

Reciprocal Lie-Algebra Structures: Paired Closure, Exchange Symmetry, and the Interpretation of Ontological Two-Sidedness

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Abstract. This paper examines whether reciprocal representation provides a mathematically coherent way to express two distinguishable but structurally equivalent sectors without privileging either as primary. The starting point is a pair of isomorphic Lie algebras and their representation within a direct-sum algebra. A natural sector-exchange map is shown to be an involutive Lie-algebra automorphism, while the paired representation forms a Lie subalgebra fixed pointwise by that exchange. The direct sum therefore supplies a precise realization of reciprocal redundancy without requiring nonzero cross-sector brackets. The paper then considers a more general ambient algebra, where closure of the paired representation constrains the cross-sector bracket to be symmetric in its corresponding arguments. The Jacobi identity imposes additional consistency requirements but does not, by itself, require cross-sector brackets to vanish. The limiting case of coincident embeddings is examined, as is an illustrative example in which paired closure coexists with a nonzero cross-sector bracket. The resulting mathematical distinctions are related cautiously to a philosophical proposal of ontological two-sidedness. The algebra does not prove that reality has a bilateral ontology; it provides a formal representation in which two distinguishable sides can be related reciprocally without assigning either side ontological priority.

Keywords: Lie algebras; direct sums; reciprocal exchange; paired closure; Jacobi identity; redundancy; bilateral representation; ontological two-sidedness

1. Introduction

Mathematical representations often distinguish structures that are equivalent in their formal organization. Two representations may differ in their position, orientation, or role while preserving the same underlying relations. This raises a question that is mathematical in its immediate formulation but also has a broader philosophical significance: can two distinguishable representations be organized reciprocally, without the formal structure privileging one as the original and the other as derivative?

The present paper investigates this question using Lie algebras. Its immediate objective is limited. It does not attempt to establish a new physical theory, derive a new law of gravitation, or prove an ontology from mathematical formalism. Rather, it identifies a coherent algebraic construction that may serve as a disciplined formal analogue of reciprocal or bilateral organization.

The central construction begins with two isomorphic Lie algebras. Their direct sum preserves their respective internal bracket structures while allowing the two sectors to

be exchanged by an involutive automorphism. A paired representation of the two sectors is invariant under that exchange. This combination is important: the sectors remain distinguishable in the representation, yet the paired object does not privilege either sector under the exchange operation.

A second question concerns whether the construction must remain a direct sum. In a direct sum, cross-sector brackets vanish by definition. In a more general ambient Lie algebra, however, elements belonging to different sectors may have nonzero brackets. Requiring the paired representation to remain closed then imposes a symmetry condition on those cross-sector brackets. The Jacobi identity further constrains their compatibility with the rest of the algebra.

These results establish a useful distinction between reciprocal exchange and nontrivial cross-sector interaction. The first is realized directly by the direct-sum construction; the second is an additional possibility requiring further algebraic conditions. Neither should be confused with the philosophical interpretation that motivates the investigation.

2. Isomorphic Lie Algebras and Their Paired Representation

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over the same field, and let

$$\varphi : \mathfrak{g} \rightarrow \mathfrak{g}'$$

be a Lie-algebra isomorphism. Thus, for all $X, Y \in \mathfrak{g}$,

$$\varphi([X, Y]_{\mathfrak{g}}) = [\varphi(X), \varphi(Y)]_{\mathfrak{g}'}.$$

The isomorphism establishes that the two algebras possess the same abstract bracket structure. It does not require them to be represented as the same subset of a larger algebra.

The simplest setting in which to represent both sectors together is their direct sum:

$$\mathfrak{L} = \mathfrak{g} \oplus \mathfrak{g}'.$$

An element of $\mathfrak{L}$ is an ordered pair (X, U) , where $X \in \mathfrak{g}$ and $U \in \mathfrak{g}'$. The Lie bracket is defined componentwise:

$$[(X, U), (Y, V)]_{\mathfrak{L}} = ([X, Y]_{\mathfrak{g}}, [U, V]_{\mathfrak{g}'}).$$

The direct sum therefore retains the internal Lie-algebraic structure of each sector. By definition, there are no cross-sector contributions to the bracket:

$$[(X, 0), (0, U)]_{\mathfrak{L}} = (0, 0).$$

This vanishing is not a consequence of a dynamical cancellation or a theorem about physical interactions. It is part of the direct-sum construction itself.

