ and $V = \mathrm{pt} \times X$, so $H^2 = V^2 = 0$ and $H \cdot V = 1$, and $(aH + bV) \cdot H = b$. Let $\Gamma_F = \{(x, Fx)\}$, let $\Gamma_F^t = \sigma(\Gamma_F) = \{(Fy, y)\}$ with σ the swap, and let Δ be the diagonal. Put

$$W := \langle H, V \rangle^{\perp} \subset \mathrm{NS}(X \times X)_{\mathbb{R}}, \quad \langle x, y \rangle_W := -x \cdot y.$$

By Hodge index [63, V Thm. 1.9], $\langle \cdot, \cdot \rangle_W$ is positive semidefinite on W , and positive definite modulo numerical equivalence (Lemma 14, §5(b),(c)).

PROPOSITION 20.1.1 (the Hecke span; [THM]). *Write $\Delta \equiv H + V + \delta$, $\Gamma_F \equiv H + pV + \gamma_F$ and $\Gamma_F^t \equiv pH + V + \gamma_V$ with $\delta, \gamma_F, \gamma_V \in W$. The Gram matrix of $(\delta, \gamma_F, \gamma_V)$ for $\langle \cdot, \cdot \rangle_W$ is*

$$G_p = \begin{pmatrix} 2g & s_1 & s_1 \\ s_1 & 2gp & s_2 \\ s_1 & s_2 & 2gp \end{pmatrix} \succeq 0.$$

PROOF. The intersection numbers are $\Gamma_F \cdot H = p$, $\Gamma_F \cdot V = 1$, $\Delta^2 = 2 - 2g$, $\Gamma_F^2 = p(2 - 2g)$ and $\Delta \cdot \Gamma_F = N_1$ [63, V Ex. 1.10]. The swap σ exchanges H and V and fixes Δ . So $\Gamma_F^t \cdot H = 1$, $\Gamma_F^t \cdot V = p$, $(\Gamma_F^t)^2 = \Gamma_F^2$ and $\Delta \cdot \Gamma_F^t = N_1$. Finally, $\Gamma_F \cap \Gamma_F^t = \{(x, Fx) : F^2x = x\}$ has N_2 points. The intersection is transversal: Γ_F has horizontal tangent ($dF = 0$) and Γ_F^t has vertical tangent. So $\Gamma_F \cdot \Gamma_F^t = N_2$. Subtract the $\langle H, V \rangle$ parts, using $(aH + bV)(cH + dV) = ad + bc$:

- $\delta^2 = -2g$ and $\gamma_F^2 = \gamma_V^2 = -2gp$;
- $\delta \cdot \gamma_F = \delta \cdot \gamma_V = N_1 - p - 1 = -s_1$;
- $\gamma_F \cdot \gamma_V = N_2 - p^2 - 1 = -s_2$.

Hodge index gives $G_p \succeq 0$. □

20.2. The two-boost theorem

Let B_η act on $\langle H, V \rangle$ by $H \mapsto e^{-\eta}H$, $V \mapsto e^\eta V$, an isometry since $(aH + bV)^2 = 2ab$, and let $\eta = \eta_p = \frac{1}{2} \log p$. This is Lemma 14, §6 and the Schmid note, §5 (where $B_\eta = \text{diag}(e^{-\eta}, e^\eta)$).

THEOREM 20.2.1 (two Lorentz transformations; [THM], given Eichler–Shimura [CITED]). (a)

$\Gamma_F/\sqrt{p} = B_\eta(H + V) + \gamma_F/\sqrt{p}$ with $\|\gamma_F/\sqrt{p}\|_W = \|\delta\|_W = \sqrt{2g}$. Hence there is an isometry R_p of W , a rotation in the plane $\langle \delta, \gamma_F \rangle$ and the identity on its orthogonal complement, with

$$\Gamma_F/\sqrt{p} = (B_\eta \oplus R_p) \Delta,$$

where the rotation angle Θ_p satisfies $\cos \Theta_p = \langle \delta, \gamma_F \rangle_W / (2g\sqrt{p}) = s_1 / (2g\sqrt{p})$. Likewise $\Gamma_F^t/\sqrt{p} = (B_{-\eta} \oplus R'_p)\Delta$. Both $B_{\pm\eta} \oplus R$ lie in $O(1, 1) \times O(W)$, the isometry group of the splitting.

(b) The Eichler–Shimura congruence $T_p \equiv \Gamma_F + \Gamma_F^t$ [47, 135] gives

$$\frac{T_p}{\sqrt{p}} \equiv 2 \cosh \eta (H + V) + \tau_p, \quad 2 \cosh \eta = \frac{p+1}{\sqrt{p}}, \quad \tau_p = \frac{\gamma_F + \gamma_V}{\sqrt{p}} \in W,$$

and

$$\tau_p^2 = -\frac{4gp + 2s_2}{p} = -\frac{2}{p} \sum_{j=1}^g a_j(p)^2.$$

(c) (pure boost) $\tau_p \equiv 0$ if and only if $a_j(p) = 0$ for all j , i.e. $\text{Tr}(T_p^2 | S_2(\Gamma_0(N))) = 0$.

(d) ($g = 1$) $\gamma_V \equiv a_p \delta - \gamma_F$, so $R'_p = R_p^{-1}$, $\Theta_p = \theta_p$ with $a_p = 2\sqrt{p} \cos \theta_p$, and

$$\frac{T_p}{\sqrt{p}} \equiv (B_\eta \oplus R_{\theta_p} + B_{-\eta} \oplus R_{-\theta_p}) \Delta = 2 \cosh \eta (H + V) + 2 \cos \theta_p \delta.$$

PROOF. (a) Use $(H + pV)/\sqrt{p} = e^{-\eta}H + e^\eta V$. Norms come from Prop. 20.1.1. Two vectors of equal length in a Euclidean space are related by a rotation in the plane they span; the angle follows from G_p . The same holds for Γ_F^t with $B_{-\eta}$. (b) The first identity holds because $B_\eta + B_{-\eta} = 2 \cosh \eta$ on $H + V$. For the second, $\tau_p^2 = (\gamma_F^2 + \gamma_V^2 + 2\gamma_F \cdot \gamma_V)/p$. By the factorization above, $\{\alpha_i\} = \{\alpha_j, p/\alpha_j\}$ with $\alpha_j + p/\alpha_j = a_j(p)$. So $s_2 + 2gp = \sum_j (\alpha_j + p/\alpha_j)^2 = \sum_j a_j(p)^2$. (c) $\tau_p \in W$, and Hodge index says a class in W with square 0 is numerically trivial. (d) The

class $x = \gamma_F + \gamma_V - a_p \delta$ has $\langle x, x \rangle_W = 4p + 2s_2 - 4a_p s_1 + 2a_p^2 = 0$, since $s_1 = a_p$ and $s_2 = a_p^2 - 2p$. So $x \equiv 0$ by Hodge index. In the plane $\langle \delta, \gamma_F \rangle$, $R_\theta^{-1} \delta = 2 \cos \theta \delta - R_\theta \delta$. $\square$

COROLLARY 20.2.2 (weight-2 Ramanujan; [THM], classical). $|\cos \Theta_p| \leq 1$, i.e. $|s_1| \leq 2g\sqrt{p}$. Applied to F^m for all m this gives $|\alpha_i| = \sqrt{p}$. For $g = 1$, the determinant of the (δ, γ_F) block of G_p gives $4p - a_p^2 \geq 0$, strict because $4p$ is not a square. In the form of Lemma 19, Prop. 1.2, this 2×2 block is twice the degree form $f_p = (1, -a_p, p)$, with discriminant $a_p^2 - 4p < 0$: f_p lies inside the cone.

This is Weil's proof [158], [63, V Ex. 1.10] combined with Eichler–Shimura, in the notation of Lemma 14. A ghost (Lemma 14, §5(c); Lemma 15) would be a positive direction in W . Then δ and $\gamma_F/\sqrt{p}$ could be related by a boost rather than a rotation, with $|\cos \Theta_p|$ replaced by $\cosh$, and the bound would fail. “No ghosts” is Hodge index on W .

PROPOSITION 20.2.3 (power Gram = Lemma 15's matrix; [THM]). For $m \geq n \geq 0$, $\Gamma_{F^m} \cdot \Gamma_{F^n} = p^n N_{m-n}$. Hence

$$\frac{\Gamma_{F^m} \cdot \Gamma_{F^n}}{p^{(m+n)/2}} = 2 \cosh((m-n)\eta) - \frac{s_{|m-n|}}{p^{|m-n|/2}},$$

which is the $C \times C$ Gram of Lemma 15, §1. For $g = 1$ this is $2 \cosh(k\eta) - 2 \cos(k\theta_p)$ with $k = m - n$, of signature $(1, 3)$ for every size $M \geq 4$.

PROOF. $\Gamma_{F^m} = (1 \times F^n)_* \Gamma_{F^{m-n}}$, and $(1 \times F^n)^* \Gamma_{F^m} = p^n \Gamma_{F^{m-n}}$ because $1 \times F^n$ is purely inseparable of degree p^n . The projection formula gives $\Gamma_{F^m} \cdot \Gamma_{F^n} = \Gamma_{F^{m-n}} \cdot p^n \Delta$. The $\langle H, V \rangle$ part is $p^m + p^n$. For the signature: $2 \cosh(k\eta) = e^{m\eta} e^{-n\eta} + e^{-m\eta} e^{n\eta}$ has signature $(1, 1)$, and $2 \cos(k\theta) = 2 \operatorname{Re}(e^{im\theta} e^{-in\theta})$ is positive of rank 2 when $\theta \notin \pi\mathbb{Z}$. The four exponentials $e^{\pm\eta}, e^{\pm i\theta}$ are distinct, so the vectors are independent once $M \geq 4$. $\square$

[NUM] For $X_0(32)$ and $p \in \{3, 5, 7, 13, 19, 29\}$ with $M = 8$, the eigenvalue signs are $(+, -, 0) = (1, 3, 4)$. This answers the ranking item's question of whether the hyperbolic part is $2 \cosh(\frac{1}{2} \log p)$ and the zero part is unitary (rank $2g$, a rotation): it is, and the conductor N enters only through g and the set of good primes.

20.3. CM calibration

Data [CITED]+[NUM], LMFDB verified. For $N \in \{27, 32, 36, 49\}$, $X_0(N)$ has genus 1 and is the optimal curve of modular degree 1:

- $27a1 = 27.a3$: $[0, 0, 1, 0, -7]$, CM by $\mathbb{Q}(\sqrt{-3})$;
- $32a1 = 32.a4$: $[0, 0, 0, 4, 0]$, CM by $\mathbb{Q}(i)$;
- $36a1 = 36.a4$: $[0, 0, 0, 0, 1]$, CM by $\mathbb{Q}(\sqrt{-3})$;
- $49a1 = 49.a4$: $[1, -1, 0, -2, -1]$, CM by $\mathbb{Q}(\sqrt{-7})$.

$X_0(64)$ has genus 3 and is isomorphic to the Fermat quartic $x^4 + y^4 = z^4$ [136, 48]. Here $S_2(\Gamma_0(64)) = \mathbb{C}f_{64} \oplus \mathbb{C}f_{32}(q) \oplus \mathbb{C}f_{32}(q^2)$ (LMFDB: new dimension 1 with 64.2.a.a, CM -4 ; cusp dimension 3), and $64a1 = 64.a3$ is $[0, 0, 0, -4, 0]$.

PROPOSITION 20.3.1 (CM rotation angles; [CITED]+[THM]). Let $E = X_0(N)$ have CM by $\mathcal{O} \subset K$, and let $p \nmid N$ with $p \geq 5$.

- (a) If p is inert in K , E_p is supersingular [45], so $a_p = 0$. Then $\theta_p = \frac{\pi}{2}$, $F^2 = -p$, and $F/\sqrt{p}$ acts on $\langle \delta, \gamma_F \rangle$ as a complex structure. In NS terms: $\gamma_F \perp \delta$, $\gamma_V = -\gamma_F$, and $T_p \equiv (p+1)(H+V)$, a pure pair of boosts. The lattice $\mathbb{Z}\delta + \mathbb{Z}\gamma_F$ is $\mathbb{Z}[\sqrt{-p}]$ with norm form $2(m^2 + pn^2)$. It does not see K .

- (b) If p splits, E_p is ordinary and $F = \pi \in \mathcal{O}$ with $\pi\bar{\pi} = p$ [45]. Then $\theta_p = \arg \pi$, the lattice $\mathbb{Z}\delta + \mathbb{Z}\gamma_F \cong \mathbb{Z}[\pi] \subset \mathcal{O}$ is scaled by 2, and $4p - a_p^2 = |d_K|m^2$ with $m \in \mathbb{Z}$. Ramanujan is elementary here: $|a_p| = |\pi + \bar{\pi}| \leq 2|\pi| = 2\sqrt{p}$. The norm $\|\cdot\|$ of Lemma 19 is supplied by $N_{K/\mathbb{Q}}$.
- (c) For $X_0(64)$ and $p \equiv 3 \pmod{4}$, all three forms have $a_p = 0$, so $\tau_p \equiv 0$ (Thm. 20.2.1(c)). This is a genus-3 pure boost. For the twist relation, $a_{64}(p) = \left(\frac{2}{p}\right)a_{32}(p)$, so $\tau_p^2 = -6a_{32}(p)^2/p$, and the rotation angles on the three isotypic planes are θ, θ, θ or $\theta, \theta, \pi - \theta$.

PROOF. (a), (b): Deuring's reduction theorem, plus Thm. 20.2.1(d). For $32a1$ and $64a1$ ($y^2 = x^3 \pm 4x$), (a) is also elementary for $p \equiv 3 \pmod{4}$: $x \mapsto -x$ and $\left(\frac{-1}{p}\right) = -1$ give $\sum_x \left(\frac{x^3 \pm 4x}{p}\right) = 0$. In (b), $(\pi - \bar{\pi})^2 = a_p^2 - 4p$, and $\pi - \bar{\pi} \in \sqrt{d_K}\mathbb{Z}$ for every $\pi \in \mathcal{O}_K$, so $4p - a_p^2 = |d_K|m^2$. (c): Thm. 20.2.1(c), with $a_{32}(p) = 0$ by (a) and $a_{64}(p) = \pm a_{32}(p)$ from the twist. The twist relation is checked numerically below; it identifies f_{64} with a quadratic twist of f_{32} (unramified outside 2), which is [NUM] here and not proved. $\square$

Checks [NUM] (lemma20_checks; PARI ellap, all good $p < 10^6$).

- (i) CM structure. For $27a1, 32a1, 36a1, 49a1, 64a1$: inert $\Rightarrow a_p = 0$, and split $\Rightarrow a_p \neq 0$ with $(4p - a_p^2)/|d_K|$ a perfect square. There are 0 failures over the about 78,500 good primes $3 \leq p < 10^6$ per curve; inert/split counts are 39,322/39,175 for $\mathbb{Q}(i)$, 39,265/39,231 for $\mathbb{Q}(\sqrt{-3})$ and 39,300/39,196 for $\mathbb{Q}(\sqrt{-7})$.
- (ii) NS Gram of $X_0(32)$, brute-force N_2 over $\mathbb{F}_{p^2}$ for $p \leq 37$. Here $s_2^{\text{count}} = a_p^2 - 2p$, and G_p has eigenvalues $\{0, *, *\}$ with null vector $(-a_p, 1, 1)$ to machine precision, as in Thm. 20.2.1(d). At inert p , $(T_p/\sqrt{p})^2 = 2(p+1)^2/p$ exactly; for example $p = 7$ gives $18.2857 = 2 \cdot 64/7$, with transverse part 0.
- (iii) $X_0(64) = F_4$, brute force. $s_1^{\text{count}} = a_{64} + 2a_{32}$ for all 45 odd $p \leq 200$, and $s_2^{\text{count}} = (a_{64}^2 - 2p) + 2(a_{32}^2 - 2p)$ for all 13 odd $p \leq 43$. $\tau_p^2 = 0$ at every inert p , and $\tau_p^2 < 0$ at split p (e.g. -4.8 at $p = 5$). The rank of G_p is 2 for $p \equiv 3 \pmod{4}$ and $p \equiv 1 \pmod{8}$, and 3 for $p \equiv 5 \pmod{8}$: the angles θ and $\pi - \theta$ then span an extra direction. The twist relation holds for all 78,497 odd $p < 10^6$.

20.4. The rapidity picture in $\mathbb{H}$ and the angle distributions

In the binary-form model (Lemma 14, §5(d); Lemma 19, Prop. 1.2), the degree form $f_p = x^2 - a_p xy + py^2$, half the (δ, γ_F) block of G_p , is a point $\tau_{f_p} = \alpha_p = \sqrt{p}e^{i\theta_p}$ of $\mathbb{H}$. Every such point lies on the geodesic $\mathcal{G}_p = \{|\tau| = \sqrt{p}\}$, which meets the imaginary axis perpendicularly at $i\sqrt{p}$.

PROPOSITION 20.4.1 (Frobenius as a right triangle; [THM]). *Let s_p be the signed hyperbolic distance from $i\sqrt{p}$ to α_p along $\mathcal{G}_p$. Then*

$$d_{\mathbb{H}}(i, i\sqrt{p}) = \eta_p = \frac{1}{2} \log p, \quad \tanh s_p = \cos \theta_p = \frac{a_p}{2\sqrt{p}}, \quad \cosh d_{\mathbb{H}}(i, \alpha_p) = \cosh \eta_p \cosh s_p.$$

So the Eisenstein trace $(p+1)/\sqrt{p} = 2 \cosh \eta_p$ and the cuspidal trace $a_p/\sqrt{p} = 2 \tanh s_p$ are the two legs of a right triangle, and the third relation is hyperbolic Pythagoras. Ramanujan is equivalent to $s_p \in \mathbb{R}$, i.e. $\alpha_p \in \mathbb{H}$. A ghost ($|a_p| > 2\sqrt{p}$, as in Lemma 15's model) has $\alpha_p \in \mathbb{R}$, off $\mathbb{H}$.

PROOF. $d(iy_1, iy_2) = |\log(y_2/y_1)|$. For $z = Re^{i\theta}$, $\cosh d(iR, z) = 1 + |z - iR|^2/(2R \cdot R \sin \theta) = 1/\sin \theta$, so $\tanh^2 s = \cos^2 \theta$, with the sign of $\cos \theta$. Also $\cosh d(i, \alpha) = 1 + |\alpha - i|^2/(2 \operatorname{Im} \alpha) = (p+1)/(2\sqrt{p} \sin \theta)$. [NUM]: the identity holds to $4 \cdot 10^{-15}$ (pythagoras). $\square$

PROPOSITION 20.4.2 (distributions in the rapidity variable; [THM] change of variables + [CITED]). *If θ has the Sato–Tate law $\frac{2}{\pi} \sin^2 \theta d\theta$, then $s = \operatorname{artanh} \cos \theta$ has density $\frac{2}{\pi} \operatorname{sech}^3 s$. If θ is uniform, s has density $\frac{1}{\pi} \operatorname{sech} s$. Hence:*

- for non-CM $E/\mathbb{Q}$ (Sato–Tate [5]), $s_p \sim \frac{2}{\pi} \operatorname{sech}^3 s ds$;
- for CM $E/\mathbb{Q}$ (Hecke [68, 67] equidistribution of $\arg \pi$; inert primes have density $\frac{1}{2}$), $s_p \sim \frac{1}{2} \delta_0 + \frac{1}{2\pi} \operatorname{sech} s ds$.

The moments of $x = \tanh s = \cos \theta$ are $E[x^2] = \frac{1}{4}$ in both cases, while $E[x^4] = \frac{1}{8}$ (ST) vs $\frac{3}{16}$ (CM) and $E[x^6] = \frac{5}{64}$ vs $\frac{5}{32}$.

PROOF. $x = \cos \theta = \tanh s$, $\sin \theta = \operatorname{sech} s$, and $|d\theta| = \operatorname{sech} s ds$. □

Boost reading. The mean transverse square of $T_p/\sqrt{p}$ for $g = 1$ is $E[\tau_p^2] = -8E[x^2] = -2$ for both laws. CM and non-CM differ only at fourth order, through how often α_p runs out toward the ideal boundary of $\mathcal{G}_p$: tails $e^{-|s|}$ (CM) vs $e^{-3|s|}$ (ST). The inert atom sits at the summit $s = 0$.

curve	n	$P(a_p = 0)$	$E[x^2]$	$E[x^4]$	$E[x^6]$	KS
11a1 (non-CM)	78473	0.0012	0.2496	0.1245	0.0776	0.0019 vs ST
37a1 (non-CM)	78473	0.0010	0.2499	0.1249	0.0781	0.0020 vs ST
27a1 (CM −3)	78473	0.5002	0.2499	0.1874	0.1560	0.0028 split vs unif.
32a1 (CM −4)	78473	0.5009	0.2492	0.1868	0.1556	0.0034 split vs unif.
36a1 (CM −3)	78473	0.5002	0.2495	0.1869	0.1556	0.0034 split vs unif.
49a1 (CM −7)	78473	0.5006	0.2494	0.1870	0.1558	0.0043 split vs unif.
64a1 (CM −4)	78473	0.5009	0.2492	0.1868	0.1556	0.0033 split vs unif.
predicted ST / CM		0 / $\frac{1}{2}$	$\frac{1}{4}$ / $\frac{1}{4}$	0.125 / 0.1875	0.0781 / 0.1563	

TABLE 20.1. [NUM] $100 < p < 10^6$, $x = a_p/2\sqrt{p} = \tanh s_p$. The KS distance of CM curves against the Sato–Tate law is 0.250–0.252 (the inert atom). For non-CM curves, $P(a_p = 0) > 0$ reflects the supersingular primes, which are infinite in number [50] but of density 0.

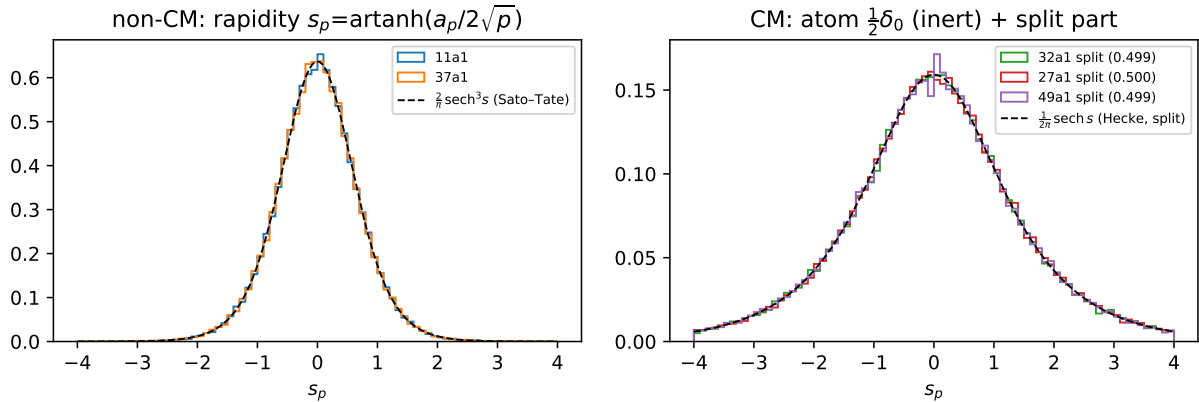


FIGURE 20.1. [NUM] Histograms of $s_p = \operatorname{artanh}(a_p/2\sqrt{p})$, $100 < p < 10^6$. Left: 11a1, 37a1 against $\frac{2}{\pi} \operatorname{sech}^3 s$. Right: the split primes of 32a1, 27a1, 49a1 (normalized by all primes) against $\frac{1}{2\pi} \operatorname{sech} s$; the inert half is the atom at $s = 0$, not drawn.

20.5. What this gives, and what transfers to $\text{Spec } \mathbb{Z}$

Beyond Eichler–Shimura + Weil: nothing for Ramanujan. Thm. 20.2.1 and Cor. 20.2.2 are a Lorentzian bookkeeping of the classical proof: Eichler–Shimura places T_p in $\text{NS}(X \times X)_{\mathbb{R}}$, and Hodge index on W (Weil) bounds the angle. The CM calibration shows the two regimes cleanly. At inert primes the transverse part vanishes ($\theta = \frac{\pi}{2}$, pure boost). At split primes the rotation is the Hecke character angle, and the norm comes free from K . That is the elementary case of Lemma 19’s “ $\|a\|$ ”. The new items are packaging and checks: Thm. 20.2.1(c),(d), Prop. 20.4.1 and the sech-laws. None proves anything not already known.

What transfers.

- (i) *The boost part, exactly* [THM], a trivial identity. The Eisenstein eigenvalue of T_p is $\sigma_1(p) = p + 1$, and $\zeta(s)\zeta(s-1)$ has Satake parameters $p^{\mp 1/2}$ after normalization, i.e. $e^{\mp \eta_p}$: a boost. The Weil explicit-formula pole kernel for ζ at $\sigma = \frac{1}{2}$ is $e^{u/2} + e^{-u/2} = 2 \cosh(u/2)$ (Lemma 18, Prop. 2.3). At the position $u = \log p$ of the prime p it equals $2 \cosh \eta_p = (p+1)/\sqrt{p}$. So “Eisenstein = boost” on $X_0(N)^2$ and “pole term = hyperbolic plane” on the ζ side (Lemma 14, §5(g); Lemma 15) are the same function sampled at $u = \log p$.
- (ii) *The rotation part, not at all.* ζ is a GL_1 object, and the Satake parameter of the trivial character at p is 1. Local temperedness is automatic and carries no information. RH is not a statement about any single p . The per- p argument uses a curve over $\mathbb{F}_p$ whose Hodge index does the work. Lemma 19 calls such per- p data shadows; for $\text{Spec } \mathbb{Z}$ no curve over $\mathbb{F}_p$ carries the zeros of ζ . The global analogue of “Hodge index on W ” is positivity of Weil’s form on the transverse (zero) part, which is [RH-EQUIV] (Lemma 15, Thm. 1.3; Lemma 19, Thm. 3.2).
- (iii) *Ghosts in GL_2* [ANALOGY]. A non-tempered cusp form would have $|a_p| > 2\sqrt{p}$, i.e. a transverse boost, $\alpha_p \notin \mathbb{H}$. For holomorphic forms this is excluded: in weight 2 by Cor. 20.2.2, in higher weight by Deligne [41]. For Maass forms it is the open Ramanujan–Selberg conjecture (Lemma 19, Rem. 3.4: Kim–Sarnak 7/64). That is a different statement from RH.

Obstruction. The two-boost decomposition is *local in p* : for each p it lives on $X_0(N)_{\mathbb{F}_p}^2$. The Weil-type content sits in the transverse factor W , which is definite by geometry. For ζ the boost factor exists (pole term), but the transverse factor is exactly the object whose definiteness is RH. No finite-level Hecke structure supplies it, since GL_1 has no cuspidal transverse part at any single p .

Rigor status. [THM]: Prop. 20.1.1, Thm. 20.2.1 given Eichler–Shimura, Cor. 20.2.2, Prop. 20.2.3, Prop. 20.4.1, and the change of variables in Prop. 20.4.2. [CITED]: Eichler–Shimura and Igusa; Deuring (Prop. 20.3.1(a),(b)); Hecke and BLGHT (the limiting laws); $X_0(64) \cong F_4$; LMFDB data. [NUM]: all checks, the twist relation $a_{64} = (\frac{2}{p})a_{32}$, Table 20.1 and Fig. 20.1. [ANALOGY]: (iii). [RH-EQUIV]: the global transverse positivity in (ii).

- QUESTION 20.5.1. (1) Should the TAA boost parameter for ζ be $\eta_p = \frac{1}{2} \log p$ per prime, so that the explicit formula is a sum of boosts sampled at $u = \log p$, or the single flow parameter u of Lemma 7? Prop. 20.4.1 suggests the former is just $\mathcal{G}_p$ ’s height.
- (2) Is the rapidity variable s_p , with sech-laws, useful for TAA, e.g. as a “transverse rapidity” feeding the Lemma 12/18 ghost models with realistic θ -statistics? Or is θ_p the right variable?
- (3) Is a Hecke-equivariant version on $X_0(N)^2$ (isotypic δ_f , angles $\theta_p(f)$ per newform, with the multiplicity-2 oldform planes as in $X_0(64)$) wanted? It is routine.

CHAPTER 21

Lemma 21: Certifying Weil positivity from the zero side

A no-go at $A \approx 2.11$, and a three-band hybrid with ceiling $\log 61$

Standalone note `lemma21.tex`, October 21, 2026.

Summary. Verdict. The *pure* zero-side route does not beat the current certified window $A = 1.6$. Suppose the only inputs are RH up to $H = 3\,000\,175\,332\,800$ (Platt–Trudgian), the explicit counting bound for $S(T)$ (Hasanalizade–Shen–Wong), and the zero-free region (Mossinghoff–Trudgian–Yang). Then no certificate of $\mathcal{Q}(F, F) \geq 0$ for *all* F supported in a window of length A can exist once $A > A_{\text{crit}}(H) = \pi D(H)/(R(H) - \frac{1}{2}) = 2.113$. The reason is explicit: a zero configuration that satisfies every one of these constraints, together with an F in the window, gives $\mathcal{Q} < 0$. The model sufficient condition reaches only $A_{\text{low}} = 0.39$; this would rise to 1.75 if all zeros above H were known to satisfy $|\beta - \frac{1}{2}| \leq 0.01$. The obstruction is not the size of a single off-line contribution. It is that the *positions* of zeros above H are unknown. F can put the background zeros on the nodes of $\tilde{F}$, and then one ghost with arbitrarily small $\delta > 0$ wins. Restricted (Sobolev or Gevrey) classes can certify large A , but they are provably blind to zeros above H , so as evidence for RH they are vacuous.

What does work (J.W.’s lithography idea, done honestly): a *three-band hybrid*. Use the prime side on low frequencies, the zero side (which is ≥ 0 below H) on a middle band $[T_1, T_2] \subset [0, H)$, and the prime side again above T_2 , where the archimedean term dominates the aligned prime waves. This removes the requirement that the Galerkin band reach $T \sim e^{S(A)}$, which is doubly exponential in A . The ceiling is $A < \log 61 - \varepsilon = 4.11 - \varepsilon$ at $H = 3 \cdot 10^{12}$, or $A < \log 47 - \varepsilon = 3.85 - \varepsilon$ with Platt’s stored zeros. The cost depends on the explicit sampling inequality for the zeros in the middle band. With the crude max-gap (Gröchenig) inequality, which is rigorous, the gain starts only at $A \gtrsim 2.9$. With a near-density sampling constant (measured, not proved), $A \approx 3.85$ costs about 10^7 core-seconds [EST]. For comparison, a prime-side Toeplitz/Levinson rewrite gives 2.1–2.8, and the Slepian basis gives a factor of about 40. None of these lengths comes near the Lemma 12 exclusion lengths.

Labels (as in Lemmas 17–20): [THM] proved here, or classical with proof; [CITED]; [NUM]; [HEUR]; [EST] cost or size estimate; [RH-EQUIV], equivalent to RH by Weil’s criterion. Numerics: `num/lemma21_core`, `_counter`, `_cost`, `_gaps`, `_hybrid (.py/.out/.json)`. Checkpoint: `num/lemma21_progress.md`. References: `lit/lemma21/verified_refs.md`.

RH-equivalent statements. “ $\mathcal{Q}(F, F) \geq 0$ for every $F \in C_c^\infty(\mathbb{R})$ ” is [RH-EQUIV] (Weil). For a *fixed* window length A , positivity is not RH-equivalent; it is a finite, checkable statement (Yoshida; Lemmas 10 and 16). Every certificate in this note is of the fixed- A kind. “RH below H ” is a theorem [124] and is used as one.

21.1. Setting

We use the conventions of Lemma 1. $\widetilde{F}(z) = \int F(u) e^{izu} du$; $H_F = F * F^*$, so $\widetilde{H_F}(z) = \widetilde{F}(z) \overline{\widetilde{F}(\bar{z})}$; and $\gamma_\rho = -i(\rho - \frac{1}{2})$. The explicit formula reads

$$(21.1.1) \quad \mathcal{Q}(F, F) = \sum_{\rho} m_{\rho} \widetilde{H_F}(\gamma_{\rho}) = \mathcal{W}[\widetilde{H_F}],$$

where $\mathcal{W}[g] = \frac{1}{2\pi} \int g \Psi dt - \sum_n \frac{\Lambda(n)}{\sqrt{n}} (\check{g}(\log n) + \check{g}(-\log n)) + (\text{pole terms})$ and

$\Psi(t) = \operatorname{Re} \psi(\frac{1}{4} + \frac{it}{2}) - \log \pi$. A window of length A means $\operatorname{supp} F \subset [-\frac{A}{2}, \frac{A}{2}]$, so $\widetilde{F}|_{\mathbb{R}} \in \operatorname{PW}_{\omega}$ with $\omega = A/2$.

LEMMA 21.1.1 (per-zero form). [THM] *A zero $\rho = \frac{1}{2} + \delta + i\gamma$ (with its reflection $1 - \bar{\rho}$) contributes $\operatorname{Re} \widetilde{H_F}(\gamma + i\delta)$ to the real part of (21.1.1). For centered F ,*

$$\operatorname{Re} \widetilde{H_F}(t + i\delta) = |\widehat{C}_{\delta}(t)|^2 - |\widehat{S}_{\delta}(t)|^2, \quad C_{\delta} = F \cosh(\delta u), \quad S_{\delta} = F \sinh(\delta u).$$

On-line zeros ($\delta = 0$) contribute $|\widehat{F}(\gamma)|^2 \geq 0$. An off-line contribution is indefinite (signature $(1, 1)$), and $|\widehat{S}_{\delta}| \leq \sinh(\delta A/2) \|F\|_1$.

PROOF. $e^{(iz)u}$ with $z = t + i\delta$ splits as $e^{itu}(\cosh \delta u - \sinh \delta u)$. The cross terms are imaginary after symmetrisation over $\rho, 1 - \bar{\rho}$. $\square$

Inputs. (I1) [124], Thm. 1: every zero with $0 < \gamma \leq H = 3\,000\,175\,332\,800$ is on the line; these are the first 12 363 153 437 138 zeros, all simple [CITED]. Positions were stored and isolated only up to $3.061 \cdot 10^{10}$ (Platt [123]). (I2) $N(T) = \theta(T)/\pi + 1 + S(T)$, with $|S(T)| \leq R(T) = \min\{0.1038 \log T + 0.2573 \log \log T + 8.3675, 0.1095 \log T + 0.2042 \log \log T + 3.0305\}$ for $T \geq e$ ([64], Cor. 1.4), and $|S| \leq 2.5167$ for $T \leq 3.06 \cdot 10^{10}$ [CITED]. *Flag:* [64], footnote 2, reports errors in Trudgian 2014 [151] and Platt–Trudgian 2015. The constants 0.112/0.278/3.385 used in Lemmas 12 and 18 come from [151]; those tables should be re-run with R_{HSW} . The effect is small: at $3 \cdot 10^{12}$, $R_{\text{HSW}} = 6.862$. (I3) $\beta < 1 - 1/(5.558691 \log |\gamma|)$ [108], so $\delta_{\max}(H) = 0.49374$. The mean density is $D(T) = \log(T/2\pi)/2\pi$, with $D(H) = 4.280$.

21.2. The no-go theorem

THEOREM 21.2.1 (counting-only certificates stop at A_{crit}). [THM] *Let $T_0 \geq H$, $D = D(T_0)$, $R = R(T_0)$, and $A_{\text{crit}} = \pi D/(R - \frac{1}{2})$. For every $A > A_{\text{crit}}$ and every $\delta \in (0, \delta_{\max}]$ there is a configuration Z with the following properties. It equals the true zeros below H . It is symmetric. It has simple zeros. It satisfies (I2)–(I3) on $[H, \infty)$, up to an error $o(1)$ in the local counting near T_0 . Its only off-line pair is at a height $\gamma_0 \approx T_0$, with offset δ . Finally, there is an F in a window of length A with $\mathcal{Q}_Z(F, F) = \sum_{\rho \in Z} \widetilde{H_F}(\gamma_{\rho}) < 0$. Hence no argument that uses only (I1)–(I3) about the zeros above H can certify $\mathcal{Q} \geq 0$ on windows of length $A > A_{\text{crit}}(H) = 2.113$.*

CONSTRUCTION. Take $\psi(u) = (1 - u^2/r^2)_+^m$ and $F(u) = e^{-i\gamma_0 u}(\psi(u - c) - \psi(u + c))$. Then $\widetilde{F}(t) = 2i \sin(c(t - \gamma_0)) \widehat{\psi}(t - \gamma_0)$, the window has length $A = 2(c + r)$, and $\widetilde{F}$ vanishes on the lattice $\gamma_0 + \frac{\pi}{c}\mathbb{Z}$. Place the background zeros near T_0 as clumps on that lattice, with clump size $\approx \pi D/c$ rounded to integers, so that the true counting function is followed in staircase fashion. The staircase deviation from θ/π stays $\leq R$ if $\pi D/c \leq 2R - 1$, that is, if $A \geq A_{\text{crit}} + 2r$. Choosing $2c\gamma_0 \in \pi\mathbb{Z}$ and putting the off-line pair at $\gamma_0 \pm i\delta$, the ghost contributes $-4(\cosh 2c\delta - 1)I^2$ with $I = \int \psi \cosh \delta u$; this is strictly negative for every $\delta > 0$. The clumps contribute 0. Splitting each clump into simple zeros at spacing ϵ costs $O(\epsilon^2)$. Zeros far from γ_0 (including all those below H) see $|\widehat{\psi}(t - \gamma_0)|^2$, which decays polynomially of order $2m + 2$; this leak is made negligible by taking $\gamma_0 - H$ large. $\square$

Explicit instance [NUM] (`lemma21_counter`, local quadratic model of θ at $\gamma_0 = H + 2 \cdot 10^6$, background over $\pm 10^6$). Every case has `count_ok` (maximum staircase deviation $\leq R$) and minimum clump size ≥ 11 .

c-fraction	r	δ	A	ghost	spread	leak	$\mathcal{Q}/\ F\ ^2$
0.97	0.05	0.25	2.279	$-7.07 \cdot 10^{-4}$	$9.0 \cdot 10^{-11}$	$1.4 \cdot 10^{-43}$	$-1.42 \cdot 10^{-2}$
0.97	0.05	10^{-3}	2.279	$-1.10 \cdot 10^{-8}$	$9.0 \cdot 10^{-11}$	$1.4 \cdot 10^{-43}$	$-2.2 \cdot 10^{-7}$
0.97	0.20	0.25	2.579	$-1.13 \cdot 10^{-2}$	$1.4 \cdot 10^{-9}$	10^{-49}	$-5.7 \cdot 10^{-2}$
0.90	0.05	0.25	2.448	$-8.25 \cdot 10^{-4}$	$9.1 \cdot 10^{-11}$	$1.4 \cdot 10^{-43}$	$-1.66 \cdot 10^{-2}$

A_{crit} at other heights (HSW constants): 2.160 (10^{13}), 2.397 (10^{16}), 2.646 (10^{20}), 3.068 (10^{30}), 3.958 (10^{100}). The limit as $T \rightarrow \infty$ is $1/(2 \cdot 0.1038) = 4.817$. Since S is unbounded (Tsang [152]: $S(t) = \Omega_{\pm}((\log t / \log \log t)^{1/3})$), no bounded R is admissible, and the threshold is a property of the counting information, not of δ .

REMARK 21.2.2. The adversarial Z is not claimed to be the zeta zeros. The theorem says what the information (I1)–(I3) can and cannot certify. The true zeros satisfy many more constraints, above all the explicit formula itself, with its prime side. Section 21.7 brings those constraints back in.

21.3. A sufficient condition in the model

PROPOSITION 21.3.1 (model positivity). [THM] (*for the translation-invariant model*) Assume every zero has $|\delta| \leq \delta_{\max}$, the distinct ordinates have maximal gap $g \leq 2R/D$, and the counting function satisfies the upper sampling bound $U = D + 4\omega R + 4\sqrt{\omega R D}$. (The upper bound is proved here, Gallagher-type.) Then $\mathcal{Q}_Z(F, F) \geq 0$ for all F in the window $A = 2\omega$, provided that for some $\eta \in (0, 1)$

$$(1 - \eta) \frac{(1 - g\omega/\pi)^2}{g} > U \left[(\eta^{-1} - 1)(\cosh(\delta_{\max}\omega) - 1)^2 + \sinh^2(\delta_{\max}\omega) \right].$$

PROOF. Lower sampling (Wirtinger plus Voronoi cells, Gröchenig [58]): $\|f - \sum f(x_k)\chi_k\| \leq \frac{g}{\pi}\|f'\| \leq \frac{g\omega}{\pi}\|f\|$, so $\sum_k |f(x_k)|^2 \geq g^{-1}(1 - g\omega/\pi)^2\|f\|^2$. Apply this to $f = \widehat{C}_{\delta}$ after the bound $|\widehat{C}_{\delta}|^2 \geq (1 - \eta)|\widehat{F}|^2 - (\eta^{-1} - 1)|\widehat{C}_{\delta} - \widehat{F}|^2$, and bound the S_{δ} -terms and the $C_{\delta} - F$ terms by upper sampling (Lemma 21.1.1). $\square$

Result [NUM] (`lemma21_core`, $T = H$):

$\delta_{\max}$	0.49374 (MTY)	0.25	0.1	0.01	10^{-3}	$\rightarrow 0$
A_{low}	0.3925	0.6025	0.9668	1.7453	1.9349	$\pi D/R = 1.9594$

The band $A_{\text{low}} \leq A \leq A_{\text{crit}}$ (from 1.96 to 2.11 as $\delta_{\max} \rightarrow 0$) is the slack between the Gröchenig constant and the staircase construction. *Conclusion:* the pure zero-side route gives 0.39 unconditionally. Even an unproved strip $|\delta| \leq 0.01$ above H would give only 1.75, and the absolute ceiling is 2.11. It never beats the prime-side 1.6 by more than about 0.5.

21.4. Comparison with Lemmas 12 and 18

Lemma 12's detection length $A_0(\delta = \frac{1}{2}, 3 \cdot 10^{12}) = 12.3$ and Lemma 18's $A^{\text{det}} \sim c\sqrt{b/\varepsilon}$ are *detection* lengths for one specified ghost in a *generic* background (true or GUE-like zeros), and they are [HEUR]. Theorem 21.2.1 is the worst case over backgrounds: the adversary nulls the background on the nodes of $\widetilde{F}$. The gap 12.3 vs 2.11 is the price of not knowing where the zeros above H are. The two statements are consistent. Detection needs $\mathcal{Q} < 0$ for *some* F when a ghost exists, while certification needs $\mathcal{Q} \geq 0$ for *all* F against all admissible configurations.

21.5. Restricted classes

PROPOSITION 21.5.1 (insensitivity). [THM] *Let $\mathcal{F}_{k,M} = \{F : \text{supp } F \subset [-\frac{A}{2}, \frac{A}{2}], \|F^{(k)}\|_2 \leq M\|F\|_2\}$. Then $|\sum_{\gamma > H} \text{Re } \widetilde{H}_F(\gamma_\rho)| \leq C_{A,k} M^2 e^{A\delta_{\max}} (\log H) H^{1-2k} \|F\|^2$ for every configuration satisfying (I2)–(I3).*

This follows by partial summation against $N(T)$, using $|\widetilde{F}(t + i\delta)| \leq e^{\delta A/2} \|F^{(k)}\|_1 |t|^{-k}$. Consequently a certificate on $\mathcal{F}_{k,M}$ (or on Gevrey classes $\|F^{(j)}\| \leq Mj!^s$, which reach $A \approx s^{-1} \log H - 2.4$ [EST]) holds for *every* configuration that agrees with the zeros below H . It carries no information about RH above H , and so is vacuous as RH evidence. Positivity on growing windows can only mean something if the class is closed under modulation $F \mapsto e^{i\gamma u} F$, and Theorem 21.2.1 exploits exactly that closure.

21.6. Prime-side costs: (b) Slepian basis, (c) Toeplitz/Levinson

Lemma 16's certificate needs a Galerkin band $|t| \leq T$ with $\log T = S(A) + 2.356$, where $S(L) = \sum_{n < e^L} 2\Lambda(n)/\sqrt{n} \sim 4e^{L/2}$. The margin follows $\log \lambda^* \approx 15.111 - 10.969 e^A$ (fit to Lemma 10/16 margins), and the measured cost grows as $N^{5.44}$. Here N is the Legendre degree, determined by $x^N/(2N+1)!! < 10^{-35}$ with $x = TA/2$.

- (b) Prolate (Slepian) basis: $N_{\text{slep}} \approx 2c/\pi + \pi^{-2} \log(1/\varepsilon) \log c$ [89], with $c = TA/2$. This is about half the Legendre degree, a factor of about $2^{5.44} \approx 40$ in cost [EST]. The exponent stays doubly exponential in A .
- (c) Toeplitz plus rank 2: on a shift-invariant spline basis, the translation-invariant part (Ψ and the prime terms) gives a Toeplitz matrix, and the pole terms add rank 2. Levinson's $O(N^2)$ [93] replaces $O(N^3)$ (and the measured $N^{5.44}$). Entry assembly then dominates. [EST]

A	T	N_{leg}	bits	Legendre (today)	Slepian	Toeplitz/Levinson
1.6	201	284	176	$1.3 \cdot 10^4$	$3.2 \cdot 10^2$	7.5
2.0	3690	5082	215	$1.2 \cdot 10^{11}$	$2.0 \cdot 10^9$	$4.1 \cdot 10^3$
2.4	$5.3 \cdot 10^4$	$8.7 \cdot 10^4$	272	$8.6 \cdot 10^{17}$	$1.4 \cdot 10^{16}$	$1.8 \cdot 10^6$
3.0	$4.8 \cdot 10^6$	$9.7 \cdot 10^6$	416	$2.4 \cdot 10^{29}$	$3.8 \cdot 10^{27}$	$4.3 \cdot 10^{10}$

Costs are in core-seconds [EST]. The reachable window at a fixed budget is: 10^4 core-s, Legendre 1.60 / Slepian 1.61 / Toeplitz 2.10; 10^7 , 1.61/1.95/2.56; 10^9 , 1.95/1.95/2.83. The steps come from prime powers entering at $A = \log n$. Toeplitz changes the polynomial factor, not the double exponential $T = e^{S(A)+2.36}$.

21.7. The three-band hybrid

Fix $0 < T_1 < T_2 < H$ and $\varepsilon > 0$. Let $k_\varepsilon = c_p \text{sinc}^{2p}(\varepsilon t/4p)$, normalised so that $\int k_\varepsilon = 1$. Then $k_\varepsilon \geq 0$ and $\widehat{k_\varepsilon}$ is supported in $[-\varepsilon/2, \varepsilon/2]$. Put $m = \mathbf{1}_{[T_1, T_2] \cup [-T_2, -T_1]} * k_\varepsilon$, so that $0 \leq m \leq 1$. Take $\nu = \mathbf{1}_{[-T_2', T_2']} * k_\varepsilon$, and set $\chi_{\text{mid}} = m^2$, $\chi_{\text{low}} = (1 - m^2)\nu$, $\chi_{\text{high}} = (1 - m^2)(1 - \nu)$. All three are ≥ 0 on $\mathbb{R}$, are entire of exponential type $\leq \frac{3}{2}\varepsilon$, and sum to 1.

THEOREM 21.7.1 (three-band identity). [THM] *For $F \in C_c^\infty$ in the window A ,*

$$\mathcal{Q}(F, F) = \mathcal{W}[\chi_{\text{low}} \widetilde{H}_F] + \sum_{\rho} m_{\rho} \chi_{\text{mid}}(\gamma_{\rho}) \widetilde{H}_F(\gamma_{\rho}) + \mathcal{W}[\chi_{\text{high}} \widetilde{H}_F],$$

where each $\mathcal{W}$ -term involves only $n < e^{A+2\varepsilon}$. Suppose $T_2 + \Delta \leq H$ and the tail of k_ε beyond distance Δ is below the margin. Then the middle sum equals $\sum_{0 < |\gamma| \leq H} |m(\gamma) \widetilde{F}(\gamma)|^2 \geq 0$, up to a

tail error. Moreover $|(\chi_{\text{high}} \widetilde{H_F})(x)| \leq \frac{1}{2\pi} \int \chi_{\text{high}} |\widetilde{F}|^2$, so

$$\mathcal{W}[\chi_{\text{high}} \widetilde{H_F}] \geq \left(\min_{t \geq T'_2 - \Delta} \Psi(t) - \mathcal{S}(A + 2\varepsilon) \right) \frac{1}{2\pi} \int \chi_{\text{high}} |\widetilde{F}|^2 - (\text{tail}).$$

PROOF. Apply (21.1.1) to the test function whose transform is $\chi \widetilde{H_F}$; its inverse transform is $\check{\chi} * H_F$, which is supported in $[-A - \frac{3}{2}\varepsilon, A + \frac{3}{2}\varepsilon]$. For real γ , $\chi_{\text{mid}}(\gamma) \widetilde{H_F}(\gamma) = |m(\gamma) \widetilde{F}(\gamma)|^2$. The high-band bound is the alignment bound of Lemma 10 applied to a nonnegative weight. Pole terms carry $\chi(\pm i/2)$, which is tiny for the mid and high bands. Zeros above H (possibly off-line) see $|\chi_{\text{mid}}(\gamma + i\delta)| \leq e^{\varepsilon\delta} \sup_{|s| \geq H - T_2} k_\varepsilon$ -tails, which can be summed against $N(T)$. $\square$

Certificate. $f = m\widetilde{F} \in \text{PW}_{(A+\varepsilon)/2}$. A lower sampling inequality

$\sum_{T_1 \leq |\gamma| \leq T_2} |f(\gamma)|^2 \geq L \|f\|^2 - (\text{tail})$ makes the middle band coercive. To get it, complete the zeros below T_1 by an artificial lattice, on which $|f|^2 \leq \text{tail}$. The high band is coercive with $\beta = \Psi(T'_2) - \mathcal{S}(A + 2\varepsilon) > 0$ once $T'_2 \gtrsim e^{\mathcal{S}(A)+2.36}$. What remains is a *low-band* Galerkin problem $\mathcal{W}[\chi_{\text{low}} \widetilde{H_F}] - c \cdot \frac{1}{2\pi} \int \chi_{\text{low}} |\widetilde{F}|^2$, with $c = \min(L, \beta)$, on $|t| \lesssim T_{\text{low}} = T_1 + 2\Delta$. This is exactly Lemma 10's structure with $(T, \beta_T) \rightarrow (T_{\text{low}}, c)$, but T_{low} is no longer tied to $e^{\mathcal{S}(A)}$. The precision is still set by $\lambda^* \approx e^{-11e^A}$, which is intrinsic. With $p \approx \frac{1}{2} \log(1/\lambda^*)$, the tails need $\Delta \approx e \log(1/\lambda^*)/\varepsilon$.

COROLLARY 21.7.2 (ceiling). [THM] *The hybrid needs $T'_2 \leq H - \Delta$ with $\Psi(T'_2) > \mathcal{S}(A + 2\varepsilon)$, that is, $\mathcal{S}(A + 2\varepsilon) + 2.36 < \log H$. At $H = 3 \cdot 10^{12}$ this gives $A < \log 61 - 2\varepsilon = 4.111 - 2\varepsilon$. If the middle-band sampling is certified from stored zero positions ($T_2 \leq 3.06 \cdot 10^{10}$), it gives $A < \log 47 - 2\varepsilon = 3.850 - 2\varepsilon$. As $H \rightarrow \infty$, $A_{\text{max}} \approx 2 \log(\frac{1}{4} \log H)$. (Compare: pure zero side 2.11; prime side today 1.6.)*

Sampling constants [NUM] (Odlyzko's first 10^5 zeros [114], not rigorous). The largest normalised gap above $t = 100$ is $\kappa = 2.805$, and the maximal gap above $t = 10^3, 10^4, 5 \cdot 10^4$ is 2.59, 2.18, 1.88. The empirical lower frame bound of $\{\gamma\}$ on $\text{PW}_{A/2}$ (minimum eigenvalue of the sinc Gram matrix on 300-unit blocks) is much better than the max-gap bound:

block	$2\pi D$	$A = 2.0$	$A = 3.0$	$A = 4.0$	$A = 5.0$	Gröchenig, $A = 3$
$[10^3, 1300]$	5.07	0.509	0.289	0.109	0.034	0
$[10^4, 10300]$	7.37	0.850	0.607	0.334	0.104	0.001
$[7 \cdot 10^4, 70300]$	9.32	1.125	0.921	0.670	0.431	0.028

Hybrid cost [EST] (lemma21_hybrid, Toeplitz/Levinson on the low band, $\varepsilon = 0.1$). “Crude” uses the rigorous max-gap criterion, $T_1 = 2\pi e^{\kappa(A+\varepsilon)/(1-\theta)}$ with $\theta = 0.3$. “Sharp” assumes a near-density inequality,¹ $T_1 = 2\pi e^{1.3(A+\varepsilon)}$, consistent with the table above.

A	T (prime)	cost	T_{low} crude	cost	T_{low} sharp	cost
2.4	$5.3 \cdot 10^4$	$1.7 \cdot 10^6$	$1.5 \cdot 10^5$	(no gain)	$5.9 \cdot 10^3$	$2.2 \cdot 10^4$
3.0	$4.7 \cdot 10^6$	$4.3 \cdot 10^{10}$	$1.6 \cdot 10^6$	$4.7 \cdot 10^9$	$1.2 \cdot 10^4$	$2.5 \cdot 10^5$
3.5	$7.8 \cdot 10^8$	$3.0 \cdot 10^{15}$	$1.2 \cdot 10^7$	$6.6 \cdot 10^{11}$	$2.0 \cdot 10^4$	$1.9 \cdot 10^6$
3.85	$2.6 \cdot 10^{10}$	$6.4 \cdot 10^{18}$	$4.7 \cdot 10^7$	$2.2 \cdot 10^{13}$	$2.8 \cdot 10^4$	$7.8 \cdot 10^6$
4.1	$1.2 \cdot 10^{12}$	$2.2 \cdot 10^{22}$	$1.3 \cdot 10^8$	$2.6 \cdot 10^{14}$	$3.7 \cdot 10^4$	$2.1 \cdot 10^7$

In addition, the middle band must be certified. In the crude version this is the maximal gap over $[T_1, T_2]$, read off Platt's database. In the sharp version it means blockwise Gram eigenvalues. These need only double precision with directed rounding, since $L = O(1)$, but there

¹[Vol. 2 editorial note] Chapter 24 (Prop. 24.2.1) shows that a frame bound with sharp band edges collapses; the “sharp” column implicitly assumes padding.

are about $N(T_2) \approx 10^{11}$ zeros at $A = 3.85$, roughly 10^{16} flops [EST]. The rigorous localisation lemma for the blockwise version (a partition of PW mass with Fejér-type windows) is not written; it is [HEUR]. With the crude criterion the gain begins at $A \approx 2.9$. Since the record normalised gap up to $3 \cdot 10^{10}$ is larger than 2.8, the crude column is optimistic.

21.8. Obstruction, questions, status

Obstruction. Counting information is translation-invariant and modulation-blind, and the Weil form on a window is modulation-covariant. So zeros above H enter through their *positions*, and those are unknown. The prime side is the only global information that pins them down. That is why the zero side can help only in a band where positions are known (below H) and the prime side must close the tail. The doubly exponential cost of Lemma 10 is then the price of the *low* band only, and through $\lambda^* \approx e^{-11e^A}$, i.e. a precision of $\approx 16e^A$ bits and a band $T_{\text{low}} \approx e \log(1/\lambda^*)/\varepsilon$, both *single* exponential in A (given a sharp sampling inequality).

QUESTION 21.8.1. Is there an explicit sampling theorem for $\{\gamma\}$ on $[T_1, T_2]$ at bandwidth $\omega < \pi D(T_1)(1 - \eta)$, using local counting (Turing’s method) instead of the maximal gap? Something Kadec/Avdonin-type “ $\frac{1}{4}$ in the mean” is what would make the sharp column rigorous.

