Arithmetic Unitarity III

In Search of Positivity — G5, Polarization, and Hilbert versus Krein

From the Axioms G1–G6 to the Verschiebung

Thermodynamics of Arithmetic Action, Lemmas 30–47

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Abstract

Volume I (ai.viXra:2610.0045v2) showed that the Riemann Hypothesis (RH) is equivalent to unitarity (the absence of negative-norm "ghost" directions) of the logarithmic scaling flow for the Weil form $\mathcal{Q}$. Volume II measured that obstruction on the Weil form itself. This volume asks the next question: *which structure could supply the positivity?* For curves over $\mathbb{F}_q$ the answer is known. The proof of Weil's theorem needs (i) a commutative algebra containing Frobenius, (ii) an involution with $FF^\dagger = q$, (iii) a trace for which the involution is positive, and (iv) Albert's algebraic step. Chapter 30 shows that ζ already has (i), (ii) and (iv), canonically, and isolates (iii) as axiom G5: a positivity of the zero-side trace that holds *for a geometric reason*, built without the zeros (G6). The chapters then test candidate sources of G5: thermal, modular and Kepler structures (Ch. 31); Osterwalder–Schrader reflection positivity (Ch. 32); causality, Kramers–Kronig and passivity (Chs. 33, 35, 36); de Branges/Krein lines reconstructed from the prime side, with the official transmission-line form of G5 (Chs. 34, 37); Galois symmetry, Iwasawa theory and Bost–Connes (Chs. 38, 39); polarization, Krein unitarity and the two stages of ghost birth (Chs. 40, 41); half-sided modular inclusions and the missing rung (Ch. 42); holomorphic flows, cavities, and boson/fermion second quantizations (Chs. 43–45); and the Verschiebung and the reflection $\mathbb{N} \to \mathbb{Z}$ (Chs. 46, 47). One conclusion recurs as a theorem in Chapters 42, 46 and 47: *a reflection gives the functional equation; positivity needs a polarization.* Algebra gives Krein unitarity; geometry (a positive polarization) gives Hilbert unitarity. Every statement carries a claim label (legend below). **RH is not proved.** No candidate tested here supplies G5, and for most of them we prove why not.

Status of this volume

This volume was assembled on 22 October 2026 from the standalone notes TAA Lemmas 30–47, using their latest versions (Lemma 34 v2, Lemma 36 v2, Lemma 37 v3, Lemma 42 v2; Lemmas 43–47 as compiled on 22 October 2026). Every supplement to these lemmas is already part of its note, for example Lemma 36 §7 (Sommerfeld) and the revision sections of Lemmas 34 and 37. The working note on the author's earlier skin-effect papers (`notes/skin_effect_prior.md`) is not a numbered lemma and is not included. The note texts are reproduced verbatim except for (i) chapter-prefixed labels and canonical citation keys; (ii) macro names that mean different things in different notes, renamed inside the outlier chapters without changing the printed symbol (Appendix C); (iii) "Questions for J.W." renamed "Open questions"; (iv) two typesetting repairs that change no text (Appendix A, D-items); and (v) footnotes marked **[Vol. 3 editorial note]**, which point to

later corrections. One labelled **Vol. 3 editorial remark** in the Introduction collects a dictionary proposed by the author; it is not part of any note. A second, **[Vol. 3 editorial addition, from Lemma 50]**, condenses the spinor material of TAA Lemma 50 (not one of Chapters 30–47). All corrections are collected in Appendix A. The one computation still running is a *placeholder* (Appendix B; macro file `pending.tex`). No value has been filled in by estimate or extrapolation.

Statement of Intellectual Honesty

The Riemann Hypothesis is NOT proved in this volume.
Nothing here implies it. No result here should be quoted as progress "towards a proof" beyond what is listed in Part 1.

Numbers refer to this volume. Chapter k is the standalone note "TAA Lemma k". A statement numbered $k.s.n$ is statement $s.n$ of that note. References to Lemmas 1–13 are to Volume I, and to Lemmas 14–29 to Volume II.

Claim labels

The notes use one tag system with a few local variants. Each tag belongs to exactly one of six volume classes.

Volume class	Meaning	Tags used in the chapters
Theorem	Proved: here, or a cited published theorem used as stated. Sub-kinds are kept visible.	[THM]; "[THM] from [CITED]" (elementary consequence of cited results, proof given); cited: [CITED], [CITED, secondary] (checked only through a named secondary source); [BIB] (bibliographic record verified, content standard background); definitions [DEF] (not claims)
Restatement	Equivalent to RH, GRH or a known statement; no new content.	[RESTATEMENT]
Analogy	Structural parallel or heuristic argument; proves nothing.	[ANALOGY], [HEURISTIC], [SKETCH] (formal computation, domain questions not settled), [ESTIMATE] (heuristic estimate, Ch. 35)
Conjecture	A stated expectation, with evidence.	[CONJ], [CONJECTURAL] (conjectural framework of a cited author, Ch. 30); items (Sn) in Part 2
Open	A question or gap explicitly left open.	[OPEN] (Chs. 31–47 except 38); [OPEN ≡ RH] (Ch. 30: open and equivalent to RH); Open problem and Question environments; "Open questions" sections
Numerical	Floating-point or multiprecision computation; not a proof.	[NUM]; and the placeholder of Appendix B

Local variants. In Chapter 38 the tag [OBSERVATION] marks an observation (a reading of proved facts), not an open problem. In Chapter 36 [PARTIAL] means "unconditional but partial" (class Theorem); in Chapter 43 [PRIOR ART] means "known in the literature, re-derived here" (class Theorem, no novelty claimed). A statement with two tags (e.g. [THM]+[NUM]) is a theorem whose numerical illustration is not part of the proof. No statement of this volume is interval-certified: the high-precision ball-arithmetic computations of Chapters 34 and 37 give *Ritz-sound* bounds whose matrix entries carry no rigorous error bound, and they are labelled [NUM] by their own notes.

Part 1. What we have proven

1a. Theorems with complete proofs in this volume. Where a note says that one analytic step is only sketched, that is repeated here.

30 Thm. 30.2.2 (spectral Albert step: a trace with positive weights on a finite spectrum with involution ι has signature $(\#\mathrm{Fix}\,\iota + p,\ p)$, so positivity forces $\iota = \mathrm{id}$); Prop. 30.3.2 ($\delta_\lambda * \delta_\lambda^\sharp = \lambda$, the exact analogue of $FF^\dagger = q$); Prop. 30.3.1 (on characters $\sharp$ induces the mirror $\rho \mapsto 1 - \bar{\rho}$); Thm. 30.6.2 (G1–G3 and G5 imply RH; this is the logic, G5 itself is open). Thm. 30.3.3 is [Restatement].

31 Prop. 31.2.2 (dissipation is line-blind); Lemma 31.3.2 ($\langle \xi, J\xi \rangle \geq 0$ is false; the correct natural-cone forms); Prop. 31.3.3 (no modular G5 on a commutative algebra); Prop. 31.6.3; Thm. 31.6.5 ($\xi(1+it)$ is a KMS_1 two-point function). Thm. 31.3.7 is [Restatement].

32 Thm. 32.5.1 ($\mathcal{Q}$ is exactly translation invariant, so half-line positivity is full Weil positivity); Thm. 32.5.4 (Weil positivity is a domination of the OS form, not reflection positivity); Thm. 32.5.5; Prop. 32.6.1; Thm. 32.7.2 (the Euclidean kernel of ζ; item (v) [Restatement]); Cor. 32.7.4.

33 Thm. 33.2.1 (the pole of ζ is a gain Lorentzian); Thm. 33.3.2; Thm. 33.4.2, Thm. 33.5.1, Thm. 33.6.2 (the de Branges threshold is $\Theta - \frac{1}{2}$), Thm. 33.7.1, all unconditional. Causality is free; passivity is RH.

34 Thm. 34.3.4; Prop. 34.3.5 (the prime-side kernel of Suzuki's line); Thm. 34.5.1; Thm. 34.6.2 (negative index $= \nu N_+(\frac{1}{2} + \omega)$; one analytic step sketched); Thm. 34.6.4 (contour shift in (i) sketched; (M) removed by Ch. 36); Prop. 34.8.1.

35 Thm. 35.2.1 (exact zero-free formula for $(-1)^n S_+^{(n)}$); Thm. 35.4.1 (no sign check below order $4.7 \cdot 10^{12}$ can fail); Prop. 35.4.4; Prop. 35.6.1 (complete monotonicity is domination; last clause [Restatement]).

36 Thm. 36.4.3 (absorption–phase sum rule); Thm. 36.5.4; Thm. 36.7.1, Cor. 36.7.2; Prop. 36.7.4 (where the "+1" of $N(T)$ comes from); Prop. 36.7.7; Thm. 36.7.9 (exact escape count, removing hypothesis (M) of Ch. 34); Thm. 36.7.15.

37 Thm. 37.2.2 (passivity criterion: $\sup_t \|\mathcal{K}[t]\| = \|\Theta\|_{H^\infty(\mathbb{C}_+)}$); Cors. 37.2.3, 37.2.4 (the uniform strict form $\sup_t < 1$ is false); Thm. 37.3.2; Cor. 37.3.3 (official G5 line $\iff$ RH; logic proved, content [Restatement]); Thm. 37.4.1 (exponential line); Props. 37.7.1 (no Euler factor is Schur), 37.7.3; Thm. 37.7.4; Thm. 37.10.1 (each cascade margin depends on the next prime only).

38 Thm. 38.2.1 (one global rapidity; Galois and Iwasawa sit in the compact factor); Prop. 38.2.4; Prop. 38.2.5; Thm. 38.3.1; Prop. 38.5.3 (BC time is dual to rapidity); Prop. 38.8.3.

39 Thm. 39.2.1; Prop. 39.3.2 (BC is a Galois-twisted product over primes); Prop. 39.3.5; Thm. 39.4.2; Thm. 39.7.3; Thm. 39.7.5 (no half-sided structure at any finite Galois level).

40 Thm. 40.2.1 (the Hodge split is a fundamental decomposition of a Krein space); Thm. 40.4.1; Cor. 40.4.3; Thm. 40.5.2 (Pick kernel = zero-space Krein form); Thm. 40.6.3; Thm. 40.6.8 (separation criterion for the two stages); Prop. 40.6.10 (scope as in Ch. 42, E3); Thm. 40.6.11.

41 Thm. 41.2.1 (canonical coincidence of the two stages); Thm. 41.2.3; Thm. 41.4.1; Cor. 41.4.2; Thm. 41.6.1; Thm. 41.6.2.

42 Thm. 42.2.1 (free positivity is Hankel/OS); Thm. 42.2.2 and Cor. 42.3.2 (the half-sided-inclusion lead fails, unconditionally); Thm. 42.3.1; Thm. 42.4.4; Thm. 42.6.3; Thm. 42.7.2 (Hilbert versus Krein for $\mathrm{SL}_2(\mathbb{R})$); Prop. 42.7.5; Thm. 42.8.2 (Newton above Hodge in genus 1 is integrality); Thm. 42.8.5.

43 Thm. 43.2.4; Thm. 43.4.1 (the Riccati generator of an elliptic curve over $\mathbb{F}_q$ is the transpose of Ch. 42's quarter turn); Thm. 43.4.4 (easy half); Thm. 43.5.1. Thm. 43.2.1 is prior art (Broughan), re-derived.

44 Thm. 44.1.1 and Cor. 44.2.3 (the Gaussian-beam form is Ch. 42's polarization); Thm. 44.2.2; Thm. 44.2.4 (rational Gouy phase $\iff$ supersingular); Thm. 44.4.1; Thm. 44.5.4.

45 Prop. 45.1.1; Thm. 45.2.1; Thm. 45.2.2 (two second quantizations; given Weil's theorem); Thm. 45.4.1; Thm. 45.6.3; Thm. 45.6.4 (a ghost pair is a negative-norm two-fermion state for every invariant metric; a reformulation); Cor. 45.6.5.

46 Thm. 46.1.3; Thm. 46.1.4 (index $1+k$ of the abstract Néron–Severi lattice of any reciprocal polynomial); Prop. 46.1.6; Thm. 46.6.1 ($FV = q$ alone gives only Krein unitarity); Prop. 46.6.3 (V realizes the functional equation; RH clause [Restatement]); Thm. 46.6.4; Prop. 46.6.6; Prop. 46.6.9.

47 Thm. 47.1.2 (minimal unitary dilation of the BC isometries); Cor. 47.1.4; Prop. 47.1.5; Thm. 47.2.3 (no global Dirac reflection for two or more primes); Thm. 47.2.7; Thm. 47.3.1 and Cor. 47.3.4 (Toeplitz $\cap$ Hankel is the fermionic cube, which excludes Weil's form); Thm. 47.4.4(a). Prop. 47.5.2 is [Restatement].

1b. Results resting on cited theorems (complete given the citations).

30 Prop. 30.2.1 (the four ingredients of the curve proof, after Milne).

33 Thm. 33.4.1 (Lagarias) and Cor. 33.4.3.

34 Thm. 34.3.2 (Suzuki's Hamiltonian).

38 Thm. 38.5.1 (Bost–Connes); Thms. 38.4.2, 38.5.4, 38.5.6.

39 Thm. 39.3.1; Thm. 39.4.1 (part (b) proved here); Thm. 39.7.2 (Borchers, Wiesbrock).

40 Prop. 40.2.3; Thm. 40.3.1 (Weil 1948, after Milne); Thm. 40.3.5.

46 Thm. 46.1.1; Thm. 46.3.1 (Deligne's Main Lemma); Thm. 46.6.5 (Connes–Consani, $\sigma_n\tilde{\rho}_n = n$); Thm. 46.6.7 (Borger: only Artin–Tate motives descend to $\mathbb{F}_1$).

47 Prop. 47.1.1; Thm. 47.2.5 (boson–fermion correspondence); Thm. 47.4.2 (standard; extra inputs isolated).

1c. Computer-assisted, certified. None. Every computation in this volume is [NUM]. The most careful ones (Arb Galerkin and ball-arithmetic cascades in Chs. 34 and 37; exact-arithmetic inertia checks of Néron–Severi Grams in Ch. 46) are reported with their error budgets in the chapters, but they are not counted as theorems here.

Part 2. What we suspect to be true

Each item is labelled and its evidence stated. None is used in any proof of Part 1.

(S1) **Cascade margins tend to** 0 [Conjecture]. At fixed admissible (ω, ν) with $\omega \le \frac{1}{2}$, the margins $t_*(P) - \frac{1}{2}\log p^+(P)$ of the Euler-truncated lines have infimum 0 over P. Under RH they stay positive for every P; for $\omega \ge \frac{1}{2}$ positivity is unconditional (Thm. 37.7.4). This is the conjecture of Ch. 37, §37.7. *Evidence:* margins decrease smoothly in p^+ along the whole sequence to $P = 127$, with no gap dependence (Thm. 37.10.1 and [NUM]). *Not decided:* whether the decay slows beyond $p^+ \approx 70$. At $P = 97$ the $r36p14$ margin is 7.3% below the $r30p12$ value, so $r30$ is not converged there (Ch. 37, v3). The deciding run is placeholder P1 (Appendix B).

(S2) **Onset cusps of Suzuki's line** [Conjecture]. Every integer n produces a cusp $\frac{64\pi}{3} b_n (t - t_n)_+^{3/2}$ in $\log Z_0$ (Conj. 34.8.2). *Evidence:* the first-order trace law (Prop. 34.8.1) and the high-precision numerics of Ch. 34, §34.8. The conjecture revises Conj. 34.5.2.

(S3) **One passive line per Galois sector** [Conjecture; RH-content Restatement]. Lemma 37's construction carries over to $\Lambda(s, \chi)$ sector by sector, and "all sector lines passive" is GRH for χ (Conj. 38.8.5). *Evidence:* unimodularity of Θ_χ on $\mathbb{R}$ is proved; the rest is the $\chi = 1$ case.

(S4) **An archimedean "main conjecture" shape** [Conjecture, Deninger's]. A $C_{\mathbb{Q}}$-equivariant H^1 with sectors H^1_χ whose regularized determinants are the completed L-functions (Conj. 38.4.4). *Evidence:* the Iwasawa main conjecture (Mazur–Wiles) on the p-adic side; consistency with Connes' spectral realization (Thm. 38.4.2).

(S5) **Finite-field Broughan criterion** [Conjecture]. Off-circle source/sink pairs of the $\sharp$-reversible field over $\mathbb{F}_q$ are detected by a pole separatrix (Ch. 43, the conjecture after Thm. 43.4.4). *Evidence:* the easy half (Thm. 43.4.4) and the numerics of Ch. 43.

(S6) **The window separation of the two stages is a regularization effect** [Numerical]. Birth-point data predict the collision threshold by the cube-root law $\delta_c \simeq (\kappa_v/2\beta_v)^{1/3}$ to within 0.1 decades in every converged window (Ch. 41). *Evidence:* Lemma 27's window pencils; the theorem holds under the stated hypotheses.

(S7) **The positive H^1 for ζ, if it exists, is a cokernel that carries a polarization** [Open]. For curves, the positive H^1 is recovered from divisors through dilation, the quotient by principal divisors, and the theta polarization (Thm. 47.4.2); for $\mathbb{Q}$ the quotient collapses (Thm. 47.4.4), which is where Connes' cokernel takes over. Lemma 46's Open Problem 46.6.8 asks for such an H^1. Caution: such a structure exists iff RH, so its existence is not evidence. Only a construction independent of the zeros would be.

(S8) **Analogies kept as analogies** [Analogy]. The thermal/Unruh, Kepler/Kustaanheimo–Stiefel and Doppler readings (Ch. 31); the network and Foster readings (Ch. 33); the optical cavity (Ch. 44); Zeeman and spin (Ch. 45); the Dirac sea (Ch. 47).

Part 3. Our final goal

The goal of the TAA programme is unchanged from Volumes I and II: to *derive* RH as a unitarity (ghost-freedom) theorem. In the language of this volume the target is axiom G5 of Chapter 30: a positivity of the zero-side trace $\mathrm{Tr}_{\mathcal{Z}}(f * f^\sharp) \geq 0$ that holds for a geometric reason, built without reference to the zeros (G6). Its official form is the passive transmission line of Chapter 37: $\|\mathcal{K}^{\omega,\nu}[t]\| < 1$ for every t, from the Euler product and the Γ-factor alone. Its structural form is a *positive polarization*. Chapters 40, 42, 46 and 47 show that the involution, the functional equation, Krein unitarity and even unitary dilations are available for free; what is missing is a positive form of Hodge–Riemann (Rosati) type on an arithmetic H^1 for which the Verschiebung is the adjoint of Frobenius.

Status. RH is **not** proven here. Every chapter does one of three things. It *restates* G5 in a sharper form (Chs. 30, 34, 37, 40, 46). It *proves that a candidate source fails*, unconditionally (Chs. 31, 32, 33, 35, 38, 39, 42, 43, 44, 45, 47). Or it *proves a structural theorem that holds equally for fakes* (Chs. 36, 41, 46), and so cannot distinguish ζ from a function with off-line zeros.

Place in the series

The series is organized in volumes of about 10–20 lemmas, each titled by its overarching thread. Lemmas 14–47 are split into two volumes, with the break after Lemma 29. This volume is the second of the two; it has 18 chapters.

Volume	Title	Contents and status
I	Arithmetic Unitarity I: Where the Riemann Hypothesis Lives in the Scaling Flow	Lemmas 1–13; ai.viXra:2610.0045v2
II	Arithmetic Unitarity II: The Light Cone of the Weil Form	Lemmas 14–29 (submitted to ai.viXra, October 2026): what the Weil form can *see*
III	Arithmetic Unitarity III: In Search of Positivity — G5, Polarization, and Hilbert versus Krein	Lemmas 30–47 (this volume): which structure could *supply* positivity

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Introduction

The thread of this volume. Volumes I and II located the Riemann Hypothesis in one place: the positivity of the Weil form $\mathcal{Q}$, that is, the absence of a negative-norm (ghost) direction of the scaling flow. They also measured that obstruction from many sides. This volume asks where such a positivity could *come from*. Its title is the question.

The axioms (Ch. 30). For a curve $C/\mathbb{F}_q$, Weil's proof uses four ingredients (Prop. 30.2.1). Chapter 30 lists them as axioms G1–G6 for $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$: algebra, Frobenius relation, Lefschetz, duality, geometric positivity, and non-circularity. The convolution algebra $C_c^\infty(\mathbb{R}_+^\times)$ with $f^\sharp(x) = x^{-1}\overline{f(1/x)}$ satisfies G1–G4 canonically, with $\delta_\lambda * \delta_\lambda^\sharp = \lambda$ in place of $FF^\dagger = q$ (Prop. 30.3.2). Albert's step carries over verbatim (Thm. 30.2.2), and G1–G3 together with G5 imply RH (Thm. 30.6.2). So the whole problem is G5, a positivity of the zero-side trace for a geometric reason, subject to G6 (built without the zeros).

The G5 line (Chs. 34, 37). Chapter 34 reconstructs the de Branges/Krein line of ξ from the prime side (Suzuki's Hamiltonian) and computes its negative index, $\nu N_+(\frac{1}{2}+\omega)$ (Thm. 34.6.2). Chapter 37 makes G5 *official*: a passive transmission line, $\|\mathcal{K}^{\omega,\nu}[t]\| < 1$ for every t, built from the Euler product and Γ alone. It shows this is equivalent to RH (Cor. 37.3.3) and that its uniform strict form is false (Cor. 37.2.4). Its exponential form is $\Theta_c = \exp(-2c\,\xi'/\xi)$ (Thm. 37.4.1). It reduces G5 to a countable list of finite-prime statements: each truncated line must survive until the next prime arrives (Thms. 37.7.4, 37.10.1).

Thermal and causal candidates (Chs. 31–33, 35, 36). Modular theory, KMS states, Bisognano–Wichmann, Kepler/Kustaanheimo–Stiefel and Doppler pictures each give *some* positivity, and $\xi(1+it)$ is a genuine KMS_1 two-point function (Thm. 31.6.5). None of them is G5 (Ch. 31). Chapter 32 shows that Weil positivity is not reflection positivity but a *domination* of the Osterwalder–Schrader form (Thm. 32.5.4). Chapter 33 shows that causality (Kramers–Kronig) is free and passivity is RH. Chapter 35 shows that a Post–Widder sign check from the prime side is blind below order $4.7\cdot 10^{12}$ (Thm. 35.4.1). Chapter 36 is the ledger: every positivity available at the obstruction, what each supplies, and the one missing statement.

Galois (Chs. 38, 39). The global rapidity and the Iwasawa parameter live in one idele class group, in different factors (Thm. 38.2.1). Bost–Connes time is dual to rapidity, and no finite Galois level carries a half-sided structure (Thm. 39.7.5).

The polarization diamond and Krein unitarity (Chs. 40–42). On a curve the Hodge split is the fundamental decomposition of a Krein space, and positivity is a Galois twist of it by c_∞ (Thm. 40.2.1). The Pick kernel of ξ'/ξ is the zero-space Krein form (Thm. 40.5.2). Ghost birth has two stages, degeneration and collision, whose separation is analysed in Chapter 41. Chapter 42 tries the last rung, half-sided modular inclusions. Their free positivity is of Hankel type (Thm. 42.2.1) and never equals Weil's form (Thm. 42.2.2). For $\mathrm{SL}_2(\mathbb{R})$, Krein invariance is automatic while Hilbert invariance needs a polarization (Thm. 42.7.2).

Flows, cavities, Fock spaces (Chs. 43–45). The same dichotomy reappears in three physical dressings. Reversible holomorphic flows: the Riccati generator of an elliptic curve is the transpose of Chapter 42's quarter turn (Thm. 43.4.1). Optical cavities: the Gaussian-beam form is the polarization (Thm. 44.1.1). Boson/fermion second quantizations: a ghost pair is a negative-norm two-fermion state for every invariant metric (Thm. 45.6.4).

Hilbert versus Krein, the Verschiebung, and reflection (Chs. 46, 47). Over the whole tower of curves, an abstract Néron–Severi lattice exists for *any* reciprocal polynomial, with positive index $1+k$, k the number of ghost pairs (Thm. 46.1.4). The Verschiebung $V = qF^{-1}$ realizes the functional equation for real and fake polynomials alike (Prop. 46.6.3). Chapter 47 reflects $\mathbb{N}$ into $\mathbb{Z}$ (minimal unitary dilation, Dirac sea, boson–fermion correspondence) and finds that reflection produces Hankel forms, whose overlap with Toeplitz forms is the fermionic cube, which excludes Weil's kernel (Thm. 47.3.1, Cor. 47.3.4).

The standing theorem. One statement is proved three times, in three settings, and is the main lesson of the volume:

A reflection gives the functional equation. Positivity needs a polarization.

- **Ch. 42 (operator algebras, SL_2).** The positivity that a half-sided inclusion supplies for free is a Hankel (Osterwalder–Schrader) form, Thm. 42.2.1, and no such form is Weil's, Thm. 42.2.2. For $g \in \mathrm{SL}_2(\mathbb{R})$, $g \neq \pm 1$, the Krein form H is always invariant. A positive-definite invariant form exists in the elliptic case, the polarization b_g built from J_g; in the hyperbolic case every invariant Hermitian form is indefinite or zero (Thm. 42.7.2).
- **Ch. 46 (curves, abstractly).** For any monic reciprocal P, $F = T$ and $V = qT^{-1}$ satisfy $FV = q$; $T \mapsto q/T$ is an involution, and the trace form makes $F/\sqrt{q}$ an isometry of signature $(D_{\mathrm{on}} + k, k)$: Krein, not Hilbert, unitarity, Thm. 46.6.1. V realizes the functional equation; RH is the statement that V acts as c_∞ on the spectrum, Prop. 46.6.3.
- **Ch. 47 ($\mathbb{Q}_p$ and $\mathbb{F}_q$).** Every form obtained by reflections and compressions from an invariant form is Toeplitz or Hankel, and the two classes overlap only on square-periodic kernels, Thm. 47.3.1. For curves, the global reflection $D \mapsto K - D$ gives the functional equation through Riemann–Roch, while positivity comes from the theta polarization, Thm. 47.4.2. For $\mathbb{Q}$ the quotient step collapses, Thm. 47.4.4.

In the language of Chapter 30, the reflection is G4 (duality, automatic) and the polarization is G5. The theorem explains why every free construction tested in this volume reaches the functional equation and stops there.

Vol. 3 editorial remark: transposition and precession. [Vol. 3 editorial remark.] *This is a dictionary proposed by the author on 22 October 2026, written out for this volume. It is not part of any chapter. Each line is a theorem (elementary) or a restatement, checkable from the cited results.*

Let α, β be a pair of Frobenius eigenvalues of a curve over $\mathbb{F}_q$ paired by the functional equation, $\alpha\beta = q$, i.e. eigenvalues of F and of $V = qF^{-1}$ on one eigenvector (Prop. 46.6.3).

(i) *Log-modulus coordinates.* Put $(u, v) = (\log_q|\alpha|, \log_q|\beta|)$. Then $FV = q$ forces $u + v = 1$: every pair lies on the anti-diagonal. [Theorem]

(ii) *Transposition.* Exchanging F and V (the Rosati transpose $F^\dagger = V$, realized by $\sharp_0$ in Thm. 46.6.1) is reflection across $u = v$. Its only fixed point on the anti-diagonal is $(\frac{1}{2}, \frac{1}{2})$. RH for the curve $\iff$ every pair sits at $(\frac{1}{2}, \frac{1}{2})$. [Restatement; Prop. 46.6.3(c), Prop. 47.5.2]

(iii) *Phase fibre.* Over $(\frac{1}{2}, \frac{1}{2})$ the fibre is the circle $|\alpha| = \sqrt{q}$. This is the amoeba (phase-tropical) reading of $\alpha\beta = q$: the log-modulus map forgets the phases. [Theorem]

(iv) *Counter-rotation.* On the circle, $\alpha = \sqrt{q}\, e^{i\theta}$ and $\beta = \sqrt{q}\, e^{-i\theta} = \bar{\alpha}$: F and V rotate in opposite senses. The product $\alpha\beta = q$ *collapses* the phase (magnitude only). The sum $\alpha + \beta = 2\sqrt{q}\cos\theta = a_p$ *keeps* it, as a standing wave. [Theorem]

(v) *Ghosts.* An off-circle pair is $\alpha = \sqrt{q}\, e^{\eta + i\theta}$, $\beta = \sqrt{q}\, e^{-\eta - i\theta}$ with $\eta \neq 0$ (a boost). The product is still q, so *the product is blind to ghosts.* They show only in the sum, $\alpha + \beta = 2\sqrt{q}\cosh(\eta + i\theta)$. For $\alpha + \beta$ to be real with $\eta \neq 0$ one needs $\theta \in \{0, \pi\}$; in genus 1 this gives $|a| = 2\sqrt{q}\cosh\eta > 2\sqrt{q}$ at that prime, which violates Hasse. [Theorem]

(vi) *The p-adic* $(\frac{1}{2},\frac{1}{2})$ *is a different point.* Supersingularity (Newton slopes $\frac{1}{2},\frac{1}{2}$) is a p-adic statement and does not force $|\alpha|=\sqrt{q}$. The fake $a=13$, $p=13$ has Dieudonné slopes $\frac{1}{2},\frac{1}{2}$, yet its roots $(13\pm\sqrt{117})/2$ are real and off the circle (Prop. 46.6.9(b)). [Theorem]

(vii) *Stage and players.* The reflection (the functional equation) builds the *stage*: the half-turn $\rho\mapsto 1-\rho$ and the line $\operatorname{Re}\rho=\frac{1}{2}$ that it fixes together with conjugation (Prop. 47.5.2). The *players* are the zeros. In the explicit formula each zero $\rho=\beta+i\gamma$ contributes a phasor $x^{\rho}=x^{1/2}\,e^{(\beta-\frac{1}{2})\log x}\,e^{i\gamma\log x}$, a rotation $e^{i\gamma\log x}$ times a boost $e^{(\beta-\frac{1}{2})\log x}$. RH $\iff$ every player is a pure rotation ($\beta=\frac{1}{2}$). [Restatement]

The remark adds no claim to the chapters. It reorganizes Prop. 46.6.3, Thm. 46.6.1, Prop. 46.6.9 and Prop. 47.5.2 into one picture of the standing theorem: the transpose (reflection) fixes the anti-diagonal and its midpoint, but only a positive form pins the pair to the phase circle.

Vol. 3 editorial addition: spinors and the polarization (from Lemma 50). [Vol. 3 editorial addition, from Lemma 50.] *Condensed from TAA Lemma 50 (22 October 2026), which is not one of Chapters 30–47. Claim labels are those of Lemma 50, translated to the classes of the legend (p. iii). Nothing here is new beyond Lemma 50, and nothing here brings RH closer.* Following the author's preference, the $\frac{1}{2}$ is presented through spinors where that is literal; where it is not, a footnote says so.

(i) *Theta characteristics are spin structures.* On a closed Riemann surface Σ of genus g a spin structure is a theta characteristic, a line bundle L with $L^{\otimes 2}\cong K$ [**16**]. Then $\deg L=g-1=-\chi/2$, and the theta characteristics form a torsor under $H^1(\Sigma;\mathbb{Z}/2)\cong(\mathbb{Z}/2)^{2g}$, so there are exactly 2^{2g}. The parity $h^0(L)\bmod 2$ is deformation invariant, with $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd ones [**16, 203**]. The spinor bundle is $K^{1/2}\oplus K^{-1/2}$, of total degree 0; on S^2 it is $\mathcal{O}(-1)\oplus\mathcal{O}(1)$, and only the half $K^{-1/2}$ has degree $\chi/2=1$. [Theorem; counts [BIB] [**16, 203**], checked [NUM] via Arf invariants for $g\le 4$]

(ii) *Spinor holonomy is half the angle.* Around a loop bounding a disc D, Levi-Civita transport rotates vectors by $\int_D K\,dA$ (local Gauss–Bonnet); on $K^{\mp 1/2}$ the holonomy is $e^{\pm\frac{i}{2}\int_D K\,dA}$. On a loop bounding nothing, the spinor holonomy is *a* square root of the vector holonomy, and the choice of signs is the spin structure. On S^2 this is Berry's spin-$\frac{1}{2}$ phase $-\frac{1}{2}\Omega$ [**31**]. [Theorem; Berry [BIB], [NUM]]

(iii) *Forms* $=S\otimes S^*$*, but Gauss–Bonnet does not need spinors.* $\mathrm{Cl}(T^*M)\otimes\mathbb{C}\cong\Lambda^\bullet T^*M\otimes\mathbb{C}$ and $d+d^*$ is its Dirac operator; on a spin manifold $\mathrm{Cl}\otimes\mathbb{C}\cong\operatorname{End}S=S\otimes S^*$, and χ, σ, $\hat{A}$ are indices of one Dirac-type operator with different gradings and twists [**14, 173**]. The statements for χ and σ hold whether or not M is spin ($\mathbb{CP}^2$ is not), so Gauss–Bonnet and the signature theorem do not depend on spinors; and in dimension 4, χ even does not imply spin. [Theorem ([BIB])]

(iv) $\Lambda^\bullet H^1=H^*(\operatorname{Jac}C)$ *is the spinor module.* For a curve C of genus g, $H^*(\operatorname{Jac}C)\cong\Lambda^\bullet H^1$ is the irreducible module, of dimension 2^{2g}, of $\mathrm{Cl}(H^1\oplus H^{1\vee})$ with its split symmetric form. The symplectic cup product alone gives a *Weyl* (bosonic) algebra; a Clifford algebra needs a symmetric form, built from ω and a Lagrangian L, whose module is $\Lambda^\bullet L$ (2^g-dimensional), with $\Lambda^\bullet H^1\cong S_L\otimes S_L^*$. Over $\mathbb{C}$, with $J=*$ the Hodge star, the spinor module of $\mathrm{Cl}(H^1_{\mathbb{R}},g_J)$ is $\Lambda^\bullet H^{0,1}\cong\bigoplus_q H^{0,q}(\operatorname{Jac}C)$, and its Hilbert structure is induced by the polarization: *a fermionic vacuum on H^1 is a choice of compatible J, and it is a Hilbert vacuum exactly when J is a polarization.* [Theorem]

(v) *RH for curves as a vacuum.* For a reciprocal P with semisimple normalized Frobenius $\gamma=F/\sqrt{q}$ on $V=\mathbb{R}[T]/(P)$ with its invariant symplectic form: every root has $|\alpha|=\sqrt{q}$ $\iff$ there is a Frobenius-invariant *positive* compatible J, i.e. a Frobenius-invariant positive fermionic vacuum on $\Lambda^\bullet(V,J)$ $\iff$ γ fixes a point of Siegel space. With distinct roots, on the line there are 2^g invariant J and exactly one is positive, selected by the Krein signs. A real ghost admits no invariant J at all, and complex quadruples admit only indefinite ones. This is not

a new proof: over $\mathbb{F}_q$ Weil's positivity comes from the Rosati involution, and the vacuum is a consequence. [Restatement (equivalent to Weil's RH for curves); proof Theorem] For the genus-2 curve $y^2 = x^5 + 3x^3 + x + 7$ over $\mathbb{F}_{11}$ the four invariant J have signatures $(4,0),(2,2),(2,2),(0,4)$. [[NUM]]

(vi) *Compact versus split, not spin, separates Weil from fake.* After the half twist (a choice of $\sqrt{q}$ over $\mathbb{F}_q$, a $\mathbb{Z}/2$-torsor; over $\mathbb{Q}$ no half twist exists), the normalized Frobenius of an elliptic curve is $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$ in $SU(2) = \mathrm{Spin}(3)$: $H^1(\frac{1}{2})$ is the spin-$\frac{1}{2}$ representation[1], and $\mathrm{Sym}^2 H^1(1)$ is the $SO(3)$ vector, rotated by 2θ; flipping $\sqrt{q} \mapsto -\sqrt{q}$ is the kernel of $SU(2) \to SO(3)$ [Theorem; Sato–Tate [CITED] [24]]. A ghost Frobenius is hyperbolic in the split form $SL_2(\mathbb{R}) = \mathrm{Spin}(2,1)_0$, which is *also* a spin group, with the same kernel $\{\pm 1\}$ and the same $\sqrt{q}$ sign. So the double cover, the half twist and the "720°" structure are present for Weil curves and fakes alike; what separates them is the compact versus split real form, i.e. whether the invariant Hermitian form on $H^1(\frac{1}{2})$ is definite. [Theorem] Calling Frobenius a "rotation" with a 720° return is [Analogy]; and H^1 is literally a spinor only for $g \leq 2$ ($\mathrm{USp}(4) \cong \mathrm{Spin}(5)$).

(vii) *Spinors do not create positivity.* A Fock space over a Krein space of signature (n_+, n_-), $n_- \geq 1$, is Krein, of total signature $(2^{n-1}, 2^{n-1})$. Second quantization, fermionic or bosonic, transports the one-particle signature and never repairs it; positivity must be put in through J. [Theorem]

(viii) *TAA phrasing (adopted 22 October 2026).* **The missing polarization = the missing invariant positive vacuum.** For curves over $\mathbb{C}$ this is a theorem (the vacuum $J = *$ is positive by Riemann's bilinear relations); for curves over $\mathbb{F}_q$ it is a restatement of Weil's theorem, item (v); for ζ it is an analogy. Its precise form is open: RH would follow from a positive J on a symplectic "$H^1(\overline{\mathrm{Spec}\,\mathbb{Z}})$" commuting with Deninger's flow [91], and given the first two ingredients, the third *is* RH. In infinite dimensions inequivalent vacua also exist (Shale–Stinespring), so a superselection sector must be chosen as well. [Theorem / Restatement / Analogy; open part [OPEN]]

In the language of Chapter 30 the vacuum J is G5. The addition changes no chapter: it rewrites the polarization of Thm. 47.4.2 and the Krein dichotomy of Thm. 46.6.1 in fermionic terms, and it records that the spinor language locates G5 without supplying it.

Chapter by chapter.

Ch. 30, A positive involution for $\mathrm{Spec}\,\mathbb{Z} \times \mathrm{Spec}\,\mathbb{Z}$?: The Rosati step and its ζ-analogue; axioms G1–G6; G5 open, and equivalent to RH.

Ch. 31, Entropy, modular theory, Kepler, Doppler.: Is G5 thermal? Tomita–Takesaki, KMS, Bisognano–Wichmann, Kustaanheimo–Stiefel, Doppler, OS. Each supplies a positivity that is not G5.

Ch. 32, Positivity element by element.: The OS template against Weil's form: translation invariance, the Euclidean (Cramér) kernel, Wick rotation; Weil positivity is domination.

Ch. 33, Dispersion, causality, passivity.: Anomalous dispersion at $x = 0$; Kramers–Kronig, positive-real functions, Lagarias, Foster, de Branges. Causality is free, passivity is RH.

Ch. 34, Inverse spectral reconstruction of the ξ-line.: Suzuki's Hamiltonian from the prime side, its numerical inverse, the arithmetic signature, the Pontryagin case (v2: high precision).

Ch. 35, Post–Widder sign check from the prime side.: An exact zero-free formula for $(-1)^n S_+^{(n)}$ and a blindness theorem.

Ch. 36, The positivity ledger.: Every positivity at the obstruction, what each supplies, and the one missing statement (v2: Sommerfeld integrality).

[1] **[Vol. 3 editorial note.]** This $\frac{1}{2}$ is spinorial in the literal sense. The $\frac{1}{2}$ of Riemann's proof is different: θ has weight $\frac{1}{2}$ because it lives on the metaplectic double cover $\mathrm{Mp}_2(\mathbb{R})$, and that $\frac{1}{2}$ is the zero-point weight of the *bosonic* oscillator vacuum (Mehler), on the Weyl-algebra side of the Weyl/Clifford parallel [204, 283, 241]; it is not a spin-$\frac{1}{2}$ (Lemma 50 §3, Prop. 3.4–3.5 and Rem. 3.6). Moreover $SL_2(\mathbb{R})$ is itself $\mathrm{Spin}(2,1)_0$, so Mp_2 is a cover *of* a spin group.

Ch. 37, G5 as a passive line (v3).: The official target: equivalences, the exponential line, the dictionary, proof strategies, the finite Euler cascade.

Ch. 38, Rapidity and the Iwasawa parameter.: One idele class group; Galois symmetry, Bisognano–Wichmann rapidity, Bost–Connes; where the dictionary breaks.

Ch. 39, The prime clock.: Scaling time, the Bost–Connes Galois tower, relative entropy, causal polarization.

Ch. 40, Polarization as Galois-twisted positivity.: Hodge–Riemann, Rosati, Krein unitarity; one theorem behind Weil's proof, the L_4/Routh kernel and the ζ ghost.

Ch. 41, Do the two stages of ghost birth coincide?: Degeneration, collision, and the detuning that separates them.

Ch. 42, The missing rung.: Half-sided modular inclusions, the orthogonality pincer, the hyperbolic dictionary, Hilbert versus Krein for SL_2, Taylor–Wiles at the nexus (v2).

Ch. 43, Reversible holomorphic flows.: $\dot{w} = i\Xi(w)$: Fiedler, Broughan's prior art, $\sharp$-reversible fields over $\mathbb{F}_q$, Nash centers.

Ch. 44, Arithmetic cavities and apertures.: Ray-transfer matrices, Gouy phases, apertures, the Planck gas of Frobenius modes.

Ch. 45, Bosons, fermions, spin and ghosts.: Two second quantizations of ζ, the supersymmetric primon gas, $SU(2)$ multiplets of Frobenius, the Fock-metric failure of an off-line zero.

Ch. 46, What forces Frobenius to be unitary.: Weil, Deligne and Katz–Sarnak over the tower; Hilbert versus Krein; the Verschiebung.

Ch. 47, Reflecting $\mathbb{N}$ into $\mathbb{Z}$.: Dilation of the BC/Verschiebung isometries, the per-prime Dirac sea, boson–fermion, Toeplitz versus Hankel, and the finite-field test.