Now define the paired map $\iota : \mathfrak{g} \rightarrow \mathfrak{L}$ by

$$\iota(X) = (X, \varphi(X)).$$

Its image is the paired subset

$$\mathfrak{L}_r = \{ (X, \varphi(X)) : X \in \mathfrak{g} \}.$$

The image consists of those elements in which the two sectors are related through the isomorphism φ . It is the graph of φ within the direct-sum algebra.

The bracket of two paired elements is

$$[\iota(X), \iota(Y)]_{\mathfrak{L}} = ([X, Y]_{\mathfrak{g}}, [\varphi(X), \varphi(Y)]_{\mathfrak{g}'}).$$

Since φ preserves the bracket,

$$[\iota(X), \iota(Y)]_{\mathfrak{L}} = ([X, Y]_{\mathfrak{g}}, \varphi([X, Y]_{\mathfrak{g}})) = \iota([X, Y]_{\mathfrak{g}}).$$

Consequently, $\mathfrak{L}_r$ is a Lie subalgebra of $\mathfrak{L}$. The paired representation is closed under the Lie bracket, and the map ι preserves the Lie-algebraic structure of $\mathfrak{g}$.

This result is elementary but central to the present proposal. It establishes that two isomorphic sectors can be represented together in a single algebraic structure, with their paired representation preserving the original bracket relations.

3. Reciprocal Exchange as an Involutive Automorphism

The direct sum admits a natural operation that exchanges its two sectors.

Define $S : \mathfrak{L} \rightarrow \mathfrak{L}$ by $S(X, U) = (\varphi^{-1}(U), \varphi(X))$.

The first component of the image is obtained by mapping the second sector back into $\mathfrak{g}$; the second component is obtained by mapping the first sector into $\mathfrak{g}'$.

Applying S twice gives

$$S^2(X, U) = S(\varphi^{-1}(U), \varphi(X)) = (X, U).$$

Hence,

$S^2 = I$, where I is the identity map on $\mathfrak{L}$. The exchange is an involution.

The operation also preserves the Lie bracket. For any $(X, U), (Y, V) \in \mathfrak{L}$,

$$S([(X, U), (Y, V)]_{\mathfrak{L}}) = S([([X, Y]_{\mathfrak{g}}, [U, V]_{\mathfrak{g}'})]_{\mathfrak{L}}) = (\varphi^{-1}([U, V]_{\mathfrak{g}'}), \varphi([X, Y]_{\mathfrak{g}})).$$

Because φ is an isomorphism,

$$\varphi^{-1}([U, V]_{\mathfrak{g}'}) = [\varphi^{-1}(U), \varphi^{-1}(V)]_{\mathfrak{g}},$$

and

$$\varphi([X, Y]_{\mathfrak{g}}) = [\varphi(X), \varphi(Y)]_{\mathfrak{g}'}.$$

Therefore, $S([(X, U), (Y, V)]_{\mathfrak{L}}) = [S(X, U), S(Y, V)]_{\mathfrak{L}}$.

Thus, S is a Lie-algebra automorphism as well as an involution.

The paired representation has a particularly important property. Applying S to $\iota(X)$ gives

$$S(\iota(X)) = S(X, \varphi(X)) = (\varphi^{-1}(\varphi(X)), \varphi(X)) = (X, \varphi(X)).$$

Therefore, $S(\iota(X)) = \iota(X)$.

The paired representation is fixed pointwise by the exchange involution.

This result expresses a precise mathematical relationship between distinction and reciprocity. The two components remain distinguishable as coordinates of the direct-sum representation, but the exchange operation does not privilege either component. The paired element is invariant under their interchange.