QUESTION 21.8.2. Does J.W. want the hybrid implemented at $A = 2.4$ or 3.0 (sharp column $\lesssim 3 \cdot 10^5$ core-s), with the middle band certified from Platt’s zeros up to $5 \cdot 10^4$ – $5 \cdot 10^6$ (few million zeros)? This would be the first certificate past the prime-side reach.

QUESTION 21.8.3. Re-run the Lemma 12/18 tables with HSW in place of Trudgian 2014 (footnote 2 of [64])?

Statement	status	value
Thm 21.2.1 (no-go)	[THM] + [NUM] instance	$A_{\text{crit}}(H) = 2.113$
Prop. 21.3.1 (model positivity)	[THM] (model)	$A_{\text{low}} = 0.39$ (MTY), 1.75 ($\delta \leq 0.01$)
Prop. 21.5.1 (restricted classes)	[THM]	vacuous above H
Slepian / Toeplitz gains	[EST]	$\times 40$; $A \leq 2.1$ – 2.8
Thm 21.7.1 (three-band identity)	[THM]	ceiling $\log 61$ ($\log 47$ stored)
Hybrid costs, sharp sampling	[EST]/[HEUR]	$A = 3.85$ at $\sim 10^7$ core-s
Full Weil positivity	[RH-EQUIV]	not addressed

CHAPTER 22

Lemma 22: The prolate degeneration test

What Connes–Consani–Moscovici prove, the window dictionary, and why prolate objects see ghosts no earlier than window positivity

Standalone note `lemma22.tex`, October 21, 2026.

Summary. We separate what Connes–Moscovici (CM) and Connes–Consani(–Moscovici) (CC, CCM) prove from what they observe numerically or conjecture. One correction is needed to the usual paraphrase. The ultraviolet match with ζ is carried by the *negative* (Sonin) spectrum of the prolate operator W_λ . The low-lying zeros come from the Weil quadratic form QW_λ via prolate vectors, ζ -cycles and, most recently, ground-state determinants. The dictionary to TAA is exact: $\lambda^2 = d = e^A$, and QW_λ on $L^2([\lambda^{-1}, \lambda], d^*u)$ is Lemma 17’s level-II window action S_A , with the same normalization (checked). We then run the degeneration test on three levels. (i) W_λ contains no arithmetic data, so it is ghost-blind by construction. Its UV agreement concerns $N(T)$, which counts zeros in the whole strip. Numerically, at $\lambda = \sqrt{2}$ its even negative-eigenvalue count stays within 1 of $N_\zeta(T)$ for $T \leq 600$. (ii) For the Weil form restricted to any subspace, including the near-radical (“prolate”) subspace, min-max forbids earlier detection. Numerically the near-radical subspace of dimension $K(d)$ already detects at exactly the same window as the full space, in all 20 test cases. (iii) For the CCM ground-state construction, whose $\hat{\xi}$ has only real zeros, a ghost pair becomes visible in the zero approximations at the same window at which $\lambda_{\min} < 0$, for every δ tested. The near-radical dimension is $K(d) = 2d - \frac{7}{4} \pm 0.7$ for $2 \leq d \leq 13.5$. This is (Shannon number in $\log x$) $- 2N(2\pi d)$, and it agrees with CC’s prolate count $2\lambda^2 - 2$ up to $\frac{1}{4}$. Hence Lemma 12’s “excess $2d$ ” is the additive Slepian count. **Verdict: the prolate operator does not detect ghosts faster than window positivity. The best a prolate object can do is detect at the same window, and the Weil-form constructions achieve this.** Along the way we found a sign bug in `num/lemma11_ghost.py`. Lemma 11’s ghost table is therefore the “extra quadruple” model, not the stated “replacement” model; see §22.5.

Labels (as in Lemmas 17, 19, 20): [THM] proved here, or classical with proof; [CITED]; [NUM]; [ANALOGY]; [HEUR]; [RH-EQUIV], equivalent to RH. Numerics: `num/lemma22_*.py/.out/.json`; `log num/lemma22_progress.md`. References: `lit/lemma22/verified_refs.md` (Crossref/arXiv checked; PDFs and texts in `lit/lemma22/`).

22.1. What is proved and what is not

CM [34] (arXiv title “Prolate spheroidal operator and Zeta”). Let $W_\lambda \xi = -\partial_x((\lambda^2 - x^2)\partial_x \xi) + (2\pi\lambda x)^2 \xi$ on $L^2(\mathbb{R})$.

- [CITED] Thm. 2.6: there is a unique self-adjoint extension W_{sa} commuting with P_λ and $\hat{P}_\lambda$, the cut-offs to $[-\lambda, \lambda]$ in x and in Fourier space. It commutes with Fourier transform, and its spectrum is discrete and unbounded on both sides.

Cor. 2.7: even eigenfunctions behave like $\sin(2\pi\lambda x)/x$ at ∞ , odd ones like $\cos(2\pi\lambda x)/x$.

Cor. 3.2: eigenfunctions with negative eigenvalue lie in the Sonin space.

- [CITED, SEMICLASSICAL] Prop. 4.2: the semiclassical number of negative eigenvalues $\alpha \geq -E^2$ in each parity is $2\sigma(E, \lambda)$, with $2\sigma = \frac{E}{\pi}(\log \frac{E}{2\pi} - 1 + \log 4 - 2\log \lambda) + O(1)$. The constant λ^2 printed in (23) is an additive $O(1)$ term; see (25) and Thm. 6.1. The paper calls this “the semiclassical approximation”, so it is not a theorem about the true count.
- [CITED] Thm. 6.1 ($\lambda = \sqrt{2}$): a Dirac-type square root $2\hat{D}$ has spectrum in $\mathbb{R} \cup i\mathbb{R}$, and $N(E) = \frac{E}{2\pi}(\log \frac{E}{2\pi} - 1) + O(1)$. The proof invokes Prop. 4.2.
- What is *not* claimed: no eigenvalue-by-eigenvalue match with zeros, and no statement involving ζ beyond the smooth leading term of $N(T)$.

CC, ζ -cycles [27].

- [CITED, NUM] QW_λ on $L^2([\lambda^{-1}, \lambda], d^*u)$ has $\approx 2\lambda^2$ extremely small eigenvalues, e.g. $2.389 \cdot 10^{-48}$ at $\lambda^2 = 11$.
- [CITED, NUM] The eigenvectors are approximated by “prolate vectors” $E(f)(u) = u^{1/2} \sum_{n \geq 1} f(nu)$, with f an even prolate function $\text{PS}_{2m,0}(2\pi\lambda^2, x/\lambda)$, $m \leq \nu(\lambda^2) \approx 2\lambda^2$, corrected so that $f(0) = \hat{f}(0) = 0$. The text records “ $\nu(\mu) = 2\mu - 1$ works well when μ is a small half integer”.
- [CITED, THM] The main theorem relates ζ -cycles to critical zeros.
- [NUM] 31 zeros are reproduced from spectral data.

CCM [25].

- [CITED] A semilocal prolate operator: the square of the scaling operator plus the grading by orthogonal polynomials.
- [CITED] Proved: stability of the semilocal Sonin space when places are added, and its relation to Hilbert spaces of entire functions.
- The roles of the two parts of the spectrum are stated explicitly: the positive part gives the low-lying zeros, the negative part (Sonin) the UV behaviour.

CCM, Zeta spectral triples [24] (Nov. 2025).

- [CITED] Thm. 1.1/5.10. Let ϵ_N be the smallest eigenvalue of the truncated form QW_λ^N , *assumed* simple, with eigenvector ξ *assumed* even. Then a rank-one perturbation of the scaling operator $D_{\log}$ is self-adjoint, its regularized determinant is $\propto \hat{\xi}$, and $\hat{\xi}$ has only real zeros. The real-zero mechanism is the Carathéodory–Fejér type theorem of [36].
- [NUM] The zeros approximate $\gamma_1, \dots, \gamma_{50}$ extremely well; with primes ≤ 13 the error at γ_1 is $2.5 \cdot 10^{-55}$.
- [RH-EQUIV]: “a rigorous proof of this convergence would establish the Riemann Hypothesis”. Section 8 lists the missing steps: simplicity and evenness of the ground state, and the approximation of ξ_λ by k_λ .

REMARK 22.1.1 (correction to the paraphrase). “UV spectrum $\sim$ squares of zero ordinates” refers to the *negative* Sonin eigenvalues α of W_{sa} on $|x| > \lambda$, with $2\sqrt{-\alpha}$ playing the role of γ . The match is semiclassical, i.e. at the level of counting functions. The IR (low-lying zeros) does not come from the spectrum of W_λ . It comes from QW_λ : its tiny eigenvalues and their prolate eigenvectors (CC), ζ -cycles (CC), and ground-state determinants (CCM 2025). The positive spectrum of W_λ on $|x| > \lambda$ replicates the spectrum on $[-\lambda, \lambda]$. On J , CM §7.2 relate it to the *trivial* zeros.

22.2. The window dictionary

PROPOSITION 22.2.1 (dictionary; [THM] change of variables, [NUM] normalization). *Put $x = e^u$, $\lambda^2 = d = e^A$, $h = A/2$. The map $f \mapsto F = f \circ \exp$ is unitary from $L^2([\lambda^{-1}, \lambda], d^*u)$ onto*

d	2	2.5	3	4	5	6	e^2	8	10	$e^{2.4}$
$K(d)$ (even+odd)	1+1	2+1	2+2	3+3	4+4	5+5	7+6	7+7	9+9	10+10
$2d - \frac{7}{4}$	2.25	3.25	4.25	6.25	8.25	10.25	13.03	14.25	18.25	20.30
$\log_{10} \lambda^*$ (even)	-2.9	-5.0	-7.3	-12.1	-17.0	-22.1	-29.2	-32.4	-42.8	-48.2

TABLE 22.1. [NUM] Near-radical dimension $K(d) = \#\{\text{eigenvalues of } \mathcal{Q}|_{\mathcal{D}_A} < 10^{-1/2}\}$, counted from the full spectra of the exact Galerkin matrices (Lemma 11, 800 bits; 100 vs 120 modes per parity agree). Above the cascade the spectrum has a plateau of $O(1)$ eigenvalues (≈ 0.5 – 1.5). Over all 20 windows $2 \leq d \leq 13.46$: $|K(d) - (2d - \frac{7}{4})| \leq 0.69$, mean -0.27 . Source: `num/lemma22_count.out`.

$L^2([-h, h], du) = \mathcal{D}_A$ (Lemma 11’s window). It carries CC’s form [27, (2.11)] to the Weil form $\mathcal{Q}(F) = \sum_\rho \widehat{H}(\gamma_\rho)$ of Lemmas 1/11. Hence:

- (a) the eigenvalues of QW_λ are the stationary values $\lambda_k(A)$ of Lemma 17’s window action S_A (level II), so $\min \text{spec } QW_\lambda = \lambda^*(d)$, and the number of negative eigenvalues is the index κ_A ;
- (b) Weil positivity of QW_λ for all λ , i.e. $\kappa_A = 0$ for all A , is [RH-EQUIV] (Weil, Bombieri [14]).

Numerical check of the normalization [NUM]:

- the per-zero term is $T_\gamma[i, j] = 2 \operatorname{Re} \widehat{e}_i(\gamma) \overline{\widehat{e}_j(\gamma)}$;
- for $F = e_0$ at $d = 3$, $\mathcal{Q}(e_0) = 0.12233$, and the 1000-zero sum is 0.11945, with the remainder being the tail;
- $\mathcal{Q} - T_{\gamma_1} \succeq 0$ holds at $d = 3$ ($\lambda_{\min} = 8.8 \cdot 10^{-11}$) and at $d = e^2$ ($3.7 \cdot 10^{-34}$, 40 modes), provided γ_1 is exact. With the 21-digit γ_1 of `zeros1000.txt`, the error $6.6 \cdot 10^{-21}$ produces a spurious eigenvalue $-1.9 \cdot 10^{-21}$ (`num/lemma22_Tcheck2.py`).

At $d = 11$ our exact-entry Galerkin value is $\lambda^*(11) \leq 9.145 \cdot 10^{-49}$ (even, 120 modes; `num/lemma22_Q11_eigs.out`). It is a Rayleigh–Ritz value, hence an upper bound up to rounding. CC report $2.389 \cdot 10^{-48}$, a factor 2.61 larger. We found no normalization difference, so we read their number as a less converged truncation. We cannot reconstruct their truncation parameter, so this is an observation, not a contradiction.

HEURISTIC 22.2.2 (two counts of the near radical; [HEUR], [NUM] Table 22.1). Two counts give the same leading term.

- (a) *Additive, Slepian*. The even functions concentrated in $[-\lambda, \lambda]$ with Fourier transform in $[-\lambda, \lambda]$ number $\approx 2\lambda^2$ (Shannon number $4\lambda^2$, halved by parity) [139, 88]. Imposing $f(0) = \widehat{f}(0) = 0$ and applying E gives about $2\lambda^2 - 2 = 2d - 2$ prolate vectors (CC: $\nu(\mu) - 1$ with $\nu = 2\mu - 1$).
- (b) *Logarithmic, Lemma 12 Heur. 6.3*. $\mathcal{D}_A$ has Shannon number $(A/\pi)T_{\text{res}} = 2d \log d$ up to height $T_{\text{res}} = 2\pi d$. A near-radical vector must vanish at the $2N(T_{\text{res}}) = 2d \log d - 2d + \frac{7}{4} + O(S)$ points $\pm\gamma$. That leaves $2d - \frac{7}{4}$.

The data favour (b) over (a) by about $\frac{1}{4}$, but the threshold ambiguity is ± 0.7 . Either way, Lemma 12’s “excess $2d$ ” is the additive prolate count: the near radical is $E(\text{prolates})$. Here the Landau–Pollak count of Lemma 18 ($\kappa \approx A|S|/2\pi$) is the log-variable Shannon number of the region $S = \{w_\sigma < 0\}$. At $\sigma = \frac{1}{2}$ that region degenerates into the zero set, and (b) is the corresponding “Shannon minus constraints” count.

22.3. Level 1: the operator W_λ is ghost-blind

PROPOSITION 22.3.1 ([THM], elementary). *W_{sa} , its spectrum and its Sonin eigenspaces depend on λ alone. Let Z be any multiset in the critical strip with the same counting function $N(T)$ as the true zero set, i.e. with zeros allowed off the line. No spectral quantity of W_λ distinguishes Z from the zeros of ζ . In particular $N(T) = \#\{0 < \gamma \leq T\}$ counts zeros irrespective of $\text{Re } \rho$: by Riemann–von Mangoldt, $N(T) = \theta(T)/\pi + 1 + S(T)$ in every ghost world. The UV agreement in [34] is with θ/π up to $O(1)$.*

So the degeneration test at level 1 has a trivial negative answer: ghosts do not perturb W_λ at all. For completeness we checked the UV statement itself [NUM] (`num/lemma22_cm.py`). We used modified Prüfer shooting on (λ, ∞) , with the regular solution at $x = \lambda$ [34, (16)] and the condition $\xi \sim \sin(2\pi\lambda x)/x$ (even) or $\cos(2\pi\lambda x)/x$ (odd) at ∞ . The asymptotic phase is converged to 10^{-4} .

- For $\lambda = \sqrt{2}$ and $\lambda = 1$, both parities, the count of negative eigenvalues with $\alpha \geq -E^2$ is within 1 of $\frac{E}{\pi}(\log \frac{E}{2\pi} - 1 + \log 4 - 2\log \lambda)$ for $40 \leq E \leq 300$ (e.g. 340 vs 339.9 and 406 vs 406.1 at $E = 300$).
- At $\lambda = \sqrt{2}$ the even count $N_W(T) = \#\{2\sqrt{-\alpha} \leq T\}$ against $N_\zeta(T)$ is 20/21, 79/79, 201/202, 340/341 at $T = 80, 200, 400, 600$.
- The individual values are not zeros: $2\sqrt{-\alpha} = 12.55, 19.42, 24.70, 29.26, 33.37, \dots$ against $14.13, 21.02, 25.01, 30.42, 32.94, \dots$

22.4. Levels 2 and 3: Weil-form prolate objects

Ghost sensitivity can only enter through QW_λ . We use Lemma 11’s exact Galerkin matrices $\mathcal{Q}$ and the models below. With $T_z[i, j] = \text{Re}(\hat{e}_i(z)\hat{e}_j(-z) + \hat{e}_i(-z)\hat{e}_j(z))$ and $\gamma = \gamma_k$ exact (from `mpmath.zetazero`):

$$\mathcal{Q}_\delta^{\text{rep}} = \mathcal{Q} - T_\gamma + T_{\gamma+i\delta} + T_{\gamma-i\delta}, \quad \mathcal{Q}_\delta^{\text{add}} = \mathcal{Q}_\delta^{\text{rep}} + 2T_\gamma, \quad \mathcal{Q}_\delta^{\text{on}} = \mathcal{Q} - T_\gamma + T_{\gamma+\delta} + T_{\gamma-\delta}.$$

$\mathcal{Q}_\delta^{\text{rep}}$ replaces the zero at γ by an off-line quadruple at distance δ . $\mathcal{Q}_\delta^{\text{add}}$ adds the quadruple and keeps γ . $\mathcal{Q}_\delta^{\text{on}}$ is the RH-consistent control, with two on-line zeros at $\gamma \pm \delta$, so $\mathcal{Q}_\delta^{\text{on}} \succeq 0$.

PROPOSITION 22.4.1 (no subspace detects earlier; [THM], min-max). *For every subspace $V \subset \mathcal{D}_A$ and every symmetric form $\mathcal{B}$: $\min_V \mathcal{B}/\|\cdot\|^2 \geq \lambda_{\min}(\mathcal{B}|_{\mathcal{D}_A})$. Hence a negative direction found inside the prolate span, or inside any span of prolate vectors, exists in $\mathcal{D}_A$ at the same A . Detection by any subspace test never precedes $A^{\det} = \inf\{A : \kappa_A \geq 1\}$.*

PROPOSITION 22.4.2 (real zeros cannot exhibit a ghost; [CITED]+[THM]). *Assume the hypotheses of [24, Thm. 1.1]: the lowest eigenvalue is simple and its eigenvector even. The argument applies to $\mathcal{B} - \epsilon_N$ for any Toeplitz-type form $\mathcal{B}$ on the window, so ghost models included, and it does not use positivity of $\mathcal{B}$. Therefore $\hat{\xi}$ has only real zeros in a ghost world as well. An off-line pair at $\gamma \pm i\delta$ can only appear as real zeros, and in our runs as one real zero near γ . Ghosts are seen only indirectly, as a failure of convergence.*

Caveat. We did not re-verify the Carathéodory–Fejér hypotheses for Legendre truncations. The real-rootedness of our computed $\hat{\xi}$ is observed, not proved: all roots were found by sign changes, and the root counts are consistent.

PROPOSITION 22.4.3 (the prolate subspace and ground-state zeros detect exactly when positivity does; [NUM]). *Take the ghost at γ_1 or γ_4 , with $\delta \in \{0.4, 0.1, 10^{-2}, 10^{-4}, 10^{-8}\}$, both parities, on Lemma 11’s grid of windows $2 \leq d \leq e^{2.6}$.*

- (a) Near-radical subspace. *Let V_K be the span of the eigenvectors of $\mathcal{Q}$ with eigenvalue $< 10^{-1/2}$, so $\dim V_K = K(d) \approx d - \frac{7}{8}$ per parity. Then $\lambda_{\min}(\mathcal{B}|_{V_K})$ and $\lambda_{\min}(\mathcal{B})$ first*

δ	0.4	0.1	10^{-2}	10^{-4}
first d with $\lambda_{\min}(\mathcal{Q}_\delta^{\text{rep}}) < 0$ (even)	≤ 2.5	e	3.32	4
first d with $\Delta_2 > \epsilon_2$ (ground-state zeros)	≤ 2.5	e (3.0)	3.32 (10.9)	4 (1.7)
Δ_2/ϵ_2 one grid step earlier	–	0.37	0.42	$1.2 \cdot 10^{-3}$
$\Delta_2, \dots, \Delta_6$ at $d = 6$	10^{-1}	10^{-2}	10^{-4}	10^{-8} – 10^{-5}

TABLE 22.2. [NUM] Ghost at γ_1 , model $\mathcal{Q}_\delta^{\text{rep}}$, even sector, windows $2.5 \leq d \leq e^{2.6}$ (Lemma 11 grid, $L \geq 0.9$); ratios Δ_2/ϵ_2 at detection in parentheses. Here $\epsilon_k(d) = |z_k(\mathcal{Q}) - \gamma_k|$ is the accuracy of the k -th real zero of $\hat{\xi}$ for the true form, and $\Delta_k = |z_k(\mathcal{Q}_\delta^{\text{rep}}) - z_k(\mathcal{Q})|$ is the shift caused by the ghost. For example, $\epsilon_1 = 2.5 \cdot 10^{-19}$, $\epsilon_2 = 1.8 \cdot 10^{-16}$, $\epsilon_6 = 9 \cdot 10^{-10}$ at $d = 6$; for $4 \leq d \leq 8$, $\epsilon_1 \approx (1.7-9) \cdot 10^3 \lambda^*$. For $d \geq 9$, ϵ_k reaches the $\approx 3 \cdot 10^{-29}$ floor set by the stored root precision. The ground-state zero test switches on at the same grid window as positivity in all four cases. Source: `num/lemma22_xizeros_an.out`.

become negative at the same window in every case. After detection the minima agree in sign; the median relative difference is $2.5 \cdot 10^{-4}$, and the largest is 0.61 (γ_4 , $\delta = 0.4$, where the ghost direction is only partly inside V_K). Even the single Rayleigh quotient of the ground state of $\mathcal{Q}$ changes sign at the same grid point, though only at the 10^{-12} – 10^{-9} level. (Table 22.3; `num/lemma22_ghostsub_an.out`.)

- (b) Ground-state zeros (the CCM 2025 construction, Table 22.2). The ghost shift Δ_k of the real zeros of $\hat{\xi}$ exceeds their accuracy ϵ_k at the same grid window at which $\lambda_{\min}(\mathcal{Q}_\delta^{\text{rep}}) < 0$. After detection $\Delta_k \approx \delta^2$ and stops decreasing in d , while $\epsilon_k(d)$ keeps falling super-exponentially. With $\mathcal{Q}_\delta^{\text{on}}$, $\hat{\xi}$ shows the two real zeros $\gamma \pm \delta$ once they are resolved (e.g. 13.7347251, 14.5347251 at $d = 5$, $\delta = 0.4$). With $\mathcal{Q}_\delta^{\text{rep}}$ it shows one.

HEURISTIC 22.4.4 (why the thresholds coincide; [HEUR]). Write $G = \hat{F}$. Then $T_{\gamma+i\delta} + T_{\gamma-i\delta} - 2T_\gamma = -4\delta^2|G'(\gamma)|^2 + O(\delta^4)$ on vectors with $G(\gamma) = 0$. Both tests compare this δ^2 -term with the size of the true form on the near radical. Positivity fails when $\lambda^* \lesssim \delta^2|G'(\gamma)|^2$. The ground state, and hence $\hat{\xi}$, switches from “vanish at γ ” to “exploit the ghost” at the same point. This is the doubly-logarithmic law of Lemmas 11/12, $-\log \lambda^*(d) \approx 2 \log \frac{1}{\delta} + O(1)$. Since $\epsilon_k \propto \lambda^*$ here, the zero test cannot beat it.

22.5. Erratum to Lemma 11’s ghost numerics

The bug. `num/lemma11_ghost.py` computes $\hat{e}_n(z)$ with $\sqrt{\pi/(2w)} J_{n+1/2}(w)$, $w = zh$. For real $w < 0$ the principal branches do not cancel, and $\hat{e}_n(-x)$ comes out with the wrong sign (checked against direct quadrature). Non-real $z = \gamma \pm i\delta$ are unaffected. Consequently the real- γ term enters as $-T_\gamma$, and the model actually computed was $\mathcal{Q}_\delta^{\text{add}} = \mathcal{Q} + T_\gamma + T_{\gamma+i\delta} + T_{\gamma-i\delta}$: a quadruple added on top of the simple zero at γ . The text of Lemma 11 (abstract (c), Table “ghost”) describes $\mathcal{Q}_\delta^{\text{rep}}$.

- $\mathcal{Q}_\delta^{\text{add}}$ is still a legitimate ghost model.
- Since $\mathcal{Q}_\delta^{\text{rep}} = \mathcal{Q}_\delta^{\text{add}} - 2T_\gamma$ and $T_\gamma \succeq 0$, Lemma 11’s d_{ghost} are upper bounds for the replacement model.
- Table 22.3 shows both models. On our 20-window grid their first-detection windows coincide in all 20 cases, so the bug does not change Lemma 11’s table at this resolution. Lemma 11’s own grid (h -step 0.05) has since been re-run (`errata/vol1_ch11_erratum.pdf`): 59 of 60 entries are unchanged, and the one change is $k = 1$, $\delta = 0.4$, odd, $2.72 \rightarrow 2.46$.
- Lemma 12’s doubly-log fit was made on the $\mathcal{Q}_\delta^{\text{add}}$ data.

k	δ	Q_δ^{rep} full	Q_δ^{rep} on V_K	ground-state RQ	Q_δ^{add} (L11 model)
1	0.4	2.50/2.50	2.50/2.50	2.50/2.50	2.50/2.50
1	0.1	2.72/3.00	2.72/3.00	2.72/3.00	2.72/3.00
1	10^{-2}	3.32/4.00	3.32/4.00	3.32/4.00	3.32/4.00
1	10^{-4}	4.00/4.48	4.00/4.48	4.00/4.48	4.00/4.48
1	10^{-8}	6.00/6.00	6.00/6.00	6.00/6.00	6.00/6.00
4	0.4	4.00/4.00	4.00/4.00	4.00/4.00	4.00/4.00
4	0.1	4.00/4.48	4.00/4.48	4.00/4.48	4.00/4.48
4	10^{-2}	4.48/4.95	4.48/4.95	4.48/4.95	4.48/4.95
4	10^{-4}	6.00/6.00	6.00/6.00	6.00/6.00	6.00/6.00
4	10^{-8}	7.39/8.00	7.39/8.00	7.39/8.00	7.39/8.00

TABLE 22.3. [NUM] First window d (even/odd) on Lemma 11’s grid at which the ghost is detected (negative value), ghost at γ_k , exact ordinates, branch-fixed $\hat{e}_n$, 100 modes per parity. Columns 3–5: replacement model on the full space, on the near-radical subspace V_K , and the Rayleigh quotient of the ground state of Q . Column 6: the model Lemma 11 actually computed. Source: num/lemma22_ghostsub_an.out.

- Lemmas 10/12/16/18 do not use this routine. lemma10_zeroside.py evaluates only at $t > 0$, and lemma12_sanitary.py uses $|w|$.
- Lemma 11/12 files are not edited here. The fix is $\sqrt{\pi/2}/\sqrt{w}$.

22.6. Verdict

Does the prolate operator detect ghosts faster than window positivity? No.

- (1) W_λ is ghost-blind by construction (Prop. 22.3.1). Its UV agreement concerns $N(T)$, which is the same in every ghost world.
- (2) On the Weil side, min-max forbids earlier detection by any subspace, so “prolate vectors” can at best tie (Prop. 22.4.1). Numerically they do tie, and the near-radical space of dimension $2d - \frac{7}{4}$ suffices (Prop. 22.4.3(a)).
- (3) The CCM ground-state zeros are real by construction (Prop. 22.4.2). They see a ghost only as stalled convergence, at the same window as positivity (Prop. 22.4.3(b)).

What prolates do give TAA:

- (a) a compression: all ghost-detection information at window A lives in a $\approx 2e^A$ -dimensional space, which is explicitly approximable by $E(\text{PS}_{2m})$;
- (b) an identification: Lemma 12’s “excess $2d$ ” is the additive Slepian count;
- (c) a reading of CCM 2025 in Lemma 19’s language. The real-zero mechanism (Carathéodory–Fejér) is the analogue of the “volume” or sign-change side, which is blind to off-line quadruples. The convergence to Ξ , which would give RH, is the “area” side [RH-EQUIV].

Obstruction. Every prolate construction that knows about ζ does so through QW_λ , and its detection power is bounded by $\lambda^*(d) = e^{-2\pi^2 d(1+o(1))}$ (Lemma 11). A route to RH through prolates must prove a statement that holds for all λ . Examples are the convergence $\hat{\xi}_\lambda \rightarrow \Xi$ [24], or simplicity and evenness of the ground state together with $k_\lambda \approx \xi_\lambda$. Each of these is at least as strong as controlling λ^* for all d [RH-EQUIV]. The finite- λ structure adds no leverage over window positivity.

Rigor status.

- [THM]: Prop. 22.2.1 (change of variables), Prop. 22.3.1, Prop. 22.4.1, and the erratum mechanism (branch analysis, checked numerically).
- [CITED]: §22.1 and Prop. 22.4.2 (CCM Thm. 1.1 under its stated hypotheses).
- [NUM]: the normalization check, Table 22.1, the W_λ counts, Prop. 22.4.3, Tables 22.2, 22.3. These are mpmath 110 digits on exact-entry (800-bit) Galerkin matrices with 100–120 modes per parity, not interval-certified. Windows with $d \geq e^{2.4}$ are not fully converged (100 $\rightarrow$ 120 modes changes λ^* by 5%), and the root grid is 0.1.
- [HEUR]: Heuristics 22.2.2, 22.4.4.
- [RH-EQUIV]: Weil positivity for all λ ; CCM’s convergence.

- QUESTION 22.6.1. (1) Should the TAA window basis be switched to CC prolate vectors $E(\text{PS}_{2m,0})$, i.e. use the $\approx 2d$ -dimensional near radical as the certificate space, for the Lemma 16 interval certificates at larger d ? The ghost information is all there (Prop. 22.4.3(a)). The cost is the high-precision prolate evaluation.
- (2) Is the count $2d - \frac{7}{4}$ (Shannon minus constraints) or $2d - 2$ (Slepian minus $f(0), \hat{f}(0)$) the right one? Distinguishing them needs a threshold-free count, e.g. the index of a Landau–Widom plunge fit.
- (3) Lemma 11’s ghost table answers the “extra quadruple” model. Do you want Lemma 11 re-issued with the replacement model (Table 22.3), or an erratum note only?

CHAPTER 23

Lemma 23: Tropicalizing $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$

The magnitude survives, the positivity does not

Standalone note `tropical_note.tex`, October 21, 2026.

Summary. We test J.W.’s proposal to treat $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ tropically. First the dictionary is corrected. Under Maslov dequantization, multiplication becomes *addition* of logarithms, and addition becomes *max*. The tropical limit of the zero side keeps exactly $\text{Re } \rho$ and discards the phases. RH itself is therefore a tropical statement ($\Theta = \frac{1}{2}$). What is not tropical is every known mechanism that *forces* it.

Decisive test (function fields). On $E/\mathbb{F}_5$, $C_2/\mathbb{F}_7$ and on fake Weil polynomials, the following do *not* separate real curves from fakes:

- p -adic Newton polygons;
- archimedean tropical roots;
- Yu’s tropical PSD cone, which consists exactly of the single-shadow bounds;
- complex-tropical (Viro $\mathcal{TC}$) positivity.

The norm $|\alpha| = \sqrt{q}$ is recovered only by (a) the limit in n of *all* shadows, (b) phase averaging (Ronkin/Jensen), or (c) positivity over a genuine field ($\mathbb{C}_h \cong \mathbb{C}$, $h > 0$), i.e. Rosati.

The Lemma 15 window Gram matrices (for $h \geq 0.1$) are numerically Lorentzian quadratics in the sense of Brändén–Huh. On all windows this is RH-equivalent.

Nothing here bears on RH beyond reformulation. Plain tropicalization of $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ cannot yield a Hodge index theorem.

Labels: [THM] proved here or classical with proof; [CITED]; [NUM]; [ANALOGY]; [DEDUCTION]: our inference from cited universal properties; [RH-EQUIV]. Numerics: `num/tropical_checks.py/.out/.json`, `num/tropical_window_lorentz.py/.out/.json`. References: `lit/tropical/verified_refs.md`.

23.1. Dequantization, done correctly; the zero side keeps only $\text{Re } \rho$

Put $x = e^{a/h}$, $y = e^{b/h}$. Then $h \log(xy) = a + b$ *exactly*, and $h \log(x + y) = h \log(e^{a/h} + e^{b/h}) \rightarrow \max(a, b)$ as $h \rightarrow 0$ [94]. So the rule is $\times \mapsto +$ and $+$ $\mapsto \max$. J.W.’s version (“multiplication $\rightarrow$ min/max, addition $\rightarrow$ multiplication”) is the same idea shifted down one level. Read on logarithms it is right: $\max$ replaces $+$ and $+$ replaces $\times$. Two consequences are used throughout:

- The phase of x disappears in the limit, since $h \log |e^{(a+i\phi)/h}| = a$.
- Cancellation becomes invisible: $x - x$ and $x + x$ have the same tropical value.

Viro’s hyperfields [153] keep track of what is lost. The triangle hyperfield sends $a \boxplus b$ to the interval $[|a - b|, a + b]$; the complex tropical $\mathcal{TC}$ sends a sum to its terms of maximal modulus. Viro [153, §9.3] realizes $\mathcal{TC}$ as the limit $h \rightarrow 0$ of the fields $\mathbb{C}_h = (\mathbb{C}, +_h, \times)$, where $a +_h b = S_h^{-1}(S_h a + S_h b)$ and $S_h(z) = |z|^{1/h} z / |z|$. Each $\mathbb{C}_h$ is a copy of $\mathbb{C}$.

PROPOSITION 23.1.1 (the tropical value of the zero side; [CITED]+[THM]). *Let $\Theta = \sup \text{Re } \rho$. Then $\limsup_{x \rightarrow \infty} \log |\psi(x) - x| / \log x = \Theta$.*

PROOF. The bound $\leq$ is $\psi(x) = x + O(x^\Theta \log^2 x)$ [106, Ex. 13.1.1]. The bound $\geq$ is $\psi(x) - x = \Omega_\pm(x^{\Theta-\varepsilon})$ [106, Thm. 15.2]. $\square$

Reading. In $\psi(x) - x = -\sum_\rho x^\rho/\rho + \dots$ take $h = 1/\log x$. Then $h \log |x^\rho| = \operatorname{Re} \rho$, so the tropical sum is $\max \operatorname{Re} \rho = \Theta$. The phases $\operatorname{Im} \rho \cdot \log x$ are exactly what is discarded. RH says $\Theta = \frac{1}{2}$, which is a purely tropical statement. Note, however, that the proof of “ $\geq$ ” uses Landau’s lemma for integrands of *one sign* [106, Lemma 15.1]. Even extracting the tropical value needs one bit of phase: a sign.

PROPOSITION 23.1.2 (Lemma 10/16 is the triangle-hyperfield bound, and it is sharp; [THM]+[NUM]). *Let $C(t) = \sum_{n < d} a_n \cos(t \log n)$ with $a_n = 2\Lambda(n)n^{-1/2} \geq 0$. The pointwise bound $C(t) \leq A_d = \sum a_n$ is the upper endpoint of the triangle-hyperfield sum $\boxplus_n a_n$, i.e. the bound that forgets every phase. For every T , $\sup_{t \geq T} C(t) = A_d$.*

PROOF. The $\log p$ ($p < d$) are $\mathbb{Q}$ -linearly independent. By Kronecker, $t \log p \in 2\pi\mathbb{Z} + (-\delta, \delta)$ holds for all $p < d$ simultaneously at arbitrarily large t . Then $t \log p^k = k t \log p$ is also near $2\pi\mathbb{Z}$, so prime powers align automatically. This is Zhu’s Lemma 3.2 [165]. $\square$

[NUM] Window 1.6 ($n = 2, 3, 4$): $A_d = 2.94197353$, and the sup found on $[200, 200 + 10^6]$ is 2.94197350. $C > A_d - 0.05$ holds on a fraction 0.0037 of t . So the Lemma 10 constant is optimal. (We checked this explicitly: a per-prime “sharpening” fails for the reason Zhu retracts in his Rem. 3.3.) Any pointwise-in- t method is exactly tropical, and the double exponential $T_1 = 2\pi e^{A_d}$ is intrinsic to that method class. Lowering it needs phase information: a Diophantine non-alignment statement [165, Rem. 1.6], or averaging. The Bohr mean square of C is 1.525, against $A_d^2 = 8.65$.

23.2. The decisive test on $C \times C/\mathbb{F}_q$

Let $P(T) = \prod_{i=1}^{2g} (1 - \alpha_i T)$ and $s_n = \sum \alpha_i^n$. The examples are:

- two real curves: $E/\mathbb{F}_5$ with $P = 1 + 3T + 5T^2$, and $C_2/\mathbb{F}_7$ with $P = 1 + 3T + 4T^2 + 21T^3 + 49T^4$;
- three *fakes*, each satisfying the functional equation $\alpha \mapsto q/\alpha$: the Lemma 19 coin $1 - 22T^2 + 49T^4$; the genus-one ghost $(1 - T)(1 - 5T)$; and the Lemma 19 non-real quadruple ($q = 7$, moduli $e^{\pm 0.3\sqrt{7}}$).

Each test below records what it outputs, compared with the truth $|\alpha|/\sqrt{q}$.

case	true $ \alpha /\sqrt{q}$	p -adic slopes	arch. trop. roots	Ronkin	NS (n_+, n_-, n_0)	first k : PSD / trop-PSD
$E/\mathbb{F}_5$	1, 1	0, 1	0.745, 1.342	0.99, 1.00	(1, 3, 3)	never / never
$C_2/\mathbb{F}_7$	1 ⁴	0, 0, 1, 1	0.88, 1, 1, 1.13	0.99, 1.00	(1, 5, 3)	never / never
coin $q=7$	0.599 ² , 1.668 ²	0, 0, 1, 1	0.564 ² , 1.773 ²	0.59–1.67	(3, 3, 3)	3 / 3
ghost $q=5$	0.447, 2.236	0, 1	0.373, 2.683	0.447, 2.22	(2, 2, 3)	2 / 2
quadruple	0.741 ² , 1.350 ²	–	0.44, 0.64, 1.57, 2.26	0.74, 1.35	(3, 3, 3)	3 / 4

[NUM] The “NS” column is the Gram matrix of $H, V, \Gamma_0, \dots, \Gamma_{n-1}$, built from $\Gamma_m \cdot \Gamma_k = q^{\min(|m-k|, 1-s_{|m-k|})}$ and similar entries. “Ronkin” is the phase average $\frac{1}{2\pi} \oint \log |P(re^{i\phi})| d\phi$ (Jensen), read off at its corners; it is evaluated on a grid, hence the 0.99.

- *p-adic Newton polygon*: identical for the real curve and its fake (coin vs C_2 ; ghost vs E). It sees $\operatorname{ord}_p \alpha$, not $|\alpha|_\infty$.
- *Archimedean tropical roots* ($-\log$ of the slopes of $\max_k \log |P_k| + k \log |T|$): these misreport even real curves. For E they give $3/\sqrt{5}$ and $(5/3)/\sqrt{5}$, a false “off-circle” pair, because the cancellation that produces $|\alpha| = \sqrt{q}$ is invisible tropically.
- *Ronkin*: exact, but it integrates over full circles, i.e. over all phases.

THEOREM 23.2.1 (all shadows $\Rightarrow$ norm is a Maslov limit in n ; [THM]). *Let $\alpha_1, \dots, \alpha_r \in \mathbb{C}^\times$ and $s_n = \sum_i \alpha_i^n$. Then*

$$\limsup_{n \rightarrow \infty} |s_n|^{1/n} = \max_i |\alpha_i|.$$

If moreover the multiset $\{\alpha_i\}$ is stable under $\alpha \mapsto q/\alpha$, then $\min |\alpha_i| = q/\max |\alpha_i|$, and

$$\sup_n |s_n| q^{-n/2} < \infty \iff \text{all } |\alpha_i| = \sqrt{q}.$$

PROOF. Group equal values: $\beta_1, \dots, \beta_m$ are distinct, with multiplicities $m_j \geq 1$. Then

$$\sum_{n \geq 1} s_n T^n = \sum_j m_j \beta_j T / (1 - \beta_j T).$$

This is a rational function with a pole at every $1/\beta_j$: the partial fractions have distinct poles and nonzero residues, so none cancels. Its radius of convergence is therefore $\min_j |\beta_j|^{-1}$.

Cauchy–Hadamard gives $\limsup |s_n|^{1/n} = \max |\beta_j|$.

Duality maps the largest modulus to the smallest, which gives the formula for $\min |\alpha_i|$. If $|s_n| \leq C q^{n/2}$ for all n , then $\max |\alpha_i| \leq \sqrt{q}$, so $\min |\alpha_i| = q/\max |\alpha_i| \geq \sqrt{q}$, and all moduli equal $\sqrt{q}$. Conversely, $|s_n| \leq r q^{n/2}$. $\square$

Maslov reading [THM]. With $h = 1/n$, $h \log |s_n| = h \log |\sum e^{n \log \alpha_i}|$. Its tropical limit is $\max_i \log |\alpha_i|$, and $\leq$ holds for every n , since $|s_n| \leq r \max |\alpha_i|^n$. The statement is a *limsup*, not a limit: phases decide which n realize it (the coin has $s_{\text{odd}} = 0$). [NUM]: $\max_{400 \leq n < 600} |c_n|^{1/n}$ with $c_n = s_n q^{-n/2}$ is 1.0017 and 1.0033 for E and C_2 ; 1.6713 for the coin (true 1.6684); 2.2361 for the ghost.

Answer to J. W.’s framing. In $(\log |x|, \arg x)$, the tropical (amoeba) coordinate of a Frobenius eigenvalue is $\log |\alpha| = \frac{1}{2} \log q$. This is the norm, $\|F\|^2 = \deg F = q$ in Lemma 19. The trace $a = \alpha + \bar{\alpha} = 2\sqrt{q} \cos \theta$ has $\log |a| = \frac{1}{2} \log q + \log |2 \cos \theta|$, which mixes magnitude and phase: Lemma 19’s “shadow”. Theorem 23.2.1 shows that shadows determine the norm *only in the tropical limit in n* , an infinite amount of data.

- PROPOSITION 23.2.2 (tropical positivity = single shadows; [CITED]+[THM]+[NUM]). (a) [164, Thm. 1] *The valuations of PSD matrices without zero entries over real Puiseux series are exactly the symmetric X with $x_{ii} + x_{jj} \leq 2x_{ij}$. Equivalently, these are the X for which the tropical determinant is attained at the identity. Only the 2×2 conditions survive.*
- (b) *For the Toeplitz sections $T_k = (c_{i-j})_{i,j < k}$, the archimedean shadow of (a) is $|c_n| \leq c_0 = 2g$ for $n < k$: the single-shadow Weil bounds. If $|c_n| < c_0$ for $0 < n < k$, then every principal minor of T_k evaluates in $\mathcal{TC}$ to $c_0^m > 0$. Indeed the identity is the unique dominant permutation term, since every other term contains an off-diagonal factor of modulus $< c_0$. So $\mathcal{TC}$ -positivity, and a fortiori any hyperfield positivity that factors through $\mathcal{TC}$, coincides with (a) for non-tied data.*
- (c) [NUM] *Lemma 19 coin family $N = 1, 2, 5, 10, 20$ (passes the Weil bound for every $n \leq N$). The first non-PSD section is $k = 3, 4, 4, 4, 5$. The first tropically non-PSD section is $k = 3, 5, 11, 21, 42$.*

So the target (iii), a positivity on *finitely many* phase-aware shadows that forces $|\alpha| = \sqrt{q}$, is met by field-level PSD (Rosati / Toeplitz, $k \leq 5$ in every test). It is not met by any $h = 0$ object: $\mathbb{T}$, Yu’s cone, triangle, $\mathcal{TC}$, or the phase hyperfield, whose sign/phase data carry no magnitude. Because $\mathbb{C}_h \cong \mathbb{C}$ for $h > 0$ (Viro), “phase-tropical positivity at finite h ” is just classical positivity in new coordinates.

PROPOSITION 23.2.3 (Lorentzian reading; [CITED]+[THM]+[NUM]). Brändén–Huh [16, Def. 2.1, Lemma 2.5]: *a quadratic form with nonnegative coefficients is Lorentzian iff its Hessian has at most one positive eigenvalue (with M -convex support), iff it is real stable.*

- (a) For E and C_2 , the NS Gram on $\langle H, V, D_m \rangle$, $D_m = \Gamma_m + t(H + V)$ ($t = 0, 1$), rescaled by $q^{-m/2}$, has nonnegative entries, M -convex support and $n_+ = 1$, so it is Lorentzian. For the fakes $n_+ = 2, 3$ and it is not. Lorentzian here = Hodge index on the span = (hyperbolic plane) $\oplus$ (negative-definite transverse part) = Rosati positivity.
- (b) Every tropical (magnitude-only) test of Lorentzian-ness is blind. The 2×2 reverse Cauchy–Schwarz conditions pass for all fakes. M -concavity of normalized log-coefficients [16, Cor. 3.16], the tropical sufficient condition, fails already for the real curves (132 and 513 failing triples).
- (c) [NUM] The Lemma 15 window Grams $G = (\ell(c_i - c_j))$ with centers $c_i = i \log q$, $q \in \{2, 3\}$, $n \in \{4, 8\}$, have, for $h \in \{0.1, 0.25, 0.5\}$, all entries > 0 and $n_+ = 1$ (inertia certified in Lemma 15). They are therefore Lorentzian quadratics and real-stable polynomials. At $h = 0.05$ the diagonal $\ell(0) = -0.0162 < 0$, so the polynomial reading fails although $n_+ = 1$ still holds.
- (d) [RH-EQUIV] By Lemma 15, Thm. 15.2 ($\sup n_+ = 1 + N_+$), “all window Grams have $n_+ \leq 1$ ” is RH. Lorentzian-ness on all positive-entry windows is a reformulation, and no new leverage.

23.3. Literature: what is proved, what is missing

Connes–Consani.

- The arithmetic site [28, 22] is the topos $\widehat{\mathbb{N}^\times}$ with structure sheaf $\mathbb{Z}_{\max}$. Its points over $\mathbb{R}_+^{\max}$ are $\mathcal{Y}$, and $\mathbb{R}_+^\times$ acts as Frobenius. The square carries a one-parameter semigroup of Frobenius correspondences, and its structure sheaf “involves Newton polygons” [22].
- The scaling site [23] is $[0, \infty) \rtimes \mathbb{N}^\times$ with piecewise-affine convex sheaf. It is a tropical curve, and Riemann–Roch (real dimensions) is proved on each periodic orbit $C_p = \mathbb{R}_+^*/p^\mathbb{Z}$.
- Riemann–Roch for $\overline{\text{Spec } \mathbb{Z}}$ (integer dimensions, over the sphere spectrum) is proved in [26]. That is a *curve*.
- The strategy [30]: $\text{RH} \iff s(f, f) \leq 0$ for f with $\int f d^*u = \int f du = 0$ (eq. 12), with $s(f, f) = D \bullet D$ for $D = \int f(\lambda) \Psi_\lambda d^* \lambda$ on the square (eq. 14). Step 3, Hodge–Riemann–Roch on the square of the complex lift, is called “the most problematic”. The tropical descent uses Jessen’s function $\varphi(\sigma) = \lim_{\frac{1}{2T}} \int \log |f(\sigma + it)| dt$, whose slope jumps give the *relative frequency* of zeros (eqs. 22–23). This is the almost-periodic Ronkin function [119].

Missing: the positivity $D \bullet D \leq 0$ on the square. The Jessen/Ronkin level is phase-averaged, so it sees densities only. A density-zero set of off-line zeros is invisible to it, exactly as in §23.2, where the Ronkin function is exact only because it is evaluated on full circles. The hyperring of adèle classes $\mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times$ [29] keeps the multiplicative structure with a hyper-addition. It is the most promising phase-retaining model, but no positivity theorem on it is known to us.

Tropical Hodge theory.

- AHK [1] prove Hodge–Riemann relations for matroids. These are combinatorial *positivity* theorems, but in a setting where the Kähler package comes from a fan, with no Frobenius.
- Amini–Piquerez [2, Thm. 1.5]: on a smooth projective tropical surface, $H_{\text{trop}}^{1,1}$ has signature $(1 + b_2, h^{1,1} - 1 - b_2)$. By IKMZ [75, Cor. 2], for tropical limits $b_2 = \dim H_{0,2} = h^{0,2} = p_g$.
- Babae–Huh [4] build a tropical surface in $\mathbb{R}^4$ whose intersection form violates the Hodge-index signature, and Shaw notes that tropical $H_{1,1}$ need not satisfy Hodge index [134].

So tropical $H^{1,1}$ absorbs p_g extra positive directions, those that classically live in $H^{2,0} \oplus H^{0,2}$, i.e. in the phases. [ANALOGY] This parallels Lemma 15's $\sup n_+ = 1 + N_+$: tropicalization can create timelike directions precisely where phase was discarded. For a tropical $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ one would need $b_2 = 0$ and a realization theorem. Neither is available.

$\mathbb{F}_1$.

- Soulé [140] records Manin's question about the fibre product and works with varieties of finite type, which $\text{Spec } \mathbb{Z}$ is not.
- Deitmar [40]: $\mathbb{F}_1$ -rings are monoids and $A \otimes_{\mathbb{F}_1} \mathbb{Z} = \mathbb{Z}[A]$.
- Borger [15]: $\mathbb{F}_1$ -algebras are Λ -rings, $\mathbb{Z}$ is initial, and colimits of Λ -rings have underlying rings as in Ring. Hence [DEDUCTION] $\text{Spec } \mathbb{Z} \times_{\mathbb{F}_1} \text{Spec } \mathbb{Z} = \text{Spec } \mathbb{Z}$ in Borger's theory: *the surface collapses to the diagonal*. In Deitmar's theory the only model of $\mathbb{Z}$ is $\mathbb{F}_1$ itself, with the same collapse. Borger states that his theory is silent at the archimedean place.
- Lorscheid's blueprints [96] interpolate between monoids and rings but do not supply a surface with an intersection form.

The nondegenerate candidate remains the Connes–Consani square.

Periods. Riemann's bilinear relations, $P\Delta^{-1t}P = 0$ and $-iP\Delta^{-1t}P > 0$ [19], come from $-i \int \overline{df} \wedge df > 0$. This uses the complex structure, i.e. a phase. On abelian varieties it becomes positivity of the Rosati trace form, the Rosati involution being adjunction for the Hermitian form $c_1(L)$ [10, Ch. 5], [102, Thm. I.14.3]. Deninger [43] conjectures a Hodge $*$ on H_{dyn}^1 with $\text{tr}(f \cup *f')$ positive and the generator Θ commuting with $*$. That would force $\Theta = \frac{1}{2} + A$, A skew, which is RH. The Betti–de Rham comparison is thus exactly where positivity enters, and it enters through a phase (Hodge $*$, i). Kontsevich–Zagier periods add nothing here: they concern which numbers are periods, not positivity.

23.4. Conclusions, ranking, questions

Verdict (B). Plain tropicalization of $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ *cannot* yield Hodge index. Its finite positivity is Yu's 2×2 cone, i.e. single shadows (Prop. 23.2.2). It misreports real curves (§23.2), and tropical surfaces already violate or shift Hodge index by p_g . What survives is the *signature*, a sign-level datum, and the *norm* as an infinite limit (Thm. 23.2.1).

New for TAA:

- (1) the dictionary of Prop. 23.1.1–23.1.2: Lemma 10 is the triangle-hyperfield bound and is sharp;
- (2) Thm. 23.2.1 as a Maslov limit, with duality;
- (3) Yu's theorem identifying tropical PSD with Lemma 19 single shadows, and the $\mathcal{TC}$ extension;
- (4) window Grams are Lorentzian quadratics.

Ranked candidates (score = 0.6 sig+0.4 prob, each on $[0, 1]$):

- (1) Positivity on the CC hyperring $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\times}$ in finite- h coordinates. Test: do Lemma 15 Grams remain Lorentzian under the $\mathbb{C}_h$ deformation of the test space? sig .8, prob .15, **0.54**. [RH-EQUIV] if all windows.
- (2) Stability-preserving operators (BH toolbox) acting on window Grams as a route to monotonicity in the window. sig .5, prob .3, **0.42**.
- (3) A tropical $b_2 = 0$ criterion: compute $H_{0,2}$ of a tropical model of the CC square. sig .6, prob .15, **0.42**.
- (4) Phase-tropical Lemma 10 (Diophantine non-alignment, Zhu Rem. 1.6). sig .4, prob .35, **0.38**. Not RH-equivalent; it only improves T_1 .
- (5) $\mathbb{F}_1$ -products (Borger/Deitmar). sig .2, prob .05, **0.14**: they collapse.

QUESTION 23.4.1. (1) Should “Lemma 23” be read as closing the tropical route, keeping only Lorentzian/stability as a toolbox? (2) Is it acceptable to replace J.W.’s “Hodge index on all windows” by “window Grams are real-stable quadratics”, which is an equivalent statement in BH language? (3) Do you want item 1 (finite- $\hbar$ deformation) run numerically next?

CHAPTER 24

Lemma 24: Lemma 21 follow-ups

Corrected counting constants, a certified padded frame bound for zeta zeros, and why the hybrid certificate was not run

Standalone note `lemma24.tex`, October 21, 2026.

Summary. Verdict.

- (1) *Corrected constants, done.* Lemmas 12 and 18 used Trudgian’s 2014 counting bound, which Hasanalizade–Shen–Wong report to be erroneous. With a corrected, HSW-based bound the Lemma 12 thresholds *improve* at low height and *worsen slightly* at high height. All recomputed rows are Arb-certified. The headline Lemma 12 value is unchanged in substance: $A_0(\frac{1}{2}, 3 \cdot 10^{12}) = 12.29 \rightarrow 12.32$. Lemma 18’s Theorem 5.1 survives with more room: the tail lower bound rises from 0.0564 to 0.0717 against the target 0.0462. Lemma 18’s detection tables do not use the counting bound at all.
- (2) *Frame theorem, partly done.* The zero data are rigorous: Platt’s LMFDB zeros to height 446000, decoded exactly. On these zeros I certify in Arb a finite, padded lower frame bound $L \geq 0.60, 0.58, 0.60, 0.55$ at $A = 2.4, 2.6, 2.8, 3.0$. I do *not* prove the requested analytic Kadec/Avdonin-type theorem for all of PW_ω : one approximation lemma is left open (Lemma 24.2.4). I also find that a frame bound with *sharp* band edges collapses to about 10^{-4} or less. Padding is essential, and Lemma 21’s “sharp” cost column implicitly assumed it.
- (3) *Hybrid certificate, not run.* The requested condition, that (2) succeed, holds only partially. Separately, the existing (Legendre) certificate code projects to about $2 \cdot 10^8$ hours on this machine at $A = 2.4$, far beyond the 24-hour stop rule. A Toeplitz/Levinson rewrite would project to about 0.8–9 hours for $A = 2.4$ –3.0 [EST], but that code does not exist.¹

Labels as in Lemmas 17–21: [THM]; [THM,CA] computer-assisted proof (Arb ball arithmetic on verified data); [CITED]; [NUM] floating-point; [EST]; [RH-EQUIV]. Nothing here is RH-equivalent; RH below H is a theorem [124]. Files: `num/lemma24_*`, `num/platt/`, `lit/lemma24/verified_refs.md`, checkpoint `num/lemma24_progress.md`.

24.1. (1) Corrected zero-counting bound

Lemmas 12 and 18 used $|N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e}| \leq 0.112 \log T + 0.278 \log \log T + 3.385$. These are Trudgian’s 2014 constants [151], which [64], footnote 2, reports to be in error (“the ranges of parameters ... were not verified properly”).

LEMMA 24.1.1 (corrected bound). [THM] *For $T > 0$ let $N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} = \frac{7}{8} + S(T) + E(T)/\pi$. Then $|E(T)| \leq \frac{3}{16T} + \frac{1}{3T} < \frac{0.53}{T}$. Consequently $|N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e}| \leq R_{\text{cor}}(T)$, where*

$$R_{\text{cor}}(T) = \begin{cases} 3.3917 + 0.17/T, & 0 < T \leq T_B = 30\,610\,046\,000, \\ 3.9055 + 0.17/T + 0.1095 \log T + 0.2042 \log \log T, & T > T_B. \end{cases}$$

¹[Vol. 2 editorial note] Superseded: a Toeplitz/Levinson certificate pipeline was written after this note. Its logged results (write-up pending) are listed in Appendix B; they are not claimed as theorems of this volume.