Conventions of this volume. Chapter k is the note "TAA Lemma k", and its statement $s.n$ is cited as $k.s.n$ in this volume (e.g. Thm. 46.6.1). Inside a chapter, "Lemma m" refers to the note m: Chapter m of this volume if $30 \leq m \leq 47$, otherwise Volume I ($m \leq 13$) or Volume II ($14 \leq m \leq 29$). Notation follows each chapter. Where two chapters use one symbol differently, Appendix C lists both. Each chapter keeps its own bibliography keys, merged into one verified list at the end of the volume (canonical keys; see `bib_verification.md`).

CHAPTER 30

Lemma 30: Is the right target a positive involution

on a commutative algebra attached to $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$*? The Rosati step, its* ζ*-analogue, and a list of axioms*

Standalone note `lemma30.tex`, October 21, 2026.

Summary. *Tags.* [THM] proved here; [CITED] checked at the source (`lit/lemma30/verified_refs.md`); [DEF] definition; [ANALOGY] analogy only; [NUM] float64 numerics (`num/lemma30_checks.py`); [RESTATEMENT] restatement of a known equivalence; [CONJECTURAL] conjectural framework of the cited author; [OPEN ≡ RH] open and equivalent to RH.

Verdict. *Half yes.* The positive involution is the right *shape*, but not the missing *piece*. Of the four ingredients of the curve proof, (i) a commutative algebra containing Frobenius, (ii) an involution with $FF^\dagger = q$, (iii) a trace for which the involution is positive, (iv) the algebraic step "positive $\Rightarrow$ $\dagger$ is complex conjugation on every factor", the ζ-side has (i), (ii) and (iv) *already*, canonically and unconditionally. The algebra is the convolution algebra $\mathcal{C}$ of the scaling group $\mathbb{R}_+^\times$ (or $\mathcal{S}(C_\mathbb{Q})$). The involution is $f^\sharp(x) = x^{-1}\overline{f(1/x)}$, which is the standard group-algebra involution in half-density coordinates. It satisfies $\delta_\lambda * \delta_\lambda^\sharp = \lambda$ exactly as $FF^\dagger = q$ (Prop. 30.3.2). It is already positive for the Haar trace. The algebraic step transfers *exactly* once it is written on the spectrum: positivity of a trace with positive multiplicities forces the involution induced on its support to be the identity, and the induced involution on characters is the mirror $\rho \mapsto 1-\bar\rho = J_{\mathrm{FE}}$ (Thm. 30.2.2, Prop. 30.3.1). For curves this is literally Milne's proof of $|\alpha| = \sqrt{q}$. Everything therefore sits in (iii): positivity of the *specific* trace $\mathrm{Tr}_\mathcal{Z}(g) = \sum_\rho m_\rho \hat g(\rho)$ (the "trace on H^1"). That positivity is Weil's criterion, i.e. RH [RESTATEMENT]. In the curve case (iii) is *not* algebra; it is supplied by geometry, namely an ample class (Milne Thm. I.14.3) or the Hodge index theorem on $C\times C$. So the correct target is *a geometric source for the positivity of* $\mathrm{Tr}_\mathcal{Z}$: an object on which $\mathrm{Tr}_\mathcal{Z}$ is a trace of the form $\mathrm{Tr}(aa^\sharp) = \langle D, D\rangle$ for a pairing whose positivity is proved *without the zeros* (axioms G5–G6, §30.6). The involution is a corollary of that object, not a substitute for it. A purely algebraic "positive involution on a commutative algebra attached to $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$" cannot be the target: in infinite dimensions positivity of an involution is *not intrinsic*; it depends on which trace, and choosing the trace is the whole problem (Rem. 30.2.4).

Candidates. None supplies (iii). Connes' adèle-class trace formula supplies (i), (ii) and the trace with its explicit formula, and its global validity is equivalent to RH (positive characteristic: [**75**, Thm. 5]). Connes–Consani–Marcolli state the Weil form exactly as "$\langle f, f^\sharp\rangle \geq 0$" [**66**, Cor. 6.3] and leave the analogue of principal divisors open. Bost–Connes supplies a $\mathbb{Q}$-structure, Galois symmetry and *positive* KMS states, but they live on another algebra and see $\zeta(\beta)$ only for real $\beta > 1$. The arithmetic and scaling sites supply the square and Frobenius correspondences Ψ_λ; the missing piece, in the authors' words, is H^1 / intersection theory / Riemann–Roch on the square. Deninger conjectures the Hodge-$*$ positivity that would give (iii), and proves the toy model. Λ-rings give $\mathbb{Z}\otimes_{\mathbb{F}_1}\mathbb{Z} = \mathbb{Z}$.

Added sections (J. W.'s follow-ups). §30.7: a "trace-free" reformulation. The scalar part of the loss operator is fixed by the functional equation and RH is "trace-free part

= 0", but every functional-equation-symmetric *linear* test is blind to it. Every equality-type criterion we checked (Balazard–Saias–Yor, Burnol/Nyman), and Li's, hides a non-negativity. The complete coordinate- and scale-free invariant of the form (Sylvester) is its signature. §30.8: the strip is a hyperbolic plane in which $\operatorname{Re} s = \frac{1}{2}$ is a geodesic and the mirror is the reflection; the distance of a zero from the line is $\operatorname{arsinh}|\tan \pi\delta|$. "Hyperbolic" there (Lobachevsky) and in Lemma 27 (split quadratic form) are different notions; we flag the pun.

30.1. Notation and conventions

Zeros: $\rho = \beta + it = \frac{1}{2} + i\gamma_\rho$, $\delta = \beta - \frac{1}{2}$, multiplicity m_ρ, mirror $\rho^* = 1 - \bar\rho$ (NOTATION.md, Lemmas 1, 25, 27). Multiplicative conventions follow Connes–Consani [**71**, eq. (1)]: for $f \in \mathcal{C} := C_c^\infty(\mathbb{R}_+^\times)$,

$$\hat f(s) = \int_0^\infty f(x) x^{s-1}\, dx, \qquad (f * g)(x) = \int_0^\infty f(y) g(x/y)\, \tfrac{dy}{y}, \qquad f^\sharp(x) = x^{-1}\overline{f(x^{-1})}.$$

This $\sharp$ is [**66**, (6.9)] (written there without the bar, for the real case), the $\tilde g$ of [**70**, (12)] (real g), and $\bar g^\sharp$ of [**71**, (2)]. In the half-density coordinate $F(u) = e^{u/2} f(e^u)$ of Lemma 1, $f * g \leftrightarrow F * G$ (additive convolution), $f^\sharp \leftrightarrow F^*(u) = \overline{F(-u)}$, and $\hat f(\rho) = \widetilde F(\gamma_\rho)$, $\widetilde F(z) = \int F e^{izu} du$ [THM] (substitution $x = e^u$). So $\sharp$ is the canonical involution of the group algebra of $(\mathbb{R}, +)$ transported by the algebra automorphism "multiply by the character $x^{1/2}$". The *zero-side trace* is the distribution

$$\operatorname{Tr}_{\mathcal{Z}}(g) := \sum_\rho m_\rho\, \hat g(\rho) \qquad (g \in \mathcal{C}),$$

and Lemma 1's Weil form is $\mathcal{Q}(F, F) = \operatorname{Tr}_{\mathcal{Z}}(f * f^\sharp)$, because $\widehat{f * f^\sharp}(\rho) = \hat f(\rho)\overline{\hat f(1-\bar\rho)}$ (Prop. 30.3.1). The explicit formula [**71**, eq. (1)]

$$\operatorname{Tr}_{\mathcal{Z}}(g) = \hat g(0) + \hat g(1) - \sum_v W_v(g)$$

writes $\operatorname{Tr}_{\mathcal{Z}}$ as "pole terms − prime/archimedean terms". For curves this is $\operatorname{Tr}(F^n \mid H^1) = \operatorname{Tr}(\cdot \mid H^0) + \operatorname{Tr}(\cdot \mid H^2) - \#\mathrm{Fix}$. Sign conventions differ between sources: [**75**] makes the prime side Δ of positive type, and [**70, 71**] write $\sum_v W_v(g * \bar g^\sharp) \le 0$ on test functions vanishing at $0, 1$. We use the zero side, $\operatorname{Tr}_{\mathcal{Z}}(f * f^\sharp) \ge 0$, as in Lemmas 1, 15, 25.

30.2. The curve argument, decomposed

Let $A/\mathbb{F}_q$ be an abelian variety (for a curve C, $A = J$), λ a polarization given by an ample divisor D, $\dagger$ the Rosati involution on $\mathrm{End}^0(A) = \mathrm{End}(A) \otimes \mathbb{Q}$, π the Frobenius endomorphism, and $\operatorname{Tr} = \operatorname{Tr}(\cdot \mid V_\ell A)$.

Proposition 30.2.1 (the four ingredients; [CITED]). *(i)* Commutative algebra containing Frobenius. $K_\pi := \mathbb{Q}[\pi] \subset \mathrm{End}^0(A)$ *is a finite-dimensional commutative* $\mathbb{Q}$*-algebra, stable under* $\dagger$ *because* $\pi^\dagger = q\pi^{-1}$ [**194**, Lemma II.1.2 and proof of II.1.3].

(ii) Involution with $FF^\dagger = q$. $\pi^\dagger \circ \pi = q_A$ [**194**, Lemma II.1.2]. *The proof uses only* $\pi^* D \sim qD$*, a property of Frobenius and the divisor, not of positivity.*

(iii) Positive trace. $(\alpha, \beta) \mapsto \operatorname{Tr}(\alpha \circ \beta^\dagger)$ *is positive definite on* $\mathrm{End}^0(A)$ [**194**, Thm. I.14.3], [**202**, Ch. IV, §21].

(iv) Algebraic step. *If a finite-dimensional commutative* $\mathbb{Q}$*-algebra* K *carries an involution* $\dagger$ *with* $\operatorname{Tr}(aa^\dagger) > 0$ *for* $a \neq 0$*, then* K *is reduced,* $K \otimes \mathbb{R}$ *is a product of copies of* $\mathbb{R}$ *and* $\mathbb{C}$*, and* $\dagger$ *preserves every factor, acting as the identity on real factors and as complex conjugation on complex ones* [**194**, proof of Lemma II.1.3, Rem. II.1.4]. *Equivalently,* K *is a product of totally real fields with* $\dagger = \mathrm{id}$ *and CM fields with* $\dagger =$ *CM conjugation. This is*

the commutative case of Albert's classification of division algebras with positive involution [**3, 4, 2**], [**202**, §21, pp. 193–203] *(section and pages as cited in* [**116**]*; not read by us).*

Then (ii)+(iv) give $|\iota(\pi)|^2 = \iota(\pi\pi^\dagger) = q$ *for every* $\iota : K_\pi \to \mathbb{C}$*, i.e.* $|\alpha| = \sqrt{q}$ *for every root* [**194**, Thm. II.1.1(b)].

Where geometry enters: only in (iii). Steps (i), (ii), (iv) are formal or algebraic. (iii) is proved in [**194**, Thm. I.14.3] from the identity

$$\mathrm{Tr}(\alpha \circ \alpha^\dagger) = \frac{2g}{(D^g)}\,(D^{g-1} \cdot \alpha^* D),$$

plus the fact that D is *ample* and $\alpha^* D$ is *effective*, so the intersection number is positive [CITED]. For a curve one may instead use the surface $C \times C$. By Lemma 27 Prop. 2.1 (from [**193**]), $\mathrm{NS}(X)_{\mathbb{R}} = \langle H, V\rangle \oplus T$ and $D^2 = -\mathrm{Tr}(\alpha_D \alpha_D^\dagger \mid V_\ell J)$ on T. So positivity in (iii) *is* the Hodge index theorem (Castelnuovo–Severi), i.e. negative definiteness of T [**193**, Thm. 1.5]; this is the route of [**126**, V, Ex. 1.10] recommended in [**194**, Ch. III, §11]. In both routes the positive input is a *positivity of intersection numbers of algebraic cycles*, i.e. *integers* attached to an *ample* or *effective* class. That is where the $\mathbb{Q}$-structure (indeed the $\mathbb{Z}$-structure) earns its keep: the trace is an integer-valued intersection number, and its sign comes from effectivity.

The algebraic step on the spectrum.

THEOREM 30.2.2 (spectral form of (iv); [THM]). *Let S be a finite set, $\iota : S \to S$ an involution, $m : S \to (0,\infty)$ with $m \circ \iota = m$. On $K = \mathbb{C}^S$ put $a^\sharp(s) = \overline{a(\iota s)}$ (a conjugate-linear algebra involution) and $\tau(a) = \sum_s m_s a(s)$. Then the Hermitian form $h(a,b) = \tau(ab^\sharp)$ has signature*

$$(n_+, n_-) = (\#\mathrm{Fix}(\iota) + p,\ p), \qquad p = \#\{\text{2-cycles of } \iota\}.$$

In particular $\sharp$ is positive for τ iff $\iota = \mathrm{id}$, iff every character $\chi_s : a \mapsto a(s)$ is a $*$-character ($\chi_s(a^\sharp) = \overline{\chi_s(a)}$). The conclusion does not depend on the weights $m > 0$.

PROOF. $h(a,a) = \sum_s m_s\, a(s)\overline{a(\iota s)}$ splits over ι-orbits. A fixed point contributes $m_s|a(s)|^2$. A 2-cycle $\{s, \iota s\}$ contributes $2m_s\,\mathrm{Re}\big(a(s)\overline{a(\iota s)}\big)$, a hyperbolic plane of signature $(1,1)$ with null coordinate lines. Finally $\chi_s(a^\sharp) = \overline{a(\iota s)}$, which equals $\overline{\chi_s(a)}$ for all a iff $\iota s = s$. □

COROLLARY 30.2.3 (Milne's step on the roots; [THM] from [CITED]). *Let $K_\pi \otimes \mathbb{C} \cong \mathbb{C}^S$, S the distinct roots of the minimal polynomial (when K_π is reduced), with weights m_α = multiplicity of α in the characteristic polynomial on $V_\ell A$. Extend $\dagger$ conjugate-linearly. Then $\mathrm{Tr}(ab^\dagger)$ is the form of Thm. 30.2.2 with*

$$\iota(\alpha) = q/\bar\alpha.$$

So $\iota = \mathrm{id}$ iff $|\alpha|^2 = q$ for all α. The negative index of the trace form equals the number of mirror pairs $\{\alpha, q/\bar\alpha\}$ with $\alpha \neq q/\bar\alpha$.

PROOF. For $a = \sum c_k \pi^k$, $a^\dagger = \sum \bar c_k (q/\pi)^k$ by (ii), so $\chi_\alpha(a^\dagger) = \sum \bar c_k (q/\alpha)^k = \overline{a(q/\bar\alpha)}$. The trace on V_ℓ of a polynomial in π is $\sum_\alpha m_\alpha a(\alpha)$ [**194**, 10.20]. If K_π has nilpotents, positivity already fails [**194**, proof of II.1.3]. □

Check [NUM]. Lemma 19's integer "coin" $X^4 - 22X^2 + 49$ ($q = 7$) has real roots $\pm(3 \pm \sqrt{2})$ and ι swaps $3 + \sqrt{2} \leftrightarrow 3 - \sqrt{2}$ (also for the negatives): two mirror pairs, and the computed trace form has signature $(2,2)$. Over $\mathbb{Q}$, $X^4 - 22X^2 + 49 = (X^2 - 6X + 7)(X^2 + 6X + 7)$, each factor $\mathbb{Q}(\sqrt{2})$ is totally real, and $\dagger$ acts on it by the nontrivial Galois automorphism, which is exactly what (iv) forbids. The genuine $X^2 - 2X + 5$ ($q = 5$) gives $(2,0)$. The fake $X^2 - 6X + 5$ (roots $1, 5$) gives $(1,1)$. The count "negative index = number of mirror pairs" is the finite twin of Lemma 1(vi), $\kappa(\mathcal{Q}) = N_+$.

REMARK 30.2.4 (positivity is intrinsic in finite dimension). By Thm. 30.2.2, for a finite-dimensional commutative semisimple algebra the property "$\sharp$ is positive for *some* faithful trace with positive weights" does not depend on the weights; it is a property of $(K, \sharp)$ alone. This is why "positive involution" is a meaningful *algebraic* notion in Albert's setting. In infinite dimensions it is not intrinsic. For $\mathcal{C}$ with $\sharp$, the Haar trace $\tau_1(g) = g(1)$ gives $\tau_1(f * f^\sharp) = \int_0^\infty |f(x)|^2 dx \geq 0$ (also [**66**, Prop. 6.4(2)]), so $\sharp$ *is* positive for τ_1. Yet the same $\sharp$ is positive for $\mathrm{Tr}_{\mathcal{Z}}$ only under RH. The trace carries the information.

30.3. The ζ-analogue

PROPOSITION 30.3.1 (characters and the mirror; [THM]). *For $s \in \mathbb{C}$ let $\chi_s(f) = \hat{f}(s)$. Then $\chi_s(f * g) = \chi_s(f)\chi_s(g)$ and*

$$\chi_s(f^\sharp) = \overline{\chi_{1-\bar{s}}(f)}.$$

Hence the transpose of $\sharp$ on characters is the mirror $s \mapsto 1 - \bar{s}$, which on $\mathcal{Z}$ is Lemma 25's J_{FE}. χ_s is a $*$-character iff $\mathrm{Re}\, s = \frac{1}{2}$.

PROOF. Multiplicativity is Fubini with $x = yz$. With $y = 1/x$, $\widehat{f^\sharp}(s) = \int_0^\infty \overline{f(y)}\, y^{-s}\, dy = \overline{\int f(y) y^{(1-\bar{s})-1} dy}$. Distinct s give distinct characters: pick f with $\hat{f}(s) \neq 0$ (Lemma 1, auxiliary lemma on nonvanishing); if $\chi_s = \chi_{s'}$ then applying both to the dilates $\delta_\lambda * f$ gives $\lambda^s \hat{f}(s) = \lambda^{s'} \hat{f}(s')$ for all λ, so $s = s'$. Hence $\chi_s \circ \sharp = \overline{\chi_s}$ iff $s = 1 - \bar{s}$. □

PROPOSITION 30.3.2 ((ii) holds tautologically; [THM]). *Extend $\sharp$ and $*$ to compactly supported measures (with $\hat{\mu}(s) = \int x^s\, d\mu$ for $d\mu = f\, d^*x$). For the point mass δ_λ, $\lambda \in \mathbb{R}_+^\times$,*

$$\delta_\lambda^\sharp = \lambda\, \delta_{1/\lambda}, \qquad \delta_\lambda * \delta_\lambda^\sharp = \lambda\, \delta_1.$$

So "scaling by λ" satisfies Milne's relation $FF^\dagger = q$ with $q = \lambda$, for every λ. Its normalization $\lambda^{-1/2}\delta_\lambda$ is $\sharp$-unitary.

PROOF. $\widehat{\delta_\lambda}(s) = \lambda^s$, so by Prop. 30.3.1 $\widehat{\delta_\lambda^\sharp}(s) = \overline{\lambda^{1-\bar{s}}} = \lambda \cdot \lambda^{-s}$. □

THEOREM 30.3.3 (the ζ-version of (iii) is Weil's criterion; [RESTATEMENT]). *The following are equivalent: (a) RH; (b) $\mathrm{Tr}_{\mathcal{Z}}(f * f^\sharp) \geq 0$ for all $f \in \mathcal{C}$; (c) the involution induced by $\sharp$ on the support $\mathcal{Z}$ of $\mathrm{Tr}_{\mathcal{Z}}$ is the identity, i.e. $J_{\mathrm{FE}} = \mathrm{id}$. Moreover the negative index of $f \mapsto \mathrm{Tr}_{\mathcal{Z}}(f * f^\sharp)$ equals N_+, the number of 2-cycles of J_{FE} (distinct zeros with $\beta > \frac{1}{2}$).*

SOURCE. (a)⇔(b) is Weil's criterion [**284, 286**], [**38**], proved in our normalization as Lemma 1(iv)(a)⇔(b), with the index count Lemma 1(vi). (a)⇔(c) is Prop. 30.3.1. In CCM's notation this is exactly [**66**, Prop. 6.2, Cor. 6.3]: "RH for all Grössencharakter L-functions $\iff$ $\langle f, f^\sharp \rangle \geq 0$" for the trace pairing on the cokernel H^1. Connes [**75**, §VIII, Thm. 5] proves, in positive characteristic, that the global trace formula is equivalent to RH, and the introduction of [**75**] states the corresponding reduction for number fields. □

Flag. Theorem 30.3.3 is the infinite-dimensional Theorem 30.2.2 with $S = \mathcal{Z}$, $\iota = J_{\mathrm{FE}}$. The only extra analysis is interpolation: test functions whose transforms isolate finitely many zeros with control of the rest (Lemma 1, proof of (vi)). The "positive involution on $\mathcal{C}/\ker \mathrm{Tr}_{\mathcal{Z}}$" formulation adds nothing to Weil's criterion; it is the same statement.

ingredient	curve / abelian variety	ζ
(i) algebra	$K_\pi=\mathbb{Q}[\pi]$, finite-dim over $\mathbb{Q}$	$\mathcal{C}$ (or $\mathcal{S}(C_{\mathbb{Q}})$), infinite-dim over $\mathbb{C}$; "K_π" $\leftrightarrow \mathcal{C}/\ker(\mathrm{ev}_{\mathcal{Z}})$
(ii) $FF^\dagger=q$	$\pi^\dagger\pi=q$ ([**194**, II.1.2])	$\delta_\lambda * \delta_\lambda^\sharp=\lambda$ (Prop. 30.3.2) [THM]
spectrum, induced involution	roots α, $\alpha\mapsto q/\bar\alpha$	zeros ρ, $\rho\mapsto 1-\bar\rho=J_{\mathrm{FE}}$ [THM]
trace	$\mathrm{Tr}(\cdot\mid V_\ell)$, $=H^1$-Lefschetz trace	$\mathrm{Tr}_{\mathcal{Z}}$, = "$H^1$"-trace via explicit formula
(iii) positivity	ample D / Hodge index on $C\times C$ [CITED]	[OPEN $\equiv$ RH] (Thm. 30.3.3)
(iv) algebraic step	Albert / Milne II.1.3	Thm. 30.2.2 on $\mathcal{Z}$ + interpolation [THM]
conclusion	$\lvert\alpha\rvert=\sqrt{q}$	$\operatorname{Re}\rho=\frac{1}{2}$

30.4. What transfers, what does not, and what replaces Albert

PROPOSITION 30.4.1 (transfer ledger). *(a)* Transfers exactly *[THM]: (i) as an algebra with involution; (ii) (Prop. 30.3.2); the induced involution on the spectrum (Prop. 30.3.1); the orbit decomposition into fixed points (positive lines) and 2-cycles (hyperbolic planes), which is Lemma 27 Prop. 1.1; the count "negative index = number of 2-cycles".*

(b) Does not transfer: finiteness. *$\mathcal{Z}$ is infinite and $\mathrm{Tr}_{\mathcal{Z}}$ is a distribution, not a finite sum and not a state; there is no C^*-completion in which $\mathrm{Tr}_{\mathcal{Z}}$ is a priori continuous and positive. Replacement: interpolation and density (Lemma 1), or the Bochner–Schwartz theorem (Rem. 30.4.2).*

(c) Does not transfer: the $\mathbb{Q}$-structure and integrality. *Albert's theorem is about $\mathbb{Q}$-algebras, and the curve's trace values $\mathrm{Tr}(aa^\dagger)$ are (up to $2g/(D^g)$) intersection numbers of cycles, i.e. integers whose sign comes from effectivity. Note that the algebraic step itself does* not *need $\mathbb{Q}$: Thm. 30.2.2 works over $\mathbb{R}$ or $\mathbb{C}$. What needs $\mathbb{Q}$ is the* proof of (iii). *On $\mathcal{C}$ there is no lattice, no integrality of $\mathrm{Tr}_{\mathcal{Z}}(f*f^\sharp)$ and no effective cone. Bost–Connes provides a $\mathbb{Q}$-structure, but on a different algebra (§30.5).*

(d) Does not transfer: a finite-dimensional representation carrying the trace. *For curves,* Tr *is the trace of an honest representation on $V_\ell J$, compatible with the Weil pairing, and † is the adjoint for the polarization. For ζ the representation "H^1" exists as a cokernel (Connes' absorption spectrum;* [**75**], [**66**, §4, §6]*), but its Hilbert structure is the unknown. Connes' Hilbert-space realization sees only critical zeros, and only for a Sobolev weight $\delta>1$* [**75**, Thm. 1].

REMARK 30.4.2 (what replaces Albert; [CITED]+[THM]). For a positive trace on the group algebra the algebraic step is automatic. A positive-definite distribution on $\mathbb{R}$ is the Fourier transform of a nonnegative tempered measure (Bochner–Schwartz, [**276**, p. 125], as quoted in [**211**]). In half-density coordinates such a measure lives on the *unitary* characters, i.e. on $\operatorname{Re}s=\frac{1}{2}$: these are exactly the $*$-characters of Prop. 30.3.1. So "Albert for $\mathcal{C}$" reads: *every positive trace is supported on $*$-characters.* The CM/totally-real dichotomy survives in degenerate form: among the $*$-characters only $\chi_{1/2}$ is real on real-valued f (the "totally real" factor, $\sharp$ acting trivially there); every $\chi_{1/2+it}$, $t\neq 0$, pairs with $\chi_{1/2-it}$ under complex conjugation of real data, a "CM" factor [ANALOGY]. Lemma 1's proof of Weil's criterion avoids Bochner–Schwartz; either route works.

REMARK 30.4.3 (circularity). By Lemma 1(iv)(c), positivity of $\mathrm{Tr}_{\mathcal{Z}}$ on $(\mathcal{C},\sharp)$ is equivalent to a GNS Hilbert space on which the scaling flow is unitary. So "find the Hilbert space/positive involution" and "prove (iii)" are the *same* problem. Defining the algebra as $\mathcal{C}/\ker(\mathrm{ev}_{\mathcal{Z}})$ or the Hilbert space as $\ell^2(\mathcal{Z},m)$ uses the zeros and is [CIRCULAR if used as input]. This is Connes' remark

on Goldfeld's inner product: its positivity "is of course equivalent to the positivity of the Weil distribution ... but this does not give any clue how to prove this positivity" [**75**, p. 50].

30.5. Candidate geometric sources

For each candidate we record whether it supplies (i) algebra $\ni$ Frobenius, (ii) $FF^\sharp = q$, (iii) positivity of the trace that equals $\mathrm{Tr}_{\mathcal{Z}}$, and what is missing according to the source.

(A) Connes: adèle class space, trace formula, absorption spectrum [CITED]. (i) $\mathcal{S}(C_{\mathbb{Q}})$ acting on functions on $X = \mathbb{A}/\mathbb{Q}^\times$; (ii) the scaling action, as in Prop. 30.3.2; trace: the global trace formula's spectral side is $\mathrm{Tr}_{\mathcal{Z}}$, with critical zeros as an absorption spectrum and noncritical zeros as resonances via their harmonic measure [**75**, abstract, §VIII Lemma 3]. The semilocal trace formula is proved [**75**, Thm. 4, §VII]. (iii) Missing: the global formula is *equivalent* to RH (positive characteristic: [**75**, Thm. 5]). The partial positive result is Connes–Consani's archimedean theorem: $W_\infty(g * g^*) \geq \mathrm{Tr}(\vartheta(g) S \vartheta(g)^*)$ for $\operatorname{supp} g \subset [2^{-1/2}, 2^{1/2}]$ with vanishing conditions [**71**, Thm. 1]. Positivity is supplied there by a *compression of the scaling action* (Sonin space), i.e. genuinely Hilbert-space positivity, but only locally.

(B) Bost–Connes system and the CCM dual system [CITED]. BC supplies a C^*-dynamical system with partition function $\zeta(\beta)$, symmetry group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{cycl}}/\mathbb{Q})$, and a phase transition with spontaneous symmetry breaking at $\beta = 1$ [**44**]. Its natural positive functionals are KMS_β states. They are positive *by definition*, which is the one place in the landscape with built-in positivity *and* a $\mathbb{Q}$-structure with Galois symmetry. But they are states on the BC algebra, not traces on $\mathcal{C}$, and they see ζ only at real $\beta > 1$, where there are no zeros. We found no trace on the BC algebra equal to the Weil distribution in [**44, 66**]. In [**66**] the explicit formula arises as a Lefschetz trace on the *cokernel* of the restriction map from the dual system (adèles to idèles), and the positivity is *restated*, not proved: [**66**, Cor. 6.3] is our Thm. 30.3.3. [**66**, Prop. 6.4] shows that the trace pairing has a large radical (functions extending to adèles), playing the role of trivial correspondences, and [**66**, §1] "leaves open the more difficult question of identifying the correct analog of the principal divisors". Supplies: (i), (ii), a $\mathbb{Q}$-structure nearby. Missing: (iii), and a bridge from KMS positivity to $\mathrm{Tr}_{\mathcal{Z}}$.

(C) Arithmetic site, its square, the scaling site [CITED]. The square of the arithmetic site exists and carries Frobenius correspondences Ψ_λ, $\lambda \in \mathbb{R}_+^\times$ [**69, 64**], and $s(f,f) = D \bullet D$ for $D = \int f(\lambda)\Psi_\lambda d^*\lambda$ [**70**, (14)]. Supplies (i), (ii) geometrically. Their strategy is Weil's *Riemann–Roch* route, not Rosati: assume $s(f,f) > 0$ and use the existence half of Riemann–Roch on the square to make $\pm D$ effective, contradicting degree 0 [**70**, §3.1], [**62**, §2.3]. Missing, in their words: "an appropriate definition of the sheaf cohomology ... H^1" on the square of the Scaling Site, "still open" [**70**, §1]; "what is missing is an intersection theory and a Riemann–Roch theorem on the square of the arithmetic site" [**62**, §4]; step 3 (HRR on the complex lift) is "the most problematic" [**70**, §3.4]. In TAA terms they aim at (iii) via RR + effectivity, i.e. the *second* geometric route of §30.2. (Lemma 23: tropical Hodge index can fail in general, so the output must come from RR, not from a tropical HI theorem.)

(D) Deninger's foliated dynamical systems [CONJECTURAL]. Deninger conjectures $H^0 = \mathbb{R}$, H^1 infinite-dimensional with the zeros as the spectrum of the generator Θ, and $H^2 = \mathbb{R}(-1)$ with Poincaré duality. He then argues: *if* there is a Hodge $*$-operator with $\langle f, f' \rangle = \mathrm{tr}(f \cup *f')$ positive definite on H^1 and Θ commuting with $*$, then $\Theta = \frac{1}{2} + A$ with A skew, "hence the Riemann conjecture would follow" [**91**, §3, pp. 168–169]. Proved toy model [CITED]: for a Riemannian foliation by surfaces on a compact 3-manifold with a dense leaf and a flow conformal on TF with factor $e^{\alpha t}$, $\Theta = \frac{\alpha}{2} + S$, S skew [**87**, Thm. 4.1]. Deninger adds that such a conformal metric probably does not exist for number fields, and that one will need Kähler identities [**87**, after (19)]. Here (iii) is the *definition* of the Hodge scalar product, $*$ is the Rosati-type adjoint, and "Θ commutes with $*$" is the analogue of (ii). Supplies: the full formal template. Missing: the space.

(E) Λ-rings / $\mathbb{F}_1$ [CITED]. Borger's motivation is that $\operatorname{Spec}\mathbb{Z}\otimes_{\mathbb{F}_1}\mathbb{Z}$ should be a surface with intersection theory [41, Intro]. But colimits of Λ-rings agree with those of rings [41, §2.3], so the coproduct of $\mathbb{Z}$ with itself is $\mathbb{Z}\otimes\mathbb{Z}=\mathbb{Z}$ [THM] (deduction, Lemma 23). Borger himself notes the theory "says nothing about the archimedean place" [41, Intro]. Connes reports the same collapse for Durov's monads, while in Segal Γ-rings $\mathbb{Z}\otimes_{\mathbb{S}}\mathbb{Z}$ is non-trivial [62, §4], [63]. Supplies none of (i)–(iii) for ζ as it stands.

candidate	(i)	(ii)	(iii)	missing (per source)
Connes trace formula	yes	yes	$\equiv$RH	global trace formula; Hilbert structure on H^1
CC21 archimedean	yes	yes	local only	all finite primes; uniform support
BC / CCM dual system	yes	yes	no	principal divisors; KMS$\to$ $\mathrm{Tr}_{\mathcal{Z}}$ bridge
arithmetic/scaling site	yes (square)	yes (Ψ_λ)	no	H^1, intersection theory, RR on square
Deninger	conj.	conj.	conj. (Hodge $*$)	the foliated space itself
Λ-rings / $\mathbb{F}_1$	no	no	no	non-trivial product; archimedean place

30.6. The target as a list of axioms

Definition 30.6.1 (geometric positivity source, GPS; [DEF]). A GPS for ζ is a tuple $(\mathsf{X},\mathsf{H},\pi,\sharp_{\mathsf{X}},\langle\,,\rangle)$: an object X of some geometric category (the would-be $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$, or X with a Frobenius flow), a space H (the would-be H^1), an action π of $\mathcal{C}$ (or $\mathcal{S}(C_{\mathbb{Q}})$) on H, and a pairing, subject to:

Axiom G1 (algebra). $\pi:\mathcal{C}\to\mathrm{End}(\mathsf{H})$ is an algebra map, commutative image. *Known* (e.g. H = Connes' cokernel).

Axiom G2 (Frobenius relation). There is an involution $\sharp_{\mathsf{X}}$ on $\pi(\mathcal{C})$, induced by transposition of correspondences on X, with $\pi(f^\sharp)=\pi(f)^{\sharp_{\mathsf{X}}}$, so that $\pi(\delta_\lambda)\pi(\delta_\lambda)^{\sharp_{\mathsf{X}}}=\lambda$. *Known algebraically* (Prop. 30.3.2); *geometrically* (as transpose of Ψ_λ) in [64, 66].

Axiom G3 (Lefschetz = explicit formula). A trace Tr_{H} is defined on $\pi(\mathcal{C})$ with $\mathrm{Tr}_{\mathsf{H}}\circ\pi=\mathrm{Tr}_{\mathcal{Z}}$. *Known as a formula* (explicit formula); *known as a cohomological trace* on the cokernel [66, §6]; as a Hilbert-space trace, global case $\equiv$ RH [75, Thm. 5].

Axiom G4 (duality). The induced involution on the spectrum of $\pi(\mathcal{C})$ is J_{FE} (functional equation = Poincaré duality). *Known* (FE; Prop. 30.3.1).

Axiom G5 (geometric positivity). For all $a\in\pi(\mathcal{C})$, $\mathrm{Tr}_{\mathsf{H}}(aa^{\sharp_{\mathsf{X}}})=c\,\langle D_a,D_a\rangle$ with $c>0$, where $a\mapsto D_a$ is linear into a space of cycles/classes on X (modulo a radical such as [66, Prop. 6.4]) and $\langle\,,\rangle$ is positive semidefinite there *for a geometric reason* (ampleness/effectivity, Hodge index, Kähler/Hodge-$*$, or Riemann–Roch existence plus degree). [OPEN $\equiv$ RH] given G3.

Axiom G6 (non-circularity). X, H, $\langle\,,\rangle$ and the proof of positivity in G5 are constructed without reference to $\mathcal{Z}$ (no $\ell^2(\mathcal{Z})$, no $\mathrm{ev}_{\mathcal{Z}}$, no Weil form as input). *Meta-axiom*; it is what separates a proof from Thm. 30.3.3.

Theorem 30.6.2 (GPS $\Rightarrow$ RH; [THM]). *G1, G2, G3, G5 imply RH. G4 and G6 are not needed for the implication; G4 is automatic, and G6 is needed for the result to have content.*

Proof. By G3, G2, G5, $\mathrm{Tr}_{\mathcal{Z}}(f*f^\sharp)=\mathrm{Tr}_{\mathsf{H}}(\pi(f)\pi(f)^{\sharp_{\mathsf{X}}})=c\langle D,D\rangle\geq 0$. Apply Thm. 30.3.3. □

Remark 30.6.3 (sanity check on curves; [CITED]). For $C/\mathbb{F}_q$: $\mathsf{X}=C\times C$, $\mathsf{H}=V_\ell J$, π generated by Γ_F, $\sharp_{\mathsf{X}}$ = transpose, which is the Rosati involution [193, fn. 7, Thm. 1.27]; G3 is the Lefschetz formula; G5 is $-D^2=\mathrm{Tr}(\alpha_D\alpha_D^\dagger)\geq 0$ on T (Hodge index), Lemma 27 Prop. 2.1; G6 holds. Note that the finite-dimensional Albert step is *not* an axiom: it is replaced by Thm. 30.3.3, which needs no finiteness.

REMARK 30.6.4 (categorical reading; [DEF]/[ANALOGY]). Write $\mathbf{Alg}^*_\tau$ for commutative $*$-algebras with a trace, morphisms preserving both. The Weil datum $(\mathcal{C}, \sharp, \mathrm{Tr}_\mathcal{Z})$ is an object there, and Theorem 30.3.3 says it is *positive* iff RH. A GPS is a factorization $(\mathcal{C}, \sharp, \mathrm{Tr}_\mathcal{Z}) \to (\pi(\mathcal{C}), \sharp_\mathsf{X}, \mathrm{Tr}_\mathsf{H})$ through the image of a functor $\mathbf{Geo} \to \mathbf{Alg}^*_\tau$ (correspondences on $\mathsf{X} \mapsto$ their action on H^1, transpose $\mapsto \sharp$), whose image consists of positive objects for reasons internal to **Geo** (polarized objects). Positivity is reflected by trace-preserving $*$-morphisms, and nothing else is needed. In this language J. W.'s question becomes: *find* **Geo** *with a polarization theory and an object whose* H^1 *realizes* $\mathrm{Tr}_\mathcal{Z}$. For curves, **Geo** = Chow motives over $\mathbb{F}_q$ with the Hodge index theorem (the standard conjecture of Hodge type for surfaces, a theorem there).

30.7. A trace-free, scale-free formulation? (J. W.)

(a) The loss operator. On the pair channel of an off-line zero (Lemma 25 Thm. 3.2) the loss operator is $\Lambda = \mathrm{diag}(\beta, 1-\beta) = \frac{1}{2}I + \mathrm{diag}(\delta, -\delta)$. In operator form on $\mathcal{H}_\mathcal{Z}$, $\boldsymbol{\beta} = \frac{1}{2} + \boldsymbol{\delta}$, $J_{\mathrm{FE}}\boldsymbol{\beta}J_{\mathrm{FE}} = 1 - \boldsymbol{\beta}$ (Lemma 25 Thm. 1.2) [CITED] (project). The scalar part $\mu = \frac{1}{2}$ is forced by the functional equation *for every zero*; RH $\iff$ $\boldsymbol{\delta} = 0$, i.e. the trace-free part vanishes [RESTATEMENT]. In telegrapher terms this is the Heaviside condition $RC = LG$, i.e. $R/L = G/C$. That condition is a ratio, so it is scale-free [THM].

(b) Linear tests are blind; detection is second order.

PROPOSITION 30.7.1 ([THM]). *Let φ be any linear functional on operators of $\mathcal{H}_\mathcal{Z}$ (bounded, or defined on the relevant span) that is functional-equation symmetric, $\varphi(J_{\mathrm{FE}}XJ_{\mathrm{FE}}) = \varphi(X)$. Then $\varphi(\boldsymbol{\delta}) = 0$, and more generally φ kills every J_{FE}-odd operator. The lowest-order J_{FE}-even quantity built from $\boldsymbol{\delta}$ is $\boldsymbol{\delta}^2 = \boldsymbol{\kappa}_H$ (Lemma 25). On the pair plane $P_\rho = \mathrm{span}(e_\rho, e_{\rho^*})$ with the Krein form $[e_\rho, e_{\rho^*}]_\mathcal{Z} = m$, $[e_\rho, e_\rho]_\mathcal{Z} = 0$ (Lemma 27 Prop. 1.1),*

$$[\boldsymbol{\delta}x, \boldsymbol{\delta}x]_\mathcal{Z} = -\delta^2[x, x]_\mathcal{Z} \qquad (x \in P_\rho).$$

So the "energy" of the trace-free part is indefinite and vanishes on the null vectors e_ρ, e_{ρ^*}, which are exactly the ghost directions. Hence "trace-free part = 0" becomes a scalar test only through a definite *norm, and on $\mathcal{H}_\mathcal{Z}$ the existence of a flow-invariant definite norm is equivalent to RH (Lemma 25 Thm. 1.2) [RESTATEMENT].*

PROOF. $J_{\mathrm{FE}}\boldsymbol{\delta}J_{\mathrm{FE}} = -\boldsymbol{\delta}$ gives $\varphi(\boldsymbol{\delta}) = \varphi(J_{\mathrm{FE}}\boldsymbol{\delta}J_{\mathrm{FE}}) = -\varphi(\boldsymbol{\delta})$. For $x = ae_\rho + be_{\rho^*}$, $\boldsymbol{\delta}x = \delta(ae_\rho - be_{\rho^*})$, so $[\boldsymbol{\delta}x, \boldsymbol{\delta}x]_\mathcal{Z} = -2m\delta^2 \,\mathrm{Re}(a\bar{b}) = -\delta^2[x, x]_\mathcal{Z}$. [NUM] check: `num/lemma30_checks.py` items 3. □

(c) "Equality" criteria hide a nonnegativity.

(1) **Balazard–Saias–Yor** [CITED] [21]:

$$\frac{1}{2\pi}\int_{\mathrm{Re}\, s = 1/2} \frac{\log|\zeta(s)|}{|s|^2}\,|ds| = \sum_{\mathrm{Re}\,\rho > 1/2} \log\Big|\frac{\rho}{1-\rho}\Big|$$

(multiplicities counted), so RH $\iff$ the integral is 0. With $s = \frac{1}{2} + it$, $|s|^2 = \frac{1}{4} + t^2$. Hidden nonnegativity [THM]: $|\rho|^2 - |1-\rho|^2 = 2\beta - 1 > 0$ for $\beta > \frac{1}{2}$, so every term is > 0. Since $|1-\rho| = |\rho^*|$, the term is $\log(|\rho|/|\rho^*|)$, a J_{FE}-*odd* quantity, summed over the half of each orbit selected by $\beta > \frac{1}{2}$. Selecting that half is taking an absolute value, i.e. a nonlinear (sign) operation, consistent with Prop. 30.7.1.