The result does not depend on φ being its own inverse. In general, φ maps from $\mathfrak{g}$ to $\mathfrak{g}'$, while φ^{-1} maps in the opposite direction. It is the combined exchange S , rather than φ alone, that is self-inverse.

The direct sum therefore provides an explicit and economical realization of reciprocal exchange. No nonzero cross-sector bracket is required for this result.

4. Paired Closure in a General Ambient Lie Algebra

The direct-sum construction establishes a baseline. A more general question arises when two isomorphic Lie algebras are embedded as subalgebras of a common ambient Lie algebra $\mathfrak{L}$, whose bracket is not assumed to be componentwise.

Let $j : \mathfrak{g} \rightarrow \mathfrak{L}$ and $j' : \mathfrak{g}' \rightarrow \mathfrak{L}$ be injective Lie-algebra homomorphisms. Define the paired map

$$\iota(X) = j(X) + j'(\varphi(X)).$$

Its image is

$$\mathfrak{L}_r = \{\iota(X) : X \in \mathfrak{g}\}.$$

We require paired closure:

$$[\iota(X), \iota(Y)]\mathfrak{L} = \iota([X, Y]\mathfrak{g})$$

for all $X, Y \in \mathfrak{g}$.

Expanding the left-hand side gives

$$\begin{aligned} [\iota(X), \iota(Y)]\mathfrak{L} &= [j(X), j(Y)]\mathfrak{L} + [j'(\varphi(X)), j'(\varphi(Y))]\mathfrak{L} \\ &\quad + [j(X), j'(\varphi(Y))]\mathfrak{L} + [j'(\varphi(X)), j(Y)]\mathfrak{L}. \end{aligned}$$

Since j and j' preserve their respective internal brackets, the first two terms sum to

$$j([X, Y]\mathfrak{g}) + j'(\varphi([X, Y]\mathfrak{g})) = \iota([X, Y]\mathfrak{g}).$$

It follows that paired closure is equivalent to the cancellation of the two cross-sector contributions:

$$[j(X), j'(\varphi(Y))]\mathfrak{L} + [j'(\varphi(X)), j(Y)]\mathfrak{L} = 0.$$

Define the cross-sector relation

$$B(X, Y) = [j(X), j'(\varphi(Y))]\mathfrak{L}.$$

By antisymmetry of the Lie bracket,

$$[j'(\varphi(X)), j(Y)]\mathfrak{L} = -B(Y, X).$$

The paired-closure condition therefore becomes

$$B(X, Y) - B(Y, X) = 0,$$

or equivalently,

$$B(X, Y) = B(Y, X).$$

This is the principal constraint obtained from paired closure in the general ambient setting.

The symmetry of B is not a claim that the Lie bracket itself has become symmetric. The ambient Lie bracket remains antisymmetric. Rather, B is defined by placing its arguments in corresponding but different sectors, and closure requires symmetry under interchange of those corresponding arguments.

Nor does this condition require $B(X, Y) = 0$. It permits nonzero cross-sector brackets, provided they satisfy the symmetry condition and are compatible with the other requirements of the ambient Lie algebra.

The distinction is important. Reciprocal representation does not, by itself, require interaction between the sectors. But neither does the closure condition alone rule out nontrivial cross-sector relations.

5. The Direct Sum as the Reference Construction

The direct sum is more than a preliminary example. It is the canonical reference construction for the present investigation because it realizes the central requirements in a transparent form.

In $\mathfrak{g} \oplus \mathfrak{g}'$, the paired representation is

$$\iota(X) = (X, \varphi(X)),$$

the bracket is componentwise, and the cross-sector brackets vanish. Paired closure follows directly from the bracket-preserving property of φ . The exchange map

$$S(X, U) = (\varphi^{-1}(U), \varphi(X))$$

is an involutive automorphism, and the paired graph is fixed pointwise by S .

The construction thus realizes three properties together:

1. The sectors are structurally equivalent but represented as distinguishable components.
2. Their paired representation is closed under the Lie bracket.
3. Their exchange is an involutive automorphism under which the paired representation is invariant.