PROOF. $N = \theta/\pi + 1 + S$ and $\theta(t) = \operatorname{Im} \log \Gamma(\frac{1}{4} + \frac{it}{2}) - \frac{t}{2} \log \pi$. Put $z = \frac{1}{4} + \frac{it}{2}$. Stirling's formula with no correction terms has remainder $|R_1(z)| \leq \sec^2(\frac{1}{2} \operatorname{ph} z)/(12|z|) \leq 1/(3t)$ (DLMF 5.11(ii) [113]), since $\operatorname{ph} z < \pi/2$ and $|z| \geq t/2$. Exactly,

$$\operatorname{Im}[(z - \frac{1}{2}) \log z - z] = \frac{t}{2} \log \frac{t}{2} - \frac{t}{2} - \frac{\pi}{8} + \frac{t}{4} \log(1 + \frac{1}{4t^2}) + \frac{1}{4} \arctan \frac{1}{2t}.$$

The last two terms lie in $[0, \frac{3}{16t}]$. For $|S|$, use $|S| \leq 2.5167$ on $[0, T_B]$ ([64], (1.7), from Platt's database) and [64], Cor. 1.4, above T_B . Of the two forms in Cor. 1.4, the second is the smaller one for $\log T < 936$. $\square$

At $T = 10^3$, $R_{\text{cor}} = 3.392$ against the old 4.696. At $3 \cdot 10^{12}$, $R_{\text{cor}} = 7.737$ against 7.536. At 10^{20} , 9.730 against 9.607. So the correction helps below T_B , where Platt's database applies, and hurts slightly above it. HSW's global Cor. 1.2 (constant term 9.3675) would be worse everywhere in this range, so it is not used.

Lemma 12 rerun [THM,CA]. `num/lemma24_112A0.py` is `lemma12_A0.py` with R replaced by R_{cor} . The acb integrands are split where an argument crosses T_B , so each piece stays analytic. Every row goes through the same search and then 300-bit Arb re-verification (`lemma24_112verify.py`). Table 24.1 gives each certified A_0 with its change from Lemma 12's certified value; "pend." rows are still running at the time of writing (resumable, `num/lemma24_112_A0.jsonl`).

TABLE 24.1. Certified $A_0(\delta, \gamma_0)$ with R_{cor} ; in parentheses, the change from Lemma 12 (Trudgian 2014). Source `num/lemma24_112_certified.json`, `lemma24_112_deltas.json`.

$\gamma_0 \setminus \delta$	0.5	0.25	0.1	0.05	0.01
γ_1 glob.	9.14 (-0.22)	16.36 (-0.44)	37.94 (-1.12)	73.88 (-2.26)	358.92 (-11.88)
γ_1 loca.	9.68 (-0.20)	17.16 (-0.58)	40.18 (-1.20)	78.36 (-2.40)	381.52 (-12.50)
10^3 glob.	9.92 (-0.36)	17.92 (-0.70)	41.64 (-1.96)	79.34 (-4.34)	371.54 (-24.82)
10^3 loca.	10.40 (-0.32)	18.90 (-0.74)	43.96 (-2.20)	83.96 (-4.62)	395.14 (-26.74)
$3 \cdot 10^{12}$ glob.	12.32 (+0.02)	22.72 (+0.04)	52.10 (+0.14)	98.96 (+0.32)	456.02 (+2.00)
$3 \cdot 10^{12}$ loca.	12.84 (+0.10)	23.78 (+0.04)	54.54 (+0.14)	103.70 (+0.34)	pend.
10^{20} glob.	13.12 (+0.02)	24.32 (+0.02)	55.60 (+0.06)	105.14 (+0.16)	482.64 (+0.94)
10^{20} loca.	13.56 (+0.00)	25.22 (+0.00)	58.24 (+0.06)	110.10 (+0.16)	pend.
10^{100} glob.	16.12 (+0.00)	pend.	pend.	pend.	pend.
10^{100} loca.	pend.	pend.	pend.	pend.	pend.

The pattern is monotone. Below T_B (at γ_1 and 10^3) A_0 drops by 0.2–26.7; the effect is largest at small δ , where tails accumulate many R -terms. Above T_B it rises by 0.00–2.00. Lemma 12's conclusion, that detection lengths are ≥ 9 and far above every certified window, is unchanged.

Lemma 18. The A_σ^{det} tables are floating-point numerics on actual zeros and contain no R , so they are unchanged. The only R -dependent step is Thm. 5.1(c), the tail $g(x) = \frac{1}{2}[M(x+H) - M(x-H) - R(x+H) - R(x-H)]/(1+H^2)$ with $X = 5000$, $H = 10$. With R_{cor} , $g(X) = 0.07167$ (Lemma 18 had 0.05640). g is increasing on each regime; at the jump at T_B the conservative value is $g \geq 0.2802$. Both exceed the target 0.0462, so Theorem 5.1 stands [THM,CA] (`lemma24_118tail.json`).

24.2. (2) Sampling the zeros: data, a certified padded frame bound, and what is missing

Data [CITED]+[THM,CA]. LMFDB distributes Platt's zeros [123, 148], isolated to $\pm 2^{-102}$ with completeness checked by a rigorous Turing method. I downloaded `zeros_{14,5000,26000,236000}.dat`; their md5 sums match LMFDB's `md5.txt`. The files

were decoded exactly ($\gamma = t_0 + Z \cdot 2^{-101}$, rational arithmetic), giving all 721 913 zeros with $0 < \gamma \leq 446\,000$. Block headers $N(t_0)$ are contiguous, and the first 10^5 agree with LMFDB's served list to 0.0. Odlyzko's table differs from them by at most $3.0 \cdot 10^{-9}$ and is used for nothing below.

Setting. Let $\omega = (A + \varepsilon)/2$ with $\varepsilon = 0.1$. Fix a band $I = [T_1, T_2]$ of length 300 at the Lemma 21 “sharp” heights $T_1 = 5900, 7400, 9300, 11500$ for $A = 2.4, 2.6, 2.8, 3.0$, and a padded band $I_p = [T_1 - p, T_2 + p]$. The model space V_M consists of the sums $f = \sum_j c_j e^{i\xi_j t}$ with ξ_j on a uniform grid of $M \approx 1.5 |I| \omega / \pi$ points in $[-\omega, \omega]$. Write $G = V^* V$ with $V_{kj} = e^{i\xi_j \gamma_k}$ ($\gamma_k \in I_p$) for the sampling form, and $B_{jl} = \int_I e^{i(\xi_l - \xi_j)t} dt$ for the mass form.

PROPOSITION 24.2.1 (edge collapse; padding). [NUM] *The minimal generalised eigenvalue L of (G, B) on the well-conditioned range of B behaves as follows (verified zeros, float64):*

A ($2\pi D(T_1)$)	$p = 0$	$p = 5$	$p = 10$	$p = 40$
2.4 (6.84)	$1.6 \cdot 10^{-4}$	0.642	0.660	0.668
2.6 (7.07)	$4.2 \cdot 10^{-5}$	0.632	0.636	0.640
2.8 (7.30)	$4.7 \cdot 10^{-3}$	0.643	0.654	0.662
3.0 (7.51)	$2.6 \cdot 10^{-5}$	0.595	0.603	0.608

With padding $p \geq 5$, $L/D \approx 0.50$ – 0.61 , stable in p . Without padding L collapses, and the collapse is erratic in band length (Remark 24.2.5).

THEOREM 24.2.2 (certified padded frame bound). [THM, CA]² *Let $Q \in \mathbb{C}^{M \times r}$ be the fixed float64 matrix spanning the range of B with eigenvalues at least $10^{-10} \max \text{spec } B$; its entries are exact binary rationals. Let $W = \{\sum_j c_j e^{i\xi_j t} : c \in Q\mathbb{C}^r\}$, and take $p = 10$. Then for every $f \in W$,*

$$\sum_{\gamma \in I_p} |f(\gamma)|^2 \geq L \int_I |f|^2, \quad L = \begin{cases} 0.60 & A = 2.4 \text{ } (K = 349, \text{ } M = 187, \text{ } r = 133), \\ 0.58 & A = 2.6 \text{ } (361, 201, 143), \\ 0.60 & A = 2.8 \text{ } (372, 215, 152), \\ 0.55 & A = 3.0 \text{ } (383, 230, 162). \end{cases}$$

PROOF (COMPUTER-ASSISTED). Each zero is entered as an Arb ball of radius 2^{-100} , and ξ_j and Q exactly. G and B (the latter in closed form) are computed in 128-bit ball arithmetic, and so is $S = Q^*(G - LB)Q$. An LDL^T factorisation of the real $2r \times 2r$ form of the Hermitian S has every pivot provably > 0 ; the minimal pivot lower bounds are 6.41, 10.6, 27.3, 25.6. Hence $S \succeq 0$. (num/lemma24_frame_arb.py, .json1; under 5 s each.) $\square$

Controls (same code, $A = 2.4$). $L = 0.70$ fails, as it must, since it lies above the float minimum 0.660. $L = 0.65$ passes. With $p = 0$, $L = 0.60$ fails after 13 pivots. So the certificate is sharp: the true minimum lies in $[0.65, 0.70)$.

PROPOSITION 24.2.3 (what the hybrid actually needs). [THM] *In a Galerkin certificate for Lemma 21, Thm. 7.1, with a finite window basis $\{b_i\}_{i \leq N}$, the middle-band term restricted to the basis is the quadratic form $\sum_{0 < |\gamma| \leq H} |m(\gamma)|^2 |\sum_i a_i \widehat{b_i}(\gamma)|^2$. It is PSD and can be computed exactly from verified zeros, so no frame theorem is needed for it. A lower frame bound enters only through the truncation tail $F_\perp = F - P_N F$: the mid band must be coercive, $\sum |m\widehat{F}_\perp(\gamma)|^2 \geq L \int |m\widehat{F}_\perp|^2 - (\text{tail})$, uniformly over the infinite-dimensional complement.*

²[Vol. 2 editorial note] The Arb enclosure first used for this theorem entered float64-rounded zeros as balls of radius 2^{-100} that need not contain them. It was re-run with exact rationals in Chapter 24A, §24A.6; the theorem stands unchanged.

LEMMA 24.2.4 (open: model-space approximation). (Needed to turn Theorem 24.2.2 into a statement about all of PW_ω .) *There are explicit $\eta(M, r, p) \rightarrow 0$ with the following property. For every $f \in \text{PW}_\omega$ there is $w \in W$ such that both $\sum_{\gamma \in I_p} |f - w|^2(\gamma)$ and $\int_I |f - w|^2$ are $\leq \eta \|f\|_{L^2(I_{p'})}^2$. Then $\sum_{I_p} |f|^2 \geq (\sqrt{L} - c\sqrt{\eta})^2 \int_I |f|^2 - \dots$*

This is a prolate-truncation statement. It needs explicit bounds on the prolate eigenvalues beyond the Shannon number $|I|\omega/\pi$ (Landau–Widom asymptotics [89] are not explicit; Bonami–Karoui-type bounds are) and an aliasing estimate for the ξ -grid. **I have not proved it.** Without it, Theorem 24.2.2 is rigorous for W only.

REMARK 24.2.5 (why not Avdonin). The oversampling ratio here is $2\pi D(T_1)/(A + \varepsilon) = 2.74, 2.62, 2.52, 2.42$. That is far from critical density, which is where Kadec/Avdonin “ $\frac{1}{4}$ in the mean” conditions bite. Kadec’s sup-norm condition fails anyway, since normalised gaps reach 2.8 (Lemma 21). The binding difficulties are edges and approximation, not density. On edges: PW_ω functions can put mass near ∂I while the samples that control them lie outside I . That is the $p = 0$ collapse, and it is why the same band gives $L = 0.0038$ at length 400 but 0.054 at length 800 (lemma24_frame_float.json). *Correction to Lemma 21:* its “sharp” column used interior (padded) Gram constants, which is right, but the mid-band partition m must then overlap the padding. This costs a few units of T and does not change any order of magnitude.

24.3. (3) Hybrid certificate: projection and stop

The requested condition, that (2) succeed, holds only partially: Lemma 24.2.4 is open. Independently, here is the cost projection, using Lemma 21’s cost model (fit to the measured Lemma 16 runs: $N^{5.44}$ scaling, λ^* fit) at the Lemma 21 sharp T_{low} :

A	T_{low}	N_{leg}	existing Legendre code	Toeplitz/Levinson (not implemented)	bits
2.4	5910	9716	$2.0 \cdot 10^8$ h	0.76 h	272
2.6	7420	13176	$1.3 \cdot 10^9$ h	1.7 h	311
2.8	9260	17685	$8.0 \cdot 10^9$ h	3.9 h	358
3.0	11500	23534	$4.8 \cdot 10^{10}$ h	8.8 h	416

Hours are wall-clock on this 8-core machine, [EST], from num/lemma24_proj.json. With the code that exists, the first step ($A = 2.4$) exceeds the 24-hour limit by a factor of about 10^7 , so **stop**. The Toeplitz column is an estimate for code that has not been written. It would also need (i) a rigorous truncation coupling for the spline basis and (ii) Lemma 24.2.4. The certificate is therefore not attempted, and no certified window beyond 1.6 is claimed.

24.4. Status, obstruction, questions

Item	status	value
Lemma 24.1.1 (corrected R)	[THM]	3.392 below T_B
L12 A_0 rerun	[THM,CA] (rows certified)	$\Delta A_0 \in [-26.7, +2.0]$; headline +0.02
L18 Thm 5.1 tail	[THM,CA]	$g(X) = 0.0717 > 0.0462$
Platt data decode	[CITED]+[THM,CA]	721 913 zeros to 446 000
Thm 24.2.2 (padded frame on W)	[THM,CA]	$L \geq 0.55$ –0.60
Prop. 24.2.1 (edge collapse)	[NUM]	$L \lesssim 5 \cdot 10^{-3}$ at $p = 0$
Lemma 24.2.4 (all of PW_ω)	open	—
Hybrid certificate	not run (stop rule)	—

Obstruction. The sampling side is in good shape: real zeros, certified constants of order one, no density problem. Two things block a rigorous hybrid: an explicit prolate-truncation lemma (analysis) and a structured, high-precision Galerkin solver (engineering).

QUESTION 24.4.1. Shall I (a) prove Lemma 24.2.4 via explicit prolate eigenvalue bounds, or (b) write the Toeplitz/Levinson certificate code (projected < 1 h at $A = 2.4$)? (b) is the larger lever, but without (a) its tail coupling must use the crude Gröchenig constant, $L \approx 0.004$ at $A = 2.4$ (Lemma 21).

QUESTION 24.4.2. Should Lemma 12's table (or Vol. 1) adopt R_{cor} ? The low-height rows improve; the $3 \cdot 10^{12}$ headline changes by $+0.02$.

CHAPTER 24A

Lemma 24, supplement: Lemma 3.4 proved

An explicit local sampling inequality on Platt's zeros for all of PW_ω

Standalone note `lemma24_L34.tex`, October 21, 2026.

Summary. Verdict. Lemma 3.4 of Lemma 24 is *proved*, in the only form in which it can be true. A lower bound $\sum_{\gamma \in I_p} |f(\gamma)|^2 \geq L \int_I |f|^2$ with no outside-mass term is false for every finite sampling set (Proposition 24A.1.1). The correct statement carries an explicitly small penalty for the mass of f outside I . Theorem 24A.5.1 [THM,CA]: for every $f \in \text{PW}_\omega$, $\omega = (A + 0.1)/2$, with Platt's zeros in I_p (padding $p = 10$),

$$\left(\sum_{\gamma \in I_p} |f(\gamma)|^2 \right)^{1/2} \geq \sqrt{L} \|f\|_{L^2(I)} - C_{\text{out}} \|f\|_{L^2(\mathbb{R} \setminus I)},$$

$$L = 0.630, 0.609, 0.610, 0.567 \quad (A = 2.4, 2.6, 2.8, 3.0), \quad C_{\text{out}} = 0.0029.$$

With $p = 5$ the values are 0.623, 0.608, 0.607, 0.568. Trading a larger penalty ($C_{\text{out}} = 0.09$) gives 0.648, 0.617, 0.625, 0.573. The proof does not use prolate eigenvalue bounds at all. It uses an explicit Kaiser–Bessel reproducing kernel and Poisson periodisation, which reduce everything *exactly* to a finite trigonometric model space, plus an Arb certificate on that whole space. All error constants are below 10^{-5} .

Gap: the result is per block. Assembling many blocks into the hybrid needs a separate step, described in §24A.7. **Side finding:** the Arb enclosure used for Lemma 24, Theorem 3.2, is invalid as written (float64-rounded zeros with a 2^{-100} radius). Re-run with exact rationals, the theorem stands unchanged (§24A.6).

Labels as in Lemma 24: [THM]; [THM,CA] computer-assisted proof (Arb ball arithmetic on verified data); [CITED]; [NUM]. Nothing here is RH-equivalent. The inequality is about the *ordinates* as a point set, and RH is not used. Files: `num/lemma24_L34_*` (cert, consts, thm32recheck, sanity), checkpoint `num/lemma24_L34_progress.md`, references `lit/lemma24/verified_refs.md`. `lemma24.tex` is not edited; this file is a separate supplement.

24A.1. Setting, and why the statement must carry a penalty

PW_ω is the space of $f \in L^2(\mathbb{R})$ with $\hat{f}(\xi) = \int f(t) e^{-i\xi t} dt$ supported in $[-\omega, \omega]$. As in Lemma 24, §2, take $\omega = (A + \varepsilon)/2$ with $\varepsilon = 0.1$. This is the class of $m\hat{F}$ in Lemma 21, §6. The band is $I = [T_1, T_1 + 300]$ with $T_1 = 5900, 7400, 9300, 11500$ for $A = 2.4, 2.6, 2.8, 3.0$, and the padded band is $I_p = [T_1 - p, T_1 + 300 + p]$. Γ_p is the set of Platt's zeros in I_p , each the exact stored rational within 2^{-102} of the true ordinate [148, 123]; write $S(f) = \sum_{\gamma \in \Gamma_p} |f(\gamma)|^2$. Membership is unambiguous, since no stored zero lies within $5 \cdot 10^{-3}$ of an end of I_p . Completeness of Γ_p is not needed: dropping points only lowers S . Shift time by $T_c = T_1 + 150$, so that $I = [-a, a]$ with $a = 150$, $I_p = [-b, b]$ with $b = a + p$, and $\|\cdot\|_E = \|\cdot\|_{L^2(E)}$.

PROPOSITION 24A.1.1 (no penalty-free version). [THM] *For any finite $\Gamma \subset \mathbb{R}$, any interval I and any $\omega > 0$, there is $f \in \text{PW}_\omega$ with $f|_\Gamma = 0$ and $\|f\|_I > 0$.*

PROOF. The evaluation map $\text{PW}_\omega \rightarrow \mathbb{C}^\Gamma$ has a kernel of finite codimension, so the kernel is nonzero. A nonzero $f \in \text{PW}_\omega$ is entire, so it cannot vanish on I . □

So Lemma 24's Lemma 3.4 must be read as a statement with an outside-mass term, as its trailing “ $\dots$ ” indicated. A float test confirms this: within a span of sines, $\min S/\|f\|_I^2$ is 0 to working precision (`lemma24_L34_sanity.py`). The useful question is how *small* the penalty can be made, and the answer below is $C_{\text{out}} \approx 3 \cdot 10^{-3}$.

24A.2. An explicit reproducing kernel

Fix $\tau = 0.05$, $\sigma = \tau/2$, $\beta = 14$, $\Omega' = \omega + \tau/2$ and $\Omega = \omega + \tau$. Let $\rho(\xi) = Z^{-1}I_0(\beta\sqrt{1 - (\xi/\sigma)^2})$ on $|\xi| \leq \sigma$ and 0 elsewhere, with $Z = 2\sigma \sinh \beta/\beta$. Define $\hat{k} = \mathbf{1}_{[-\Omega', \Omega']} * \rho$, which is the Kaiser–Bessel (“ I_0 –sinh”) smoothing of an indicator [79]. Then

$$0 \leq \hat{k} \leq 1, \quad \hat{k} = 1 \text{ on } [-\omega, \omega], \quad \text{supp } \hat{k} = [-\Omega, \Omega], \quad \hat{k}(\pm\Omega) = 0, \quad k(x) = \frac{\sin \Omega' x}{\pi x} \check{\rho}(x),$$

where $\check{\rho}(x) = \int \rho(\xi) e^{i\xi x} d\xi$.

LEMMA 24A.2.1 (transform pair). [THM] + [CITED]

$$\check{\rho}(x) = \frac{\beta}{\sinh \beta} \cdot \frac{\sin \sqrt{\sigma^2 x^2 - \beta^2}}{\sqrt{\sigma^2 x^2 - \beta^2}}$$

(read as $\sinh \sqrt{\beta^2 - \sigma^2 x^2} / \sqrt{\beta^2 - \sigma^2 x^2}$ when $\sigma|x| < \beta$). Hence for $y > \beta/\sigma$,

$$k^*(y) := \sup_{|x| \geq y} |k(x)| \leq \frac{\beta / \sinh \beta}{\pi y \sqrt{\sigma^2 y^2 - \beta^2}}, \quad \kappa_1(y) := \int_{|x| \geq y} |k| \leq \frac{2}{\pi \sinh \beta} \arcsin \frac{\beta}{\sigma y}.$$

PROOF. It suffices to show

$$\int_{-1}^1 I_0(\beta \sqrt{1 - u^2}) \cos(yu) du = \frac{2 \sin \sqrt{y^2 - \beta^2}}{\sqrt{y^2 - \beta^2}}.$$

Expand I_0 by DLMF 10.25.2. By DLMF 10.9.4 with $\nu = k + \frac{1}{2}$,

$$\int_{-1}^1 (1 - u^2)^k \cos(yu) du = \sqrt{\pi} k! J_{k+1/2}(y) / (y/2)^{k+1/2}.$$

Integrating termwise (the convergence is uniform) gives

$$\sqrt{\pi} (2/y)^{1/2} \sum_{k \geq 0} \frac{(\beta^2/2y)^k}{k!} J_{k+1/2}(y).$$

The multiplication theorem DLMF 10.23.1 (for J with the upper signs there is no restriction on λ), with $\lambda^2 = 1 - \beta^2/y^2$, sums this to $\lambda^{-1/2} J_{1/2}(\lambda y)$. By DLMF 10.16.1 the whole expression equals $2 \sin(\lambda y)/\lambda$, which is even in λ , so the choice of branch does not matter. Both envelope factors decrease in $|x|$, and $|\sin| \leq 1$. Finally $\int_y^\infty dx / (x \sqrt{\sigma^2 x^2 - \beta^2}) = \beta^{-1} \arcsin(\beta/\sigma y)$. A 40-digit quadrature check of the identity agrees to $7.5 \cdot 10^{-16}$. $\square$

24A.3. Exact reduction to a finite model space

Let $D = 600$, $c = b + D$, $J = [-c, c]$ and $P = b + c + D$, so $P = 1520$ for $p = 10$ and 1510 for $p = 5$. Put $\Delta = 2\pi/P$, $V = \text{span}\{e^{ij\Delta t} : |j\Delta| < \Omega\}$ with $M = \dim V$, $U = [b, P - b]$, and $N = [-(P - b), P - b]$.

THEOREM 24A.3.1 (reduction). [THM] Suppose that for all $w \in V$,

$$(24A.3.1) \quad S(w) + \mu \|w\|_U^2 \geq L \|w\|_I^2.$$

Then every $f \in \text{PW}_\omega$ satisfies

$$S(f)^{1/2} \geq (\sqrt{L} - \eta_J) \|f\|_I - \sqrt{2\mu} \|f\|_{N \setminus I_p} - \eta_J \|f\|_{J \setminus I} - (\sqrt{2\mu} + \eta_{\text{far}}) \|f\|_{\mathbb{R} \setminus J},$$

where $\eta_J = \sqrt{\mu} \varepsilon_U + (\sqrt{2aL} + \sqrt{K}) \varepsilon_{\text{al}}$, $\eta_{\text{far}} = \sqrt{L} \kappa_1(D) + \sqrt{\kappa_1(D) K k^*(D)}$, $K = \#\Gamma_p$, and, with $s_0 = \sqrt{1 - (\beta/\sigma D)^2}$ and $c_0 = \frac{\beta/\sinh \beta}{\pi \sigma s_0}$,

$$\varepsilon_{\text{al}} = 2\sqrt{2c} c_0 \left(\frac{1}{D^2} + \frac{\pi^2}{6P^2} \right), \quad \varepsilon_U = 2\sqrt{(P-2b)2c} c_0 \left(\frac{1}{(2b+D)^2} + \frac{\pi^2}{6P^2} \right).$$

PROOF. (i) Since $\hat{k} = 1$ on $\text{supp } \hat{f}$, $f = f * k$ pointwise. Put $h = (f \mathbf{1}_J) * k$ and $e_{\text{far}} = (f \mathbf{1}_{\mathbb{R} \setminus J}) * k$, so that $f = h + e_{\text{far}}$. (ii) *Periodisation.* $h \in L^1$ is continuous with $|h(x)| \leq \|f\|_J \sqrt{2c} k^*(\text{dist}(x, J)) = O(x^{-2})$ (Cauchy–Schwarz). So $w(t) := \sum_m h(t + mP)$ converges uniformly, and its Fourier coefficients are $\hat{h}(j\Delta)/P$. Here $\hat{h} = f \mathbf{1}_J \hat{k}$ vanishes for $|\xi| \geq \Omega$. Hence w is a trigonometric polynomial in V ; this is the Poisson summation step. (iii) *Errors on $[-b, b]$.* $f - w = e_{\text{far}} - e_{\text{al}}$ with $e_{\text{al}}(t) = \sum_{m \neq 0} h(t + mP)$. For $|t| \leq b$ the distance from $t + mP$ to J is at least $D + (|m| - 1)P$. Since $k^*(y) \leq c_0/y^2$ for $y \geq D$, this gives $|e_{\text{al}}(t)| \leq \varepsilon_{\text{al}} \|f\|_J$. By Cauchy–Schwarz, $|e_{\text{far}}(t)|^2 \leq \kappa_1(D) \int_{\mathbb{R} \setminus J} |f(u)|^2 |k(t - u)| du$. Therefore $S(e_{\text{far}}) \leq \kappa_1 K k^*(D) \|f\|_{\mathbb{R} \setminus J}^2$ and $\|e_{\text{far}}\|_I \leq \kappa_1 \|f\|_{\mathbb{R} \setminus J}$. (iv) *The penalty term.* On U , $w = h + h(\cdot - P) + \sum_{m \neq 0, -1} h(\cdot + mP)$. The two leading terms live on the disjoint arcs $[b, P - b]$ and $[b - P, -b]$ of $N \setminus I_p$, and $h = f - e_{\text{far}}$ with $\|e_{\text{far}}\|_{L^2(\mathbb{R})} \leq \|f\|_{\mathbb{R} \setminus J}$ because $|\hat{k}| \leq 1$. The remaining terms are at distance at least $jP + 2b + D$ from J . So $\|w\|_U \leq \sqrt{2}(\|f\|_{N \setminus I_p} + \|f\|_{\mathbb{R} \setminus J}) + \varepsilon_U \|f\|_J$. (v) Combine $S(f)^{1/2} \geq S(w)^{1/2} - S(f - w)^{1/2}$, $S(w)^{1/2} \geq \sqrt{L} \|w\|_I - \sqrt{\mu} \|w\|_U$ (from (24A.3.1)), $\|w\|_I \geq \|f\|_I - \|f - w\|_I$ and $\|f\|_J \leq \|f\|_I + \|f\|_{J \setminus I}$. $\square$

24A.4. The finite certificate on all of V

THEOREM 24A.4.1 (certified (24A.3.1)). [THM,CA] *Inequality (24A.3.1) holds on all of V for the following values. The Platt balls are exact rationals of radius 2^{-100} ; there is no subspace restriction, unlike Lemma 24, Thm. 3.2.*

A	$p = 10$						$p = 5$		
	K	M	$\mu=10^{-6}$	10^{-5}	10^{-4}	10^{-3}	K	M	$\mu=10^{-6}$
2.4	349	629	0.631	0.638	0.646	0.649	339	625	0.624
2.6	361	677	0.610	0.613	0.616	0.618	349	673	0.609
2.8	372	725	0.611	0.616	0.621	0.626	361	721	0.608
3.0	383	775	0.568	0.571	0.573	0.574	371	769	0.569

PROOF (COMPUTER-ASSISTED). $S(w) = c^* G c$, $\|w\|_I^2 = c^* B_a c$ and $\|w\|_U^2 = c^*(PI - B_b)c$, where $(B_h)_{jl} = 2 \sin((l - j)\Delta h)/((l - j)\Delta)$. The Hermitian matrix $\mathcal{S} = G + \mu(PI - B_b) - LB_a$ is computed in 128-bit Arb. A float64 Cholesky factor R of $\mathcal{S}_{\text{float}} - sI$ is taken, with $s = \frac{1}{2} \lambda_{\min}(\mathcal{S}_{\text{float}})$, and its entries are used as exact binary rationals. Arb then bounds $\|\mathcal{S} - sI - R^* R\|_F \leq 10^{-8}$ in every case, while $s \geq 4.8 \cdot 10^{-4}$. So $\mathcal{S} \succeq (s - \|\cdot\|_F)I \succ 0$. Each case takes 5–10 s (lemma24_L34_cert.py, .jsonl). The table entry is $\lfloor 1000(L_{\text{float}}^* - 0.002) \rfloor / 1000$. $\square$

Controls. $L = 0.66$ at $\mu = 10^{-3}$ and $L = 0.70$ at $\mu = 10^{-6}$ (both above the float optimum) are rejected. With no padding ($p = 0$) the float optimum collapses to $L^* = 0.072$ even with $\mu = 10^{-3}$, which reproduces Lemma 24’s edge collapse.

24A.5. Main result

For $\beta = 14$, $\tau = 0.05$ and $D = 600$ (so $\sigma D/\beta = 1.07$), Arb gives the following bounds, uniformly over all rows:

$$k^*(D) \leq 2.3 \cdot 10^{-9}, \quad \kappa_1 \leq 1.3 \cdot 10^{-6}, \quad \varepsilon_{\text{al}} \leq 2.3 \cdot 10^{-7},$$

$$\varepsilon_U \leq 4.3 \cdot 10^{-6}, \quad \eta_J \leq 7.6 \cdot 10^{-6}, \quad \eta_{\text{far}} \leq 2.1 \cdot 10^{-6}$$

(lemma24_L34_consts.py, .json).

THEOREM 24A.5.1 (Lemma 24, Lemma 3.4, proved form). [THM,CA] For every $f \in \text{PW}_{(A+0.1)/2}$,

$$S(f)^{1/2} \geq \sqrt{L} \|f\|_{L^2(I)} - C_{\text{out}} \|f\|_{L^2(\mathbb{R} \setminus I)}, \quad C_{\text{out}} = 2\sqrt{2\mu} + \eta_J + \eta_{\text{far}},$$

with the values below. The finer form of Theorem 24A.3.1 also holds: its penalty outside $J = [T_c - c, T_c + c]$, $c = p + 750$, has coefficient only $\sqrt{2\mu} + 2 \cdot 10^{-6}$.

L for	$A = 2.4$	2.6	2.8	3.0
$p = 10, \mu = 10^{-6}, C_{\text{out}} = 0.0028$	0.630	0.609	0.610	0.567
$p = 5, \mu = 10^{-6}, C_{\text{out}} = 0.0028$	0.623	0.608	0.607	0.568
$p = 10, \mu = 10^{-4}, C_{\text{out}} = 0.028$	0.645	0.615	0.620	0.572
$p = 10, \mu = 10^{-3}, C_{\text{out}} = 0.090$	0.648	0.617	0.625	0.573
L24 Thm. 3.2 (W only)	0.60	0.58	0.60	0.55

Squared form: for any $\theta \in (0, 1)$, $S(f) \geq (1 - \theta)L\|f\|_I^2 - (\theta^{-1} - 1)C_{\text{out}}^2\|f\|_{\mathbb{R} \setminus I}^2$. For example, with $\theta = 0.1$ and $\mu = 10^{-6}$: $S(f) \geq 0.9L\|f\|_I^2 - 7.3 \cdot 10^{-5}\|f\|_{\mathbb{R} \setminus I}^2$.

PROOF. Combine Theorems 24A.3.1 and 24A.4.1. Here $(\sqrt{L_{\text{cert}}} - \eta_J)^2 \geq L_{\text{cert}} - 1.2 \cdot 10^{-5}$, rounded down to three decimals. Each of the three outside norms is at most $\|f\|_{\mathbb{R} \setminus I}$. The squared form follows from $2XY \leq \theta X^2 + Y^2/\theta$, and it is trivial when the right-hand side is negative. $\square$

REMARK 24A.5.2. (a) The values exceed Lemma 24's W -only constants, because V is larger and the penalty μ absorbs the unsampled arc. The extra band $\tau = 0.05$ costs about 0.01 (float: $L^* = 0.660$ at $\tau = 0$ against 0.652). (b) A float test on 210 sinc-span functions, including the adversarial minimisers of $S/\|f\|_I^2$, found $\min\{\text{lhs}/\text{rhs}\} = 1.28$, with no violation [NUM]. (c) Prolate theory (Slepian–Landau–Pollak; Landau–Widom [89]) is not used. The periodisation route needs no eigenvalue bounds, and its errors are explicit kernel tails. No Bonami–Karoui constant is used, so none is cited.

24A.6. Side finding: the enclosure in Lemma 24, Thm. 3.2

lemma24_frame_arb.py enters each zero as `arb(float(g))` with radius 2^{-100} . The stored `zeros_upto.npy` values are float64 roundings, wrong by up to $\frac{1}{2}\text{ulp} \approx 4.5 \cdot 10^{-13}$ at $t \approx 10^4$, so the balls need not contain the zeros. The proof of Thm. 3.2 is therefore incomplete as written. Re-run with exact rationals (lemma24_L34_thm32recheck.py, everything else identical), all four cases certify again at $L = 0.60, 0.58, 0.60, 0.55$, with minimal pivots 6.41, 10.6, 27.3, 25.6.

Thm. 3.2 stands. Lemma 24's float tables (Prop. 3.1) are [NUM] and unaffected.

24A.7. What is proved, and what is still missing

Proved. Lemma 3.4 in a block form, with explicit constants, for all of PW_ω , $p \in \{5, 10\}$, and $A \in \{2.4, 2.6, 2.8, 3.0\}$ at the Lemma 21 sharp heights (Theorem 24A.5.1).

Missing for the three-band hybrid (Lemma 21, Thm. 7.1; Lemma 24, Prop. 3.3).

- (i) *Assembly.* The hybrid needs coercivity on the whole middle band $[T_1, T_2]$, which is far longer than 300. Summing Theorem 24A.5.1 over n blocks gives

$$2S_{\text{mid}} \geq (1 - \theta) \min_j L_j \|f\|_{\text{mid}}^2 - n(\theta^{-1} - 1)C_{\text{out}}^2\|f\|_{\mathbb{R}}^2,$$

since the padded blocks overlap at most twice. The penalty grows linearly in n , because $\|f\|_{\mathbb{R} \setminus I_j}$ is global. With $\mu = 10^{-6}$ and $\theta = 0.1$ this stays below 0.1 only for $n \lesssim 1400$ blocks (length about $4 \cdot 10^5$). A *localised* penalty would remove the n , and the far coefficient is

already $\sqrt{2\mu} + 2 \cdot 10^{-6}$. Doing so needs one further estimate: replacing $\|e_{\text{far}}\|_{L^2(\mathbb{R})}$ in step (iv) by a distance-weighted bound. That estimate is not written.

- (ii) *More blocks.* Only the four Lemma 21 blocks are certified. Each further block is a 5–10 s run, but the zeros are needed: `num/platt/` holds them up to 446 000, and LMFDB has the rest.
- (iii) *The hybrid itself.* The Toeplitz/Levinson low-band code (Lemma 24, §3) is still unwritten.

QUESTION 24A.7.1. Should the next step be the localised-penalty estimate in (i), which makes the assembly n -independent, or the Toeplitz/Levinson low-band certificate? The latter is now the only analytic-free obstacle for $A = 2.4$ with a middle band of at most 1400 blocks.

Lemma 24, supplement b: a multi-block sampling bound

on Platt's zeros with constants independent of the number of blocks

Standalone note `lemma24_L34b.tex`, October 21, 2026.

Summary. Verdict. The far-field estimate is now distance-weighted, and the block inequality of `lemma24_L34` (“L34”) has been rebuilt on a partition of unity. Together these give a middle-band sampling bound whose constants do *not* depend on the number n of blocks (Theorem 24B.2.3, [THM]). Its finite input, one penalised inequality per block, is Arb-certified for *every* block of the Lemma 21 middle band at $A = 2.4, 2.6, 2.8$ and 3.0 .¹ For all $f \in \text{PW}_{(A+0.1)/2}$ and $\theta \in (0, 1)$,

$$\sum_{\gamma \in [T_1-30, T_2+30]} |f(\gamma)|^2 \geq (1 - \theta) L_{\min} \|f\|_{L^2[T_1+10, T_2-10]}^2 - (\theta^{-1} - 1) \Lambda^2 \|f\|_{L^2(\mathbb{R})}^2,$$

with $L_{\min} = 0.485$ on $[5900, 220700]$ (716 blocks) at $A = 2.4$, $L_{\min} = 0.475$ on $[7400, 311900]$ (1015 blocks) at $A = 2.6$, $L_{\min} = 0.450$ on $[9300, 4759800]$ (15835 blocks) at $A = 2.8$, $L_{\min} = 0.470$ on $[11500, 17597200]$ (58619 blocks) at $A = 3.0$, and $\Lambda^2 \leq 9.60 \cdot 10^{-5}$ in all four cases. For example, at $\theta = 0.05$ this reads $S_{\text{mid}} \geq 0.4589 \|f\|_{\text{mid}}^2 - 0.00183 \|f\|_{\mathbb{R} \setminus \text{mid}}^2$ at $A = 2.4$, with $\text{mid} = [T_1 + 10, T_2 - 10]$. The price of tiling is $L_{\min} \approx 0.49$, against 0.63 for one isolated block (L34). The cause is the low blocks just above T_1 ; the median block has $L \approx 0.86$ ($A \leq 2.6$) and 1.149 ($A = 2.8$, where most blocks are high and dense). At $A = 2.8$, $\theta = 0.05$ gives $(c_{\text{mid}}, c_{\text{out}}) = (0.4256, 0.00183)$. At $A = 3.0$ the run is complete: all 58619 blocks on $[11500, 17597200]$ are certified, with $L_{\min} = 0.470$, median 1.270, $\Lambda^2 \leq 9.60 \cdot 10^{-5}$, and $\theta = 0.05$ gives $(c_{\text{mid}}, c_{\text{out}}) = (0.4446, 0.00183)$.

Labels. [THM] proved; [THM,CA] computer-assisted (Arb on verified data); [DEF] definition; [NUM] floating point; [ANALOGY]. Nothing here is RH-equivalent. Files: `num/lemma24_L34b_*` (zeros, cert, consts, tables; blocks.jsonl, consts.json); `handoff num/lemma24_L34_for_certificate.md`. Neither `lemma24.tex` nor its scripts were touched.

24B.1. Blocks and weights

DEFINITION 24B.1.1. [DEF] Let $a = 150$, $r = 30$, $s = 10$, and let $c_j = T_1 + 150 + 300j$ for $j = 0, \dots, n-1$, so that $T_2 = T_1 + 300n$. In the shifted variable $t - c_j$ the *sample weight* is $w_j(t) = \min\{1, \max\{0, (a + r - |t|)/(2r)\}\}$ and the *mass weight* is $\psi_j = \frac{1}{2s} \mathbf{1}_{[-a, a]} * \mathbf{1}_{[-s, s]}$. Both are trapezoids. Adjacent trapezoids add to 1 on their overlap, so

$$\sum_j w_j \leq 1 \text{ on } \mathbb{R}, \quad \sum_j \psi_j = 1 \text{ on } [T_1 + s, T_2 - s], \quad \sum_j \psi_j \leq 1.$$

Put $S_j(f) = \sum_{\gamma} w_j(\gamma) |f(\gamma)|^2$, summed over Platt's zeros (exact stored rationals, radius 2^{-100}). Let K_j be the number of zeros in $\text{supp } w_j$, of length $2b$ with $b = a + r = 180$, and $\|f\|_{\psi_j}^2 = \int \psi_j |f|^2$.

¹[Vol. 2 editorial note] These are middle-band *sampling* inequalities. They are not Weil positivity on windows of length 2.4–3.0. Certified window positivity in this volume stops at $A = 1.6$ (Chapter 16). See Appendix A, item B1..

The sample weights overlap the neighbouring blocks by $r = 30$ on each side; the mass weights overlap by only $s = 10$. This asymmetry is the *padding*, now compatible with a partition of unity. (With no padding at all the single-block constant collapses: L34 found $L^* = 0.07$ at $p = 0$.) A float scan over eleven pairs (r, s) with $r \leq 80$, $s \leq 20$, at the four lowest blocks, picked $(30, 10)$ [NUM]. Isolated blocks give 0.63 but cannot be summed. Halving the weights on shared strips gives 0.39. Counting shared zeros twice costs a factor 2.

24B.2. The per-block inequality with a localised far field

The kernel k (Kaiser–Bessel, $\beta = 14$, $\tau = 0.05$), the bounds k^* , κ_1 and the periodic model space V are as in L34, §§2–3. The parameters are now $D = 600$, $c = b + D = 780$, $P = b + c + D = 1560$, $J_j = c_j + [-c, c]$, $N_j = c_j + [-(P - b), P - b]$ and $N'_j = c_j + [-(P - b + D), P - b + D]$, and the arcs are $A_j = c_j + ([b, P - b] \cup [b - P, -b]) \subset N_j \setminus \text{supp } w_j$.

THEOREM 24B.2.1 (local block inequality). [THM] *Suppose that for all $v \in V$ (shifted to c_j)*

$$(24B.2.1) \quad S_j(v) + \mu \|v\|_{[b, P-b]}^2 \geq L_j \|v\|_{\psi_j}^2.$$

Then every $f \in \text{PW}_\omega$ satisfies $S_j(f)^{1/2} \geq \sqrt{L_j} \|f\|_{\psi_j} - E_j(f)$, where

$$E_j = \eta_J \|f\|_{J_j} + \sqrt{2\mu} (\|f\|_{N_j \setminus \text{supp } w_j} + \|f\|_{N'_j \setminus J_j} + \|e_j^{(2)}\|_{A_j}) + \sqrt{L_j} \|e_{\text{far}, j}\|_{\psi_j} + S_j(e_{\text{far}, j})^{1/2}.$$

*Here $e_{\text{far}, j} = (f \mathbf{1}_{\mathbb{R} \setminus J_j}) * k$, $e_j^{(2)} = (f \mathbf{1}_{\mathbb{R} \setminus N'_j}) * k$, and $\eta_J = \sqrt{\mu} \varepsilon_U + (\sqrt{2aL_j} + \sqrt{K_j}) \varepsilon_{\text{al}}$, with $\varepsilon_{\text{al}}, \varepsilon_U$ as in L34, Thm. 3.1, at the new (b, c, P) .*

PROOF. This is L34, Thm. 3.1, steps (i)–(v), with three changes. (a) $\|\cdot\|_I$ becomes $\|\cdot\|_{\psi_j}$ and S becomes S_j . Since $0 \leq \psi_j \leq \mathbf{1}_{[-(a+s), a+s]}$, $\int \psi_j = 2a$, $0 \leq w_j \leq \mathbf{1}_{[-b, b]}$ and $a + s < b$, every pointwise and L^2 error bound of L34 on $[-b, b]$ applies verbatim. (b) In step (iv), $\|e_{\text{far}}\|_{A_j}$ is no longer bounded by $\|f\|_{\mathbb{R} \setminus J}$. Instead split $f \mathbf{1}_{\mathbb{R} \setminus J} = f \mathbf{1}_{N' \setminus J} + f \mathbf{1}_{\mathbb{R} \setminus N'}$. Plancherel with $|\hat{k}| \leq 1$ bounds the first part by $\|f\|_{N' \setminus J}$, and the second part is $e^{(2)}$. (c) The far-field terms are kept as norms of e_{far} and $e^{(2)}$, not bounded yet; they are summed over j in Lemma 24B.2.2. $\square$

LEMMA 24B.2.2 (distance-weighted far field; n -independent sums). [THM] *Write $m(R) = \lfloor 2R/300 \rfloor + 1$, the maximal number of the intervals $c_j + [-R, R]$ containing a point. For every $f \in L^2(\mathbb{R})$,*

$$\begin{aligned} \sum_j \|f\|_{J_j}^2 &\leq m(c) \|f\|^2, \quad \sum_j \|f\|_{N_j \setminus \text{supp } w_j}^2 \leq m(P - b) \|f\|^2, \quad \sum_j \|f\|_{N'_j \setminus J_j}^2 \leq m(P - b + D) \|f\|^2, \\ \sum_j \|e_j^{(2)}\|_{A_j}^2 &\leq m(P - b) \kappa_1^2 \|f\|^2, \quad \sum_j L_j \|e_{\text{far}, j}\|_{\psi_j}^2 \leq L_{\max} \kappa_1^2 \|f\|^2, \\ \sum_j S_j(e_{\text{far}, j}) &\leq 2K_{\max} \kappa_1 c_0 \left(\frac{1}{D^2} + \frac{1}{300D} \right) \|f\|^2, \end{aligned}$$

where $\|f\| = \|f\|_{L^2(\mathbb{R})}$, $\kappa_1 = \kappa_1(D)$, and c_0 is as in L34 ($k^(y) \leq c_0/y^2$ for $y \geq D$).*

PROOF. The first three are overlap counts. For the far fields, Cauchy–Schwarz gives $|e(t)|^2 \leq \kappa_1 \int_E |f(u)|^2 |k(t - u)| du$, where E is the excluded set and $|t - u| \geq D$ whenever t lies in the relevant support and $u \in E$. Exchanging the sum over j with the u -integral leaves, for each fixed u , a weight that is summed over the blocks.

- For $e^{(2)}$ the weight is $\sum_j \int_{A_j} |k(t - u)| \mathbf{1}_{|t-u| \geq D} dt \leq m(P - b) \kappa_1$, since the $A_j \subset N_j$ overlap at most $m(P - b)$ times.
- For e_{far} on ψ_j the weight is $\int (\sum_j \psi_j)(t) |k(t - u)| \mathbf{1}_{|t-u| \geq D} dt \leq \kappa_1$.

- For the samples the weight is $\sum_{j:u \notin J_j} \sum_{\gamma} w_j(\gamma) |k(\gamma - u)| \leq \sum_{j:u \notin J_j} K_j k^*(\text{dist}(u, \text{supp } w_j))$.
On each side of u , the i -th such block lies at distance at least $D + 300i$, and
 $\sum_{i \geq 1} (D + 300i)^{-2} \leq \int_0^\infty (D + 300x)^{-2} dx = 1/(300D)$.

□

THEOREM 24B.2.3 (multi-block bound, n -independent). [THM] Suppose (24B.2.1) holds for every block $j = 0, \dots, n-1$. Put $L_{\min} = \min_j L_j$, $L_{\max} = \max_j L_j$, $K_{\max} = \max_j K_j$, and

$$\begin{aligned} \Lambda &= \eta_J \sqrt{m(c)} + \sqrt{2\mu} \left(\sqrt{m(P-b)} + \sqrt{m(P-b+D)} + \sqrt{m(P-b)} \kappa_1 \right) \\ &\quad + \sqrt{L_{\max}} \kappa_1 + \left(2K_{\max} \kappa_1 c_0 \left(\frac{1}{D^2} + \frac{1}{300D} \right) \right)^{1/2} \end{aligned}$$

(with η_J evaluated at $L_{\max}$ and $K_{\max}$). Then for every $f \in \text{PW}_\omega$ and every $\theta \in (0, 1)$,

$$\begin{aligned} \sum_{\gamma \in [T_1-r, T_2+r]} |f(\gamma)|^2 &\geq (1-\theta) \sum_j L_j \|f\|_{\psi_j}^2 - (\theta^{-1} - 1) \Lambda^2 \|f\|_{L^2(\mathbb{R})}^2 \\ &\geq (1-\theta) L_{\min} \|f\|_{[T_1+s, T_2-s]}^2 - (\theta^{-1} - 1) \Lambda^2 \|f\|^2. \end{aligned}$$

Λ does not depend on n .

PROOF. Per block, $S_j \geq ((\sqrt{L_j} \|f\|_{\psi_j} - E_j)_+)^2 \geq (1-\theta) L_j \|f\|_{\psi_j}^2 - (\theta^{-1} - 1) E_j^2$. Minkowski's inequality in $\ell^2(j)$ gives $(\sum_j E_j^2)^{1/2} \leq \sum_{\text{terms}} (\sum_j (\cdot)^2)^{1/2} \leq \Lambda \|f\|$ by Lemma 24B.2.2. Finally $\sum_j S_j \leq \sum_{\gamma} (\sum_j w_j(\gamma)) |f(\gamma)|^2$, with $\sum_j w_j \leq 1$ supported in $[T_1 - r, T_2 + r]$, and $\sum_j \psi_j = 1$ on $[T_1 + s, T_2 - s]$. □

REMARK 24B.2.4. (a) The negative ordinates $-\gamma$ give the same bound for $f(-\cdot) \in \text{PW}_\omega$. Summing over both signs doubles the Λ^2 term. (b) The left side is at most the full zero sum $\sum_{0 < |\gamma| \leq H} |f(\gamma)|^2$, which is what Lemma 21's middle term is when $f = m\hat{F}$. (c) Λ is dominated by the two $\sqrt{2\mu}$ overlap terms, about 0.0098 at $\mu = 10^{-6}$. Lowering μ lowers it like $\sqrt{\mu}$, at a small cost in L_j (L34, Thm. 4.1).

24B.3. Zero data above 446 000

DEFINITION 24B.3.1. [DEF] For $t \in [446000, 19346000]$ the zeros are the nine Platt files `zeros_446000.dat`, $\dots$, `zeros_17246000.dat` [148], md5-verified against LMFDB's `md5.txt`. They are decoded exactly as before ($\gamma = t_0 + Z/2^{101}$, integer Z) by a vectorised decoder (`num/lemma24_L34b_zeros2.py`). It does 32-bit-limb cumulative sums with carry normalisation, so no rounding occurs. On `zeros_236000.dat` it reproduces the scalar decoder of the earlier files bit for bit (363 820 zeros).

Checks [NUM]. Within and across all 13 files, block ranges $[t_0, t_1]$ and zero indices $[N_0, N_1]$ are contiguous, and the stored values strictly increase. At every file start T , $N_0 - (\vartheta(T)/\pi + 1) \in (-0.2, 0.6)$, consistent with $N(T)$ (that is, $|S(T)| < 1$). In total there are 42 199 876 zeros in the nine files. One anomaly: `zeros_2546000.dat` (md5 correct) carries 58 740 bytes after its 1001 declared blocks. They are a copy of the last block header followed by 4516 identical 13-byte records, a constant filler that would run past t_1 . LMFDB's reader only visits declared blocks, so the decoder accepts exactly this pattern and ignores it. As in L34, completeness is not needed for the lower bound: missing points only lower S_j .

24B.4. Certification of every block

THEOREM 24B.4.1 (all blocks). [THM, CA] Inequality (24B.2.1) with $\mu = 10^{-6}$ holds for every block listed below. The proof is as in L34, Thm. 4.1: Arb $\mathcal{S} = G_w + \mu(P\mathbf{I} - B_b) - L_j B_\psi$, where

$(B_\psi)_{jl} = \hat{\psi}((l-j)\Delta)$ and $\hat{\psi}(x) = 2 \sin(ax) \sin(sx)/(sx^2)$, a float Cholesky factor, and a rigorous Frobenius bound on the residual. Each sample weight is evaluated in ball arithmetic at the zero's ball. The $A = 2.8$ blocks use the faster but equally rigorous variant of Remark 24B.4.2.

A	band	n	all cert.	$L_{\min}$	median	$L_{\max}$	$K_{\max}$	worst blocks	$T_{\text{left}}(L_j)$
2.4	[5900, 220700]	716	yes	0.485	0.856	1.012	600	6500 (0.485), 7700 (0.516), 8600 (0.524)	
2.6	[7400, 311900]	1015	yes	0.475	0.862	1.056	620	7700 (0.475), 8600 (0.495), 7400 (0.518)	
2.8	[9300, 4759800]	15835	yes	0.450	1.149	1.384	777	11700 (0.450), 10800 (0.480), 9600 (0.493)	
3.0	[11500, 17597200]	58619	yes	0.470	1.270	1.524	852	15400 (0.470), 14800 (0.474), 13600 (0.485)	

A	$\Lambda^2 \leq$	$c_{\text{mid}}(\theta = 0.05)$	$c_{\text{out}}(\theta = 0.05)$
2.4	$9.587 \cdot 10^{-5}$	0.4589	0.00183
2.6	$9.588 \cdot 10^{-5}$	0.4494	0.00183
2.8	$9.596 \cdot 10^{-5}$	0.4256	0.00183
3.0	$9.599 \cdot 10^{-5}$	0.4446	0.00183

Λ terms ($A = 3.0$): $2.79e-05$, $4.47e-03$, $5.29e-03$, $5.70e-09$, $1.57e-06$, $3.87e-06$; $m(c), m(P-b), m(P-b+D) = 6, 10, 14$.

The blocks are contiguous, and none failed. Each certified L_j is $\lfloor 1000(L_j^* - 0.002) \rfloor / 1000$. Every residual bound is $< 10^{-9}$ against a shift $s_j \geq 3.0e - 04$. No stored zero lies within 10^{-20} of a weight breakpoint; the minimum distance is $1.4e - 06$. Every residual bound is at most $3.3e - 09$. Blocks for $A \leq 2.6$ took 5–56 s each (`lemma24_L34b_cert.py`). Blocks for $A = 2.8$ took a median of 0.7 s (`lemma24_L34b_cert2.py`). Both scripts are resumable.

REMARK 24B.4.2 (Toeplitz variant, [THM,CA]). On the uniform grid $\xi_j = j\Delta$, all three forms are Toeplitz: $(G_w)_{jl} = \sigma_{l-j}$ with $\sigma_d = \sum_{\gamma} w(\gamma) e^{id\Delta(\gamma-c)}$, and likewise B_b and B_ψ . So $\mathcal{S} - \sigma I$ is determined by its symbol t_d , $0 \leq d \leq 2n$, which is computed in Arb (128 bits) from the exact zeros. The powers z^d of $z = e^{i\Delta(\gamma-c)}$ are formed by multiplication and re-anchored by a direct exponential every 16 steps; this controls the growth of rectangular complex balls. Let $\hat{t}_d$ be the double nearest the midpoint of t_d , and $\rho_d \geq |t_d - \hat{t}_d|$, rounded up. With R a float Cholesky factor of $\hat{\mathcal{S}} - \sigma I$, the product R^*R is formed from the four real products $R_r^T R_r$, $R_i^T R_i$, $R_r^T R_i$ and its transpose. These carry the a-priori bound $|\mathfrak{fl}(X^T Y) - X^T Y| \leq \gamma_M |X|^T |Y| + M\eta$, valid for any summation order and with FMA (IEEE double, round-to-nearest, $\gamma_M = Mu/(1 - Mu)$, $u = 2^{-53}$) [69]. The final subtractions get a further γ_6 factor, and the Frobenius sum is inflated by $1 + 4(2M^2 + 4)u$. The certificate is $\|E\|_F^{\text{up}} < \sigma$, exactly as before. Cross-check [NUM]: on 60 randomly chosen blocks with $A \in \{2.4, 2.6\}$, both scripts give identical K_j , identical certified L_j and identical verdicts, with $|L_{\text{old}}^* - L_{\text{new}}^*| \leq 2 \cdot 10^{-12}$.