(2) **Nyman–Beurling, Burnol's form** [CITED] [53, Thm. 1.2]: $\|P(\mathbf{1})\| = \prod_{\mathrm{Re}\,\rho > 1/2} |(1-\rho)/\rho|$, with P the orthogonal projection onto the Nyman–Beurling subspace. RH $\iff$ $\|P\mathbf{1}\| = \|\mathbf{1}\| = 1$. The nonnegativity is a Hilbert norm (a Blaschke product of modulus ≤ 1).

(3) **Li** [CITED] [178]: RH $\iff$ $\lambda_n \geq 0$ for all $n \geq 1$, $\lambda_n = \frac{1}{(n-1)!}\frac{d^n}{ds^n}[s^{n-1}\log\xi(s)]_{s=1}$; in zero form $\lambda_n = \sum_\rho[1-(1-1/\rho)^n]$ (secondary). This is an inequality criterion. Structure [THM]: $w = 1-1/\rho$ maps $\operatorname{Re}\rho = \frac{1}{2}$ to $|w| = 1$, and the mirror $\rho \mapsto \rho^*$ to $w \mapsto 1/\bar{w}$, so on-line zeros give $1-\cos n\theta_\rho \geq 0$, while an off-line pair contributes $w^n + \bar{w}^{-n}$-type terms with one exponentially growing modulus, a discrete boost (Lemma 27). It is a Carathéodory–Toeplitz positivity (Lemma 19).

Every equality form we checked is $\sum$(nonnegative per-orbit penalty) = 0, or a norm identity. None is a linear identity among FE-symmetric quantities, as Prop. 30.7.1 predicts.

(d) Signature is the coordinate- and scale-free invariant.

Proposition 30.7.2 ([THM] / [CITED]). *For nondegenerate Hermitian forms on a finite-dimensional space, the inertia (n_+, n_-) is a complete invariant under congruence (Sylvester), and it is unchanged by positive rescaling. So it is the complete coordinate- and scale-free invariant. In infinite dimensions the negative index κ (Pontryagin) is the invariant used throughout TAA. RH $\iff$ $\kappa(\mathcal{Q}) = 0$ (Lemma 1(vi)) $\iff$ the transverse part (complement of the pole plane) of Lemma 15's $\mathcal{L}$ has $n_+ = 1$ exactly, i.e. the zero sector is definite [RESTATEMENT]. On a family of windows κ_A can change only where the form degenerates ($\det G_A = 0$, Lemma 27 Thm. 4.4). Definite-type eigenvalues are structurally stable (Krein–Gelfand–Lidskii, Lemma 27 §4). So the signature is the trace-free, scale-free quantity, and it is locally constant: the only way it can be certified is globally (all windows), which is the content of Lemmas 10–19.*

(e) GR analogy. [ANALOGY]. Einstein's condition $\mathrm{Ric} = \Lambda_{\mathrm{cc}} g$ is "trace-free Ricci $\mathrm{Ric}_0 = \mathrm{Ric} - \frac{\mathcal{R}_{\mathrm{sc}}}{n} g = 0$", with the scalar part free, just as $\mu = \frac{1}{2}$ is free and forced. The parallel with (b) is exact at the level of linear algebra [THM]. In Riemannian signature $|\mathrm{Ric}_0|^2 \geq 0$ certifies $\mathrm{Ric}_0 = 0$. In Lorentzian signature it does not: null dust $\mathrm{Ric}_0 \propto k \otimes k$ with k null is nonzero, trace-free, and has $|\mathrm{Ric}_0|^2 = (k\cdot k)^2 = 0$. This is the same phenomenon as null e_ρ. No dynamical content is claimed.

30.8. Hyperbolic orthogonality (J. W.)

(a) The strip as a hyperbolic plane.

Proposition 30.8.1 ([THM]). *$w = e^{i\pi s}$ maps $S = \{0 < \operatorname{Re} s < 1\}$ biholomorphically onto the upper half-plane, and pulls back $|dw|/\operatorname{Im} w$ to $\pi|ds|/\sin(\pi \operatorname{Re} s)$. Under this map: $\operatorname{Re} s = \frac{1}{2}$ goes to the imaginary axis (a geodesic); the mirror $s \mapsto 1-\bar{s}$ goes to $w \mapsto -\bar{w}$ (the reflection in that geodesic); vertical translation $s \mapsto s+ic$ goes to $w \mapsto e^{-\pi c}w$ (a hyperbolic translation along it); and*

$$\sinh d\big(\tfrac{1}{2}+\delta+it,\ \{\operatorname{Re} s = \tfrac{1}{2}\}\big) = |\tan \pi\delta|,$$

independent of t. So RH $\iff$ every zero is at hyperbolic distance 0 from the critical geodesic [RESTATEMENT].

Proof. $w = e^{-\pi t}e^{i\pi\sigma}$, $\operatorname{Im} w = e^{-\pi t}\sin\pi\sigma > 0$, $|dw| = \pi|w||ds|$. The distance from w to the imaginary axis satisfies $\sinh d = |\operatorname{Re} w|/\operatorname{Im} w = |\cot\pi\sigma| = |\tan\pi\delta|$. [NUM]: agreement to 5×10^{-15} for $\delta \in \{0.01, \ldots, 0.49\}$ and three heights; pull-back ratio $1 - 1.3\times10^{-7}$ at step 10^{-7} (item 1). □

(b) What RH does not say. RH puts zeros on the geodesic; it says nothing about spacing [THM] (any discrete set on the line is consistent with it). Statistically the unfolded spacings follow GUE: Montgomery's pair correlation, derived under RH for restricted support [196], and Odlyzko's computations near the 10^{12}-th zero, where "agreement ... is generally good, and improves at larger heights" [214] [CITED].

(c) Orthogonality in the Weil/Krein picture. On $\mathcal{H}_\mathcal{Z}$ with $[x,y]_\mathcal{Z} = \langle J_{\mathrm{FE}} x, y\rangle$ (Lemma 25 Def. 1.1), $[e_\rho, e_\sigma]_\mathcal{Z} = m_\rho\,\delta_{\sigma,\rho^*}$ [THM]. So: distinct J_{FE}-orbits are automatically orthogonal; an on-line e_ρ is positive ($[e_\rho,e_\rho]_\mathcal{Z} = m_\rho$); an off-line e_ρ is null (self-orthogonal), lies on the light cone of the Minkowski plane P_ρ, and pairs only with its mirror. Hence RH $\iff$ no zero direction is self-orthogonal $\iff$ $\{e_\rho/\sqrt{m_\rho}\}$ is an orthonormal frame for $[\cdot,\cdot]_\mathcal{Z}$ [RESTATEMENT]. Status: rigorous on $\mathcal{H}_\mathcal{Z}$ (a space defined from $\mathcal{Z}$, so [CIRCULAR if used as input] as an input) and in Lemma 27's finite zero model (Prop. 1.1, 3.2). The identification of $\mathcal{H}_\mathcal{Z}$ with a space of test functions is formal off RH. *Pun flagged:* "hyperbolic" in (a) is Lobachevsky geometry (curvature -1); in (c) it is a split quadratic form (signature $(1,1)$). They agree only in that both measure departure from $\delta = 0$: the distance $\operatorname{arsinh}|\tan\pi\delta|$ versus the boost rapidity $\delta\log x$ (Lemma 27 Thm. 1.3). We know no theorem relating them [ANALOGY].

(d) Twistor real structures. [ANALOGY]. In twistor theory a real structure is an anti-holomorphic involution; for Euclidean signature ([**15**]), real points of the base correspond to lines invariant under a fixed-point-free antiholomorphic involution of twistor space (see [**279**], [**225**] for the Minkowski version). We have not checked these statements at page level. The mirror $s \mapsto 1-\bar{s}$ is an antiholomorphic involution of $\mathbb{C}$ whose fixed set is the critical line, and RH says every zero is a real point. The analogy goes no further: the twistor involution has no fixed points, whereas ours does, and nothing forces zeros to be real points.

30.9. Open questions

Q1. Do you accept the reformulation "target = a GPS (G1–G6), with G5 the only open axiom" in place of "a positive involution on a commutative algebra"? The involution is already there (Prop. 30.3.2, Rem. 30.2.4).

Q2. Which positivity mechanism should G5 imitate: (a) ampleness (Milne I.14.3, intersection with an ample class); (b) Hodge index on a surface (Castelnuovo–Severi); (c) Riemann–Roch existence plus degree (Weil/Connes–Consani); (d) Hodge-$*$/Kähler (Deninger)? For curves these give the same theorem, but they suggest different objects for ζ.

Q3. Is a positivity source with a $\mathbb{Q}$-structure worth pursuing, by bridging the Bost–Connes KMS positivity (built in, Galois-equivariant) to $\mathrm{Tr}_\mathcal{Z}$? We found no such bridge in the sources; it would be a concrete Lemma-sized literature-and-test task.

Q4. Should Vol. 2 adopt $f^\sharp(x) = x^{-1}\overline{f(1/x)}$ (CCM/CC) alongside Lemma 1's $F^*(u) = \overline{F(-u)}$, with the dictionary of §30.1?

Q5. For §30.7: is the signature/index (with Lemma 27's stability) the "trace-less, coordinate-free" object you wanted, or did you intend a condition that is *linear* in the data? Prop. 30.7.1 rules out FE-symmetric linear ones.

Summary of status

statement	status	source
Prop. 30.2.1 four ingredients; geometry only in (iii)	cited	Milne AV, Mumford, Albert
Thm. 30.2.2 spectral Albert step, signature $(\#\mathrm{Fix}+p, p)$	thm	here
Cor. 30.2.3 $\iota(\alpha) = q/\bar{\alpha}$; coin $(2, 2)$	thm, num	here, Lemma 19
Rem. 30.2.4 positivity intrinsic only in finite dim; Haar trace	thm	here, CCM Prop 6.4
Prop. 30.3.1 transpose of $\sharp$ = mirror	thm	here
Prop. 30.3.2 $\delta_\lambda * \delta_\lambda^\sharp = \lambda$	thm	here
Thm. 30.3.3 positivity of $\mathrm{Tr}_{\mathcal{Z}} \iff$ RH	restatement	Weil, Lemma 1, CCM Cor 6.3
§30.5 candidates: none supplies (iii)	cited	Con99, CCM09, BC95, CC, Den, Bor
Thm. 30.6.2 GPS $\Rightarrow$ RH	thm (conditional)	here
Prop. 30.7.1 FE-symmetric linear tests blind; Krein energy indefinite	thm	here, L25, L27
BSY / Burnol / Li hide nonnegativity	cited + thm	BSY99, Bur, Li97
Prop. 30.8.1 strip hyperbolic, $\sinh d = \lvert\tan \pi\delta\rvert$	thm, num	here
GUE spacing	cited	Mon73, Odl87
twistor, GR	analogy	—

CHAPTER 31

Lemma 31: Is the geometric positivity source (G5) thermal? Entropy, Tomita–Takesaki, Kustaanheimo–Stiefel, Doppler, Bisognano–Wichmann, Penrose and Osterwalder–Schrader

Standalone note `lemma31.tex`, October 21, 2026.

Summary. Tags. [THM] proved here; [CITED] checked at the source (`lit/lemma31/verified_refs.md`); [CITED, secondary] checked only through a named secondary source; [DEF] definition; [ANALOGY] analogy only; [NUM] numerical check (`num/lemma31_checks.py`, `num/lemma31_xi.py`); [RESTATEMENT] equivalent restatement of RH/G5; [HEURISTIC] heuristic, not a theorem.

Verdict. J. W.'s intuition is right about *shape* and wrong about *source*. The thermal, entropic, Kepler and Doppler structures he names all exist for ζ, most of them unconditionally and several exactly. None supplies G5, and each fails for a reason that can be stated as a theorem.

(1) There are two positivities. *Arrow-of-time positivity* (dissipation, entropy production, contraction semigroups, spectrum conditions, positivity of Lévy/Gibbs measures, KL inequalities) is free for ζ. It is Lemma 25's (P0)–(P1) plus the Euler product. It is also *line-blind* (Prop. 31.2.2, Rem. 31.2.4). *State positivity* of Weil's functional is G5 = RH [RESTATEMENT].

(2) Tomita–Takesaki turns a state into a flow. It never produces the state. A modular realization of the Weil form (Def. 31.3.6) implies RH (Thm. 31.3.7), but the work is done by the positivity of the state, so this is G5 restated [RESTATEMENT]. Two points are new. (a) If the scaling flow is modular *with zeros as modular eigenvalues*, RH follows already from self-adjointness of $\log \Delta$. (b) A commutative algebra has trivial modular flow, so a modular G5 must be noncommutative (Prop. 31.3.3). Bost–Connes KMS states are genuine and zero-free (G6 holds), but their positivity is a positive measure on $\{\log n\}$, the prime side. Weil needs positivity of the *zero* measure, and positivity does not cross the Fourier duality of the explicit formula (Prop. 31.3.10). Naive OS fails because the prime atoms are negative (Lemma 25). Connes' unitary realization sees only critical zeros.

(3) KS is exact, with $\omega^2 = -E/2$ (Prop. 31.4.1, [NUM]). The Kepler symmetry algebra is $\mathfrak{so}(4)/\mathfrak{e}(3)/\mathfrak{so}(3,1)$ (verified symbolically). "On-line = rotation, off-line = boost" is exact at the level of Lemma 27's pencil. But an off-line zero is a *complex* energy, and Kepler has none. The analogy stops exactly at RH.

(4) Doppler is exact [THM]: an off-line pair's null components scale by $e^{\pm\eta}$, $\eta = \delta \log x$. RH $\Leftrightarrow$ no Doppler shift [RESTATEMENT]. The pole is the one free boost, $\eta = \frac{1}{2}\log x$. Orbits are Rindler hyperbolas.

(5) Bisognano–Wichmann matches the off-line plane exactly as *spacetime* linear algebra (two null eigendirections scaled by $e^{\pm\eta}$). It does not match the Hilbert space where BW boosts act unitarily, and supplying that space is G5. Borchers needs positive-energy translations. In the type I BC phase none exist (Prop. 31.6.3). The KMS halfway analogy becomes precise and new (Thm. 31.6.5): $\xi(1+it)$ is the two-point function of a positive KMS_1 (detailed-balance) measure $\propto \Phi(\lambda/2)e^{\lambda/2}d\lambda$,

the functional equation is the KMS condition, and the midpoint is $\Xi(t)$. This is unconditional, so it is again the wrong positivity.

(6) Penrose/OS: the "timelike (Milne) zone" of an off-line plane is the *negative* cone of $\mathcal{Q}$, the ghosts (Prop. 31.7.1). Twistor positive frequency is "indefinite form + a flow-invariant positive half". Off the line no such half exists, and its existence for all pairs is RH (Prop. 31.7.2, [RESTATEMENT]). The only transform that *creates* Hilbert positivity is OS reconstruction, which only transports reflection positivity of the input. The missing reflection-positive input is G5 (Prop. 31.7.3). We found no refereed Weil↔reflection-positivity literature.

Nothing here proves RH.

31.1. Conventions

We use the notation of Lemmas 1, 25, 27 and 30:

- $\widetilde{F}(z) = \int F(u)e^{izu}du$ and $F^*(u) = \overline{F(-u)}$, so $\widetilde{F^*}(z) = \overline{\widetilde{F}(\bar{z})}$;
- $\rho = \frac{1}{2} + \delta_\rho + it_\rho = \frac{1}{2} + i\gamma_\rho$ with $\gamma_\rho = t_\rho - i\delta_\rho$, and $\rho^* = 1 - \bar{\rho}$;
- $\mathcal{Q}(F,G) = \sum_\rho m_\rho \widetilde{F}(\gamma_\rho)\overline{\widetilde{G}(\bar{\gamma}_\rho)}$ (Lemma 1);
- $\mathcal{H}_\mathcal{Z} = \ell^2(\mathcal{Z}, m)$, $J_{\mathrm{FE}}e_\rho = e_{\rho^*}$, $[x,y]_\mathcal{Z} = \langle J_{\mathrm{FE}}x, y\rangle$;
- the half-density scaling flow is $U_u e_\rho = x^{\rho-1/2}e_\rho = e^{iu\gamma_\rho}e_\rho$, $x = e^u$, $U_u = e^{iu\gamma}$;
- $P_\rho = \mathrm{span}\{e_\rho, e_{\rho^*}\}$ is the pair plane (Lemma 27);
- G1–G6 are Lemma 30's axioms (G5: geometric positivity source; G6: non-circularity);
- $\Lambda(n)$ is von Mangoldt and Φ is the Jacobi-theta kernel.

31.2. Two positivities: the arrow of time versus states

Definition 31.2.1 ([DEF]). *Arrow-of-time positivity* is positivity attached to a semigroup or to a one-sided spectral condition. Examples: a Lyapunov/storage function decreasing along a contraction semigroup, a generator with half-line spectrum (spectrum condition, passivity), or a positive coefficient (Lévy, Dirichlet, Gibbs) measure of a generating function on a half-plane of convergence. *State positivity* is $\omega(a^*a) \geq 0$ for a *specified* functional. For ζ that functional is Weil's $\mathcal{W}$ on $(C_c^\infty(\mathbb{R}), *, ^*)$, and the condition is $\mathcal{Q}(F,F) \geq 0$.

The two are logically independent. KMS states and the vacuum are time-symmetric, and a dissipative system need not carry a positive invariant state.

Proposition 31.2.2 (dissipation is line-blind; [THM]). *Let $\mathcal{Z}$ be a finite multiset in $0 < \mathrm{Re}\, s < 1$, invariant under $\rho \mapsto \bar{\rho}$ and $\rho \mapsto 1-\rho$, with $N_+ \geq 1$ off-line pairs. Build $\mathcal{H}_\mathcal{Z}$, $\boldsymbol{\beta} = \mathrm{Re}\,\rho$ (diagonal), $Z(\tau) = e^{-\tau\boldsymbol{\beta}}$ and $\mathcal{E}(x) = \frac{1}{2}\|x\|^2$ as in Lemma 25 §1.*

(a) Then $0 < \boldsymbol{\beta} < 1$, Z is a strict contraction semigroup, and $\frac{d}{d\tau}\mathcal{E}(Z(\tau)x) = -\langle \boldsymbol{\beta} Z(\tau)x, Z(\tau)x\rangle < 0$ for $x \neq 0$.

(b) At the same time the Krein form has N_+ negative squares, one per hyperbolic pair plane (Lemma 27 Prop. 2.1).

So no implication "zero-side arrow-of-time positivity ⇒ G5" holds.

Proof. (a) is Lemma 25 Prop. 1.6(a), which uses only $\mathrm{spec}\,\boldsymbol{\beta} \subset (0,1)$. (b) is Lemma 27 Prop. 2.1. □

Proposition 31.2.3 (thermodynamic positivity of the zeta distribution; [THM] with BC facts [CITED]). *For $\beta > 1$ let $P_\beta(n) = n^{-\beta}/\zeta(\beta)$. This is the Gibbs distribution of $H\varepsilon_n = (\log n)\varepsilon_n$ on $\ell^2(\mathbb{N}^*)$, with partition function $\zeta(\beta)$ [44, §5, Thm. 25].*

(a) $t \mapsto \zeta(\beta+it)/\zeta(\beta)$ is positive definite, with Bochner measure $\nu_\beta = \sum_n P_\beta(n)\delta_{\log n}$ (for the variable $-t$).

(b) The mean energy is $\langle H\rangle_\beta = -\zeta'/\zeta(\beta) = \sum \Lambda(n)n^{-\beta}$. *The variance is* $(\log\zeta)''(\beta) = \sum \Lambda(n)(\log n)n^{-\beta} > 0$.

(c) $\mathrm{KL}(P_{\beta'}\|P_\beta) = \log\zeta(\beta) - \log\zeta(\beta') + (\beta-\beta')\langle H\rangle_{\beta'} \geq 0$, *i.e. the tangent-line inequality for the convex function* $\log\zeta$. *(For these commuting density matrices this is the relative entropy of* [7]*.)*

(d) $\log\zeta(\beta+it) - \log\zeta(\beta) = \sum_n \frac{\Lambda(n)}{\log n} n^{-\beta}(n^{-it}-1)$ *is a Lévy–Khintchine exponent with positive Lévy measure. Hence* $\mathrm{Re}[3\log\zeta(\sigma) + 4\log\zeta(\sigma+it) + \log\zeta(\sigma+2it)] \geq 0$, *the inequality behind (P1).*

[NUM]: Toeplitz matrices on a 41-point grid are PSD at $\beta = 1.5, 2, 3$ *(minimum eigenvalues* $7.9\cdot 10^{-3}$, $2.6\cdot 10^{-5}$, $3.1\cdot 10^{-10}$*). Mean energy and KL agree with direct series to* 10^{-9}.

Proof. (a) $\zeta(\beta+it)/\zeta(\beta) = \sum P_\beta(n)e^{-it\log n}$; apply Bochner. (b) Differentiate $\log\zeta$, the log-partition function. (c) $\log(P_{\beta'}/P_\beta) = (\beta-\beta')\log n + \log\zeta(\beta) - \log\zeta(\beta')$; average under $P_{\beta'}$ and use Gibbs' inequality. (d) Take log of the Euler product and use $3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2$. □

Remark 31.2.4 (why these cannot give G5).

(i) [THM] Fix $a > 0$. $L_a(s) = \zeta(s)\zeta(s+a)$ has an Euler product with positive log-coefficients $\frac{\Lambda(n)}{\log n}(1+n^{-a})$ and satisfies (a)–(d) verbatim for $\beta > 1$. Its nontrivial zeros lie on $\mathrm{Re}\, s = \frac{1}{2}$ and $\mathrm{Re}\, s = \frac{1}{2} - a$. So (a)–(d) do not locate zeros on one line.

(ii) [HEURISTIC] With the functional equation added, the general form of "(a)–(d) + FE ⇒ RH" is GRH for the Selberg class, which is open, so no formal non-implication is claimed. What we can state is the scope of the method: Lévy positivity is used on $\sigma > 1$ and gives zero-free regions near $\mathrm{Re}\, s = 1$ (Lemma 25 (P1)–(P2)). It controls zeros only through boundary values as $\sigma \downarrow 1$.

(iii) [THM] (a)–(d) are positivities of measures on $\{\log n\}$. Weil's $\mathcal{W}$ restricted to $u \neq 0$ is a *negative* measure there, with atoms $-\Lambda(n)n^{-1/2}$ (Lemma 25 Prop. 6.4(b)). Its positive-definiteness corresponds to positivity of the zero measure $\sum m_\rho\delta_{\gamma_\rho}$ (Prop. 31.3.10).

Remark 31.2.5 (entropy and the arrow of time; [ANALOGY] except the identity). Bost–Connes record $\log\mathrm{Tr}\, e^{-\beta H} = \sup_\varphi(S(\varphi) - \beta\varphi(H))$ [44, §1] [CITED]. For ζ this is Prop. 31.2.3(c) in Legendre form [THM]. The "arrow of time" is $Z(\tau)$ with its monotone storage energy. RH needs a *time-symmetric* positivity: a state invariant under scaling, as the vacuum is invariant under time translations. Entropy *production* is the wrong half of thermodynamics. Equilibrium (KMS) positivity is the relevant half (§31.3, Thm. 31.6.5).

31.3. Tomita–Takesaki theory as a candidate G5 mechanism

31.3.1. Standard facts.

Definition 31.3.1 ([CITED]; [264], [263], [47], as summarized in [72, §2–3]). Let M act on $\mathcal{H}$ with cyclic separating Ω, $\omega = \langle\Omega, \cdot\,\Omega\rangle$ faithful normal. The closure of $S : x\Omega \mapsto x^*\Omega$ is $S = J\Delta^{1/2}$, with J an antiunitary involution and $\Delta > 0$. Then $JMJ = M'$ and $\Delta^{it}M\Delta^{-it} = M$. The state ω is KMS at $\beta = 1$ for the modular flow, and the modular flow is the unique flow with this property ([72, §3], with [123] for KMS). The natural cone is $\mathcal{P}^\natural = \overline{\{xJxJ\Omega\}}$. Araki's relative entropy $S(\phi|\psi) \geq 0$ uses relative modular operators [7, 6].

Lemma 31.3.2 (natural-cone positivity, and a correction to the brief; [THM], [NUM]). $\langle\Omega, xJxJ\Omega\rangle = \langle x^*\Omega, Jx\Omega\rangle = \|\Delta^{1/4}x^*\Omega\|^2 \geq 0$. *In contrast,* $\langle\xi, J\xi\rangle \geq 0$ *is* false *in general. Since* $\langle\lambda\xi, J\lambda\xi\rangle = \bar\lambda^2\langle\xi, J\xi\rangle$, *the expression is not even real:* $\xi = i\Omega$ *gives* -1. *The correct quadratic positivities are* $\langle\xi, \Delta^{1/2}\xi\rangle \geq 0$ *and the self-duality of* $\mathcal{P}^\natural$.

Proof. $J\Omega = \Omega$ and $Jx\Omega = J(J\Delta^{1/2}x^*\Omega) = \Delta^{1/2}x^*\Omega$. [NUM]: Gibbs state on $M_4(\mathbb{C})$, identity to 10^{-15}, $\|S - J\Delta^{1/2}\| \approx 3\cdot 10^{-15}$, and $\langle i\Omega, Ji\Omega\rangle = -1$. □

PROPOSITION 31.3.3 (no modular G5 on a commutative algebra; [THM] from [CITED]). *If M is commutative, every faithful normal state is a trace and its modular group is trivial. So the scaling flow can be a nontrivial modular flow only on a noncommutative algebra.*

PROOF. A trace satisfies KMS for the trivial flow, and the KMS flow is unique (Def. 31.3.1). □

This sharpens Lemma 30 [THM]. A "positive involution on a commutative algebra" cannot realize G5 as modular/thermal time. Bost–Connes and the adele class space are noncommutative; Lemma 30's convolution algebra is not.

31.3.2. Antiunitaries and the spectrum.

PROPOSITION 31.3.4 (Klein four-group; [THM], [NUM]). *Let $\boldsymbol{\gamma}$ have a complete eigensystem e_ρ ($\boldsymbol{\gamma} e_\rho = \gamma_\rho e_\rho$), $U_u = e^{iu\boldsymbol{\gamma}}$, and let Θ be antiunitary with $\Theta U_u \Theta^{-1} = U_{\varepsilon u}$, $\varepsilon = \pm 1$. Then $\Theta\boldsymbol{\gamma}\Theta^{-1} = -\varepsilon\boldsymbol{\gamma}$ and $\Theta \ker(\boldsymbol{\gamma} - \lambda) = \ker(\boldsymbol{\gamma} + \varepsilon\bar{\lambda})$. On zeros:*

- *$\varepsilon = +1$ (Θ commutes with the flow) gives $\rho \mapsto \bar{\rho}$;*
- *$\varepsilon = -1$ (Θ reverses the flow) gives $\rho \mapsto 1 - \bar{\rho} = \rho^*$;*
- *a* linear *map anticommuting with $\boldsymbol{\gamma}$ gives $\rho \mapsto 1 - \rho$.*

With the identity these form the Klein four-group of symmetries of $\mathcal{Z}$.

PROOF. $\Theta e^{iu\boldsymbol{\gamma}}\Theta^{-1} = e^{-iu\Theta\boldsymbol{\gamma}\Theta^{-1}}$. If $\boldsymbol{\gamma} e = \lambda e$, then $\boldsymbol{\gamma}\Theta e = -\varepsilon\bar{\lambda}\Theta e$. Finally use $\gamma_{\bar{\rho}} = -\overline{\gamma_\rho}$, $\gamma_{\rho^*} = \overline{\gamma_\rho}$ and $\gamma_{1-\rho} = -\gamma_\rho$. [NUM]: random check, error 0. □

REMARK 31.3.5 ("$J = J_{\mathrm{FE}}$" made precise; [THM]). A modular conjugation commutes with its modular flow. If the scaling flow were modular, J would act on zeros by *reality*, $\rho \mapsto \bar{\rho}$. The mirror $\rho \mapsto \rho^*$ is the *flow-reversing* antiunitary. On test functions it is $F \mapsto F^*$, which is Lemma 30's $f^\sharp$, the $*$-operation of the group algebra ($\delta_\lambda^* = \lambda\delta_{1/\lambda}$). The linear Krein symmetry factors as $J_{\mathrm{FE}} = \Theta_{\mathrm{mirror}} \circ C$, with C the antiunitary reality map $e_\rho \mapsto e_{\bar{\rho}}$. So the steering statement "antiunitarity composes J_{FE} with conjugation to give the mirror" holds, with this assignment:

- mirror = the flow-reversing antiunitary = the algebra's $*$;
- modular J = the flow-commuting antiunitary = reality.

Under RH, $\rho^* = \rho$ and both act on zeros like the identity (reality).

31.3.3. Modular realizations of the Weil form.

DEFINITION 31.3.6 ([DEF]). A *modular realization of the Weil form* consists of a standard pair (M, Ω), a constant $c \neq 0$, and a linear map $F \mapsto a_F$ from $C_c^\infty(\mathbb{R})$ into operators affiliated with M, with $\Omega \in \operatorname{dom} a_F$, such that:

(MR1) $\mathcal{Q}(F, G) = \langle a_F\Omega, a_G\Omega\rangle$;
(MR2) $\Delta^{it} a_F \Omega = a_{T_{ct}F}\Omega$;
(MR3) $a_{F^*}\Omega = a_F^*\Omega$.

The *natural-cone variant* replaces (MR1) by $\mathcal{Q}(F, F) = \langle \Omega, a_F J a_F J\Omega\rangle$.

THEOREM 31.3.7 ([THM]; [RESTATEMENT]). *(a) Either form of (MR1) gives $\mathcal{Q} \geq 0$, hence RH by Weil's criterion (Lemma 1). Neither (MR2) nor (MR3) is used.*
(b) Suppose $\Delta^{it} = U_{ct}$ on an invariant subspace containing eigenvectors e_ρ for every zero. Then RH holds.
(c) (MR2)–(MR3) are consistent with Prop. 31.3.4.
So a modular realization is G5 restated (via (a)) or RH restated (via (b)). It counts as progress only if it is constructed under G6.

PROOF. (a) $\mathcal{Q}(F, F) = \|a_F\Omega\|^2$, or $= \|\Delta^{1/4} a_F^*\Omega\|^2$ by Lemma 31.3.2. (b) Δ^{it} is unitary, so $|e^{ict\gamma_\rho}| = 1$ for all t, which gives $\delta_\rho = 0$. (c) Apply Prop. 31.3.4 with $\varepsilon = +1$ to J and $\varepsilon = -1$ to *. □

REMARK 31.3.8 (G6; converse). GNS of $\mathcal{Q}$ presupposes $\mathcal{Q} \geq 0$, so it is circular [RESTATEMENT]. Bost–Connes KMS states and Connes' L^2 scaling action are non-circular, but neither is known to satisfy (MR1). *Converse sketch, not proved here.* Under RH, U is unitary on the GNS space of $\mathcal{Q}$ and reality gives J with $J\boldsymbol{\gamma}J = -\boldsymbol{\gamma}$. Then $(e^{c\boldsymbol{\gamma}}, J)$ determines a standard subspace, and second quantization gives a von Neumann algebra with these modular data [**181**] [CITED, secondary]. We do not claim (MR1) for it.

31.3.4. Candidates.

PROPOSITION 31.3.9 (Bost–Connes). *The following facts are [CITED] from* [**44**]*:*

- *the BC algebra is the Hecke algebra of $P_{\mathbb{Z}}^{+} \subset P_{\mathbb{Q}}^{+}$, with P the $ax+b$ group (§2);*
- *$\sigma_t(\mu_n) = n^{it}\mu_n$ and $\sigma_t(e(y)) = e(y)$ (§5 (2));*
- *for $0 < \beta \leq 1$ there is a unique KMS_β state, a type III_1 factor state (Thm. 5(a), §7, Cor. 29);*
- *for $\beta > 1$ the extremal states $\varphi_{\beta,\alpha} = \zeta(\beta)^{-1}\operatorname{Tr}(\pi_\alpha(\cdot)e^{-\beta H})$ are of type I_∞ and are indexed by embeddings $\mathbb{Q}^{\mathrm{cycl}} \to \mathbb{C}$ (Thm. 5(b), Thm. 25).*

The following are [THM].

*(a) In the GNS space of $\varphi_{\beta,\alpha}$ (Hilbert–Schmidt operators), $Jk = k^*$ and $\Delta^{1/2}k = e^{-\beta H/2}ke^{\beta H/2}$ ([**72**, §4.1 (39)–(44)] [CITED]). So $\log\Delta$ has pure point spectrum in $\beta\log\mathbb{Q}_+^\times$.*

(b) Vector states give positive-definite functions $t \mapsto \langle\xi, \Delta^{it}\xi\rangle$ with Bochner measure on $\beta\log\mathbb{Q}_+^\times$. If $\mathcal{W}$ were such a function (after rescaling time by c), then every zero would be on the line and every ordinate would lie in the countable group $c\beta\log\mathbb{Q}_+^\times$. No such relation is known. We neither exclude nor claim it.

(c) The zeros enter only as zeros of the continuation of the partition function $\zeta(\beta)$ to complex β, where there is no Gibbs state. In statistical-mechanics terms they are Fisher-type zeros [ANALOGY].

So BC provides G1 (noncommutativity), G6, and genuine thermal positivity, but on the prime side.

PROOF. (a) Δ^{it} acts diagonally on $\varepsilon_n \otimes \bar{\varepsilon}_m$ with eigenvalue $(m/n)^{i\beta t}$ (up to the sign convention). (b) Spectral theorem. (c) $\operatorname{Tr} e^{-\beta H} = \zeta(\beta)$. □

PROPOSITION 31.3.10 (the two positivities are Fourier-dual; [THM]). *Prop. 31.2.3(a) says: a positive measure on $\{\log n\}$ $\Leftrightarrow$ positive-definiteness in t. Weil's criterion says: a positive zero measure $\sum m_\rho\delta_{\gamma_\rho}$ on $\mathbb{R}$ (all γ_ρ real) $\Leftrightarrow$ positive-definiteness of $\mathcal{W}$. The explicit formula writes $\mathcal{W}$ as a signed measure on $\{\pm\log n\}$ with negative atoms, plus pole and archimedean terms. A positive measure on one side gives positive-definiteness on the other, never a positive measure there ($\hat{\delta}_a$ is a phase). So BC positivity does not transport to Weil positivity through Fourier duality.*

PROPOSITION 31.3.11 (naive OS in modular language; [THM] from Lemma 25). *Take $E_+ = C_c^\infty(0,\infty)$, $\theta u = -u$ and the kernel $\mathcal{W}$. Then OS positivity $\langle\theta f, f\rangle_{\mathcal{W}} \geq 0$ is the half-line analogue of $\langle\Omega, aJaJ\Omega\rangle \geq 0$. It fails because of the negative atom $-\frac{\log 2}{\sqrt{2}}\delta_{\log 2}$: [NUM] values $-2.50, -11.08, -35.58, -121.35$ (Lemma 25 Prop. 6.4(b)). In modular terms, the prime atoms are negative contributions in the would-be cone.*

REMARK 31.3.12 (Connes' adele class space). Connes' scaling action on L^2_δ is unitary, and its spectrum consists of the critical zeros only [**75**, Thm. 1] [CITED]. This is Thm. 31.3.7(b) in another form: unitary realizations cannot contain off-line zeros as eigenvalues, so RH becomes a completeness statement [RESTATEMENT]. [THM]: the additive translations do not descend to $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$, since $x \mapsto x+b$ commutes with $x \mapsto qx$ for all q only if $b = 0$. So the adele class space has the "a" of $ax+b$ and not the "$+b$".

REMARK 31.3.13 (thermal time; standard subspaces; [ANALOGY]). The Connes–Rovelli thermal time hypothesis [72] [CITED] would say: *the scaling flow is the thermal time of a state on arithmetic observables.* That is (MR2), which is as strong as RH once zeros are modular eigenvalues (Thm. 31.3.7(b)). The closest structural bridge is Neeb–Ólafsson's reflection-positive one-parameter groups [206, 207] together with standard subspaces [181]. We found no application of these to ζ.

31.4. Kepler, the Kustaanheimo–Stiefel lift, and the pencil

PROPOSITION 31.4.1 (KS lift and Sundman time; [THM] (computation) and [NUM]). *For $z \in \mathbb{C}^2$ let $x = (2\,\mathrm{Re}\,\bar z_1 z_2, 2\,\mathrm{Im}\,\bar z_1 z_2, |z_1|^2 - |z_2|^2)$. This is the Hopf map, and $|x| = |z|^2 =: r$ [163, 252]. Let $z'' = -\omega^2 z$ in s, with fibre constraint $\mathrm{Im}\langle z, z'\rangle = 0$, and let $dt = r\,ds$ (Sundman time). Then $\ddot x = -kx/r^3$ with*

$$k = 2(|z'|^2 + \omega^2|z|^2), \qquad E = \tfrac12|\dot x|^2 - k/r = -2\omega^2, \quad \textit{i.e.} \quad \omega^2 = -E/2.$$

So $E<0$ gives a real oscillator, $E>0$ an inverted oscillator ($\omega^2<0$), and $E=0$ a free particle.

PROOF SKETCH. Under the constraint $|dx/ds|^2 = 4r|z'|^2$, so $|\dot x|^2 = 4|z'|^2/r$ and $Er = 2|z'|^2 - k$. Using the conserved $C = |z'|^2 + \omega^2 r$ and $k = 2C$ gives $E = -2\omega^2$. The force law is the standard KS identity [252]. [NUM]: integrated orbits conserve E with error $\sim 10^{-8}$ and match $-2\omega^2$; the Newton residual is $\le 6\cdot 10^{-8}$. □

REMARK 31.4.2 (coupling-constant metamorphosis). Hietarinta–Grammaticos–Dorizzi–Ramani introduced the "new-time" transformation that exchanges a coupling constant with the energy [129] [CITED] (abstract). The Kepler↔oscillator case ($k \leftrightarrow$ oscillator energy, $E \leftrightarrow \omega^2$) is stated in [270] [CITED, secondary], and Prop. 31.4.1 exhibits it. J. W.'s "time brought into the system" is the Sundman step $dt = r\,ds$. [ANALOGY]: $ds = dt/r$ has the shape of $du = dx/x$, so $u = \log x$ is the "Sundman time" of the scaling problem, turning a multiplicative group into a translation group.

PROPOSITION 31.4.3 (Kepler algebra; [THM], sympy). *Let $L = r \times p$, $A = p \times L - mk\hat r$ and $H = p^2/2m - k/r$. Then*

$$\{L_i, L_j\} = \epsilon_{ijk}L_k, \quad \{A_i, L_j\} = \epsilon_{ijk}A_k, \quad \{A_i, A_j\} = -2mH\epsilon_{ijk}L_k, \quad \{A, H\} = 0.$$

With $D = A/\sqrt{2m|E|}$, $\{D_i, D_j\} = -\mathrm{sgn}(E)\epsilon_{ijk}L_k$. This gives $\mathfrak{so}(4)$ for $E<0$, $\mathfrak{so}(3,1)$ for $E>0$, and $\mathfrak{e}(3)$ for $E=0$. The historical attribution to Fock [102] and Bargmann [22] follows [290] [CITED, secondary].

PROPOSITION 31.4.4 (dictionary with Lemma 27; labelled).

Kepler/KS	*ζ (pencil $\mu = \gamma^2$)*	*status*
$E<0$, $\omega^2>0$, $SO(4)$	*on-line: $\mu = t^2 > 0$, rotation x^{it}*	*algebra [THM], physics [ANALOGY]*
$E>0$, $\omega^2<0$, $SO(3,1)$	*pole pair: $\mu = -\frac14$, pure boost*	*algebra [THM], physics [ANALOGY]*
$E=0$, $E(3)$	*ghost birth* $\det G_A = 0$ *(Lemma 27)*	*[ANALOGY]*
none *(E real)*	*off-line, $t \neq 0$: $\mu = t^2 - \delta^2 - 2it\delta \notin \mathbb{R}$*	*[THM]; "resonance" [ANALOGY]*

[THM]: for $t \neq 0$, μ is real iff $\delta = 0$, and then $\mu > 0$. All nontrivial zeros have $t \neq 0$, so the only real negative μ is the pole pair's. Where the analogy stops. *$\omega^2 = -E/2$ is real because E is an eigenvalue of a self-adjoint operator. "μ real" for ζ is RH [RESTATEMENT]. KS does not make a non-self-adjoint generator self-adjoint. Fock–Bargmann distinguish real energies, and RH excludes complex ones.*

31.5. Doppler factors and Rindler worldlines

PROPOSITION 31.5.1 (Doppler; [THM], [NUM]). *Let $\delta = \delta_\rho \neq 0$ and $u = \log x$. On P_ρ the flow U_u multiplies e_ρ and e_{ρ^*} by $e^{itu}e^{\pm\eta}$, $\eta = \delta u$ (Lemma 27 Thm. 2.3(a)). Here $e^{\pm\eta} = \sqrt{(1 \pm v)/(1 \mp v)}$ with $v = \tanh\eta$. These are the relativistic Doppler factors of the light-cone components $k^\pm$ under a boost of rapidity η. [NUM]: error $2.5 \cdot 10^{-14}$; $|U_u e_\rho| = x^\delta$; Krein-unitarity to 10^{-16}.*

- *(a) RH $\Leftrightarrow$ $\eta_\rho \equiv 0$ for every zero (no Doppler shift) [RESTATEMENT] (Lemma 27 Thm. 2.3(d)).*
- *(b) The pole pair $\{0, 1\}$ is the one unconditional boost, with $\eta = \frac{1}{2}\log x$ and Doppler factor $\sqrt{x}$. Its unnormalized modes x^1, x^0 are split symmetrically by $x^{-1/2}$.*
- *(c) For $v = ae_\rho + be_{\rho^*}$ (a, b real, $ab \neq 0$) the orbit $ae^\eta e_\rho + be^{-\eta}e_{\rho^*}$ satisfies $[v, v]_{\mathcal{Z}} = 2mab$ for all u. In null coordinates $x^+x^- = ab$, which is a Rindler hyperbola with rapidity $\eta = \delta u$ linear in u. Its proper time is $\sqrt{|ab|}\,\eta$ and its proper acceleration is $1/\sqrt{|ab|}$ [THM]. Reading u as proper time is [ANALOGY].*

REMARK 31.5.2 (Unruh; BC phase transition; [ANALOGY]). By Bisognano–Wichmann/Unruh, a uniformly accelerated observer sees the vacuum as thermal at $T = a/2\pi$ [**273**], [**72**, (57)] [CITED]. One could assign "temperature $\delta/2\pi$ per unit u" to an off-line zero, but no theorem transfers: Unruh's temperature belongs to a *state*, and the zero side has none (G5). Identification [THM]: the pole channel $\delta = \frac{1}{2}$ and the BC critical point $\beta = 1$ [**44**, Thm. 5] are both the pole $s = 1$. For $\beta > 1$ the phase is type I, with discrete modular spectrum and no half-sided translations (Prop. 31.6.3). For $0 < \beta \leq 1$ it is the unique III_1 phase. That this explains the critical line is [ANALOGY] only.