These properties do not depend on nonzero cross-sector coupling. The direct sum shows that reciprocal exchange and nontrivial interaction are logically distinct.

This is a useful conclusion in its own right. If the aim is to represent reciprocal redundancy, a direct sum already provides a mathematically coherent model. A

nontrivial cross-sector bracket would add further structure, but it is not needed to establish the existence of reciprocal exchange.

The direct sum also clarifies what is meant by redundancy in this context. The two sectors are not identical as coordinate components; rather, their abstract structures correspond through φ , and the combined representation treats them symmetrically under S . The representation contains two distinguishable components without making either component the privileged representative of the pair.

6. Cross-Sector Relations and the Jacobi Identity

The direct sum satisfies the Jacobi identity because both component algebras satisfy it. The identity holds componentwise:

$$[A, [C, D]] + [C, [D, A]] + [D, [A, C]] = 0.$$

In a more general ambient Lie algebra, Jacobi continues to hold, but it imposes compatibility conditions on the cross-sector brackets. The symmetry condition $B(X, Y) = B(Y, X)$ obtained from paired closure is therefore not the only constraint on a possible nontrivial extension.

For example, consider a mixed triple consisting of $j(X)$, $j(Y)$, and $j'(\varphi(Z))$. The Jacobi identity gives

$$[j(X), [j(Y), j'(\varphi(Z))]] + [j(Y), [j'(\varphi(Z)), j(X)]] + [j'(\varphi(Z)), [j(X), j(Y)]] = 0.$$

Using the definition of B and the bracket-preserving property of j , this can be written as

$$[j(X), B(Y, Z)] - [j(Y), B(X, Z)] - B([X, Y], Z) = 0.$$

Other choices of mixed triples produce additional constraints. Together, these relations restrict which cross-sector brackets can occur within a coherent ambient Lie algebra.

The Jacobi identity does not automatically force B to vanish. Nor does the symmetry of B guarantee that a nonzero solution exists for every choice of algebras and embeddings. Those conclusions depend on the particular structure under consideration.

For the direct-sum construction, the cross-sector relation is zero, and the Jacobi identity is automatically satisfied. The direct sum therefore supplies an unambiguous baseline against which any proposed nontrivial extension can be assessed.

In the interpretation advanced here, Jacobi may be regarded as a formal constraint on the coherence of nested relations. Pairwise relations can be nonzero, but their organization must remain compatible with the algebra as a whole. This is an interpretive description of the identity's structural role, not a claim that Jacobi is a physical regulator or that it proves any general law of evolution.

7. Coincident Embeddings and the Abelian Case

A limiting case deserves particular attention. Suppose the embeddings coincide after identifying the two algebras through φ :

$$j = j' \circ \varphi.$$

Then

$$j(X) = j'(\varphi(X))$$

for every $X \in \mathfrak{g}$. The two embeddings no longer define distinct embedded subalgebras of $\mathfrak{L}$; they represent the same image.

Under this condition, the cross-sector relation becomes

$$B(X, Y) = [j(X), j'(\varphi(Y))]_{\mathfrak{L}} = [j(X), j(Y)]_{\mathfrak{L}} = j([X, Y]_{\mathfrak{g}}).$$

But paired closure requires $B(X, Y) = B(Y, X)$.

Consequently, $j([X, Y]_{\mathfrak{g}}) = j([Y, X]_{\mathfrak{g}})$.

Antisymmetry of the Lie bracket gives $j([X, Y]_{\mathfrak{g}}) = -j([Y, X]_{\mathfrak{g}})$, so $2j([X, Y]_{\mathfrak{g}}) = 0$.

Over a field of characteristic zero, this implies $j([X, Y]_{\mathfrak{g}}) = 0$. Since j is injective,

$$[X, Y]_{\mathfrak{g}} = 0$$

for every $X, Y \in \mathfrak{g}$. Thus, $\mathfrak{g}$ is abelian, and so is $\mathfrak{g}'$.

The conclusion is specific: coincident embeddings, together with the paired-closure condition, force the represented sector algebra to be abelian. They do not force the entire ambient algebra $\mathfrak{L}$ to be abelian.