24B.5. The final uniform inequality

COROLLARY 24B.5.1 (Lemma 21 middle band). [THM,CA] Let $f \in \text{PW}_{(A+0.1)/2}$ and $\theta \in (0, 1)$.

- $A = 2.4$, $T_1 = 5900$, $T_2 = 220700$:
 $\sum_{\gamma \in [5870, 220730]} |f(\gamma)|^2 \geq (1 - \theta) 0.485 \|f\|_{[5910, 220690]}^2 - (\theta^{-1} - 1) 9.59 \cdot 10^{-5} \|f\|_{L^2(\mathbb{R})}^2$.
- $A = 2.6$, $T_1 = 7400$, $T_2 = 311900$:
 $\sum_{\gamma \in [7370, 311930]} |f(\gamma)|^2 \geq (1 - \theta) 0.475 \|f\|_{[7410, 311890]}^2 - (\theta^{-1} - 1) 9.59 \cdot 10^{-5} \|f\|_{L^2(\mathbb{R})}^2$.
- $A = 2.8$, $T_1 = 9300$, $T_2 = 4759800$:
 $\sum_{\gamma \in [9270, 4759830]} |f(\gamma)|^2 \geq (1 - \theta) 0.450 \|f\|_{[9310, 4759790]}^2 - (\theta^{-1} - 1) 9.60 \cdot 10^{-5} \|f\|_{L^2(\mathbb{R})}^2$.
- $A = 3.0$, $T_1 = 11500$, $T_2 = 17597200$:
 $\sum_{\gamma \in [11470, 17597230]} |f(\gamma)|^2 \geq (1 - \theta) 0.470 \|f\|_{[11510, 17597190]}^2 - (\theta^{-1} - 1) 9.60 \cdot 10^{-5} \|f\|_{L^2(\mathbb{R})}^2$.

At $\theta = 0.05$ the coefficients are $(0.4589, 0.00183)$, $(0.4494, 0.00183)$, $(0.4256, 0.00183)$ and $(0.4446, 0.00183)$. Here $c_{\text{mid}} = (1 - \theta)L_{\min} - (\theta^{-1} - 1)\Lambda^2$ multiplies $\|f\|_{\text{mid}}^2$ and $c_{\text{out}} = (\theta^{-1} - 1)\Lambda^2$ multiplies $\|f\|_{\mathbb{R} \setminus \text{mid}}^2$.

Each T_2 covers Lemma 21's $T'_2 = e^{S_{\text{align}}(A+2\varepsilon)+2.36}$, which is $2.205 \cdot 10^5$, $3.119 \cdot 10^5$, $4.7595 \cdot 10^6$ and $1.75971 \cdot 10^7$ at $A = 2.4, 2.6, 2.8, 3.0$, computed from $S_{\text{align}}(L) = \sum_{n < e^L} 2\Lambda(n)/\sqrt{n}$.

24B.6. What remains for the full hybrid

- (i) $A = 3.0$. The zeros are now available and decoded (§3). The run on [11500, 17597200] (58 619 blocks, same scheme, `lemma24_L34b_cert2.py`, resumable) is complete: all 58 619 blocks are certified, with $L_{\min} = 0.470$ (worst blocks at $T_{\text{left}} = 15400, 14800, 13600$), median 1.270 and $K_{\max} = 852$; see the tables in §5.
- (ii) *Absorbing* $\Lambda^2 \|f\|^2$. The penalty is global but tiny, about $1.8 \cdot 10^{-3}$ at $\theta = 0.05$. In the hybrid it must be paid by the low and high bands: $\|f\|^2 = \frac{1}{2\pi} \int |m\hat{F}|^2$, which is at most $\frac{1}{2\pi} \int (\chi_{\text{mid}} + \dots) |\hat{F}|^2$. This is bookkeeping for the certificate code (handoff file), not new analysis.
- (iii) *The low-band Toeplitz/Levinson certificate* (another task) and the coupling of the three bands in Lemma 21, Thm. 7.1. The coupling also includes the k_ε tails and the off-line zeros above H ; it is not addressed here.

CHAPTER 25

Lemma 25: The telegrapher components of the zero-waves x^ρ

The operator model, the passivity ladder, and the functional-equation channel

Standalone note lemma25.tex, October 21, 2026 (revised: operator reading first; passivity ladder).

Summary. J. W. asked whether the polar (“phase-tropical”) coordinates $(\log |x|, \arg x)$ of the zero-waves x^ρ break down exactly into the components of Heaviside’s distortionless line. They do, and the cleanest statement is an operator statement, which now comes first (§25.1). On the zero space $\mathcal{H}_Z = \ell^2(\mathcal{Z}, m)$ the multiplication operators $\beta = \operatorname{Re} \rho$ (loss), $t = \operatorname{Im} \rho$ (phase) and the functional-equation involution $J_{\text{FE}} : \rho \mapsto 1 - \bar{\rho}$ satisfy $J_{\text{FE}} \beta J_{\text{FE}} = 1 - \beta$. That one identity is the whole content of the functional equation here: the mean loss rate is $\mu = \frac{1}{2}$ for every line (the half-density, Heaviside’s substitution), J_{FE} exchanges the series and shunt branches, and the residual $\delta = \beta - \frac{1}{2}$ is J_{FE} -odd while $\kappa_H = \delta^2$ is J_{FE} -even, so the functional equation cannot remove it. Weil’s form is the Krein form $\langle J_{\text{FE}} x, y \rangle$, the scaling generator is $\gamma = t - i\delta$, and three energy functionals separate cleanly: the storage energy is positive and decreasing (passivity), the reduced Caldirola–Kanai energy is conserved iff $\delta = 0$, and the Weil energy is conserved always and definite iff $\delta = 0$. Per functional-equation pair the line is exactly the uniform telegrapher mode with $\{R/L, G/C\} = \{\beta, 1 - \beta\}$ and Heaviside residual $\kappa_H = \delta^2$, so the line of ρ is distortionless iff $\operatorname{Re} \rho = \frac{1}{2}$. The slogan “passivity = PNT” is replaced by a precise *passivity ladder*. $R, G \geq 0$ is free: it follows from the Euler product and the functional equation, with no PNT. $R, G > 0$ is $\zeta(1 + it) \neq 0$; it implies the PNT by Wiener–Ikehara, and the PNT implies it by an elementary argument. A logarithmic margin is the de la Vallée Poussin region, with error $x \exp(-c\sqrt{\log x})$. A uniform margin ($\Theta < 1$) is open, and is equivalent to a power saving. $\delta = 0$ is RH, with error $O(\sqrt{x} \log^2 x)$ (von Koch). The transfer-function reading, the curve case (where Rosati positivity supplies $\delta = 0$), and the comparison of Weil positivity with Osterwalder–Schrader positivity follow. **Nothing here proves RH.**

Labels. [DEF]: definition (a choice of words or of normalization, nothing to prove); [THM]: proved here (or classical, proof indicated); [CITED]: published result used as stated; [NUM]: numerical or symbolic check; [ANALOGY]: structural analogy, not a derivation; [RH-EQUIV]: equivalent to RH by Weil’s criterion or by definition, so a reformulation, not progress; [OPEN]: an open problem.

Conventions. $u = \log x$, $D = d/du$. A nontrivial zero is written $\rho = \beta_\rho + it_\rho$ with $\beta_\rho = \operatorname{Re} \rho$, $t_\rho = \operatorname{Im} \rho$, and $\delta_\rho = \beta_\rho - \frac{1}{2}$. In the project’s notation (NOTATION.md, Lemma 1) $\rho = \frac{1}{2} + i\gamma_\rho$, so $\gamma_\rho = t_\rho - i\delta_\rho$. Lemma 17’s “ γ ” in $x^\rho = e^{\beta u} e^{i\gamma u}$ is our t_ρ . Telegrapher quantities carry the subscript “tel”: $\gamma_{\text{tel}} = \alpha_{\text{tel}} + i\beta_{\text{tel}}$. Bold symbols ($\beta, t, \delta, \gamma, \rho$) are the multiplication operators by the corresponding numbers on the zero space (Def. 25.1.1). The functional equation $\xi(s) = \xi(1 - s)$ and $\bar{\xi}(\bar{s}) = \xi(s)$ [39, Ch. 8] make the zero set invariant under $\rho \mapsto \rho^* := 1 - \bar{\rho}$ (reflection in the critical line), with $m_{\rho^*} = m_\rho$. This map *preserves* t_ρ and sends $\beta \mapsto 1 - \beta$, $\delta \mapsto -\delta$, $\gamma \mapsto \bar{\gamma}$. $\Theta = \sup_\rho \operatorname{Re} \rho \in [\frac{1}{2}, 1]$ as in Lemma 1. $\mathcal{Z} = \mathcal{Z}_0 \sqcup \mathcal{Z}_+ \sqcup \mathcal{Z}_-$ splits the distinct zeros by $\delta = 0$, $\delta > 0$, $\delta < 0$ (Lemma 1, proof of (vi)); $\mathcal{Z}_- = \{\rho^* : \rho \in \mathcal{Z}_+\}$ and $N_+ = \#\mathcal{Z}_+$.

25.1. The operator reading

This section states the lemma as a statement about operators. Everything after it is a restriction to one functional-equation pair (§25.2), a coordinate reading (§25.3), or an assembly (§25.4). Every component is marked [DEF], [THM] or [ANALOGY]. The symbolic identities are checked in `num/lemma25_operator.py/.out`.

25.1.1. State space, Weil form, generator.

- DEFINITION 25.1.1 (zero space and its operators; [DEF]). (i) *State space*. $\mathcal{H}_{\mathcal{Z}} := \ell^2(\mathcal{Z}, m)$, the sequences $x = (x_\rho)$ indexed by the *distinct* nontrivial zeros with $\|x\|^2 = \sum_\rho m_\rho |x_\rho|^2 < \infty$, and inner product $\langle x, y \rangle = \sum_\rho m_\rho x_\rho \bar{y}_\rho$. The basis vectors are e_ρ . The splitting of $\mathcal{Z}$ gives $\mathcal{H}_{\mathcal{Z}} = \mathcal{H}_0 \oplus \mathcal{H}_+ \oplus \mathcal{H}_-$ with $\mathcal{H}_\bullet = \ell^2(\mathcal{Z}_\bullet, m)$. This is the Hilbert space of Lemma 1, Thm. 2.1(v), but used here *without* assuming RH. Off RH its inner product is a choice: any other positive weights would do equally well (Remark 25.1.3).
- (ii) *Phase operator* $\mathbf{t}$: multiplication by t_ρ , with domain $\text{dom } \mathbf{t} = \{x : \sum m_\rho t_\rho^2 |x_\rho|^2 < \infty\}$.
- (iii) *Loss operator* β : multiplication by β_ρ . *Residual* $\delta := \beta - \frac{1}{2}$. *Heaviside residual operator* $\kappa_{\mathcal{H}} := \delta^2$.
- (iv) *Wave generator* $\rho := \beta + i\mathbf{t}$ and *scaling generator* $\gamma := \mathbf{t} - i\delta$, both with domain $\text{dom } \mathbf{t}$. So $\rho = \frac{1}{2} + i\gamma$, and γ is multiplication by γ_ρ (Lemma 1's A , but now without RH).
- (v) *Functional-equation involution* $(J_{\text{FE}}x)_\rho := x_{\rho^*}$.
- (vi) *Weil (Krein) form* $[x, y]_{\mathcal{Z}} := \langle J_{\text{FE}}x, y \rangle = \sum_\rho m_\rho x_\rho \bar{y}_\rho$.
- (vii) *Evaluation* $\text{ev} : \mathcal{A} \rightarrow \mathcal{H}_{\mathcal{Z}}$, $\text{ev}F := (\tilde{F}(\gamma_\rho))_\rho$, on Lemma 1's test space $\mathcal{A} \supset C_c^\infty(\mathbb{R})$.

THEOREM 25.1.2 (structure of the zero space; [THM], (f) [RH-EQUIV]). (a) $\mathbf{t}$ is self-adjoint (unbounded). $\beta, \delta, \kappa_{\mathcal{H}}$ are bounded and self-adjoint, $\beta, \delta, \kappa_{\mathcal{H}}, \mathbf{t}$ commute, and ρ, γ are closed normal operators. Moreover

$$\sigma(\beta) = \overline{\{\beta_\rho\}} \subset [1 - \Theta, \Theta] \subset [0, 1], \quad \|\delta\| = \Theta - \frac{1}{2}, \quad \|\kappa_{\mathcal{H}}\| = (\Theta - \frac{1}{2})^2.$$

- (b) J_{FE} is a self-adjoint unitary involution, $J_{\text{FE}}\mathcal{H}_\pm = \mathcal{H}_\mp$ and $J_{\text{FE}}|_{\mathcal{H}_0} = 1$. Further $J_{\text{FE}}\beta J_{\text{FE}} = 1 - \beta$, $J_{\text{FE}}\delta J_{\text{FE}} = -\delta$, $J_{\text{FE}}\mathbf{t} J_{\text{FE}} = \mathbf{t}$, $J_{\text{FE}}\kappa_{\mathcal{H}} J_{\text{FE}} = \kappa_{\mathcal{H}}$, $\frac{1}{2}(\beta + J_{\text{FE}}\beta J_{\text{FE}}) = \frac{1}{2}$.
- (c) $(\mathcal{H}_{\mathcal{Z}}, [\cdot, \cdot]_{\mathcal{Z}})$ is a Krein space with fundamental symmetry J_{FE} . Its negative index is $\dim \ker(J_{\text{FE}} + 1) = N_+$.
- (d) γ is J_{FE} -self-adjoint: $J_{\text{FE}}\gamma^* J_{\text{FE}} = \gamma$, i.e. $[\gamma x, y]_{\mathcal{Z}} = [x, \gamma y]_{\mathcal{Z}}$ on $\text{dom } \mathbf{t}$. Hence $e^{iu\gamma}$ preserves $[\cdot, \cdot]_{\mathcal{Z}}$ for every $u \in \mathbb{R}$, and $\|e^{iu\gamma}\| = e^{|u|(\Theta - \frac{1}{2})}$.
- (e) $\text{ev}(\mathcal{A}) \subset \text{dom } \mathbf{t}$, and for $F, G \in \mathcal{A}$ and $u \in \mathbb{R}$:

$$\mathcal{Q}(F, G) = [\text{ev}F, \text{ev}G]_{\mathcal{Z}}, \quad \text{ev}(T_u F) = e^{iu\gamma} \text{ev}F, \quad \text{ev}(D_{\text{sc}} F) = \gamma \text{ev}F \quad (F \in C_c^\infty).$$

So the Weil form is the pull-back of the Krein form, the scaling flow is $e^{iu\gamma}$, and its generator D_{sc} (Lemma 17) is γ .

- (f) [RH-EQUIV] The following are equivalent: RH; $\delta = 0$; $\kappa_{\mathcal{H}} = 0$; $J_{\text{FE}} = 1$; $[\cdot, \cdot]_{\mathcal{Z}}$ is positive definite; γ is self-adjoint; $e^{iu\gamma}$ is unitary on $\mathcal{H}_{\mathcal{Z}}$; there is a positive definite bounded Hermitian form on $\mathcal{H}_{\mathcal{Z}}$ invariant under all $e^{iu\gamma}$.

PROOF. (a) These are multiplication operators by sequences, real for $\mathbf{t}, \beta, \delta, \kappa_{\mathcal{H}}$ and bounded for the last three. $\sigma(\beta)$ is the closure of the set of values. The bound $1 - \Theta \leq \beta_\rho \leq \Theta$ is the definition of Θ plus the symmetry $\beta \mapsto 1 - \beta$. Also $\sup |\delta_\rho| = \Theta - \frac{1}{2}$ by the same symmetry. $\Theta \leq 1$ is (P0) of Prop. 25.1.8. (b) $\rho \mapsto \rho^*$ is an involutive bijection of $\mathcal{Z}$ with $m_{\rho^*} = m_\rho$. It fixes $\mathcal{Z}_0$ and swaps $\mathcal{Z}_\pm$. Then $(J_{\text{FE}}\beta J_{\text{FE}})_\rho = \beta_{\rho^*} x_\rho = (1 - \beta_\rho) x_\rho$ and $t_{\rho^*} = t_\rho$. The other identities follow. On one off-line pair, in the basis (e_ρ, e_{ρ^*}) , this is $\sigma_1 \text{diag}(\beta, 1 - \beta) \sigma_1 = 1 - \text{diag}(\beta, 1 - \beta)$ (check file, items 1–4). (c) A self-adjoint unitary J

makes $\langle Jx, y \rangle$ a Krein form, with maximal negative subspaces of dimension $\dim \ker(J+1)$. Here $\ker(J_{\text{FE}}+1)$ consists of the vectors with $x_{\rho^*} = -x_\rho$. These vanish on $\mathcal{Z}_0$ and form one line in each off-line pair, and there are $\#\mathcal{Z}_+ = N_+$ pairs. (d) γ^* is multiplication by $\bar{\gamma}_\rho = \gamma_{\rho^*}$, so $J_{\text{FE}}\gamma^*J_{\text{FE}}$ is multiplication by $\bar{\gamma}_{\rho^*} = \gamma_\rho$. For the flow, $|e^{iu\gamma_\rho}| = e^{u\delta_\rho}$, and $\overline{e^{iu\gamma_{\rho^*}}} = e^{-iu\gamma_\rho}$ gives invariance (check items 5–6). The norm is $\sup_\rho e^{u\delta_\rho} = e^{|u|\sup|\delta|}$. (e) By Lemma 1, Lemma 1.2, $(1+|\gamma|^2)|\tilde{F}(\gamma)| \leq \|F\|_{\mathcal{A}}$ on $|\operatorname{Im} \gamma| \leq \frac{1}{2}$. Also $\sum m_\rho(1+|\gamma_\rho|^2)^{-1} < \infty$ (Lemma 1, §1). Together these give $\sum m_\rho(1+t_\rho^2)|\tilde{F}(\gamma_\rho)|^2 < \infty$. Lemma 1's definition $\mathcal{Q}(F, G) = \sum m_\rho \tilde{F}(\gamma_\rho) \tilde{G}(\bar{\gamma}_\rho)$ is $\sum m_\rho x_\rho \bar{y}_{\rho^*} = \langle x, J_{\text{FE}}y \rangle = [x, y]_{\mathcal{Z}}$. The flow identity is Lemma 1, Thm. 2.1(ii), and $\widetilde{D_{\text{sc}}F}(z) = z\tilde{F}(z)$ (Lemma 17). (f) $\delta = 0 \iff \text{all } \delta_\rho = 0 \iff \text{RH}$. By (b), $\delta = 0$ forces $\mathcal{Z}_\pm = \emptyset$, so $J_{\text{FE}} = 1$. Conversely $J_{\text{FE}} = 1$ forces $\mathcal{Z}_+ = \emptyset$. A Krein form is definite iff its fundamental symmetry is 1. γ is self-adjoint, and $e^{iu\gamma}$ is unitary, iff all γ_ρ are real. Finally, if h is bounded, Hermitian and invariant, then $h(e_\rho, e_\rho) = h(e^{iu\gamma}e_\rho, e^{iu\gamma}e_\rho) = e^{2u\delta_\rho}h(e_\rho, e_\rho)$ for all u . So $h(e_\rho, e_\rho) = 0$ when $\delta_\rho \neq 0$, and h is not definite. Under RH, $h = \langle \cdot, \cdot \rangle$ works. $\square$

REMARK 25.1.3 (what is canonical; [THM] for the facts). Canonical (fixed by ζ): $\mathbf{t}$, β , J_{FE} , the Krein form $[\cdot, \cdot]_{\mathcal{Z}}$, γ , and the decomposition $\mathcal{H}_0 \oplus \mathcal{H}_+ \oplus \mathcal{H}_-$ into spectral subspaces of δ . Not canonical off RH: the *definite* inner product of $\mathcal{H}_{\mathcal{Z}}$. Any weights $w_\rho > 0$ give a Hilbert space on which the same multiplication operators act with the same spectra. Statements (P0)–(P3) below are spectral statements about β , so they do not depend on this choice. Under RH the choice is forced: it is $[\cdot, \cdot]_{\mathcal{Z}}$ itself.

25.1.2. The telegrapher components.

DEFINITION 25.1.4 (telegrapher mode in operator form; [DEF], classical). For a uniform line with per-unit-length $L, C > 0$ and $R, G \geq 0$, the Fourier mode e^{ikx} of $V_x = -LI_t - RI$, $I_x = -CV_t - GV$ is the system $M\dot{X} = -(\Sigma + ik\sigma_1)X$ on $X = (I, V) \in \mathbb{C}^2$, with

$$M = \operatorname{diag}(L, C) \text{ (storage)}, \quad \Sigma = \operatorname{diag}(R, G) \text{ (dissipation)}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ (spatial coupling)}.$$

Write $M^{-1}\Sigma = \mu + d\sigma_3$ with $\mu = \frac{1}{2}(\frac{R}{L} + \frac{G}{C})$, $d = \frac{1}{2}(\frac{R}{L} - \frac{G}{C})$, $\sigma_3 = \operatorname{diag}(1, -1)$. The Heaviside residual is $\kappa_H = d^2$.

[THM] (classical; check items 9–11). The storage energy $\frac{1}{2}X^*MX$ satisfies $\frac{d}{dt}\frac{1}{2}X^*MX = -(R|I|^2 + G|V|^2) \leq 0$, because $ik\sigma_1$ is skew-Hermitian. After $X = e^{-\mu t}Y$, the generator of Y is $-(d\sigma_3 + ikM^{-1}\sigma_1)$. This is skew-adjoint for $\langle M\cdot, \cdot \rangle$, for any k , iff $d = 0$, i.e. iff $RC = LG$. So, in operator language, Heaviside's condition says that *the dissipation is a scalar multiple of the storage*, $\Sigma = \mu M$. Then the line is $e^{-\mu t}$ times a unitary group. Prop. 25.6.1 gives the same statement for the field.

- DEFINITION 25.1.5 (telegrapher components of the zero space; [DEF]; the electrical names are [ANALOGY]). (i) *Storage* L, C : $M = 1$, i.e. $L = C = 1$. This is a normalization: the uniform mode never sees L and C separately (Prop. 25.2.3(iii)).
- (ii) *Dissipation*: $\Sigma := M\beta = \beta$.
- (iii) *Series and shunt branches*: $\mathcal{H}_+$ (zeros with $\beta > \frac{1}{2}$) is the series branch, and $\mathcal{H}_- = J_{\text{FE}}\mathcal{H}_+$ is the shunt branch. $R := \beta|_{\mathcal{H}_+}$ and $G := \beta|_{\mathcal{H}_-}$, so on the pair of $\rho \in \mathcal{Z}_+$, $R/L = \beta_\rho$ and $G/C = 1 - \beta_\rho$. On $\mathcal{H}_0$ the two branches coincide, with $R/L = G/C = \frac{1}{2}$. Calling $\mathcal{H}_+$ “series” is a convention; the splitting itself is canonical.
- (iv) *Generator and orientation*: the explicit-formula waves are $e^{u\rho}$ (growth in u). The telegrapher (decaying) orientation is $\tau = -u$, with contraction semigroup $Z(\tau) := e^{-\tau\rho}$, $\tau \geq 0$. Its phase factor $e^{-i\tau t}$ is unitary and its loss factor is $e^{-\tau\beta}$. The orientation is a convention (Prop. 25.2.3(i)).

- (v) *Mean rate and residual*: $\mu := \frac{1}{2}(\beta + J_{\text{FE}}\beta J_{\text{FE}})$ and $\delta = \frac{1}{2}(\beta - J_{\text{FE}}\beta J_{\text{FE}})$ (the J_{FE} -even and J_{FE} -odd parts of β).
- (vi) *Spatial coupling $ik\sigma_1$* : no counterpart. Only the uniform mode $k = 0$ is realized by ζ in this lemma [ANALOGY]. A second, spatial direction would be new structure (Question 4).

PROPOSITION 25.1.6 (what each component is, in operator language; [THM]). (a) Passivity $\Sigma \geq 0$, i.e. $R, G \geq 0$, is $0 \leq \beta \leq 1$. It holds unconditionally ((P0) of Prop. 25.1.8). Equivalently, $Z(\tau)$ is a contraction semigroup.

- (b) $\mu = \frac{1}{2}$: by Thm. 25.1.2(b), $\mu = \frac{1}{2} \cdot 1$ as an operator, for every zero, unconditionally. This is the functional equation. The Heaviside substitution is the factorization

$$Z(\tau) = e^{-\tau/2} e^{-\tau(\delta+it)}, \quad \text{equivalently} \quad e^{u\rho} = e^{u/2} e^{iu\gamma},$$

and $e^{-u/2}$ is the half-density normalization $E(u) = e^{-u/2}(\psi(e^u) - e^u)$ of Lemma 17.

- (c) $\kappa_{\text{H}} = \delta^2$: the reduced (demodulated, Heaviside-substituted) envelope $W(u) = e^{u\delta}x$ satisfies $W'' = \kappa_{\text{H}}W$, so κ_{H} is the (tachyonic) mass-squared operator of the envelope, with $\|\kappa_{\text{H}}\| = (\Theta - \frac{1}{2})^2$. κ_{H} is J_{FE} -even, so the functional equation is compatible with every value of it.
- (d) Functional equation = series/shunt duality. J_{FE} maps $\mathcal{H}_+$ onto $\mathcal{H}_-$ and $J_{\text{FE}}\beta J_{\text{FE}} = 1 - \beta$. On the pair of ρ this is $\sigma_1 \text{diag}(R, G) \sigma_1 = \text{diag}(G, R)$, the telegrapher duality $R \leftrightarrow G$, $L \leftrightarrow C$ (with $L = C = 1$). The two descriptions agree because $R/L + G/C = 2\mu = 1$.
- (e) One channel. On an off-line pair $\{\rho, \rho^*\}$, ρ restricts to $\text{diag}(\rho, 1 - \bar{\rho})$. Its characteristic polynomial $\det(D - \rho|_O)$ is the operator P_ρ of Def. 25.2.1, and demodulating by $e^{-it\rho u}$ gives the normal form of Thm. 25.2.2 (check items 7–8).
- (f) Explicit formula. With $\text{tr}_m X = \sum_\rho m_\rho X_{\rho\rho}$ for diagonal X , Davenport's explicit formula (Thm. 25.4.1) reads, for $x = e^u > 1$,

$$E(u) = - \lim_{T \rightarrow \infty} \text{tr}_m(\mathbf{1}_{|t| < T} e^{iu\gamma} \rho^{-1}) - e^{-u/2} \log 2\pi - \frac{1}{2} e^{-u/2} \log(1 - e^{-2u}).$$

PROOF. (a) Thm. 25.1.2(a), and $\|e^{-\tau\beta}\| = e^{-\tau \inf \sigma(\beta)} \leq 1$. (b) Thm. 25.1.2(b), and $\rho = \frac{1}{2} + \delta + it$. (c) W is $e^{u\delta\rho}x_\rho$ componentwise. (d) Thm. 25.1.2(b) and check item 1. (e) Check items 7–8, and Thm. 25.2.2(a). (f) $e^{iu\gamma\rho}/\rho = e^{(\rho - \frac{1}{2})u}/\rho$ with weight m_ρ , which is Thm. 25.4.1 term by term with the same symmetric truncation. $\square$

25.1.3. Three energy functionals.

PROPOSITION 25.1.7 (storage, reduced, Weil; [THM], the “iff” clauses [RH-EQUIV]). For $x \in \mathcal{H}_{\mathcal{Z}}$ and $\tau \geq 0$:

- (a) Storage (physical) energy. $\mathcal{E}_{\text{st}}(x) := \frac{1}{2}\|x\|^2$ (with $M = 1$). Then $\mathcal{E}_{\text{st}}(Z(\tau)x) = \frac{1}{2} \sum m_\rho e^{-2\tau\beta\rho} |x_\rho|^2$ is positive and nonincreasing, and $\frac{d}{d\tau} \mathcal{E}_{\text{st}}(Z(\tau)x) = -\langle \beta Z(\tau)x, Z(\tau)x \rangle \leq 0$. It is definite but not conserved: it is a Lyapunov function, which is passivity. It depends on the conventional choice of M (Remark 25.1.3).
- (b) Reduced (Caldirola–Kanai) energy. $\mathcal{E}_{\text{CK}}(\tau; x) := e^{2\mu\tau} \mathcal{E}_{\text{st}}(Z(\tau)x) = \frac{1}{2} \sum m_\rho e^{-2\tau\delta\rho} |x_\rho|^2$. It is definite, and it is conserved for all x iff $\delta = 0$, i.e. iff RH. In general $e^{-2\tau(\Theta - \frac{1}{2})} \leq \mathcal{E}_{\text{CK}}(\tau; x)/\mathcal{E}_{\text{CK}}(0; x) \leq e^{2\tau(\Theta - \frac{1}{2})}$. This is the first-order form of Lemma 17, Prop. “CK” [18, 80]. The second-order form is Prop. 25.3.2(c).
- (c) Weil energy. $[x, x]_{\mathcal{Z}}$ is conserved under $e^{iu\gamma}$ for all x and u (Thm. 25.1.2(d)), and it is definite iff RH (Thm. 25.1.2(f)). On $\text{ev}F$ it is $\mathcal{Q}(F, F)$.
- (d) Noether charge. $\mathcal{E}[F] = \mathcal{Q}(\text{D}_{\text{sc}}F, F) = [\gamma \text{ev}F, \text{ev}F]_{\mathcal{Z}}$. Under RH its spectrum is $\{\gamma_\rho\}$, unbounded in both directions (Remark 25.6.2(i)).

So no functional is both definite and conserved unless RH (Thm. 25.1.2(f), last item). Passivity concerns (a), the only one that is automatically definite, and it is free. RH is the statement that (b) is conserved, equivalently that (c) is definite. Both say $\delta = 0$.

PROOF. $Z(\tau)$ acts diagonally by $e^{-\tau\rho}$, with $|e^{-\tau\rho}|^2 = e^{-2\tau\beta\rho}$. Differentiate termwise, which is justified since β is bounded. For (b), $2\mu = 1$. (c), (d): Thm. 25.1.2(d),(e). $\square$

25.1.4. The passivity ladder.

PROPOSITION 25.1.8 (passivity ladder). *Each rung is a statement about the spectrum of the bounded self-adjoint loss operator β , equivalently about $\|\delta\| = \Theta - \frac{1}{2}$, or about the growth bound $\|e^{iu\gamma}\| = e^{|u|(\Theta - \frac{1}{2})}$ of the scaling flow. Since $J_{FE}\beta J_{FE} = 1 - \beta$ and J_{FE} commutes with $\mathbf{t}$, a lower bound for β in terms of $\mathbf{t}$ gives the same bound for $1 - \beta$. One inequality states each rung.*

- (P0) [THM] Passive: $0 \leq \beta \leq 1$, i.e. $R, G \geq 0$, i.e. $0 \leq \beta_\rho \leq 1$. This follows from the Euler product and the functional equation alone, with no input from the prime number theorem.
- (P1) [THM], [CITED] Strictly passive: $\ker \beta = \ker(1 - \beta) = 0$, i.e. $R, G > 0$ (injective), i.e. $0 < \beta_\rho < 1$ for every ρ . This is equivalent to $\zeta(1 + it) \neq 0$ for all real $t \neq 0$ (Hadamard, de la Vallée Poussin; [39, Ch. 13]).
 - (P1) $\Rightarrow$ PNT $\psi(x) \sim x$, by the Wiener–Ikehara Tauberian theorem [72], [160, §10]. This is the nontrivial direction.
 - PNT $\Rightarrow$ (P1), by the elementary argument below. This is the easy direction.
- (P2) [THM], [CITED] Logarithmic margin: there is $c > 0$ with $\beta \leq 1 - c/\log(|\mathbf{t}| + 2)$ (equivalently $\beta \geq c/\log(|\mathbf{t}| + 2)$). This is a restatement of the de la Vallée Poussin zero-free region $\beta \leq 1 - c/\log(|\mathbf{t}| + 2)$ [39, Ch. 13]. It implies $\psi(x) = x + O(x \exp(-c'\sqrt{\log x}))$ [39, Ch. 18].
- (P3) [OPEN] Uniform margin (exponential stability): $\beta \geq \varepsilon$ for some $\varepsilon > 0$, i.e. β boundedly invertible, i.e. $\|\delta\| < \frac{1}{2}$, i.e. $\Theta < 1$. More precisely, for $\theta \in [\frac{1}{2}, 1)$,

$$\Theta \leq \theta \iff \psi(x) - x = O_\varepsilon(x^{\theta+\varepsilon}) \text{ for every } \varepsilon > 0,$$

and $\Theta \leq \theta$ gives $\psi(x) - x = O(x^\theta \log^2 x)$. This is open for every $\theta < 1$ (the quasi-RH, cf. Lemma 18, Thm. 3.3).

- (P4) [RH-EQUIV] Distortionless: $\delta = 0$, i.e. $\beta = \frac{1}{2}$, i.e. $R/L = G/C$ on every line, i.e. RH. This is equivalent to $\psi(x) = x + O(\sqrt{x} \log^2 x)$ (von Koch [154]), and also to $\psi(x) = x + O_\varepsilon(x^{1/2+\varepsilon})$ for all $\varepsilon > 0$ (the case $\theta = \frac{1}{2}$ of (P3)).

With the labelling of Def. 25.1.5(iii), $R/L = \beta_\rho > \frac{1}{2}$ on $\mathcal{H}_+$ holds automatically. So (P0)–(P3) are really statements about the shunt branch, $G/C = 1 - \beta_\rho$: it is ≥ 0 for (P0), > 0 for (P1), $\geq c/\log(|t_\rho| + 2)$ for (P2), and $\geq \varepsilon$ for (P3). In telegrapher terms, (P1) and (P3) are the standard distinction between asymptotically stable (every mode decays: β injective) and exponentially stable (uniform decay rate: β invertible). For infinitely many modes the two differ, and the gap between them is exactly where the open problem $\Theta < 1$ sits.

PROOF. (P0). For $\sigma > 1$, $\zeta(s) \prod_p (1 - p^{-s}) = 1$ with an absolutely convergent product of nonzero factors, so $\zeta(s) \neq 0$. Also $\Gamma(s/2) \neq 0$ and $s(s-1) \neq 0$, so $\xi(s) \neq 0$ for $\sigma > 1$. By $\xi(s) = \xi(1-s)$, also $\xi(s) \neq 0$ for $\sigma < 0$. The nontrivial zeros are the zeros of ξ [39, Ch. 8, Ch. 12].

(P1) $\Leftrightarrow \zeta(1+it) \neq 0$. A nontrivial zero with $\beta = 1$ is a zero on $\sigma = 1$. One with $\beta = 0$ is the reflection ρ^* of one with $\beta = 1$. ($s = 1$ is not a zero of ξ , since $\xi(1) = \frac{1}{2}$.)

(P1) $\Rightarrow$ PNT. Apply Wiener–Ikehara to $a_n = \Lambda(n) \geq 0$, $\sum a_n n^{-s} = -\zeta'/\zeta(s)$ ($\sigma > 1$). Under (P1), $-\zeta'/\zeta(s) - \frac{1}{s-1}$ extends continuously to $\sigma \geq 1$, so $\psi(x) \sim x$.

PNT $\Rightarrow$ (P1). By partial summation, $-\zeta'/\zeta(s) = s \int_1^\infty \psi(x) x^{-s-1} dx$ for $\sigma > 1$. Write $\psi(x) = x + r(x)$ with $r(x) = o(x)$. Then $s \int_1^\infty x \cdot x^{-s-1} dx = \frac{s}{s-1}$ (check item 12). For each $\epsilon > 0$

we have $|\int_1^\infty r(x)x^{-s-1}dx| \leq C_\epsilon + \epsilon/(\sigma - 1)$. Suppose $\zeta(1 + it_0) = 0$ with $t_0 \neq 0$ and multiplicity $m \geq 1$. Then $(\sigma - 1)(-\zeta'/\zeta)(\sigma + it_0) \rightarrow -m$ as $\sigma \downarrow 1$, while the right-hand side gives $\limsup_{\sigma \downarrow 1} (\sigma - 1)|\zeta'/\zeta(\sigma + it_0)| \leq \epsilon|1 + it_0|$, because $\frac{s}{s-1}$ stays bounded at $s = 1 + it_0$. Letting $\epsilon \rightarrow 0$ gives a contradiction.

(P2). The first statement is the zero-free region rewritten as a functional-calculus inequality between the commuting multiplication operators β and t . The second is Davenport's Ch. 18, eq. (1), deduced there from Ch. 13 and the truncated explicit formula of Ch. 17.

(P3), “ $\Rightarrow$ ”. Davenport's truncated explicit formula [39, Ch. 17, eq. (9)] gives $\psi(x) = x - \sum_{|t_\rho| \leq T} x^\rho/\rho + O(1 + x \log^2(xT)/T)$. Use $|x^\rho| \leq x^\Theta$ and $\sum_{|t_\rho| \leq T} 1/|\rho| \ll \log^2 T$ (from $N(T)$, [39, Ch. 15]), and take $T = x$. This gives $\psi(x) - x = O(x^\Theta \log^2 x)$. “ $\Leftarrow$ ”. If $\psi(x) - x = O(x^{\theta+\epsilon})$, then $-\zeta'/\zeta(s) - \frac{s}{s-1} = s \int_1^\infty (\psi(x) - x)x^{-s-1}dx$ converges absolutely and is holomorphic on $\sigma > \theta + \epsilon$. So ζ has no zeros there. Let $\epsilon \rightarrow 0$.

(P4). $\Theta = \frac{1}{2}$ in (P3). The bound $O(\sqrt{x} \log^2 x)$ is von Koch's [154], and it is also the case $\Theta = \frac{1}{2}$ of the proof of (P3) “ $\Rightarrow$ ”. $\square$

REMARK 25.1.9 (why not “passivity is the prime number theorem”; [THM] for the logic). (i)

Which passivity. Passivity in the engineering sense ($R, G \geq 0$, (P0)) is free. It follows from the Euler product and the functional equation, and it is far weaker than the PNT. Only *strict* passivity (P1) is tied to the PNT.

- (ii) “*Equivalent*” is *vacuous between proved theorems*. (P1) and the PNT are both theorems, and in classical logic any two theorems are equivalent. What has content is the direction and the cost of each implication. (P1) $\Rightarrow$ PNT needs a Tauberian theorem: Wiener–Ikehara, or the explicit formula with a zero-free region. PNT $\Rightarrow$ (P1) is a few lines. The precise sentence is therefore: *strict passivity of every zero-line is the analytic input of the prime number theorem*.
- (iii) *Termwise is not enough*. In the half-density normalization the PNT reads $E(u) = o(e^{u/2})$. (P1) makes each channel term $e^{iu\gamma_\rho}/\rho$ of Prop. 25.1.6(f) individually $o(e^{u/2})$, since $|e^{iu\gamma_\rho}| = e^{u\delta_\rho}$ with $\delta_\rho < \frac{1}{2}$. But (P1) allows $\sup \delta_\rho = \frac{1}{2}$, so the infinite sum is not controlled term by term. Controlling it is exactly the Tauberian step, which is not a telegrapher argument. For the same reason the telegrapher picture proves none of (P1)–(P4); it only organizes them.
- (iv) *Where unitarity enters*. (P0)–(P3) are dissipativity and stability statements about $Z(\tau)$ in the storage energy, which is a conventional Hilbert structure (Remark 25.1.3). (P4) is the only rung at which the canonical Krein form becomes a Hilbert form and the scaling flow becomes unitary (Thm. 25.1.2(f)). In this precise sense the ladder separates “RH is a statement about unitarity” from the PNT, whose analytic input is strict dissipativity.
- (v) *Error terms in general (pointer only)*. Ingham's direction (zero-free region $\Rightarrow$ error term) and its converses due to Turán and Staś, with loss in the constants, are stated for general regions $\beta \leq 1 - \eta(|t|)$ in Pintz [122]. Pintz writes them in terms of $\omega(x) = \min_{t \geq 1} (\eta(t) \log x + \log t)$. We verified the bibliographic record of [122] but read its content only through a secondary summary, so nothing here depends on it. The rung (P2) uses only Davenport.

25.2. One channel: the telegrapher normal form of a functional-equation pair

This section writes out Prop. 25.1.6(e): the restriction of the operator model of §25.1 to one pair $\{\rho, \rho^*\}$, where every operator is a 2×2 matrix.

DEFINITION 25.2.1 ([DEF]). For a zero ρ let

$P_\rho = (D - \rho)(D - (1 - \bar{\rho})) = D^2 - (1 + 2it_\rho)D + (\beta_\rho(1 - \beta_\rho) - t_\rho^2 + it_\rho)$. Its kernel is spanned by $e^{\rho u}$ and $e^{(1-\bar{\rho})u}$. If $\delta_\rho = 0$, it is $(D - \rho)^2$ with kernel $\{e^{\rho u}, ue^{\rho u}\}$.

THEOREM 25.2.2 (telegrapher normal form of a zero; [THM]). *Let $\rho = \beta + it$ be a nontrivial zero, $\delta = \beta - \frac{1}{2}$.*

- (a) Demodulation. $e^{-itu} P_\rho e^{itu} = (D - \beta)(D - (1 - \beta)) = D^2 - D + \beta(1 - \beta)$, which has real coefficients. The common phase factor e^{itu} can be removed exactly because the partners share t .
- (b) Normal form. In the reversed variable $\tau = -u$ the demodulated operator is $D_\tau^2 + D_\tau + \beta(1 - \beta) = (D_\tau + \beta)(D_\tau + 1 - \beta)$. This is the uniform-mode operator of Prop. 25.3.2(b) with $LC = 1$ and

$$\frac{R}{L} = \beta, \quad \frac{G}{C} = 1 - \beta \quad (\text{or the swap}).$$

Both are ≥ 0 by (P0) of Prop. 25.1.8 (Euler product and functional equation only), so the line is passive. Both are > 0 by (P1) ($\zeta(1 + it) \neq 0$ [39, Ch. 13]), so it is strictly passive. Neither statement involves the PNT; see Remark 25.1.9.

- (c) Heaviside substitution. $\mu = \frac{1}{2}(\frac{R}{L} + \frac{G}{C}) = \frac{1}{2}$ for every zero, and $e^{-\mu\tau} = e^{u/2}$. Writing the demodulated envelope as $V = e^{u/2}W$ gives

$$W'' = \kappa_H W, \quad \kappa_H = \left(\frac{R/L - G/C}{2} \right)^2 = \frac{(2\beta - 1)^2}{4} = \delta^2.$$

Equivalently, $(R/L - G/C)^2 = (2\beta - 1)^2 = 4\delta^2$.

- (d) Distortionless $\iff$ critical line. The line of ρ satisfies Heaviside's condition $R/L = G/C$ iff $\kappa_H = 0$ iff $\text{Re } \rho = \frac{1}{2}$. Then the two time constants coincide ($\frac{R}{L} = \frac{G}{C} = \frac{1}{2}$), the operator is $(D - \rho)^2$, and the envelope is free ($W'' = 0$). If $\delta \neq 0$, the envelope satisfies the tachyonic Klein–Gordon equation $W'' = \delta^2 W$. This is literal after demodulation, but the full mode $e^{\rho u}$ still oscillates at frequency t , which is Lemma 19's correction: a complex pair, not a tachyon proper.
- (e) Spectral parameter. $\rho(1 - \rho) = \frac{1}{4} + \gamma_\rho^2$ with $\gamma_\rho = t - i\delta$, and $\beta(1 - \beta) = \frac{1}{4} - \delta^2$.

PROOF. Everything is expansion of polynomials in D , checked symbolically (check file, §3).

(a): $e^{-itu} D e^{itu} = D + it$, so $(D + it - \rho)(D + it - (1 - \bar{\rho})) = (D - \beta)(D - (1 - \beta))$. (b): substitute $D = -D_\tau$ and compare with $LC(D_\tau + R/L)(D_\tau + G/C)$. The unordered pair of roots determines $\{R/L, G/C\}$. (c): Prop. 25.3.2(a) with $LC = 1$. (d), (e): direct. $\square$

PROPOSITION 25.2.3 (what is forced and what is conventional; [THM]). *Suppose we require a functional-equation pair to be written as the uniform mode of a constant-coefficient telegrapher line after removing a common phase.*

- Forced: the demodulating phase e^{itu} (the only common phase of the pair); the unordered pair $\{R/L, G/C\} = \{\beta, 1 - \beta\}$ (the roots); $\mu = \frac{1}{2}$ (functional equation, unconditional); $\kappa_H = \delta^2$.
- Conventional: (i) the orientation $\tau = -u$. In u itself the normal form has $R/L = -\beta$, $G/C = -(1 - \beta)$, a line with “negative resistance” (gain), and nothing else changes. (ii) Which root is R/L (series) and which is G/C (shunt). The functional equation $\rho \leftrightarrow 1 - \bar{\rho}$ swaps them. That is the series/shunt (impedance/admittance) duality $R \leftrightarrow G$, $L \leftrightarrow C$ of the telegrapher equation, so neither choice is canonical. (iii) LC (the speed) and $Z_0 = \sqrt{L/C}$: the uniform mode never sees x , so these are invisible and we set $L = C = 1$. Then “ $Z_0 \mapsto 1/Z_0$ ” under the swap is also convention.
- Not available: the conjugation pairing $\{\rho, \bar{\rho}\}$. It gives a real underdamped mode with mean rate β (not fixed by the functional equation) and leaves R/L , G/C individually undetermined. Only the functional-equation pairing gives a universal μ .

PROPOSITION 25.2.4 (relation to Lemma 17's Lax–Phillips $+\frac{1}{4}$; [THM]). *In $r = \log y$ the cusp operator $y^2 \partial_y^2$ is $D(D - 1)$, with exponents $\{0, 1\}$. That is a line with $\{R/L, G/C\} = \{0, 1\}$,*

$\mu = \frac{1}{2}$, $\kappa_H = \frac{1}{4}$, which is Lemma 17, Prop. “cusp”: the bare half-density mass term $\frac{1}{4}$. Lax–Phillips’s $L = y^2\Delta + \frac{1}{4}$ [91] restricts to $(D - \frac{1}{2})^2$, the distortionless line with coincident time constants. The demodulated zero line of Thm. 25.2.2 is

$$(D - \beta)(D - (1 - \beta)) = D(D - 1) + \beta(1 - \beta) = (D - \frac{1}{2})^2 - \delta^2.$$

So a zero adds $\beta(1 - \beta) = \frac{1}{4} - \delta^2$ to the bare cusp line where Lax–Phillips add $\frac{1}{4}$. The cancellation is exact iff $\delta = 0$. In words: $D(D - 1) + c$ has roots $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, so $c = \frac{1}{4}$ is the largest shift with real roots and the unique one with coincident roots. Lax–Phillips’s $+ \frac{1}{4}$ is that shift, and RH says every zero line attains it. Also, $Ly^s = (s - \frac{1}{2})^2 y^s$, so on y^ρ the Lax–Phillips eigenvalue is $(\rho - \frac{1}{2})^2 = -\gamma_\rho^2$. That is real (and ≤ 0) for every nonreal on-line zero, and nonreal for an off-line zero.

PROOF. $y^2\partial_y^2 = \partial_r^2 - \partial_r$ (Lemma 17); the rest is algebra (§3 of the check file). $\square$

REMARK 25.2.5 (caveat on the Lax–Phillips match). The match is an identity between operators in one variable after the identification $r \leftrightarrow u$, and that identification is conventional. In Lax–Phillips theory the zeros of ζ appear as resonances at $s = \rho/2$, not as y^ρ (Lemma 17, Remark “bounce”). The Lax–Phillips-native statement is Lemma 7, Thm. 4.1(i): the resonances decay at a common rate iff RH. That is again “equal attenuation for all frequencies”, but it is a different realization of the same condition, not a consequence of Prop. 25.2.4.

PROPOSITION 25.2.6 (the channel energy: distorted $\Rightarrow$ hyperbolic; [THM]). Let $\delta \neq 0$ and let $\mathcal{V} = \{W = ae^{\delta u} + be^{-\delta u}\}$ be the envelope space of a line, with the scaling flow $(T_s W)(u) = W(u - s)$.

- (a) The Heaviside energy $\frac{1}{2}(W'^2 - \delta^2 W^2)$ is conserved and equals $-2\delta^2 ab$. It is a form of signature $(1, 1)$, and the pure modes $e^{\pm\delta u}$ are neutral.
- (b) More generally, every T -invariant Hermitian form on $\mathcal{V}$ is $c a \bar{b} + \bar{c} \bar{a} b$. So it is hyperbolic or zero, and never definite.
- (c) If $\delta = 0$ the envelope of the explicit-formula wave is constant (a only), T acts trivially, and $m|a|^2$ with $m > 0$ is invariant and definite.

This is the telegrapher form of the Krein-space fact used in Lemmas 17 and 19 (non-real frequency $\Rightarrow$ neutral eigenvectors in a $(1, 1)$ plane). The Weil form’s contribution of an off-line pair, $m_\rho(\tilde{F}(\gamma_\rho)\tilde{F}(\bar{\gamma}_\rho) + \text{c.c.})$ (Lemma 18, Thm. 2.1), has exactly the shape (b).

PROOF. (a): substitute (checked symbolically). (b): T_s acts by $\text{diag}(e^{-\delta s}, e^{\delta s})$, so invariance kills the $|a|^2$ and $|b|^2$ coefficients. (c): clear. $\square$

25.3. The transfer-function reading

This is the coordinate reading of §25.1: J.W.’s polar pair $(\log |x|, \arg x)$ evaluated on single waves. It is equivalent to the operator reading but carries less structure: the Krein form and J_{FE} only appear implicitly, through the partner of each zero.

25.3.1. The classical telegrapher line. A uniform line with series resistance $R \geq 0$ and inductance $L > 0$, and shunt conductance $G \geq 0$ and capacitance $C > 0$ (per unit length), obeys $v_{xx} = LCv_{tt} + (RC + LG)v_t + RGv$. A time-harmonic wave $e^{i\omega t}V(x)$ has $V'' = \gamma_{\text{tel}}^2 V$ with

$$\gamma_{\text{tel}}(\omega)^2 = (R + i\omega L)(G + i\omega C) = RG - \omega^2 LC + i\omega(RC + LG),$$

with $\gamma_{\text{tel}} = \alpha_{\text{tel}} + i\beta_{\text{tel}}$, $\alpha_{\text{tel}} \geq 0$, $\beta_{\text{tel}} > 0$ for $\omega > 0$.

PROPOSITION 25.3.1 (Heaviside’s condition; [THM], classical). For $\omega > 0$ the following are equivalent:

- (i) the attenuation $\alpha_{\text{tel}}(\omega)$ does not depend on ω ;
- (ii) the phase velocity $\omega/\beta_{\text{tel}}(\omega)$ does not depend on ω (no dispersion);
- (iii) $R/L = G/C$ (equivalently $RC = LG$).

In that case $\gamma_{\text{tel}} = \sqrt{LC}(R/L + i\omega)$, $\alpha_{\text{tel}} = R\sqrt{C/L}$, $\beta_{\text{tel}} = \omega\sqrt{LC}$, and the characteristic impedance $Z_0 = \sqrt{(R + i\omega L)/(G + i\omega C)} = \sqrt{L/C}$ is real and constant. A matched line of length ℓ has transfer function $H(\omega) = e^{-\gamma_{\text{tel}}\ell}$, so in polar coordinates

$$\log |H| = -\alpha_{\text{tel}} \ell \quad (\text{attenuation}), \quad \arg H = -\beta_{\text{tel}} \ell \quad (\text{phase}).$$

PROOF. Write $\gamma_{\text{tel}}^2 = (\alpha_{\text{tel}}^2 - \beta_{\text{tel}}^2) + 2i\alpha_{\text{tel}}\beta_{\text{tel}}$ and compare with the formula above: $\alpha_{\text{tel}}^2 - \beta_{\text{tel}}^2 = RG - \omega^2 LC$ and $2\alpha_{\text{tel}}\beta_{\text{tel}} = \omega(RC + LG)$. (i) $\Rightarrow$ (iii): let $\alpha_{\text{tel}} \equiv a$. If $a = 0$, then $RC + LG = 0$, so $R = G = 0$ and (iii) holds. If $a > 0$, then $\beta_{\text{tel}} = \omega(RC + LG)/2a$, and $a^2 - \omega^2(RC + LG)^2/4a^2 = RG - \omega^2 LC$ for all ω . Comparing coefficients gives $a^2 = RG$ and $(RC + LG)^2 = 4a^2 LC = 4RGLC$, i.e. $(RC - LG)^2 = 0$. (ii) $\Rightarrow$ (iii): if $\beta_{\text{tel}} = b\omega$, then $\alpha_{\text{tel}} = (RC + LG)/2b$ is constant, and we are back in (i). (iii) $\Rightarrow$ (i),(ii): $(R + i\omega L)(G + i\omega C) = LC(R/L + i\omega)(G/C + i\omega) = LC(R/L + i\omega)^2$. The statement about Z_0 follows in the same way, and H is the standard matched-line transfer function. All identities were checked symbolically (`num/lemma25_checks.out`, §1). Historical source: [66, Vol. II, pp. 119–154]. Textbook treatment: [126, §2.7, “The distortionless line”]. $\square$

So J. W.’s coordinate pair $(\log |x|, \arg x)$, which is the pair (Log, Arg) of amoeba/phase-tropical geometry [101], applied to a line’s transfer function is literally (attenuation, phase) times length. Heaviside’s condition says the first coordinate is flat in ω and the second is linear in ω .

PROPOSITION 25.3.2 (Heaviside substitution and the uniform mode; [THM]). *Let $\mu = \frac{1}{2}(\frac{R}{L} + \frac{G}{C})$ and $\kappa_H = (\frac{1}{2}(\frac{R}{L} - \frac{G}{C}))^2$. Then:*

- (a) $v = e^{-\mu t}w$ turns the telegrapher equation into $LC w_{tt} - w_{xx} - LC \kappa_H w = 0$. The mass term is $LC \kappa_H = \frac{(RC-LG)^2}{4LC}$, as in Lemma 17, Prop. “telegrapher”.
- (b) The uniform (x -independent, $k = 0$) mode obeys $LC(D_t + \frac{R}{L})(D_t + \frac{G}{C})v = 0$. Its two characteristic exponents are $-R/L$ and $-G/C$: one time constant from the series branch and one from the shunt branch. Their mean is $-\mu$, and half their difference squared is κ_H .
- (c) The mode $w = e^{ikx}f(t)$ has $LC f'' + (k^2 - LC \kappa_H)f = 0$ and the conserved reduced energy $\mathcal{E}_k = \frac{1}{2}(LC f'^2 + (k^2 - LC \kappa_H)f^2)$. So $\mathcal{E}_k \geq 0$ for every k iff $\kappa_H = 0$ iff $RC = LG$. If $\kappa_H > 0$, $\mathcal{E}_k$ is indefinite exactly for $k^2 < LC \kappa_H$.

PROOF. (a) is direct substitution, checked symbolically. (b): $LCs^2 + (RC + LG)s + RG = (Ls + R)(Cs + G)$. (c) follows from (a) by separation of variables. The physical stored energy $\frac{1}{2}(LI^2 + CV^2)$ of a passive line is always positive, so it is the *reduced* energy (in Caldirola–Kanai variables [18, 80]; Lemma 17, Prop. “CK”) whose positivity is equivalent to Heaviside’s condition. $\square$

25.3.2. The zeta dictionary: polar coordinates of a zero-wave. In the explicit formula [39, Ch. 17], each nontrivial zero contributes $-x^\rho/\rho$ to $\psi(x)$. In $u = \log x$,

$$x^\rho = e^{\beta_\rho u} e^{it_\rho u}, \quad \log |x^\rho| = \beta_\rho u, \quad \arg x^\rho = t_\rho u \pmod{2\pi}.$$

After the half-density normalization $E(u) = e^{-u/2}(\psi(e^u) - e^u)$ of Lemma 17, Thm. “arithmetic distortionless”, the wave becomes $e^{(\rho - \frac{1}{2})u} = e^{\delta_\rho u} e^{it_\rho u}$.