31.6. Rindler, Bisognano–Wichmann, Borchers

(1) Rindler modes = Mellin modes.

PROPOSITION 31.6.1 ([THM]). *In $\mathbb{R}^{1,1}$ with $x^\pm = x^1 \pm x^0$, the boost is $B_\eta(x^+, x^-) = (e^\eta x^+, e^{-\eta}x^-)$. On the wedge ($x^+ > 0$), the functions $f_\omega = (x^+)^{i\omega}$ satisfy $f_\omega \circ B_\eta = e^{i\omega\eta}f_\omega$: they are Rindler modes. With $x^+ = x$ they are the Mellin modes $x^{i\gamma}$, $\omega = \gamma$.*

- *(a) An on-line zero has a real Rindler frequency $\omega = t$.*
- *(b) An off-line zero has a complex frequency $\omega = t - i\delta$. Under the boost of rapidity $\log x$ its mode picks up $x^{it}x^\delta$, the Doppler factor of Prop. 31.5.1. The mirror partner picks up $x^{-\delta}$.*

PROOF. $|e^{i\omega\eta}| = e^{\eta\delta}$ for $\omega = t - i\delta$. □

(2) Bisognano–Wichmann. [CITED] via [**33, 34**], with content [CITED, secondary] from [**72**, §4.3 (55)] and [**39**]: for the vacuum and the wedge algebra of a Wightman field, Δ^{it} is the boost of rapidity $\mp 2\pi t$ (convention-dependent) and $J = \Theta U(R(\pi))$ (CPT times a rotation). On spacetime the null directions scale by $e^{\pm 2\pi t}$. The BW 1976 abstract (checked) relates complex Lorentz transformations to TCP and duality.

PROPOSITION 31.6.2 (exact match, and what it is a match *to*; [THM]). *$(P_\rho, [\cdot,\cdot]_{\mathcal{Z}})$ has signature $(1, 1)$ with null basis e_ρ, e_{ρ^*}, and $U_u|_{P_\rho} = x^{it}B(\delta u)$. Under $\delta u = 2\pi t$ and $(e_\rho, e_{\rho^*}) \mapsto (\partial_+, \partial_-)$, this is the BW boost acting on the* spacetime *plane $(\mathbb{R}^{1,1}, ds^2)$, an isometry of an indefinite form. BW also implement the boost* unitarily *on a Hilbert space with a positive invariant state. The zero side has the spacetime half and lacks the Hilbert half. Supplying the Hilbert half is G5 [RESTATEMENT].*

(3) Antiunitarity. See Prop. 31.3.4 and Rem. 31.3.5 [THM]. BW's J commutes with the wedge boosts ($\varepsilon = +1$), so on zeros it would act by reality. The mirror is the case $\varepsilon = -1$.

(4) Natural cone versus Weil. *Conditional theorem* (Thm. 31.3.7; [THM], [RESTATEMENT]). Suppose there is a cyclic-separating realization in which the modular flow is the scaling flow, J and the mirror are as in Prop. 31.3.4, and $\mathcal{Q}(F, F) = \langle\Omega, a_F J a_F J\Omega\rangle$. Then Weil positivity holds, hence RH. This is an **equivalent restatement of G5**, not progress.

(5) The key gap: Borchers and half-sided inclusions. [CITED]. *Borchers* [40] (abstract checked): suppose (M,Ω) is standard and $U(a)$ is a translation group with positive generator, $U(a)\Omega=\Omega$ and $U(a)MU(-a)\subset M$ for $a\geq 0$. Then the modular group acts on U by dilations: $\Delta^{it}U(a)\Delta^{-it}=U(e^{-2\pi t}a)$, and $JU(a)J=U(-a)$. *Wiesbrock* [289] (abstract checked): half-sided modular inclusions $N\subset M$ give such a U with generator $\frac{1}{2\pi}(\log\Delta_N-\log\Delta_M)\geq 0$, and M must be of type III_1. So "boost = modular flow" is *derived* from positive-energy translations plus a positive state.

Proposition 31.6.3 (no Borchers translations with discrete modular spectrum; [THM]). *If* $\log\Delta$ *has pure point spectrum (for example, the BC states with* $\beta>1$*, Prop. 31.3.9(a)), then* $P\geq 0$ *and* $\Delta^{it}P\Delta^{-it}=e^{-2\pi t}P$ *imply* $P=0$.

Proof. For an eigenvector ξ of Δ^{it}, $e^{-2\pi t}\langle\xi,P\xi\rangle=\langle\xi,P\xi\rangle$ for all t. So $\langle\xi,P\xi\rangle=0$ and $P^{1/2}\xi=0$. Eigenvectors span $\mathcal{H}$. □

Remark 31.6.4 (what fails in each setting).
(i) *Bost–Connes* [THM] from [CITED]. BC is built on $ax+b$, but the roles are transposed. In Borchers the modular group *is* the dilation and it rescales the translations. In BC the time evolution fixes the translations $e(y)$ and gives the dilation isometries characters n^{it}: it is the dual grading by $\log n$. Moreover $\{e(y):y\in\mathbb{Q}/\mathbb{Z}\}$ is torsion, with no positive-generator one-parameter subgroup. For $\beta>1$, Prop. 31.6.3 applies. For $0<\beta\leq 1$ (III_1) Wiesbrock's type obstruction is absent, but no half-sided structure is known.
(ii) *Adele class space*: the $+b$ does not descend (Rem. 31.3.12).
(iii) *Zero space* [THM]. Any $P\geq 0$ that is a function of $\boldsymbol{\gamma}$ commutes with U, so the Borchers relation forces $P=0$. Lemma 25's free spectrum condition (P0) is commutative and cannot be a Borchers translation.

Gap, summarized [RESTATEMENT]. BW/Borchers derive the boost law from positive energy plus a positive state. ζ has a free (commutative) positive energy and the boost. It lacks the state, and the state is G5.

(6) KMS halfway point versus $\sigma=\frac{1}{2}$: made precise.

Theorem 31.6.5 (ξ is a KMS_1 two-point function; [THM], [NUM]). *Let* $\Phi(u)=\sum_{n\geq 1}(2\pi^2n^4e^{9u}-3\pi n^2e^{5u})e^{-\pi n^2e^{4u}}$. *Then* Φ *is even and* $\int_0^\infty\Phi(u)\cos(zu)\,du=\frac{1}{8}\xi(\frac{1}{2}+\frac{iz}{2})$ [236, (1),(3)] *[CITED].*

(a) $\Phi>0$ *on* $\mathbb{R}$.
(b) $F(t):=\xi(1+it)=\int e^{i\lambda t}d\mu(\lambda)$ *with* $d\mu=2\Phi(\lambda/2)e^{\lambda/2}d\lambda\geq 0$. *So* F *is positive definite:* $F(t)=\langle 1,e^{it\lambda}1\rangle_{L^2(\mu)}$.
(c) (Detailed balance = KMS$_1$*.)* $d\mu(-\lambda)=e^{-\lambda}d\mu(\lambda)$. *Equivalently,* F *is bounded and analytic on* $0\leq\operatorname{Im}z\leq 1$ *with* $F(t+i)=F(-t)$. *This is the KMS condition at* $\beta=1$*, and here it is* $\xi(s)=\xi(1-s)$.
(d) The midpoint is $F(t+\frac{i}{2})=\Xi(t)=\xi(\frac{1}{2}+it)=\int e^{i\lambda t}e^{-\lambda/2}d\mu$*, also positive definite. It is the analogue of* $\langle\xi,\Delta^{1/2}\xi\rangle\geq 0$.

[NUM]: $\Xi(0)=0.497120778\ldots$, $\Xi(3)$, $\Xi(10)$, $\xi(1+\frac{i}{2})$ *and* $\xi(1+2i)$ *match the integrals to* $\sim 10^{-16}$*; the evenness error is* $6\cdot 10^{-16}$.

Proof. (a) For $u\geq 0$ the n-th term is $\pi n^2e^{5u}e^{-\pi n^2e^{4u}}(2\pi n^2e^{4u}-3)>0$, since $2\pi>3$. Then use evenness. (b) By evenness, $\xi(\frac{1}{2}+iz)=4\int_{\mathbb{R}}\Phi(u)e^{2izu}du$. Put $z=t-\frac{i}{2}$ and $\lambda=2u$. (c) Evenness gives $e^{-\lambda}d\mu(\lambda)=d\mu(-\lambda)$, so $F(t+i)=\int e^{i\lambda t}e^{-\lambda}d\mu=F(-t)$. Analyticity follows from the super-exponential decay of Φ. (d) Take $z=t$ in (b). □

Remark 31.6.6 (reading). [THM] for the logic, [ANALOGY] for the physics. J. W.'s "$\Delta^{1/2}$ halfway $=\sigma=\frac{1}{2}$" is literally true: the KMS strip is the critical strip, the KMS relation is the FE,

and the midpoint is the critical line. But (a)–(d) are *unconditional*: they give positive-definiteness of Ξ, coming from $\Phi > 0$. RH is that Ξ has only real zeros, a Laguerre–Pólya property that $\Phi > 0$ does not imply. Via de Bruijn–Newman RH is $\Lambda_{\mathrm{dBN}} \le 0$, and $\Lambda_{\mathrm{dBN}} \ge 0$ is a theorem [**236**]. So the thermal structure of ξ is free, which is exactly why it is not G5.

31.7. Penrose and Osterwalder–Schrader (J. W.'s last question)

J. W.: "Rindler is spacelike; transform the Mellin modes and antiunitarity to the timelike zone via a Penrose transform. Wouldn't that give positivity?"

(a) Rindler versus Milne.

PROPOSITION 31.7.1 ([THM]). *In $(\mathbb{R}^{1,1}, ds^2 = dx^+dx^-)$ the boost generator is $K = x^+\partial_+ - x^-\partial_-$, with $ds^2(K) = -x^+x^-$. The dilation generator is $\mathcal{D} = x^+\partial_+ + x^-\partial_-$, with $ds^2(\mathcal{D}) = x^+x^-$.*

- *(a) In the Rindler wedge ($x^+x^- > 0$) boost orbits are timelike and dilation orbits spacelike.*
- *(b) In the Milne cone ($x^+x^- < 0$) boost orbits are spacelike (along the hyperbolic slices) and dilation orbits timelike.*
- *(c) The ζ action on an off-line plane is* boost-type*: it scales the two null directions oppositely. The dilation part of $x^\rho = x^{1/2}x^\delta x^{it}$ is the common factor $x^{1/2}$, which the FE fixes (Lemma 25: $J_{\mathrm{FE}}\beta J_{\mathrm{FE}} = 1-\beta$) and the half-density normalization removes. In Lemma 30 §7 terms, dilation is the trace part and boost the trace-free part.*
- *(d) With $ds^2 \leftrightarrow [\cdot,\cdot]_{\mathcal{Z}}$ on P_ρ ($[v,v]_{\mathcal{Z}} = 2mab$), the wedge is $\mathcal{Q}$-positive and the Milne cone is* $\mathcal{Q}$-negative*, the ghost cone. So "moving to the timelike zone" of an off-line plane means moving into the ghost directions. It cannot create positivity. The invariant statement is: signature $(1,1)$, with both cones flow-invariant.*

PROOF. The tangents are $(x^+, \mp x^-)$; read off the signs. (c) is Lemma 27 Thm. 2.3(a). (d) is Lemma 27 Prop. 2.1(c). [NUM]: tangent Krein norm $= -2m\delta^2 ab$. □

(b) Twistors: an indefinite form plus a positive half. [CITED, secondary] via the review [**13**]: $\mathbb{T} = \mathbb{C}^4$ has a Hermitian form of signature $(2,2)$, preserved by $SU(2,2)$, which is locally $SO(4,2)$. $\mathbb{PT}$ splits into $\mathbb{PT}^\pm$ and $\mathbb{PN}$ (light rays). Solutions of the massless field equations that extend holomorphically to the future tube correspond to cohomology on twistor space (Eastwood–Penrose–Wells, as reported in [**13**]). Forward-tube analyticity is the spectrum condition. Original sources: [**225, 279**] (bibliographic only).

PROPOSITION 31.7.2 (positive frequency in Krein form; [THM]). *(a) Let $H = H^*$, $\ker H = 0$, and $[x,y]_H = \langle x, \mathrm{sgn}(H)y\rangle$. Then e^{itH} preserves both forms, and $[\cdot,\cdot]_H$ is positive on the flow-invariant subspace $\{H > 0\}$. This is the Klein–Gordon structure. The positive half is invariant* because $\mathrm{sgn}(H)$ *commutes with the flow.*

- *(b) For $\delta \neq 0$, the only U-invariant lines in P_ρ are $\mathbb{C}e_\rho$ and $\mathbb{C}e_{\rho^*}$, both null. So there is no flow-invariant positive line. A flow-invariant maximal positive subspace of the zero sector exists iff RH [RESTATEMENT] (equivalently $[U, J_{\mathrm{FE}}] = 0$, Lemma 27 Thm. 2.3(d)).*
- *(c) The ζ-analogue of forward-tube analyticity is half-plane analyticity and the Euler product on $\mathrm{Re}\, s > 1$. Its zero-side content is (P0)–(P1), which is free (Lemma 25). So the Penrose "positive half" is a spectrum condition, not Hilbert positivity, and does not supply G5.*

PROOF. (a) Direct. (b) For $u \neq 0$, $U_u|_{P_\rho}$ has the distinct eigenvalues $e^{iut \pm \delta u}$, so its invariant lines are the eigenlines. They are null by Lemma 27 Prop. 2.1(a). (c) Lemma 25 Prop. 1.6. □

(c) The two QFT positivities and OS reconstruction. [CITED], standard content of [**253**] (bibliographic data checked, page numbers not). Wightman QFT has a *spectrum condition* (spectrum in the forward cone, i.e. forward-tube analyticity) and *Hilbert positivity*. The first corresponds to passivity, which is free for ζ. RH corresponds to the second [RESTATEMENT]. The transform that *produces* Hilbert positivity is OS reconstruction [**217, 216**] [CITED]: from Schwinger functions with reflection positivity on positive-time test functions one obtains a Hilbert space, a positive Hamiltonian and Wightman functions. OS II "extend[s] and correct[s]" the 1973 paper (abstract checked).

PROPOSITION 31.7.3 (Weil as a θ-form; the needs of an OS structure). *(i) [THM]. With $\theta F = F^*$ and $\langle\langle f, g\rangle\rangle := \iint f(x)g(y)\mathcal{W}(x+y)\,dx\,dy$,*

$$\mathcal{Q}(F,F) = \iint F(x)\overline{F(y)}\,\mathcal{W}(x-y)\,dx\,dy = \langle\langle F, \theta F\rangle\rangle \quad (F \in C_c^\infty(\mathbb{R})).$$

This has the OS shape with reflection = mirror, but without *a half-space restriction (Lemma 25 Prop. 6.4(a)).*

(ii) [THM]. With $E_+ = C_c^\infty(0,\infty)$ and kernel $\mathcal{W}$, reflection positivity fails (Prop. 31.3.11).

(iii) [DEF]. An OS structure for ζ needs:

- *(OS1) a Euclidean space E and a kernel S;*
- *(OS2) a reflection θ;*
- *(OS3) a half-space E_+ with a translation semigroup preserving it;*
- *(OS4) $\langle \theta f, f\rangle_S \geq 0$ on E_+;*
- *(OS5) an identification of the reconstruction with Weil's space and the scaling flow, so that $\mathcal{Q} \geq 0$ on all of $C_c^\infty(\mathbb{R})$.*

The G5 piece is (OS4)+(OS5) *[RESTATEMENT]. (OS1)–(OS3) are cheap, and OS then yields a positive generator, which is the free half. The Hilbert positivity of the output* is *the reflection positivity of the input. Any success needs a new pair (S,θ), not $\mathcal{W}$ with $u \mapsto -u$.*

Literature (searched 2026-10-21). We found **no refereed work** relating Weil positivity to OS reflection positivity, and none constructing an OS structure for ζ. The closest structural bridge is Neeb–Ólafsson [**206, 207**] (reflection positivity ↔ Lax–Phillips), which has no arithmetic content. One unrefereed item exists: D. Toupin, "Haar Positivity: From Weil to Wightman" (PhilArchive TOUHPF; Zenodo 10.5281/zenodo.19987449, 2026). It asserts that the local terms of the explicit formula are ≥ 0, which is false (the prime atoms are negative; Lemma 25). It is *not* cited as support.

31.8. Open questions

QUESTION 31.8.1. The thermal structure you predicted exists exactly (Thm. 31.6.5): $\beta = 1$, the critical strip as KMS strip, and $\sigma = \frac{1}{2}$ as midpoint. It is free and therefore not G5. Should Volume 2 record it[1] as the "thermal reading of the functional equation"?

QUESTION 31.8.2. By Thm. 31.3.7(b) and Prop. 31.3.3, a modular G5 needs a noncommutative algebra on which the scaling flow is thermal time and the zeros are modular eigenvalues. Should we adopt "G5′: a non-circular standard pair whose modular flow restricts to the scaling flow on a subspace detecting all zeros" as an equivalent [RESTATEMENT] but more structural target?

QUESTION 31.8.3. BC at $0 < \beta \leq 1$ (type III_1) is the only phase not excluded by Prop. 31.6.3. Is there a continuous positive-energy $ax + b$ structure in a III_1 completion?

[1] **[Vol. 3 editorial note]** With the approved two-volume split, the series is Vol. I (Lemmas 1–13), Vol. II (Lemmas 14–29) and Vol. III (Lemmas 30–47, this volume). "Volume 2" in this question meant the next volume of the series; for Lemmas 30–47 that is this volume. See Appendix A, item D1.

QUESTION 31.8.4. OS with the reflection "elsewhere": is there a kernel on a two-variable space (e.g. $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$) with a factor-swapping reflection whose reconstruction is Weil's space?

QUESTION 31.8.5. Do you accept "off-line zero = complex-energy resonance of a non-self-adjoint Sundman-regularized problem" as the physical picture? Under it, RH $\Leftrightarrow$ self-adjointness of the regularized generator, which is again G5 [RESTATEMENT].

Summary: new versus restatement

statement	status	note
Prop. 31.2.2, Rem. 31.2.4: line-blindness	[THM] (new framing)	ghosts with strict passivity; $\zeta(s)\zeta(s+a)$
Prop. 31.2.3: BC thermodynamics	[THM], known in substance	prime side, free
Lemma 31.3.2: $\langle \xi, J\xi \rangle \not\geq 0$	[THM]	corrects the brief
Prop. 31.3.3: commutative $\Rightarrow$ trivial flow	[THM] from [CITED]	sharpens Lemma 30
Prop. 31.3.4: Klein four-group	[THM] (new here)	J = reality, mirror = $*$
Thm. 31.3.7: modular realization $\Rightarrow$ RH	[RESTATEMENT]	G5 restated
Prop. 31.3.10: Fourier-dual positivities	[THM] (new framing)	why BC cannot transfer
Props. 31.4.1–31.4.4: KS, LRL, dictionary	[THM] (classical) / [ANALOGY]	stops at complex energy
Prop. 31.5.1: Doppler	[THM]; (a) [RESTATEMENT]	pole: $\sqrt{x}$
Props. 31.6.1, 31.6.2: Rindler = Mellin, BW match	[THM]; G5 part [RESTATEMENT]	spacetime half only
Prop. 31.6.3: no Borchers for $\beta > 1$	[THM] (new, elementary)	discrete modular spectrum
Thm. 31.6.5: ξ is KMS_1	[THM] (new framing of Riemann's theta proof)	free, hence not G5
Prop. 31.7.1: Milne = ghost cone	[THM] (new)	answers J. W.
Prop. 31.7.2: no invariant positive half	[THM]; (b) [RESTATEMENT]	Penrose = spectrum condition
Prop. 31.7.3: G5 = (OS4)+(OS5)	[THM], [DEF], [RESTATEMENT]	no literature

CHAPTER 32

Lemma 32: What "positivity in the RH sense" requires, element by element. The Osterwalder–Schrader template, Weil's form, the Euclidean (Cramér) kernel, Wick rotation, polarizations, and capitulation

Standalone note `lemma32.tex`, October 21, 2026.

Summary. Tags. [THM] proved here; [CITED] checked at the source (`lit/lemma32/verified_refs.md`); [CITED, secondary] checked only through a named secondary source; [DEF] definition; [RESTATEMENT] equivalent to RH/G5, no progress; [ANALOGY] analogy only; [OPEN] open; [NUM] numerical (`num/lemma32_checks.py`, 107 614 Platt zeros up to height $8 \cdot 10^4$).

Verdict. Weil's form $\mathcal{Q}$ already *is* an OS form, but a degenerate one: its reflection is the mirror $\rho \mapsto \rho^* = 1 - \bar{\rho}$, and its "positive-time" space is everything. The one OS element it lacks is a *proper* positive-time subspace on which positivity can be checked and from which the full form is reconstructed. Every candidate for that subspace either is vacuous or fails. The one that works is built from the zeros.

(1) *Half-spaces are vacuous for $\mathcal{Q}$ itself* (Thm. 32.5.1, [THM]). $\mathcal{Q}$ is exactly translation invariant on the log-scale line, pole terms included, so "$\mathcal{Q} \geq 0$ on functions supported in a half-line" is *equivalent* to full Weil positivity. The only nontrivial support restriction is the window length. Positivity on short windows is known (Yoshida, Bombieri, Burnol, Suzuki, and our certified bounds), and it does not propagate to long windows by invariance (Lemma 29's exact pair threshold).

(2) *The genuine OS form of $\mathcal{W}$ fails, and Weil is a domination condition, not an RP condition* (Thm. 32.5.4, [THM]). For real F supported in (a, ∞), $\mathcal{Q}(F \pm R_a F) = 2(\mathcal{Q}(F) \pm \mathrm{OS}_a(F))$. Hence Weil positivity is equivalent to $|\mathrm{OS}_a(F)| \leq \mathcal{Q}(F)$ on the positive half. OS positivity $\mathrm{OS}_a \geq 0$ is not needed and is false unconditionally (Lemma 25, Prop. 6.3(b)). [NUM]: $\mathrm{OS}_a/\mathcal{Q}$ ranges over $(-0.83, 0.89)$ on our test bumps, always inside $(-1, 1)$.

(3) *What $\mathcal{W}$ looks like as a Euclidean kernel on the positive half* (Thm. 32.5.5, [THM]): $\mathrm{OS}_0(F) = |\int F e^{u/2}|^2 - \sum_{k \geq 1} |\int F e^{-(2k+\frac{1}{2})u}|^2 - \sum_n \Lambda(n) n^{-1/2} (F \star F)(\log n)$. The pole is one RP-positive "tachyonic" mode. The archimedean place is a tower of RP-*negative* Euclidean masses $2k + \frac{1}{2}$, which is a Pauli–Villars ghost tower [ANALOGY]. The primes are atoms of no Laplace type. The nontrivial zeros do not appear as Euclidean masses at all.

(4) *A θ-invariant positive-time space forces $\theta = \mathrm{id}$* (Prop. 32.6.1, [THM]). In finite zero models the maximal flow-invariant θ-nonnegative subspace has the off-line zeros as its *null space*: the OS quotient silently deletes them. RH $\iff$ the OS null space is 0 [RESTATEMENT].

(5) *The correct Euclidean object exists and is new to this series* (Thm. 32.7.2, [THM] + [RESTATEMENT]). Let $S_+(\tau) = \sum_{t_\rho > 0} m_\rho e^{-\tau \gamma_\rho}$ (Cramér's function on the imaginary axis). The spectrum condition (holomorphy in a half-plane) is *free*. The Weil

kernel is its boundary value, $\mathcal{W} = 2\,\mathrm{Re}\,V(\cdot + i0)$. Moreover: RH $\iff$ S_+ is completely monotone $\iff$ S_+ is reflection positive in the OS/Widder sense. Under RH the OS reconstruction is ℓ^2 of the upper zeros with $H = \mathrm{diag}(t_\rho) \geq 0$, and Wick rotation gives back the scaling flow. Euclidean time is $\tau = \arg x$. This is the single missing element, *G5 localized*, and it is zero-built (G6 fails).

(6) *Polarization (steering 1).* A polarization has two independent halves: integrality (Bohr–Sommerfeld/Kostant–Souriau prequantization) and positivity (Kähler/Hodge–Riemann). Weil positivity is the positivity half. For curves integrality is automatic. For $\mathrm{Spec}\,\mathbb{Z}$ the integral half already fails at ∞ ($\widehat{\mathrm{Pic}} \cong \mathbb{R}$; the $\log p$ are $\mathbb{Q}$-independent). Finite truncations are finite-dimensional Krein spaces, real but not *naturally* integral. "Leech as one IR step" is a metaphor (Rem. 32.8.3).

(7) *Wick rotation $\leftrightarrow$ principalization/capitulation* (§32.9): [ANALOGY], weak. Both kill a kernel, but capitulation carries no form and no positivity, and three theorems cut against the specific claim. The 691 class sits in the minus part, which does not capitulate along the cyclotomic $\mathbb{Z}_{691}$-tower. Opening the Hilbert class field *multiplies* the positivity conditions ($\zeta_H = \prod_\chi L(s,\chi)$). The "$u_4 = 0$ is a Wick rotation" statement of 2608.0081 is a gauge choice, not a Wick rotation.

Nothing here proves RH. The new theorems are Thms. 32.5.1, 32.5.4, 32.5.5, 32.7.2 and Prop. 32.6.1. All are elementary. Thm. 32.7.2(v) is a restatement. We did not search the literature for earlier statements of Thm. 32.7.2(v). It is elementary and may well be known, so we claim no priority.

32.1. Conventions and what is cross-referenced

We use the notation of Lemmas 1, 25, 30 and 31 (see `NOTATION.md`):

- $\widetilde{F}(z) = \int F(u)e^{izu}du$ and $F^*(u) = \overline{F(-u)}$;
- $\rho = \frac{1}{2} + i\gamma_\rho$, $\gamma_\rho = t_\rho - i\delta_\rho$, $\rho^* = 1 - \bar{\rho}$, so that $\gamma_{\rho^*} = \bar{\gamma}_\rho$;
- $\mathcal{Q}(F,G) = \sum_\rho m_\rho \widetilde{F}(\gamma_\rho)\overline{\widetilde{G}(\bar{\gamma}_\rho)} = \iint F(x)\overline{G(y)}\mathcal{W}(x-y)\,dx\,dy$, with $\mathcal{W} = \sum_\rho m_\rho e^{i\gamma_\rho u}$ in $\mathcal{D}'(\mathbb{R})$;
- multiplicatively, $F(u) = e^{u/2}f(e^u)$, $\hat{f}(\rho) = \widetilde{F}(\gamma_\rho)$, $f^\sharp(x) = x^{-1}\overline{f(1/x)}$;
- $\mathcal{H}_\mathcal{Z} = \ell^2(\mathcal{Z}, m)$, $J_{\mathrm{FE}}e_\rho = e_{\rho^*}$, $[x,y]_\mathcal{Z} = \langle J_{\mathrm{FE}}x, y\rangle$, $U_u e_\rho = e^{iu\gamma_\rho}e_\rho$;
- $N_+ = \#\{\rho : \delta_\rho > 0\}$;
- G1–G6 are Lemma 30's axioms (G5 geometric positivity $\equiv$ RH, G6 non-circularity).

$\mathcal{W}$ is real and even, because the zero set is invariant under $\rho \mapsto \bar{\rho}$ and $\rho \mapsto 1-\rho$. Hence $\mathcal{Q}(A+iB) = \mathcal{Q}(A) + \mathcal{Q}(B)$ for real A, B: the cross terms cancel by evenness. Most statements can therefore be made for real F. For $\mathrm{supp}\,F \subset (a,\infty)$ put

$$(R_a F)(u) = F(2a-u), \qquad \mathrm{OS}_a(F) = \iint F(x)F(y)\,\mathcal{W}(x+y-2a)\,dx\,dy \quad (F \text{ real}).$$

$\mathcal{E}_+(a) = C_c^\infty((a,\infty);\mathbb{R})$ is the positive-time space, and $\mathcal{E}_+ = \mathcal{E}_+(0)$.

Not repeated here. Lemma 31 already contains the following:

- the two-positivities dichotomy (§2 there);
- the Milne-zone/ghost identification (Prop. 7.1);
- "no flow-invariant positive half off the line" (Prop. 7.2);
- the list (OS1)–(OS5) with G5 = (OS4)+(OS5) (Prop. 7.3);
- the KMS form of ξ (Thm. 6.5).

Lemma 25 Prop. 6.3 contains the failure of naive RP from the $\log 2$ atom. Here we *refine* these: (OS3), the half-space, is analysed exactly (§32.5), and a correct Euclidean kernel is supplied (§32.7).

32.2. The OS template

DEFINITION 32.2.1 (reflection positivity; [DEF], [217, 216, 112, 207]). A *reflection positive Hilbert space* is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$ with the following properties [207, Def. 2.1.1]:

- $\mathcal{E}$ is a Hilbert space;
- θ is a unitary involution;
- $\mathcal{E}_+ \subset \mathcal{E}$ is a closed subspace;
- $\langle \eta, \eta \rangle_\theta := \langle \eta, \theta\eta \rangle \geq 0$ for $\eta \in \mathcal{E}_+$.

With $\mathcal{N} = \{\eta \in \mathcal{E}_+ : \langle \eta, \eta \rangle_\theta = 0\}$, the *OS space* $\widehat{\mathcal{E}}$ is the completion of $\mathcal{E}_+/\mathcal{N}$. In Euclidean field theory, $\mathcal{E}$ is generated by fields at Euclidean points, θ is time reflection, $\mathcal{E}_+$ is generated by fields at positive time, and the positivity is OS axiom (E2)/(RP) for the Schwinger functions [217, 216], [112, Ch. 6, 19].

PROPOSITION 32.2.2 (the OS machine; [CITED]). *(a) θ-nonnegative subspaces are exactly the graphs of contractions from a subspace of $\mathcal{E}^{\theta=1}$ to $\mathcal{E}^{\theta=-1}$* [207, Lemma 2.2.1]. *If $\mathcal{E}_+$ and $\theta\mathcal{E}_+$ are both θ-positive, then $\mathcal{E}_+ \cap \theta\mathcal{E}_+ = \mathcal{E}^{\theta=1} \cap \mathcal{E}_+$* [207, Rem. 2.2.2].

(b) (Semigroup.) Let $(U_t)_{t\in\mathbb{R}}$ be unitary with $U_t\mathcal{E}_+ \subset \mathcal{E}_+$ for $t \geq 0$ and $\theta U_t \theta = U_{-t}$. Then the OS transforms $\widehat{U}_t$ form a strongly continuous semigroup of hermitian contractions on $\widehat{\mathcal{E}}$ [207, Prop. 4.2.2]. *Hence $\widehat{U}_t = e^{-tH}$ with $H = H^* \geq 0$, and $U_t^c = e^{itH}$ is its Wick rotation* [207, Ex. 3.3.5, Def. 4.2.3].

(c) (Kernels.) Let φ be continuous on an interval (a, b). The kernel $\varphi(\frac{x+y}{2})$ is positive definite iff $\varphi(x) = \int_{\mathbb{R}} e^{-\lambda x} d\mu(\lambda)$ with $\mu \geq 0$ (Widder) [207, Thm. 4.1.1], [288]. *On $(0, \infty)$, φ is completely monotone iff it is a Laplace transform of a positive measure on $[0, \infty)$ (Bernstein) iff it is decreasing and positive definite on the $*$-semigroup $((0, \infty), \mathrm{id})$* [207, Thm. 4.1.2], [28].

(d) (Reflection positive functions.) $\psi : \mathbb{R} \to \mathbb{C}$ is reflection positive for $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id})$ when both *kernels $\psi(x - y)$ (on $\mathbb{R}$) and $\psi(x + y)$ (on $\mathbb{R}_+$) are positive definite* [207, Def. 4.1.3]. *If φ is bounded and positive definite on $([0, \infty), \mathrm{id})$, then $\psi(t) = \varphi(|t|)$ is reflection positive* [207, Cor. 4.3.3].

(e) Every strongly continuous semigroup of hermitian *contractions has a Euclidean realization (minimal unitary dilation)* [207, Prop. 4.3.5].

(Numbering is that of arXiv:1802.09037v1, which we read in full for these items.) Two things matter below. First, Bochner positivity $[\psi(x-y)]$ and OS/Widder positivity $[\psi(x+y)]$ are distinct conditions. Weil's criterion is the *first*. Second, the semigroup is hermitian on $\widehat{\mathcal{E}}$ *because* of RP. Contractivity alone is the spectrum condition, which is the free passivity of Lemma 25 (P0)–(P1). Hermitian-ness is the Hilbert-space positivity. Lemma 31 §2 calls these arrow-of-time and state positivity. We keep them apart throughout.

32.3. Element 1: Euclidean space and the time direction

PROPOSITION 32.3.1 (the scaling line is Minkowskian for ζ; [THM]). *On $\mathcal{H}_{\mathcal{Z}}$ the scaling flow $U_u = e^{iu\gamma}$ is a one-parameter group. Its spectrum is $\{\gamma_\rho\}$, which is symmetric under $\gamma \mapsto -\bar{\gamma}$ and unbounded above and below. It is therefore not of the form e^{iuH} with $H \geq 0$, so it is not the Wick rotation of an OS semigroup on the full zero space. Its positive-frequency part $\{t_\rho > 0\}$ is a canonical half.*

PROOF. $\rho \mapsto \bar{\rho}$ sends $\gamma \mapsto -\bar{\gamma}$. Zeros have $|t_\rho| \to \infty$ in both directions, and no zero has $t_\rho = 0$: for $0 < \sigma < 1$, $\zeta(\sigma) = (1 - 2^{1-\sigma})^{-1}\eta(\sigma) < 0$, because the alternating series $\eta(\sigma) = \sum(-1)^{n-1}n^{-\sigma}$ is > 0 and $1 - 2^{1-\sigma} < 0$. □

So $u = \log x$ is *real time*, the Minkowski direction. This is consistent with Lemma 25 Prop. 6.3(a): Weil positivity produces a *unitary* group. The idele class group $C_{\mathbb{Q}} = \mathbb{R}_+^\times \times \widehat{\mathbb{Z}}^\times$ adds

only a compact (spatial/internal) factor (Lemma 28 Part A). The Euclidean time is identified in §32.7: it is $\tau = \arg x$, the imaginary log-scale direction. Status: [THM] for the identification, [DEF] for the choice.

32.4. Element 2: the reflection

There are three involutions, and it matters which one plays θ:

(i) the *reality* $\rho \mapsto \bar{\rho}$ (antiunitary, commutes with the flow; Lemma 31 Klein-four theorem);
(ii) the *mirror* $\rho \mapsto \rho^* = 1 - \bar{\rho}$ (antiunitary, reverses the flow; it is the $*$ of the convolution algebra, $f \mapsto f^\sharp$);
(iii) the *functional equation* J_{FE} = mirror∘reality, which is linear, unitary, and an involution on $\mathcal{H}_{\mathcal{Z}}$.

On the test-function side, the time reflection $u \mapsto -u$ (i.e. $x \mapsto 1/x$) composed with conjugation is $F \mapsto F^*$, and $\widetilde{F^*}(\gamma) = \overline{\widetilde{F}(\bar{\gamma})}$. Weil's form is

$$\mathcal{Q}(F,G) = [\mathrm{ev}F, \mathrm{ev}G]_{\mathcal{Z}}, \qquad \mathrm{ev}F = (\widetilde{F}(\gamma_\rho))_\rho,$$

i.e. $\langle J_{\mathrm{FE}} x, y\rangle$: a θ-twisted inner product with $\theta = J_{\mathrm{FE}}$ (Lemma 25; Lemma 31 Prop. 7.3). Status [THM]: Weil's form is an OS-type θ-form with $\theta = J_{\mathrm{FE}}$ on zero space, equivalently the mirror on test functions. What is missing is element 3.

32.5. Element 3 (central): the positive-time subspace

32.5.1. Half-lines are vacuous.

THEOREM 32.5.1 (translation invariance; [THM]). *Let* $(T_b F)(u) = F(u-b)$. *Then* $\mathcal{Q}(T_b F, T_b G) = \mathcal{Q}(F,G)$ *exactly, for every* $b \in \mathbb{R}$. *The pole contributions are invariant too. Consequently, for any half-line* $I = (a,\infty)$ *or* $(-\infty, a)$,

$$\mathcal{Q} \geq 0 \text{ on } C_c^\infty(I) \iff \mathcal{Q} \geq 0 \text{ on } C_c^\infty(\mathbb{R}) \iff RH.$$

PROOF. $\widetilde{T_b F}(z) = e^{izb}\widetilde{F}(z)$, so each term picks up $e^{i\gamma b}\overline{e^{i\bar{\gamma}b}} = e^{i\gamma b}e^{-i\gamma b} = 1$. In the explicit formula (Lemma 1) the pole terms are $\widetilde{F}(\pm i/2)\overline{\widetilde{G}(\mp i/2)}$ and pick up $e^{-b/2}e^{b/2} = 1$; multiplicatively, $\lambda^{1/2}\lambda^{-1/2}$. Every $F \in C_c^\infty(\mathbb{R})$ is a translate of an element of $C_c^\infty(I)$. The last equivalence is Weil's criterion. □

REMARK 32.5.2 (correction to the task's expectation). One might expect the pole or archimedean terms to break invariance and make the half-line restriction nontrivial. They do not: $\mathcal{Q}$ is exactly invariant. What the archimedean and prime terms break is the *sign* of the OS form (Thm. 32.5.5), not invariance.

The only nontrivial support condition is therefore the *window length* A ($\mathrm{supp}\, F$ in an interval of length A; Lemma 24's model spaces). Here the literature is precise:

- RH $\iff$ $\mathcal{Q} \geq 0$ on $C_c^\infty(-a,a)$ for every a, and positivity holds for small a: Yoshida [**300**] [CITED, secondary] (content via [**38, 258**]; Yoshida not opened).
- Bombieri re-proved it and showed the minimum is attained [**38**] [CITED].
- Burnol: there is $c > 1$ with positivity for $\mathrm{supp}\, f \subset [1/c, c]$ [**54**] [CITED].
- Suzuki: the lowest eigenvalue is continuous in a, which gives another proof of Yoshida [**258**, Thm. 1.3] [CITED].
- Connes–Consani prove positivity of the *archimedean* contribution for $\mathrm{supp}\, f \subset [2^{-1/2}, 2^{1/2}]$ [**71**, Thm. 1] [CITED].
- Our own certified positive lower bounds on two windows are in Lemmas 10 and 16 (Arb).

PROPOSITION 32.5.3 (small windows do not propagate; [THM] via Lemma 29). *Positivity on all windows of length $\le A_0$ plus translation invariance does not imply positivity on longer windows. In the shifted family of Lemma 29, a single pair with parameters (ε, w) is invisible on every window of length $\le A^{\mathrm{pair}}(\varepsilon, w) = \frac{2}{\sqrt{\varepsilon w}} \arcsin \frac{w-\varepsilon}{w+\varepsilon}$ and visible beyond it (Lemma 29, exact pair threshold).*

So the positive-time space that would make RH an OS statement cannot be a half-line, and windows give a filtration (negative index nondecreasing in A) rather than an RP reduction.