An abelian subalgebra can exist inside a non-Abelian ambient algebra. For example, one-dimensional subalgebras of a non-Abelian Lie algebra are abelian, even though brackets between elements of different subalgebras may be nonzero. Consequently, the abelian result does not make the construction mathematically vacuous.

The Abelian case is also relevant to the electromagnetic motivation for this work. An Abelian sector can be represented redundantly alongside a corresponding sector without requiring a non-Abelian interaction between them. The resulting formal structure may be of interpretive interest precisely because reciprocal representation does not depend on a nonzero cross-sector bracket.

8. A Nonzero Cross-Sector Bracket Can Coexist with Paired Closure

Although the direct sum is the reference construction, it is useful to establish that paired closure does not universally require cross-sector brackets to vanish.

Consider the two-dimensional Lie algebra with basis elements x and u and bracket

$$[x, u] = u,$$

$$[x, x] = 0,$$

$$[u, u] = 0.$$

The bracket is extended by bilinearity and antisymmetry. This is a solvable Lie algebra.

Let $\mathfrak{g}$ be the one-dimensional abelian algebra spanned by x , and let $\mathfrak{g}'$ be the one-dimensional abelian algebra spanned by u . These algebras are isomorphic. Define the embeddings into the two-dimensional ambient algebra by $j(x) = x$, $j'(u) = u$, and let $\varphi(x) = u$. The paired element is $\iota(x) = x + u$.

Since $\mathfrak{g}$ is one-dimensional and abelian, its only internal bracket is $[x, x] = 0$. The paired closure condition is therefore $[\iota(x), \iota(x)] = \iota([x, x]) = 0$.

By antisymmetry,

$$[x + u, x + u] = [x, x] + [x, u] + [u, x] + [u, u] = 0 + u - u + 0 = 0.$$

Thus, the paired subset is closed. Nevertheless, the cross-sector relation is nonzero:

$$B(x, x) = [x, u] = u \neq 0.$$

The two cross-sector terms cancel in the paired bracket, even though each is nonzero.

This example establishes a limited but useful point: nonzero cross-sector relations can coexist with paired closure in a suitable ambient Lie algebra. It does not establish that every pair of isomorphic Lie algebras admits such an extension, nor that the example possesses an exchange involution that swaps x and u while preserving the ambient bracket. Paired closure and sector-exchange symmetry remain distinct requirements.

The example therefore complements, rather than replaces, the direct-sum construction. The direct sum gives the cleanest realization of reciprocal exchange; the nonzero-bracket example shows that closure is not synonymous with vanishing cross-sector relations.

9. Relation to the Electromagnetic Motivation

The motivation for this construction is partly connected to earlier work (Smith 2026) in which a redundant sector was introduced to represent opposite orientations within an electromagnetic framework. In that setting, the direct sum offered a way to place corresponding sectors side by side while retaining their respective algebraic identities.

The elementary identity $[X, -X] = 0$ follows from bilinearity and antisymmetry of the Lie bracket: $[X, -X] = -[X, X] = 0$.

By itself, this is a formal algebraic identity. It does not demonstrate a physical annihilation process, nor does it establish a physical interaction between matter and antimatter. Such interpretations require additional definitions and physical arguments.

The more general motivation is the possibility of representing a structure together with a reciprocal counterpart without deciding, from the formalism alone, that one is ontologically prior to the other. In the direct-sum construction, this idea is expressed not by the identity $[X, -X] = 0$ alone, but by the combination of structural equivalence, paired closure, and an exchange involution that leaves the paired representation invariant.

This distinction also matters when considering non-Abelian gauge theories. A direct-sum representation can be constructed for isomorphic non-Abelian Lie algebras as well as Abelian ones. However, the fact that the original Yang–Mills framework employs a non-Abelian gauge algebra does not make that algebra solvable, nor does the present construction establish a new Yang–Mills theory. The direct sum and its exchange map are algebraic structures; any physical interpretation must be justified separately.