PROPOSITION 25.3.3 (spatial / transfer-function reading; [THM] for the identities). *Read u as line length and $\omega := t_\rho$ as frequency. Write $H_\rho(u) = e^{(\rho - \frac{1}{2})u} = e^{-\gamma_{\text{tel}} u}$ with $\gamma_{\text{tel}}(\rho) = -\delta_\rho - it_\rho$,*

i.e. attenuation $\alpha_{\text{tel}} = -\delta_\rho$ and phase constant $\beta_{\text{tel}} = -t_\rho$ (the sign only fixes the direction of travel). Then:

- (a) Phase leg. $\beta_{\text{tel}} = -\omega$ is exactly linear, with unit phase velocity, for every zero. This is automatic: the frequency is defined as $\text{Im } \rho$, so the phase leg carries no information about RH.
- (b) Attenuation leg. The functional-equation partner $1 - \bar{\rho}$ has the same frequency t_ρ and the opposite attenuation $+\delta_\rho$. So the “dispersion relation” $\omega \mapsto \alpha_{\text{tel}}$ read off the zero set takes two values $\pm\delta_\rho$ at $\omega = t_\rho$ unless $\delta_\rho = 0$.
- (c) Hence the zero set defines a distortionless line in Heaviside’s sense (single-valued α_{tel} , independent of ω) iff all $\delta_\rho = 0$, i.e. RH [RH-EQUIV]. Moreover the constant is then forced to be 0 (a lossless normalized line, i.e. growth rate exactly $\frac{1}{2}$ before normalization), not just “some constant”.

PROOF. (a) is the definition. (b): $1 - \bar{\rho} = (1 - \beta_\rho) + it_\rho$, so $\delta_{1-\bar{\rho}} = -\delta_\rho$. (c): a single-valued constant α_{tel} requires $\delta = -\delta$ at every occupied frequency. $\square$

REMARK 25.3.4 (which coordinate is which). This makes J.W.’s decomposition exact. $\arg x^\rho$ is the *phase* (Heaviside’s β_{tel}), and it is distortionless for free. $\log |x^\rho|$ is the *attenuation* (Heaviside’s α_{tel}), and it carries everything. Its average over a functional-equation pair is always $\frac{1}{2}$ (the half-density). Its spread within the pair is $2|\delta|$. This also sharpens the halfway/norm discussion of Lemma 19. The shadow (trace) mixes both coordinates, the norm is the Log coordinate alone, and the functional equation already fixes the mean of Log over each pair. RH is the statement that the spread vanishes.

25.4. Assembly: the explicit formula as a superposition of lines

Call a *channel* an orbit O of $\rho \mapsto 1 - \bar{\rho}$ on the nontrivial zeros: $O = \{\rho\}$ if $\delta_\rho = 0$, and $O = \{\rho, 1 - \bar{\rho}\}$ otherwise. All zeros in a channel share t_O and $|\delta_O|$.

THEOREM 25.4.1 (multichannel line; [THM], last clause [RH-EQUIV]). For $x = e^u > 1$ (with ψ_0 the averaged ψ), Davenport’s explicit formula [39, Ch. 17], read in channels, is

$$E(u) = e^{-u/2}(\psi_0(e^u) - e^u) = -\lim_{T \rightarrow \infty} \sum_{|t_O| < T} e^{it_O u} w_O(u) - e^{-u/2} \log 2\pi - \frac{1}{2} e^{-u/2} \log(1 - e^{-2u}),$$

with $w_O = \sum_{\rho \in O} m_\rho e^{\delta_\rho u} / \rho$ and $(D^2 - \delta_O^2) w_O = 0$. So each channel is the Heaviside-reduced envelope of the line of Thm. 25.2.2, carried at frequency t_O . Moreover:

- (a) Pole channel. The carrier e^u (pole $s = 1$) and the constant $-\log 2\pi$ (from $s = 0$) form the exponent pair $\{1, 0\}$ at $t = 0$. That is the line with $\{R/L, G/C\} = \{1, 0\}$, $\delta = \frac{1}{2}$, $\kappa_H = \frac{1}{4}$, which is exactly the bare cusp line of Prop. 25.2.4. It is the one distorted channel present unconditionally. It sits exactly on the boundary of passivity ($G/C = 0$, (P0) of Prop. 25.1.8). So (P1) says precisely that no zero-line is as distorted as the pole line.
- (b) Trivial zeros $-2n$ are unpaired (their reflections $1 + 2n$ are not zeros). They are the archimedean (Γ -factor) term, not lines of this type.
- (c) Count. Each distorted nontrivial channel contains exactly one zero with $\text{Re } \rho > \frac{1}{2}$. So the number of distorted channels is $1 + N_+$ (pole channel included). By Lemma 15 (“Lorentzian index”) and Lemma 17, Thm. “ ζ : distortion” (iii), this is the number of timelike directions of $\mathcal{L}$. The timelike directions of the Lorentzian Weil form are the distorted lines.
- (d) [RH-EQUIV] All nontrivial channels distortionless ($R/L = G/C$ for every line) $\iff$ RH. This is the definition of RH written in channels. It is no new leverage, and it is equivalent to Lemma 17, Thm. “arithmetic distortionless”.

PROOF. Group the zeros in Davenport’s symmetric sum by height. Partners share t , so the grouping is compatible with the truncation $|t| < T$. For fixed t there are finitely many zeros, so each group is a finite sum of envelopes in $\ker(D^2 - \delta^2)$. A zero of multiplicity m gives $m x^\rho / \rho$, since $-\zeta'/\zeta$ has simple poles, so no $ue^{\rho u}$ terms appear. (a), (b) read off the remaining terms. (c): the count follows from the definitions and the cited lemmas. $\square$

[NUM] (`num/lemma25_checks.out`, §4.) Primes versus channels (1000 on-line channel pairs + pole + trivial): $E = -0.6439$ vs -0.6315 at $x = 100.5$; -0.1207 vs -0.1218 at 1000.5 ; 0.1290 vs 0.1785 at $10^4 + 0.5$; 0.1615 vs 0.1676 at $10^5 + 0.5$. The differences are truncation at 1000 zeros, consistent with Lemma 17’s rms 0.031. A hypothetical channel $\rho = 0.6 + 1000i$ ($\kappa_H = 0.01$) has envelope $3.1 \cdot 10^{-3}$, 0.15 and 22.0 at $u = 10, 50, 100$. That is the growing envelope $W'' = \delta^2 W$. Numerically all first five zeros satisfy $|\zeta(1 - \bar{\rho})| < 2 \cdot 10^{-30}$ (30 digits), as the functional equation requires.

25.5. Curves over $\mathbb{F}_q$: the same lines, with $\delta = 0$ proved

Let $X/\mathbb{F}_q$ have Frobenius eigenvalues $\alpha_1, \dots, \alpha_{2g}$ and put $u = n \log q$. Write $\rho_i = \log \alpha_i / \log q$ (any branch). Then $\alpha_i^n = e^{\rho_i u}$, with rate $\beta_i = \log |\alpha_i| / \log q$ and phase $\arg \alpha_i$ per step.

PROPOSITION 25.5.1 ([THM]). (a) *Poincaré duality (the functional equation of Z_X) makes $\{\alpha_i\}$ invariant under $\alpha \mapsto q/\bar{\alpha}$. The partner has the same argument and rate $1 - \beta$, so each pair is a line with $\{R/L, G/C\} = \{\beta, 1 - \beta\}$ and $\mu = \frac{1}{2}$.* (b) *On the sampled line, $x_n = a\alpha^n + b(q/\bar{\alpha})^n$ demodulates to $w_n = x_n e^{-in \arg \alpha} q^{-n/2}$, which obeys the discrete telegrapher equation*

$$w_{n+2} - 2w_{n+1} + w_n = \kappa_H^{\text{disc}} w_{n+1}, \quad \kappa_H^{\text{disc}} = 4 \sinh^2(\tfrac{1}{2} \delta \log q) = \delta^2 (\log q)^2 + O(\delta^4).$$

(c) *The pole channel $\{q^n, 1\}$ has rates $\{1, 0\}$ and $\delta = \frac{1}{2}$, as for ζ .* (d) *Weil’s theorem [158] (equivalently Rosati positivity, Lemma 19, or Hodge index plus Cauchy–Schwarz, Lemma 17, Thm. “distortionless = Hodge index”) gives $\delta_i = 0$ for all i . Every line of a curve is distortionless.*

PROOF. (a): $q/\bar{\alpha} = |\alpha|^{-2} q \alpha$ is a positive multiple of α . (b): the demodulated roots are $q^{\pm \delta}$, with sum $2 \cosh(\delta \log q)$ and product 1. (c), (d): as stated. $\square$

TABLE 25.1. [NUM] Lines of the Lemma 3/19 examples (`num/lemma25_checks.out`, §5). “partner err” = $\min_j |q/\bar{\alpha}_i - \alpha_j|$; residual = max error of the discrete telegrapher recurrence (b) on the channel of largest $|\alpha|$, $n \leq 11$.

example	$ \alpha /\sqrt{q}$	$\beta, 1 - \beta$	$\arg \alpha = \arg(q/\bar{\alpha})$	δ	$\kappa_H = \delta^2$	κ_H^{disc}	partner err / resid.
$E/\mathbb{F}_5 (1 + 3T + 5T^2)$	1	0.5, 0.5	± 2.306111	0	0	0	$2 \cdot 10^{-16} / 1 \cdot 10^{-15}$
$C_2/\mathbb{F}_7 (1 + 3T + 4T^2 + 21T^3 + 49T^4)$	1, 1	0.5, 0.5	$\pm 1.183200, \pm 2.808119$	0	0	0	$1 \cdot 10^{-15} / 5 \cdot 10^{-15}$
coin $1 - 22T^2 + 49T^4$	1.668416, 0.599371	0.763052, 0.236948	0, π	± 0.263052	0.069196	0.267787	$4 \cdot 10^{-15} / 1 \cdot 10^{-12}$
quadruple $ \alpha /\sqrt{7} = e^{\pm 0.3}$, $\arg = \pm 1$	1.349859, 0.740818	0.654170, 0.345830	± 1.000000	± 0.154170	0.023768	0.090677	0 / $3 \cdot 10^{-15}$

The coin’s lines are at phase 0 and π , i.e. zero frequency. Its envelope is a real exponential even before demodulation, the one case where “tachyon” is literal (cf. Lemma 19(b)). The quadruple passes every “volume” test of Lemma 19 §4 and is caught here as two distorted lines with $\kappa_H = 0.0238$. The real curves have every line distortionless.

25.6. Energy positivity versus norm positivity

This section separates three positivity notions that the transmission-line language tends to merge. The first is the physical energy of a line, which is always positive. The second is a Hamiltonian bounded below (the spectrum condition). The third is a positive-definite inner product (no negative-norm states, i.e. unitarity). Weil positivity is the third.

PROPOSITION 25.6.1 (energy compartments; [THM], classical). *Take the telegrapher equations $V_x = -LI_t - RI$, $I_x = -CV_t - GV$ on a closed (periodic) line, with energy $\mathcal{E} = \frac{1}{2} \int (LI^2 + CV^2) dx$, split into the magnetic part $\mathcal{E}_M = \frac{1}{2} \int LI^2$ and the electric part $\mathcal{E}_E = \frac{1}{2} \int CV^2$.*

- (a) $\mathcal{E}$ is positive definite for all $R, G \geq 0$, whether or not the line is distortionless. It satisfies $\dot{\mathcal{E}} = - \int (RI^2 + GV^2) dx = -\frac{2R}{L}\mathcal{E}_M - \frac{2G}{C}\mathcal{E}_E$.
- (b) In the uniform mode the compartments decouple and decay at the rates $2R/L$ (magnetic) and $2G/C$ (electric).
- (c) There is a constant c with $\dot{\mathcal{E}} = -2c\mathcal{E}$ for every state iff $R/L = G/C = c$. Then $\mathcal{E}(t) = e^{-2\mu t}\mathcal{E}(0)$ exactly, with $\mu = R/L$. So distortionless means precisely that the two energy compartments decay at equal rates: the dissipation is proportional to the stored energy, independently of how the energy is shared between the magnetic and electric parts. If $R/L \neq G/C$, then $e^{2\mu t}\mathcal{E}$ oscillates as the energy moves between the compartments.

PROOF. (a): multiply the equations by I and V , add, and integrate; the flux $\int (VI)_x$ vanishes on a closed line. (b): put $\partial_x = 0$. (c): the state (I, V) at a fixed time is arbitrary, so $\int (RI^2 + GV^2) = c \int (LI^2 + CV^2)$ for all (I, V) forces $R = cL$ and $G = cC$. Conversely, $(I, V) = e^{-ct}(\tilde{I}, \tilde{V})$ with $(\tilde{I}, \tilde{V})$ solving the lossless equations, whose energy is conserved. [NUM] (num/lemma25_energy.out): for the $k = 3$ mode, $e^{2\mu t}\mathcal{E}$ is constant (0.500000) for $R = G = 0.5$, $L = C = 1$, and ranges over $[0.4412, 0.5769]$ for $R = 0.9$, $G = 0.1$. $\square$

Zeta reading. In the channel normal form (Thm. 25.2.2, with $\tau = -u$), the two compartments of the line of ρ decay at 2β and $2(1 - \beta)$. RH is “equal compartment decay for every channel”. Which compartment is magnetic is the conventional series/shunt choice of Prop. 25.2.3. This is a restatement and adds no leverage.

REMARK 25.6.2 (energy bounded below vs. positive-definite norm; [THM] for the facts, the reading is ours). In quantum theory two different positivity conditions are imposed. One is the *spectrum condition*: the generator of time translations is ≥ 0 on the state space. The other is *positivity of the metric*: the inner product is positive definite, i.e. there are no negative-norm states. The second is what makes time evolution unitary in a Hilbert space. Pais–Uhlenbeck-type theories (Lemma 17) can trade one for the other (Lemma 17, Ostrogradsky section). For ζ :

- (i) The relevant “energy” is Lemma 17’s Noether charge $\mathcal{E}[F] = \mathcal{Q}(\mathbf{D}_{\text{sc}}F, F)$. Under RH its spectrum is $\{\gamma\}$, which is symmetric under $\gamma \mapsto -\gamma$, so it is *never* bounded below. The spectrum condition is not the right target.
- (ii) Weil positivity, $\mathcal{Q}(F, F) \geq 0$ for all F [157] (Lemma 1), is positivity of the *metric*. By Lemma 19, Thm. 3.2(iii), it is equivalent to a Hilbert space on which the scaling flow is unitary [RH-EQUIV].
- (iii) Naive term-by-term energy positivity fails. In Lemma 17’s field form $\mathcal{S} = \text{pole} + \langle F, \mathbf{K}(D)F \rangle - (\text{primes})$, the kinetic symbol $\mathbf{K}(r) = \text{Re } \psi(\frac{1}{4} + \frac{ir}{2}) - \log \pi$ is *negative* for $|r| < 6.28984$. For a Gaussian, the three pieces are $3.344 - 2.238 - 1.106 \approx 0$ (Lemma 17, Thm. “field form”), so positivity of $\mathcal{Q}$ is a

cancellation between indefinite pieces. At the archimedean place alone, Connes–Consani prove $W_\infty(g * g^*) \geq \text{Tr}(\vartheta(g)\mathbf{S}\vartheta(g)^*) \geq 0$ only under a support restriction ($\text{supp } g \subset [2^{-1/2}, 2^{1/2}]$) and vanishing conditions on $\hat{g}$ [33, Thm. 1]. Local positivity is therefore not available in general. An unrefereed manuscript found in our search (§25.6.2) asserts that the local terms are “always ≥ 0 ”. That is contradicted by $K < 0$ near $r = 0$.

- (iv) The telegrapher analogue of this split is Prop. 25.6.1(a) versus Prop. 25.3.2(c). The physical energy $\mathcal{E}$ is positive for every line, distorted or not, so it is the wrong positivity. The Heaviside-reduced energy is positive iff the line is distortionless. Only the second corresponds to Weil positivity. On the zero space this is Prop. 25.1.7: storage energy (a) versus reduced energy (b) and Weil energy (c).

25.6.1. Osterwalder–Schrader reflection positivity. In Osterwalder–Schrader (OS) positivity [115] there is a time reflection θ , and the condition is imposed *only on test functions supported in positive time*: $\langle \theta f, f \rangle_S \geq 0$. For a translation-invariant two-point kernel s on $\mathbb{R}$ and $\theta u = -u$, this reads

$$\text{RP}_s(f) = \iint \overline{f(x)} f(y) s(x+y) dx dy \geq 0 \quad (\text{supp } f \subset (0, \infty)).$$

The OS reconstruction then gives a Hilbert space with a contraction semigroup e^{-tH} , $H \geq 0$. That is the *spectrum condition*, obtained from reflection positivity [115]; [111, Chs. 2–4]. Weil’s form, written in the conventions of Lemma 17 ($F^*(u) = \overline{F(-u)}$), is

$$\mathcal{Q}(F, F) = \mathcal{W}(F * F^*) = \iint F(x) \overline{F(y)} \mathcal{W}(x-y) dx dy \quad (\text{all } F \in C_c^\infty(\mathbb{R})).$$

PROPOSITION 25.6.3 (shape comparison; [THM]). (a) *Weil positivity is positive-definiteness of a distribution on the group $(\mathbb{R}, +)$ (the convolution-square or Bochner–GNS shape), required for all supports. Since $\mathcal{W}$ is even (functional equation), $\mathcal{Q}(F, F) = \iint F(x) (\theta F)(y) \mathcal{W}(x+y)$. That is formally the OS expression, but evaluated on the pair $(F, \theta F)$ with no support restriction, which is only a change of variables. The positivity it encodes produces a unitary group (Lemma 19), not a contraction semigroup with $H \geq 0$.*

- (b) *The genuine OS condition for the kernel $s = \mathcal{W}$ and $\theta u = -u$ fails. Take $f \geq 0$ a bump of width ε centred at $c = \frac{1}{2} \log 2$. The atom $-\frac{\log 2}{\sqrt{2}} \delta_{\log 2}$ of $\mathcal{W}$ on $(0, \infty)$ contributes*

$$-\frac{\log 2}{\sqrt{2}} \int f(x) f(\log 2 - x) dx \asymp -\varepsilon^{-1},$$

while the smooth part (pole kernel minus archimedean kernel) stays bounded. Hence

$$\text{RP}_\mathcal{W}(f) \rightarrow -\infty.$$

The same holds at every prime power and for any reflection $u \mapsto 2a - u$, by translation invariance. [NUM]: $\text{RP} = -2.50, -11.08, -35.58, -121.35$ for $\varepsilon = 0.1, 0.03, 0.01, 0.003$ (smooth part +1.18; num/lemma25_energy.out).

- (c) *So Weil positivity is not an instance of OS reflection positivity for the reflection $u \mapsto -u$ of the scaling line. Structurally it is the analogue of positivity of the Minkowski-side (Wightman) inner product, or of a positive-definite function on a group. OS positivity is the Euclidean device for producing such a positive inner product from a kernel that need not be positive-definite. If an OS route to RH exists, it would need a different kernel and reflection whose OS reconstruction yields Weil’s Hilbert space. None is known (§25.6.2). [ANALOGY] for this last reading.*

PROOF. (a): substitute $y \mapsto -y$ and use $\mathcal{W}(-u) = \mathcal{W}(u)$. (b): on $(0, \infty)$, $\mathcal{W}$ equals $2 \cosh(u/2)$ (pole) minus the archimedean kernel k_∞ (smooth for $u > 0$) minus $\sum_n \Lambda(n) n^{-1/2} \delta_{\log n}$ (Lemma 17, local-kernel form). For a bump of width ε , $\iint f(x) f(y) \delta(x + y - \log 2) = \int f(x) f(\log 2 - x) dx = \int f^2 \asymp \varepsilon^{-1}$, while the smooth part is $O(\|f\|_1^2) = O(1)$. Only $n = 2$ meets $x + y \in (\log 2 - 2\varepsilon, \log 2 + 2\varepsilon)$ once $\varepsilon < \frac{1}{2} \log \frac{3}{2}$. (c) summarizes (a) and (b). $\square$

25.6.2. What the literature contains (web-verified, 2026-10-21). We searched for any published theorem or discussion relating Weil positivity to OS reflection positivity, including Connes, Jaffe, Neeb–Ólafsson, Yoshida and Burnol.

- **No refereed theorem was found** that identifies Weil positivity with, or derives it from, OS reflection positivity.
- **Closest structural bridge (cited, not about ζ).** Neeb–Ólafsson connect reflection positivity on $\mathbb{R}$ with Lax–Phillips outgoing realizations of unitary one-parameter groups [110]. Their book develops this, with Lax–Phillips scattering in its chapter “Reflection positivity on the real line” [111]. Since ζ has a Lax–Phillips realization (Pavlov–Faddeev; Lemmas 7, 17), this is the place where an OS-type statement for ζ could be formulated. We found no paper that does it.
- **Connes–Consani [33]** give a “potential conceptual reason” for Weil positivity via positivity of a trace of the scaling action compressed to Sonin space (archimedean place, small support). The mechanism is a trace and a prolate/Toeplitz analysis, not OS reflection positivity.
- **Jaffe [77]** reviews reflection positivity with no arithmetic application. **Yoshida [163]** studies Hermitian forms attached to zeta functions (Weil’s form). We verified the bibliographic record only and found no indication that it uses reflection positivity. For **Burnol** we found only expository commentary on the Weil explicit formula (M. Watkins’ web pages), with no OS link.
- **Unrefereed material.** Two PhilArchive manuscripts (ids TOUHPF, TOULMA) state the Weil/OS analogy informally (“positivity turns correlation data into a Hilbert space”). One of them asserts that the local terms are nonnegative, which is false (Remark 25.6.2(iii)). A 2025 preprint (Preprints.org 202507.1370 / ai.viXra 2507.0079) claims a proof of RH “via OS analytics”. It is not peer-reviewed, and we do not rely on or endorse it. None of these is cited as support.

25.7. What is missing

- (1) **The functional equation fixes the mean and not the spread.** In Heaviside’s language the functional equation fixes $\mu = \frac{1}{2}$ for every line (Thm. 25.2.2(c)) unconditionally. It says nothing about κ_H . In operator form (Prop. 25.1.6(b),(c)): the functional equation fixes the J_{FE} -even part $\mu = \frac{1}{2}$ of β , and the obstruction is the J_{FE} -odd part δ , whose square κ_H is J_{FE} -even and therefore invisible to the functional equation. Distortionless lines need the two time constants $R/L = \beta$ and $G/C = 1 - \beta$ to *coincide*, and no structure on ζ is known that equalizes them.
- (2) **Equal partner rates are equivalent to positivity.** By Prop. 25.2.6, a line admits a positive flow-invariant energy iff it is distortionless. So “every line has equal partner rates” is equivalent to the existence of one positive invariant energy on the whole multichannel line, i.e. Weil positivity of $\mathcal{Q}$ [157] (Lemma 1). Lemma 19, Thm. 3.2(iii) gives the Hilbert-space form [RH-EQUIV]. Prop. 25.3.2(c) shows that asking for the reduced energy to be positive is the Heaviside condition itself, so the telegrapher picture cannot supply that positivity without circularity.

- (3) **For curves the positivity comes from geometry.** For curves the energy is the Rosati form (Lemma 19) or the negative-definite transverse part of $\text{NS}(X)_\mathbb{R}$ (Lemma 17), both from an ample class on $X \times X$. For ζ the analogous surface or polarization is the open item “ ζ -side body / geometric transverse spacelikeness” of the ranking.
- (4) **Physical lines are made distortionless by design, and ζ has no designer.** Heaviside’s practical remedy was to add series inductance (loading) until $R/L = G/C$. The closest arithmetic analogue is the de Bruijn–Newman heat flow H_t (Lemma 2): by definition of Λ_{dBN} , all zeros of H_t are real (“on the line”) exactly for $t \geq \Lambda_{\text{dBN}}$. Rodgers–Tao [129] prove $\Lambda_{\text{dBN}} \geq 0$: ζ needs nonnegative loading, and RH says zero loading suffices. [ANALOGY]: the heat flow does not act on a line’s κ_H by any formula derived here.

25.8. Open questions

Answered in this revision (removed from the list): which reading is primary (the operator reading, now §25.1), and whether passivity should be stated as the unconditional half (it is now the passivity ladder, Prop. 25.1.8, with Remark 25.1.9).

- (1) Should Volume 2 keep the electrical names for the functional equation (*series/shunt duality*, $R \leftrightarrow G$, $L \leftrightarrow C$), or state it only as $J_{\text{FE}}\beta J_{\text{FE}} = 1 - \beta$ with $J_{\text{FE}}\mathcal{H}_+ = \mathcal{H}_-$ (Prop. 25.1.6(d))? The operator identity is forced. Which branch is called “series” is a convention.
- (2) Is the loading-coil / heat-flow analogy (§25.7(4)) worth a quantitative test? The test would track κ_H of the complex zero pairs of H_t for $t < 0$ as they collide.
- (3) Is a targeted note worth writing? It would formulate an OS-type positivity for the Pavlov–Faddeev/Lax–Phillips realization of ζ (Lemmas 7, 17), using Neeb–Ólafsson’s dictionary between reflection positivity and outgoing realizations [110]. Prop. 25.6.3 shows that the naive reflection $u \mapsto -u$ on the scaling line fails, so the reflection would have to live elsewhere.
- (4) (new) The telegrapher has a spatial coupling $ik\sigma_1$ (modes $k \neq 0$). ζ realizes only the uniform mode $k = 0$ (Def. 25.1.5(vi)). Should we look for a second, “spatial” variable that would realize $k \neq 0$? Candidates are the second factor of $\text{Spec } \mathbb{Z} \times \text{Spec } \mathbb{Z}$ (Lemma 23) or the x -direction of Lax–Phillips. A genuine $k \neq 0$ structure is the one place where the telegrapher could carry information that the zero set alone does not.
- (5) (new) The first open rung of the ladder is (P3), $\Theta < 1$ (exponential stability, β invertible), which is strictly weaker than RH. Should it become an explicit intermediate target in the ranking? Every positivity route we have (Lemmas 15, 18, 19, 26) is calibrated at $\sigma = \frac{1}{2}$. Lemma 18 already gives the σ -family whose threshold is Θ .

Status summary

Statement	Status	Note
Def 25.1.1 zero space $\mathcal{H}_Z$, $[\cdot, \cdot]_Z$, ev	[DEF]	definite inner product
Thm 25.1.2 Krein structure, $J_{\text{FE}}\beta J_{\text{FE}} = 1 - \beta$, $\mathcal{Q} = \text{ev}^*[\cdot, \cdot]_Z$, flow $e^{iu\gamma}$	[THM]; (f) [RH-EQUIV]	conventional off RH Lemma 1; sympy items 1–8
Def 25.1.4, 25.1.5 telegrapher components $(M, \Sigma, \text{series/shunt}, Z(\tau))$	[DEF]; names [ANALOGY]	$k \neq 0$ has no ζ counterpart
Prop 25.1.6 $\mu = \frac{1}{2}$ (FE), κ_H J_{FE} -even, FE = series/shunt duality	[THM]	
Prop 25.1.7 storage / reduced CK / Weil energies	[THM]; iff's [RH-EQUIV]	no functional definite and conserved unless RH
Prop 25.1.8 passivity ladder (P0)–(P4)	(P0)–(P2) [THM]/[CITED]; (P3) [OPEN]; (P4) [RH-EQUIV]	Davenport Chs. 13, 17, 18; Wiener–Ikehara; von Koch
Rem 25.1.9 not “passivity = PNT”	[THM] (logic)	Pintz: pointer only
Prop 25.3.1 Heaviside condition, transfer function	[THM] (classical)	Heaviside; Pozar §2.7
Prop 25.3.2 substitution, uniform mode, reduced energy	[THM]	sympy
Prop 25.3.3 polar coordinates = (attenuation, phase)	[THM]; (c) [RH-EQUIV]	phase leg automatic
Thm 25.2.2 normal form $R/L = \beta$, $G/C = 1 - \beta$, $\mu = \frac{1}{2}$, $\kappa_H = \delta^2$	[THM]	$R, G \geq 0$ free; > 0 is (P1)
Prop 25.2.3 forced vs conventional	[THM]	
Prop 25.2.4 LP $+\frac{1}{4} = \text{maximal shift}$; exact iff $\delta = 0$	[THM]	identification $r \leftrightarrow u$ conventional
Prop 25.2.6 distorted $\Rightarrow$ hyperbolic invariant forms	[THM]	Krein, telegrapher form
Thm 25.4.1 multichannel explicit formula; count $1 + N_+$	[THM]; (d) [RH-EQUIV]	Davenport Ch. 17; Lemmas 15, 17
Prop 25.5.1 curves, discrete telegrapher, $\delta = 0$	[THM], [CITED] (Weil)	Table 25.1 [NUM]
Prop 25.6.1 equal compartment decay $\iff RC = LG$	[THM], [NUM]	classical
Rem 25.6.2 spectrum condition vs metric positivity; $K < 0$	[THM]/[CITED]	Lemma 17; CC21
Prop 25.6.3(a),(b) Weil $\neq$ OS-RP; RP fails for $\mathcal{W}$	[THM], [NUM]	(c) [ANALOGY]
§25.7(4) loading coil $\leftrightarrow$ dBN flow	[ANALOGY]	Rodgers–Tao

Files. num/lemma25_checks.py/.out/.json, num/lemma25_energy.py/.out, num/lemma25_operator.py/.out (new: items 1–12 of §25.1); lit/lemma25/verified_refs.md. Reused: num/zeros1000.txt. No shared macro changes (local macros only). Previous version: archive/lemma25.before-operator-first-20261021-202059.tex/.pdf.

CHAPTER 26

Lemma 26: Window Gram matrices under the Viro/Connes–Consani deformation $\theta_{1/\hbar}$

The Weil index lives only at $\hbar = 1$

Standalone note `lemma26.tex`, October 21, 2026.

Summary. Verdict. The deformation adds nothing beyond Lemma 15, and this is proved in both readings.

- (T) Transport the Weil form to Viro’s field $\mathbb{C}_{\hbar} \cong \mathbb{C}$. Its inertia is *independent* of $\hbar$ (Thm. 26.1.3).
- (N) Keep the numbers and change the arithmetic. The Gram matrix becomes the entrywise signed power $G_{\hbar} = \theta_{1/\hbar}(G)$. By Connes–Consani, θ_{λ} ranges over *all* automorphisms of their hyperfield $\mathbb{R}^{\text{convex}}$. Then:
 - $\hbar = 1$ is Lemma 15’s Gram.
 - As $\hbar \rightarrow 0$ the inertia is fixed by dominant permutations and signs; for Toeplitz data this is Yu’s tropical PSD test (single shadows, Lemma 23).
 - For positive-entry Grams, Garg–Aujla gives monotonicity: Lorentzian at some $\hbar < 1$ implies Lorentzian at $\hbar = 1$. So $\hbar < 1$ is a strictly *weaker* certificate and $\hbar > 1$ a *weaker* detector.
 - The Weil index is not definable over $\mathbb{R}^{\text{convex}}$ (Thm. 26.3.2).

Numerics (certified Arb inertia; 72 windows $\times$ 14 values of $\hbar \times 5$ injected ghosts, plus function-field controls).

- The true ζ Grams lose Lorentzian-ness for $\hbar < 1$ (43 of 72 windows already at $\hbar = 0.9$). The threshold $\hbar_* \rightarrow 1$ as the window grows, explained to first order by near-null negative directions.
- Ghosts *can* be separated from the truth on smaller windows at some $\hbar \neq 1$, after the fact. But the test is basis-dependent and raises false alarms on the true ζ form. It also flags real curves over $\mathbb{F}_q$, where RH is a theorem.

Not a detector. **No bearing on RH beyond reformulation.** Candidate 1 of Lemma 23 is closed.

Labels: [THM], [CITED], [NUM] (certified Arb inertia unless noted “float”), [HEUR], [RH-EQUIV]. We write $\hbar$ for the Viro deformation parameter. Lemma 15’s B-spline bandwidth is written h_B , and windows are $V(h_B, s, n) = \text{span}\{\varphi_{ks}\}_{k < n}$ with length $\ell_V = (n - 1)s$. Code: `num/lemma26_*.py`; data `num/lemma26_*.json(1)`; references `lit/lemma26/verified_refs.md`.

26.1. The construction, from the Connes–Consani sources

What exists.

- (a) [29]: a quotient R/G of a ring by a subgroup of $R^{\times}$ is a hyperring extension of the Krasner hyperfield $\mathbf{K} = \{0, 1\}$ iff $G \cup \{0\}$ is a subfield. Hence $\mathbf{H} = \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\times}$ is a hyperring over $\mathbf{K}$.
- (b) [21]: the trace-formula reading of the explicit formula expresses $N(q)$ as an intersection number for the scaling action on $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\times}$. They also construct the hyperfield

$\mathbb{R}^{\text{convex}} = (\mathbb{R}, +_c, \cdot)$ (Thm. 5.6). It is an extension of the sign hyperfield $\mathbf{S}$, with $x +_c y = \{\alpha x + \beta y : \alpha, \beta > 0, \text{sign}(\alpha x + \beta y) = \alpha \text{sign } x + \beta \text{sign } y\}$. Its automorphisms are exactly θ_λ , $\lambda \in \mathbb{R}^\times$, with $\theta_\lambda(x) = x^\lambda$ for $x > 0$ and θ_λ the identity on $\mathbf{S}$ (Prop. 5.7). Hence $\theta_\lambda(x) = \text{sign}(x)|x|^\lambda$.

- (c) [153, §9.3]: $S_{\hbar}(z) = |z|^{1/\hbar} z/|z|$, and $\mathbb{C}_{\hbar} = (\mathbb{C}, +_{\hbar}, \times)$ with $a +_{\hbar} b = S_{\hbar}^{-1}(S_{\hbar}a + S_{\hbar}b)$ is “a copy of $\mathbb{C}$ ”. On $\mathbb{R}$, $S_{\hbar} = \theta_{1/\hbar}$.

What does not exist (as far as the cited sources go): a Weil form *valued* in $\mathbf{H}$, $\mathbf{K}$ or $\mathbb{R}^{\text{convex}}$. The Weil form, and Lemma 15’s $L = P - Q$, is real-valued on test functions. A “positivity on the hyperring at deformation $\hbar$ ” must therefore act on the *scalars*. Below are the only two natural ways to do this.

DEFINITION 26.1.1. Let $G = (L(\varphi_i, \varphi_j))_{i,j < n}$ be a window Gram matrix (Lemma 15) and $\hbar > 0$. Set $\mathbb{R}_{\hbar} = (\mathbb{R}, +_{\hbar}, \times)$, with order transported by $S_{\hbar}$; this is the usual order, since $S_{\hbar}$ is odd and increasing.

- (T) The *transported form* is $L^{(\hbar)}(f, g) = S_{\hbar}^{-1}L(S_{\hbar}f, S_{\hbar}g)$ over $\mathbb{R}_{\hbar}$.
(N) The *window Gram at deformation $\hbar$* is G read as a matrix over $\mathbb{R}_{\hbar}$. Its $\mathbb{R}_{\hbar}$ -inertia is the inertia of $G_{\hbar} := \theta_{1/\hbar}(G) = (\text{sign } g_{ij}|g_{ij}|^{1/\hbar})$ over $\mathbb{R}$ (Lemma 26.1.2).

LEMMA 26.1.2 ([THM]). $S_{\hbar} : \mathbb{R}_{\hbar} \rightarrow \mathbb{R}$ is an isomorphism of ordered fields. A symmetric matrix M over $\mathbb{R}_{\hbar}$ has $\mathbb{R}_{\hbar}$ -inertia equal to the $\mathbb{R}$ -inertia of $S_{\hbar}(M)$ (entrywise).

PROOF. $S_{\hbar}$ is multiplicative, additive by the definition of $+_{\hbar}$, bijective, and increasing. Sylvester’s law of inertia holds in any ordered field and is preserved by ordered-field isomorphisms applied entrywise. $\square$

THEOREM 26.1.3 (reading (T) is $\hbar$ -invariant; [THM]). The Gram of $L^{(\hbar)}$ on $S_{\hbar}^{-1}V$ is $S_{\hbar}^{-1}(G)$, so its $\mathbb{R}_{\hbar}$ -inertia is that of G for every $\hbar > 0$. In particular, Lemma 15’s Thm. 15.2 ($\sup_V n_+ = 1 + N_+$) holds verbatim for every $\hbar$.

PROOF. Lemma 26.1.2 applied to $M = S_{\hbar}^{-1}(G)$. $\square$

PROPOSITION 26.1.4 (covariance of (N); [THM]+[NUM]). Let P be a permutation matrix and D diagonal with nonzero entries. Then $\theta_\lambda(DPGP^TD) = |D|^\lambda \Sigma P \theta_\lambda(G) P^T \Sigma |D|^\lambda$ with $\Sigma = \text{sign } D$, so inertia is preserved under monomial changes of basis. It is not preserved in general. [NUM] On $V(0.1, 0.5, 8)$ at $\hbar = 0.9$, the basis φ_k gives inertia $(1, 7, 0)$, while the basis $\psi_k = \varphi_k + \varphi_{k+1}$ of the same subspace gives $(3, 5, 0)$. Rescaling by 2^k leaves $(1, 7, 0)$ unchanged. On $V(0.25, 0.5, 8)$: $(3, 5, 0)$ versus $(4, 4, 0)$.

So $G_{\hbar}$ for $\hbar \neq 1$ is a function of the *basis*, not of the form on V . This is the first obstruction: whatever $n_+(G_{\hbar})$ measures, it is not a property of the Weil form.

26.2. Where the limits land

$\hbar = 1$. $G_1 = G$, Lemma 15’s Gram, which is computed from the prime side. [NUM] We cross-checked it against the zero side $P(d) - \sum_{\gamma} 2\hat{F}(\gamma)^2 \cos(d\gamma)$, using Odlyzko’s first 10^5 zeros [114]. For $h_B \in \{0.1, 0.25, 0.5\}$ and $d \in \{0, 0.5, 1, 2\}$, $|\text{prime} - \text{zero}| \leq 3.5 \times 10^{-12}$.

PROPOSITION 26.2.1 ($\hbar \rightarrow 0$: dominant permutations and signs; [THM]). Let G be real symmetric with nonzero entries. Suppose that for each leading block $G^{(k)}$ the maximum $M_k = \max_{\sigma \in S_k} \prod_i |g_{i\sigma(i)}|$ is attained by a unique σ_k . Put $\varepsilon_k = \text{sgn}(\sigma_k) \prod_i \text{sign } g_{i\sigma_k(i)}$. Then for all sufficiently small $\hbar$, $G_{\hbar}$ has

$$n_- = \#\{\text{sign changes in } (1, \varepsilon_1, \dots, \varepsilon_n)\}, \quad n_+ = n - n_-.$$

PROOF. With $p = 1/\hbar$, $\det G_h^{(k)} = \sum_{\sigma} \operatorname{sgn} \sigma \prod_i \operatorname{sign}(g_{i\sigma(i)}) |g_{i\sigma(i)}|^p = \varepsilon_k M_k^p (1 + O(r_k^p))$ with $r_k < 1$. So for large p every leading minor is nonzero with sign ε_k . Jacobi’s signature rule then gives the count. $\square$

Reading. The limit depends only on $(\log |g_{ij}|)$ (max-plus data) and $(\operatorname{sign} g_{ij})$ (sign-hyperfield data). For Toeplitz sections with $|c_m| < c_0$ for $m \neq 0$, $\sigma_k = \operatorname{id}$ and $\varepsilon_k = +1$, so G_h is PSD for small $\hbar$. This is exactly Lemma 23 Prop. 2.2 (Yu’s tropical PSD cone [164], i.e. single shadows). [NUM] Real curves: the Toeplitz sections of $E/\mathbb{F}_5$ ($n \leq 12$) are PSD for $\hbar \leq 0.1$, and those of $C_2/\mathbb{F}_7$ for $\hbar \leq 0.3$. For the Lemma 19 coin family $N = 20$, the first non-PSD size is $k = 4$ at $\hbar = 1$, then 5, 6, 9, 12, 22, 37 at $\hbar = 0.9, 0.5, 0.3, 0.2, 0.1, 0.05$. It drifts toward Yu’s $k = 42$ (Lemma 23). The opposite limit $\hbar \rightarrow \infty$ ($\lambda \rightarrow 0$) gives $G_h \rightarrow \operatorname{sign}(G)$: the sign hyperfield $\mathbf{S}$ alone.

26.3. Monotonicity and the $\mathbb{R}^{\operatorname{convex}}$ obstruction

THEOREM 26.3.1 ([CITED]+[THM]). Garg–Aujla [55]: if A is symmetric with all entries positive and exactly one positive eigenvalue, then $A^{\circ r}$ has exactly one positive eigenvalue for every $r \in (0, 1]$. Hence, for a positive-entry Gram G :

- (a) $\Lambda(G) = \{\hbar > 0 : n_+(G_h) = 1\}$ is an up-set: if $\hbar_1 \in \Lambda(G)$ and $\hbar_2 > \hbar_1$, then $G_{\hbar_2} = (G_{\hbar_1})^{\circ(\hbar_1/\hbar_2)}$, so $\hbar_2 \in \Lambda(G)$. Put $\hbar_*(G) = \inf \Lambda(G)$.
- (b) Certificates. $n_+(G_h) = 1$ at some $\hbar < 1$ implies $n_+(G) = 1$. It is a stronger hypothesis for the same conclusion, so it never certifies a window Lemma 15 cannot.
- (c) Detectors. $n_+(G) \geq 2$ implies $n_+(G_h) \geq 2$ for all $\hbar \leq 1$, so a true ghost is never lost below 1. For $\hbar > 1$, $n_+(G_h) \geq 2$ implies $n_+(G) \geq 2$: a sound but weaker detector.
- (d) For $\hbar < 1$, $n_+(G_h) \geq 2$ implies nothing about G . False alarms occur (§26.4).

THEOREM 26.3.2 ($\mathbb{R}^{\operatorname{convex}}$ obstruction; [THM]+[NUM]). Let $\mathcal{P}$ be any property of finite real symmetric matrices that is expressible by a first-order formula, with parameters in $\mathbf{S}$, in the language of the hyperfield $\mathbb{R}^{\operatorname{convex}}$ (multiplication, the hypersum relation $z \in x +_c y$, constants $0, \pm 1$), applied to the entries. Then $\mathcal{P}(G) \iff \mathcal{P}(\theta_\lambda(G))$ for all $\lambda > 0$. “Exactly one positive eigenvalue” is not invariant under θ_λ . Hence the Lorentzian/Weil index is not an $\mathbb{R}^{\operatorname{convex}}$ -property, and a fortiori not an $\mathbf{H}$ - or $\mathbf{K}$ -property.

PROOF. Automorphisms preserve the truth of first-order formulas. By [21, Prop. 5.7], every θ_λ is an automorphism of $\mathbb{R}^{\operatorname{convex}}$ fixing $\mathbf{S}$. Non-invariance: [NUM] $V(0.1, 0.5, 8)$ has $n_+(G) = 1$ but $n_+(G_{0.8}) = 3$, both certified. $\square$

This is the precise form of “ $\mathbb{C}_\hbar \cong \mathbb{C}$, so nothing new”. Positivity is a field-level notion. In the field it is invariant under (T); in the hyperfield it cannot be stated, because the hyperfield’s own symmetries destroy it.

26.4. Numerics

Setup. Grid: $h_B \in \{0.05, 0.1, 0.25\}$, $s \in \{0.25, 0.5, 0.75, 1, \log 2, \log 3\}$, $n \in \{4, 8, 12, 16\}$, and $\hbar \in \{4, 2, 1.5, 1, 0.9, \dots, 0.1, 0.05\}$. Ghosts are injected by Lemma 15’s (= Lemma 11/18) quadruple model, $L_{\text{gh}} = L - \sum_{\pm g_0 \pm ib} e^{id\gamma} \widehat{F}(\gamma)^2$, for $(g_0, b) \in \{(30, .1), (30, .25), (30, .4), (14.13, .1), (100, .1)\}$. Inertia comes from Arb ball LDL (ldl_inertia); signed powers are taken on balls excluding 0. All entries are positive for $h_B \geq 0.1$; for $h_B = 0.05$, $\ell(0) = -0.0162$.

(i) Truth.

- For every window and every $\hbar \geq 1$, $n_+ \leq 1$ (0 of 72 flagged), and Thm. 26.3.1 shows zero violations.

- For $\hbar < 1$, the true Gram is flagged ($n_+ \geq 2$) on 43, 48, 54, 59 of 72 windows at $\hbar = 0.9, 0.8, 0.5, 0.3$, with n_+ up to 13.
- Bisection gives $\hbar_*$. For $h_B = 0.1$, $s = 0.5$: $\hbar_* \in [0.4005, 0.4006]$, $[0.8639, 0.8640]$, $[0.9961, 0.9962]$, $[0.9998, 0.9999]$ for $n = 4, 8, 12, 16$. For $h_B = 0.5$, every window gives $\hbar_* > 0.9988$, and most give $\hbar_* > 0.99994$.
- [HEUR] Explanation: first order. $G^{\circ(1+\epsilon)} = G + \epsilon G \circ \log^\circ G + O(\epsilon^2)$, so a negative eigenvalue λ_k crosses 0 near $\epsilon_k = -\lambda_k/v_k^T(G \circ \log^\circ G)v_k$. The prediction $1/(1 + \min \epsilon_k)$ matches the observed $\hbar_*$ to $\lesssim 10^{-3}$ for $n \geq 8$; e.g. $V(0.1, 0.5, 12)$: predicted 0.99611, observed $[0.99607, 0.99613]$. The window Grams have negative eigenvalues tending to 0 (e.g. -9×10^{-5} there; -9×10^{-8} at $V(0.25, 0.25, 12)$). Any non-congruence perturbation tips them over, so $\hbar_* \rightarrow 1$.

(ii) **Ghosts.** The table gives the smallest window length ℓ_V (over the grid, all h_B) at which the ghost Gram has $n_+ \geq 2$ while the true Gram has $n_+ \leq 1$, both certified.

ghost (g_0, b)	$\hbar = 1$ (Lemma 15)	best $\hbar < 1$	best $\hbar > 1$
(14.13, 0.1)	1.75 ($h_B = .25$)	0.75 ($\hbar = 0.5, 0.4$; $h_B = .1$)	1.5 ($\hbar = 4$; $h_B = .1$)
(30, 0.1)	7.5 ($h_B = .1$)	1.75 ($\hbar = 0.9$; $h_B = .1$)	1.75 ($\hbar = 4$; $h_B = .05$)
(30, 0.25)	7.5 ($h_B = .1$)	1.75 ($\hbar = 0.9$; $h_B = .1$)	1.75 ($\hbar = 4$; $h_B = .05$)
(30, 0.4)	3.75 ($h_B = .1$)	1.75 ($\hbar = 0.9$; $h_B = .1$)	1.75 ($\hbar = 2, 4$; $h_B = .05$)
(100, 0.1)	not detected	not detected	not detected

So, after the fact, $\hbar \neq 1$ separates ghost from truth on *smaller* windows. This is the outcome J. W. asked about. It is *not* a detector:

- (1) For $\hbar < 1$ the same test flags the true ζ form on most windows: $V(0.1, 0.25, 12)$ already has $n_+ = 3$ at $\hbar = 0.9$. The separating windows are exactly those where the truth happens to survive, and there is no theorem saying $n_+(G_\hbar) \geq 2 \Rightarrow$ off-line zero (Thm. 26.3.1(d)).
- (2) The test is basis-dependent (Prop. 26.1.4).
- (3) The $\hbar > 1$ wins occur only at $h_B = 0.05$, where entries are not all positive, so Garg–Aujla does not apply. There the same test flags the true ζ form. With $s = 0.02$, $\ell(d) < 0$ for $d \leq 0.04$, and $V(0.05, 0.02, 4)$ has $n_+ = 0$ at $\hbar = 1$ but 2 at $\hbar = 1.5, 2$. $V(0.05, 0.02, 8)$ has $n_+ = 1$ at $\hbar = 1$ but 3 at $\hbar = 1.5, 2, 4$. All certified. The wins come from the ghost making individual entries negative (e.g. $\ell_{\text{gh}}(1.25) = -0.0031$), a sign-hyperfield event, and the true $\ell(d)$ has such events too.

(iii) **Function fields**, where RH is a theorem. We use the Γ -Gram $q^{\min} N_{|m-n|}$ rescaled by $q^{-m/2}$, and the Toeplitz sections $c_{j-j'}$ (Lemma 12).

- Real curves, Toeplitz: PSD at $\hbar = 1$ for all $n \leq 12$. False alarms at every $\hbar \neq 1$ in the middle range: E fails from $n = 3$ for $\hbar \in \{1.5, 2, 4\}$ and from $n = 5$ for $\hbar \in [0.4, 0.9]$; C_2 fails from $n = 4$ ($\hbar = 2, 4$) and from $n = 10$ ($\hbar \in [0.5, 0.9]$).
- Real curves, Γ -Gram: Lorentzian for all $\hbar \geq 1$. It fails at $\hbar = 0.9$ from $n = 5$ (E) and $n = 8$ (C_2).
- Fakes, Γ -Gram: the coin is detected at $\hbar = 1$ from $n = 4$, at $\hbar = 1.5$ only from $n = 8$, and never at $\hbar = 2, 4$ ($n \leq 12$). The off-circle quadruple: from $n = 5$ at $\hbar = 1$, never for $\hbar > 1$.

In every control, $\hbar = 1$ is the unique value that is both sound and sharp.

26.5. Verdict, rigor, obstruction, questions, ranking

Verdict. Refuted as a source of new leverage.

- (T) is exactly Lemma 15 for every $\hbar$ (Thm. 26.1.3).

- (N) is basis-dependent. It is a weaker certificate for $\hbar < 1$ and a weaker detector for $\hbar > 1$ on positive-entry Grams (Thm. 26.3.1), and an unsound detector otherwise (§26.4).
- Its $\hbar \rightarrow 0$ limit is the single-shadow / tropical test of Lemma 23 (Prop. 26.2.1).
- The Weil index is not expressible over $\mathbb{R}^{\text{convex}}$, $\mathbf{H}$ or $\mathbf{K}$ (Thm. 26.3.2).

Rigor.

- Proved: Lemma 26.1.2, Thms. 26.1.3, 26.3.1 (given [55]) and 26.3.2, Props. 26.1.4 and 26.2.1.
- Certified numerics: all inertias quoted.
- Heuristic: the first-order explanation of $\hbar_* \rightarrow 1$ (numerically confirmed), and the claim $\hbar_* \rightarrow 1$ along exhausting windows.

Obstruction. The ordered-field structure is what positivity uses. θ_λ is a field automorphism only for $\lambda = 1$, while it is an automorphism of $\mathbb{R}^{\text{convex}}$ for every λ . [RH-EQUIV]:

“ $n_+(G_1(V)) \leq 1$ for all V ” is RH (Lemma 15). No $\hbar \neq 1$ version is known to be RH-equivalent or even RH-implied: for $\hbar < 1$ it fails on the true ζ form.

QUESTION 26.5.1. (1) Did J. W. intend a hyperring-*valued* Weil form? None exists in [21, 29]; constructing one would be new mathematics. (2) Close candidate 1 and move on to Brändén–Huh stability operators acting on the *form*, which are congruence-covariant, rather than on matrix entries?

Ranking update (0.6 sig + 0.4 prob):

- (1) Lemma 23 candidate 1 (this lemma): **closed**. Re-scored sig .2 (target inexpressible over $\mathbb{R}^{\text{convex}}$), prob .02: 0.13 (was 0.54).
- (2) Brändén–Huh stability-preserving *linear* operators on test spaces: 0.42, unchanged. Lemma 26 shows the operators must act on the form, not entrywise.
- (3) Tropical $b_2 = 0$ for the CC square: 0.42, unchanged.
- (4) Phase-tropical Lemma 10: 0.38, unchanged.

Lemma 27: Ghost birth in the window Weil form and Krein collisions

The hyperbolic plane of an off-line pair, the boost, and the two-stage mechanism

Standalone note `lemma27.tex`, October 21, 2026 (rev. 2: Addendum A thresholds, Addendum B Lagrange dictionary).

Summary. J.W.’s L_4/L_5 intuition says that a ghost (a negative square of the window Weil form) is born the way an elliptic equilibrium loses stability, by a Krein collision. We test this rigorously. An off-line pair $\rho, \rho^* = 1 - \bar{\rho}$ spans a hyperbolic plane of Weil’s Krein form, and the half-density scaling flow acts on it as a Lorentz boost of rapidity $\delta \log x$ composed with a rotation of angle $\gamma \log x$; on-line zeros are pure rotations, and RH is equivalent to the scaling flow commuting with the fundamental symmetry J_{FE} (§27.1, proved). The curve analogue is the decomposition $NS(C \times C)_{\mathbb{R}} = \langle H, V \rangle \oplus T$ with $x^2 = -\text{Tr}(\alpha_x \alpha_x^\dagger | V_\ell J)$ on T (§27.2, cited). We then define the *frequency pencil* (K_A, G_A) , $K_A(F, G) = Q(F', G')$, $G_A(F, G) = Q(F, G)$ on a window. In a finite zero model its eigenvalues are exactly γ_ρ^2 , where $\gamma_\rho = t - i\delta$ is the Lemma 25 generator, so a rotation that acquires a boost is an eigenvalue that leaves the real axis (proved). **Verdict.** The Krein-collision picture holds, but in two stages, not one. Ghost *birth* (the jump of κ_A) is *not* a Krein collision: by Sylvester’s law it can only happen where G_A is singular, and a pencil eigenvalue passes through ∞ and returns with negative Krein type. The *boost* is acquired later, at a genuine opposite-signature Krein collision of that negative-type eigenvalue with the positive-type eigenvalue γ^2 . After the collision the complex pair agrees with $(\gamma \pm i\delta)^2$ to 6–7 digits. κ_A does not count collisions: $\kappa_A = n_{\mathbb{C}} + n_{-}$ (complex pairs plus pending negative-type real eigenvalues), an exact consequence of the canonical form, confirmed in all 1012 + 282 runs (including the 92 pure- ζ runs, where $\kappa_A = 0$). On the canonical zero space the double on-line zero is of definite type, so it can leave the axis only because the Krein space itself degenerates.¹ L_4 remains an analogy that shares the same theorem (definite-type stability). **Nothing here proves RH.**

0. Verdict and status labels

Labels: [THM] proved here; [CITED] a verified external theorem (`lit/lemma27/verified_refs.md`); [DEF] definition; [NUM] numerical (high precision, not certified); [ANALOGY]; [CONJ].

- (i) *Is ghost birth exactly an opposite-signature collision?* **No.** The jump $\kappa_A : 0 \rightarrow 1$ happens only when $\det G_A = 0$ (Lemma 27.4.1, [THM]). At that moment one eigenvalue of the pencil goes through ∞ and comes back as a real eigenvalue of *negative* type ([NUM], all runs). The opposite-signature collision comes second, and it is where the eigenvalue leaves the real axis. That is the moment the rotation acquires a boost (Theorem 27.4.4).

¹[Vol. 2 editorial note] Lemma 41 (Vol. III) refines this remark: in a normalized chart the same instant is a mixed-type collision. The two descriptions are one event in two charts, not a contradiction. See Appendix A.

- (ii) *Does κ_A count collisions?* **No**, but exactly: $\kappa_A = n_{\mathbb{C}} + n_-$ + (Jordan corrections) (Proposition 27.4.2, [THM] from [CITED]). Completed collisions ($n_{\mathbb{C}}$) and pending ones (n_-) count equally.
- (iii) *What transfers?* Three theorems transfer verbatim, because the window pencil *is* a selfadjoint matrix in a finite-dimensional Krein space: the canonical form with sign characteristic [57, Ch. 5], [98, Thm. 2.2]; definite-type stability (Krein, Gelfand–Lidskii) [85, 56], [98, Thm. 4.7]; and Sylvester inertia. Pontryagin’s theorem [125], [74] is the infinite-dimensional form of (ii).
- (iv) *What is new?* (a) A spectral detector: off-line zeros appear as non-real pencil eigenvalues $\approx \gamma_\rho^2$, and the window recovers δ from $\text{Im } \mu = 2t\delta$. (b) The leading negative weight of a nearly double zero is $2m\kappa_H = 2m\delta^2$ (Lemma 25’s Heaviside residual) on the derivative functional (Proposition 27.3.3). (c) The exact count identity. There is no new stability theorem for ζ . The definite-type theorem only says that a window with $\kappa_A = 0$ is robust, and that is already obvious.