32.5.2. The OS parity identity.

THEOREM 32.5.4 (Weil = domination of the OS form; [THM]). *Let $F \in \mathcal{E}_+(a)$ be real. Then:*

(a) $\mathcal{Q}(R_aF) = \mathcal{Q}(F)$, $\mathcal{Q}(F, R_aF) = \mathrm{OS}_a(F)$, and $\mathcal{Q}(F \pm R_aF) = 2\big(\mathcal{Q}(F) \pm \mathrm{OS}_a(F)\big)$;

(b) for real $G \in C_c^\infty$ with even/odd parts G_e, G_o about a, $\mathcal{Q}(G) = \mathcal{Q}(G_e) + \mathcal{Q}(G_o)$;

(c) the following are equivalent: (i) RH; (ii) $\mathcal{Q} \ge 0$ on $\mathcal{E}_+(a)$; (iii) the Gram matrix $\left(\begin{smallmatrix} \mathcal{Q}(F) & \mathrm{OS}_a(F) \\ \mathrm{OS}_a(F) & \mathcal{Q}(F) \end{smallmatrix}\right) \succeq 0$ for all $F \in \mathcal{E}_+(a)$, i.e. $|\mathrm{OS}_a(F)| \le \mathcal{Q}(F)$;

(d) OS reflection positivity, $\mathrm{OS}_a \ge 0$ on $\mathcal{E}_+(a)$, is false *unconditionally (Lemma 25 Prop. 6.3(b)). It is therefore neither necessary nor sufficient for (i).*

PROOF. (a) $R_a = T_{2a} \circ (u \mapsto -u)$; $\mathcal{W}$ is even and $\mathcal{Q}$ is T-invariant. Next, $\mathcal{Q}(F, R_aF) = \iint F(x)F(2a-y)\mathcal{W}(x-y)$; substituting $y \mapsto 2a - y$ gives $\mathrm{OS}_a(F)$. Bilinearity gives the rest. (b) $\mathcal{Q}(G_e, G_o) = \mathcal{Q}(R_aG_e, R_aG_o) = -\mathcal{Q}(G_e, G_o)$. (c) (i)⇒(iii): apply Weil to $F \pm R_aF$ and use (a). (iii)⇒(ii) is trivial. (ii)⇒(i) is Thm. 32.5.1. □

So OS asks positivity of the *off-diagonal* entry of the $(F, \theta F)$ Gram matrix, while Weil asks positivity of the diagonal and then *implies* that it dominates the off-diagonal. This is the exact sense in which "half-space positivity plus invariance" relates to full Weil positivity: it is equivalent, and the reflection plays no role.

32.5.3. The Euclidean content of $\mathcal{W}$ on the positive half.

THEOREM 32.5.5 (OS spectral decomposition; [THM]). *On $u > 0$, as distributions,*

$$\sum_\rho m_\rho e^{i\gamma_\rho u} = e^{u/2} - \sum_{k\ge1} e^{-(2k+\frac12)u} - \sum_{n\ge2} \Lambda(n) n^{-1/2} \delta_{\log n}.$$

This is Lemma 25's "$2\cosh(u/2) - k_\infty$−primes" with $k_\infty(u) = \sum_{k\ge0} e^{-(2k+\frac12)u}$. Hence for real $F \in \mathcal{E}_+$, with $(F \star F)(v) = \int F(x)F(v-x)dx$,

$$\mathrm{OS}_0(F) = \Big(\int F e^{u/2}\Big)^2 - \sum_{k\ge1} \Big(\int F e^{-(2k+\frac12)u}\Big)^2 - \sum_n \frac{\Lambda(n)}{\sqrt{n}} (F \star F)(\log n).$$

PROOF. Differentiate the von Mangoldt explicit formula $\psi(x) = x - \sum_\rho x^\rho/\rho - \log 2\pi - \frac12 \log(1 - x^{-2})$ $(x > 1)$ in the distributional sense. Then multiply by $x^{1/2}$ and set $x = e^u$: $d\psi = \sum \Lambda(n)\delta_n$, $\frac{d}{dx}\frac12 \log(1-x^{-2}) = \frac{x^{-3}}{1-x^{-2}} = \sum_{k\ge1} x^{-2k-1}$, and $x\,dx$-densities become du. Then substitute into $\iint F(x)F(y)\mathcal{W}(x+y)$; each exponential factorizes. [NUM]: zero side ($\le 8 \cdot 10^4$) and prime side agree to $\le 10^{-14}$ in all five test cases (Table 32.1). □

Reading. In OS language, a Euclidean kernel $\int e^{-\lambda(x+y)} d\mu(\lambda)$ is RP iff $\mu \ge 0$ (Prop. 32.2.2(c)). Term by term:

- *Pole*: one mode with $\mu = +\delta_{-1/2}$. It is RP-positive and rank one, with negative "mass" $-\frac12$ (exponentially *growing*): a tachyon [ANALOGY].
- *Archimedean place*: masses $2k + \frac12$, $k \ge 1$, all with weight -1. It is RP-negative: a tower of Pauli–Villars-type ghosts [**220**] [ANALOGY].

- *Primes*: atoms at $x+y=\log n$. They are not Laplace transforms at all, and they drive $\mathrm{OS}_0 \to -\infty$ on narrow bumps (Lemma 25).
- The *nontrivial zeros* appear only on the Minkowski side, as oscillatory exponents. A zero would become a Euclidean mass only after $u \mapsto i\tau$ (§32.7).

This is the requested comparison of "a Euclidean field on a positive half" with Weil's archimedean term. The archimedean term *is* a Euclidean (Laplace-type, completely monotone) kernel on the positive half, $k_\infty = \sum_{k\ge 0} e^{-(2k+\frac12)u}$, and it is RP-positive as a kernel. But it enters $\mathcal{W}$ with a minus sign. So, as a Euclidean field theory, the archimedean place contributes a *negative-metric* free field ([THM] for the sign, [ANALOGY] for the field language). CC21's archimedean positivity is a different mechanism (trace of the compressed scaling action on Sonin space), and it is limited to small support.

TABLE 32.1. [NUM] OS decomposition for bumps $F = (1-((u-c)/r)^2)_+^6$ (exact $\widetilde{F}$ via $J_{m+1/2}$, quadrature check $6\cdot 10^{-17}$).

c	r	pole	triv.	primes	OS_0	$\mathcal{Q}$	$\mathrm{OS}_0/\mathcal{Q}$
1.0	0.3	0.1139561	0.0002988	0.1383676	−0.0247103	0.0298250	−0.829
0.9	0.25	0.0715726	0.0003413	0.0381874	0.0330438	0.0372342	0.887
$\frac12\log 6$	0.2	0.0456010	0.0002209	0.0114039	0.0339762	0.0437596	0.776
1.5	0.4	0.3344021	0.0000441	0.3340161	0.0003419	0.0155466	0.022
$\frac12\log 2$	0.3	0.0592868	0.0106253	0.0730084	−0.0243470	0.0298250	−0.816

For bumps at $c=\frac12\log n$ with $r\in\{0.2,0.1,0.05,0.02\}$, the ratio is negative for prime powers $n=2,3,4,5,7$ (e.g. $n=2$: $-0.614,-0.392,-0.275,-0.191$) and positive for $n=6$ ($\Lambda=0$: $+0.776,\dots,+0.019$). Always $|\mathrm{OS}_0|<\mathcal{Q}$, as Thm. 32.5.4(c) predicts under RH. Rows 1 and 5 have the same r and identical $\mathcal{Q}$, which checks translation invariance.

32.6. Elements 4 and 5: the RP inequality and the null quotient

Work in a finite zero model: a finite multiset $\mathcal{Z}$ in the strip, symmetric under $\rho\mapsto\bar\rho, 1-\rho$, with $\mathcal{H}_\mathcal{Z}=\mathbb{C}^\mathcal{Z}$, $\theta=J_{\mathrm{FE}}$, and the Krein form $[x,y]=\langle\theta x,y\rangle$.

PROPOSITION 32.6.1 (the OS quotient deletes the off-line zeros; [THM]). *(a) (θ-invariant domain.) If $\mathcal{E}_+$ is θ-invariant and θ-nonnegative, then $\theta=\mathrm{id}$ on $\mathcal{E}_+$. Indeed, for $x\in\mathcal{E}_+$, $y=x-\theta x\in\mathcal{E}_+$ has $\langle y,\theta y\rangle=-\|y\|^2\ge 0$. Weil uses $\mathcal{E}_+=$ everything (ev is onto $\mathbb{C}^\mathcal{Z}$ by Lagrange interpolation in Paley–Wiener space, even from a single short interval). Hence Weil positivity in a model $\iff$ $\theta=I$ $\iff$ no off-line zeros [RESTATEMENT].*

(b) (Index.) $\kappa := \dim \mathcal{H}_\mathcal{Z}^{\theta=-1} = N_+$, the number of negative squares, one per pair plane (Lemma 27).

(c) (Invariant halves.) Let $\mathcal{E}_+$ be U-invariant and θ-nonnegative. For all but countably many u, U_u has distinct eigenvalues, so $\mathcal{E}_+=\mathrm{span}\{e_\rho:\rho\in S\}$ for some S. Nonnegativity forces S to contain at most one of each off-line pair $\{\rho,\rho^*\}$. A maximal such $\mathcal{E}_+$ is $\mathrm{span}\{e_\rho:\delta_\rho=0\}\oplus\mathrm{span}\{e_\rho:\rho\in S'\}$, where S' picks one member of each pair (2^{N_+} choices). Its dimension is $\#\{\delta=0\}+N_+$, which is the maximal dimension of a nonnegative subspace in a Krein space of index N_+.

(d) (Null space.) For each such maximal invariant $\mathcal{E}_+$, the OS null space is $\mathcal{N}=\mathrm{span}\{e_\rho:\rho\in S'\}$, so $\widehat{\mathcal{E}}=\mathrm{span}\{e_\rho:\delta_\rho=0\}$. The OS reconstruction sees exactly the critical zeros. RH $\iff$ $\mathcal{N}=0$ [RESTATEMENT].

PROOF. (c) If $e^{iu\gamma}=e^{iu\gamma'}$ with $\gamma\ne\gamma'$, then $u(\gamma-\gamma')\in 2\pi\mathbb{Z}$, which happens only for $u=0$ when $\gamma-\gamma'\notin\mathbb{R}$, and for countably many u otherwise. Invariant subspaces of a diagonalizable

operator with simple spectrum are spans of eigenvectors. $[e_\rho, e_\sigma] = m_\rho \delta_{\sigma,\rho^*}$, so for an off-line pair $[e_\rho - e_{\rho^*}, e_\rho - e_{\rho^*}] = -2m_\rho < 0$. Hence S cannot contain a whole off-line pair. For $x = a + b$ with a on-line and $b \in \operatorname{span} S'$ we get $[x,x] = [a,a] = \|a\|^2 \geq 0$, since $[a,b] = 0 = [b,b]$. A maximal nonnegative subspace in a Krein space of dimension n and index κ has dimension $n - \kappa$; here $n - \kappa = \#\{\delta = 0\} + 2N_+ - N_+$. (d) follows from the same computation. □

Comparison with Lemma 15 $(1 + N_+)$. Lemma 15's Lorentzian count is $n_+ = 1 + N_+$: the pole is the $+1$ and each off-line pair adds one direction of the "wrong" sign. In the OS picture each off-line pair contributes one null vector. So the N_+ is the same, and the $+1$ is the pole, which is RP-positive in Thm. 32.5.5. Pontryagin's theorem (invariant maximal nonnegative subspaces for Krein-selfadjoint operators) is realized here explicitly by $\operatorname{span}\{e_\rho : \delta_\rho > 0\}$ plus the on-line part, with no appeal to the general theorem. Prop. 32.6.1(c) is the finite form of Lemma 31 Prop. 7.2. The infinite statement for ζ is a restatement. It parallels Connes' theorem that the unitary trace-formula realization sees only critical zeros [**75**].

32.7. Elements 6 and 7: semigroup, Wick rotation, and the Cramér kernel

Definition 32.7.1 ([DEF]). For $\operatorname{Im} z > 0$ put $V(z) = \sum_{t_\rho > 0} m_\rho e^{i\gamma_\rho z}$, and for $\operatorname{Re}\tau > 0$ put $S_+(\tau) = V(i\tau) = \sum_{t_\rho>0} m_\rho e^{-\tau\gamma_\rho}$. Cramér's function [**77**] is $\sum_{t_\rho>0} e^{\rho z} = e^{z/2}V(z)$. For $f \in C_c^\infty(0,\infty)$ put $\mathcal{L}f(z) = \int f(x)e^{-zx}dx$ and

$$\mathcal{R}(f) = \iint \overline{f(x)} f(y)\, S_+(x+y)\, dx\, dy.$$

Theorem 32.7.2 (Euclidean kernel of ζ; [THM], with (v) [RESTATEMENT]). *(i) (Spectrum condition, free.) V converges absolutely and locally uniformly on $\operatorname{Im} z > 0$, so it is holomorphic there. Unconditionally, $|e^{i\gamma z}| = e^{-t\operatorname{Im} z + \delta \operatorname{Re} z} \leq e^{-t\operatorname{Im} z + |\operatorname{Re} z|/2}$.*

(ii) (Boundary value.) In $\mathcal{D}'(\mathbb{R})$, $V(\cdot + i0) = \sum_{t>0} m_\rho e^{i\gamma_\rho u}$ and $\mathcal{W} = V(\cdot + i0) + \overline{V(\cdot + i0)} = 2\operatorname{Re} V(\cdot + i0)$.

(iii) (RP form is Krein-shaped.) $\mathcal{R}(f) = \sum_{t_\rho>0} m_\rho \overline{\mathcal{L}f(\bar\gamma_\rho)}\, \mathcal{L}f(\gamma_\rho) = \sum_{t_\rho>0} m_\rho \mathcal{L}f(\gamma_\rho)\overline{\mathcal{L}f(\gamma_{\rho^*})}$, using $\overline{\mathcal{L}f(\bar\gamma)} = \mathcal{L}\bar f(\gamma)$. This is Weil's shape, restricted to positive frequencies, with $\mathcal{L}$ in place of $\widetilde{\ }$ and the mirror as reflection.

(iv) (Asymptotics, free.) $S_+(\tau) = -\frac{\gamma_E + \log 2\pi\tau}{2\pi\tau} + \frac{7}{8} + o(1)$ as $\tau \downarrow 0$. [NUM]: the residual at $\tau = 0.1, 0.01, 0.002$ is $-0.0749, -0.00730, -0.00144$, about -0.72τ.

*(v) ([RESTATEMENT]) The following are equivalent: (a) RH; (b) S_+ is completely monotone on $(0,\infty)$; (c) $\mathcal{R}(f) \geq 0$ for all $f \in C_c^\infty(0,\infty)$ (OS/Widder reflection positivity of S_+); (d) $\psi(t) = S_+(|t|)$ satisfies the $\psi(x+y)$-half of Def. 4.1.3 of [**207**] (equivalent to (c)). Under (a), ψ is moreover a reflection positive function in the full sense: $\psi(x-y)$ is also positive definite.*

(vi) (Reconstruction under RH.) If RH holds, $f \mapsto (\sqrt{m_\rho}\mathcal{L}f(t_\rho))_{t_\rho>0}$ induces a unitary $\widehat{\mathcal{E}} \cong \ell^2(\{t_\rho > 0\})$. Translation T_s, $s \geq 0$, becomes e^{-sH} with $H = \operatorname{diag}(t_\rho) > 0$, and the Wick rotation e^{iuH} is the scaling flow U_u on the positive-frequency zero sector.

Proof. (i) $\sum_{t>0} m_\rho e^{-ty} < \infty$ for $y > 0$, since $N(T) = O(T\log T)$. (iv) Write $N = M + R$ with $M(t) = \frac{t}{2\pi}\log\frac{t}{2\pi e}$, so $M'(t) = \frac{1}{2\pi}\log\frac{t}{2\pi}$ and $\int_0^\infty e^{-\tau t}dM = -\frac{\gamma_E + \log 2\pi\tau}{2\pi\tau}$. Then $\int e^{-\tau t}dR = \tau\int_0^\infty R(t)e^{-\tau t}dt$ (with $R(0) = 0$), and $R = \frac{7}{8} + S(t) + O(1/t)$ (Riemann–von Mangoldt). Since $\int_0^T S = O(\log T)$, the Abel mean of S vanishes and the Abel mean of R is $\frac{7}{8}$. (ii) For $\phi \in C_c^\infty$, $\int V(x+iy)\phi(x)dx = \sum m_\rho e^{-\gamma_\rho y}\widetilde{\phi}(\gamma_\rho)$, with $|e^{-\gamma y}| = e^{-ty} \leq 1$ and $\sum|\widetilde{\phi}(\gamma_\rho)| < \infty$. Let $y \downarrow 0$. The zeros with $t < 0$ are $\bar\rho$, with $\gamma_{\bar\rho} = -\bar\gamma_\rho$, so their sum is the complex conjugate, and no zero has $t = 0$ (Prop. 32.3.1). (iii) Fubini: $\iint \bar f(x)f(y)e^{-\gamma(x+y)} = \overline{\mathcal{L}f(\bar\gamma)}\mathcal{L}f(\gamma)$, and $\bar\gamma_\rho = \gamma_{\rho^*}$.

(v) (a)⇒(b),(c): under RH, $\gamma_\rho = t_\rho > 0$, so $S_+ = \int e^{-\tau\lambda} d\nu$ with $\nu = \sum m\delta_{t_\rho} \geq 0$, and $\mathcal{R}(f) = \sum m|\mathcal{L}f(t_\rho)|^2$. Also $e^{-t|u|}$ is positive definite (Fourier transform $2t/(t^2+\xi^2) > 0$), which gives the full RP of ψ. (b)⇒(c) is Bernstein. (c)⇔(d) by definition.

(c)⇒(a): $\mathcal{R} \geq 0$ on C_c^∞ is equivalent to positive-definiteness of the continuous kernel $S_+(x+y)$ (approximate identities). By Widder (Prop. 32.2.2(c), applied to $\varphi(s) = S_+(2s)$), $S_+(\tau) = \int_{\mathbb{R}} e^{-\lambda\tau} d\mu(\lambda)$ on $(0,\infty)$ with $\mu \geq 0$. For real $a > 0$ the integral converges, so it converges absolutely on $\operatorname{Re}\tau = a$ with $|\int e^{-\lambda(a-ib)} d\mu| \leq S_+(a)$. Both sides are holomorphic on $\operatorname{Re}\tau > 0$ (the left by (i)), so they agree there. Fix $a > 0$ and put $G(b) = S_+(a-ib)$ for $b \geq 0$. Then $|G| \leq S_+(a)$, and

$$G(b) = \sum_{t_\rho > 0} c_\rho e^{ib\gamma_\rho}, \quad c_\rho = m_\rho e^{-a\gamma_\rho} \neq 0, \quad \sum |c_\rho| < \infty, \quad |e^{ib\gamma_\rho}| = e^{b\delta_\rho} \leq e^{b/2}.$$

For $\operatorname{Re} w > \frac{1}{2}$, termwise integration gives $\int_0^\infty e^{-wb} G(b) db = \sum_\rho c_\rho/(w - w_\rho)$ with $w_\rho = i\gamma_\rho = \delta_\rho + it_\rho$. The left side is holomorphic on $\operatorname{Re} w > 0$ since G is bounded. The right side is meromorphic on $\mathbb{C}$, with a simple pole of residue $c_\rho \neq 0$ at each w_ρ: the series converges uniformly on compacts avoiding the discrete set $\{w_\rho\}$. By the identity theorem they agree on $\{\operatorname{Re} w > 0\} \setminus \{w_\rho\}$, so no w_ρ has $\operatorname{Re} w_\rho = \delta_\rho > 0$. By the functional equation no $t > 0$ zero has $\delta < 0$, and by conjugation the same holds for $t < 0$.

(vi) The map is isometric by (iii). Its range is dense: if $y \in \ell^2$ is orthogonal to the range, then $\int f(x) \sum \overline{y}_\rho \sqrt{m_\rho} e^{-t_\rho x} dx = 0$ for all f. The continuous function $\sum \overline{y}_\rho \sqrt{m_\rho} e^{-t_\rho x}$ (absolutely convergent for $x > 0$) then vanishes, and uniqueness of absolutely convergent generalized Dirichlet series gives $y = 0$. Finally $\mathcal{L}(T_s f)(t) = e^{-st}\mathcal{L}f(t)$. □

Remark 32.7.3 (what is free and what is not). The OS semigroup $\operatorname{diag}(e^{-s\gamma_\rho})$ is a contraction on $\ell^2(t > 0)$ unconditionally, since $|e^{-s\gamma}| = e^{-st}$. This is the spectrum condition, and it is free. It is *hermitian* iff all γ_ρ are real iff RH. This is Prop. 32.2.2(b) read backwards, and it is the precise relation to the passivity ladder. Lemma 25's (P0)–(P1) contraction $e^{-\tau\beta}$ contracts in the $\operatorname{Re}\rho$ direction. The Cramér semigroup contracts in the $\operatorname{Im}\rho$ direction. Both are free, and neither is hermitian off the line.

Corollary 32.7.4 (Euclidean time = $\arg x$; [THM]). *Write $x = e^{u+i\tau}$. Then $x^{\rho-1/2} = e^{i\gamma u}e^{-\gamma\tau}$, so $S_+(\tau)$ is the positive-frequency zero sum at $|x| = 1$, $\arg x = \tau$. The Wick rotation $u \mapsto i\tau$ is rotation of the multiplicative variable onto the unit circle. It is Fourier-dual to the rotation $\sigma \leftrightarrow t$ of the s-plane in Lemma 31's KMS statement. This matches Lemma 25's polar coordinates $(\log|x|, \arg x)$.*

Remark 32.7.5 (prime side). Cramér [**77**], Guinand and Illies [**135**] (abstract read), and Jorgenson–Lang [**142**] show that V, after removing an explicit $\log z$ term, continues meromorphically with simple poles tied to prime powers, and satisfies a Guinand-type functional equation [CITED, secondary]. So S_+ has an explicit-formula expression. Whether its complete monotonicity can be read off from the prime side is exactly the open content (§32.11).

Proposition 32.7.6 (the free Euclidean partner; [THM] via Lemma 31). $\xi(\frac{1}{2}+\tau) = 4\int \Phi(u) e^{2\tau u} du$ *with $\Phi > 0$ (Lemma 31 Thm. 6.5). So $\tau \mapsto \xi(\frac{1}{2}+\tau)$ is exponentially convex (Widder-RP) unconditionally. [NUM]: Hankel matrices $[\xi(\frac{1}{2}+\tau_i+\tau_j)]$ on $\tau \in \{0, 0.25, \ldots, 2\}$ (mpmath, 80 digits) are positive definite, with minimum eigenvalue $9.09 \cdot 10^{-22}$. This is positivity of the theta density Φ, not of the zero measure: the wrong positivity (Lemma 31 §2).*

32.8. Polarization: Riemann forms, Sommerfeld, and which half matters

Definition 32.8.1 (Riemann form; [DEF], [**194**, §I.2]). Let $X = V/\Lambda$ be a complex torus. A *Riemann form* is an alternating $E : \Lambda \times \Lambda \to \mathbb{Z}$ such that:

- (compatibility) $E_{\mathbb{R}}(iv, iw) = E_{\mathbb{R}}(v, w)$;

- (positivity) the Hermitian form $H(v,w) = E_{\mathbb{R}}(iv,w) + iE_{\mathbb{R}}(v,w)$ is positive definite.

(Hermitian $H \leftrightarrow$ alternating $E = \operatorname{Im} H$ with $E(iv,iw) = E(v,w)$ is [194, Lemma 2.4].)

PROPOSITION 32.8.2 (the two halves; [CITED]/[THM]). *(a) (Integral half = prequantization.)* $H^2(X,\mathbb{Z}) = \mathrm{Alt}^2(\Lambda,\mathbb{Z})$*, so a constant 2-form* $\omega = E_{\mathbb{R}}$ *has integral class iff* $E(\Lambda,\Lambda) \subset \mathbb{Z}$*. Equivalently, its flux through every lattice 2-torus* $T(\lambda,\mu)$ *is* $E(\lambda,\mu) \in \mathbb{Z}$*. That is the Bohr–Sommerfeld/Kostant–Souriau condition for a prequantum line bundle* [160] *([THM] for the torus identification).*

(b) (Positive half = positive Kähler polarization.) $H > 0$ *iff* $g = E_{\mathbb{R}}(J\cdot,\cdot)$ *is a positive* J*-compatible metric. Under Kodaira, an integral Kähler class is ample* [158]*; a torus is algebraic iff it admits a Riemann form* [194, Thm. 2.8]*, and* E_λ *is a nondegenerate Riemann form iff* λ *is ample* [194]*. For ample* L*,* $\chi(L) = (D^g)/g!$ *and* $\deg \phi_L = \chi(L)^2$ [194, Thm. 11.1]*: the quantum Hilbert space* $H^0(X,L)$ *has dimension* $\chi(L)$*.*

(c) (Independence.) Integral, compatible, indefinite*:* $E = E_1 \oplus (-E_2)$ *on* $E_1 \times E_2$ *(elliptic curves).* Positive, compatible, non-integral*:* cE_0 *with* $c \notin \mathbb{Q}$ *on an elliptic curve. For* $g \geq 2$*, a generic torus has no nonzero integral compatible form but carries positive flat Kähler metrics (Siegel;* [194, Rem. 2.7]*).*

(d) (Integral + positive ⇒ Rosati.) For a polarization by ample D*,* $\mathrm{Tr}(\alpha\alpha^\dagger) = \frac{2g}{(D^g)}(D^{g-1} \cdot \alpha^* D) > 0$ [194, Thm. 14.3]*. Positivity of the Rosati involution is the input to Weil's proof of RH for curves over* $\mathbb{F}_q$ *via Jacobians (Milne's historical notes* [194]*).*

(e) (Curves.) For a curve C*, the intersection form on* $H_1(C,\mathbb{Z})$ *is a Riemann form for* $J(C)$ [194, Thm. 18.5]*. Integrality is automatic (cup product into* $H^2(C,\mathbb{Z}) = \mathbb{Z}$*). Positivity is supplied by the complex structure* J*: Riemann's bilinear relations,* $\frac{i}{2}\int \omega \wedge \bar{\omega} > 0$*.*

J. W.'s question: is the Hermitian form the Sommerfeld integral condition? Half of it. The Sommerfeld/Bohr quantization condition is the *integrality* of $E = \operatorname{Im} H$, i.e. quantized flux. The *positivity* of $H = E(i\cdot,\cdot) + iE$ is a separate condition: choosing a positive polarization (a Kähler structure compatible with E). Prop. 32.8.2(c) shows that neither implies the other. [THM].

32.8.1. The Spec $\mathbb{Z}$ target as a triple. Following `briefs/prequantization_brief.md` (not edited), the target is

$$(\Lambda,\ E : \Lambda \times \Lambda \to \mathbb{Z} \text{ integral},\ J \text{ with } E(x,Jx) > 0),$$

with the curve case realized by $(H_1(C,\mathbb{Z}), \text{cup}, \text{Hodge } J)$, or by $\mathrm{NS}(C \times C)$ with the Hodge index form. Status by half:

- *Integral half at the finite places: present.* Arakelov divisors are integral at p.
- *Integral half at* ∞*: absent.* $\widehat{\mathrm{Pic}}(\mathrm{Spec}\,\mathbb{Z}) \cong \mathbb{R}$ via $\widehat{\deg}$ [THM]. Indeed, $\mathrm{div}(a) = \sum_p v_p(a)[p] - \log|a|[\infty]$ has degree 0 by the product formula. Any $\sum n_p[p] + x[\infty]$ is equivalent to $(x + \sum n_p \log p)[\infty]$, by subtracting $\mathrm{div}(\prod p^{n_p})$. So the class group is $\mathbb{R}$, the archimedean coefficient is real, and integrality is lost exactly at ∞ [208].
- *No single rescaling makes the prime side rational* [THM]: $\{\log p\}$ is $\mathbb{Q}$-linearly independent (unique factorization). The atoms of $\mathcal{W}$ sit at $\log n$ with weights $\Lambda(n)/\sqrt{n}$.
- *Positive half: this is G5.* Weil positivity is precisely "$E(x,Jx) > 0$" with no integral lattice in sight.
- *Candidates for* J *at* ∞: the archimedean Euler factor and its Deninger regularized-determinant interpretation [90] [ANALOGY]. The brief's Q-D ("Sommerfeld at ∞": a line bundle at ∞ fixing x) asks for the *integral* half there. Prop. 32.8.2(c) says that even a positive answer to Q-D would not give positivity.
- *Arithmetic surfaces*: Faltings' arithmetic Hodge index theorem [98] [CITED, secondary] and Hriljac's height pairing [134] (bibliographic only) give integral-plus-signature structures

on arithmetic surfaces over $\operatorname{Spec}\mathcal{O}_K$, not on "$\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$". No transfer to Weil's form is known to us.

32.8.2. Finite truncations, the Leech remark, and integrality.

Remark 32.8.3 (Leech; [ANALOGY]). "The Leech lattice as one IR step from an infinite-rank UV object" is a metaphor. No known sequence of truncations of the Weil form has a rank-24 member with an integral (let alone even unimodular) structure. We know of no construction that places Λ_{24} among Weil-form truncations.

Proposition 32.8.4 (truncations; [THM]). *Truncations of $\mathcal{Q}$, by height T (zeros with $|t_\rho|\le T$) or by window A (Lemma 24's model spaces), are finite-dimensional Krein spaces. For the height cutoff the dimension is $N(T)\sim\frac{T}{2\pi}\log T$, and the index is the number of off-line pairs below T. As $T\to\infty$ (respectively $A\to\infty$) they converge to the full form. Their Gram entries are real numbers built from* $\log p$, $\Lambda(n)/\sqrt{n}$ *and archimedean integrals: real, not integral. However,* any *nondegenerate real symmetric form on $\mathbb{R}^n$ is $\mathbb{Z}$-valued on* some *basis (Sylvester: diagonalize and rescale to ±1). So the honest statement is the following: integrality fails at finite cutoffs* only as the absence of a natural lattice compatible across cutoffs. *In the curve case such a lattice exists ($s_n=\#C(\mathbb{F}_{q^n})$ data are integers and* $\mathrm{NS}(C\times C)$ *is a lattice).*

Question 32.8.5 ([OPEN]). Does some natural truncation of $\mathcal{Q}$ carry an integral structure? That is: a $\mathbb{Z}$-lattice Λ_T in the truncated test space on which a suitably normalized $\mathcal{Q}$ is $\mathbb{Z}$-valued, with $\Lambda_T\subset\Lambda_{T'}$ compatible for $T<T'$. Known integral structures of arithmetic intersection type (Faltings, Hriljac) live on arithmetic surfaces and are not known to us to restrict to Weil's form. For $\operatorname{Spec}\mathbb{Z}$ itself: *none known to us.*

Which half matters for RH. The positivity half. Integrality without positivity gives nothing: the integral indefinite example of Prop. 32.8.2(c). Positivity alone is RH (G5). Integrality matters for RH only through the mechanism of the curve proof, where it is what makes positivity *provable*: an ample class in a lattice, with $\#C(\mathbb{F}_{q^n})\in\mathbb{Z}$ feeding Castelnuovo–Severi. That is the "Sommerfeld at ∞" hope, [ANALOGY].

32.9. Element 8: Wick rotation and principalization/capitulation

32.9.1. J. W.'s hypothesis, quoted exactly.

From **TAA Monograph II, "The Adelic Field Quantization" (archive 2608.0072v1), §3.3 "Clairaut's Theorem as the Nilpotent Gauge Invariant ($d^2U=0$)", p. 9**:

> "The sub-quantum engine executes this truncation via the Wick-Principalization Transform. First, the system performs a Wick Rotation ($t\to-i\tau$), flipping the metric signature from Hyperbolic Lorentzian $(+,-,-,-)$ to Elliptic Euclidean $(+,+,+,+)$. This operation acts as an instantaneous "Apoapsis Shift," transforming the wildly vibrating, precessional wave into a frozen, Euclidean statistical instanton. Second, the transform opens up the Hilbert Class Field (H)—the maximal unramified abelian extension of the base field. By injecting complex algebraic extension roots directly into the field structure, the non-principal ideal loop of the χ_{11} eigenspace is completely unwound, cleared of its branch cuts, and flattened into a perfectly smooth, principal ideal generated by a single global coordinate."

Related passages: **2608.0081v1, §5.2 "Fibration Inversion, Wick Rotation, and Stokes' Flow", p. 10**: "This parameter freeze $[u_4=0]$ is the literal, first-principles execution of the Wick Rotation ($t\to-i\tau$). By forcing $u_4=0$, the metric signature of the flow equations flips dynamically from Hyperbolic Lorentzian to Elliptic Euclidean…" and "…the non-principal ideal loop of the Cl^-_{691} defect is completely unwound… flattened into a perfectly smooth, principal ideal generated by a single global coordinate axis." **2609.0007v1, §13.2, p. 27**: "…At the 691 unipotent puncture

where unique factorization shatters, the universe performs a cosmic Wick rotation to open up the Hilbert Class Field, unrolling and flattening non-principal ideal loops into principal ones."

32.9.2. The theorems.

PROPOSITION 32.9.1 ([CITED]). *(a) (Principal ideal theorem.) Every ideal of K becomes principal in the Hilbert class field H. Hilbert conjectured this; Artin reduced it to a group-theoretic statement about the transfer* [**12**]*, and Furtwängler proved it* [**108**].

(b) (Hilbert 94.) For cyclic unramified L/K, $[L : K]$ divides the order of the capitulation kernel $\ker(\mathrm{Cl}_K \to \mathrm{Cl}_L)$ [**35**, Intro.].

(c) (Idelic description.) $\ker j \cong C_L^G/C_K$*, i.e. capitulation measures the failure of Galois descent for idele classes* [**35**, Thm. 1.1] *(arXiv).*

(d) (Minus parts do not capitulate.) For $p \neq 2$ and a CM field containing μ_p, there is no capitulation on the minus part of the p-class groups along the cyclotomic $\mathbb{Z}_p$-extension. This is "well known" per [**58**, §2]*. For $p = 2$ it fails in general; see* [**201**].

(e) (Greenberg/Jaulent.) Greenberg's conjecture (vanishing of λ, μ for totally real fields [**115**]*) is equivalent to capitulation of the logarithmic class group (Jaulent, as reported in* [**113**]*) [CITED, secondary].*

(f) (691.) $B_{12} = -691/2730$*. By Herbrand–Ribet* [**235**], [**280**, Ch. 6] *[CITED, secondary] (statement via the Wikipedia article), the ω^{-11}-eigenspace ($= \omega^{679}$, an* odd *character) of the 691-class group of $\mathbb{Q}(\mu_{691})$ is nonzero. So the TAA "$\chi_{11}/\mathrm{Cl}_{691}^-$ defect" lies in the* minus *part. (The sign of the exponent depends on the idempotent convention; the parity does not.)*

(g) (Artin factorization.) $\zeta_H(s) = \prod_{\chi \in \widehat{\mathrm{Cl}_K}} L(s, \chi)$ *for the Hilbert class field* [**208**]*. For $K = \mathbb{Q}$,* $\mathrm{Cl} = 1$ *and $H = \mathbb{Q}$.*

32.9.3. Assessment.

(1) *What matches* [ANALOGY]. Both the OS quotient and capitulation are "passing to a larger/new object kills a kernel": null vectors $\mathcal{N}$ in OS, the capitulation kernel in class field theory. Complex conjugation on a CM field is an involution with a $\pm$ splitting, like θ.

(2) *What does not match* [THM]. Capitulation carries no bilinear form and no positivity. There is nothing for it to make positive. The OS quotient is defined by a form, while capitulation is a map of finite abelian groups. Wick rotation (analytic continuation $t \to -i\tau$) is reversible, whereas both the OS quotient and capitulation discard information. A Wick rotation is not a field extension: it changes the real form of a complexified group/time, not the base field.

(3) *The specific 691 claim* [THM] via Prop. 32.9.1(d),(f). The 691 class is in the minus part, and the minus part does *not* capitulate along the cyclotomic $\mathbb{Z}_{691}$-tower, the natural arithmetic "time". It dies only in the Hilbert class field (Furtwängler), which is unramified, not cyclotomic. So the "Euclidean" step that unwinds it is not a tower/flow step.

(4) *Positivity* [THM] via (g). Opening H replaces one zeta function by a product of h_K L-functions. RH for ζ_H is RH for all of them. Capitulation thus *adds* positivity conditions rather than supplying one. Over $\mathbb{Q}$ there is nothing to capitulate.

(5) $u_4 = 0$ [THM]. In Kustaanheimo–Stiefel coordinates (Lemma 31 §4), $u_4 = 0$ is a choice of section of the $U(1)$ fibre (gauge fixing of the bilinear constraint). It does not continue time to imaginary values, and it changes no signature: the KS energy relation $\omega^2 = -E/2$ is unchanged. Calling it a Wick rotation is not correct.

(6) *Verdict.* [ANALOGY], weak. Where the analogy could be made precise is the *null quotient*: both kill something. In the zeta setting, what OS kills are exactly the off-line zeros (Prop. 32.6.1(d)). That is RH restated, not principalization.

32.10. Concluding table

Element	OS	ζ counterpart	Status	Contribution to positivity
1 Euclidean time	$x^0 \to$ Euclidean τ	$\tau = \arg x$ (Cor. 32.7.4); $u = \log x$ is real time (Prop. 32.3.1)	[THM], [DEF]	none (setting)
2 Reflection θ	time reflection	mirror $\rho \mapsto \rho^*$; $\theta = J_{\mathrm{FE}}$ on $\mathcal{H}_{\mathcal{Z}}$	[THM]	defines the form; its -1 space is the ghosts
3 $\mathcal{E}_+$	positive-time fields	half-line: vacuous (Thm. 32.5.1); windows: filtration; Cramér $\mathcal{E}_+$	[THM]	Weil = domination $\lvert\mathrm{OS}_a\rvert \le \mathcal{Q}$ (Thm. 32.5.4)
4 RP inequality	$\langle\theta\eta, \eta\rangle \ge 0$	for $\mathcal{W}$: false (primes, trivial-zero ghosts); for S_+: $\iff$ RH	[THM], [RESTATEMENT]	*the* missing element
5 Null quotient	$\mathcal{E}_+/\mathcal{N}$	$\mathcal{N}$ = off-line zeros (Prop. 32.6.1); N_+ vs Lemma 15's $1 + N_+$	[THM] (finite), [RESTATEMENT]	RH $\iff \mathcal{N} = 0$
6 Semigroup	e^{-tH}, $H \ge 0$	$\mathrm{diag}(e^{-s\gamma_\rho})$: contraction free; hermitian $\iff$ RH	[THM]	spectrum condition free; hermitian-ness = G5
7 Wick rotation	$e^{-tH} \to e^{itH}$	$S_+(\tau) \to V(u + i0)$, $\mathcal{W} = 2\,\mathrm{Re}\,V$	[THM]	transports positivity; creates none
8 Capitulation	–	principal ideal thm, minus-part non-capitulation, $\zeta_H = \prod L$	[ANALOGY] (weak)	none; adds conditions
Polarization	–	integral half fails at ∞; positive half = G5	[THM], [RESTATEMENT], [OPEN]	positivity half = RH

32.11. The single missing element, and open questions

The missing element ([RESTATEMENT]). A *zero-free* (G6) construction of a positive measure $\nu \ge 0$ on $(0, \infty)$ with

$$\sum_{t_\rho > 0} m_\rho e^{-\tau\gamma_\rho} = \int e^{-\tau\lambda} d\nu(\lambda) \qquad (\tau > 0),$$

i.e. complete monotonicity of the Euclidean kernel S_+, proved from the prime side. Equivalently: an OS positive-time space whose reconstruction is Weil's Hilbert space. Everything else in the OS template is present unconditionally: the reflection, the spectrum condition, the Wick rotation, the boundary-value identity $\mathcal{W} = 2\,\mathrm{Re}\,V$. This is G5 *localized*: (OS4) for the specific kernel S_+.

Question 32.11.1 ([OPEN]). Cramér/Guinand/Illies give V an explicit prime-side continuation. Is there a prime-side expression for the Bernstein measure test $(-1)^n S_+^{(n)} \ge 0$, even asymptotically as $n \to \infty$ (Post–Widder inversion), that does not reintroduce the zeros?

Question 32.11.2 ([OPEN]). Is there a reflection "elsewhere"? That is, a pair $(\theta, \mathcal{E}_+)$ with $\theta \ne$ mirror (e.g. the p-adic $x \mapsto 1/x$ at a finite place, or the Sonin/prolate pair of CC21) whose OS quotient is Weil's Hilbert space. By Thm. 32.5.1 it cannot be a half-line in $\log x$. By Prop. 32.6.1(a) it cannot be θ-invariant.

QUESTION 32.11.3 ([OPEN], = Q. 32.8.5 and brief Q-D). Is there a natural integral lattice on truncations of $\mathcal{Q}$, compatible across cutoffs, whose "Sommerfeld at ∞" would supply the integrality half (and hence a curve-type route to positivity)? None is known to us.

QUESTION 32.11.4 (obstruction, [THM]). Any OS route must explain the sign of the archimedean tower in Thm. 32.5.5. On the positive half, $\mathcal{W}$ is (tachyon) − (Pauli–Villars ghosts) − (prime atoms). A Euclidean structure that reconstructs Weil's space must make this combination dominated by $\mathcal{Q}$ (Thm. 32.5.4). It cannot make it positive.

Status summary

Theorems (proved here): Thm. 32.5.1; Thm. 32.5.4; Thm. 32.5.5; Prop. 32.6.1(a)–(d) (finite models); Thm. 32.7.2(i)–(iii),(vi) and the equivalence in (v); Cor. 32.7.4; Prop. 32.8.4; $\widehat{\mathrm{Pic}}(\mathrm{Spec}\,\mathbb{Z}) \cong \mathbb{R}$; $\mathbb{Q}$-independence of $\log p$; the parts of the capitulation assessment that follow from Prop. 32.9.1. **Restatements**: Weil positivity = θ-trivial OS; $\mathcal{N} = 0$; complete monotonicity of S_+. **Analogies**: tachyon/Pauli–Villars reading; Deninger J; Leech; Wick $\leftrightarrow$ capitulation (weak). **Open**: §32.11. **Numerics**: Table 32.1, S_+ asymptotics, ξ Hankel test. Floating-point Hankel tests of S_+ were at noise level ($\sim 10^{-20}$) and are *not* claimed.

CHAPTER 33

Lemma 33: The anomalous-dispersion pattern of $\zeta(\frac{1}{2}+ix)$ near $x=0$, and causality versus passivity. Kramers–Kronig, positive-real functions, Lagarias' positivity, Foster reactances, de Branges and Krein realizations

Standalone note `lemma33.tex`, October 21, 2026.