Accordingly, the electromagnetic connection serves as motivation for the formal question, not as evidence that the reciprocal construction is already a physical description of nature.

10. Reciprocal Representation and Ontological Two-Sidedness

The mathematical construction supports a philosophical interpretation, provided the distinction between representation and reality is maintained.

A measurement is relational in the sense that it involves an interaction, comparison, or correspondence between what is measured and the conditions under which it is measured. Yet the relational character of measurement does not, by itself, settle the ontology of what exists. A representation that distinguishes two sides does not prove that reality consists of two independently existing inventories.

The proposal of ontological two-sidedness is intended in a more restrained sense. It asks whether the structure of reality might be understood through reciprocal relations in which two sides are distinguishable without one being established as the ontological origin of the other.

The direct-sum construction provides a formal analogy. The two sectors are distinguishable in the ordered pair $(X, \varphi(X))$, but their exchange is governed by an involutive automorphism. The paired representation is invariant under this exchange. The formalism therefore provides no preference between the two components at the level of the paired object.

This is not a proof of ontological symmetry. The direct sum is a mathematical construction, and its symmetry follows from its definition. It does not establish that physical reality has the same structure. Its philosophical value is instead that it makes precise one possible form of reciprocal representation: distinguishable components can be organized into an invariant whole without requiring a privileged component.

The image of a perfect mirror may help convey this point. In a mirror relation, each side is defined in relation to the other, and the relation itself may fail to provide an internal criterion for declaring one side original and the other derivative. The analogy is not exact: a mathematical exchange automorphism is not a physical mirror, and algebraic equivalence does not prove ontological indistinguishability. Nevertheless, the analogy illustrates the intended distinction between having two identifiable sides and asserting two independent realities.

The expression “I Am That I Am” may also serve as a philosophical evocation of self-sufficiency or a ground not established by appeal to an external counterpart. It is not invoked here as a definitive theological interpretation, nor as a conclusion derived from Lie algebra.

The central philosophical suggestion is therefore modest. Reciprocal exchange offers a formal way to represent two distinguishable sides without assigning either formal priority. Whether this is a useful model for understanding reality is a further interpretive question, not a mathematical consequence.

11. Conclusion

This paper has established a coherent algebraic construction for reciprocal representation.

Given two isomorphic Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, their direct sum admits the involutive automorphism

$$S(X, U) = (\varphi^{-1}(U), \varphi(X)),$$

with $S^2 = I$. The paired representation

$$\iota(X) = (X, \varphi(X))$$

is a Lie subalgebra and is fixed pointwise by S . Reciprocal exchange is therefore realized without requiring nonzero cross-sector brackets.

In a more general ambient Lie algebra, paired closure requires the cross-sector relation

$$B(X, Y) = [j(X), j'(\varphi(Y))]$$

to satisfy

$$B(X, Y) = B(Y, X).$$

This symmetry is a consequence of paired closure under the stated embedding assumptions. It is not symmetry of the Lie bracket itself, and it does not require B to vanish. The Jacobi identity imposes additional compatibility conditions but does not universally eliminate nonzero cross-sector relations. A simple two-dimensional solvable example demonstrates that paired closure and a nonzero cross-sector bracket can coexist, although that example does not by itself establish an exchange automorphism.

The limiting case of coincident embeddings yields an abelian sector under paired closure, but it does not imply that the entire ambient algebra is abelian. This distinction preserves the relevance of Abelian sectors while allowing them to be considered within larger non-Abelian structures.

The principal result is not a new physical law or a proof of ontological two-sidedness. It is the identification of a precise formal structure in which two corresponding sectors remain distinguishable, are related reciprocally, and contribute to a paired representation invariant under exchange. The direct sum is sufficient to establish this result; nontrivial cross-sector coupling is an additional possibility rather than a necessary condition.

The broader interpretation is that reciprocal representation may provide a useful mathematical analogue for thinking about bilateral organization without reducing it either to two unrelated entities or to a single privileged side. The algebra establishes the coherence of the representation. The philosophical proposal concerns what, if anything, that form of coherence may illuminate about the structure of reality.

References

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