27.1. The hyperbolic plane of an off-line pair and the boost

Notation is that of Lemma 25 §1. $\mathcal{H}_{\mathcal{Z}} = \ell^2(\mathcal{Z}, m)$ with orthonormal-up-to-weight basis e_ρ , $\|e_\rho\|^2 = m_\rho$. $\mathbf{J}_{\text{FE}} e_\rho = e_{\rho^*}$ with $\rho^* = 1 - \bar{\rho}$. The Krein (Weil) form is $[x, y]_{\mathcal{Z}} = \langle \mathbf{J}_{\text{FE}} x, y \rangle$. The generator is $\gamma e_\rho = \gamma_\rho e_\rho$, where $\gamma_\rho = t_\rho - i\delta_\rho$ and $\rho = \frac{1}{2} + \delta_\rho + it_\rho$. The half-density scaling flow is $U_u = e^{iu\gamma}$, $u = \log x$, i.e. $U_u e_\rho = x^{\rho-1/2} e_\rho$ [DEF] (Lemma 25, Def. 1.1, Thm. 1.2).

PROPOSITION 27.1.1 (hyperbolic plane; [THM]). *Let $\delta_\rho \neq 0$ and $P_\rho = \text{span}\{e_\rho, e_{\rho^*}\}$, with $m = m_\rho = m_{\rho^*}$.*

- (a) $[e_\rho, e_\rho]_{\mathcal{Z}} = [e_{\rho^*}, e_{\rho^*}]_{\mathcal{Z}} = 0$ and $[e_\rho, e_{\rho^*}]_{\mathcal{Z}} = m$. So P_ρ is a hyperbolic plane: both basis directions are null, and Q pairs them.
- (b) $f_\pm = (e_\rho \pm e_{\rho^*})/\sqrt{2m}$ satisfy $\mathbf{J}_{\text{FE}} f_\pm = \pm f_\pm$ and $[f_\pm, f_\pm]_{\mathcal{Z}} = \pm 1$, $[f_+, f_-]_{\mathcal{Z}} = 0$. So $\mathbf{J}_{\text{FE}}|_{P_\rho}$ is the fundamental symmetry of the Krein plane $(P_\rho, [\cdot, \cdot]_{\mathcal{Z}})$: $[x, y]_{\mathcal{Z}} = \langle \mathbf{J}_{\text{FE}} x, y \rangle$, and $\langle \cdot, \cdot \rangle$ is the associated Hilbert form.
- (c) In evaluation coordinates $a = \tilde{F}(\gamma_\rho)$, $b = \tilde{F}(\bar{\gamma}_\rho)$, the pair’s contribution to $Q(F, F)$ is $m(a\bar{b} + b\bar{a}) = \frac{m}{2}(|a + b|^2 - |a - b|^2)$ (Lemma 1(vi)).

PROOF. $[e_\rho, e_\rho]_{\mathcal{Z}} = \langle e_{\rho^*}, e_\rho \rangle = 0$ since $\rho^* \neq \rho$, and $[e_\rho, e_{\rho^*}]_{\mathcal{Z}} = \langle e_{\rho^*}, e_{\rho^*} \rangle = m$. Then (b) is a 2×2 computation, and (c) is the polarization identity. $\square$

REMARK 27.1.2 (sign convention; important). For Q itself the sum f_+ is positive and the difference f_- negative. The project’s *geometric* convention (Lemmas 14, 15) uses $\mathcal{L} = \mathcal{P} - Q$, which matches the intersection form: the pole terms give the one timelike (positive) direction, and RH makes the zero sector spacelike. In that convention the signs flip. The *difference* f_- is the timelike direction, and it is the ghost: “an extra timelike direction in $\langle H, V \rangle^\perp$ ” (Lemma 14, discussion after Castelnuovo–Severi; §27.2 here). The steering phrase “sum timelike, difference spacelike” is correct for Q and reversed for $\mathcal{L}$. We state both and use Q -signs below.

THEOREM 27.1.3 (boost $\circ$ rotation; [THM]). (a) On P_ρ , $U_u = x^{it} B(\delta \log x)$ with $x = e^u$, $t = t_\rho$, $\delta = \delta_\rho$, and $B(\eta)e_\rho = e^\eta e_\rho$, $B(\eta)e_{\rho^*} = e^{-\eta} e_{\rho^*}$. In the $\mathbf{J}_{\text{FE}}$ -eigenbasis $f_\pm$,

$$B(\eta) = \begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix},$$

which is a Lorentz boost of rapidity $\eta = \delta \log x$ in $SO(1, 1)$. The factor $x^{it} = e^{it \log x}$ is a rotation (a $U(1)$ phase) that commutes with B .

- (b) U_u is $\mathbf{J}_{\text{FE}}$ -unitary for every real u : $[U_u x, U_u y]_{\mathcal{Z}} = [x, y]_{\mathcal{Z}}$. Also $\mathbf{J}_{\text{FE}} B(\eta) \mathbf{J}_{\text{FE}} = B(-\eta)$. The functional equation reverses the boost and fixes the rotation.

- (c) On an on-line zero ($\delta_\rho = 0$) the line $\mathbb{C}e_\rho$ is J_{FE} -fixed and positive, and $U_u = x^{it}$ is a pure rotation. On the pole pair $\{0, 1\}$ ($t = 0$, $\delta = \mp \frac{1}{2}$) the flow is a pure boost of rapidity $\frac{1}{2} \log x$.
- (d) The following are equivalent: (1) RH (every $\delta_\rho = 0$); (2) $U_u J_{FE} = J_{FE} U_u$ for all u ; (3) U_u is unitary for the Hilbert form $\langle \cdot, \cdot \rangle$; (4) the zero sector contains no hyperbolic plane, i.e. $[\cdot, \cdot]_Z$ is positive definite on it; (5) no boost occurs: every U_u is a rotation.

PROOF. (a) $U_u e_\rho = e^{iu(t-i\delta)} e_\rho = e^{iut} e^{u\delta} e_\rho$, and $\gamma_{\rho^*} = t + i\delta$ gives $U_u e_{\rho^*} = e^{iut} e^{-u\delta} e_{\rho^*}$. Changing to the basis $f_\pm$ gives the matrix. (b) $[Ue_\rho, Ue_{\rho^*}]_Z = e^{iut} e^{u\delta} \overline{e^{iut} e^{-u\delta}} m = m$, and the null entries stay null. J_{FE} swaps $e_\rho \leftrightarrow e_{\rho^*}$, hence $\eta \leftrightarrow -\eta$. (c) is immediate. (d) A J_{FE} -unitary U commutes with J_{FE} iff it is Hilbert-unitary, because $U^{[*]} = J_{FE} U^* J_{FE}$. By (a)–(b), U_u commutes with J_{FE} on P_ρ iff $B(\delta u) = B(-\delta u)$ for all u , iff $\delta = 0$. Finally (4) $\Leftrightarrow$ (1) is Proposition 27.1.1 together with Lemma 25 Thm. 1.2. $\square$

Status. (d) is a restatement of RH, not progress toward it. Its value is that it puts the ghost, the boost and the failure of $[U, J_{FE}] = 0$ in one sentence: *RH is the statement that the scaling flow is a rotation group, i.e. unitary, on the zero sector* (J.W.’s thesis, operator form).

27.2. The curve case

Let C be a smooth projective curve of genus g over $\overline{\mathbb{F}}_q$, J its Jacobian, $X = C \times C$, $H = \{pt\} \times C$, $V = C \times \{pt\}$, and for a divisor D write $d_1 = D \cdot V$, $d_2 = D \cdot H$ (the notation of [103, §1]).

- PROPOSITION 27.2.1 ([CITED]). (a) $H^2 = V^2 = 0$ and $H \cdot V = 1$, so $\langle H, V \rangle$ is a hyperbolic plane. $NS(X)_{\mathbb{R}} = \langle H, V \rangle \oplus T$ with $T = \langle H, V \rangle^\perp$, and T is negative definite (Hodge index; Castelnuovo–Severi $D^2 \leq 2d_1 d_2$) [103, Thm. 1.5], Lemma 14(d).
- (b) $T \otimes \mathbb{Q}$ is the space of correspondences modulo divisors of valence zero [103, p. 11]. Via $D \mapsto \alpha_D \in \text{End}^0(J)$ the transpose $D \mapsto D'$ becomes the Rosati involution $\dagger$ [103, footnote 7 and Thm. 1.27], [109, Ch. IV].
- (c) For $D \in T$ (so $d_1 = d_2 = 0$),

$$D^2 = -\sigma(D \circ D') = -\text{Tr}(\alpha_D \alpha_D^\dagger | V_\ell J),$$

where $\sigma(D) = d_1 + d_2 - D \cdot \Delta$ is Weil’s trace and $\sigma(D \circ D') = 2d_1 d_2 - D^2$ [103, p. 11, eq. (8) ff.]. Equivalently, it is -2Re Tr on the g -dimensional analytic representation. So $c = 1$ with the trace on the rank- $2g$ module $V_\ell J$. Its positivity is Rosati positivity [102, Thm. I.14.3], [103, Thm. 1.27].

Checks [THM]. $\Delta_0 = \Delta - H - V \in T$ has $\Delta_0^2 = (2 - 2g) - 4 + 2 = -2g = -\text{Tr}(1 | V_\ell)$. For Frobenius, $\Gamma_0^2 = q(2 - 2g) - 2q = -2gq = -\text{Tr}(FF^\dagger)$, since $FF^\dagger = q$.

COROLLARY 27.2.2 (the curve flow is boost $\oplus$ rotation; [THM] from [CITED]). The normalized Frobenius $\Gamma_F / \sqrt{q}$ acts as $B_\eta \oplus R$ with $e^\eta = \sqrt{q}$ (Lemma 20). The factor B_η is a boost of $\langle H, V \rangle$, $H \mapsto e^{-\eta} H$, $V \mapsto e^\eta V$. The factor $R = F / \sqrt{q}$ is an isometry of the definite space T ($RR^\dagger = 1$), hence a rotation. Per unit $u = \log q^n$ the rapidity is $\frac{1}{2}$, i.e. $\delta = \frac{1}{2}$, exactly the pole channel of Theorem 27.1.3(c). A definite space admits no boosts, so RH for curves is “no hyperbolic plane in T ”. A ζ -ghost would be precisely a hyperbolic plane P_ρ inside the zero sector.

	curve $C/\mathbb{F}_q$	ζ (zero space $\mathcal{H}_Z$)
hyperbolic plane, pure boost	$\langle H, V \rangle, e^\eta = \sqrt{q}$	pole pair $\{0, 1\}$, rapidity $\frac{1}{2} \log x$
transverse sector	T , form $-\text{Tr}(\alpha\alpha^\dagger)$	on-line $e_\rho, [\cdot, \cdot]_Z > 0$
flow on sector	$F/\sqrt{q}$, Rosati-unitary (rotation)	x^{it} (rotation)
fundamental symmetry	$D \mapsto D'$ (Rosati $\dagger$)	J_{FE}
ghost	impossible (Hodge index)	off-line P_ρ : boost $\delta \log x$

The table is [CITED]/[THM] row by row. One caveat: in the curve case the symmetry $\dagger$ acts on *operators*, while J_{FE} acts on *vectors*. They match because J_{FE} implements $\gamma \mapsto \gamma^{[*]}$ (Lemma 25, $J_{\text{FE}}\gamma^*J_{\text{FE}} = \gamma$). The row “fundamental symmetry” is therefore an [ANALOGY] at the level of objects, though precise at the level of the involution induced on generators.

27.3. The frequency pencil

DEFINITION 27.3.1 ([DEF]). For the window $A = [-h, h]$ and the basis $\phi_n(u) = (1 - (u/h)^2)P_n(u/h)$ of fixed parity (Lemma 11, `num/lemma27_krein.py`), set $\mathbf{G}_A = (Q(\phi_i, \phi_j))$ and $\mathbf{K}_A = (Q(\phi'_i, \phi'_j))$, both computed from the same (ghost-modified) Weil matrix. The *frequency pencil* is $\mathbf{K}_A x = \mu \mathbf{G}_A x$. When $\mathbf{G}_A$ is invertible, $\mathbf{A} = \mathbf{G}_A^{-1} \mathbf{K}_A$ is $\mathbf{G}_A$ -selfadjoint. A real eigenvalue μ has Krein type $\text{sign}(x^* \mathbf{G}_A x)$; n_- is the number of real eigenvalues of negative type and $n_{\mathbb{C}}$ the number of pairs $\mu, \bar{\mu}$ with $\text{Im } \mu > 0$.

PROPOSITION 27.3.2 (finite zero model; [THM]). Let $Q(F, G) = \sum_{j=1}^M m_j \tilde{F}(z_j) \overline{\tilde{G}(\bar{z}_j)}$ with $\{z_j\}$ closed under conjugation, $m_j > 0$, $m_{\bar{j}} = m_j$. Suppose the M test functions have $E = (\tilde{\phi}_n(z_j))$ invertible. Then $\mathbf{G}_A^{-1} \mathbf{K}_A = E^{-1} \text{diag}(z_j^2) E$. Its eigenvalues are exactly $z_j^2 = \gamma_\rho^2$, real z_j have positive Krein type, and each conjugate pair $t \mp i\delta$ gives $\mu = (t \mp i\delta)^2 = t^2 - \delta^2 \mp 2it\delta$.

PROOF. $\tilde{F}'(z) = \mp iz \tilde{F}(z)$, and the conjugated factor contributes $\overline{\mp i\bar{z}} = \pm iz$, so the j -th term of $Q(F', G')$ is z_j^2 times that of $Q(F, G)$. Write $\mathbf{G}_A = E^* W E$ and $\mathbf{K}_A = E^* Z^* W Z E$ with $Z = \text{diag}(z_j)$ and W the pairing matrix $j \leftrightarrow \bar{j}$. Then $W^{-1} Z^* W = Z$ (this is $J_{\text{FE}} \gamma^* J_{\text{FE}} = \gamma$), so $\mathbf{G}_A^{-1} \mathbf{K}_A = E^{-1} Z^2 E$. For real z_j the eigenvector is $E^{-1} e_j$ with $\mathbf{G}_A$ -norm $m_j > 0$. The proof is checked numerically in `num/lemma27_finite_model.py`. $\square$

So in the finite model the pencil *is* the compression of γ^2 . A real eigenvalue is a rotation, $\sqrt{\mu} = t$. A non-real one is a rotation with a boost: $\sqrt{\mu} = t - i\delta$, and $\text{Im } \mu / (2 \text{Re } \sqrt{\mu})$ is the rapidity per unit u . The phrase “rotation acquiring a boost” literally means “a pencil eigenvalue leaving the real axis”.

PROPOSITION 27.3.3 (the splitting double zero; [THM]). Let a double on-line zero at γ split as $\gamma \pm i\delta$ (the replacement model of Lemma 11, `lemma11_ghost_fixed.py`). Its contribution to $Q(F, F)$ changes from $2m|f|^2$ to

$$2m \text{Re}(\tilde{F}(\gamma + i\delta) \overline{\tilde{F}(\gamma - i\delta)}) = 2m(|f|^2 - \delta^2 |f'|^2) + O(\delta^4), \quad f = \tilde{F}(\gamma), \quad f' = \tilde{F}'(\gamma)$$

(for real-on- $\mathbb{R}$ transforms). The rank-1 positive form becomes a rank-2 form of signature $(1, 1)$. Its negative direction is the derivative functional $F \mapsto \tilde{F}'(\gamma)$ with weight $-2m\delta^2 = -2m\kappa_{\text{H}}$, where $\kappa_{\text{H}} = \delta^2$ is the Heaviside residual of Lemma 25.

PROOF. Taylor expand: $(f + i\delta f' - \frac{\delta^2}{2} f'') \overline{(f - i\delta f' - \dots)}$ for $\tilde{F}(\bar{z}) = \overline{\tilde{F}(z)}$. The $O(\delta)$ term is $i\delta(f\bar{f}' + f'\bar{f}) \in i\mathbb{R}$, and the $O(\delta^2)$ terms give $-\delta^2 |f'|^2 - \delta^2 \text{Re}(f\bar{f}'')$. Completing the square, $|f|^2 - \delta^2 \text{Re}(f\bar{f}'') = |f - \frac{\delta^2}{2} f''|^2 + O(\delta^4)$, so that term only moves the positive direction. The leading term that creates a negative direction is $-\delta^2 |f'|^2$. $\square$

So the window has a ghost ($\kappa_A \geq 1$) as soon as $2m\delta^2|f'|^2$ beats the positive margin of G_A in the direction where f is small and f' is large. Birth is therefore a threshold phenomenon, and its threshold falls quickly as the window grows (Table 27.1).

27.4. The Krein-space theorems that transfer

LEMMA 27.4.1 (Sylvester inertia; [THM]). *Let $G_A(\delta)$ be continuous. Then $\kappa_A(\delta)$ is constant on every interval where $G_A(\delta)$ is invertible. Hence ghost birth occurs only at δ_* with $\det G_A(\delta_*) = 0$. If $K_A(\delta_*)$ is nonzero on $\ker G_A(\delta_*)$, then at δ_* the pencil has the eigenvalue $\mu = \infty$, and a finite eigenvalue escapes to or returns from $\pm\infty$.*

PROOF. Eigenvalues of a Hermitian matrix depend continuously on it, and a sign change requires a zero eigenvalue. The pencil $K_A - \mu G_A$ with singular G_A has $\det$ of degree $< \dim$ in μ , which means an eigenvalue at ∞ . $\square$

PROPOSITION 27.4.2 (count identity; [THM] from [CITED]). *Let G_A be invertible with κ_A negative eigenvalues, and $A = G_A^{-1}K_A$. By the canonical form for G_A -selfadjoint matrices [57, Ch. 5], [98, Thm. 2.2], (A, G_A) is unitarily similar to a direct sum of real blocks $(J_n(\lambda), \varepsilon S_n)$ and non-real blocks $(J_n(\lambda) \oplus J_n(\bar{\lambda}), S_{2n})$. Counting the negative eigenvalues of εS_n ($\lfloor n/2 \rfloor$, plus 1 if n is odd and $\varepsilon = -1$) and of S_{2n} (n) gives*

$$\kappa_A = \sum_{\operatorname{Im} \lambda > 0} m_{\text{alg}}(\lambda) + \sum_{\lambda \in \mathbb{R}} \sum_{\text{blocks}} \left(\lfloor \frac{n}{2} \rfloor + [n \text{ odd}, \varepsilon = -1] \right).$$

In the semisimple case, $\kappa_A = n_{\mathbb{C}} + n_-$.

This is the finite-dimensional case of Pontryagin's theorem [125] (a selfadjoint operator in Π_κ has a κ -dimensional nonpositive invariant subspace) and of its refinement in [74].

THEOREM 27.4.3 (definite-type stability; [CITED]). *A real eigenvalue of a G_A -selfadjoint matrix that is semisimple and of definite type (all signs equal) stays real and semisimple under all sufficiently small selfadjoint perturbations. One of mixed type (both signs, or a Jordan block) can be pushed off the axis by an arbitrarily small one. This is Krein [85] and Gelfand–Lidskii [56] (periodic Hamiltonian systems); the finite-dimensional statement is [98, Thm. 4.7] (Hamiltonian form, after Mehrmann–Xu) and [57, Ch. 9]. The Hamiltonian form transfers: $\mathcal{H}$ is $\mathcal{J}$ -Hamiltonian iff $i\mathcal{H}$ is $i\mathcal{J}$ -selfadjoint.*

To apply it along a path where both G_A and K_A vary, write $G_A(\delta) = P(\delta)^* S P(\delta)$ on an interval where G_A is invertible, with S a fixed signature matrix and P continuous. Then PAP^{-1} is an S -selfadjoint path in one fixed Krein space [THM].

THEOREM 27.4.4 (two-stage mechanism; [THM] given the stated hypotheses, plus [NUM]). *Along $\delta \mapsto (K_A(\delta), G_A(\delta))$ starting from $\kappa_A(0) = 0$:*

- (a) (birth) κ_A first becomes 1 at a zero of $\det G_A$ (Lemma 27.4.1). It is not a collision.
- (b) (no premature boost) While $\kappa_A = 0$ every real eigenvalue is of positive type, and the spectrum is real (Theorem 27.4.3, trivially, since $G_A > 0$).
- (c) (boost) A non-real pair $\mu, \bar{\mu}$ can appear only at a real eigenvalue of mixed type. Generically, i.e. with semisimple eigenvalues before the collision, that is an opposite-signature Krein collision. By Proposition 27.4.2 it does not change κ_A : the transition is $(n_{\mathbb{C}}, n_-) = (0, 1) \rightarrow (1, 0)$.

[NUM] In every run, birth and boost are separated: $(\kappa_A, n_{\mathbb{C}}, n_-)$ goes $(0, 0, 0) \rightarrow (1, 0, 1) \rightarrow (1, 1, 0)$. The born eigenvalue enters from $-\infty$ ($-9.5 \cdot 10^4, -8.2 \cdot 10^5$), climbs the real axis, and collides with the positive-type eigenvalue $\approx \gamma_1^2 = 199.7904548$. The emerging pair matches $(\gamma_1 \pm i\delta)^2$.

REMARK 27.4.5 (canonical level; [THM]). In $\mathcal{H}_{\mathcal{Z}}$ itself a double on-line zero is a semisimple eigenvalue of γ of definite (positive) type and multiplicity 2. By Theorem 27.4.3 it *cannot* leave the real axis under a small perturbation inside a fixed Krein space. It leaves anyway because the Krein space changes: e_ρ and e_{ρ^*} become distinct vectors, and the inertia of the pair jumps from $(2, 0)$ to $(1, 1)$. So at the canonical level, ghost birth is a metric degeneration and not a Krein collision. The window splits this single event into the two stages above.

27.5. Numerics

Script `num/lemma27_krein.py` (mpmath, 110 digits). It uses the Lemma 11 window Weil matrices and the replacement model $Q - T_{\gamma_k} + T_{\gamma_k+i\delta} + T_{\gamma_k-i\delta}$, with $\delta = 0$ meaning the double zero $Q + T_{\gamma_k}$. The *scan* covers $h \in \{0.10, \dots, 1.20\}$ (23 values), parity 0/1, $N \in \{20, 30\}$, $k \in \{0 \text{ (pure } \zeta), 1, 2\}$ and $\delta \in \{0, 10^{-4}, 10^{-2}, 0.1, 0.4\}$, for 1012 runs (`lemma27_scan.json`). The *fine sweep* covers 47 values $\delta \in \{0\} \cup [10^{-6}, 0.56]$ for $h \in \{0.5, 0.7, 1.0\}$, $k = 1$, $N = 20$, for 282 runs (`lemma27_dsweep.json`).

- $\kappa_A = n_{\mathbb{C}} + n_-$ in all 1294 runs. Outcomes: $(0, 0, 0)$ 607, $(1, 0, 1)$ 79, $(1, 1, 0)$ 326 (scan). Pure ζ ($k = 0$, 92 runs) always gives $(0, 0, 0)$, as RH-consistency requires. $N = 20$ and $N = 30$ always agree on κ_A . They disagree on $(n_{\mathbb{C}}, n_-)$, i.e. $(0, 1)$ versus $(1, 0)$, in 5 of 506 cases: $h = 0.6$ (parity 1, $\delta = 0.1$), and $h = 0.9, 0.95, 0.95, 1.0$ ($\delta = 10^{-4}$), all with $k = 2$, and all at a collision threshold.
- The sequence is monotone in δ in every fine-sweep series, with no reverse transitions.
- At $\delta = 0$ the lowest eigenvalues are 199.790454, 441.926150, 625.542996 for $h = 1$, i.e. $\gamma_1^2, \gamma_2^2, \gamma_3^2$ to 7–9 digits ([NUM] for Proposition 27.3.2 in the window).
- After the collision, e.g. $h = 1$, $\delta = 0.0562$: $\mu = 199.78729255 \pm 1.58970709i$, against $(\gamma_1 \pm i\delta)^2 = 199.787293 \pm 1.589707i$. At $\delta = 0.316$: $199.69045468 \pm 8.93959172i$, against $199.690455 \pm 8.939592i$. The window reads off δ from $\text{Im } \mu / 2\gamma_1$.
- (v1 bullet withdrawn; see the erratum in §27.9. The value at $h = 1$, parity 0, $\delta = 4.2 \cdot 10^{-6}$ is $\text{Im } \mu = 9.66 \cdot 10^{-5}$, not 9.66: a string-truncation bug, not a conditioning artifact.)

h	parity	birth $\delta_b \in$	collision $\delta_c \in$
0.5	0	$(3.16, 4.22) \cdot 10^{-2}$	$(0.42, 0.56)$
0.5	1	$(0.133, 0.178)$	> 0.56
0.7	0	$(4.22, 5.62) \cdot 10^{-5}$	$(5.62, 7.50) \cdot 10^{-3}$
0.7	1	$(2.37, 3.16) \cdot 10^{-4}$	$(1.78, 2.37) \cdot 10^{-2}$
1.0	0	$(0, 10^{-6})$	$(3.16, 4.22) \cdot 10^{-6}$
1.0	1	$(0, 10^{-6})$	$(1.00, 1.33) \cdot 10^{-5}$

TABLE 27.1. Thresholds for the first zero, $k = 1$ (fine sweep, [NUM]). Both thresholds fall rapidly with h , and δ_c/δ_b shrinks.

REMARK 27.5.1 ([CONJ]). $\delta_b(A), \delta_c(A) \rightarrow 0$ as $A \rightarrow \infty$, and the two stages merge into the canonical metric degeneration. This would be consistent with Lemma 1(vi) (for large truncations the ghost count is half the number of off-line zeros).

27.6. L_4 , Routh, MacKay: what is analogy

At L_4 of the restricted three-body problem the linearization is Hamiltonian. Routh's critical ratio $27\mu(1 - \mu) = 1$ [130] is where two imaginary eigenvalue pairs of opposite Krein signature collide and leave the axis [97]. What is shared with §27.4 is a *theorem*: definite-type stability

and the mixed-type collision (Theorem 27.4.3) hold for both, so that part of J.W.’s intuition is literally correct for stage (c). What is not shared: the window pencil has no mechanics, no mass ratio, and no Hamiltonian, and stage (a) (metric degeneration) has no counterpart at L_4 , where the symplectic form never degenerates. The L_4 label is [ANALOGY]; the collision theorem is [CITED].

27.7. Open questions

- Q1. Should Volume 2 call the jump of κ_A “ghost birth” (metric degeneration) and the collision “boost onset”? The steering identifies them, but rigorously they are different events.
- Q2. Conventions: present the hyperbolic plane with Q -signs (sum positive) or with $\mathcal{L}$ -signs (difference timelike, matching Lemmas 14–15)? We recommend $\mathcal{L}$ -signs in Volume 2 for the curve parallel.
- Q3. (Partly answered in §27.9.) $\delta_b \rightarrow 0$ is proved super-polynomially; numerically both thresholds are doubly exponential, with slope ratio ≈ 1.6 . Still open: a proof of the doubly exponential rate (a sampling or hole-effect lower bound for the derivative Christoffel function), and a proof that $\delta_c \rightarrow 0$ outside the reduced model.
- Q4. Is the pencil detector useful in practice? Certify (in Arb) that the window pencil on true ζ data has only real, positive-type eigenvalues near γ_k^2 for $k \leq K$. That would be a window-level “rotation certificate”, equivalent in content to the Lemma 15 inertia certificates but with eigenvalue *locations*.
- Q5. In the curve case, is there a pencil on T (e.g. $\text{Tr}(\alpha F (\alpha F)^\dagger)$ against $\text{Tr}(\alpha \alpha^\dagger)$) whose eigenvalues are the $|\alpha_i|^2$? That would make Corollary 27.2.2 a pencil statement and the dictionary exact.

27.8. Summary

Statement	status	source
Prop. 27.1.1 hyperbolic plane, J_{FE} fundamental symmetry	[THM]	Lemma 25
Thm. 27.1.3 boost/rotation; $\text{RH} \Leftrightarrow [U_u, J_{\text{FE}}] = 0$	[THM] (restatement)	–
Prop. 27.2.1 $\text{NS}(X)_{\mathbb{R}} = \langle H, V \rangle \oplus T$, $x^2 = -\text{Tr}(\alpha \alpha^\dagger)$	[CITED]	Milne, Mumford
Cor. 27.2.2 Frobenius = boost $\oplus$ rotation	[THM] from [CITED]	Lemma 20
Prop. 27.3.2 pencil = compression of γ^2 (finite model)	[THM]	–
Prop. 27.3.3 negative weight $2m\kappa_H$	[THM] (leading order)	–
Lemma 27.4.1 birth only at $\det G_A = 0$	[THM]	Sylvester
Prop. 27.4.2 $\kappa_A = n_{\mathbb{C}} + n_{-}$ (+Jordan)	[THM] from [CITED]	GLR, Mehl, Pontryagin
Thm. 27.4.3 definite-type stability	[CITED]	Krein, Gelfand–Lidskii
Thm. 27.4.4 two-stage mechanism	[THM]+[NUM]	–
$\mu \approx (\gamma \pm i\delta)^2$ in the window; thresholds	[NUM]	§27.5
$\delta_b, \delta_c \rightarrow 0$	[CONJ]	–
L_4/Routh	[ANALOGY] (shared theorem [CITED])	Routh, MacKay

27.9. Addendum A (rev. 2): the two thresholds as the window grows

This addendum answers open question Q3. We use the replacement model at $\gamma = \gamma_1$ and write $d = e^{2h}$ for Lemma 11’s cusp height. $\delta_b(h)$ is the first δ with $\kappa_A = 1$ and $\delta_c(h)$ the first δ with

$n_{\mathbb{C}} = 1$. **Verdict.** Both thresholds tend to 0, and both do so doubly exponentially in h , i.e. exponentially in d . The rates differ, so the ratio δ_c/δ_b is unbounded, and it too grows doubly exponentially. The two stages merge at $\delta = 0$ in absolute terms. In window size they stay a bounded distance apart: for a fixed small δ the boost needs a cusp height about $1.6\times$ larger ($\Delta h \approx 0.24$); the reduced model predicts $3/2$ ($\Delta h = \frac{1}{2} \log \frac{3}{2} \approx 0.20$). What is proved: an exact birth formula (Proposition 27.9.1); unconditional super-polynomial decay of δ_b on the full window space (Theorem 27.9.3); and the full two-threshold law in a reduced two-mode model (Theorem 27.9.5). The doubly exponential rates themselves are [NUM], and conditional on Zhu's law they are consistent with Lemma 12.

27.9.1. Birth is a Christoffel threshold. Let R be the window Gram of $Q - T_\gamma$ (the background, i.e. γ_1 removed). Let $X, Y, Z \in \mathbb{R}^N$ be the window vectors of $F \mapsto \tilde{F}(\gamma), \tilde{F}'(\gamma), \tilde{F}''(\gamma)$, phase-normalized to be real in each parity sector. Put $R_+ = R + 4XX^\top$.

PROPOSITION 27.9.1 (birth formula; [THM] + [NUM]). *Assume $R > 0$ (hypothesis (H_R)): it holds if the background zeros are on the line, and in every computed window $\lambda_{\min}(R) > 0$). Then*

$$G_A(\delta) = R_+ + \delta^2(-4YY^\top - 2(XZ^\top + ZX^\top)) + O(\delta^4).$$

For the truncation $G^b(\delta) = R_+ - 4\delta^2YY^\top$ one has exactly $\det G^b(\delta) = \det R_+ (1 - \delta^2/\varepsilon)$ and $\kappa(G^b(\delta)) = [\delta^2 > \varepsilon]$, where

$$\varepsilon(h) = \frac{1}{4Y^\top R_+^{-1}Y} = \frac{1}{4} \left(\sup_{F \in \text{window}} \frac{|\tilde{F}'(\gamma)|^2}{R(F) + 4|\tilde{F}(\gamma)|^2} \right)^{-1}$$

is a derivative Christoffel function. [NUM] In every run (Q100: $N = 20, 30, 40, 49$; Q240: $N = 40-100$) the bisected δ_b equals $\sqrt{\varepsilon}$ to better than 1%. In the Q240 runs κ_A is 0 at $0.99\sqrt{\varepsilon}$ and 1 at $1.01\sqrt{\varepsilon}$.

PROOF. Taylor expansion as in Proposition 27.3.3, with the factor 4 from $T_z + T_{\bar{z}} = 4 \operatorname{Re}(\tilde{F}(z)\overline{\tilde{F}(\bar{z})})$. Then the matrix determinant lemma. The inertia follows from the Haynsworth additivity $\operatorname{In}(R_+ - 4\delta^2YY^\top) = \operatorname{In}(R_+) + \operatorname{In}(1 - 4\delta^2Y^\top R_+^{-1}Y) - \operatorname{In}(1)$. $\square$

LEMMA 27.9.2 (monotonicity in N ; [THM]). *The bases are nested $(\phi_0, \phi_2, \dots)$, so $\kappa_A^{(N)} \leq \kappa_A^{(N+1)} \leq \kappa_A^{\text{full}}$: a negative subspace of a compression is negative for the larger form. Hence $\delta_b^{(N)}(h)$ is nonincreasing in N , and every computed $\delta_b^{(N)}$ is an upper bound for the full-window $\delta_b^{\text{full}}(h)$.*

There is no such monotonicity for δ_c , since $n_{\mathbb{C}}$ of a compression is not monotone in general.

THEOREM 27.9.3 (unconditional super-polynomial decay of δ_b ; [THM]). *Let the window space be all real $F \in C_c^\infty(-h, h)$ of fixed parity. Let $Q_\delta(F) = R(F) + c \operatorname{Re}(\tilde{F}(\gamma + i\delta)\overline{\tilde{F}(\gamma - i\delta)})$ with $c > 0$ and $R(F) = c \sum_{\rho \neq \rho_1} m_\rho |\tilde{F}(\gamma_\rho)|^2$. This is the zero-side form of the replacement model, by Weil's explicit formula, under RH for the background zeros. Then for every $k \geq 2$ there is C_k with $\delta_b^{\text{full}}(h) \leq C_k h^{-k}$ for $h \geq 1$.*

PROOF. Take $\Psi \in C_c^\infty(-1, 1)$ even with $\Psi \geq 0$ and $\int \Psi = 1$. Then $\tilde{\Psi}$ is entire, real on $\mathbb{R} \cup i\mathbb{R}$, and satisfies $\tilde{\Psi}(iy) = \int \Psi \cosh(yu) du \geq 1$ and $|\tilde{\Psi}(w)| \leq C_k(1 + |w|)^{-k}$ for $|\operatorname{Im} w| \leq 1$. Set $\Phi_h(z) = (z - \gamma)\tilde{\Psi}(h(z - \gamma))$. This is the transform of a function supported in $[-h, h]$: dilation gives the support, and the factor $(z - \gamma)$ is a derivative times a modulation, which keeps it. Let $\tilde{F}_h(z) = \Phi_h(z) + \Phi_h(-z)$ in the even sector, or $i(\Phi_h(z) - \Phi_h(-z))$ in the odd sector. Then F_h is real, has the right parity, and is supported in $(-h, h)$.

- *Ghost term.* $\tilde{F}_h(\gamma \pm i\delta) = \pm i\delta P + e_\pm$ with $P = \tilde{\Psi}(ih\delta) \geq 1$ and $|e_\pm| = |\Phi_h(-\gamma \mp i\delta)| \leq E := C_k(2\gamma + 1)(1 + 2\gamma h)^{-k}$. So the pair term is at most $c(-\delta^2 P^2 + 2\delta P E + E^2)$.
- *Background.* For $\gamma_\rho \neq \gamma$ we have $|\Phi_h(\pm\gamma_\rho)| \leq C_k |\pm\gamma_\rho - \gamma|^{1-k} h^{-k}$. The gap $\min_{\rho \neq \rho_1} |\gamma_\rho - \gamma_1| > 0$ and $\sum_\rho m_\rho |\gamma_\rho|^{-2} < \infty$ give $R(F_h) \leq C'_k h^{-2k}$.

Hence for $4E \leq \delta \leq 1$, $Q_\delta(F_h) \leq -\frac{c}{2}\delta^2 + cE^2 + C'_k h^{-2k} < 0$ once $\delta > C_k h^{-k}$, so $\kappa \geq 1$. $\square$

Status. This bound is far from sharp: the data are doubly exponential. The sharp rate is a sampling problem, namely the hole effect below Nyquist density (Lemma 11 §3). Conditionally on a lower bound $\lambda_*(d) \geq e^{-\varkappa d}$ for the window margin (Zhu's law, $\varkappa = 2\pi^2$), Lemma 12 Prop. 6.1 gives the *lower* bound $\log \frac{1}{\delta_b} \leq \frac{\varkappa}{2}d + O(\log h)$. That bound is conditional; the matching upper bound is heuristic.

27.9.2. The reduced two-mode model.

DEFINITION 27.9.4 ([DEF]). Let φ be the rotation mode ($x = 1, y = 0$, no background) and ψ the ghost mode ($x = 0, y = 1$, background $R(\psi)/4 = \varepsilon$, background frequency $\omega^2 = R_K(\psi)/R(\psi)$). Keep the pair terms of Proposition 27.3.3 exactly and drop $O(\delta^4)$ inside K_A . Then

$$G = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon - \delta^2 \end{pmatrix}, \quad K = \begin{pmatrix} c & -2\gamma\delta^2 \\ -2\gamma\delta^2 & \varepsilon\omega^2 - c\delta^2 \end{pmatrix}, \quad c = \gamma^2 - \delta^2.$$

THEOREM 27.9.5 (two thresholds in the model; [THM]). Put $\nu = \mu - c$ and $\Delta = \omega^2 - c > 0$. Then $\det(K - \mu G) = 0$ iff $(\varepsilon - \delta^2)\nu^2 - \varepsilon\Delta\nu - 4\gamma^2\delta^4 = 0$, so:

- $\delta_b = \sqrt{\varepsilon}$. At δ_b the leading coefficient vanishes and one root passes through ∞ ; it returns from $-\infty$ because $\Delta > 0$, with negative type.
- Complex roots occur iff $16\gamma^2\delta^4(\delta^2 - \varepsilon) > \varepsilon^2\Delta^2$. The left side increases from 0 to ∞ on $\delta > \delta_b$, so δ_c is unique, and $\delta_c = (\varepsilon\Delta/4\gamma)^{1/3}(1 + O((\varepsilon/\Delta^2)^{1/3}))$ as $\varepsilon \rightarrow 0$.
- $\delta_c/\delta_b = (\Delta/4\gamma)^{1/3}\varepsilon^{-1/6}(1 + o(1)) \rightarrow \infty$. If Δ stays bounded above and below, $\log \delta_c / \log \delta_b \rightarrow \frac{2}{3}$.
- For $\delta \gg \delta_c$: $\mu = c + \frac{\varepsilon\Delta}{2(\varepsilon - \delta^2)} \pm 2i\gamma\delta(1 + O(\varepsilon/\delta^2))$, so $\text{Im } \mu \rightarrow 2\gamma\delta$, i.e. $\mu \rightarrow (\gamma \pm i\delta)^2$ (boost rapidity δ).

PROOF. Expand the determinant. The discriminant is $\varepsilon^2\Delta^2 + 16\gamma^2\delta^4(\varepsilon - \delta^2)$. For $\delta^2 > \varepsilon$ it is negative iff (b) holds. Solving with $\delta^2 \gg \varepsilon$ gives (b)–(d). The types follow from Proposition 27.4.2, since $\kappa(G) = 1$ for $\delta > \delta_b$. $\square$

In the window, Δ computed from the Christoffel minimizer ψ grows like N^2 at fixed h . This is junk at high frequency, and the measured δ_c converges in N . So we define the *effective* Δ_{eff} by inverting (b) exactly on the measured pair (ε, δ_c) : $\Delta_{\text{eff}} = 4\gamma\delta_c^2\sqrt{\delta_c^2 - \varepsilon}/\varepsilon$. The model is [ANALOGY]-level, a structural normal form: it fixes the *shape* of the law, and Δ_{eff} is a fitted, slowly varying parameter.

27.9.3. Numerics (well-conditioned). Scripts: `num/lemma27_thresholds.py` (dps 140; bisection in $\log \delta$, 40 steps for δ_b by an LDL inertia count, 26 steps for δ_c by n_C), `lemma27_dc49.py` ($N = 49$, the maximum for Q100), and `lemma27_birthconv.py` (Lemma 11's 240-mode matrices, 800-bit data, dps 200, $N \leq 100$). Q100 and Q240 agree at $N = 40$ to all printed digits.

- *Birth law.* Fit on $h \in [0.7, 1.2]$: $\log \frac{1}{\delta_b} = 5.83d - 14.0$ (parity 0; max residual 0.30) and $5.67d - 15.3$ (parity 1; 0.39). The local slope rises slowly, $5.76 \rightarrow 5.97$ over $d = 5\text{--}11$, i.e. $\varkappa_{\text{eff}} = 2a \approx 11.5\text{--}12$. Conditional asymptotics (Zhu): $\text{slope} \rightarrow \pi^2 \approx 9.87$. Not reached.

h	d	$\log_{10} \delta_b$ (p0/p1)	$\log_{10} \delta_c$ (p0/p1)	θ (p0)	$\log_{10} \Delta_{\text{eff}}$	N (δ_c/δ_b)
0.5	2.72	-1.40/ -0.84	-0.36/ -0.01	0.26	3.47	40/40
0.6	3.32	-2.63/ -1.95	-1.12/ -0.69	0.43	3.65	40/40
0.7	4.06	-4.31/ -3.53	-2.18/ -1.65	0.51	3.85	40/40
0.8	4.95	-6.43/ -5.55	-3.53/ -2.94	0.55	4.03	40/40
0.9	6.05	-9.17/ -8.17	-5.26/ -4.59	0.57	4.29	49/100
1.0	7.39	-12.54/ -11.46	-7.44/ -6.72	0.59	4.46	49/100
1.1	9.03	-16.73/ -15.57	$\leq -9.95/ \leq -9.29$	-	-	49/100
1.2	11.02	-21.91/ -20.65	$\leq -11.87/ \leq -10.53$	-	-	49,40/100

TABLE 27.2. Thresholds at γ_1 ([NUM]). N -stability: δ_b changes by ≤ 0.004 decades from $N = 80$ to 100 for $h \leq 1.2$. At $h = 1.3$ it is not converged ($-27.15 \rightarrow -28.33$). δ_c changes by ≤ 0.016 decades from $N = 40$ to 49 for $h \leq 1.0$. At $h = 1.1, 1.2$ it still moves (0.56, 1.06 decades), so only the $N = 49$ values are shown, as presumptive upper bounds. In Q100 the δ_b values for $h \geq 1.1$ are not converged at $N \leq 49$, so θ is omitted there.

- *Collision law.* Fit on $h \in [0.7, 1.0]$ (converged): $\log \frac{1}{\delta_c} = 3.64d - 9.8$ (parity 0; max residual 0.09) and $3.50d - 10.5$ (parity 1; 0.12). The slope ratio $a_b/a_c = 1.60$ and 1.62 (model: $3/2$). θ rises toward $a_c/a_b \approx 0.62$ (model: $2/3$). Δ_{eff} grows only slowly ($10^{3.9} \rightarrow 10^{4.5}$), which is why the exponent stays slightly below $2/3$.
- *Ratio.* $\log(\delta_c/\delta_b) \approx (a_b - a_c)d \approx 2.2d$, so the ratio is unbounded and grows doubly exponentially in h .
- *Boost read-out.* At $\delta = 10^{1/2}\delta_c$ the pair matches $(\gamma_1 \pm i\delta)^2$. For example, at $h = 1.1$, $N = 30$: $\text{Im } \mu = 1.03496 \cdot 10^{-6}$ against $1.03548 \cdot 10^{-6}$.

Relation to Lemma 11/12's doubly-log law. δ_b is *the same event* as Lemma 11's d_{ghost} , the first negative eigenvalue of the ghost model. Inverting our fit gives $d_b(\delta) = 2.40 + 0.172 \log \frac{1}{\delta}$ (parity 0) and $2.69 + 0.176 \log \frac{1}{\delta}$ (parity 1), against Lemma 12's $d_{\text{ghost}} = 2.28 + 0.192 \log \frac{1}{\delta}$. That fit used the e_n basis, a coarser d -grid and the minimum over parities. Agreement is within Lemma 12's residual 0.24 in d ([NUM]). The *boost* law is the new analogue: $d_c(\delta) = 2.70 + 0.275 \log \frac{1}{\delta}$ (parity 0) and $3.00 + 0.286 \log \frac{1}{\delta}$ (parity 1). Its slope is $1.6 \times$ Lemma 12's, and the model predicts $\frac{3}{2} \times$. Both are doubly-log in the window. A boost is therefore detected at about 1.6 times the cusp height of the ghost, or $\Delta h \approx \frac{1}{2} \log 1.6 \approx 0.24$ later in window half-width, for every small δ .

Erratum to §27.5 (v1). The “transitional outlier” ($h = 1$, parity 0, $\delta = 4.2 \cdot 10^{-6}$, “ $\text{Im } \mu = 9.66$ ”) was a *reporting bug*: the printed string `9.66230522e-5` was truncated to 8 characters and lost its exponent. Recomputed at dps 140 and 220 (identical), $\text{Im } \mu = 9.6623 \cdot 10^{-5}$. This is below the asymptotic $2\gamma\delta = 1.19 \cdot 10^{-4}$, as expected just past the collision, since $\delta_c = 3.5 \cdot 10^{-6}$ there. So there is no conditioning artifact; the v1 bullet is withdrawn.

27.10. Addendum B: a Lagrange-point dictionary

27.10.1. Setting and the equilibrium condition. For the planar circular restricted three-body problem in the rotating frame (normalization of [100, Ch. II.G], [99]), the primaries $1 - \mu$ and μ sit at $(-\mu, 0)$ and $(1 - \mu, 0)$, and

$$H = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y - U, \quad U = \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}, \quad \Omega = \frac{1}{2}(x^2 + y^2) + U.$$

PROPOSITION 27.10.1 (equilibria; [THM], classical). $\nabla H = 0$ iff $p = (-y, x)$, i.e. zero velocity in the rotating frame, and $\nabla \Omega = 0$, i.e. $\nabla U = -(x, y)$. In the inertial frame this says: the net

gravitational acceleration equals $-\mathbf{r}$, where $\mathbf{r}$ is the position from the barycenter. So it is radial, directed to the barycenter, with the centripetal magnitude $|\mathbf{r}|$ ($\omega = 1$). Its tangential component is zero, and it is orthogonal to the velocity $\boldsymbol{\omega} \times \mathbf{r}$. The condition is first order ($\nabla = 0$) and holds at all five points: L_1 – L_3 (Euler) and $L_{4,5}$ (Lagrange). We checked it symbolically at $L_4 = (\frac{1}{2} - \mu, \frac{\sqrt{3}}{2})$ (`num/lemma27_L4.py`).

Clarifying J.W.’s phrasing. The force is *collinear* with the radius vector from the barycenter. The orthogonality is to the velocity, i.e. the tangent, and it is *Euclidean*. Nothing hyperbolic enters the force balance. The hyperbolic or “self-orthogonal” geometry sits in the *Krein form*. In the window it is the off-line direction e_ρ , which is Q -null ($[e_\rho, e_\rho]_{\mathcal{Z}} = 0$, Proposition 27.1.1). At L_4 it is the eigenvectors of a post-Routh quadruple, which are S -null because $v^* S v = 0$ for every non-real eigenvalue of an S -selfadjoint operator [THM].

27.10.2. Linearization at L_4 and Routh’s value as a Krein collision. Let $S = \text{Hess } H(L_4)$ and $A = \mathcal{J}S$ the linear Hamiltonian vector field.

THEOREM 27.10.2 ([THM]; computed symbolically, agrees with [100, 130]). (a)

$$\text{Hess } \Omega(L_4) = \begin{pmatrix} 3/4 & \frac{3\sqrt{3}}{4}(1-2\mu) \\ \frac{3\sqrt{3}}{4}(1-2\mu) & 9/4 \end{pmatrix},$$

with determinant $\frac{27}{4}\mu(1-\mu) > 0$ and trace 3. So it is positive definite for $0 < \mu < 1$, and L_4 is a strict maximum of the effective potential $V_{\text{eff}} = -\Omega$.

- (b) S has signature $(2, 2)$ for every $0 < \mu < 1$. In the Lagrangian coordinates $(q, \dot{q})$ the energy is $\frac{1}{2}|\dot{q}|^2 - \frac{1}{2}q^T \text{Hess } \Omega q$, and the linear change $(q, p) \mapsto (q, \dot{q})$ preserves inertia. So H_2 is indefinite, and Lagrange–Dirichlet does not apply.
- (c) $\det(A - \lambda) = \lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu)$. So $\lambda^2 = \frac{1}{2}(-1 \pm \sqrt{1 - 27\mu(1-\mu)})$, and the spectrum is on $i\mathbb{R}$ iff $27\mu(1-\mu) \leq 1$, i.e. $\mu \leq \mu_R = \frac{1}{2}(1 - \frac{\sqrt{69}}{9}) = 0.0385208965$ (Routh [130]).
- (d) iA is S -selfadjoint, since $S(iA) = iS\mathcal{J}S$ is Hermitian. So §27.4 applies verbatim, in a Krein space of signature $(2, 2)$. For $\mu < \mu_R$ the eigenvalues of iA are $\pm\omega_f, \pm\omega_s$, all semisimple. Their Krein types are mixed: the fast mode is positive and the slow mode negative ([NUM] below). This gives $\kappa(S) = 2 = n_-$, the count identity of Proposition 27.4.2.
- (e) At $\mu = \mu_R$ the two modes collide at $\omega = 1/\sqrt{2}$, and for $\mu > \mu_R$ they form a quadruple $\pm a \pm ib$. This is an opposite-signature Krein collision [THM]. If the colliding types were equal, Theorem 27.4.3 would keep the spectrum on the axis for μ slightly above μ_R , contradicting (c). Mixed type is thus forced, and the numerics confirm it.
- (f) As $\mu \downarrow 0$: $\omega_s = \sqrt{27\mu/4}(1 + O(\mu)) \rightarrow 0$, and S degenerates. At $\mu = 0$ its eigenvalues are $-2.303, 0, 1.303, 2$ (inertia $(2, 1) + \text{null}$). So the negative-type slow mode is born from frequency 0 through a degeneration of the Krein metric S . This is the exact structural counterpart of window stage 1, where G_A degenerates and the eigenvalue passes through ∞ ; frequency 0 in iA corresponds to ∞ in the inverted pencil.

μ	ω_f (type)	ω_s (type)	In S	spectrum
Sun–Jupiter $9.54 \cdot 10^{-4}$	0.996757 (+)	0.080469 (–)	(2, 2)	on $i\mathbb{R}$
Earth–Moon 0.01215	0.954503 (+)	0.298200 (–)	(2, 2)	on $i\mathbb{R}$
$\mu_R - 10^{-6}$	0.708870 (+)	0.705340 (–)	(2, 2)	on $i\mathbb{R}$
$\mu_R + 10^{-6}$	quadruple $\pm 0.001765 \pm 0.707109i$		(2, 2)	Hamiltonian–Hopf
0.1	quadruple $\pm 0.37378 \pm 0.79982i$		(2, 2)	unstable

Collinear points (Earth–Moon, [NUM]). At L_1, L_2, L_3 ($x = 0.83692, 1.15568, -1.00506$) the eigenvalues are ± 2.93205 and $\pm 2.33438i$; ± 2.15868 and $\pm 1.86265i$; ± 0.17787 and $\pm 1.01042i$.

In $S = (3, 1)$. The real pair is a saddle, a pure boost, and is unstable for every μ . In the iA picture it is one non-real pair, so $\kappa(S) = 1 = n_{\mathbb{C}}$, again the count identity.

27.10.3. The dictionary.

Restricted 3-body (rotating frame)	Window Weil pencil / zero space	status
equilibrium $\nabla H = 0$ (first order, automatic at L_1 – L_5)	FE kills the linear δ term: δ is J_{FE} -odd, the $O(\delta)$ part of the pair is purely imaginary (Prop. 27.3.3; Lemma 25)	each side [THM]; the match “first order free, physics in second order” is [ANALOGY]
$H_2 = \frac{1}{2}S$, Krein metric of iA	G_A , Krein metric of $G_A^{-1}K_A$	same structure: selfadjoint in a finite-dim Krein space [THM]
Lagrange–Dirichlet: S definite $\Rightarrow$ stable	$\kappa_A = 0$ (RH-like window): $G_A > 0$, spectrum real, all positive type, strongly stable	same theorem (definite metric) [THM]/[CITED]
$\mu \downarrow 0$: S degenerates, slow mode born at frequency 0 with negative type	stage 1: $\det G_A = 0$, eigenvalue through ∞ , negative type	same mechanism (metric degeneration, Lemma 27.4.1); the parameter map $\mu \leftrightarrow \delta$ is [ANALOGY]
$0 < \mu < \mu_R$: spectrally stable, H_2 indefinite, mixed Krein types	$\delta_b < \delta < \delta_c$: $\kappa_A = 1$, all eigenvalues real, one negative-type ghost, no boost	same canonical form and count [THM]; $\kappa = 2$ vs 1 (real Hamiltonian $\pm$ doubling)
Routh $27\mu(1 - \mu) = 1$: opposite-signature collision, Hamiltonian–Hopf	$\delta = \delta_c$: opposite-signature collision, boost onset	same theorem (Thm. 27.4.3) [THM]/[CITED]
quadruple $\pm a \pm ib$ (rotation + boost)	$(\gamma \pm i\delta)^2$ (rotation γ , rapidity δ)	[ANALOGY] (different operators)
collinear saddle $\pm\sigma$ (pure boost)	pole pair $\{0, 1\}$: pure boost, rapidity $\frac{1}{2}$	[ANALOGY]
eigenvectors of the quadruple S -null	e_ρ Q -null	same fact (non-real $\Rightarrow$ neutral) [THM]

What is the same theorem and what is analogy. The shared mathematics is exactly §27.4: the canonical form with sign characteristic, the count $\kappa = n_{\mathbb{C}} + n_-$, definite-type stability versus mixed-type collision, and Sylvester inertia for metric degeneration. Both systems are selfadjoint operators in finite-dimensional Krein spaces, so those statements hold *verbatim* on both sides. Everything else is analogy: the operators, the parameters (μ against δ and h), the mechanics (there is no Hamiltonian flow behind the window pencil), and the value of Routh’s constant. Nonlinear stability at L_4 (KAM and Moser, away from the resonances; Lyapunov-stable even at μ_R [132]) has *no* counterpart: the window problem is linear.

CHAPTER 28

Lemma 28: (A) spatial modes for the zero telegrapher line;

(B) $\Theta < 1$ as a target: verification, obstructions, translations, and a pole-dominance theorem for short windows

Standalone note `lemma28.tex`, October 21, 2026.

Summary. *Tags.* [THM] proved here; [CITED] quoted from a source checked at the source (`lit/lemma28/verified_refs.md`); [DEF] definition; [ANALOGY] analogy only; [NUM] float64 numerics, not certified; [RESTATEMENT] a restatement of a known equivalence (no new content); $[\equiv \text{GRH for the family}]$ equivalent to GRH for the family; [CONJECTURAL] conjectural framework.