Summary. Tags. [THM] proved here; [CITED] checked at the source (`lit/lemma33/verified_refs.md`); [CITED, secondary] checked only through a named secondary source; [DEF] definition; [RESTATEMENT] equivalent to RH (or to a named open rung), no progress; [ANALOGY] analogy only; [OPEN] open; [NUM] numerical (`num/lemma33_checks.py`, `num/lemma33_disc.py`; 107 614 Platt zeros up to height $8\cdot10^4$).

Verdict. J. W.'s observation is correct and has an exact explanation. Causality is *not* the missing link: it is free. The missing link is *passivity*, and passivity is RH.

(1) *The dispersion pattern* (Thm. 33.2.1, [THM]). Near $x=0$, $\zeta(\frac{1}{2}+ix)=\frac{1}{s-1}+\gamma+O(|s-1|)$, and on the line $\frac{1}{s-1}=-\frac{2}{1-2ix}$. This is a single-pole (Debye) Lorentzian: absorption $-2/(1+4x^2)$ with HWHM $\frac{1}{2}$ (the distance from the line to $s=1$), dispersion $-4x/(1+4x^2)$, and *negative* strength: a gain line. Re and Im form an exact Hilbert pair (Kramers–Kronig holds). The remainder is governed by the Stieltjes constants: the error drops $0.0376\to0.0012\to0.00005$ at $x=0$ as γ_1,γ_2 are added ([NUM], Table 33.1, Fig. 33.1). The pole is Lemma 32's tachyonic mode $e^{u/2}$ seen in frequency, and it is Lemma 15's "+1".

(2) *Causality versus passivity* (§33.3). For the "ζ-network" $\mathsf{Z}(p)=\frac{\xi'}{\xi}(\frac{1}{2}+p)$, causality in the support sense (impulse response on $u>0$, Kramers–Kronig) holds unconditionally: the impulse response is Weil's distribution $\mathcal{W}$ restricted to $u>0$ (Thm. 33.3.2, [THM]). Passivity (positive-real on $\operatorname{Re}p>0$) is RH (Lagarias, Hinkkanen [CITED]). In finite models Weil's form is exactly twice the absorbed energy of that network (Prop. 33.3.4, [THM]), so Weil positivity = passivity [RESTATEMENT].

(3) *Ladder* (§33.4). New, unconditional *shadow theorem* (Thm. 33.4.2, [THM]): $\operatorname{Re}\frac{\xi'}{\xi}(s)>0$ for $\sigma>\frac{1}{2}$ outside the half-discs $|s-(\frac{1}{2}+it_\rho)|<\delta_\rho$ of the off-line zeros. Consequences: positive-real for $\sigma>\frac{1}{2}$, $|t|\le3\cdot10^{12}-\frac{1}{2}$; positive-real right of an explicit zero-free curve. Each rung P0–P4 of Lemma 25 becomes a positive-real region, and $\Theta=\inf\{\theta:\operatorname{Re}\frac{\xi'}{\xi}>0 \text{ on } \sigma>\theta\}$ exactly (Lagarias' Thm. 1.1 with θ, [CITED]; Thm. 33.4.4). Electrically, $\Theta-\frac{1}{2}$ is the least uniform distortionless loss that makes the ζ-network passive [ANALOGY].

(4) *Lossless boundary and Foster* (§33.5, [THM]). Unconditionally, $\frac{\xi'}{\xi}(\frac{1}{2}+it)=-i\,\Xi'(t)/\Xi(t)$ is purely imaginary, and the boundary absorption spectrum is $\pi\sum_{\delta_\rho=0}m_\rho\delta_{t_\rho}\ge0$: off-line zeros are *invisible* on the line. Under RH, Z is an infinite Foster reactance (series of parallel LC tanks), and the zeros of Ξ' are real and interlace with those of Ξ ([NUM]: all 37 gaps below 120).

(5) *Realization* (§33.6). $E = \Xi + i\Xi'$ is built from ξ without the zeros, and it is Hermite–Biehler $\iff$ RH ([THM] proof, [RESTATEMENT]). New: Lagarias' functions $E_h(z) = \xi(\frac{1}{2} + h - iz)$ are de Branges for every $h \geq \Theta - \frac{1}{2}$, and $\Theta - \frac{1}{2}$ is the exact threshold (Thm. 33.6.2, [THM]). Under RH there is a unique positive Krein string realizing the zeros ([CITED, secondary], [RESTATEMENT]). "Built without the zeros" means: construct the string/Hamiltonian from the Euler product and Γ-data with its positivity manifest. Nothing in hand does this.

(6) *Link to Lemma 32* (§33.7, [THM]). Unconditionally $\mathsf{Z}(p) = 2\int_0^\infty S_+(\tau)\sin(p\tau)\,d\tau$ for $|\operatorname{Im} p| < t_1$. Complete monotonicity of S_+ and positive-realness of Z are the Laplace and Stieltjes faces of one statement: the zero measure ν is a positive measure on $(0,\infty)$. J. W.'s "half-turn" reading (Prop. 33.7.2) and his "arrow of time" question (Rem. 33.7.3) are treated there.

Nothing here proves RH. The new theorems (Thms. 33.2.1, 33.3.2, 33.4.2, 33.5.1, 33.6.2, 33.7.1; Props. 33.3.4, 33.7.2) are elementary, and we claim no priority for them.

33.1. Conventions and cross-references

We follow `NOTATION.md` (Lemmas 1, 25, 30–32): $\widetilde{F}(z) = \int F(u)e^{izu}du$; $\rho = \frac{1}{2} + i\gamma_\rho$ with $\gamma_\rho = t_\rho - i\delta_\rho$, so $\operatorname{Re}\rho = \beta_\rho = \frac{1}{2} + \delta_\rho$; $\rho^* = 1 - \bar{\rho}$; $\Theta = \sup\beta_\rho$; m_ρ is the multiplicity; $\mathcal{W} = \sum_\rho m_\rho e^{i\gamma_\rho u}$ is the Weil distribution (real and even); $N_+ = \#\{\rho : \delta_\rho > 0\}$. $\Xi(z) = \xi(\frac{1}{2} + iz)$ is real entire and even, and its zeros are the γ_ρ. New for this lemma:

$$\mathsf{Z}(p) := \frac{\xi'}{\xi}(\tfrac{1}{2} + p) \quad (\text{"}\zeta\text{-network"}), \qquad m(z) := -\frac{\Xi'}{\Xi}(z), \qquad X(t) := -\frac{\Xi'}{\Xi}(t) \quad (t \in \mathbb{R}).$$

So $p = s - \frac{1}{2}$ is the Laplace (complex-frequency) variable, the critical line is the imaginary p-axis, and $\operatorname{Re} p > 0$ is the open right half-plane $\sigma > \frac{1}{2}$. Davenport [**80**, Ch. 12] gives, with the sum taken over symmetric pairs (absolutely convergent once paired),

$$\begin{aligned} \frac{\xi'}{\xi}(s) &= \sum_\rho \frac{1}{s-\rho}, \qquad \operatorname{Re}\frac{\xi'}{\xi}(s) = \sum_\rho \frac{\sigma - \beta_\rho}{|s-\rho|^2} \quad (\text{absolutely convergent}), \\ \frac{\xi'}{\xi}(s) &= \frac{1}{s} + \frac{1}{s-1} - \frac{\log\pi}{2} + \frac{1}{2}\psi(\tfrac{s}{2}) + \frac{\zeta'}{\zeta}(s). \end{aligned} \tag{33.1.1}$$

Also $\frac{\xi'}{\xi}(1-s) = -\frac{\xi'}{\xi}(s)$, $\frac{\xi'}{\xi}(\frac{1}{2}) = 0$, and $\frac{\xi'}{\xi}(1) = -B = 1 + \frac{\gamma}{2} - \frac{1}{2}\log 4\pi = 0.0230957\ldots$ [**80**, Ch. 12].

Cross-references. Lemma 25, Prop. 1.8 (passivity ladder P0–P4; there "passive" means $0 \leq \beta_\rho \leq 1$ channel by channel, which is P0); Lemma 25, Prop. (Heaviside substitution); Lemma 18, Thm. 3.3 (Θ is the Weil-positivity abscissa of the damped forms); Lemma 29, Prop. (counterexample) (positivity of $w_\sigma = 2\operatorname{Re}\frac{\xi'}{\xi}(\sigma + it)$ on one line does not exclude zeros to its right); Lemma 15 ($\sup n_+ = 1 + N_+$); Lemma 32, Thm. 5.5 ($\mathcal{W}$ on $u > 0$) and Thm. 7.2 (Cramér kernel S_+; RH $\iff$ S_+ completely monotone); Lemma 28 (telegrapher line); Lemma 31, §2 (arrow of time versus states).

Two words used precisely. *Causal*: a response that is the transform of a kernel supported on $u \geq 0$; equivalently (Titchmarsh) its boundary values are a Hilbert (Kramers–Kronig) pair. *Passive*: the response absorbs nonnegative energy for every input; for immittances, equivalent to positive-real (Def. 33.3.1).

33.2. The dispersion pattern near $x = 0$

THEOREM 33.2.1 (the pole is a gain Lorentzian; [THM]). *Let $s = \frac{1}{2} + ix$, $x \in \mathbb{R}$.*

(a) $\zeta(s) = \frac{1}{s-1} + \sum_{n\geq 0}\frac{(-1)^n\gamma_n}{n!}(s-1)^n$ *(Stieltjes constants, $\gamma_0 = \gamma$), the series being entire.*

(b) *On the line,* $\dfrac{1}{s-1} = \dfrac{-\frac{1}{2} - ix}{\frac{1}{4} + x^2} = -\dfrac{2}{1-2ix}$. *Hence*

$$\operatorname{Re}\frac{1}{s-1} = -\frac{2}{1+4x^2}, \qquad \operatorname{Im}\frac{1}{s-1} = -\frac{4x}{1+4x^2}.$$

This is the single-pole (Debye) Lorentzian with relaxation time 2, *HWHM* $\frac{1}{2}$ *in* x, *and strength* -2. *The HWHM is the distance from the critical line to the pole* $s = 1$.

(c) *(Kramers–Kronig.)* $\frac{1}{s-1} = -\int_0^\infty e^{-u/2} e^{ixu}\, du = \widetilde{K}(x)$ *with* $K(u) = -e^{-u/2}\mathbf{1}_{u>0}$. *So* $x \mapsto \frac{1}{s-1}$ *is the boundary value of a function holomorphic in* $\operatorname{Im} x > 0$, *and* $\operatorname{Im}\frac{1}{s-1} = \mathcal{H}[\operatorname{Re}\frac{1}{s-1}]$ *(Hilbert transform). The kernel is causal and decaying, and its sign is negative: a gain line.*

(d) *(Explicit remainder.) For* $\sigma \geq \frac{1}{2}$, $|\zeta(s) - \frac{1}{s-1} - \gamma| \leq |s-1|(2 + 4\max(1, |s|))$.

Proof. (a) is the Laurent expansion of ζ at its simple pole of residue 1; the coefficients are the Stieltjes constants by definition [**27**] (bibliographic). (b) is algebra. (c) $\int_0^\infty e^{-(\frac{1}{2}-ix)u} du = \frac{1}{\frac{1}{2}-ix} = -\frac{1}{s-1}$. Analyticity in $\operatorname{Im} x > 0$ holds because the only pole is at $x = -\frac{i}{2}$, and the Hilbert-pair statement is Titchmarsh's theorem [**268**, Thm. 95] [CITED, secondary]. It can also be checked directly: the Hilbert transform of $1/(1+y^2)$ is $y/(1+y^2)$, applied with $y = 2x$. (d) For $\sigma > 0$, $\zeta(s) = \frac{s}{s-1} - sI(s)$ with $I(s) = \int_1^\infty \{u\} u^{-s-1} du$, and $\gamma = 1 - I(1)$. So $R(s) := \zeta(s) - \frac{1}{s-1} - \gamma = I(1) - sI(s) = -\int_1^s \frac{d}{dw}[wI(w)]\, dw$ along the segment from 1 to s. On that segment $\operatorname{Re} w \geq \frac{1}{2}$, and $|I(w)| \leq \int_1^\infty u^{-\operatorname{Re} w - 1} du \leq 2$, $|I'(w)| \leq \int_1^\infty u^{-\operatorname{Re} w - 1} \log u\, du \leq 4$, $|w| \leq \max(1, |s|)$. □

x	$\zeta(\frac{1}{2} + ix)$	$\frac{1}{s-1} + \gamma$	err_0	err_1 $(+\gamma_1)$	err_2 $(+\gamma_2)$	bound (d)
0	-1.46035	-1.42278	0.0376	0.00116	0.000049	3.0
0.5	$-0.45930 - 0.96125i$	$-0.42278 - 1.0i$	0.0532	0.00234	0.000139	4.24
1	$0.14394 - 0.72210i$	$0.17722 - 0.8i$	0.0847	0.00597	0.00056	7.24
2	$0.44055 - 0.31165i$	$0.45957 - 0.47059i$	0.160	0.0219	0.0037	21.1
3	$0.53274 - 0.07890i$	$0.52316 - 0.32432i$	0.246	0.0533	0.0130	43.1
5	$0.70181 + 0.23104i$	$0.55741 - 0.19802i$	0.453	0.192	0.073	111
10	$1.54490 - 0.11534i$	$0.57223 - 0.09975i$	0.973	1.25	0.95	421

Table 33.1. [NUM] (`num/lemma33_checks.json`, mpmath, 30 digits). $\mathrm{err}_k = |\zeta - (\frac{1}{s-1} + \sum_{n \leq k} (-1)^n \gamma_n (s-1)^n / n!)|$; $\gamma_1 = -0.0728158$, $\gamma_2 = -0.0096904$. J. W.'s values are confirmed. The rigorous bound (d) is valid but crude. The pole term dominates for $|x| \lesssim 2$; beyond $|x| \approx 3$ the Taylor part is no longer small, and from $x \approx 14$ on the zeros take over.

Remark 33.2.2 (reading the pattern). [THM] for the identities; [ANALOGY] for the optics names.

(i) *Two orientations of time.* In the transfer-function convention (e^{-pu} with $p = s - \frac{1}{2} = ix$), the function $\frac{1}{s-1} = \frac{1}{p - \frac{1}{2}}$ is the Laplace transform of the *causal but growing* response $e^{u/2}\mathbf{1}_{u>0}$, or of the *stable but anti-causal* response $-e^{u/2}\mathbf{1}_{u<0}$. No causal stable representation exists, because the pole $p = \frac{1}{2}$ lies in $\operatorname{Re} p > 0$. Thm. 33.2.1(c) is the second representation written with $u \mapsto -u$. Either way the line is unstable or of negative strength: *a gain line.* Kramers–Kronig holds, so a Kramers–Kronig test cannot detect this. It is the textbook distinction between causality and passivity, seen on ζ itself.

(ii) *Lemma 32.* $e^{u/2}$ is exactly the pole term of $\mathcal{W}$ on $u > 0$ (Lemma 32, Thm. 5.5), the RP-positive "tachyonic" mode. J. W.'s Lorentzian is that mode in frequency.

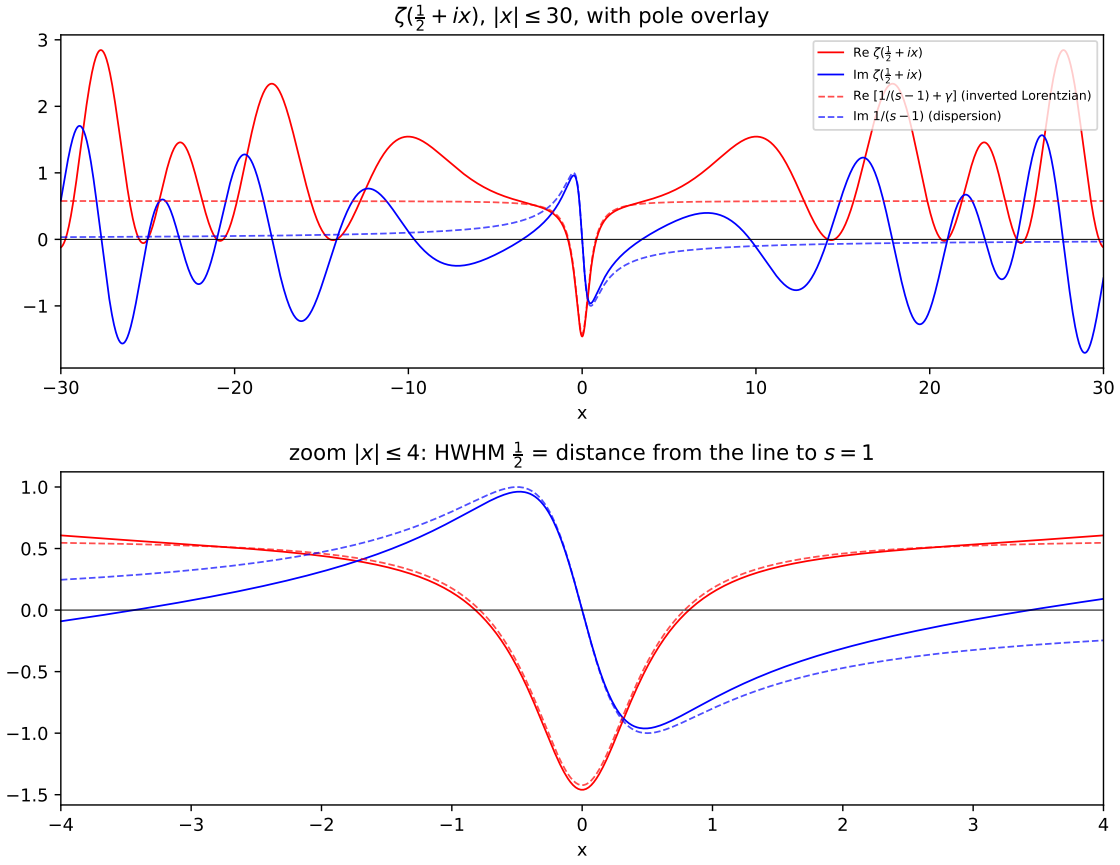

FIGURE 33.1. [NUM] J. W.'s plot reproduced: $\operatorname{Re}\zeta(\frac{1}{2}+ix)$ (red) and $\operatorname{Im}\zeta(\frac{1}{2}+ix)$ (blue) for $|x| \le 30$ (top) and $|x| \le 4$ (bottom). Dashed: the pole Lorentzian $\operatorname{Re}[\frac{1}{s-1}+\gamma]$ and the dispersion profile $\operatorname{Im}\frac{1}{s-1}$. The inverted absorption dip (depth $-2+\gamma$) and the antisymmetric dispersion swing ∓ 1 at $x = \pm\frac{1}{2}$ are the pole. The oscillations at $|x| \ge 14$ are the zeros.

(iii) *Lemma 15.* The right-half-plane singularities of the response $\frac{\zeta'}{\zeta}(\frac{1}{2}+p)$ are the pole $p = \frac{1}{2}$ and the poles $p = \rho - \frac{1}{2}$ with $\delta_\rho > 0$, so there are $1 + N_+$ of them (distinct). This is Lemma 15's index $\sup n_+ = 1 + N_+$, counted as "gain modes". The pole is the one gain mode present unconditionally; RH says it is the only one.

(iv) *Lemma 25.* There the pole channel $\beta = 1$ is the marginal channel $G/C = 1 - \beta = 0$. Marginal in the demodulated coordinates (decay rate 0 for the carrier e^u) is the same as growth $e^{u/2}$ relative to the half-density $e^{u/2}$ of the critical line: the two statements agree.

(v) *ξ removes the line.* In $\xi = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta$ the factor $s-1$ cancels the pole. On the line, $\operatorname{Re}\frac{1}{s} = -\operatorname{Re}\frac{1}{s-1} = \frac{1/2}{1/4+x^2}$, so the gain line of $\frac{1}{s-1}$ and the loss line of $\frac{1}{s}$ cancel exactly in $\frac{\xi'}{\xi}$. What is left on the line is purely reactive (Thm. 33.5.1). So the anomalous-dispersion pattern belongs to ζ, not to the completed object. The completed object's question is whether any *other* gain line exists.

33.3. Causality versus passivity

DEFINITION 33.3.1 ([DEF], classical). A function f holomorphic on $\operatorname{Re} p > 0$ is *positive-real* (PR) if $\operatorname{Re} f \ge 0$ there and $f(\bar{p}) = \overline{f(p)}$. A function g holomorphic on $\operatorname{Im} z > 0$ with $\operatorname{Im} g \ge 0$

is a *Pick* (Herglotz–Nevanlinna) function. f is PR iff $z \mapsto if(-iz)$ is Pick. A PR f is *lossless* (a reactance) if $\operatorname{Re} f = 0$ a.e. on the imaginary axis.

Cited facts.

- [**268**, Thm. 95] [CITED, secondary] (Wikipedia, "Hilbert transform"): for $f \in L^2(\mathbb{R})$ the following are equivalent: f is the boundary value of an H^2 function of the upper half-plane; $\operatorname{Im} f = \mathcal{H} \operatorname{Re} f$; the inverse Fourier transform of f vanishes on a half-line. That is causality $\iff$ analyticity $\iff$ Kramers–Kronig; see also [**212**] (bibliographic).
- [**301**] [CITED] (abstract): for linear time-invariant n-ports, passivity implies causality, and the scattering matrix is bounded-real; the immittances are PR. [**302**] [CITED] (abstract): in the distributional (convolution) setting, a linear translation-invariant operator is passive iff its immittance is PR. See also [**303**] (bibliographic).
- [**51**] [CITED, secondary] (Wikipedia, "Positive-real function"): a rational function is the driving-point impedance of a finite passive network iff it is PR. Cauer's ladder synthesis [**56**] and Darlington's lossless-two-port-plus-resistor synthesis [**78**] are cited bibliographically only.
- [**104**] [CITED, secondary] (Wikipedia, "Foster's reactance theorem"): the reactance of a lossless one-port is strictly increasing in frequency, and its poles and zeros alternate; Foster's canonical forms realize it.

The logic is one-way: *passive* $\Rightarrow$ *causal*, not conversely. Causal responses with gain (negative resistance, inverted populations) satisfy Kramers–Kronig. Thm. 33.2.1(c) is such a response.

THEOREM 33.3.2 (impulse response of the ζ-network; [THM]). *On* $(0,\infty)$ *let* $d\mathcal{W}_+(u) = e^{u/2}du - \sum_{k\ge1} e^{-(2k+\frac12)u}du - \sum_{n\ge2} \Lambda(n) n^{-1/2}\delta_{\log n}$. *This is* $\mathcal{W}|_{(0,\infty)}$ *(Lemma 32, Thm. 5.5). Then for* $\operatorname{Re} p, \operatorname{Re} q > \frac12$,

$$\mathsf{Z}(p) - \mathsf{Z}(q) = \int_{(0,\infty)} \left(e^{-pu} - e^{-qu}\right) d\mathcal{W}_+(u),$$

with the integral absolutely convergent. So Z *is, up to an additive constant (a series resistor fixed by* $\mathsf{Z}(0) = 0$*), the Laplace transform of the causal kernel* $\mathcal{W}_+$. *Formally this is* $\int_0^\infty e^{-pu} \sum_\rho m_\rho e^{i\gamma_\rho u} du = \sum_\rho \frac{m_\rho}{p - i\gamma_\rho}$, *which is* (33.1.1).

PROOF. Use (33.1.1) with $s = \frac12 + p$. The pole term gives $\int_0^\infty e^{-pu} e^{u/2} du = \frac{1}{p-1/2} = \frac{1}{s-1}$. From $\psi(z) = -\gamma + \sum_{k\ge0}(\frac{1}{k+1} - \frac{1}{k+z})$ we get $\frac1s + \frac12\psi(\frac{s}{2}) = \frac12 - \frac{\gamma}{2} + \sum_{k\ge1}(\frac{1}{2k+2} - \frac{1}{2k+s})$. Its p-differences are $-\sum_{k\ge1}(\frac{1}{p+2k+\frac12} - \frac{1}{q+2k+\frac12}) = -\int_0^\infty (e^{-pu} - e^{-qu}) \sum_k e^{-(2k+\frac12)u} du$. Near $u = 0$ the integrand is $O(u) \cdot O(1/u)$, so it is integrable. Finally $\frac{\zeta'}{\zeta}(s) = -\sum \Lambda(n) n^{-1/2} n^{-p}$ for $\operatorname{Re} p > \frac12$. The constant $-\frac12 \log \pi$ cancels in differences. □

REMARK 33.3.3 (what is free and what is not; [THM] for the facts). (i) *Support causality is free.* Every even real kernel splits as $\mathcal{W} = \mathcal{W}_+ + \check{\mathcal{W}}_+$, and $\mathcal{W}_+$ is causal by construction. Its transform is holomorphic where it converges ($\operatorname{Re} p > \frac12$, because of $e^{u/2}$). This is Lemma 32's "spectrum condition is free" in another guise.

(ii) *Stability is P3.* Z continues meromorphically to all of $\mathbb{C}$, with poles exactly at $p = \rho - \frac12$. So it is holomorphic in $\operatorname{Re} p > \Theta - \frac12$ and in no larger half-plane. In the time domain the growing pole mode $e^{u/2}$ must be cancelled by the prime atoms. To relative order $x^{\theta+\varepsilon}$ this is $\psi(x) - x = O(x^{\theta+\varepsilon})$, which is Lemma 25 (P3).

(iii) *Passivity is RH.* Z is PR on $\operatorname{Re} p > 0$ $\iff$ RH (Thm. 33.4.1 below). Causal *and* stable on $\operatorname{Re} p > 0$ is already RH, since it needs holomorphy there.

PROPOSITION 33.3.4 (Weil's form is absorbed energy; finite models; [THM]). *Let* $\{\gamma_j\}$ *be a finite multiset closed under* $\gamma \mapsto \bar\gamma$ *and* $\gamma \mapsto -\gamma$, *with weights* $m_j > 0$. *Put* $W_N(u) = \sum_j m_j e^{i\gamma_j u}$,

$h_N = W_N \mathbf{1}_{u\ge 0}$ and $Z_N(p) = \int_0^\infty e^{-pu} h_N = \sum_j \frac{m_j}{p - i\gamma_j}$. For real $F \in C_c^\infty(\mathbb{R})$,

$$\mathcal{Q}_N(F) := \iint F(x)F(y)W_N(x-y)\,dx\,dy = 2\int F(x)\,(h_N * F)(x)\,dx,$$

i.e. twice the energy absorbed by the one-port with impulse response h_N when it is driven by the current F (voltage $h_N * F$). The following are equivalent: (a) $\mathcal{Q}_N \ge 0$ (passivity); (b) all γ_j are real; (c) Z_N is PR. When they hold, Z_N is lossless: it is a Foster reactance.

Proof. W_N is continuous and even, so splitting $\{x>y\}\cup\{x<y\}$ gives the identity. (b)⇒(a): $\mathcal{Q}_N(F) = \sum m_j |\widetilde{F}(\gamma_j)|^2$. (a)⇒(b): an off-line pair makes $\mathcal{Q}_N$ indefinite (Lemma 15; Lemma 32, Prop. 6.1). (b)⇔(c): with all γ_j real, $\operatorname{Re}\frac{m}{p-i\gamma} = \frac{m\operatorname{Re} p}{|p-i\gamma|^2} > 0$. A non-real γ_j puts a pole in $\operatorname{Re} p > 0$ (one of $i\gamma_j$, $i\bar\gamma_j$ has positive real part), and near a pole $\operatorname{Re} Z_N$ takes both signs. □

For ζ itself, Weil's criterion [**284**] and Thm. 33.4.1 give *Weil positivity* $\iff$ Z *PR* $\iff$ *RH*. This is the precise sense of "passivity on $\mathcal{H}_{1/2}$ = RH" [RESTATEMENT].

33.4. Lagarias' positivity, the shadow theorem, and the ladder

Theorem 33.4.1 (Lagarias [**168**], Thms. 1.1–1.3). *[CITED]; stated earlier by Hinkkanen* [**130**] *([CITED, secondary] via* [**168**]*).*

(a) *For $\theta \ge \frac12$: all $\beta_\rho \le \theta$ $\iff$ $\operatorname{Re}\frac{\xi'}{\xi}(s) > 0$ on $\operatorname{Re} s > \theta$. (Lagarias states this for a class of admissible entire functions; ξ is one. The case $\theta = \frac12$ is: RH $\iff$ $\operatorname{Re}\frac{\xi'}{\xi} > 0$ on $\operatorname{Re} s > \frac12$ $\iff$ Z is PR.)*

(b) *Let $h_{\mathrm{L}}(\sigma) = \inf_{t\in\mathbb{R}} \operatorname{Re}\frac{\xi'}{\xi}(\sigma+it)$. Unconditionally $h_{\mathrm{L}}(\sigma) = \frac{\xi'}{\xi}(\sigma)$ for $\sigma \ge 10$.*

(c) *Under RH, $h_{\mathrm{L}}(\sigma) = \frac{\xi'}{\xi}(\sigma)$ for all $\sigma > \frac12$.*

The correction [**166**] *repairs Lemma 3.1 and a sign in eq. (3.8) of* [**168**]*; the theorems stand. Lagarias remarks that $g(\tau) = i\frac{\xi'}{\xi}(\frac12 + i\tau)$ is of Pick type exactly under RH. With $\xi(\frac12 + i\tau) = \Xi(\tau)$ the printed g equals $-m(\tau)$, so $\operatorname{Im} g \le 0$ on the upper half-plane; the Pick function is $m = -g$. Indeed $\operatorname{Im} m(x+iy) = \operatorname{Re}\frac{\xi'}{\xi}(\frac12 + y - ix)$ (from $\frac{\xi'}{\xi}(1-s) = -\frac{\xi'}{\xi}(s)$). This is only a convention.*

Theorem 33.4.2 (shadow theorem; unconditional; [THM]). *For an off-line zero ρ ($\delta_\rho \ne 0$) let $D_\rho = \{s : \sigma > \frac12,\ |s - (\frac12 + it_\rho)| < |\delta_\rho|\}$. This is the half-disc centred on the critical line at the zero's height, with the zero on its boundary arc. Then*

$$\operatorname{Re}\frac{\xi'}{\xi}(s) > 0 \qquad \text{for all } s \text{ with } \sigma > \tfrac12,\ s \notin \textstyle\bigcup_{\delta_\rho\ne 0} D_\rho,$$

and $\operatorname{Re}\frac{\xi'}{\xi}$ takes negative values in every D_ρ (near ρ itself if $\beta_\rho > \frac12$).

Proof. Group ρ with ρ^*, which has the same ordinate and $\delta_{\rho^*} = -\delta_\rho$. Put $a = \sigma - \frac12 > 0$, $d = |\delta_\rho|$ and $v = t - t_\rho$. By (33.1.1) the pair contributes

$$\frac{a-d}{(a-d)^2+v^2} + \frac{a+d}{(a+d)^2+v^2} = \frac{2a\,(a^2+v^2-d^2)}{\big((a-d)^2+v^2\big)\big((a+d)^2+v^2\big)}.$$

(Expand the numerator: $(a-d)(a+d)^2 + (a+d)(a-d)^2 = 2a(a^2-d^2)$, and the v^2 terms give $2av^2$.) This is >0 iff $a^2+v^2 > d^2$, i.e. $s \notin D_\rho$. An on-line zero contributes $\frac{a}{a^2+v^2} > 0$. All terms are positive outside the union, and there is at least one zero, so the absolutely convergent sum is >0. Inside D_ρ, take $s = \rho - \varepsilon$ ($\beta_\rho > \frac12$): $\operatorname{Re}\frac{\xi'}{\xi}(s) = -m_\rho/\varepsilon + O(1) \to -\infty$, and $|s - \frac12 - it_\rho| = \delta_\rho - \varepsilon < \delta_\rho$. □

Corollary 33.4.3 (unconditional positive-real regions; [THM] given the cited inputs). *(a) $\operatorname{Re}\frac{\xi'}{\xi}(s) > 0$ for $\sigma > \frac12$, $|t| \le H - \frac12$, where $H = 3\cdot 10^{12}$. (By* [**228**] *there are no off-line zeros with $|t_\rho| \le H$, and $|\delta_\rho| < \frac12$.)*

(b) $\operatorname{Re}\frac{\xi'}{\xi}(s) > 0$ *whenever* $|t| \ge 2$ *and* $\sigma \ge 1 - \dfrac{1}{5.558691\log(|t|+\frac12)}$. *([200] gives* $\delta_\rho < \frac12 - \eta(t_\rho)$ *with* $\eta(T) = 1/(5.558691\log T)$ *decreasing, so a shadow meeting height* t *has* $t_\rho < |t| + \frac12$ *and lies in* $\sigma < 1 - \eta(t_\rho) \le 1 - \eta(|t|+\frac12)$*.)*

(c) $\operatorname{Re}\frac{\xi'}{\xi}(\sigma) > 0$ *for real* $\sigma > \frac12$*: shadows have height* $\ge t_1 - \frac12 > 13$.

THEOREM 33.4.4 (ladder → positive-real regions; [THM], with (P3) and (P4) [RESTATEMENT]). *In Lemma 25's ladder:*

(P0) *(*$0 \le \beta \le 1$*, a theorem)* $\iff \operatorname{Re}\frac{\xi'}{\xi} > 0$ *on* $\sigma > 1$*. Here* $\mathsf{Z}(\cdot + \frac12)$ *is PR.*

(P1) *(*$0 < \beta < 1$*, a theorem)* $\iff$ $\frac{\xi'}{\xi}$ *is holomorphic with* $\operatorname{Re} > 0$ *on the closed half-plane* $\sigma \ge 1$.

(P2) *(zero-free region* $\beta \le 1 - c/\log(|t|+2)$*, a theorem)* $\Rightarrow \operatorname{Re}\frac{\xi'}{\xi} > 0$ *on* $\sigma \ge 1 - c/\log(|t|+3)$ *(and on* $\sigma > \frac12$*,* $|t| \le H - \frac12$*). Explicitly, Cor. 33.4.3(b).*

(P3) *(*$\Theta < 1$*, open)* $\iff \operatorname{Re}\frac{\xi'}{\xi} > 0$ *on some half-plane* $\sigma > \theta$ *with* $\theta < 1$*. Exactly:*

$$\Theta = \min\{\theta \ge \tfrac12 :\ \operatorname{Re}\tfrac{\xi'}{\xi} > 0 \text{ on } \sigma > \theta\}.$$

(P4) *(RH)* $\iff$ Z *is PR* $\iff$ m *is Pick.*

Combined with Lemma 18, Thm. 3.3, Θ *is simultaneously the PR abscissa of* $\frac{\xi'}{\xi}$*, the Weil-positivity abscissa of the damped forms, the abscissa of holomorphy of* $\frac{\xi'}{\xi}$ *(Rem. 33.3.3(ii)), and* $\frac12+$ *the de Branges threshold of Thm. 33.6.2.*

PROOF. (P0), (P3), (P4): Thm. 33.4.1(a) with $\theta = 1, \theta, \frac12$; the minimum is attained because the condition at $\theta = \Theta$ is (a) with $\theta = \Theta$. (P1): on $\sigma = 1$ each term $\frac{1-\beta}{|s-\rho|^2}$ is > 0 iff $\beta < 1$, and holomorphy on $\sigma = 1$ means no zero there. (P2): Thm. 33.4.2. The shadow of ρ lies in $\sigma < \beta_\rho$, $|t - t_\rho| < \frac12$. □

REMARK 33.4.5 (electrical reading of Θ; [THM] for the identity, [ANALOGY] for the names). $\mathsf{Z}(a+p)$ is the response of the same network with every mode damped by e^{-au}, i.e. with a uniform distortionless loss a added (Lemma 25, Heaviside substitution). By (P3), $\mathsf{Z}(a+\cdot)$ is PR iff $a \ge \Theta - \frac12$. So $\Theta - \frac12$ *is the least uniform distortionless loss that makes the* ζ*-network passive.* Lemma 25's channelwise "passive" (P0) is the statement at $a = \frac12$. This also explains Lemma 29's counterexample: the line $\sigma = \frac12 + a$ meets D_ρ only if $a < |\delta_\rho|$, and on such a line other zeros can mask the negativity. Positivity of w_σ on one line therefore does not exclude zeros to its right.

[NUM] *Checks* (`num/lemma33_checks.json`, `lemma33_disc.json`). (i) On the lines $\sigma = \frac12 + a$, $a \in \{0.01, 0.05, 0.2, 0.5\}$, $0 \le t \le 100$ (step 0.05), the minimum of $\operatorname{Re}\frac{\xi'}{\xi}$ is attained at $t = 0$, with values $4.62\cdot10^{-4}$, $2.31\cdot10^{-3}$, $9.24\cdot10^{-3}$, 0.0230957. This is consistent with Thm. 33.4.1(c) in the range tested, and the last value equals $-B$ to all printed digits. (ii) For ζ with a planted pair at $t_0 = 50$, $\delta_0 = 0.1$, on a 120×301 grid of $[0.5005, 0.62]\times[48.5, 51.5]$: 1305 grid points have $\operatorname{Re}\frac{\xi'}{\xi} < 0$. All of them lie in the half-disc D_ρ (1546 grid points), and none lies outside it. The pair formula of the proof agrees with direct summation (-0.427666 at $a = 0.05$, $v = 0.08$, $d = 0.1$).

33.5. The lossless boundary and Foster's theorem

THEOREM 33.5.1 (Foster form and boundary spectrum; unconditional; [THM]). *(a)* $\mathsf{Z}(p) = \displaystyle\sum_{t_\rho > 0} \frac{2m_\rho\, p}{p^2 + \gamma_\rho^2}$*, absolutely and locally uniformly convergent off the poles. (Both* ρ *and* ρ^* *occur among the* $t_\rho > 0$*.)*

(b) On the line, $\mathsf{Z}(it) = \frac{\xi'}{\xi}(\frac12 + it) = -i\frac{\Xi'}{\Xi}(t) = iX(t)$ *with* $X = -\frac{d}{dt}\log|\Xi|$ *real: the boundary is purely reactive.*

(c) *For* $\varphi \in C_c(\mathbb{R})$, $\lim_{a\downarrow 0} \int \varphi(t)\, \mathrm{Re}\, \mathsf{Z}(a+it)\, dt = \pi \sum_{\delta_\rho = 0} m_\rho\, \varphi(t_\rho)$. *The boundary absorption spectrum is the positive measure* $\pi \sum_{\delta_\rho=0} m_\rho \delta_{t_\rho}$, *and off-line zeros do not appear in it.*

(d) *Split* $\mathsf{Z} = \mathsf{Z}_{\mathrm{on}} + \mathsf{Z}_{\mathrm{off}}$ *by* $\delta_\rho = 0$ *or not.* Z_{on} *is a PR Foster reactance (the Herglotz completion of the boundary spectrum in (c)).* $\mathsf{Z}_{\mathrm{off}}$ *is purely imaginary on the line, has its poles* $p = \delta_\rho + it_\rho$ *in* $\mathrm{Re}\, p > 0$, *and is negative somewhere near each of them. So:* RH $\iff$ $\mathsf{Z}_{\mathrm{off}} = 0$ $\iff$ Z is the Foster/Herglotz completion of its own boundary data *[RESTATEMENT].*

Proof. (a) ξ is even about $\frac12$ and of order 1, and its zeros in p are $\pm i\gamma_\rho$. So $\xi(\frac12 + p) = \xi(\frac12)\prod_{t_\rho>0}(1 + p^2/\gamma_\rho^2)$ with $\sum |\gamma_\rho|^{-2} < \infty$; take the logarithmic derivative. (b) Ξ is real on $\mathbb{R}$ and $\frac{d}{dt}\log \Xi(t) = i\frac{\xi'}{\xi}(\frac12 + it)$. (c) By (33.1.1), $\mathrm{Re}\, \mathsf{Z}(a+it) = \sum_\rho m_\rho \frac{a-\delta_\rho}{(a-\delta_\rho)^2+(t-t_\rho)^2}$. Group ρ, ρ^* as in Thm. 33.4.2. For a pair with $d = |\delta_\rho| > 0$ the term is $2a(a^2+v^2-d^2)/(((a-d)^2+v^2)((a+d)^2+v^2))$. This is $O(a)$ uniformly in t for $a < d/2$, and $\le C/v^2$ for $|v| \ge 1$, $a \le 1$. On-line zeros give Poisson kernels $\frac{a}{a^2+v^2}$, with $\int \varphi(t)\frac{a}{a^2+(t-t_\rho)^2}dt \to \pi\varphi(t_\rho)$, and the same C/v^2 bound away from $\mathrm{supp}\,\varphi$. Since $\sum_\rho t_\rho^{-2} < \infty$, dominated convergence applies. (d) On $p = iy$ the pair term of an off-line pair is $\frac{-d}{d^2+v^2} + \frac{d}{d^2+v^2} = 0$. The rest is (c) and the pole argument of Thm. 33.4.2. □

Remark 33.5.2 (the answer to "is Kramers–Kronig positivity the missing link?"; [THM] for the facts). On the boundary everything is free: Z satisfies the dispersion relations (it is the transform of the causal $\mathcal{W}_+$, Thm. 33.3.2), and its absorption spectrum is positive (Thm. 33.5.1(c)). What Kramers–Kronig with positive absorption does *not* control is a term that is lossless on the boundary and active inside. That is exactly $\mathsf{Z}_{\mathrm{off}}$. Each off-line quadruple is a "non-Foster" section $\frac{2p}{p^2+\gamma^2} + \frac{2p}{p^2+\bar\gamma^2}$, realizable only with negative elements [ANALOGY]. RH asserts that no such section is present.

Theorem 33.5.3 (RH ⇒ Foster reactance and interlacing; [THM]; the converse direction of (a) is [RESTATEMENT]). *Assume RH. Then:*

(a) Z *is the impedance of an infinite Foster-I network: a series chain of parallel LC tanks, one per* $t_\rho > 0$, *with* $C_\rho = 1/(2m_\rho)$ *and* $L_\rho = 2m_\rho/t_\rho^2$ *(resonance at* t_ρ*). Conversely,* Z *of this form with real* t_ρ *is RH.*

(b) $X'(t) = \sum_\rho \frac{m_\rho}{(t-t_\rho)^2} > 0$ *off the poles (Foster's reactance theorem).*

(c) *The zeros of* Ξ' *are real. On each gap* (t_k, t_{k+1}) *between consecutive distinct zeros of* Ξ, Ξ' *has exactly one zero, and it is simple; at a zero of* Ξ *of multiplicity* m, Ξ' *has a zero of order* $m-1$. *So the zeros of* Ξ *and* Ξ' *interlace. (Also* $\Xi'(0) = 0$ *in the central gap.)*

Proof. (a) A parallel LC tank has impedance $\frac{pL}{1+p^2LC} = \frac{1}{C}\frac{p}{p^2+1/(LC)}$; compare with Thm. 33.5.1(a). (b) Differentiate $X = \sum_\rho \frac{-m_\rho}{t-t_\rho}$ (paired). (c) Near a zero t_k of multiplicity m, $X \sim -m/(t-t_k)$. So on (t_k, t_{k+1}), X increases strictly from $-\infty$ to $+\infty$ and has exactly one zero, which is simple. Off the axis, $\mathrm{Im}\, \frac{\Xi'}{\Xi}(z) = -\mathrm{Im}\, z \sum_\rho \frac{m_\rho}{|z-t_\rho|^2} \neq 0$. This is also the interlacing lemma for Hermite–Biehler functions, [**169**, Lemma 2.2] (de Branges' Lemma 5), applied to Thm. 33.6.1. □

[NUM] (`num/lemma33_checks.json`, N3 and N5). (i) Foster sum over the 107 614 Platt zeros, plus the tail $\int_T^\infty \frac{2p}{t^2}\, dN(t) \approx \frac{2p(\log(T/2\pi)+1)}{2\pi T}$ with $T = 8\cdot 10^4$, against direct mpmath evaluation of (33.1.1): the differences are $2.0\cdot10^{-11}$, $1.2\cdot10^{-10}$, $8.0\cdot10^{-10}$, $1.8\cdot10^{-9}$ at $p = 0.5,\ 0.2{+}3i,\ 0.1{+}20i,\ 1{+}50i$. (At $p = \frac12$ the value is $-B = 0.0230957$.) (ii) For all 37 gaps between consecutive zeros below $t = 120$, X has exactly one sign change and is strictly increasing on a 400-point grid. $\max |\mathrm{Re}\, \frac{\xi'}{\xi}(\frac12 + it)| = 2.4\cdot 10^{-29}$ (working precision). Whether strict monotonicity of X off the zeros *alone* implies RH:

we found no reference either way [OPEN]. (An off-line pair contributes $2m_\rho \operatorname{Re}(t-\gamma_\rho)^{-2}$ to X', which is negative for $|t-t_\rho|<|\delta_\rho|$, but other zeros may mask it.)

33.6. Realization: de Branges, Krein, Darlington, and "built without the zeros"

Recall de Branges' definition ([81], as used in [169, §2]). An entire E is *Hermite–Biehler* (HB, "de Branges") if $|E(\bar z)|<|E(z)|$ for $\operatorname{Im} z>0$. Then $E=A-iB$ with A,B real entire, the space $\mathcal{H}(E)$ exists, and the zeros of A and B are real and interlace [169, Lemma 2.2] [CITED].

THEOREM 33.6.1 (a zero-free de Branges candidate; [THM], the equivalence is [RESTATEMENT]). *Let $E(z)=\Xi(z)+i\Xi'(z)$. It is built from ξ alone, with no knowledge of the zeros. For $\operatorname{Im} z>0$ with $\Xi(z)\neq 0$,*

$$|E(z)|^2-|E(\bar z)|^2=4|\Xi(z)|^2\operatorname{Im} m(z),\qquad \operatorname{Im} m(x+iy)=\operatorname{Re}\frac{\xi'}{\xi}(\tfrac12+y-ix).$$

Hence E is HB $\iff$ RH. Under RH, $A=\Xi$ and $B=-\Xi'$. By de Branges theory ([169, §6], [CITED] as a general statement), a selfadjoint extension of multiplication by z in $\mathcal{H}(E)$ then has spectrum the zero set of A, i.e. $\{t_\rho\}$.

PROOF. Ξ and Ξ' are real entire, so $\overline{E(\bar z)}=\Xi(z)-i\Xi'(z)$. Then $|\Xi+i\Xi'|^2-|\Xi-i\Xi'|^2=4\operatorname{Im}(\Xi\overline{\Xi'})=-4|\Xi|^2\operatorname{Im}\frac{\Xi'}{\Xi}$. The second identity is the convention remark in Thm. 33.4.1. RH $\Rightarrow$ HB: Thm. 33.4.1(a) gives $\operatorname{Im} m>0$ on $\operatorname{Im} z>0$, and under RH Ξ has no zeros there, so there is nothing more to check. HB $\Rightarrow$ RH: if $\Xi(z_0)=0$ with $\operatorname{Im} z_0>0$, then $\operatorname{Im} m$ takes negative values near the pole z_0, and HB fails. □

THEOREM 33.6.2 (the de Branges threshold is $\Theta-\frac12$; unconditional; [THM]). *For $h>0$ let $E_h(z)=\xi(\frac12+h-iz)$. Lagarias [169, Lemma 6.1] [CITED] proves that E_h is HB for $h\geq\frac12$ unconditionally, and for every $h>0$ under RH. More precisely:*

(a) E_h is HB for every $h>0$ with $h\geq\Theta-\frac12$;
(b) if E_h is HB for every $h>h_1$, then $\Theta-\frac12\leq h_1$.

Hence $\Theta-\frac12=\inf\{h_1\geq 0:\ E_h$ is HB for all $h>h_1\}$. In ladder terms: P0/P1 give $h\geq\frac12$ (Lagarias), P3 is "some $h_1<\frac12$", and RH is $h_1=0$.

PROOF. Write $z=x+iy$ with $y>0$, and $\alpha=y-ix$. Then $E_h(z)=\xi(\frac12+h+\alpha)$, and $|E_h(\bar z)|=|\xi(\frac12+\alpha-h)|$ by $|\xi(1-\bar s)|=|\xi(s)|$.

Write each zero as $\frac12+w$ with $w=\delta+i\tau$. Group the Hadamard product $\xi(\frac12+u)=\xi(\frac12)\prod(1-u^2/w^2)$ over the quadruples $\{w,\bar w,-w,-\bar w\}$. This gives

$$\frac{|\xi(\frac12+\alpha+h)|}{|\xi(\frac12+\alpha-h)|}=\prod_{\text{quadruples}}\frac{|\alpha+h-w|\,|\alpha+h-\bar w|\,|\alpha+h+w|\,|\alpha+h+\bar w|}{|\alpha-h+\bar w|\,|\alpha-h+w|\,|\alpha-h-\bar w|\,|\alpha-h-w|}.$$

Pair the numerator and denominator factors in the order written, so that paired factors have equal imaginary parts. Then for each pair

$$|\alpha+h\mp w|^2-|\alpha-h\pm\bar w|^2=4y(h\mp\delta),$$

and likewise for the pairs with w and $\bar w$ exchanged. So every factor is ≥ 1 when $h\geq|\delta|$, and >1 when $h>|\delta|$. There are infinitely many on-line zeros (Hardy), and each gives a factor >1. This proves (a).

For (b), fix x and let $F(\sigma)=\log|\xi(\frac12+\sigma-ix)|$, which is even in σ. Let $\sigma_0>h_1$. Since E_{σ_0} is HB, $F(\sigma_0+y)>F(\sigma_0-y)$ for $y>0$. Letting $y\downarrow 0$ gives $F'(\sigma_0)=\operatorname{Re}\frac{\xi'}{\xi}(\frac12+\sigma_0-ix)\geq 0$ wherever F' is defined. Now suppose a zero had $\delta_0>h_1$. Take $x=-t_{\rho_0}$ and let $\sigma_0\uparrow\delta_0$. Then $F'(\sigma_0)\to-\infty$, a contradiction. □

The unconditional spaces $\mathcal{H}(E_h)$, $h \geq \frac{1}{2}$, realize the *differenced* functions A_h, B_h, which are the subject of [169]. They do not realize Ξ. Moving h from $\frac{1}{2}$ down to 0 is the same as moving the loss of Rem. 33.4.5 down to 0.

PROPOSITION 33.6.3 (Krein string; [CITED, secondary] for the correspondence, [RESTATEMENT] for the equivalence). *Under RH,* $m(z) = \sum_{t_\rho>0} \frac{2m_\rho z}{t_\rho^2 - z^2} = z\,G(z^2)$, *where*

$$G(\lambda) = \sum_{t_\rho>0} \frac{2m_\rho}{t_\rho^2 - \lambda} = \int_{(0,\infty)} \frac{d\Sigma(\xi)}{\xi - \lambda}, \qquad \Sigma = \sum_{t_\rho>0} 2m_\rho\, \delta_{t_\rho^2} \geq 0, \quad \int \frac{d\Sigma}{1+\xi} < \infty.$$

So G is a Stieltjes-class function. By Krein's correspondence (Kac–Krein [144]; we read its content only in the introduction of arXiv:1101.2989, [CITED, secondary]), there is a unique string, i.e. a nonnegative mass distribution on an interval, whose characteristic function is G. Conversely, suppose some string has characteristic function G. Then G has poles only on $(0,\infty)$, so $\gamma_\rho^2 > 0$ for every ρ, and RH holds. So RH $\iff$ "the zeros are the spectrum of a positive Krein string" [RESTATEMENT]. Equivalently, there is a canonical system with nonnegative Hamiltonian [234, 295] *(bibliographic only).*

REMARK 33.6.4 (Darlington, Cauer, Lemma 28, "built without the zeros"). [ANALOGY] except where marked.

(i) *Darlington* [78], bibliographic only. In the classical theory, a PR impedance is realized as a lossless two-port terminated in a resistor. Under RH, Z needs no resistor, because it is already lossless. The shifted $\mathsf{Z}(a+\cdot)$ with $a > 0$ is lossy, and its "termination" is the uniform loss a of Rem. 33.4.5.

(ii) *Cauer* [56], bibliographic only. The continued-fraction (ladder) expansion of a Stieltjes function is the discretized string: positive ladder elements correspond to positive masses and compliances.

(iii) *Lemma 28.* A telegrapher line with $R, L, G, C \geq 0$ is passive *by construction*: positivity is visible in the parameters, and PR follows from it. The ζ-string would be a lossless line with variable $L(x), C(x)$. To get RH this way one would have to *construct* $L(x), C(x) \geq 0$ from the Euler product and the Γ-factor, and then prove that the impedance of that line is Z, never passing through the zeros. That is the precise content of "built without the zeros". The function $E = \Xi + i\Xi'$ of Thm. 33.6.1 is built without the zeros, but its positivity (HB) is RH. So it relocates the problem; it does not solve it.

(iv) *Indefinite strings.* An off-line zero would require an "active" string. We cite no characterization of spectral data for strings or canonical systems with sign-changing mass or Hamiltonian, because we have verified none. Pontryagin-space versions of de Branges theory are not used here.

Lagarias' survey [167] treats de Branges spaces for Dirichlet L-functions; we checked only its bibliographic record and use none of its content.

33.7. Link to Lemma 32: the Cramér kernel and the zero measure

THEOREM 33.7.1 (sine transform; unconditional; [THM]). *Let* $S_+(\tau) = \sum_{t_\rho>0} m_\rho e^{-\tau\gamma_\rho}$ *(Lemma 32, Thm. 7.2; Cramér* [77]*) and let* $t_1 = 14.1347\ldots$. *For* $|\operatorname{Im} p| < t_1$,

$$\mathsf{Z}(p) = 2\int_0^\infty S_+(\tau)\sin(p\tau)\,d\tau.$$

PROOF. For $\operatorname{Re}\gamma > |\operatorname{Im} p|$, $\int_0^\infty e^{-\gamma\tau}\sin(p\tau)\,d\tau = \frac{p}{p^2+\gamma^2}$. Since $|\sin w| \leq |w|e^{|\operatorname{Im} w|}$, we have $\sum_\rho m_\rho \int_0^\infty |e^{-\gamma_\rho\tau}\sin p\tau|\,d\tau \leq \sum_\rho m_\rho \frac{|p|}{(t_\rho - |\operatorname{Im} p|)^2} < \infty$, so Fubini applies. Now use Thm. 33.5.1(a). Near $\tau = 0$, $S_+(\tau) \sim \frac{\log(1/\tau)}{2\pi\tau}$ (Lemma 32), and $\sin p\tau = O(\tau)$, so the integral converges there. $\square$

One measure, many faces ([THM] for the identities, [RESTATEMENT] for every positivity line). Let $\nu = \sum_{t_\rho>0} m_\rho \delta_{\gamma_\rho}$. Off the line it has complex support points. Unconditionally:

$$S_+(\tau) = \int e^{-\tau\lambda} d\nu, \quad \mathsf{Z}(p) = \int \frac{2p}{p^2+\lambda^2} d\nu, \quad m(z) = \int \frac{2z}{\lambda^2 - z^2} d\nu, \quad \mathcal{W} = \int 2\cos(\lambda u)\, d\nu.$$

Each positivity statement below says the same thing: *ν is a positive measure on* $(0,\infty)$, which is RH.

- S_+ completely monotone (Bernstein, Laplace face; Lemma 32, Thm. 7.2).
- Z positive-real (Lagarias; Stieltjes/Foster face).
- m Pick, equivalently E Hermite–Biehler (de Branges face).
- G a Stieltjes function, equivalently a positive Krein string exists (Krein face).
- $\mathcal{W}$ positive-definite (Bochner face; Weil).

Thm. 33.7.1 links the Laplace face to the Stieltjes face unconditionally. The sine kernel does not preserve positivity, though, so neither positivity implies the other except through ν. This is why none of these reformulations is closer to a proof than any other: each is a uniqueness theorem for one representation of the same ν.

Proposition 33.7.2 (J. W.'s "half-turn" reading of complete monotonicity; [THM], with the last clause [RESTATEMENT]). *Identify $\mathbb{C} \cong \mathbb{R}^2$, so that multiplication by i is $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$, with $S^2 = -I$.*

(a) (Minkowski side.) $\frac{d}{du} e^{i\gamma u} = i\gamma\, e^{i\gamma u}$. For real $\gamma > 0$, one derivative acts on the mode as γS: a quarter-turn with dilation γ.

(b) (Wick rotation.) Put $u = i\tau$; then $e^{i\gamma u} = e^{-\gamma\tau}$ and $\frac{d}{d\tau} = i\frac{d}{du}$. So $\frac{d}{d\tau} e^{-\tau\gamma} = i \cdot i\gamma\, e^{-\tau\gamma} = -\gamma\, e^{-\tau\gamma}$, i.e. $\gamma S^2 = -\gamma I$. One Euclidean derivative is an exact half-turn *(times $|\gamma|$) iff $-\gamma \in (-\infty, 0)$, i.e. iff γ is real and positive.*

(c) (Off-line pair.) Let $\gamma = t - i\delta$ ($t > 0$, $\delta \neq 0$) and let its partner be $\bar\gamma = t + i\delta$ (ρ and ρ^*). Write $\gamma = |\gamma| e^{-i\theta}$ with $\theta = \arctan(\delta/t)$. Each derivative multiplies the γ-mode by $-\gamma = |\gamma| e^{i(\pi-\theta)}$ and the partner by $|\gamma| e^{i(\pi+\theta)}$: a rotation by $\pi \mp \theta$, not π. The pair contributes

$$(-1)^n \frac{d^n}{d\tau^n} \Big[m\big(e^{-\tau\gamma} + e^{-\tau\bar\gamma}\big) \Big] = m\big(\gamma^n e^{-\tau\gamma} + \bar\gamma^n e^{-\tau\bar\gamma}\big) = 2m\,|\gamma|^n e^{-\tau t} \cos(n\theta - \tau\delta).$$

*(*Sign convention. *With $\gamma_\rho = t - i\delta$ the phase is $n\theta - \tau\delta$. The form $\cos(n\theta + \tau\delta)$ in the request corresponds to labelling the partner as γ. The pair is invariant under $(\delta, \theta) \mapsto (-\delta, -\theta)$, so the two agree.)*

(d) (First failure of the pair term.) For fixed $\tau \geq 0$, the pair term of $(-1)^n S_+^{(n)}(\tau)$ first becomes negative at $n_* = \min\{n \geq 0 : \cos(n|\theta| - \tau|\delta|) < 0\}$. If $0 \leq \tau|\delta| < \frac{\pi}{2}$, then $n_* = \lfloor (\frac{\pi}{2} + \tau|\delta|)/|\theta| \rfloor + 1$ (when the ratio is not an integer). Examples ([NUM]): $t = 14.1347$, $\delta = 0.1$ gives $\theta = 7.07 \cdot 10^{-3}$ and $n_* = 223$ at $\tau = 0$, 237 at $\tau = 1$; $t = 14.1347$, $\delta = 0.01$ gives $n_* = 2221$; $t = 1000$, $\delta = 0.4$ gives $n_* = 3927$ at $\tau = 0$. This is the pair term only. *For S_+ itself, Lemma 32, Thm. 7.2 guarantees that complete monotonicity fails at* some *(n, τ). The on-line background dominates for small n, and we do not give an explicit first n.*

(e) Hence RH $\iff$ every Euclidean derivative acts on every mode of S_+ as an exact half-turn [RESTATEMENT]. This is Lemma 32, Thm. 7.2 restated.

The action is a rotation with dilation*: multiplication by $-\gamma$ is a similitude of $\mathbb{R}^2$ with determinant $|\gamma|^2$, an element of $\mathbb{C}^\times = \mathbb{R}_{>0} \times SO(2)$. It is not a unimodular shear, and it lies in $SL_2(\mathbb{Z})$ only when $|\gamma| = 1$ and the angle is a multiple of $\frac{\pi}{2}$. The only exact $SL_2(\mathbb{Z})$ content is $S^2 = -I$, i.e.*

"Wick rotation ∘ Minkowski derivative = half-turn" for real γ. The link to Newton's triangle is [ANALOGY].

PROOF. (a)–(c) are differentiation and $\gamma^n = |\gamma|^n e^{-in\theta}$, $e^{-\tau\gamma} = e^{-\tau t}e^{i\tau\delta}$, so $\gamma^n e^{-\tau\gamma} = |\gamma|^n e^{-\tau t}e^{-i(n\theta-\tau\delta)}$; add the conjugate. (d) $n \mapsto n|\theta| - \tau|\delta|$ increases from $-\tau|\delta| > -\frac{\pi}{2}$ and first exceeds $\frac{\pi}{2}$ at n_*. (e) Combine (b) with Lemma 32, Thm. 7.2. □

REMARK 33.7.3 (J. W.: "isn't $\mathcal{E}_+$ the rotation itself, the arrow of time, entropy?"). [THM] for (ii); [ANALOGY] for the entropy language.

(i) In OS, $\mathcal{E}_+$ is a *subspace*: the states supported at positive Euclidean time. The motion is the semigroup $e^{-\tau H}$ acting on it. The rotation is the motion, not the subspace. The precise version of the idea is this: one-directional rotation means the positive-frequency (Hardy) half, i.e. the upper zeros $t_\rho > 0$. That is the half used by V and S_+ in Lemma 32, and it is the natural $\mathcal{E}_+$-candidate.

(ii) *One-sided rotation does not detect off-line zeros* ([THM]). An off-line zero contributes $e^{i\gamma_\rho u} = e^{it_\rho u}e^{\delta_\rho u}$. Its frequency $t_\rho > 0$ is still positive: the mode is a spiral, outward for $\delta_\rho > 0$ and inward for its partner ρ^*. The splitting by the sign of t_ρ is independent of the δ_ρ. So the positive-frequency half and its spectrum condition (holomorphy of V on $\operatorname{Im} z > 0$, Lemma 32, Thm. 7.2) hold unconditionally and carry no information about RH.

(iii) *Dissipation is free.* Entropy production, read as an inward spiral, is passivity in Lemma 25's demodulated sense: decay rates $\beta_\rho, 1-\beta_\rho \geq 0$ (P0), > 0 (P1). These are theorems. Compare Lemma 31, §2 (the arrow of time versus positivity of states).

(iv) *RH is balance, not an arrow.* RH says no mode spirals: $|e^{i\gamma_\rho u}| = 1$, zero net growth per mode, time-reversal balance (unitarity). The functional equation already balances growth *pairwise* (δ against $-\delta$); RH asks for balance mode by mode. In this lemma's language, the PR property of Z on $\operatorname{Re} p > 0$ (Lagarias) is the "lossy" side and losslessness on the line is the "balanced" side. Lossy in the interior is the arrow; Foster-lossless on the line *with no non-Foster part* (Thm. 33.5.1(d)) is RH.

33.8. Concluding table

item	label	content
dispersion at $x = 0$	[THM] [NUM]	pole Debye/Lorentzian, HWHM $\frac{1}{2}$, strength -2 (gain); Kramers–Kronig holds; Stieltjes remainder
pole = gain mode	[THM]	$e^{u/2}$ (Lemma 32); $1 + N_+$ right-half-plane singularities of $\frac{\zeta'}{\zeta}(\frac{1}{2}+p)$ (Lemma 15)
support causality of Z	[THM]	impulse response $\mathcal{W}\|_{(0,\infty)}$ (Thm. 33.3.2); free
stability of Z	[RESTATEMENT] (P3)	holomorphic exactly on $\operatorname{Re} p > \Theta - \frac{1}{2}$
passivity of Z	[CITED] [RESTATEMENT]	PR $\iff$ RH (Lagarias, Hinkkanen); Weil = energy (Prop. 33.3.4)
shadow theorem	[THM] [NUM]	negativity only in half-discs $\|s - \frac{1}{2} - it_\rho\| < \|\delta_\rho\|$
PR regions	[THM]	$\|t\| \leq 3 \cdot 10^{12} - \frac{1}{2}$; right of the MTY curve; P0–P4 mapped; Θ = PR abscissa
boundary spectrum	[THM]	lossless line; absorption $\pi \sum_{\delta=0} m\delta_t$; off-line zeros invisible
Foster/interlacing	[THM] under RH, [NUM]	series LC tanks; zeros of Ξ' real and interlacing
$E = \Xi + i\Xi'$	[THM] [RESTATEMENT]	built without zeros; HB $\iff$ RH

de Branges threshold	[THM]	E_h HB for $h \geq \Theta - \frac{1}{2}$; threshold exactly $\Theta - \frac{1}{2}$
Krein string	[CITED, secondary] [RESTATEMENT]	positive string exists $\iff$ RH
S_+ and Z	[THM] [RESTATEMENT]	sine transform; same ν; CM $\iff$ PR $\iff$ RH
half-turn	[THM] [RESTATEMENT], Newton [ANALOGY]	exact half-turn per derivative $\iff$ γ real; a similitude, in $SL_2(\mathbb{Z})$ only via $S^2 = -I$
arrow of time	[THM] (spiral) [ANALOGY]	positive frequency blind to δ; dissipation free; RH = balance

33.9. Open questions

QUESTION 33.9.1 (a string from the Euler product; [OPEN]). $E_{1/2}(z) = \xi(1 - iz)$ is de Branges unconditionally. Compute (numerically, from Taylor data of ξ at 1) the Stieltjes/Cauer continued fraction of the corresponding $G_{1/2}$, i.e. the discretized Krein string, and follow it as $h \downarrow 0$. Do the ladder elements have a prime-side expression (through (33.1.1) and the Dirichlet series of $\frac{\zeta'}{\zeta}$ at $\sigma = \frac{1}{2} + h > 1$)? Can their positivity be propagated below $h = \frac{1}{2}$? By Thm. 33.6.2, any $h < \frac{1}{2}$ reached would give $\Theta < 1$.

QUESTION 33.9.2 (quantitative shadows; [OPEN]). Thm. 33.4.2 confines the negativity of $\operatorname{Re}\frac{\xi'}{\xi}$ to half-discs of radius $|\delta_\rho|$, and the negativity set always contains a neighbourhood of ρ to its left. Give a lower bound for the area (or for the $\int(\operatorname{Re}\frac{\xi'}{\xi})_-$) inside D_ρ in terms of m_ρ, δ_ρ and the local zero density. The aim is a criterion that a finite computation on a grid (Hinkkanen-type) could use, and which is immune to Lemma 29's masking.

QUESTION 33.9.3 (gain–loss sum rules; [OPEN]). J. W.'s gain line (the pole) and the zeros' absorption lines are tied by unconditional sum rules, for example $\sum_\rho \operatorname{Re}\frac{1}{\rho} = -B = \frac{\xi'}{\xi}(1) = 0.0230957$ [**80**, Ch. 12], and more generally by the moments $\int \varphi\, d(\text{boundary spectrum})$ given by the explicit formula. Is there a Bode/Fano-type integral constraint[1] (weighted by the pole Lorentzian $\frac{1/2}{1/4+t^2}$) in which the non-Foster part Z_{off} enters with a *definite sign*? That would turn a sum rule into a zero-detector.

Status summary

- **The dispersion pattern is explained** ([THM]): it is the pole of ζ at distance $\frac{1}{2}$ from the line, a Kramers–Kronig-consistent *gain* Lorentzian. Equivalently it is Lemma 32's tachyonic mode $e^{u/2}$, Lemma 15's "+1", and Lemma 25's marginal channel. In ξ it is cancelled exactly.
- **Causality is not the missing link**: support causality and boundary Kramers–Kronig with positive absorption are unconditional (Thms. 33.3.2, 33.5.1). **Passivity is**, and passivity of Z on $\operatorname{Re} p > 0$ is RH (Lagarias; Weil = absorbed energy) [RESTATEMENT].
- **New, unconditional, elementary** ([THM], no priority claimed): the shadow theorem and its PR regions; the ladder→PR map with Θ = PR abscissa; boundary spectrum and the on/off split; the de Branges threshold $\Theta - \frac{1}{2}$; the sine-transform link $\mathsf{Z} \leftrightarrow S_+$; the half-turn computation.
- **Restatements** ([RESTATEMENT]): Z PR; m Pick; E HB; positive Krein string; Foster form; S_+ CM; half-turn per derivative; Weil = passivity.
- **Analogies** ([ANALOGY]): optics and circuit names, non-Foster sections, entropy/arrow language, Newton's triangle (except $S^2 = -I$).

[1] **[Vol. 3 editorial note]** Answered in Chapter 36, Thm. 36.4.3: yes, and it is Littlewood's lemma, so it is classical (Lemma 36, correction (d)).

- **Not verified, not cited**: Pontryagin-space de Branges theory; any statement that monotonicity of X alone implies RH; content of [**167**], [**27**], [**78**], [**56**].

CHAPTER 34

Lemma 34: The de Branges/Krein line of ξ reconstructed from the prime side. Suzuki's Hamiltonian, its numerical inverse, the arithmetic signature, and the indefinite (Pontryagin) case

Standalone note `lemma34.tex`, v1 October 21, 2026; v2 (high-precision extension and corrections) October 21, 2026 (UTC).

Summary. Statement of intellectual honesty. Nothing here proves RH. Every claim carries a tag: [THM] proved here (or an elementary consequence of cited results, proof given); [CITED] checked at the source, listed in `lit/lemma34/verified_refs.md`; [CITED, secondary] checked only through a named secondary source; [RESTATEMENT] equivalent to RH (no progress); [ANALOGY] analogy only; [NUM] numerical (`num/lemma34_*.py`, data in `num/l34/`); [CONJ] conjecture; [OPEN] open. We claim no priority for elementary results.

v2 (this revision). New §34.8: an Arb (multiprecision) Galerkin extends the line to $t = 1.5$ (integers $n \le 20$), proves the exact trace law for every onset, finds that removing *any single* atom $n \le 23$ breaks the line (always a short), and re-measures Lemma 37's cascade margins at high precision. The corrections V1–V8 are listed in §34.2. Statements changed in place are marked "(v2)". All new numbers are [NUM] in the sense of §34.8.1: Ritz-sound upper bounds, not interval certificates.

Verdict.

(1) *The object* (§34.3). The right E is Suzuki's $E^{\omega,\nu}(z) = \xi(\frac{1}{2} + \omega - iz)^\nu$ with $\nu\omega > 1$ [**257**]. Its Hamiltonian has determinant one, $H = \mathrm{diag}(1/Z_0, Z_0)$, and Z_0 is a ratio of Fredholm determinants of a Hankel operator $\mathcal{K}[t]$ whose kernel is an explicit *prime-side* sum [CITED]. In the variable t this is a lossless line with unit wave speed and impedance $Z_0(t)$, equivalently a Krein string with mass density $\mathfrak{m} = Z_0^{-2}$ in arclength $\ell = \int Z_0\, dt$ [THM]. For $\omega \ge \frac{1}{2}$ everything is unconditional. We use $(\omega, \nu) = (\frac{1}{2}, 3)$; $\nu = 1$ is excluded at $\omega = \frac{1}{2}$ by Suzuki's condition $\nu\omega > 1$.

(2) *Numerical inverse* (§34.4, [NUM]). A Galerkin scheme with exact cell integrals (Ritz property: computed eigenvalues are certified inner bounds of the spectrum) reconstructs $\log Z_0(t)$ to 4–5 digits for $t \le 0.55$. The extreme gap $1 - \|\mathcal{K}[t]\|$ collapses super-exponentially ($8.6 \cdot 10^{-2}$ at $t = 0.25$, $2.2 \cdot 10^{-4}$ at 0.5, $3.3 \cdot 10^{-11}$ at 1.0, $1.2 \cdot 10^{-21}$ at 1.5 (v2)), and beyond $t \approx 0.6$ the profile is *not* converged at double precision. (v2) The Arb Galerkin of §34.8 gives $\log Z_0$ to ~ 6 significant digits up to $t = 1.5$ ($\log Z_0(1.5) = 52.27635$). Sanity cases pass: a Laplace identity for the kernel (10^{-7}), the small-t trace law, and no Ritz value outside $(-1, 1)$ in any unconditional run, as Suzuki's Prop. 5.1 requires.

(3) *Arithmetic signature* (§34.5). It is exact but small. The line on $[0, t]$ sees exactly the integers $n < e^{2t}$ [CITED], all with coefficients $c_\nu(n) > 0$ (Jordan-totient type, *not* von Mangoldt) [THM]. On $[0, \frac{1}{2}\log 2)$ the line is purely archimedean [THM]. The atom $n = 2$ switches on continuously, with $\log Z_0^{\mathrm{full}} - \log Z_0^{\mathrm{arch}} \approx C(t - \frac{1}{2}\log 2)^\beta$ and local exponent $\beta \to 1.5$–1.6 [NUM], a square-root cusp in the reflectivity. (v2) For every $n \le 20$ the onset's leading term is the exact trace term $\frac{64\pi}{3} c_\nu(n) n^{-1/2}(t - t_n)^{3/2}$ (Prop. 34.8.1, [THM]; dominance [NUM]). The *archimedean-only* line is not passive:

its symbol is not unimodular ($\Theta_\infty(0) = (\zeta(\frac{1}{2}+\omega)/\zeta(\frac{1}{2}-\omega))^\nu$, e.g. -19.74, 1336.5, 839.7), and it breaks (Ritz value < -1) at $t_*^{\mathrm{arch}} \le 0.400$ ((v2) 0.398570), 0.394, 0.391, 0.394 for $(\omega,\nu) = (\frac{1}{2},3), (1.5,1), (0.3,4), (0.2,6)$ [NUM], shortly after the prime 2 enters at 0.3466. (v2) The v1 sentence "a prime-side description beyond $n \le 3$ is not visible" is withdrawn: at high precision every onset $n \le 20$ is resolved (§34.8).

(4) *Indefinite case* (§34.6). Under finiteness and non-degeneracy hypotheses, $E^{\omega,\nu} \in HB_\kappa$ (Kaltenbäck–Woracek) with $\kappa = \nu\, N_+(\frac{1}{2}+\omega)$, i.e. 2ν per off-line quadruple with $\beta > \frac{1}{2}+\omega$ (Thm. 34.6.2, [THM]). This is consistent with Lemma 36, Prop. 5.1 (2 per quadruple), which concerns a different form. New *escape theorem* (Thm. 34.6.4, [THM]): $\mathcal{K}[t] = U[t] + R[t]$ with $\|U[t]\| \le 1$ and $\operatorname{rk} R[t] \le \kappa$, and $n_\pm(R[t]) \le \kappa/2$. Hence at most $\kappa/2$ eigenvalues lie above 1 and at most $\kappa/2$ below -1, at every t. If $\kappa > 0$, the line breaks at a finite t_* (v1 needed a hypothesis (M) here; Lemma 36 v2, Thm. 7.9 removes it and makes the counts exact, see §34.8). Planted-zero numerics give exactly 1/1 ($\kappa = 2$) and 2/2 ($\kappa = 4$) escaping eigenvalues [NUM]. *Correction to the premise:* "RH $\Leftrightarrow m \ge 0$" is not the right formulation. Sign-indefinite strings (Eckhardt–Kostenko) have *Herglotz* Weyl functions, so they are a $\kappa = 0$ object. The $\kappa > 0$ world is that of Kaltenbäck–Woracek general Hamiltonians with inner singularities. RH $\Leftrightarrow \kappa(\omega) = 0\ \forall\omega \Leftrightarrow \mathcal{P}(E^{\omega,\nu})$ is a Hilbert space $\forall\omega \Leftrightarrow$ the positive line never breaks [RESTATEMENT].

(5) *Obstruction* (§34.7). Nothing here supplies G5. In passive-network form, G5 is: $\|\mathcal{K}^{\omega,\nu}[t]\| < 1$ for every t and all small ω, proved from the Euler product. (v2: v1 wrote $\sup_t \|\cdot\| < 1$, which is false in the passive case, where sup = 1; Lemma 37, K1. This is J.W.'s adopted official $\mathrm{G5}_{\mathrm{line}}$.) Positivity of the prime coefficients cannot suffice alone, because the archimedean part is active.

34.1. Conventions, corrections, cross-references

$s = \frac{1}{2} - iz$, $\rho = \frac{1}{2} + \delta_\rho + it_\rho$ nontrivial zeros, $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, $F^\#(z) = \overline{F(\bar z)}$. Suzuki's ω is Lagarias' (and Lemma 33's) h. $N_+(\sigma) = \#\{\rho : \operatorname{Re}\rho > \sigma\}$ with multiplicity. HB = Hermite–Biehler ($|E^\#| < |E|$ on $\mathbb{C}_+$); $\overline{HB}$ in [257] denotes HB without real zeros.

Corrections to the task premise. (C1) $E_{1/2}(z) = \xi(1-iz)$ with $\nu = 1$ violates Suzuki's condition $\nu\omega > 1$ [257, (2.8)], so his construction does not apply to it. We use $(E_{1/2})^3$. It is HB exactly when $E_{1/2}$ is, but its de Branges space is different. (C2) In the det-one gauge the Hamiltonian, and so the string mass, is positive *wherever it is defined* (Thm. 34.3.4). The RH content is *existence for all t* (no break) together with $J \to 0$, not a sign of m. See §34.6 for indefinite strings. (C3) The Kaltenbäck–Woracek series is in two journals: I–II in *Integral Equations Operator Theory*, III–VI in *Acta Sci. Math. (Szeged)*. (C4) Our earlier midpoint-Nyström code (`num/lemma34_string.py`, data `l34/str_*.json`) produced spurious eigenvalues > 1, from an $O(h^{3/2})$ cusp error. It is superseded and must not be used.

Cross-references. Lemma 33 (de Branges threshold $\Theta - \frac{1}{2}$; "built without the zeros"); Lemma 27 (birth at det = 0 vs Krein collision); Lemma 36, Prop. 5.1 (each off-line quadruple raises the index of $\mathcal{L}$ by exactly 2) and its Q on kinks of $\lambda^*(e^A)$ at $A = \log n$; Lemma 35 (prime-side Post–Widder check of S_+).

34.2. Revision note (v2): corrected errors

(V1) *Official G5 statement.* v1 (§34.7, the Corollary after Thm. 34.6.4) wrote "$\sup_t \|\mathcal{K}[t]\| < 1$". In the passive case Θ is inner and $\sup_t \|\mathcal{K}[t]\| = 1$ exactly, not attained (Lemma 37, Thm. 2.2 and correction K1). Correct form: $\|\mathcal{K}[t]\| < 1$ for every t (equivalently sup ≤ 1). Suzuki's Prop. 5.1 gives the pointwise statement, not sup < 1.

(V2) *Onset mechanism.* v1 attributed the onset exponent $\nu\omega$ to edge singularities of the resolvent, against a "naive" prediction $\nu\omega+1$. Wrong: the exponent $\nu\omega$ is already in the elementary trace term (Prop. 34.8.1). Conj. 34.5.2 is replaced by Conj. 34.8.2.

(V3) *Precision estimate.* v1 predicted gaps $\sim 10^{-12}$ at $t\approx 1.5$ and 20–30 digits needed. Actual gaps are $1.2\cdot 10^{-21}$ (top) and $6.5\cdot 10^{-19}$ (bottom), with $\det(1-\mathcal{K}[1.5])\approx 2.8\cdot 10^{-63}$; we used 512 bits.

(V4) *"Certified".* v1 called Ritz exits "certified". They are Ritz-sound given the matrix entries, but the kernel tables and quadrature have no rigorous error bounds. All such statements, in v1 and v2, are [NUM] upper bounds on break times, not interval certificates.

(V5) *Visibility of the arithmetic signature.* v1: "not visible beyond $n\le 3$". Withdrawn: all onsets $n\le 20$ are resolved (§34.8).

(V6) *Hypothesis (M).* Thm. 34.6.4(iv) and its Corollary assumed (M). Lemma 36 v2, Thm. 7.9 removes it and proves exact counts $n_\pm(R[t])=\kappa/2$. Statements are updated in place, marked v2.

(V7) *Lemma 37 Table 1 (recorded here; Lemma 37 not edited).*[1] For $t\gtrsim 0.8$ its cascade break times are too late by 0.003–0.011, more than its stated error $1.5\cdot 10^{-3}$. Corrected values are in Table 34.3, e.g. margin 0.0240 (not 0.027) for $P=3$ and 0.0131 (not 0.023) for $P=11$. Lemma 39's margin fits used the old values.

(V8) $A=2t$ *remark.* The v1 Remark in §34.5 is reconciled with Lemma 37, Prop. 5.1 in §34.8: $A=2t$ holds as an information horizon only, for prime powers.

34.3. (a) Which E, which space, which string

Definition 34.3.1 ([257], (2.4)–(2.7), (2.9), (2.10); [CITED]). For $\omega>0$, $\nu\in\mathbb{Z}_{>0}$: $E^{\omega,\nu}(z)=\xi(\frac12+\omega-iz)^\nu$, $\Theta^{\omega,\nu}=(E^{\omega,\nu})^\#/E^{\omega,\nu}=(\xi(\frac12-\omega-iz)/\xi(\frac12+\omega-iz))^\nu$. Further $K(x)=\frac{1}{2\pi}\int_{\operatorname{Im}z=v}\Theta^{\omega,\nu}(z)e^{-ixz}\,dz$ with v large, which vanishes for $x<0$, and $\mathcal{K}[t]f(x)=\mathbf{1}_{(-\infty,t)}(x)\int_{-\infty}^t K(x+y)f(y)\,dy$ on $L^2(-\infty,t)$. Finally

$$H(t)=\begin{pmatrix}1/Z_0(t)&0\\0&Z_0(t)\end{pmatrix},\qquad Z_0(t)=\Big(\frac{\det(1+\mathcal{K}[t])}{\det(1-\mathcal{K}[t])}\Big)^2,\qquad -\frac{d}{dt}\begin{pmatrix}A\\B\end{pmatrix}=z\begin{pmatrix}0&-1\\1&0\end{pmatrix}H\begin{pmatrix}A\\B\end{pmatrix}.$$

Since $K=0$ on $(-\infty,0)$, $\mathcal{K}[t]$ annihilates $L^2(-\infty,-t)$ [257, Prop. 5.1(i)]. So $\mathcal{K}[t]$ is the Hankel operator $f\mapsto\int_{-t}^t K(x+y)f(y)dy$ on $L^2(-t,t)$, which is what we discretize.

Theorem 34.3.2 (Suzuki [257]; [CITED]). *Let $\nu\omega>1$. (i) Thm. 2.1: $\mathcal{K}[t]$ is Hilbert–Schmidt and self-adjoint with continuous kernel, $\mathcal{K}[t]=0$ for $t\le 0$, and ±1 are not eigenvalues for $t\in[0,\tau)$, some $\tau>0$. (ii) Thm. 2.2: on $[0,\tau)$ the system has a solution with $A(0,z)-iB(0,z)=E^{\omega,\nu}(z)$. (iii) Prop. 5.1: if $E^{\omega,\nu}\in\overline{HB}$, then $\|\mathcal{K}[t]\|<1$ for* all *$t\ge0$, so $\tau=\infty$. (iv) Prop. 2.2: $E^{\omega,\nu}\in\overline{HB}$ for $\omega>\frac12$ unconditionally, and for $\omega\le\frac12$ under RH. (v) Thm. 2.4: RH holds iff there are (ω_n,ν_n) with $\omega_n\downarrow 0$, $\nu_n\omega_n>1$, $\det(1\pm\mathcal{K}^{\omega_n,\nu_n}[t])\neq0$ for all $t\ge0$, and $J(t;z,z)\to0$ $(t\to\infty)$ on $\mathbb{C}_+$. (vi) Locality (remark after Thm. 2.1): $\mathcal{K}[t]$ depends only on $K|_{[0,2t)}$, i.e. on the gamma factor and on finitely many prime-side coefficients.*

The chain itself is de Branges' Thm. 40 [81] ([CITED, secondary], via [257, §1]): every $E\in HB$ is the initial value of a canonical system with $J\to0$. Suzuki's contribution is an *explicit* Hamiltonian built from Θ alone, with no zeros as input.