Verdict A (spatial modes). No candidate gives “a real telegrapher line” for the *nontrivial* zeros. An exact dispersion relation exists only for the *trivial* zeros (the Γ -factor channel). There, Hecke Grössencharacters of an imaginary quadratic field shift the trivial-zero ladder linearly in the angular index $|\ell|$. Nontrivial zeros carry only a Weyl law (the density depends on the analytic conductor). The best rigorous spatial variable is the Pontryagin dual of the compact part of the idele class group, i.e. Dirichlet characters for $\mathbb{Q}$ and Hecke characters for K ; vertical twists $|\cdot|^{i\tau}$ are a gauge. With it, $k = 0$ is ζ , and the Weil form is *block-diagonal* in k (Connes). So “distortionless at every k ” is GRH for the family $[\equiv \text{GRH for the family}]$. The quadratic forms couple across k only through positivity of prime-side coefficients, which acts at the edge $\sigma \approx 1$ (Linnik’s principles). The one known mechanism in which cross- k coupling forces balance is Deligne’s tensor-power/positive-coefficient argument (Weil I, §3). Its GL_1 shadow is trivial for ζ , so this is a negative finding.

Verdict B ($\Theta < 1$). (i) Lemma 18’s $\Theta = \min\{\sigma : \kappa_{\sigma,A} = 0 \ \forall A\}$ equals the classical $\sup \text{Re } \rho$, and the minimum is attained [RESTATEMENT]. (ii) The target “ $\kappa_{\sigma,A} = 0$ for all A ” with $\sigma < 1$ is *literally* $\Theta \leq \sigma$. Zero-free regions (width $\rightarrow 0$) and density theorems (counts, not absence) cannot reach it. (iii) Density theorems do translate rigorously into bounds on the *negativity mass* of the Lagarias symbol, $\int_{-T}^T (w_\sigma)_- \leq 2\pi N(\sigma, 2T) + \text{tail}$ [THM]. Thus Ingham, Huxley, Guth–Maynard and the log-free family estimate become “the negative part of the damped symbol is a vanishing fraction” statements. (iv) **New theorem (pole dominance)** [THM]. Among all zero pairs in the closed strip, the worst damped pair kernel is the pole kernel $e^{(1-\sigma)|u|} + e^{-\sigma|u|}$, by a Schur-product argument with the positive-definite function $\cosh(bx)/\cosh x$. Hence $\kappa_{\sigma,A} = 0$ whenever $A \leq A_\sigma^{\text{pole}}$, using only $0 \leq \text{Re } \rho \leq 1$. Rigorously $A_\sigma^{\text{pole}} \geq \log \frac{\sigma}{1-\sigma}$ (Pólya), i.e. $\kappa_{\sigma,A} = 0$ for $\sigma \geq 1/(1 + e^{-A})$. Numerically $A_\sigma^{\text{pole}} \approx (2.3\text{--}3.0)/\sqrt{1-\sigma}$ [NUM], e.g. $\kappa_{\sigma,A} = 0$ for $\sigma \geq 0.948$ at $A = 10$, versus Lemma 18’s 0.99977. The theorem is data-blind and height-blind, so it cannot give $\Theta < 1$.

28.1. Setting

Notation is that of Lemma 18 (Def. 1.1 there): $\mathcal{Q}_\sigma(F) = \sum_\rho m_\rho \widehat{G}_\sigma(\gamma_\rho)$ with $G_\sigma = e^{-c|u|} H_F$, $c = \sigma - \frac{1}{2}$, $\gamma_\rho = -i(\rho - \frac{1}{2})$, and $\widehat{G}(z) = \int G(u) e^{izu} du$. $\mathcal{D}_A$ is the space of $F \in C_c^\infty$ supported in an interval of length A . $\kappa_{\sigma,A}$ is the negative index of $\mathcal{Q}_\sigma$ on $\mathcal{D}_A$, $A_\sigma^{\text{det}} = \inf\{A : \kappa_{\sigma,A} \geq 1\}$,

$\Theta = \sup \operatorname{Re} \rho$, and $w_\sigma(t) = 2 \operatorname{Re} \frac{\xi'}{\xi}(\sigma + it)$. For $\rho = \beta + i\gamma_0$ write $\delta = \beta - \frac{1}{2}$, $\varepsilon = \beta - \sigma$, $w = \beta + \sigma - 1$; for $\beta \geq \frac{1}{2}$ we have $w \geq \sigma - \frac{1}{2}$. A kernel k is *p.d. on* $(-A, A)$ if $\sum_{i,j} c_i \bar{c}_j k(x_i - x_j) \geq 0$ for all finite families with x_i in an interval of length A . We use Lemma 25's telegrapher reading: $P_\rho = (D - \rho)(D - (1 - \bar{\rho}))$, demodulated to $(D - \beta)(D - (1 - \beta))$, $LC = 1$, $R/L = \beta$, $G/C = 1 - \beta$, and the passivity ladder (P0) $0 \leq \beta \leq 1$, ..., (P3) $\Theta < 1$, (P4) $\delta = 0$ (Lemma 25, Prop. 1.8).

LEMMA 28.1.1 (pair kernel). *[THM] The zeros $\rho, 1 - \bar{\rho}$ (same γ_0 , $\beta \geq \frac{1}{2}$) together contribute $\iint F(x) \bar{F}(y) e^{i\gamma_0(x-y)} k_\delta(x-y) dx dy$ to $\mathcal{Q}_\sigma(F)$, where*

$$k_\delta(u) = 2 \cosh(\delta u) e^{-c|u|} = e^{\varepsilon|u|} + e^{-w|u|}.$$

On-line zeros ($\delta = 0$) give $k_0 = 2e^{-c|u|}$, which is p.d. on $\mathbb{R}$. The pair contribution is ≥ 0 on $\mathcal{D}_A$ iff k_δ is p.d. on $(-A, A)$. Set $A^{\text{pair}}(\varepsilon, w) := \sup\{A : k_\delta \text{ p.d. on } (-A, A)\}$.

PROOF. $\widehat{G}_\sigma(\gamma_\rho) = \int G_\sigma(u) e^{i\gamma_0 u} e^{\delta u} du$, and $1 - \bar{\rho}$ has $-\delta$, so the sum is $\int H_F(u) e^{i\gamma_0 u} k_\delta(u) du = \iint F(x) \bar{F}(y) e^{i\gamma_0(x-y)} k_\delta(x-y)$. Multiplication by $e^{i\gamma_0 x}$ is unitary on $\mathcal{D}_A$ and preserves supports, so modulation does not change positivity. $\widehat{e^{-c|u|}} = 2c/(c^2 + t^2) > 0$. $\square$

28.2. Part A: spatial modes

28.2.1. What a “real line” requires.

PROPOSITION 28.2.1 (telegrapher with a spatial term). *[THM] For the demodulated line $(D - \beta)(D - (1 - \beta))$, add a spatial term k^2 : $(\lambda + \beta)(\lambda + 1 - \beta) + k^2 = 0$ for modes $e^{\lambda\tau + ikx}$ (orientation $\tau = -u$ of Lemma 25). Then $\lambda = -\frac{1}{2} \pm \sqrt{\delta^2 - k^2}$. For $|k| > |\delta|$ every mode is damped at the common rate $\frac{1}{2}$ and oscillates at $\omega(k) = \sqrt{k^2 - \delta^2}$, which is a genuine dispersion relation. The line is distortionless ($\omega = |k|$ for every k) iff $\delta = 0$, the Heaviside condition $R/L = G/C$.*

PROOF. $(\lambda + \frac{1}{2})^2 - \delta^2 + k^2 = 0$. $\square$

So a “spatial mode” needs three things: (D1) a family indexed by k whose $k = 0$ member is ζ ; (D2) an exact relation tying channel- k spectral data to k ; and (D3) the distortionless condition at every k should be a statement about the family, ideally with a cross- k constraint that $k = 0$ alone lacks.

28.2.2. Candidate (2): vertical twists $\zeta(s + i\tau)$; Rankin–Selberg.

PROPOSITION 28.2.2 (twist is a gauge). *[THM] Let $Q^{(\tau)}$ be the form built from the zeros $\rho - i\tau$ of $\zeta(s + i\tau)$. Then $Q_\sigma^{(\tau)}(F) = \mathcal{Q}_\sigma(e^{-i\tau x} F)$. So $Q_\sigma^{(\tau)}$ and $\mathcal{Q}_\sigma$ have the same index on every window, and $\omega = \gamma_0 - \tau$ is a Galilean shift, not a dispersion relation. The mirror $\rho \mapsto 1 - \bar{\rho}$ survives the twist; the conjugation $\rho \mapsto \bar{\rho}$ does not.*

PROOF. $H_{e^{-i\tau} \cdot F} = e^{-i\tau u} H_F$, $e^{-c|u|}$ is unchanged, and $\widehat{e^{-i\tau u} G}(\gamma) = \widehat{G}(\gamma - \tau)$. The mirror image of $\rho - i\tau$ is $1 - \bar{\rho} - i\tau$, again a twisted zero. The conjugate $\bar{\rho} + i\tau$ is a twisted zero only if $\bar{\rho} + 2i\tau$ is a zero. $\square$

[CITED] Connes [35, eqs. (29)–(30)] shows that the twist $D' = D - is$, $L(\tilde{\chi}', z) = L(\tilde{\chi}, z + is)$, is a change of the splitting $C_k \cong C_{k,1} \times N$. The only cross- τ relation in TAA is Lemma 18's Cauchy average over modulations (Prop. 3.1(b) there), which is the damping. At GL_1 , Rankin–Selberg $L(\chi_1 \times \chi_2, s) = L(\chi_1 \chi_2, s)$ gives nothing new [THM] (Euler factors multiply). **Answers:** no dispersion; distortionless at every τ is RH itself [RESTATEMENT]; no new cross- τ constraint.

28.2.3. Candidates (1) and (3): characters of the idele class group.

DEFINITION 28.2.3. [DEF] $C_{\mathbb{Q}} = \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^{\times}$. Its unitary characters are $|\cdot|^{i\tau} \otimes \chi$ with χ a primitive Dirichlet character. For an imaginary quadratic K , Hecke characters have infinity type $\chi_{\infty}(z) = (z/|z|)^{\ell}$, $\ell \in \mathbb{Z}$. The *spatial variable* is the discrete part $k = \chi$ (conductor q , parity a), resp. $k = (\chi_f, \ell)$; τ is the gauge of Prop. 28.2.2.

- PROPOSITION 28.2.4. (a) [CITED] Connes [35, Thm. 1, decomposition (23)]: the spectral realization splits as $\bigoplus_{\chi \in \widehat{K}} H_{\chi}$, and the Weil distribution is the “synthesis of the explicit formulas for all L -functions with Grössencharakter”. Hence the family Weil form is block-diagonal in k , and family indices add [THM] (given block-diagonality).
- (b) [THM] $L(\bar{s}, \bar{\chi}) = \overline{L(s, \chi)}$, so conjugation maps channel χ to $\bar{\chi}$ ($k \mapsto -k$). The functional equation also relates χ and $\bar{\chi}$ [CITED] [39, Ch. “The Functional Equation of the L Functions”, pp. 65–72]. Their composite, the mirror $\rho \mapsto 1 - \bar{\rho}$, is the only symmetry that preserves k .
- (c) Exact dispersion, trivial zeros only. [CITED] For Hecke L -functions over K with gamma factor parameters $a(\chi) + b(\chi) = n_K$, the trivial zeros are at $0, -1, -3, \dots$ (order $b(\chi)$) and $-2, -4, \dots$ (order $a(\chi)$) [149, (2.4), (2.7)]. For $\mathbb{Q}$ this is the parity ladder $-a - 2n$. For K imaginary quadratic with $\chi_{\infty} = (z/|z|)^{\ell}$ the factor is $\Gamma_{\mathbb{C}}(s + |\ell|/2)$, so the trivial zeros sit at $-|\ell|/2 - n$ (secondary source, see `verified_refs`). The real part of a trivial zero moves linearly in $|\ell|$: channel ℓ starts its ladder $\frac{1}{2} + \frac{|\ell|}{2}$ further from the critical line. This is an exact k -dependent “cutoff”, and it is the only one found.
- (d) Nontrivial zeros: Weyl law only. [CITED] $N(T, \chi) = \frac{T}{\pi} \log \frac{qT}{2\pi e} + O(\log qT)$ [39, Ch. “The Number $N(T, \chi)$ ”, pp. 101–103]. The conductor shifts the density, but no zero of channel k is determined by k .
- (e) [$\equiv$ GRH for the family] Distortionless ($\delta = 0$) in every channel is GRH for Dirichlet (resp. Hecke) L -functions. This is Weil’s criterion for the family [157]; Connes [35, Thm. 5] proves the function-field case and states the number-field analogue as a conjecture.
- (f) Cross- k coupling. By (a) none enters the quadratic form. It enters only through prime-side positivity, e.g. $\sum_{\chi \bmod q} \chi(n) = \varphi(q) \mathbf{1}_{n \equiv 1}$, i.e. nonnegative coefficients of $\prod_{\chi} L(s, \chi)$. That positivity yields Linnik’s three principles [CITED] [149, (3.3)–(3.5)]: a zero-free region $\sigma \geq 1 - 0.10367/\log(q(2 + |t|))$ with at most one exceptional zero; the log-free density $\sum_{\chi} N(\sigma, T, \chi) \ll_{\epsilon} (qT)^{(12/5+\epsilon)(1-\sigma)}$; and zero repulsion. All three are edge ($\sigma \rightarrow 1$) statements. Hecke’s angular equidistribution of prime ideals [68] corresponds to (P1), $L(1 + it, \chi_{\ell}) \neq 0$, for all ℓ .

28.2.4. Candidate (4): Deninger foliations. [CONJECTURAL]/[CITED] In Deninger’s formalism [42, §3], $H_{\text{dyn}}^0 = \mathbb{R}$ with $\Theta = 0$, H_{dyn}^1 carries the nontrivial zeros, and $H_{\text{dyn}}^2 = \mathbb{R}$ with $\Theta = \text{id}$. A Hodge $*$ commuting with Θ would force $\Theta = \frac{1}{2} + A$ with A skew, which is RH. On Riemannian foliations reduced leafwise cohomology is isomorphic to leafwise-harmonic forms [42, (13)] (Álvarez López–Kordyukov). If the flow is conformal with factor $e^{\alpha t}$, then $\Theta = \frac{\alpha}{2} + S$ on $\bar{H}^1$ with S skew [42, Thm. 4.1]. [ANALOGY] This is the exact analogue of “distortionless”, and it says that the zeros are the leafwise *zero modes* (harmonic forms). In this picture the non-harmonic leafwise modes, the would-be $k \neq 0$ channels, do not appear in ζ . That explains why no spatial dispersion is visible from ζ alone. No such space is known for $\mathbb{Q}$ [42].

28.2.5. Candidate (5): $C \times C$ over $\mathbb{F}_q$. [CITED]/[ANALOGY] The second factor is not a momentum: $H^1 \otimes H^1 \cong \text{End}(H^1)$, and on the transcendental part the intersection form is $-\text{Tr}(\alpha\alpha^{\dagger})$ (Lemma 27, Prop. 2.1; Milne pRH). If a finite group G acts on C , the isotypic

decomposition of H^1 is orthogonal for the G -invariant Rosati form, so Hodge index again acts channel by channel (block-diagonal). The one known cross-index mechanism is Deligne’s [41, §3] (“catalysé par la lecture de Rankin”). It needs (i) a family over a curve with Zariski-dense symplectic monodromy (Thm. 3.2(ii)); (ii) positivity of the coefficients of $\frac{d}{dt} \log \det(1 - F_x t, \mathcal{F}^{\otimes 2k})^{-1}$ for every k (Lemmas 3.3–3.4); and (iii) the comparison of radii of absolute convergence (Lemmas 3.5–3.6), which combined with $2k \rightarrow \infty$ gives the weight bound. Its automorphic analogue is Langlands’ observation: absolute convergence of $L(s, \pi, \text{Sym}^k)$ on $\text{Re } s > 1$ for all k implies Ramanujan for GL_2 (quoted in [11, §5.2]). Newton–Thorne prove Sym^n functoriality for holomorphic newforms [112], and [76] gives $H_n(\frac{1}{2})$. **Negative finding** [THM]: for $\zeta(\text{GL}_1)$ Sym^k of a character is a character, and the zeros are not Satake parameters, so there is no tensor-power family to amplify.

28.2.6. Recommendation.

candidate	exact dispersion?	distortionless $\forall k$	cross- k constraint
(1) Dirichlet χ	trivial zeros (parity) only	GRH (Dirichlet)	edge only (Linnik)
(2) twists, RS	no (gauge)	RH	none beyond L18 Prop. 3.1(b)
(3) Hecke / $\widehat{C_K}$	trivial zeros, linear in $ \ell $	GRH (Hecke)	edge; equidistribution $\Leftrightarrow$ (P1)
(4) Deninger	conjectural (zero modes)	would be a theorem	unknown (no space)
(5) $C \times C$	no (dual slot)	Hodge index per channel	Deligne $\otimes^{2k}$ (not GL_1)

Best spatial variable: (3), the dual of the compact part of C_K (for $\mathbb{Q}$: Dirichlet characters, i.e. (1) is a special case). It is rigorous, $k = 0$ is ζ , and the channels are exactly orthogonal. It is a *waveguide with decoupled modes*: the telegrapher picture extends channel by channel, but the dispersion is exact only in the Γ -factor (trivial-zero) sector. The architecture that would make k -coupling bite is Deligne-type tensor-power amplification, which needs Deninger’s H^1 . Nothing in hand provides it.

28.3. Part B: $\Theta < 1$

28.3.1. Verification.

PROPOSITION 28.3.1. [RESTATEMENT] $\min\{\sigma \geq \frac{1}{2} : \kappa_{\sigma,A} = 0 \ \forall A\} = \sup_{\rho} \text{Re } \rho$, and the minimum is attained. Equivalently (Lemma 25, (P3)), $\Theta = \inf\{\theta : \psi(x) - x = O(x^{\theta+\epsilon}) \ \forall \epsilon\}$.

PROOF. “ $\sup$ ”: this is Lemma 18, Thm. 3.3. “Attained”: at $\sigma = \Theta$ every zero has $\varepsilon \leq 0$, so the pair kernel $e^{\varepsilon|u|} + e^{-w|u|}$ is a sum of p.d. Laplace kernels (or $1 + e^{-w|u|}$ when $\varepsilon = 0$, and 1 is p.d.). By Lemma 28.1.1, $\mathcal{Q}_{\Theta} \geq 0$ on every window. The ψ form is proved in Lemma 25. $\square$

Nothing new here: Lemma 18’s Θ is the classical one.

28.3.2. Obstructions (all [RESTATEMENT] or [CITED]).

- (O1) *The target is the hypothesis.* “ $\kappa_{\sigma,A} = 0$ for all A ” $\Leftrightarrow \Theta \leq \sigma$. So any proof must exclude every zero right of σ at every height.
- (O2) *Zero-free regions shrink.* [CITED] Landau’s method [150, Thm. 3.10]: if $\zeta(s) = O(e^{\phi(t)})$ in $1 - \theta(t) \leq \sigma \leq 2$ ($\phi, 1/\theta$ positive non-decreasing, $\theta \leq 1$, $\phi \rightarrow \infty$, plus the growth condition (3.10.1) there), then there are no zeros in $\sigma \geq 1 - A_1 \theta(2t+1)/\phi(2t+1)$. The width $\theta/\phi \rightarrow 0$. Vinogradov–Korobov in explicit form [108]: width $1/(55.241(\log |t|)^{2/3}(\log \log |t|)^{1/3})$, and $1/(5.558691 \log |t|)$ for the classical region.
- (O3) *Density theorems count, they do not exclude.* [CITED] $N(\sigma, T) \leq T^{3(1-\sigma)/(2-\sigma)+o(1)}$ (Ingham), $T^{3(1-\sigma)/(3\sigma-1)+o(1)}$ (Huxley), hence $T^{12(1-\sigma)/5+o(1)}$, and $T^{30(1-\sigma)/13+o(1)}$ (Guth–Maynard) [61, (1.2)–(1.4)]. All of them are compatible with $N(\sigma, \infty) = \infty$.

- (O4) *Each window is unconditional; uniformity is the problem.* [CITED] For every A , Lemma 18 Thm. 4.3 gives $\sigma_*(A) < 1$; Thm. 28.3.6 below improves it. Neither can be uniform in A without (O1).

28.3.3. Translations: negativity mass of the damped symbol.

PROPOSITION 28.3.2. [THM] Let $\frac{1}{2} < \sigma < 1$, $T \geq 1$, and $N(\sigma, X) = \#\{\rho : \beta > \sigma, |\gamma| \leq X\}$ with multiplicity. Assume $N(\sigma, X) = o(X)$. Then

$$\int_{-T}^T (w_\sigma(t))_- dt \leq 2\pi N(\sigma, 2T) + 4 \int_{2T}^\infty \frac{N(\sigma, X)}{X^2} dX.$$

PROOF. $w_\sigma(t) = \sum_\rho \frac{2(\sigma-\beta)}{(\sigma-\beta)^2 + (t-\gamma)^2}$ [39, Ch. 12], [87], and the series converges absolutely. Only the terms with $\beta > \sigma$ are negative, so $(w_\sigma)_- \leq \sum_{\beta > \sigma} \frac{2\varepsilon}{\varepsilon^2 + (t-\gamma)^2}$. Each such Lorentzian has total mass 2π . If $|\gamma| > 2T$, its mass on $[-T, T]$ is $\leq \int_{|s| \geq |\gamma|/2} 2\varepsilon s^{-2} ds = 8\varepsilon/|\gamma| \leq 4/|\gamma|$. Partial summation gives $\sum_{|\gamma| > 2T, \beta > \sigma} |\gamma|^{-1} = \int_{2T}^\infty X^{-1} dN(\sigma, X) \leq \int_{2T}^\infty N(\sigma, X) X^{-2} dX$, the boundary term being ≤ 0 . $\square$

COROLLARY 28.3.3. [THM] (given the cited inputs)

- (a) With [61, (1.2)–(1.4)]: $\int_{-T}^T (w_\sigma)_- \leq T^{a(\sigma)+o(1)}$, with $a = \frac{3(1-\sigma)}{2-\sigma}$ (Ingham, < 1 for every $\sigma > \frac{1}{2}$), $\frac{12}{5}(1-\sigma)$ (Huxley, < 1 iff $\sigma > \frac{7}{12}$), or $\frac{30}{13}(1-\sigma)$ (GM, < 1 iff $\sigma > \frac{17}{30}$). Compare $N(T) \sim \frac{T}{\pi} \log T$ for all zeros [151]. The negativity mass of the damped symbol is a vanishing fraction, and the exponents are exactly the density exponents.
- (b) With [108]: every zero with $\beta > \sigma$ has $|\gamma| > T_0(\sigma) := \exp(1/(5.558691(1-\sigma)))$. So $N(\sigma, 2T) = 0$ for $2T \leq T_0(\sigma)$, and on $|t| \leq T_0/2$ only the tail term remains. No zero below height $T_0(\sigma)$ contributes negativity.
- (c) Family template: for any finite set $\mathfrak{F}$ of primitive characters, $\sum_{\chi \in \mathfrak{F}} \int_{-T}^T (w_{\sigma, \chi})_- \leq 2\pi \sum_\chi N(\sigma, 2T, \chi) + \text{tail}$. With the log-free bound [149, (3.4)] this is $\ll_\epsilon (qT)^{(12/5+\epsilon)(1-\sigma)}$, which is $O(1)$ at $\sigma = 1 - c/\log(qT)$ (edge). Large-sieve averages over $q \leq Q$ (Bombieri [13], Gallagher’s density near $\sigma = 1$ [53], Montgomery [107]) plug into the same template. Their precise statements are not quoted here; only the bibliographic records were checked.

REMARK 28.3.4 (what is not proved). Lemma 18 (NUM) found $\kappa_{\sigma, A} \approx A|S|/2\pi$, with $S = \{w_\sigma < 0\}$ (Landau–Pollak counting). A rigorous link “index $\leq C \cdot A \cdot (\text{negativity mass}) + \text{boundary}$ ” for the full form $\mathcal{Q}_\sigma = \mathcal{W}^{(\sigma)} + \sum \pi$ is not established here; it is Question 28.3.8. Prop. 28.3.2 is a statement about the symbol, not about $\kappa_{\sigma, A}$.

28.3.4. The concrete lemma: pole dominance.

LEMMA 28.3.5. [THM] For $0 \leq b < 1$, $x \mapsto \cosh(bx)/\cosh x$ is positive definite on $\mathbb{R}$, with

$$\int_{\mathbb{R}} \frac{\cosh bx}{\cosh x} e^{itx} dx = \frac{\pi \cos(\pi b/2) \cosh(\pi t/2)}{|\cos(\pi(b+it)/2)|^2} > 0.$$

PROOF. For $|\operatorname{Re} s| < 1$, $\int e^{sx}/\cosh x dx = \pi \sec(\pi s/2)$ (Euler’s beta integral: substitute $y = e^{2x}$ to get $\int_0^\infty y^{(s-1)/2}/(1+y) dy = \pi/\sin(\pi(s+1)/2)$). Writing $\cosh(bx)e^{itx} = \frac{1}{2}(e^{(b+it)x} + e^{(-b+it)x})$ and using $\sec(\pi(-b+it)/2) = \sec(\pi(b+it)/2)$ gives $\pi \operatorname{Re} \sec(\pi(b+it)/2) = \pi \operatorname{Re} \cos(z)/|\cos z|^2$ with $\operatorname{Re} \cos(x+iy) = \cos x \cosh y$. The transform is positive and integrable, so Bochner applies. $\square$

THEOREM 28.3.6 (pole dominance). [THM] Let $\sigma > \frac{1}{2}$ and $P_\sigma(u) := e^{(1-\sigma)|u|} + e^{-\sigma|u|}$, which is $k_{1/2}$, the pair kernel of a zero on the edge $\beta = 1$ (the pole kernel of Lemma 18, Prop. 2.3). Put $A_\sigma^{\text{pole}} := \sup\{A : P_\sigma \text{ p.d. on } (-A, A)\}$.

- (a) For $0 \leq \delta' < \delta \leq \frac{1}{2}$, $k_{\delta'} = k_{\delta} \cdot \frac{\cosh \delta' u}{\cosh \delta u}$, so $\{A : k_{\delta} \text{ p.d.}\} \subseteq \{A : k_{\delta'} \text{ p.d.}\}$. In particular $A^{\text{pair}}(\varepsilon, w)$ is non-increasing in β at fixed σ , and $A^{\text{pair}} \geq A_{\sigma}^{\text{pole}}$ for every pair in the closed strip.
- (b) If $A \leq A_{\sigma}^{\text{pole}}$ then $\kappa_{\sigma, A} = 0$, for any spectral datum satisfying (P0) $0 \leq \text{Re } \rho \leq 1$ (with the convergence of Lemma 18, Def. 1.1).
- (c) $A_{\sigma}^{\text{pole}} \geq u^*(\sigma) := \log \frac{\sigma}{1-\sigma}$. Equivalently, $\kappa_{\sigma, A} = 0$ whenever $\sigma \geq \frac{1}{1+e^{-A}}$.
- (d) $A_{\sigma}^{\text{det}} \geq \inf_{\beta > \sigma} A^{\text{pair}}(\beta - \sigma, \beta + \sigma - 1) \geq A_{\sigma}^{\text{pole}}$, and for a single pair $A^{\text{pair}}(\varepsilon, w) \geq \frac{\log(w/\varepsilon)}{\varepsilon + w}$.
- (e) (Below Θ , conditional.) $\kappa_{\sigma, A} = 0$ whenever $\Theta \leq \sigma + \frac{2\sigma-1}{e^A-1}$.

PROOF. (a) The ratio is $g(\delta u)$ with $g(x) = \cosh(bx)/\cosh x$, $b = \delta'/\delta < 1$, which is p.d. by Lemma 28.3.5. Kernel matrices of a product are Hadamard products, so Schur's product theorem applies: p.d. on $(-A, A)$ times p.d. on $\mathbb{R}$ is p.d. on $(-A, A)$. Then $\delta \mapsto k_{\delta}$ (fixed σ) runs from k_0 to $k_{1/2} = P_{\sigma}$. (b) $\mathcal{Q}_{\sigma}$ is an absolutely convergent sum of pair contributions (Lemma 28.1.1), each ≥ 0 on $\mathcal{D}_A$ by (a). (Zeros with $\beta < \frac{1}{2}$ are the mirrors and are already counted.) (c) On $[0, \infty)$, P_{σ} is convex ($P''_{\sigma} > 0$) with $P'_{\sigma}(0) = 1 - 2\sigma < 0$ and minimum at u^* , where $(1 - \sigma)e^{(1-\sigma)u} = \sigma e^{-\sigma u}$. For $A < u^*$ define $f = P_{\sigma}$ on $[0, A]$, continue it by its tangent line at A (slope < 0) down to 0, then set $f = 0$, and extend evenly. f is even, continuous, convex and non-increasing on $[0, \infty)$, and tends to 0, so it is p.d. by Pólya's criterion [117, Thm. 1]. Since $f = P_{\sigma}$ on $[-A, A]$, P_{σ} is p.d. on $(-A, A)$ for every $A < u^*$, hence on $(-u^*, u^*)$. Finally $A \leq \log \frac{\sigma}{1-\sigma} \Leftrightarrow \sigma \geq (1 + e^{-A})^{-1}$. (d) If every pair is p.d. on $(-A, A)$ then $\kappa_{\sigma, A} = 0$, which gives the first inequality; the second is (a). The same Pólya construction for $e^{\varepsilon u} + e^{-w u}$ (derivative $\varepsilon - w < 0$ at 0, minimum at $\log(w/\varepsilon)/(\varepsilon + w)$) gives the pair bound. (e) Zeros with $\beta \leq \sigma$ give p.d. kernels. For $\sigma < \beta \leq \Theta$, (d) needs $A \leq \log(w/\varepsilon)/(2\beta - 1)$. Since $2\beta - 1 \leq 1$, $\log(w/\varepsilon) \geq A$ suffices, i.e. $\varepsilon \leq w e^{-A}$, i.e. $\beta \leq \sigma + (2\sigma - 1)/(e^A - 1)$. $\square$

σ	0.55	0.6	0.7	0.8	0.9	0.95	0.99	0.995	0.999
$\log \frac{\sigma}{1-\sigma}$ [THM]	0.2007	0.4055	0.8473	1.3863	2.1972	2.9444	4.5951	5.2933	6.9068
A_{σ}^{pole} [NUM]	0.40271	0.82208	1.79610	3.21767	6.18236	10.27676	27.56295	40.57720	96.07586
$A_{\sigma}^{\text{pole}} \sqrt{1-\sigma}$	0.270	0.520	0.984	1.439	1.955	2.298	2.756	2.869	3.038
A									
	1	1.6	2	3	5	10	15		
$\sigma_{\text{pole}}(A)$ [NUM]	0.620137	0.681983	0.717547	0.787769	0.871320	0.947946	0.972081		
Pólya $1/(1 + e^{-A})$ [THM]	0.731059	0.832018	0.880797	0.952574	0.993307	0.999955	1.000000		
Lemma 18 $\sigma_*(A)$	0.99230	0.99477	0.99627	0.99794	0.99914	0.999774	0.999898		

[NUM] Method (`num/lemma28_pole.py`): float64 Nyström on Gauss–Legendre nodes; smallest eigenvalue of P_{σ} ; bisection. Main column $n = 240$; $n = 120$ agrees to ~ 4 digits up to $\sigma = 0.97$, but at 0.999 the two give 98.12 vs 96.08 (less converged). The inverse table uses $n = 160$.

Consistency: for all 59 Lemma 18 ghost detections, $A_{\text{detect}}/A_{\sigma}^{\text{pole}} \geq 2.58$ (minimum at $\sigma = 0.76$, $\varepsilon = 0.14$, $\gamma_0 = 27.72$: 6.617 vs 2.561), as (d) requires. `num/lemma28_pair.py`: the exact A^{pair} is 2–39 $\times$ the Pólya bound, and $A^{\text{pair}} \sqrt{\varepsilon w} \in [2.4, 3.7]$ over the grid,¹ consistent with Lemma 18's $\sqrt{b/\varepsilon}$ heuristic for an isolated pair.

REMARK 28.3.7 (honest scope). (1) Theorem 28.3.6 uses only (P0). It is therefore true for every “fake ζ ” with zeros anywhere in the closed strip: it is data-blind and height-blind, and it excludes no zero. It is the short-window face of (O1) and cannot give $\Theta < 1$. (2) Rigorously it beats Lemma 18's $\sigma_*(A)$ for $A \leq 5$ (table); for $A \geq 10$ the Pólya bound is worse than $\sigma_*(A)$.

¹[Vol. 2 editorial note] Erratum (Chapter 29, §29.3.3, E2): the Nyström grid overestimates A^{pair} by up to 17.4%; the exact values lie in $[0.40, 3.10]$, always $< \pi$.

The NUM A_σ^{pole} beats $\sigma_*(A)$ at every tabulated A , by a factor 50–270 in $1 - \sigma$ ($A = 1$ to 15). Both behave like A^{-2} , with constants ≈ 9 (from $A_\sigma^{\text{pole}}\sqrt{1 - \sigma} \rightarrow 3.04$ at 0.999) vs 0.023, so the factor tends to ≈ 400 .² (3) Interpretation [ANALOGY]: the worst ghost is a zero sitting on the pole line $\beta = 1$, i.e. the damping budget $c = \sigma - \frac{1}{2}$ must beat a mode at the passivity boundary (P0). Pole dominance is the statement that passivity alone buys windows of length $\asymp (1 - \sigma)^{-1/2}$ (NUM) $\geq \log \frac{\sigma}{1 - \sigma}$ (THM). (4) Priority not checked: Lemma 28.3.5 and the Schur step are elementary and may be folklore.

28.3.5. Roadmap to uniform damped positivity at $\sigma = 1 - \epsilon$.

- (R1) Certify A_σ^{pole} (Arb + Krein extension of exponential-type kernels) to turn the NUM column into THM. Conjecture: $A_\sigma^{\text{pole}}\sqrt{1 - \sigma} \rightarrow c_\infty \in [3, 4]$.³
- (R2) Prove the index–mass link (Question 28.3.8). Combined with Cor. 28.3.3(a), it would give: for $\sigma > \frac{17}{30}$, the index of $\mathcal{Q}_\sigma$ restricted to frequency band $[T, 2T]$ is $o(\text{band dimension})$. That is “almost-everywhere damped positivity”, the strongest statement density theorems can support.
- (R3) Use Cor. 28.3.3(b) (MTY) for a height-localized $\kappa_{\sigma,A} = 0$ below $T_0(\sigma)$. This needs control of the tail term inside the window form, not only the symbol.
- (R4) The gap to $\Theta < 1$ is exactly (O1). No step above removes it. A genuine route needs a mechanism excluding zeros at all heights simultaneously, e.g. a family/tensor-power positivity (Part A, Deligne), which is not available for GL_1 .

QUESTION 28.3.8. Is there a constant C such that, for F in $\mathcal{D}_A$ with $\hat{F}$ concentrated on $[T, T + H]$, the negative index of $\mathcal{Q}_\sigma$ is $\leq C(1 + \frac{A}{2\pi} \int_{T-1}^{T+H+1} (w_\sigma)_-)$? (This would make Prop. 28.3.2 an index statement.)

QUESTION 28.3.9. Is $A_\sigma^{\text{pole}} \sim c_\infty(1 - \sigma)^{-1/2}$, and what is c_∞ ? Is there a closed form (the kernel is a sum of two exponentials, so the extension problem is of finite type)?

QUESTION 28.3.10 (for J.W.). Part A shows that a “real line” with exact dispersion lives in the trivial-zero sector: channel ℓ ’s ladder at $-|\ell|/2 - n$. Is that the spatial structure you meant, i.e. cutoff frequencies of a waveguide? Or do you want a family whose *nontrivial* zeros obey a relation like $\delta_k^2 = \delta_0^2 - k^2$? We found none, and block-diagonality says the family Weil form cannot see one.

²[Vol. 2 editorial note] Erratum (Chapter 29, E3): $c_\infty = \pi$ exactly, and the factor is $\pi^2/0.0229 \approx 431$.

³[Vol. 2 editorial note] Resolved in Chapter 29, Theorem 29.3.1: $A_\sigma^{\text{pole}} = 2 \arcsin(2\sigma - 1)/\sqrt{\sigma(1 - \sigma)}$, so $c_\infty = \pi$.

Lemma 29: (A) index versus negativity mass of the damped symbol;

$$(B) \text{ the exact pole threshold } A_{\sigma}^{\text{pole}} = \frac{2 \arcsin(2\sigma - 1)}{\sqrt{\sigma(1 - \sigma)}}$$

Standalone note `lemma29.tex`, October 21, 2026.

Summary. *Tags.* [THM] proved here; [CITED] quoted from a source checked at the source (`lit/lemma29/verified_refs.md`); [DEF] definition; [ANALOGY] analogy only; [NUM] floating-point numerics; [CERT] interval arithmetic (Arb via python-flint, 256 bits), a rigorous enclosure; [RESTATEMENT] restatement, no new content.

Verdict 1 (Question 3.8 of Lemma 28): the inequality as posed is false; a corrected form is proved. No bound $\kappa_{\sigma,A} \leq C(1 + \frac{A}{2\pi} \int (w_{\sigma})_-)$ holds for spectral data satisfying only (P0) and convergence (Prop. 29.2.4). We build data with $w_{\sigma} \geq 0$ *everywhere* and $\kappa_{\sigma,A} \geq K$ for any K at one fixed A . The obstruction is the point forms of Lemma 18, Thm. 2.1. Any valid link must therefore count zeros with $\text{Re } \rho > \sigma$, not only the negativity mass. What is proved [THM]: (i) a trace (Chebyshev) bound $\kappa \leq \frac{A}{2\pi\eta} \int (\eta - q)_+$ for every form with symbol q on a window of length A (Thm. 29.2.2). (ii) If $N_+(\sigma) < \infty$, then $\kappa_{\sigma,A} \leq N_+(\sigma) + \frac{A}{2\pi} (|\{w_{\sigma} < \eta\}| + \eta^{-1} \int (w_{\sigma})_-)$ (Cor. 29.2.6). (iii) A band-localized index bound for ζ with no hypothesis on $N_+(\sigma)$ (Thm. 29.2.9). Its right side is $|I|$ -independent apart from a fixed low-height term, and is linear in the number of zeros with $\text{Re } \rho \geq \sigma$ near the band I , plus a convergent tail. (iv) The “damped positivity at almost every height” corollary (Cor. 29.2.12). On frequency band $[T, 2T]$, window length A : at most $T^{e(\sigma)+o(1)}$ negative squares, against $\gg_A T$ available dimensions, with $e(\sigma) = \min\{\frac{3(1-\sigma)}{2-\sigma}, \frac{30}{13}(1-\sigma)\}$. *Honest note:* the fraction tends to 0 for every $\sigma > \frac{1}{2}$ already with Ingham’s exponent. Guth–Maynard’s 30/13 improves the rate for $\sigma > 0.7$, and 17/30 is only the point where the GM exponent alone drops below 1; it is not a threshold for this corollary.

Verdict 2 (Question 3.9): solved exactly, beyond the request. [THM] $A_{\sigma}^{\text{pole}} = \frac{2 \arcsin(2\sigma-1)}{\sqrt{\sigma(1-\sigma)}} = \frac{\pi-4 \arcsin \sqrt{1-\sigma}}{\sqrt{\sigma(1-\sigma)}}$, and the supremum is attained (Thm. 29.3.1). Hence $c_{\infty} = \pi$, and more precisely $A_{\sigma}^{\text{pole}} = \pi(1-\sigma)^{-1/2} - 4 + \frac{\pi}{2}(1-\sigma)^{1/2} + O(1-\sigma)$. By scaling, *every* zero pair has exact threshold $A^{\text{pair}}(\varepsilon, w) = \frac{2}{\sqrt{\varepsilon w}} \arcsin \frac{w-\varepsilon}{w+\varepsilon} < \frac{\pi}{\sqrt{\varepsilon w}}$. This replaces the Pólya bounds of Lemma 28, Thm. 3.6(c),(d). The conditional “ $\kappa_{\sigma,A} = 0$ if $\Theta \leq \sigma + \varepsilon$ ” improves from $\varepsilon \asymp e^{-A}$ to $\varepsilon \asymp A^{-2}$ (Cor. 29.3.2(e)). Independently, Arb certifies brackets of relative width $\leq 10^{-9}$ at all tabulated σ and A , and each contains the closed form [CERT]. The Lemma 28 NUM column was accurate to $\leq 7 \cdot 10^{-3}$ relative; its pair-grid claims need errata (§29.3.3).

Verdict 3 (priority). Positivity of the *undamped* ($\sigma = \frac{1}{2}$) Weil form on small support is due to Yoshida (1992). It was re-proved by Bombieri (2000) and (in Mellin form) by Burnol (2000), and unified by Suzuki (2026); see §29.4. These results use ζ ’s explicit formula. In the sources checked, we found no statement of the σ -damped, data-blind pole-dominance theorem, of the $\cosh(bx)/\cosh x$ Schur step, or of the closed-form threshold. They may be folklore (the threshold is a finite-type extension

problem), and the search was not exhaustive. Suzuki's continuity of the lowest eigenvalue is the same device as our emergence argument (proof of Thm. 29.3.1).

29.1. Setting

Notation is that of Lemma 18, Def. 1.1 and Lemma 28, §1 [RESTATEMENT].

$\mathcal{Q}_\sigma(F) = \sum_\rho m_\rho \widehat{G}_\sigma(\gamma_\rho)$ with $G_\sigma = e^{-c|u|} H_F$, $c = \sigma - \frac{1}{2}$, $\widehat{F}(z) = \int F(u) e^{izu} du$, and $\|F\|^2 = \frac{1}{2\pi} \int |\widehat{F}|^2$. $\mathcal{D}_A$ is the space of $F \in C_c^\infty$ supported in an interval of length A , and $\kappa_{\sigma,A}$ is the negative index there. Complex F are allowed, since complexification preserves inertia (Lemma 18, remark after Def. 1.1). Further, $w_\sigma(t) = \sum_\rho \frac{2(\sigma-\beta)}{(\sigma-\beta)^2 + (t-\gamma)^2}$ and $N_+(\sigma) = \#\{\text{Re } \rho > \sigma\}$. For $\rho = \beta + i\gamma$ with $\beta \geq \frac{1}{2}$, the pair $\{\rho, 1 - \bar{\rho}\}$ contributes $\iint F(x) \overline{F(y)} e^{i\gamma(x-y)} k_\rho(x-y)$ with $k_\rho(u) = e^{\varepsilon|u|} + e^{-w|u|}$, where $\varepsilon = \beta - \sigma$ and $w = \beta + \sigma - 1$ (Lemma 28, Lemma 1.1). The pole kernel is $P_\sigma = e^{(1-\sigma)|u|} + e^{-\sigma|u|}$. By *model datum* we mean any multiset of ρ with $0 \leq \text{Re } \rho \leq 1$ that is symmetric under $\rho \mapsto \bar{\rho}$ and $\rho \mapsto 1 - \bar{\rho}$ and has $\sum m_\rho (1 + \gamma^2)^{-1} < \infty$. This is exactly the generality of Lemma 28, Thm. 3.6. For ζ we write $\Omega(t) = \log(e + |t|)$.

29.2. Part A: index versus negativity mass (Question 3.8)

29.2.1. A trace bound.

LEMMA 29.2.1. [THM] *Every finite-dimensional $V \subset \mathcal{D}_A$ lies in $C_c^\infty(J)$ for a single interval J of length A .*

PROOF. Let $F_1, \dots, F_k$ be a basis and $\{x_m\}$ a countable dense subset of $\bigcup_j \{F_j \neq 0\}$. For c outside a countable union of proper subspaces of $\mathbb{C}^k$, $F = \sum c_j F_j$ has $F(x_m) \neq 0$ for all m . Then $\text{supp } F \supseteq \bigcup_j \text{supp } F_j$, and $\text{supp } F$ lies in an interval of length A . $\square$

THEOREM 29.2.2 (trace bound). [THM] *Let J be an interval of length A and q real and measurable on $\mathbb{R}$. Suppose $\mathcal{Q}(F) = \frac{1}{2\pi} \int |\widehat{F}|^2 q$ is absolutely convergent for $F \in C_c^\infty(J)$, and $g := (\eta - q)_+ \in L^1$ for some $\eta > 0$. Let $V \subset C_c^\infty(J)$ with $\mathcal{Q} < 0$ on $V \setminus 0$, and suppose that for some $\theta' \in [0, \eta)$ and measurable $E \subset \mathbb{R}$ we have $\frac{1}{2\pi} \int_E |\widehat{F}|^2 g \leq \theta' \|F\|^2$ on V . Then*

$$\dim V \leq \frac{A}{2\pi(\eta - \theta')} \int_{\mathbb{R} \setminus E} (\eta - q)_+.$$

In particular ($E = \emptyset$), the negative index of $\mathcal{Q}$ on $C_c^\infty(J)$ is at most $\frac{A}{2\pi\eta} \int (\eta - q)_+$.

PROOF. For $F \in V \setminus 0$, $q \geq \eta - g$ gives $\frac{1}{2\pi} \int |\widehat{F}|^2 g > \eta \|F\|^2$, hence $b(F) := \frac{1}{2\pi} \int_{\mathbb{R} \setminus E} |\widehat{F}|^2 g > (\eta - \theta') \|F\|^2$. The operator $B = 1_J \mathcal{F}^{-1}(g 1_{\mathbb{R} \setminus E}) \mathcal{F} 1_J \geq 0$ on $L^2(J)$ is trace class with $\text{tr } B = \frac{|J|}{2\pi} \int_{\mathbb{R} \setminus E} g$ (its kernel is $\frac{1}{2\pi} \widehat{g 1_{E^c}}(y - x)$ on $J \times J$, continuous, with diagonal $\frac{1}{2\pi} \int g 1_{E^c}$). By min-max, $\lambda_{\dim V}(B) \geq \min_{F \in V, \|F\|=1} b(F) > \eta - \theta'$. Hence $\dim V \leq \#\{j : \lambda_j(B) > \eta - \theta'\} \leq \text{tr } B / (\eta - \theta')$. $\square$

REMARK 29.2.3. Theorem 29.2.2 is the elementary trace half of eigenvalue counting for time-frequency limiting. It is the rigorous form of Lemma 18's NUM observation $\kappa \approx A|S|/2\pi$, as an upper bound only. No Landau–Pollak asymptotics are needed or used.

29.2.2. The literal inequality is false.

PROPOSITION 29.2.4 (counterexample). *[THM] Let $\frac{1}{2} < \sigma < 1$. There is $A_0 = A_0(\sigma)$ such that for every $K \in \mathbb{N}$ there is a model datum with: $w_\sigma(t) \geq 0$ for all $t \in \mathbb{R}$ (so $\int (w_\sigma)_- = 0$ on every band); $N_+(\sigma) = 2K$; and $\kappa_{\sigma, A_0} \geq K$. Consequently no inequality $\kappa_{\sigma, A} \leq C(1 + \frac{A}{2\pi} \int_I (w_\sigma)_-)$, with C independent of the datum, holds on the class of model data, whether or not F is localized in frequency.*

PROOF. *Background.* Put $b = 2\pi/h$ and take the on-line zeros $\frac{1}{2} + inh$, $n \in \mathbb{Z}$. For $F \in \mathcal{D}_A$ with $A < b$, Poisson summation applies: G_σ is Lipschitz with compact support and $\widehat{G}_\sigma = O(\gamma^{-2})$. Since $\text{supp } G_\sigma \subset [-A, A] \subset (-b, b)$, it gives $\sum_n \widehat{G}_\sigma(nh) = \frac{2\pi}{h} \sum_k G_\sigma(2\pi k/h) = b G_\sigma(0) = b \|F\|^2$. Its symbol is $\sum_n \frac{2c}{c^2 + (t-nh)^2} = \frac{b \sinh(bc)}{\cosh(bc) - \cos(bt)} \geq b \tanh(bc/2)$. *Ghosts.* Fix $\beta = \frac{1+\sigma}{2}$, so that $\varepsilon = \frac{1-\sigma}{2}$ and $w = \frac{3\sigma-1}{2}$. Add the quadruples $\beta \pm i\gamma_j$, $1 - \beta \pm i\gamma_j$ for $j = 1, \dots, K$. On $[-L, L]$ with $L = A/2$, take φ odd and real. Writing $e^{\varepsilon|u|} = 2 \cosh \varepsilon u - e^{-\varepsilon|u|}$ and using $\cosh \varepsilon(x-y) = \cosh \varepsilon x \cosh \varepsilon y - \sinh \varepsilon x \sinh \varepsilon y$ and $\widehat{e^{-w|u|}} \leq 2/w$,

$$\iint \varphi(x)\varphi(y)k(x-y) \leq -2\left(\int \varphi \sinh \varepsilon x\right)^2 + \frac{2}{w}\|\varphi\|^2.$$

For $\varphi = \sinh(\varepsilon x)1_{[-L, L]}$ this is $-2\|\varphi\|^4 + \frac{2}{w}\|\varphi\|^2$, with $\|\varphi\|^2 = \frac{\sinh(\varepsilon A)}{2\varepsilon} - \frac{A}{2}$. Choose b so large that $b \tanh(bc/2) \geq \frac{2}{\varepsilon} + 1$, and $A_0 := b/2$ so large that $2\|\varphi\|^2 > b + \frac{2}{w}$. Both are possible because $\|\varphi\|^2$ grows exponentially in A_0 . Then $m := b\|\varphi\|^2 - 2\|\varphi\|^4 + \frac{2}{w}\|\varphi\|^2 < 0$. Replace φ by an odd real $C_c^\infty(-L, L)$ approximant: all forms involved are $L^2(J)$ -continuous with bounded kernels, so $m < 0$ persists. *Gram matrix.* Let $F_j = e^{-i\gamma_j x} \varphi(x)$. Then $Q(F_i, F_j)$ is $b\langle F_i, F_j \rangle$ plus $2K$ pair terms $\int H_{F_i, F_j}(u) e^{\pm i\gamma_k u} k(u) du$, where $H_{F_i, F_j}(u) = e^{-i\gamma_i u} \int \varphi(y+u) \varphi(y) e^{-i(\gamma_i - \gamma_j)y} dy$. Let all γ_j and all gaps $|\gamma_i - \gamma_j|$ tend to infinity. By Riemann–Lebesgue and dominated convergence on $[-A, A]$, every term tends to 0 except, for $i = j$, the term $b\|\varphi\|^2$ and the pair at $+\gamma_j$, which equals $\iint \varphi \varphi k$. Hence the Gram matrix tends to $m \cdot I_K$ (or below it), and $\kappa_{\sigma, A_0} \geq K$. *Symbol.* Each ghost quadruple contributes $\geq -\frac{2\varepsilon}{\varepsilon^2 + (t \mp \gamma_j)^2}$ at $\pm \gamma_j$ (the $1 - \beta$ members give positive Lorentzians). With separations ≥ 2 the total ghost negativity at any t is $\geq -\frac{2}{\varepsilon} - 1$. So $w_\sigma \geq b \tanh(bc/2) - \frac{2}{\varepsilon} - 1 \geq 0$. $\square$

REMARK 29.2.5. Lemma 18, Cor. 2.2(a) says that each point form adds at most one negative square. Prop. 29.2.4 shows that each can actually *add* one while the symbol stays positive. It is the damped analogue of Bombieri's count at $\sigma = \frac{1}{2}$: with finitely many exceptional zeros, large truncations have exactly half as many negative eigenvalues as there are exceptional zeros [14] [CITED]. For ζ itself the counterexample says nothing. It says only that a proof of Question 3.8 for ζ that does not count off-line zeros would have to use more than (P0) together with the symbol.

29.2.3. Corrected forms.

COROLLARY 29.2.6 (finite $N_+(\sigma)$). *[THM] If $N_+(\sigma) < \infty$ and $\sigma \neq \text{Re } \rho$ for all ρ , then for every $A, \eta > 0$,*

$$\kappa_{\sigma, A} \leq N_+(\sigma) + \frac{A}{2\pi\eta} \int_{\mathbb{R}} (\eta - w_\sigma)_+ \leq N_+(\sigma) + \frac{A}{2\pi} \left(|\{w_\sigma < \eta\}| + \frac{1}{\eta} \int (w_\sigma)_- \right).$$

PROOF. Lemma 18, Cor. 2.2(a) gives $\kappa_{\sigma, A} \leq \kappa_{\sigma, A}^W + N_+(\sigma)$. Lemma 29.2.1 and Thm. 29.2.2 with $q = w_\sigma$ bound $\kappa_{\sigma, A}^W$. Finally $(\eta - w)_+ \leq \eta 1_{\{w < \eta\}} + w_-$. (If $\int (\eta - w_\sigma)_+ = \infty$ the bound is trivial.) $\square$

For ζ we cannot assume $N_+(\sigma) < \infty$, so we localize. Fix an even $\chi \in C_c^\infty(-A-1, A+1)$ with $0 \leq \chi \leq 1$ and $\chi = 1$ on $[-A, A]$.

DEFINITION 29.2.7. [DEF] A pair with $\beta \geq \sigma$ is *harmless* if $A \leq A^{\text{pair}}(\varepsilon, w)$, i.e. k_ρ is p.d. on $[-A, A]$. This includes $\beta = \sigma$, where $k_\rho = 1 + e^{-w|u|}$. Otherwise it is *visible*; by Cor. 29.3.2(d), visible means $\beta > \beta_A(\sigma)$. For visible ρ put $\Phi_\rho(\tau) = \int \chi(u)k_\rho(u)e^{i\tau u}du$ (real and even). Set $C_A := \sup_{\rho \text{ visible}} \sup_{\tau} (1 + \tau^2)|\Phi_\rho(\tau)|$ and $M_A := \sup_{\rho} \int (\Phi_\rho)_- \leq \pi C_A$. For an interval I and $H > 0$ let $N_+(I; H) = \#\{\rho : \beta \geq \sigma, \text{dist}(\gamma, I) \leq H\}$ and $S_+(I; H) = \sum_{\rho \text{ visible}, \text{dist}(\gamma, I) > H} \text{dist}(\gamma, I)^{-1}$. A complex subspace $V \subset \mathcal{D}_A$ is θ -concentrated on I if $\int_{\mathbb{R} \setminus I} |\widehat{F}|^2 \Omega \leq \theta \int |\widehat{F}|^2$ for all $F \in V$. Let $\kappa_{\sigma, A}(I; \theta)$ be the supremum of $\dim V$ over θ -concentrated V with $\mathcal{Q}_\sigma < 0$ on $V \setminus 0$.

LEMMA 29.2.8. [THM] (a) $C_A \leq 2(\|\chi k\|_{L^1} + \|(\chi k)''\|_{TV}) \leq C(\chi)(A+2)^2 e^{(A+1)/2}$, uniformly over pairs with $\frac{1}{2} \leq \sigma < \beta \leq 1$. (b) [150, Thm. 9.2] [CITED] gives C_ζ with $\sum_{\rho} (1 + (t - \gamma)^2)^{-1} \leq C_\zeta \Omega(t)$. (c) [150, Thm. 9.4] [CITED] gives H_0 such that for every $\sigma > \frac{1}{2}$ and $\eta > 0$ there is $t_1 = t_1(\sigma, \eta)$ with $n(t, H_0) := \#\{\rho : |\gamma - t| \leq H_0\} \geq \eta(1 + H_0^2)/c$ for $|t| \geq t_1$.

PROOF. (a) Use $|\Phi| \leq \|\chi k\|_1$, $|\Phi(\tau)| \leq \tau^{-2} \|(\chi k)''\|_{TV}$ and $\min(x, y\tau^{-2}) \leq 2(x+y)/(1+\tau^2)$. Here $\varepsilon \leq \frac{1}{2}$ and $w \leq 1$, so $k, |k'|, |k''| \leq e^{(A+1)/2} + 1$ on the support, and the kink at 0 adds $2|\varepsilon - w| \leq 2$ to the variation. (b) $N(t+n+1) - N(t+n) \leq C \log(|t| + |n| + e)$; sum against $(1+n^2)^{-1}$. (c) If $|N(T) - M(T)| \leq C_0 \log T$ with M the main term, then $n(t, H) \geq \frac{H}{\pi} \log \frac{t-H}{2\pi} - 2C_0 \log(2t)$ for $H \leq t/2$. Take $H_0 = 4\pi C_0 + 1$. $\square$

THEOREM 29.2.9 (band-localized index bound for ζ). [THM] Let $\frac{1}{2} < \sigma < 1$, $A > 0$, $\eta > 0$, H_0, t_1 as in Lemma 29.2.8, and $\theta' = (\eta + C_\zeta C_A)\theta < \eta$. For every interval I ,

$$\kappa_{\sigma, A}(I; \theta) \leq \frac{A}{2\pi(\eta - \theta')} \left[\eta |I \cap [-t_1, t_1]| + (2H_0\eta + M_A) N_+(I; H_0) + C_A S_+(I; H_0) \right].$$

PROOF. Hybrid symbol. Harmless pairs give forms ≥ 0 on $\mathcal{D}_A$ (Lemma 28, Lemma 1.1), so dropping them can only increase the negative index. Every other pair either has $\beta < \sigma$, giving two Lorentzians $\frac{2(\sigma-\beta)}{(\sigma-\beta)^2 + \tau^2}$ and $\frac{2w}{w^2 + \tau^2}$ with $w \geq c$ (or one, of width c , if $\beta = \frac{1}{2}$), or is visible. A visible pair's kernel may be replaced by χk_ρ , since H_F is supported in $[-A, A]$. Hence $Q'(F) = \frac{1}{2\pi} \int |\widehat{F}|^2 q$ with $q = P_{\text{good}} + \sum_{\text{vis}} \Phi_\rho(\cdot - \gamma)$, absolutely convergent by Lemma 29.2.8(a),(b) and Tonelli, and $\kappa_{\sigma, A}(I; \theta) \leq \kappa(Q'; I; \theta)$. Away from I , $(\eta - q)_+ \leq \eta + C_A C_\zeta \Omega$, and $\Omega \geq 1$, so $\frac{1}{2\pi} \int_{\mathbb{R} \setminus I} |\widehat{F}|^2 (\eta - q)_+ \leq \theta' \|F\|^2$ on V . Apply Thm. 29.2.2 with $E = \mathbb{R} \setminus I$, using Lemma 29.2.1. On I , $(\eta - q)_+ \leq (\eta - P_{\text{good}})_+ + \sum_{\text{vis}} (\Phi_\rho(\cdot - \gamma))_-$. Each good pair has a member of width in $[c, 1]$, whose Lorentzian is $\geq \frac{2c}{1+H_0^2}$ on $|t - \gamma| \leq H_0$. So $P_{\text{good}}(t) \geq \frac{c}{1+H_0^2} (n(t, H_0) - 2n_+(t, H_0))$, where n_+ counts $\beta \geq \sigma$. By Lemma 29.2.8(c), $P_{\text{good}} \geq \eta$ wherever $|t| \geq t_1$ and $n_+(t, H_0) = 0$. The exceptional set in I has measure $\leq 2H_0 N_+(I; H_0)$. Visible zeros near I contribute $\leq M_A$ each. Far ones contribute $\leq C_A \int_{\text{dist} \geq d} (1 + s^2)^{-1} ds \leq C_A/d$. $\square$

REMARK 29.2.10. Windowing (replacing k_ρ by χk_ρ , which changes the symbol but not the form on $\mathcal{D}_A$) is the device of Burnol's small-support proof [17] [CITED]. There a term $A_\varepsilon \cos(\varepsilon\tau)$, invisible for small support, is added to the symbol. Lemma 18's reflection identity (Prop. 5.2 there) is the same idea.

LEMMA 29.2.11 (available dimension). [THM] Fix $A > 0$ and $\theta > 0$. There are $c_A > 0$ and T_2 such that for $T \geq T_2$ some complex $V \subset \mathcal{D}_A$ with $\dim V \geq c_A T$ is θ -concentrated on $I_T = [T, 2T]$.

PROOF. Let $\varphi_A(x) = \psi(x/A)$ with $\psi \in C_c^\infty(0, 1)$ fixed, and $F_j = e^{-it_j x} \varphi_A$ with $t_j \in [T + \sqrt{T}, 2T - \sqrt{T}]$ spaced Δ_0/A . The Gram entries are $A|\widehat{\psi}|^2(A(t_i - t_j))$. Rapid decay gives $\text{Gram} \geq \frac{1}{2}\|\varphi_A\|^2 I$ once $\Delta_0 = \Delta_0(\psi)$ is large. By Cauchy–Schwarz, $\int_{\mathbb{R} \setminus I_T} |\sum c_j \widehat{F}_j|^2 \Omega \leq \sum |c_j|^2 \sum_j \int_{|\tau| \geq \sqrt{T}} |\widehat{\varphi_A}(\tau)|^2 \Omega(t_j + \tau) d\tau \ll_{A,N} \sum |c_j|^2 T^{1-N} \log T$, which is $\leq \theta \pi \|\varphi_A\|^2 \sum |c_j|^2 \leq \theta \int |\widehat{F}|^2$ for large T . Take $c_A = A/(2\Delta_0)$. $\square$

COROLLARY 29.2.12 (damped positivity at almost every height). *[THM] (given the cited density bounds) Let $\frac{1}{2} < \sigma < 1$ and $A > 0$. Put $\theta_0 = \frac{1}{2(1+C_\zeta C_A)}$ and $e(\sigma) = \min\{\frac{3(1-\sigma)}{2-\sigma}, \frac{30}{13}(1-\sigma)\}$ (Ingham, Guth–Maynard [61], (1.2),(1.4)) [CITED]. For $0 < \theta \leq \theta_0$ and $T \rightarrow \infty$:*

- (a) $\kappa_{\sigma,A}([T, 2T]; \theta) \ll_{\sigma,A} N(\sigma, 4T) + \int_{4T}^\infty N(\sigma, X) X^{-2} dX \leq T^{e(\sigma)+o(1)}$. Equivalently, on every θ -concentrated $V \subset \mathcal{D}_A$ the form $\mathcal{Q}_\sigma$ is ≥ 0 on a subspace of codimension $\leq T^{e(\sigma)+o(1)}$.
- (b) Such V exist with $\dim V \geq c_A T$ (Lemma 29.2.11). Hence the proportion of negative directions among frequency- $[T, 2T]$ wave packets on windows of length A is $\leq T^{e(\sigma)-1+o(1)} \rightarrow 0$.

PROOF. Thm. 29.2.9 with $\eta = 1$, $I = I_T$ and $T \geq t_1$. Zeros with $|\gamma| \leq 4T$ and $\text{dist} > H_0$ contribute $\leq 1/H_0$ each to S_+ . Zeros with $|\gamma| > 4T$ have $\text{dist} \geq |\gamma|/2$, and partial summation as in Lemma 28, Prop. 3.2 handles them. $N_+(I_T; H_0) \leq N(\sigma, 4T)$. Ingham’s exponent is < 1 for all $\sigma > \frac{1}{2}$, and GM’s is smaller for $\sigma > 0.7$. $\square$

REMARK 29.2.13 (what this is not). (1) Cor. 29.2.12 is a dimension count at each height band. It gives neither $\kappa_{\sigma,A} < \infty$ nor anything toward $\Theta < 1$. The gap is still (O1) of Lemma 28. (2) Below the MTY height $T_0(\sigma)$ (Lemma 28, Cor. 3.3(b)), $N_+(I; H_0) = 0$ for bands with $I + [-H_0, H_0] \subset [-T_0, T_0]$. The bound then reduces to $\eta|I \cap [-t_1, t_1]| + C_A S_+$. If explicit C_ζ , t_1 , C_A and an explicit density bound make it $< \eta - \theta'$, it gives $\kappa_{\sigma,A}(I; \theta) = 0$. That is the height-localized “ $\kappa = 0$ below T_0 ” of Lemma 28 R3, but it needs explicit constants that are not supplied here.

29.3. Part B: the exact pole threshold (Question 3.9)

THEOREM 29.3.1. *[THM] Let $\frac{1}{2} < \sigma < 1$, $a = 1 - \sigma$, $\omega = \sqrt{a\sigma}$ and $\alpha = \arcsin(2\sigma - 1) \in (0, \frac{\pi}{2})$. Then P_σ is p.d. on $[-A, A]$ iff $A \leq A^* := 2\alpha/\omega$. So*

$$A_\sigma^{\text{pole}} = \frac{2 \arcsin(2\sigma - 1)}{\sqrt{\sigma(1-\sigma)}} = \frac{\pi - 4 \arcsin \sqrt{1-\sigma}}{\sqrt{\sigma(1-\sigma)}},$$

and the supremum in Lemma 28’s definition is attained. For $A > A^*$ an explicit negative direction is $\mu^* + t\delta_x$ for $x \in (A^*, A]$ and small $t > 0$, with μ^* as in Lemma 29.3.4.

COROLLARY 29.3.2. *[THM]*

- (a) $A_\sigma^{\text{pole}} \sqrt{1-\sigma} \uparrow \pi$, and $A_\sigma^{\text{pole}} = \frac{\pi}{\sqrt{a}} - 4 + \frac{\pi}{2}\sqrt{a} - \frac{8}{3}a + O(a^{3/2})$. So $c_\infty = \pi$ (Lemma 28 conjectured [3, 4]).
- (b) $\sigma_{\text{pole}}(A) = \frac{1+\sin \alpha_A}{2}$, where $\alpha_A \in (0, \frac{\pi}{2})$ is the root of $A \cos \alpha = 4\alpha$; and $1 - \sigma_{\text{pole}}(A) = \frac{\pi^2}{(A+4)^2}(1 + O(A^{-1}))$.
- (c) (exact pair threshold) For $0 < \varepsilon < w$: $A^{\text{pair}}(\varepsilon, w) = \frac{2}{\sqrt{\varepsilon w}} \arcsin \frac{w-\varepsilon}{w+\varepsilon}$. For a zero, $\frac{w-\varepsilon}{w+\varepsilon} = \frac{2\sigma-1}{2\beta-1}$.
- (d) For fixed σ , $\beta \mapsto A^{\text{pair}}$ decreases from $+\infty$ ($\beta \downarrow \sigma$) to A_σ^{pole} ($\beta = 1$). Let $\beta_A(\sigma)$ be the root of $A^{\text{pair}} = A$ when $A > A_\sigma^{\text{pole}}$. Then $\beta_A(\sigma) - \sigma = \frac{\pi^2}{(2\sigma-1)A^2}(1 + o(1))$ as $A \rightarrow \infty$.

(e) (conditional, improving Lemma 28, Thm. 3.6(e)) $\kappa_{\sigma,A} = 0$ whenever $\Theta \leq \beta_A(\sigma)$ (or $A \leq A_\sigma^{\text{pole}}$). The admissible excess $\Theta - \sigma$ is $\asymp A^{-2}$, instead of $\frac{2\sigma-1}{e^A-1}$.

PROOF. (a) $\pi/2 - \alpha = \arccos(1 - 2a) = 2 \arcsin \sqrt{a}$, then expand. (b) $\sin \alpha = 2\sigma - 1$ gives $\cos \alpha = 2\omega$. (c) Rescale $u = v/(\varepsilon + w)$: $k = e^{\varepsilon'|v|} + e^{-w'|v|}$ with $\varepsilon' + w' = 1$, so $A^{\text{pair}}(\varepsilon, w) = A_{w'}^{\text{pole}}/(\varepsilon + w)$ with $w' = w/(\varepsilon + w)$. (d) Monotonicity is Lemma 28, Thm. 3.6(a); the asymptotic follows from (c) with $\varepsilon \rightarrow 0$. (e) Lemma 28, Thm. 3.6(b),(d) with (d). $\square$

29.3.1. Proof of Theorem 29.3.1. Let T_A be the operator $(Tf)(x) = \int_0^A P_\sigma(x-y)f(y)dy$ on $L^2[0, A]$. It is compact and self-adjoint, and p.d. on $[-A, A]$ is equivalent to $T_A \geq 0$ (continuous kernel).

LEMMA 29.3.3 (eigen-reduction). [THM] If $\varphi'' = s\varphi$ on $[0, A]$ with $s \notin \{a^2, \sigma^2\}$, then $T\varphi = \lambda(s)\varphi + E_1 e^{ax} + E_2 e^{-ax} + E_3 e^{-\sigma x} + E_4 e^{\sigma x}$, where $\lambda(s) = \frac{2a}{s-a^2} - \frac{2\sigma}{s-\sigma^2} = \frac{2(a-\sigma)(s+a\sigma)}{(s-a^2)(s-\sigma^2)}$ and

$$E_1 = \frac{a\varphi(0)+\varphi'(0)}{a^2-s}, \quad E_2 = e^{aA} \frac{a\varphi(A)-\varphi'(A)}{a^2-s}, \quad E_3 = -\frac{\sigma\varphi(0)-\varphi'(0)}{\sigma^2-s}, \quad E_4 = -e^{-\sigma A} \frac{\sigma\varphi(A)+\varphi'(A)}{\sigma^2-s}.$$

Consequently $\lambda < 0$ is an eigenvalue iff $\lambda = \lambda(s_1)$ with $s_1 > \sigma^2$, and the 4×4 matrix $B(\eta; A)$ of (E_k) on a basis of $\ker(D^2 - s_1)(D^2 - s_2)$ is singular. Here $\eta = s_1^{-1/2} \in (0, 1/\sigma)$ and $s_2 = a\sigma \frac{-1+(a^2+\sigma^2+a\sigma)\eta^2}{1+a\sigma\eta^2} \in (-a\sigma, a^2)$. With the basis $\cosh(\sqrt{s_2}x)$, $\sinh(\sqrt{s_2}x)/\sqrt{s_2}$, $\eta^{-1}e^{-x/\eta}$, $\eta^{-1}e^{-(A-x)/\eta}$, and the rows E_2, E_4 divided by $e^{aA}, e^{-\sigma A}$, the entries of B are continuous on $[0, 1/\sigma) \times (0, \infty)$. At $\eta = 0$, $B(0; A) = M(A)$ with

$$M(A) = \begin{pmatrix} 1 & 1/a & 1 & 0 \\ \cos \omega A + \frac{\sigma}{\omega} \sin \omega A & \frac{\sin \omega A}{\omega} - \frac{\cos \omega A}{a} & 0 & 1 \\ -1 & 1/\sigma & 1 & 0 \\ -\cos \omega A + \frac{a}{\omega} \sin \omega A & -\frac{\sin \omega A}{\omega} - \frac{\cos \omega A}{\sigma} & 0 & 1 \end{pmatrix}, \quad \det M(A) = \frac{\sin(\omega A - 2\alpha)}{\omega^3}.$$

PROOF. The identity: two integrations by parts on $[0, x]$ and $[x, A]$. It was checked symbolically for $\varphi = e^{\pm ry}$ with general a, σ, r, A (num/lemma29_symbolic3.py), and extends to all s by analyticity. If $T\varphi = \lambda\varphi$ with $\lambda \neq 0$, then $\varphi \in C^\infty$, and applying $(D^2 - a^2)(D^2 - \sigma^2)$ gives $\lambda(D^2 - a^2)(D^2 - \sigma^2)\varphi = 2(a - \sigma)(D^2 + a\sigma)\varphi$. So φ lies in the kernel for the two roots $s_{1,2}$ of $\lambda(s) = \lambda$, and then $E_k(\varphi) = 0$ because $e^{\pm ax}, e^{\pm \sigma x}$ are independent. For $\lambda < 0$: $\lambda(s) < 0$ iff $s \in (-a\sigma, a^2) \cup (\sigma^2, \infty)$, λ increases from $-\infty$ to 0^- on (σ^2, ∞) , and Vieta gives s_2 . Continuity holds because $a\eta, \sigma\eta < 1$, $a^2 - s_2 > 0$, $\sigma^2 - s_2 > 0$ and $e^{-A/\eta} \rightarrow 0$. At $\eta = 0$, $s_2 = -\omega^2$ and $a^2 + \omega^2 = a$ (since $a + \sigma = 1$). $\det M$ was verified symbolically (lemma29_symbolic.py). $\square$

LEMMA 29.3.4 (null measure). [THM] Let $\mu^* = (- (2\sigma - 1) \cos \omega y + 2\omega \sin \omega y) dy - \delta_0 + \delta_{A^*}$ on $[0, A^*]$. Then $(T\mu^*)(x) := \int P_\sigma(x-y)d\mu^*(y) = 0$ for $x \in [0, A^*]$, and $(T\mu^*)(x) = -2(2\sigma - 1)(e^{a(x-A^*)} - e^{-\sigma(x-A^*)}) < 0$ for $x > A^*$.

PROOF. Direct integration, verified symbolically in the variable α , with $a = \frac{1-\sin \alpha}{2}$, $\sigma = \frac{1+\sin \alpha}{2}$, $\omega = \frac{\cos \alpha}{2}$ (num/lemma29_symbolic.py, _symbolic2.py). Conceptually: $\lambda(-\omega^2) = 0$, and the four coefficient conditions are the rows of $M(A^*)$, which is singular. $\square$

PROOF OF THEOREM 29.3.1. ($A_\sigma^{\text{pole}} \geq A^*$, emergence.) Rescaling to $L^2[0, 1]$ makes $A \mapsto T_A$ norm-continuous, so $\lambda_{\min}(A)$ is continuous. It is non-increasing in A and ≥ 0 for $A < \log \frac{\sigma}{a}$ (Lemma 28, Thm. 3.6(c)). Suppose $A_e := \sup\{A : \lambda_{\min} \geq 0\} < A^*$. Then $\lambda_{\min}(A_e) = 0$, and there are $A_n \downarrow A_e$ with negative eigenvalues $\lambda_n \rightarrow 0^-$. Since $\lambda(\eta) \sim 2(a - \sigma)\eta^2$, $\eta_n \rightarrow 0$. By Lemma 29.3.3, $\det B(\eta_n; A_n) = 0$, so by continuity $\det M(A_e) = 0$, i.e. $\omega A_e \in 2\alpha + \pi\mathbb{Z}$. The smallest positive solution is $\omega A = 2\alpha$, which forces $A_e \geq A^*$, a contradiction. So $T_A \geq 0$ for $A < A^*$, and by continuity for $A = A^*$. ($A_\sigma^{\text{pole}} \leq A^*$.) If P_σ were p.d. on $[0, A']$ with $A' > A^*$,

then $\langle \mu, \nu \rangle = \iint P_\sigma(x - y) d\mu d\nu$ would be positive semidefinite on measures on $[0, A']$ (weak-* limits of discrete measures). By Lemma 29.3.4, $\langle \mu^*, \mu^* \rangle = \int T\mu^* d\mu^* = 0$. Cauchy–Schwarz then gives $T\mu^*(x) = \langle \mu^*, \delta_x \rangle = 0$ for all $x \leq A'$, contradicting Lemma 29.3.4. Explicitly, $\langle \mu^* + t\delta_x, \mu^* + t\delta_x \rangle = 2tT\mu^*(x) + t^2P_\sigma(0) < 0$ for small $t > 0$. $\square$

REMARK 29.3.5. The emergence step is the mechanism of Suzuki’s proof of Yoshida’s theorem: continuity of the lowest eigenvalue in the support length [144, Thm. 1.3] [CITED]. It is also the “Krein collision at $\lambda = 0$ ” picture of Lemma 27 [ANALOGY]. The new input here is that for a finite-type kernel the collision can only occur where an explicit 4×4 threshold determinant vanishes.

29.3.2. Certified table. [CERT] num/lemma29_cert.py implements Lemma 29.3.3 in Arb (256 bits; cosh, sinh by power series with rigorous tails; $\|T\| \leq 2[\frac{e^{aA}-1}{a} + \frac{1-e^{-\sigma A}}{\sigma}]$ restricts η to $\lambda(\eta) \geq -\|T\|$). *PSD at A*: the enclosure of $\det B(\eta; A)$ excludes 0 on every box of an adaptive cover of $[0, \eta_{\max}]$. *Negative at A*: a certified sign change of $\det B$ at two points of $(0, 1/\sigma)$. num/lemma29_run.py bisepts each to relative width $\leq 10^{-9}$. The inputs σ (or A) are the binary doubles nearest the decimals.

σ	certified $A_\sigma^{\text{pole}} \in$	closed form	$A_\sigma^{\text{pole}} \sqrt{1 - \sigma}$	Lemma 28 NUM
0.55	[0.40268818, 0.40268819]	0.4026881845	0.270131	0.40271
0.6	[0.82204026, 0.82204027]	0.8220402693	0.519904	0.82208
0.7	[1.79600675, 1.79600676]	1.7960067589	0.983713	1.79610
0.8	[3.21750554, 3.21750555]	3.2175055440	1.438912	3.21767
0.9	[6.18196811, 6.18196813]	6.1819681200	1.954910	6.18236
0.95	[10.27570979, 10.27570981]	10.2757098020	2.297719	10.27676
0.99	[27.54731231, 27.54731234]	27.5473123250	2.754731	27.56295
0.995	[40.52693280, 40.52693283]	40.5269328222	2.865687	40.57720
0.999	[95.39292403, 95.39292410]	95.3929240832	3.016589	96.07586
0.9999	[310.17470776, 310.17470795]	310.1747078125	3.101747	–

A	certified $\sigma_{\text{pole}}(A) \in$
1	[0.6201498959, 0.6201498960]
1.6	[0.6820002664, 0.6820002665]
2	[0.7175654295, 0.7175654296]
3	[0.7877888201, 0.7877888202]
5	[0.8713390399, 0.8713390400]
10	[0.9479627851, 0.9479627852]
15	[0.9720996485, 0.9720996487]

Every bracket contains the closed form (num/lemma29_tables.out). The pair formula, Cor. 29.3.2(c), was also certified at $(\sigma, \beta) = (0.76, 0.90), (0.6, 0.8), (0.7, 0.99)$: PSD at 0.9999 A^{pair} and negative at 1.0001 A^{pair} (num/pair_check.py). For the Lemma 18 ghost ($\sigma = 0.76$, $\varepsilon = 0.14$), $A^{\text{pair}} = 4.6556 \leq A_{\text{detect}} = 6.617$ and $A_{0.76}^{\text{pole}} = 2.5609$, as Lemma 28 required.

29.3.3. Errata to Lemma 28 (recorded here; lemma28.tex not edited).

- (E1) NUM column of Thm. 3.6: relative errors $5 \cdot 10^{-5}$ ($\sigma \leq 0.9$), $5.7 \cdot 10^{-4}$ (0.99), $7.2 \cdot 10^{-3}$ (0.999: 96.076 vs exact 95.393). The $\sigma_{\text{pole}}(A)$ row is correct to the printed ~ 5 digits.
- (E2) Pair grid (num/lemma28_pair.json): Nyström overestimates A^{pair} by up to +17.4% ($\sigma = 0.99$, $\varepsilon = 10^{-4}$: 367.79 vs exact 313.25). The claim “ $A^{\text{pair}} \sqrt{\varepsilon w} \in [2.4, 3.7]$ ” is wrong: on that grid the exact values lie in $[0.40, 3.10]$, and always $A^{\text{pair}} \sqrt{\varepsilon w} < \pi$.
- (E3) “ $c_\infty \in [3, 4]$ ”: $c_\infty = \pi$. The factor “ ≈ 400 ” (pole vs. Lemma 18 σ_*) is $\pi^2/0.0229 \approx 431$ if $A^2(1 - \sigma_*) \rightarrow 0.0229$, which is the last tabulated value. Exactly, $1 - \sigma_{\text{pole}} \sim \pi^2/(A + 4)^2$.
- (E4) The Lemma 18 consistency check ($A_{\text{detect}}/A_\sigma^{\text{pole}} \geq 2.58$) is unaffected: exact 2.5839.

29.4. Priority check (Theorem 3.6 of Lemma 28 and Theorem 29.3.1)

- Yoshida [163]: RH is equivalent to positivity of the Weil form on $C_c^\infty(-a, a)$ for every a , and positivity holds for small a [CITED] (content via [14, 144]; Yoshida not opened).
- Bombieri [14] [CITED]: the minimum is attained on $L^2[-t, t]$, Yoshida’s theorem is re-proved, and with finitely many exceptional zeros the number of negative eigenvalues of large truncations is half their number.
- Burnol [17] [CITED]: there is $c > 1$ with $Z(k) \geq 0$ for $\text{supp } g \subset [1/c, c]$, by symbol positivity plus a term invisible on small support.
- Suzuki [144] [CITED]: unifies Yoshida, Bombieri, Connes–Consani(–Moscovici). The lowest eigenvalue is continuous in a , which gives another proof of Yoshida.

All four concern $\sigma = \frac{1}{2}$ and the actual ζ (the archimedean and prime terms enter). Lemma 28’s Thm. 3.6 instead concerns $\sigma > \frac{1}{2}$ and *any* datum obeying (P0): it is data-blind, the opposite regime. We did not find, in these sources or in searches, the damped pole-dominance statement, the $\cosh(bx)/\cosh x$ Schur step, or the closed form of Thm. 29.3.1. Ingredients that are classical and used as such: Schur’s product theorem, Pólya’s criterion, Bochner, and the sech Fourier transform. Theorem 29.3.1 is a finite-type positive-definite extension problem (Krein’s setting, not used and not cited). We therefore report the results as *not found in the literature checked; possibly folklore*.

29.5. Open questions

QUESTION 29.5.1. Is the N_+ term in Thm. 29.2.9 sharp up to constants for ζ -like data, i.e. does each visible zero near the band force a negative square for some A ? (Prop. 29.2.4 shows this for separated ghosts.)

QUESTION 29.5.2. With explicit C_ζ, t_1 (Trudgian-type) and an explicit density bound, does Thm. 29.2.9 give certified $\kappa_{\sigma, A}(I; \theta) = 0$ on bands below $T_0(\sigma)$ (Lemma 28 R3)?

QUESTION 29.5.3 (for J.W.). Cor. 29.3.2(e) turns the conditional window statement into $\Theta - \sigma \lesssim \pi^2 / ((2\sigma - 1)A^2)$. Is that the “damping budget” scaling you expected from the telegrapher picture: a pair is invisible for windows shorter than $\pi / \sqrt{\varepsilon w}$, the half-period of $\cos(\sqrt{\varepsilon w} u)$?

APPENDIX A

Corrections

This appendix collects every correction that touches the material of this volume: (A) errata stated inside the notes themselves, which are reproduced verbatim in the chapters; (B) later corrections to these chapters from Lemmas 30–47 (Volume III), marked in the chapters by editorial footnotes; (C) Volume I items revisited here; (D) inconsistencies found while collating this volume, and how each is handled. A correction of a claim first made in chat is listed where that claim entered a note. No theorem of Part 1 of the Statement of Intellectual Honesty is affected by any item below.

A. Errata recorded in the chapters

- A1. **Ch. 14, Remark “conventions; correction”.** The brief’s $\Gamma_F = qH + V$ is correct only with the opposite naming $H := \text{pt} \times C$; with $H = C \times \text{pt}$ the class is $H + qV$.
- A2. **Ch. 14, “Correction of emphasis” (691).** $691 \mid h(\mathbb{Q}(\mu_{691}))$ already follows from Kummer’s criterion; Ribet’s contribution is the ω^{-11} -eigenspace.
- A3. **Ch. 14, “J.W.’s pasted heuristics, with corrections”.** Chat heuristics reproduced with corrections and the label *heuristic*.
- A4. **Ch. 18, item (5) of the summary.** Volume I’s guess $A_\sigma^{\det} \approx c/(\beta - \sigma)$ (Vol. I, Part 2, S5) is not what happens at fixed σ ; the data point to $\sqrt{b/\varepsilon}$ (see C5).
- A5. **Ch. 19, §19.4.** J.W.’s volume/area sandwich is a valid proof with a fixable gap: Schur’s inequality must be applied on $\mathbb{R}[\pi]$ with the Rosati form, not on V_ℓ . Once fixed, the volume step is redundant.
- A6. **Ch. 20.** “Null-trace” inert primes: $\gamma_F \perp \delta$ (a right-angle rotation); the vector γ_F is *not* null. This corrects the chat wording.
- A7. **Ch. 22, Remark 22.1.1.** Correction to the usual paraphrase of Connes–Moscovici: the ultraviolet match with ζ is carried by the *negative* (Sonin) spectrum of the prolate operator, not by its low-lying spectrum.
- A8. **Ch. 22, §22.5.** Sign bug in `num/lemma11_ghost.py` (wrong branch of $\sqrt{w}$ for real $w < 0$). Volume I’s Lemma 11 table describes the “added quadruple” model, not the “replacement” model. Applied in Volume I v2 (its correction items 1–2; one table entry changed, $2.72 \rightarrow 2.46$).
- A9. **Ch. 23, summary.** The dictionary in the proposal was corrected: under Maslov dequantization multiplication becomes *addition* and addition becomes *max*.
- A10. **Ch. 24, §24.1.** Volume I’s Lemma 12 and Ch. 18 used Trudgian’s 2014 bound for $S(T)$, reported erroneous by Hasanalizade–Shen–Wong. Replaced by R_{cor} (Lemma 24.1.1). Effects: the Lemma 12 thresholds improve below T_B and worsen slightly above it (headline $A_0(\frac{1}{2}, 3 \cdot 10^{12})$: $12.29 \rightarrow 12.32$); Ch. 18’s Theorem 18.4.1 stands with more room ($0.0564 \rightarrow 0.0717$ against 0.0462).
- A11. **Ch. 24, Prop. 24.2.1.** Ch. 21’s “sharp” cost column implicitly assumed padded band edges; with sharp edges the frame bound collapses to $\lesssim 10^{-4}$.

- A12. **Ch. 24A, §24A.6.** The Arb enclosure of Thm. 24.2.2 used float64-rounded zeros with radius 2^{-100} , so the balls need not contain the zeros: the proof was incomplete as written. Re-run with exact rationals: the theorem stands unchanged.
- A13. **Ch. 25, summary and Prop. 25.1.8.** The slogan “passivity = PNT” (earlier draft and chat) is withdrawn and replaced by the passivity ladder: $R, G \geq 0$ is free; $R, G > 0$ is $\zeta(1+it) \neq 0$, which is equivalent to the PNT.
- A14. **Ch. 27, §27.9, “Erratum to §27.5 (v1)”.** The “transitional outlier” $\text{Im } \mu = 9.66$ was a string-truncation bug; the value is $9.6623 \cdot 10^{-5}$. The v1 bullet on a conditioning artifact is withdrawn.
- A15. **Ch. 29, §29.3.3, errata E1–E4 to Ch. 28.** (E1) the NUM column of Thm. 28.3.6 has relative error up to $7.2 \cdot 10^{-3}$ (at $\sigma = 0.999$: 96.08 vs. exact 95.393); (E2) the pair grid overestimates A^{pair} by up to 17.4%, and the claim $A^{\text{pair}} \sqrt{\varepsilon w} \in [2.4, 3.7]$ is wrong (exact range $[0.40, 3.10]$, always $< \pi$); (E3) $c_\infty = \pi$, and “ ≈ 400 ” is $\pi^2/0.0229 \approx 431$; (E4) the Ch. 18 consistency check is unaffected (exact 2.5839). Footnotes at the three places in Ch. 28.

B. Later corrections to these chapters (Lemmas 30–47)

- B1. **“ $A \leq 2.8$ certified” (Lemma 36, item (a)).** The certified statements at $A = 2.4$ – 3.0 in Chs. 24, 24A, 24B are *middle-band sampling inequalities* (e.g. $L_{\min} = 0.450$ at $A = 2.8$). They are *not* Weil positivity on windows of that length. Certified window positivity stands at $A = 1.6$ (Ch. 16). Footnote in Ch. 24B.
- B2. **Hybrid minorant (Lemma 36, item (b)).** The hybrid Galerkin *minorant* failing at $A = 1.6$ and 2.4 (early pipeline runs) is a failure of the lower-bound form, *not* evidence that $\mathcal{Q} < 0$. (Appendix B.)
- B3. **The “+1” in $N(T)$ (Lemma 36, item (e), Prop. “where the +1 comes from”).** Ch. 14’s “the +1 comes from the pole of ζ at $s = 1$ ” is refined: it is $\frac{1}{\pi} \Delta \arg[s(s-1)] = 1$ exactly, shared between the factors s and $s-1$ of ξ (the poles of $\pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s)$ at 0 and 1), asymptotically $\frac{1}{2}$ each. Footnote in Ch. 14.
- B4. **“Maslov $-\frac{1}{8}$ ” (Lemma 36, item (f)).** Not an integer Maslov index in the standard convention ($\mu = \frac{5}{2}$ with the cutoff area, $-\frac{3}{2}$ for the whole $\frac{7}{8}$). It is a Stirling phase of $\Gamma(\frac{1}{4} + \frac{iT}{2})$. Ch. 14 already labels the Bohr–Sommerfeld count *heuristic* and the $-\frac{1}{8}$ “matched, not derived”; the theorem (Prop. 14.1.1) is unaffected. Footnote in Ch. 14.
- B5. **Ch. 27’s canonical remark (Lemma 41, C3 and C5).** Ch. 27 says the double on-line zero “can leave the axis only because the Krein space itself degenerates”; Lemma 41 says it leaves “through a mixed-type collision”. These describe the same instant in two charts, related by the singular rescaling $g = \delta \tilde{g}$. Lemma 41’s Thm. “two-stage” contains Ch. 27’s Thm. 27.9.5 (Lemma 27, Thm. 9.5) as the exact special case $k = 1$, $\beta = 2\gamma$; no change is needed there. Footnote in Ch. 27.
- B6. **“Ghost $\iff$ index 2” (chat; Lemma 46, E1).** For a reciprocal (Weil-type) polynomial with k distinct roots outside the circle, the Néron–Severi Gram has $n_+ = 1 + k$: a real ghost pair gives 2, a complex quadruple 3, not always “2”. Ch. 15’s $\sup_V n_+ = 1 + N_+$ is the correct form and is unaffected.

C. Volume I items revisited

- C1. **Vol. I, Ch. 11, ghost-injection table.** Corrected in Volume I v2 from the Ch. 22 erratum (A8). Nothing further.

- C2. **Vol. I, Ch. 12, explicit detection constants.** The counting constant is replaced by R_{cor} (A10). Certified changes are in Table 24.1; the smallest certified A_0 moves from 9.36 to 9.14 (γ_1 , global, $\delta = 0.5$). The conclusion of Vol. I, Prop. 12.5.2 (today’s certified windows exclude nothing) is unchanged. Cells still marked “pend.” are listed in Appendix B.
- C3. **Vol. I, Introduction, “ $A_0 \approx (3.7\text{--}8.4)/\delta$ ”.** The certified file gives $\delta \cdot A_0 \in [3.71, 8.06]$ (Lemma 44, which notes that STATE §20 quotes 8.4). The upper end 8.4 should read 8.06 in a future Volume I revision.
- C4. **Vol. I, front matter, Part 1, item L1.** It describes N_+ as the number of zeros with $\text{Re } \rho > \frac{1}{2}$ “with multiplicity”. Volume I’s own Chapter 1 (statement (c) and the displayed definition of N_+) counts *distinct* zeros, and that is the correct count: each distinct off-line zero contributes one hyperbolic plane whatever its multiplicity (also Lemma 40). This is an inconsistency inside Volume I, found while collating; it should be fixed in its next revision. Conventions used in this volume: Appendix C.
- C5. **Vol. I, Part 2, S5 (σ -RG law $c/(\beta - \sigma)$).** Not what happens at fixed σ (A4); replaced by S1 of this volume, with the rigorous exponent- $\frac{1}{2}$ bound and the exact pair threshold.
- C6. **Vol. I, Part 2, S1 (Zhu’s window).** Positivity on Zhu’s window $[-0.8, 0.8]$ is certified independently (Ch. 16), with constant $9.2792 \cdot 10^{-18}$, above Zhu’s $8.9 \cdot 10^{-18}$.
- C7. **Vol. I, Part 2, S2 (ghost-detection slope).** Ch. 22 shows that the Volume I fit was made on the added-quadruple model; Ch. 27, Addendum A, reproduces a doubly-logarithmic law for the birth threshold within the Volume I residual.
- C8. **Vol. I, Ch. 10 (optimality of the certificate).** Rechecked in Ch. 23 (Prop. 23.1.2: the bound is the triangle-hyperfield bound, and it is sharp). Correct as stated.

D. Inconsistencies found while collating, and their handling

- D1. $A_0 \geq 9.36$ in Ch. 16 predates the R_{cor} rerun of Ch. 24 (now 9.14). Handled by a footnote; the conclusion is unchanged.
- D2. **Ch. 24’s summary says the Toeplitz/Levinson code “does not exist”.** It was written afterwards; its results are logged but not written up. Handled by a footnote and Appendix B; nothing from it is claimed in Part 1.
- D3. **Ch. 28’s numerical A_σ^{pole} , the pair grid and the conjecture $c_\infty \in [3, 4]$** are superseded by Ch. 29’s closed form. Handled by footnotes; the Ch. 28 text is kept verbatim.
- D4. N_+ **with or without multiplicity.** Ch. 15 writes $N_+ = \#\{\rho : \text{Re } \rho > \frac{1}{2}\}$ without specifying. The equality $\sup n_+ = 1 + N_+$ requires the *distinct* count, as in Volume I, Ch. 1. Ch. 18’s $N_+(\sigma)$ counts *with* multiplicity (as the comment in `taa-macros.tex` says). Every inequality in which it appears (Cor. 18.2.2, Cor. 29.2.6) remains true, since the multiplicity count is the larger. Handled in Appendix C; no text changed.
- D5. **The working notation file** (NOTATION.md, row for A_σ^{det}) still records “ $\approx c/(\beta - \sigma)$ numerically”, contradicting Ch. 18, item (5). The notation appendix of this volume uses the corrected statement; NOTATION.md was not edited.
- D6. **Two macros with two meanings.** R_{cor} is the corrected counting error in Ch. 24 and the hyperfield $\mathbb{R}^{\text{convex}}$ in Ch. 26; S is a sans-serif S in Ch. 21 and S_{align} in Ch. 24B. They are renamed in Chs. 26 and 24B; the printed symbols are unchanged.
- D7. **The name γ .** Ch. 17 writes γ for the real ordinate t_ρ ; Volume I, Ch. 1 and Chs. 25, 27 use the complex $\gamma_\rho = t_\rho - i\delta_\rho$. Each chapter states its convention; see Appendix C.

- D8. **Bibliography.** The same works appeared under different keys in different notes (e.g. Bo00/Bombieri00/Bom00). They are merged under one key. One internal pointer (L34, uncited) was dropped. No reference had to be dropped for lack of verification (see the note before the bibliography).

APPENDIX B

Pending numerical results and logged certificates

Placeholders

The rows below are computations that were still running when this volume was built (22 October 2026). Each entry is a macro in `pending.tex`, the single placeholder file of this volume, and prints as `[pending: ·]`. **No value has been estimated, extrapolated or filled in.** A placeholder may be replaced only by the certified output of its run (status `CERTIFIED` in its log or JSON). A run that ends uncertified stays as it is and is reported as such.

ID	Quantity	Run	Value	Status
P1	certified lower bound for $\lambda^*(e^{2.4})$, (prime/zero/prime) certificate, Ch. 24 thread	hybq24_q16: $n = 11551$, $p = 69$, $T_c = 6000$, 448 bits, $q = 16$ hybrid cer-	<code>[pending: P1]</code>	running (relaunched 12:47, queue num/l24t/fixq3.sh; an earlier attempt was OOM-killed)
P2	certified lower bound for $\lambda^*(e^{1.8})$, prime-side certificate	pri18_p32: $p = 32$, $T_c = 6000$, precision 320	<code>[pending: P2]</code>	queued after P1 (retry of pri18 with larger p)
P3	uncertified cells of Table 24.1 (Lemma 12 rerun with R_{cor}), listed below	num/lemma24_l12_par.py; 300-bit Arb re-verification	<i>pend.</i>	computing / awaiting 300-bit Arb re-verification

Cells of Table 24.1 still marked “pend.” (generated from `num/lemma24_l12table.tex` at build time; when the certified JSON is updated and `num/lemma24_build.sh` regenerates that table, a rebuild of this volume shows the certified value and the cell leaves this list):

γ_0 , mode	δ	macro
$3 \cdot 10^{12}$ loca.	$\delta = 0.01$	<code>\PendLXIICellTrilllocaE</code>
10^{20} loca.	$\delta = 0.01$	<code>\PendLXIICellTwentylocaE</code>
10^{100} glob.	$\delta = 0.25$	<code>\PendLXIICellHundredglobB</code>
10^{100} glob.	$\delta = 0.1$	<code>\PendLXIICellHundredglobC</code>
10^{100} glob.	$\delta = 0.05$	<code>\PendLXIICellHundredglobD</code>
10^{100} glob.	$\delta = 0.01$	<code>\PendLXIICellHundredglobE</code>
10^{100} loca.	$\delta = 0.5$	<code>\PendLXIICellHundredlocaA</code>
10^{100} loca.	$\delta = 0.25$	<code>\PendLXIICellHundredlocaB</code>
10^{100} loca.	$\delta = 0.1$	<code>\PendLXIICellHundredlocaC</code>
10^{100} loca.	$\delta = 0.05$	<code>\PendLXIICellHundredlocaD</code>
10^{100} loca.	$\delta = 0.01$	<code>\PendLXIICellHundredlocaE</code>

The cell 10^{100} local, $\delta = 0.01$, is still being computed. The other cells have been computed but not yet re-verified in 300-bit Arb, so no value is shown.

About P2. A first run (pri18) finished but was *not* certified: the matrix part certified $\lambda_0 = 3.59 \cdot 10^{-23}$, but the tail term made the certified lower bound negative. P2 is the retry with a larger p . The failure of a lower-bound form is not evidence that $Q < 0$ (Appendix A, item B2.).

Logged certificates, write-up pending

After Chapter 24 was written, the Toeplitz/Levinson certificate pipeline (`num/124t/`) logged the following results with status `CERTIFIED [THM,CA]` in `num/lemma24_progress.md` and the run logs. They are reproduced from the logs. They are **not** theorems of this volume and are not listed in Part 1 of the Statement of Intellectual Honesty, because the method has not yet been written up and checked as a note. The hybrid runs use the mid-band constants of Ch. 24B.

Window	Certificate	Parameters	Logged bound $\lambda^*(e^A) \geq$
$A = 1.6$	prime side (validation)	$n = 253, p = 40, T_c = 200, 256$ bits	$9.775296574 \cdot 10^{-18}$
$A = 1.6$	prime side	$n = 7700, p = 28, T_c = 6000, 256$ bits	$1.464603146 \cdot 10^{-17}$
$A = 1.6$	hybrid, $q = 4$	$n = 7700, p = 28, T_c = 6000, 256$ bits	$1.457333143 \cdot 10^{-17}$
$A = 1.55$	prime side	$n = 7460, p = 28, T_c = 6000, 256$ bits	$2.287026830 \cdot 10^{-16}$
$A = 1.5$	prime side	$T_c = 6000, 256$ bits (<code>124t/runpri15_b.log</code>)	$3.237969834 \cdot 10^{-15}$
$A = 1.5$	hybrid	$n = 7300, p = 28, T_c = 6000, 256$ bits	$2.446287476 \cdot 10^{-15}$
$A = 2$	hybrid, $q = 12$	$n = 9626, p = 48, T_c = 6000, 320$ bits	$5.038361831 \cdot 10^{-30}$
$A = 2$	prime side	$n = 9626, p = 48, T_c = 6000, 320$ bits	$4.994651852 \cdot 10^{-30}$

If confirmed in a written note, the $A = 2$ rows would extend certified window positivity from 1.6 to 2. By Volume I, Ch. 12 and Table 24.1, this would still exclude no zero, since detection lengths are ≥ 9 .

APPENDIX C

Consolidated notation

Notation follows the shared file `taa-macros.tex` (snapshot in this build) and Volume I. Symbols that notes defined locally are now global macros (`volume-macros.tex`); two clashing names were renamed without changing the printed symbol (Appendix A, D6). The table lists the symbols introduced in Chapters 14–29, by chapter of first use.

Conventions that differ between chapters

Object	Convention	Chapters
Fourier–Laplace	$\tilde{F}(z) = \int F(u)e^{izu}du$ (pole functionals $\tilde{F}(\pm \frac{i}{2})$) $\hat{F}(z) = \int F(u)e^{izu}du$ $\hat{f}(\xi) = \int f(t)e^{-i\xi t}dt$ (Paley–Wiener class PW_ω)	15, 17 (as Vol. I, Ch. 1) 18, 28, 29 24, 24A, 24B
γ	complex $\gamma_\rho = t_\rho - i\delta_\rho$, $\rho = \frac{1}{2} + i\gamma_\rho$ real ordinate t_ρ , written γ	25, 27 (as Vol. I, Ch. 1) 17 (and most numerics)
N_+	number of <i>distinct</i> zeros with $\text{Re } \rho > \frac{1}{2}$ (needed for $\sup n_+ = 1 + N_+$)	15 (as Vol. I, Ch. 1)
$N_+(\sigma)$	number of zeros with $\text{Re } \rho > \sigma$, <i>with</i> multiplicity	18, 29 (upper bounds only)
$\mathcal{Q}, \mathcal{Q}$	Weil form of Vol. I, Ch. 1 (zero side); $\mathcal{Q}$ is the same form	all; 15
$\mathcal{L}, \mathcal{P}$	$\mathcal{L} = \mathcal{P} - \mathcal{Q}$; $\mathcal{P}(F, G) = a(F)\overline{b(G)} + b(F)\overline{a(G)}$	15, 17, 26
signs	$\mathcal{Q}$ -signs vs. $\mathcal{L}$ -signs are opposite on the transverse part; “timelike” refers to $\mathcal{L}$	15, 27 (Rem. on sign convention)
κ	κ_A window index; $\kappa_{\sigma,A}$ damped index; $\kappa_H = \kappa_H$ Heaviside residual	16–18; 18, 28, 29; 25

Symbols by chapter

Symbol	Macro	Meaning
<i>Ch. 14</i> $NS(X)_{\mathbb{R}}$	<code>\NSR</code>	real Néron–Severi space; intersection form of signature $(1, \rho - 1)$ (Hodge index)
η	<code>\rap</code>	rapidity of the boost $(a, b) \mapsto (e^{-\eta}a, e^{\eta}b)$ on $\langle H, V \rangle$; Frobenius: $e^{\eta} = \sqrt{q}$
$\overline{N}(T)$ H, V	<code>\Nsm</code> —	$\frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8}$ $H = C \times \text{pt}$, $V = \text{pt} \times C$, $H^2 = V^2 = 0$, $H \cdot V = 1$; $\Gamma \equiv H + qV + \gamma$, $\Delta \equiv H + V + \delta$
<i>Ch. 15</i> $a(F), b(F)$	—	$\tilde{F}(\frac{i}{2}) = \int F e^{-u/2}$, $\tilde{F}(-\frac{i}{2}) = \int F e^{u/2}$: null coordinates; $T_t :$ $(a, b) \mapsto (e^{-t/2}a, e^{t/2}b)$
$\mathcal{P}, \mathcal{L}$	<code>\Pw</code> , <code>\Lw</code>	pole form (rank 2, signature $(1, 1)$); Lorentzian Weil form $\mathcal{L} = \mathcal{P} - \mathcal{Q}$, prime side minus archimedean term

Symbol	Macro	Meaning
$g_{\text{eff}}(h)$	<code>\geff</code>	effective genus $\mathcal{Q}(\varphi_0, \varphi_0)/(2a_0^2)$ of a cubic B-spline of half-width $2h$
<i>Ch. 16</i> $\lambda^*(d)$	<code>\lamw</code>	window margin on windows of length $\log d$ (Vol. I); certified values in <code>num/lemma16_e16vals.tex</code>
<i>Ch. 17</i> $\mathcal{S}[F]$	<code>\Sact</code>	field action $\mathcal{Q}(F, F) = \text{pole} + \text{kinetic} + \text{interaction}$
$K(r)$	<code>\Kkin</code>	kinetic symbol $\text{Re } \psi(\frac{1}{4} + \frac{ir}{2}) - \log \pi = 2\theta'(r)$
$k_\infty(u)$	<code>\kloc</code>	archimedean local kernel $e^{- u /2}/(1 - e^{-2 u })$
$\mathcal{S}_A$	<code>\SA</code>	window action; stationary values $\lambda_k(A)$, number of negative ones $= \kappa_A$
$\mathcal{A}$	<code>\Apart</code>	particle action $\oint \xi dx - \int H_{\text{cl}} dt$
$D_{\text{sc}}, \mathcal{E}$	<code>\Agen, \Enoe</code>	scaling generator $i \frac{d}{du}$; Noether charge $\mathcal{Q}(D_{\text{sc}} F, F)$
μ, κ	—	telegrapher constants $\frac{RC+LG}{2LC}, \frac{(RC-LG)^2}{4LC}$; Heaviside $RC = LG \iff \kappa = 0$
<i>Ch. 18</i> $\mathcal{Q}_\sigma, \mathcal{W}^{(\sigma)}$	<code>\Qsig, \Wsig</code>	damped Weil form; Lagarias form $\frac{1}{2\pi} \int \widehat{F} ^2 w_\sigma, w_\sigma = 2 \text{Re } \frac{\xi'}{\xi}(\sigma + it)$
$\kappa_{\sigma,A}, \kappa_{\sigma,A}^W$	<code>\kapsig, \kapw</code>	negative index of $\mathcal{Q}_\sigma$ (resp. $\mathcal{W}^{(\sigma)}$) on windows of length A ; non-decreasing in A , not monotone in σ
$z_\rho^\sigma, \pi_\rho^\sigma$	<code>\zsig, \pisig</code>	$-i(\rho - \sigma)$; point form of a zero seen from σ , signature $\leq (1, 1)$
A_σ^{det}	<code>\Adet</code>	$\inf\{A : \kappa_{\sigma,A} \geq 1\}$; non-decreasing in σ (rigorous). Numerically $\asymp \sqrt{b/\varepsilon}$ at fixed σ , <i>not</i> $c/(\beta - \sigma)$
Θ	—	$\sup \text{Re } \rho = \min\{\sigma : \kappa_{\sigma,A} = 0 \ \forall A\}$
<i>Ch. 19</i> $\mathcal{B}$	<code>\RosB</code>	Rosati trace form $\text{Tr}(\varphi\psi^\dagger V_\ell J)$
s_n, c_n	—	$\sum \alpha_i^n = q^n + 1 - N_n$; $c_n = s_n/q^{n/2}$ (level- n shadow)
T_k	—	Toeplitz section $(c_{i-j})_{i,j < k}$
<i>Ch. 20</i> $\Gamma_F, \Gamma_F^t, \gamma_F, \gamma_V$	<code>\GF, ...</code>	graphs of Frobenius and Verschiebung; their transverse parts
B_η, W	<code>\Bo, \Wt</code>	boost of $\langle H, V \rangle$; $\langle H, V \rangle^\perp$ with $\langle x, y \rangle_W = -x \cdot y$
τ_p, θ_p, s_p	—	transverse part of $T_p/\sqrt{p}$; CM angle; rapidity $\tanh s_p = a_p/2\sqrt{p}$
<i>Ch. 21</i> $D(T), A_{\text{crit}}, A_{\text{low}}$	<code>\Dn, ...</code>	zero density; critical and model window lengths of the no-go theorem
S	<code>\Salign</code>	alignment sum of the prime waves (the Galerkin band must reach $T \sim e^{S(A)}$); Ch. 21
<i>Ch. 22</i> $W_\lambda, K(d)$	<code>QW_\lambda, \Wl, \QW, \Kn</code>	prolate operator; Weil quadratic form on $L^2([\lambda^{-1}, \lambda], d^*u)$; near-radical dimension
$\mathcal{Q}_\delta^{\text{rep}}, \mathcal{Q}_\delta^{\text{add}}, \mathcal{Q}_\delta^{\text{on}}$	<code>\Qrep, ...</code>	replacement, added-quadruple and on-line ghost models
<i>Ch. 23</i> $\mathbb{T}, \mathcal{T}\mathbb{C}$	<code>\Tmax, \TC</code>	tropical semifield; Viro's complex-tropical hyperfield
<i>Chs. 24, 24A, 24B</i> R_{cor}	<code>\Rc</code>	corrected explicit error in $N(T)$ (replaces Trudgian 2014)

Symbol	Macro	Meaning
$PW_\omega, \ f\ _I$	<code>\PW, \nrm</code>	Paley–Wiener space with $\text{supp } \hat{f} \subset [-\omega, \omega]$; L^2 norm on I
S_{align}	<code>\SalignB</code>	$S_{\text{align}}(L) = \sum_{n < e^L} 2\Lambda(n)/\sqrt{n}$ in Ch. 24B (renamed from <code>\Salign</code>)
$L_{\min}, \Lambda^2, c_{\text{mid}}, c_{\text{out}}$ <i>Ch. 25</i>	<code>num/lemma24_</code> <code>L34b_numbers</code>	certified block constants of the multi-block sampling bound
$\gamma_{\text{tel}} = \alpha_{\text{tel}} + i\beta_{\text{tel}}$	<code>\gtel, ...</code>	telegrapher propagation constant (never bare γ, α, β)
P_ρ, κ_H	<code>\Pch, \kap</code>	$(D - \rho)(D - (1 - \bar{\rho}))$; Heaviside residual, $= \delta^2$ for the line of ρ
$\mathcal{H}_{\mathcal{Z}}, J_{\text{FE}}, [x, y]_{\mathcal{Z}}, \text{ev}$	<code>\Hz, \Jfe, \kr,</code> <code>\ev</code>	zero space $\ell^2(\mathcal{Z}, m)$; $(J_{\text{FE}}x)_\rho = x_{1-\bar{\rho}}$; Krein form $\langle J_{\text{FE}}x, y \rangle$; evaluation $F \mapsto (\tilde{F}(\gamma_\rho))_\rho$
$t, \beta, \delta, \gamma, \rho, \kappa_H$	<code>\bt, ...</code>	multiplication operators on $\mathcal{H}_{\mathcal{Z}}$; $\rho = \frac{1}{2} + i\gamma$, $\kappa_H = \delta^2$
$\mathcal{E}_{\text{st}}, \mathcal{E}_{\text{CK}}$ (P0)–(P4)	<code>\Est, \ECK</code> —	storage and reduced Caldirola–Kanai energies passivity ladder: $\beta \in [0, 1] / \beta \in (0, 1) / \text{dlVP margin} / \Theta < 1 / \delta = 0$
<i>Ch. 26</i> $\hbar, \mathbb{R}^{\text{convex}}, n_+$	<code>\hb,</code> <code>\RcConvex,</code> <code>\npos</code>	deformation parameter; Connes–Consani hyperfield (renamed from <code>\Rc</code>); number of positive squares
<i>Ch. 27</i> G_A, K_A	<code>\GA, \KA</code>	window Grams $Q(\varphi_i, \varphi_j), Q(\varphi'_i, \varphi'_j)$; frequency pencil $K_A x = \mu G_A x$
$n_{\mathbb{C}}, n_-$	<code>\nc, \nm</code>	number of complex pairs; number of real eigenvalues of negative Krein type ($\kappa_A = n_{\mathbb{C}} + n_-$)
$\delta_b(A), \delta_c(A)$ <i>Chs. 28, 29</i>	—	birth threshold ($\det G_A = 0$); collision threshold
$k_\delta(u)$	—	pair kernel $2 \cosh(\delta u) e^{-c u } = e^{\varepsilon u } + e^{-w u }$, $\varepsilon = \beta - \sigma$, $w = \beta + \sigma - 1$
P_σ $A_\sigma^{\text{pole}}, A^{\text{pair}}$	<code>\Pk</code> <code>\Apole,</code> <code>\Apair</code>	pole kernel $e^{(1-\sigma) u } + e^{-\sigma u }$ largest A with the pole (resp. pair) kernel p.d. on $(-A, A)$; exactly $2 \arcsin(2\sigma - 1)/\sqrt{\sigma(1-\sigma)}$ (Ch. 29)
$N(\sigma, X), (w_\sigma)_-$	—	zeros with $\beta > \sigma$, $ \gamma \leq X$ (with multiplicity); negative part of the Lagarias symbol

Claim tags

See the legend in the Statement of Intellectual Honesty (p. [iii](#)).

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