Proposition 34.3.3 ($\omega=\frac12$ is unconditional; [THM] from [CITED]). *$E^{1/2,\nu}\in\overline{HB}$ for every ν. Hence for $(\frac12,3)$, $\|\mathcal{K}[t]\|<1$ and Z_0 is defined for all $t\ge0$.*

[1] **[Vol. 3 editorial note]** Since then Lemma 37 has been revised (v2, v3). Its correction K8 adopts these values, and the cascade table printed in Chapter 37 is the corrected one (Table 37.1). See Appendix A, item D2.

PROOF. $E_{1/2}$ is HB by [169, Lemma 6.1] (and Lemma 33, threshold theorem). It has no real zeros, because $\xi(1-iu)\neq 0$ (ζ has no zeros on $\operatorname{Re} s=1$). Powers preserve both properties. Apply Thm. 34.3.2(iii). Condition (2.8) is $3\cdot\frac12>1$. □

Suzuki states Prop. 2.2 with $\omega>\frac12$. The boundary value $\omega=\frac12$ needs exactly the zero-free line $\operatorname{Re} s=1$.

THEOREM 34.3.4 (line and string; [THM], elementary). *Wherever Z_0 is defined, $H\geq 0$ and $\det H=1$. With $A=V$, $B=I$, the system reads $V'=zZ_0I$, $I'=-zV/Z_0$. This is a lossless transmission line with series inductance Z_0, shunt capacitance $1/Z_0$, unit wave speed and characteristic impedance $Z_0(t)$; its local reflection coefficient is $r(t)=\frac12(\log Z_0)'(t)$. With $d\ell=Z_0\,dt$ and $\lambda=z^2$ one gets $-\frac{d^2A}{d\ell^2}=\lambda\,\mathfrak{m}(\ell)A$ with $\mathfrak{m}=Z_0^{-2}>0$, a Krein string [144] of length $L=\int_0^\infty Z_0\,dt$.*

PROOF. $JH\binom{A}{B}=\binom{-Z_0B}{A/Z_0}$. Then $\frac{dA}{d\ell}=zB$ and $\frac{dB}{d\ell}=-zA/Z_0^2$. □

So the correspondence is $E^{\omega,\nu}\leftrightarrow\mathcal{H}(E^{\omega,\nu})\leftrightarrow H=\operatorname{diag}(Z_0^{-1},Z_0)\leftrightarrow$ (line $Z_0(t)$) $\leftrightarrow$ (string $\mathfrak{m}=Z_0^{-2}$). Positivity is built in. What can fail is *existence for all t*.

PROPOSITION 34.3.5 (prime-side kernel for ζ; [THM], elementary). *For $x>0$, $K(x)=\sum_{n\leq e^x}c_\nu(n)\,n^{-1/2}\,G(x-\log n)$. Here $c_1(n)=n^\omega\prod_{p\mid n}(1-p^{-2\omega})=n^{-\omega}J_{2\omega}(n)$ (Jordan totient), $c_\nu=c_1^{*\nu}$ (Dirichlet power), and G is the inverse Laplace transform of $(\gamma(\frac12+p-\omega)/\gamma(\frac12+p+\omega))^\nu$ with $\gamma(s)=\xi(s)/\zeta(s)=(s-1)\pi^{-s/2}\Gamma(\frac{s}{2}+1)$. All $c_\nu(n)>0$. At $\omega=\frac12$, $c_1(n)n^{-1/2}=\varphi(n)/n$. Near 0^+, $G(x)\sim Cx^{\nu\omega-1}/\Gamma(\nu\omega)$.*

PROOF. Put $p=-iz$, $s=\frac12+p$. Then $\Theta^{\omega,\nu}=\big[\frac{\gamma(s-\omega)}{\gamma(s+\omega)}\big]^\nu\big[\frac{\zeta(s-\omega)}{\zeta(s+\omega)}\big]^\nu$, and $\frac{\zeta(s-\omega)}{\zeta(s+\omega)}=\sum_n n^{-s}\sum_{de=n}d^\omega\mu(e)e^{-\omega}=\sum_n c_1(n)n^{-s}$. The factor $n^{-1/2-p}$ is a shift by $\log n$. Stirling gives $(\gamma(s-\omega)/\gamma(s+\omega))^\nu\sim C'p^{-\nu\omega}$. This is Suzuki's (2.7) made explicit for ζ (his $q(n)$). □

34.4. (b) Numerical inverse spectral reconstruction

Method. We tabulate G by Talbot inversion (mpmath, 30 digits; `num/lemma34_kernel.py`, `lemma34_tab*.py`). The second antiderivative K_2 ($K_2''=K$) is tabulated directly from the Laplace transform $\widehat{G}(p)/p^2$; its spline error is $\leq 6\cdot10^{-13}$. We use a Galerkin basis of cell indicators of width h on $(-t,t)$, with entries $\langle e_i,\mathcal{K}[t]e_j\rangle=h^{-1}[K_2(c+h)-2K_2(c)+K_2(c-h)]$, $c=x_i+x_j$, exact up to table error (`num/lemma34_galerkin.py`). *Ritz property*: the Galerkin eigenvalues lie in $[\inf\sigma(\mathcal{K}[t]),\sup\sigma(\mathcal{K}[t])]$. So a Ritz value >1 (or <-1) *certifies* $\|\mathcal{K}[t]\|>1$, up to the table error. $\log Z_0=4\sum_j\operatorname{artanh}\lambda_j$.

Sanity cases ([NUM]). (1) Laplace identity $\int_0^X K(x)e^{-px}dx=\Theta^{\omega,\nu}(ip)$ (tail-corrected), relative error $1.3\cdot10^{-7}$ for $(\frac12,3)$, $(1.5,1)$ and $1.5\cdot10^{-6}$ for $(0.3,4)$, $(0.2,6)$ at $p=4,6$ (`l34/checks_laplace.json`). This checks the prime-side expansion and the G tables against ξ itself. (2) Rank-one kernel $ce^{-a(x+y)}$: eigenvalue error $-2.3\cdot10^{-6},-5.9\cdot10^{-7},-1.5\cdot10^{-7}$ at $h=\frac{1}{100},\frac{1}{200},\frac{1}{400}$, i.e. $O(h^2)$ (`l34/checks_rank1.json`). (3) Small-t trace law $\log Z_0(t)=4\operatorname{tr}\mathcal{K}[t]+O(\operatorname{tr}\mathcal{K}[t]^3)=2K_1(2t)+\dots$: relative deviation $9\cdot10^{-4}$ at $t=0.016$, growing like the cubic term (`l34/checks_trace.json`). (4) *Theorem check*: in *every* unconditional run, no Ritz value leaves $(-1,1)$. These are $(\frac12,3)$ at $h^{-1}\in\{200,400,800,1600\}$, $t\leq1.5$; $(1.5,1)$, $(0.3,4)$, $(0.2,6)$ at $h^{-1}\in\{400,800\}$; and the control run to $t=3$. The minimal gaps are $4.9\cdot10^{-8}$ to $1.9\cdot10^{-7}$. (The last two cases are $\omega<\frac12$, conditional on nothing because the computation does not use HB. They are consistent with RH but do not test it, since off-line zeros at height $\gtrsim10^{12}$ act only at $t\gtrsim\frac{\nu}{2}\log10^{12}$.)

Honest limitation. The near-unit eigenvalues ($n_{\text{near}}=1,3,4,7,10,14$ at $t=0.25,\dots,1.5$) have gaps of 10^{-7} to 10^{-8}. Each contributes $\approx 2\log(2/\text{gap})$ to $\log Z_0$, so $\log Z_0$ at $t\geq0.6$ is

t	$h=\frac{1}{200}$	$\frac{1}{400}$	$\frac{1}{800}$	$\frac{1}{1600}$	Richardson	$1-\lambda_{\max}$	$\#\{\lvert\lambda\rvert>0.9\}$
0.10	1.4689	1.4681	1.4679	1.4678	1.4678	$5.7\cdot10^{-1}$	0
0.25	4.6548	4.6544	4.6543	4.6542	4.6542	$8.6\cdot10^{-2}$	1
0.40	9.0073	9.0316	9.0391	9.0404	9.041	$1.8\cdot10^{-3}$	2
0.50	10.8131	10.9765	11.0215	11.0324	11.036	$2.2\cdot10^{-4}$	3
0.55	12.2374	12.7046	12.8467	12.8834	12.90	$6.0\cdot10^{-5}$	3
0.60	12.61	13.99	14.61	14.80	n.c.	$1.0\cdot10^{-5}$	3
1.00	13.39	14.64	16.25	17.68	n.c.	$\le 1.7\cdot10^{-7}$	7
1.50	14.68	15.51	16.29	17.65	n.c.	$\le 4.9\cdot10^{-8}$	14

TABLE 34.1. $\log Z_0(t)$ for $(\omega,\nu)=(\frac{1}{2},3)$ ([NUM]; `l34/gal_w05_h*.json`). n.c. = not converged. The gap column is from $h=\frac{1}{1600}$. Beyond $t\approx0.6$ the true gap lies below what the discretization resolves, and the computed gap still shrinks with h.

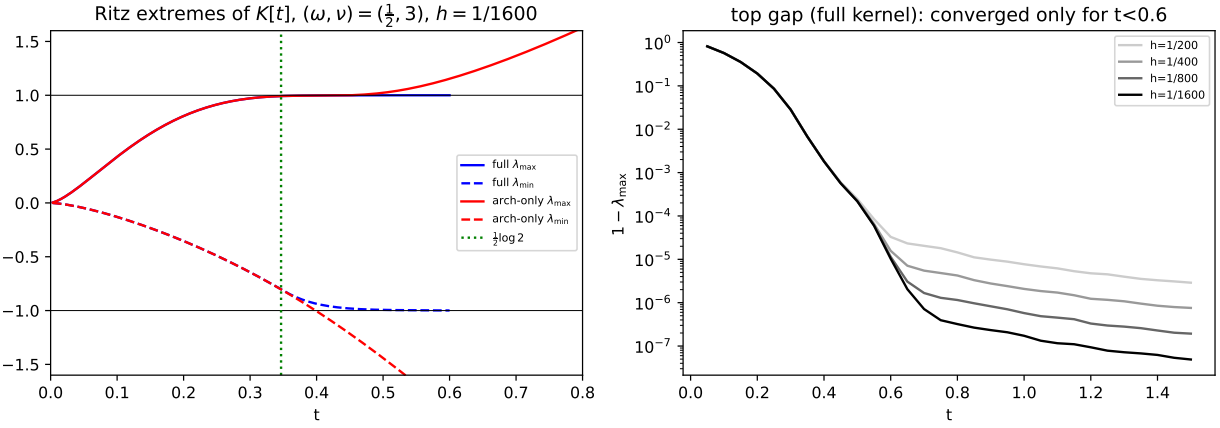

FIGURE 34.1. Left: Ritz extremes of $\mathcal{K}[t]$, full kernel (blue) vs archimedean-only kernel G (red), $h=\frac{1}{1600}$. Right: top gap $1-\lambda_{\max}$, h-refinement. [NUM]

controlled by digits that a double-precision $O(h^2)$ scheme cannot provide. The reliable profile therefore covers $t\le0.55$, i.e. the integers $n\le3$. Going further needs higher precision (Arb/mpmath Galerkin, or spectral bases adapted to the edge singularities). We estimate gaps $\sim10^{-12}$ by $t\approx1.5$, so about 20–30 digits.

34.5. (c) The arithmetic signature

THEOREM 34.5.1 (onsets; [THM] from Thm. 34.3.2(vi) and Prop. 34.3.5). *$H|_{[0,t]}$ is a function of G and $\{c_\nu(n):n<e^{2t}\}$. Every integer $n\ge2$ switches on at $t_n=\frac{1}{2}\log n$ with weight $c_\nu(n)n^{-1/2}>0$. In particular $Z_0=Z_0^{\mathrm{arch}}$ on $[0,\frac{1}{2}\log2]$, where Z_0^{arch} is built from G alone.*

So the "density" is indexed by *all* integers n, not prime powers. The weights are multiplicative Jordan-totient powers (at $\omega=\frac{1}{2}$, $\nu=3$: $(\varphi/\mathrm{id})^{*3}$, e.g. $\frac{3}{2}$ at $n=2$, 2 at $n=3$), not $\Lambda(n)$. The time-scale is $\frac{1}{2}\log n$ because $\mathcal{K}[t]$ couples $x+y\in(0,2t)$.

Shape of an onset ([NUM], `num/lemma34_onset.py`, $h=\frac{1}{1600},\frac{1}{3200}$). With $D(t)=\log Z_0^{\mathrm{full}}-\log Z_0^{\mathrm{arch}}$: $D\equiv0$ for $t<t_2$ (to 10^{-11}). The first-order formula $4\,\mathrm{tr}[(1-\mathcal{K}_{\mathrm{arch}}^2)^{-1}\delta\mathcal{K}]$ matches D to 0.3% for $t-t_2\le0.01$, and the local exponent of D is 1.54 at $t-t_2=6\cdot10^{-4}$, rising to 1.85 at 10^{-2}, independent of h (Fig. 34.2, right). A naive estimate with bounded resolvent kernel predicts exponent $\nu\omega+1=2.5$. The data instead fit $\nu\omega=1.5$.

CONJECTURE 34.5.2 ([CONJ]). At each t_n with $c_\nu(n) \neq 0$, $\log Z_0(t) = (\text{smooth}) + C_n(t - t_n)_+^{\nu\omega}(1 + o(1))$ with $C_n \propto c_\nu(n) n^{-1/2}$. The extra factor $(t - t_n)^{-1}$ relative to the naive estimate comes from inverse-square-root edge behaviour of the resolvent kernel of $\mathcal{K}[t]$ at the corner $x = y = t$. So the reflectivity $r(t)$ has a cusp $(t - t_n)_+^{\nu\omega - 1}$ at every $\frac{1}{2}\log n$.

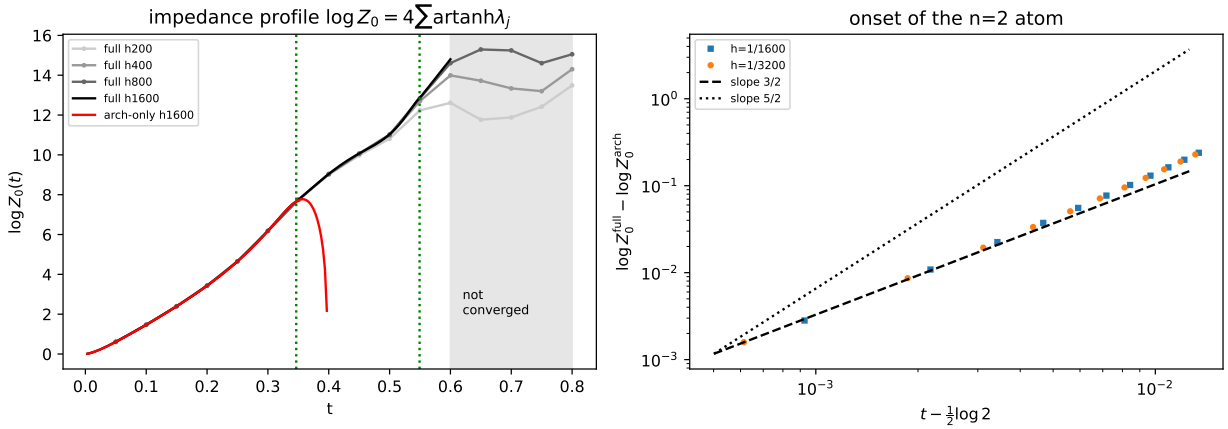

FIGURE 34.2. Left: $\log Z_0(t)$, full kernel at four resolutions and archimedean-only. Dotted: $\frac{1}{2}\log 2$, $\frac{1}{2}\log 3$. Right: onset of the $n = 2$ atom, log-log. [NUM]

Archimedean-only string. Dropping the ζ-ratio leaves $\Theta_\infty = (\gamma(s - \omega)/\gamma(s + \omega))^\nu$. On $\mathbb{R}$ this has modulus $|\Theta_\infty| = |\zeta(s + \omega)/\zeta(s - \omega)|^\nu$, not 1. Exactly, $\Theta_\infty(0) = (\zeta(\frac{1}{2} + \omega)/\zeta(\frac{1}{2} - \omega))^\nu$, which is $-19.739 = -2\pi^2$ for $(1.5, 1)$, 1336.5 for $(0.3, 4)$, 839.7 for $(0.2, 6)$, and a pole of order ν for $\omega = \frac{1}{2}$ [THM]. So the archimedean line is *active*, not passive. Numerically its Hankel sections break early: the first certified Ritz value < -1 occurs at

$$t_*^{\text{arch}} \leq 0.4000\ (\tfrac{1}{2}, 3), \quad 0.3938\ (1.5, 1), \quad 0.3906\ (0.3, 4), \quad 0.3938\ (0.2, 6)$$

(step 1/320, identical at $h = \frac{1}{800}, \frac{1}{1600}$; `134/archexit.json`) [NUM]. In all four cases this is after $t_2 = 0.3466$. The full line stays contractive, as it must. With all prime-side terms, the symbol becomes unimodular; the primes act as the matched load [ANALOGY].

REMARK 34.5.3 ([ANALOGY]). Lemma 36 asks about kinks of $\lambda^*(e^A)$ at $A = \log n$. Here the onsets sit at $2t = \log n$. If Lemma 24/36's window length A corresponds to $2t$, these are the same edge atoms seen through two different positivity functionals. This is not proved.

Answer to (c). A prime-side description of the line is *exact* (Thm. 34.5.1, Prop. 34.3.5) but *implicit*: Z_0 is a nonlinear functional (Fredholm determinants) of the prime-side kernel. A *local* prime-side description, one feature per n at $\frac{1}{2}\log n$, is *conjectured* (Conj. 34.5.2). It is visible numerically only for $n = 2$ (and marginally $n = 3$). No von Mangoldt or explicit-formula structure is visible in Z_0.

34.6. The indefinite case: Pontryagin spaces, κ, strings

DEFINITION 34.6.1 ([148], Def. 2.14; [CITED]). $E \in HB_\kappa$ if E is entire, E and $E^\#$ have no common nonreal zeros (HB2), and $K_E(w, z) = \frac{E(z)\overline{E(w)} - E^\#(z)\overline{E^\#(w)}}{2\pi i(\bar{w} - z)}$ has κ negative squares on $\mathbb{C}$. Then K_E is the reproducing kernel of a Pontryagin space $\mathcal{P}(E)$ with negative index κ [149]. HB_0 is de Branges' class (with HB2).

The inverse spectral theory of $\mathcal{P}(E)$ is that of *general Hamiltonians*: canonical systems with finitely many inner singularities [151] ([CITED, secondary] via the introduction of [148]), and maximal chains in $\mathcal{M}_\kappa$ correspond bijectively to N_κ ([150], cited as Thm. 3.6 in [148]; [CITED, secondary]). Krein–Langer [162, 161] introduced the generalized Nevanlinna classes.

THEOREM 34.6.2 (κ = off-line count; [THM], one analytic step sketched). *Fix $\omega > 0$, $\nu \geq 1$. Assume (B) $N_+(\frac{1}{2}+\omega) < \infty$ and (ND) no ρ with $\operatorname{Re}\rho = \frac{1}{2}+\omega$, and no two zeros ρ, ρ' with $t_{\rho'} = t_\rho$ and $\delta_\rho + \delta_{\rho'} = 2\omega$. Then $E^{\omega,\nu} \in HB_\kappa$ with*

$$\kappa = \nu\, N_+(\tfrac{1}{2}+\omega) = 2\nu \cdot \#\{\text{off-line quadruples with } \beta > \tfrac{1}{2}+\omega\}.$$

If $N_+(\frac{1}{2}+\omega) = \infty$, K_E has infinitely many negative squares.

PROOF. Zeros of $E^{\omega,\nu}$ are $z_\rho = -t_\rho + i(\delta_\rho - \omega)$ with multiplicity νm_ρ. Those in $\mathbb{C}_+$ are the ρ with $\delta_\rho > \omega$. Both ρ and $\bar\rho$ qualify, so there are 2 per quadruple. (ND) is exactly (HB2): a common nonreal zero means z and $-z$ are both zeros, since $E^\#(z) = E(-z)$. Let $\beta(z) = \prod (z - z_j)/(z - \bar z_j)$ over the κ zeros in $\mathbb{C}_+$, a Blaschke product of degree κ, and $\Theta = E^\#/E$. Then $K_E = E(z)\overline{E(w)}L_\Theta$ with $L_S(w,z) = \frac{1 - S(z)\overline{S(w)}}{2\pi i(\bar w - z)}$, and multiplication by $E(z)\overline{E(w)}$ does not change negative squares off the zero set. By continuity of the kernel, the count on $\mathbb{C}$ equals the count on a dense subset. *Upper bound.* $\Theta_1 := \beta\Theta$ is analytic in $\mathbb{C}_+$ and unimodular on $\mathbb{R}$. It is bounded on $\operatorname{Im} z \geq v_0$ by [257, (4.3)], and of growth $\exp(o(|z|))$ in the strip $0 < \operatorname{Im} z < v_0$ (Stirling; for $1/\zeta$ this uses (B) and standard bounds for $\log\zeta$ in zero-free regions; *this step is sketched, not written out*). Phragmén–Lindelöf gives $|\Theta_1| \leq 1$, so $L_{\Theta_1} \geq 0$. Now $L_\Theta = (L_{\Theta_1} - L_\beta)/(\beta(z)\overline{\beta(w)})$, and L_β is the kernel of the κ-dimensional space $H^2 \ominus \beta H^2$. Hence $n_-(L_\Theta) \leq \kappa$. *Lower bound.* Near each z_j of order m_j, take m_j points $z_j + \varepsilon e^{2\pi i k/m_j}$. Divide row a of the Gram matrix $[L_\Theta(z_b, z_a)]$ by $\Theta(z_a)$ and column b by $\overline{\Theta(z_b)}$. This gives $-S_\varepsilon + C_\varepsilon$, where S_ε is the Szegő Gram matrix of the κ points and $C_\varepsilon = O(\varepsilon^{2\max m_j})$ (since $|\Theta|^{-1} = O(\varepsilon^{m_j})$). Clustered Szegő Gram matrices satisfy $\lambda_{\min}(S_\varepsilon) \gtrsim \varepsilon^{2(\max m_j - 1)}$, because the confluent Gram matrix of the derivative functionals $f \mapsto f^{(k)}(z_j)$, $k < m_j$, on H^2 is positive definite. So the matrix is negative definite of size κ for small ε, and $n_- \geq \kappa$. The same argument with any finite subset of poles gives the case $N_+ = \infty$. □

COROLLARY 34.6.3 ([RESTATEMENT]). *RH $\iff$ for all $\omega > 0$: $E^{\omega,\nu} \in HB_0$, i.e. $\kappa(\omega) = 0$ and (ND) holds $\iff$ $\mathcal{P}(E^{\omega,\nu})$ is a Hilbert space for all $\omega > 0$. For $\omega \geq \frac{1}{2}$ this holds unconditionally (Prop. 34.3.3).*

Consistency with Lemma 36. Lemma 36, Prop. 5.1 shows that each off-line quadruple raises $\sup n_+(\mathcal{L}|_V)$ by exactly 2, one per mirror pair. Thm. 34.6.2 gives 2ν per quadruple for the de Branges kernel of $E^{\omega,\nu}$, so 2 at $\nu = 1$. The counts agree. The forms are different objects (Weil's form on test functions vs. the kernel of $\mathcal{P}(E^{\omega,\nu})$), and we do *not* claim an isometric identification [ANALOGY].

THEOREM 34.6.4 (escape; [THM], contour shift in (i) sketched). *Let Θ be meromorphic, unimodular on $\mathbb{R}$, with $\Theta(-z) = 1/\Theta(z)$ and $\Theta(-\bar z) = \overline{\Theta(z)}$, bounded on $\operatorname{Im} z \geq v_0$, with finitely many poles $z_j \in \mathbb{C}_+$ of total order κ and none on $\mathbb{R}$. Let K and $\mathcal{K}[t]$ be as in §34.3. This covers $E^{\omega,\nu}$ under (B), (ND), and also the planted symbols of §34.6.1. Then:*

- *(i) $\mathcal{K}[t] = U[t] + R[t]$, where $U[t]$ is the compression to $L^2(-\infty, t)$ of the unitary $f \mapsto \mathcal{F}^{-1}(\Theta \cdot \mathcal{F}\check f)$, so $\|U[t]\| \leq 1$, and $R[t]$ has kernel $R(x+y)$, $R(x) = \sum_j P_j(x) e^{-iz_j x}$ (residues, $\deg P_j < m_j$), with $\operatorname{rk} R[t] \leq \kappa$.*
- *(ii) $n_+(R[t]) \leq \kappa/2$ and $n_-(R[t]) \leq \kappa/2$ for every t. (v2: equality holds, Lemma 36 v2, Thm. 7.9.)*
- *(iii) At every t, $\#\{\lambda \in \sigma(\mathcal{K}[t]) : \lambda > 1\} \leq \kappa/2$ and $\#\{\lambda < -1\} \leq \kappa/2$.*

(iv) Assume (M): the poles of maximal imaginary part b form one pair $\{w,-\bar w\}$. (v2: (M) is unnecessary, and eventually exactly $\kappa/2$ eigenvalues lie on each side, Lemma 36 v2, Thm. 7.9.) If $\kappa>0$, then for all large t there are eigenvalues >1 and <-1. Hence there is a first $t_*<\infty$ with $\det(1-\mathcal{K}[t_*])\det(1+\mathcal{K}[t_*])=0$, and Suzuki's H cannot be continued through t_* by (2.10).

Proof. (i) Shift the contour in the definition of K from $\operatorname{Im} z=v$ down to $\mathbb{R}$. This holds in the sense of distributions, where $\Theta|_{\mathbb{R}}\in L^\infty$, and the crossed poles give R. Multiplication by a unimodular Θ composed with reflection is unitary on $L^2(\mathbb{R})$, and $\Theta(-u)=\overline{\Theta(u)}$ makes the kernel real. Compression does not increase the norm. (ii) Poles come in pairs $z,-\bar z$; none is purely imaginary for ξ, since ξ has no real zeros. A pair of order m contributes $2\operatorname{Re}[P(x+y)e^{-iw(x+y)}]$, so $\langle R[t]f,f\rangle=\sum_{\text{pairs}}2\operatorname{Re}Q(\zeta(f))$ with Q a complex quadratic form in the m functionals $\zeta_k(f)=\int y^k e^{-iwy}f$. The real form $q=\operatorname{Re}Q$ on $\mathbb{C}^m\cong\mathbb{R}^{2m}$ satisfies $q(i\zeta)=-q(\zeta)$, so $n_+(q)=n_-(q)\le m$, and pulling back does not increase $n_\pm$. Sum over pairs. (iii) $\mathcal{K}[t]\le 1+R[t]$; on the span of eigenvectors with $\lambda>1$, $R[t]>0$. (iv) Take $f(x)=\operatorname{Re}[e^{i\phi}e^{i\bar w x}]\mathbf{1}_{(t-1,t)}$. Then $\zeta_0(f)=\frac12 e^{i\phi}\int_{t-1}^t e^{2bx}dx+O(1)$ and $\|f\|^2\asymp e^{2bt}$. The leading pair gives Rayleigh quotient $\asymp|c|\,t^{m-1}e^{2bt}\cos(2\phi+\arg c)$, while other pairs contribute $O(t^{m'}e^{2b't})$ with $b'<b$, and $U[t]$ contributes $O(1)$. Choose ϕ for either sign. $\mathcal{K}[t]$ is norm-continuous in t with $\mathcal{K}[0]=0$. □

Corollary 34.6.5 ([THM]; [RESTATEMENT] in the last clause). *Under (B), (ND) ((M) not needed, v2): $\kappa(\omega)=0\iff\|\mathcal{K}[t]\|<1$ for every $t\iff\sup_t\|\mathcal{K}[t]\|\le 1$. (v2: v1 wrote that the sup is <1; it equals 1 in the passive case, see V1.) Unconditionally, if $\|\mathcal{K}[t]\|<1$ for all t, then $\kappa(\omega)\in\{0,\infty\}$. RH $\iff$ the positive line $Z_0^{\omega,\nu}$ exists on $[0,\infty)$ with $J\to 0$ for a sequence $\omega\downarrow 0$ (Suzuki, Thm. 34.3.2(v)).*

34.6.1. Planted off-line zeros. We multiply $\Theta^{1/2,3}$ by the Blaschke factor $\frac{(z-\bar w)(z+w)}{(z-w)(z+\bar w)}$, $w=a+ib$. This moves a zero pair into $\mathbb{C}_+$ and gives a toy with $\kappa=2$; Thm. 34.6.4 applies. The kernel becomes $K+K*r$ with $r(x)=\frac{2b}{a}2\operatorname{Re}[we^{-ixw}]$ (Filon quadrature, $O(h^2)$). Results with $a=14.134725$ (`l34/gal_pl_*.json`, Fig. 34.3; [NUM]):

- control $b=10^{-6}$: no exit to $t=3$ ($\lambda_{\max}\le 1-1.5\cdot10^{-7}$);
- $b=0.30$: first exit at $t=2.2$ ($h=\frac{1}{400}$) and $t=2.0$ ($h=\frac{1}{800}$). Ritz values only increase under refinement, so the certified statement is $t_*\le 2.0$; t_* is not converged;
- $b=0.15$, 0.08: exits at 2.075, 2.15 ($h=\frac{1}{400}$). The weak b-dependence reflects that the unperturbed gaps are already $\sim10^{-7}$;
- two pairs ($b=0.3$ at 14.1347, $b=0.2$ at 21.0220, $\kappa=4$): counts $1/1$ from $t=2.175$, then $2/2$ from $t=2.6$.

In every run the number of exits on each side never exceeds $\kappa/2$, and equals $\kappa/2$ at the end. This is exactly Thm. 34.6.4(iii)–(iv). The exceedances are small ($\le 1.6\cdot10^{-3}$ at $t=3$) and the onset times are discretization-sensitive.

34.6.2. Strings with sign-changing mass are *not* the indefinite case. Eckhardt–Kostenko [96, Thm. 6.1] [CITED]: generalized indefinite strings $-u''=z\,u\,\omega+z^2u\,\upsilon$ (ω a real distribution in H^{-1}_{loc}, $\upsilon\ge0$) are in bijection with *all* Herglotz–Nevanlinna functions, by reduction to de Branges' positive canonical systems. Krein's case (Stieltjes functions) is $\upsilon=0$, $\omega\ge0$ [96, Prop. 7.3]. Langer–Winkler [172] ([CITED, secondary] via [96]) cover Herglotz functions with finitely many singularities on $(-\infty,0)$. Consequently, a sign-changing mass corresponds to a *Hilbert* ($\kappa=0$) spectral problem whose Weyl function is merely not Stieltjes. Negative squares (N_κ, $\kappa>0$) require the Kaltenbäck–Woracek general Hamiltonians. So:

$$\text{RH}\iff\kappa(\omega)=0\ \forall\omega\iff\mathcal{P}(E^{\omega,\nu})\text{ Hilbert}$$
$$\iff\text{the positive line never breaks}\quad[\text{RESTATEMENT}],$$

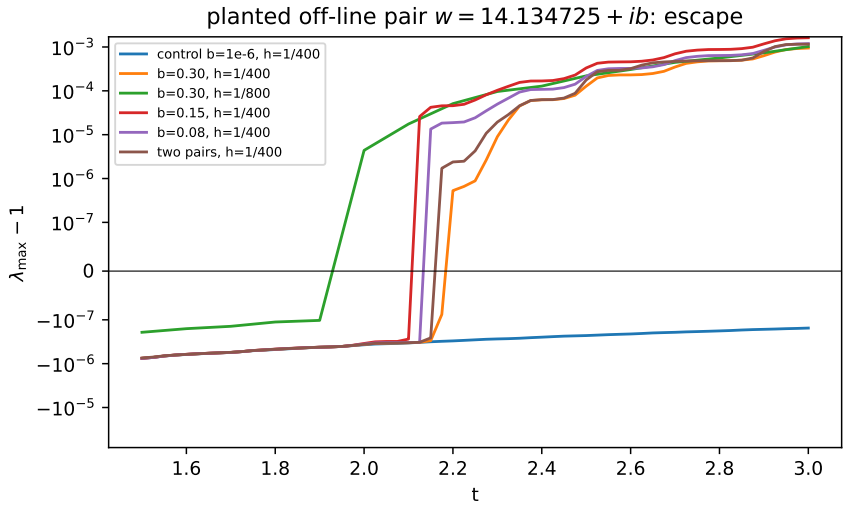

FIGURE 34.3. $\lambda_{\max} - 1$ (symlog) for planted zero pairs and control. [NUM]

whereas " $\iff$ $\mathfrak{m} \geq 0$" is *false as stated*: in the det-one gauge $\mathfrak{m} > 0$ wherever defined, and a sign-indefinite string would not encode $\kappa > 0$.

CONJECTURE 34.6.6 ([OPEN]). If $\kappa(\omega) \in (0, \infty)$, the Suzuki line continues past each break t_* (where $Z_0 \to 0$ or ∞) as a Kaltenbäck–Woracek general Hamiltonian, with an inner singularity at t_*, and the total negative index is κ. We have not verified that the KW singularity structure matches the determinant zeros.

34.6.3. Relation to Lemma 27 (Krein signatures).

- [THM] (Sylvester): the inertia of $1 \mp \mathcal{K}[t]$ changes only where $\det(1 \mp \mathcal{K}[t]) = 0$. A break is an eigenvalue of $\mathcal{K}[t]$ *passing through* ± 1; it is not a collision.
- In ω, κ jumps $0 \to 2\nu m_\rho$ exactly when ω crosses δ_ρ. At $\omega = \delta_\rho$, $E^{\omega,\nu}$ has a real zero, $\notin \overline{HB}$, and the space degenerates.

This matches Lemma 27, stage (a): birth at $\det G_A = 0$, not a collision, and "the Krein space degenerates" [ANALOGY]. The collision stage (b) of Lemma 27 would correspond here to non-real eigenvalues of the self-adjoint relation (multiplication by z) in $\mathcal{P}(E^{\omega,\nu})$, of which there are at most κ in $\mathbb{C}_+$ (Pontryagin's theorem). We have not computed them [OPEN].

34.7. (d) The obstruction and questions

G5 in passive-network form ([RESTATEMENT] for the full statement; v2 corrected per V1 and Lemma 37). *For all small* $\omega > 0$ *(some* $\nu > 1/\omega$*):* $\|\mathcal{K}^{\omega,\nu}[t]\| < 1$ *for every* $t \geq 0$*, proved from the Euler product and* Γ. (Lemma 37: $J \to 0$ is not needed for the direction $\Rightarrow$RH.) Equivalently: the lossless line with impedance $Z_0^{\omega,\nu}$, built from prime-side data, exists for all t. Nothing in this lemma supplies it.

- Positivity of all $c_\nu(n)$ is free (Prop. 34.3.5) but insufficient: the archimedean part alone is active and breaks at $t \approx 0.39$ ([NUM]).
- Contractivity is an exact cancellation of an archimedean gain ($|\Theta_\infty| \neq 1$) by the prime factor, uniformly in t. That is the Euler-product content [ANALOGY].
- Circularity flag: any proof of $\|\mathcal{K}[t]\| < 1$ via $E \in \overline{HB}$ is circular for $\omega < \frac{1}{2}$. Thm. 34.6.4 shows that, under (M), contractivity for all t *forces* $\kappa = 0$. So a direct norm bound would be a genuine route, but it is G5 itself.

QUESTION 34.7.1. Should G5 be stated officially as the uniform Hankel-contractivity of §34.7 (passive line), alongside Lemma 36's (G5a)–(G5d)? It is Lemma 36's (G5d) made concrete. *(v2) Answered:* J.W. adopted the passive line as the official G5 target, in the corrected form $\|\mathcal{K}[t]\| < 1$ for every t (Lemma 37), with the exponential line as headline.

QUESTION 34.7.2. Precision: to see $n \geq 4$ we need $\approx$ 20–30-digit Galerkin up to $t \approx 1.5$ (matrices ~ 5000). Approve a detached multiprecision run? Or an edge-adapted spectral basis first? *(v2) Done*, §34.8. It needed 512 bits, not 20–30 digits (V3).

QUESTION 34.7.3. Does the window length A of Lemmas 24/36 correspond to $2t$ here, so that the $\lambda^*(e^A)$ kinks at $A = \log n$ and our onsets at $t = \frac{1}{2}\log n$ are the same atoms? *(v2) Answered* in §34.8 together with Lemma 37, Prop. 5.1: as an information horizon only, for prime powers.

QUESTION 34.7.4. Track $t_*^{\mathrm{arch}}(\omega, \nu)$ as $\omega \downarrow 0$ ($\nu\omega$ fixed)? It is always just above $\frac{1}{2}\log 2$ in our four cases. Is the margin $t_*^{\mathrm{arch}} - \frac{1}{2}\log 2 > 0$ structural? *(v2) Answered:* yes for $\omega \geq \frac{1}{2}$ (Lemma 37, Thm. 7.4). Numerically it converges to the exponential-line value as $\omega \downarrow 0$ (0.0528 at $\beta = 1.5$), §34.8.

QUESTION 34.7.5. Pursue the KW continuation past t_* (Conjecture in §34.6), or the non-real spectrum in $\mathcal{P}(E^{\omega,\nu})$ for planted zeros (Lemma 27 stage (b))?

34.8. v2: high-precision extension to $t = 1.5$

This section is new in v2. It carries out Question 2 of v1 (multiprecision run to $t \approx 1.5$, approved by J.W.) and answers v1's Questions 3 and 4, reconciled with Lemma 37 and Lemma 36 v2.

34.8.1. Method and what "certified" means here. Solver (`num/lemma34_arb.py`). Uniform mesh of N cells on $(-t, t)$, orthonormal Legendre polynomials of degree $\leq p$ on each cell, all arithmetic in Arb balls (python-flint) at 256 bits ($t \leq 1$) or 512 bits ($t > 1$). Because $\mathcal{K}[t]$ is Hankel, each Galerkin block reduces exactly to moments $\int_0^1 K(g + h\tau)\tau^k \, d\tau$; these are computed atom by atom (Prop. 34.3.5) with Gauss–Jacobi quadrature on the cell containing the $v^{\nu\omega-1}$ singularity of $G(v - \log n)$ and geometrically refined Gauss–Legendre elsewhere. G comes from a 50-digit piecewise Chebyshev table of $g = G/v^{1/2}$ on $[0, 3.2]$ (Talbot inversion at 55 digits; validation error $2.7 \cdot 10^{-52}$; `l34/gchebhp_w0.5_nu3.json`). Then $\log Z_0 = 2\log\det(1+A) - 2\log\det(1-A) - 4\,\mathrm{tr}A + 4\,\mathrm{tr}_{\mathrm{exact}}$ (trace-corrected). Two resolutions: lvl0 $(N, p) = (\max(10, 30t), 12)$, lvl1 $(\max(14, 40t), 16)$.

Certification status (honest labelling). The Ritz argument is rigorous: the Galerkin matrix is a compression of $\mathcal{K}[t]$, so a Galerkin eigenvalue < -1 (equivalently $\det(1+A) < 0$) implies that $\mathcal{K}[t]$ itself has an eigenvalue < -1, and since $\lambda_{\min}(\mathcal{K}[t])$ is non-increasing in t (sections are compressions, Lemma 37) the true break time is *at most* that t. At the bracket ends of Table 34.3 the determinant signs were verified at lvl1 as rigorous Arb signs of the computed matrix (every ball excludes 0). What is *not* interval-certified is the matrix itself: the 50-digit G table and the quadrature carry no rigorous remainder bounds. We therefore label all results below [NUM] ("high-precision numerical, Ritz-sound modulo kernel/quadrature accuracy"). Upper bounds on break times are robust; lower bounds ("still unbroken at x_{lo}") are numerical only.

34.8.2. The line to $t = 1.5$. The gap closes super-exponentially, roughly a factor 10^2 per 0.1 in t near $t = 1.5$; v1's estimate "$\sim 10^{-12}$ by $t \approx 1.5$, so 20–30 digits" was wrong by nine orders of magnitude (correction V3). The double-precision values of v1 Table 34.1 for $t \geq 0.6$ (flagged "n.c." there) are superseded by Table 34.2; e.g. $\log Z_0(1.0) = 28.04835$, not ≈ 17.7. A second, independent double-precision solver (`num/lemma34_hp.py`) agrees: $28.0484 \pm 3 \cdot 10^{-4}$.

t	$\log Z_0$ (lvl0)	$\log Z_0$ (lvl1)	$1-\lambda_{\max}$	$1+\lambda_{\min}$
0.5	11.0362079	11.0362080	–	–
0.6	14.8694139	14.8694141	$1.0\cdot10^{-5}$	$9.5\cdot10^{-4}$
0.7	17.9471511	17.9471513	$3.9\cdot10^{-7}$	$3.7\cdot10^{-5}$
0.8	19.9061075	19.9061078	$3.7\cdot10^{-8}$	$1.8\cdot10^{-6}$
0.9	23.3650316	23.3650318	$2.4\cdot10^{-9}$	$2.6\cdot10^{-7}$
1.0	28.0483531	28.0483533	$3.3\cdot10^{-11}$	$1.0\cdot10^{-8}$
1.1	31.6210545	31.6210548	$5.3\cdot10^{-13}$	$9.1\cdot10^{-11}$
1.2	36.3627927	36.3627930	$7.9\cdot10^{-15}$	$2.7\cdot10^{-12}$
1.3	40.9965575	40.9965602	$7.1\cdot10^{-17}$	$2.5\cdot10^{-14}$
1.4	46.6837615	46.6837675	$3.9\cdot10^{-19}$	$3.1\cdot10^{-16}$
1.5	52.2763499	52.2763534	$1.2\cdot10^{-21}$	$6.5\cdot10^{-19}$

TABLE 34.2. $(\omega,\nu)=(\frac12,3)$, [NUM] (`l34/arb_runs.jsonl`, Fig. 34.4). lvl0 and lvl1 agree to $\le 4\cdot10^{-7}$ for $t\le1.2$ and $\le 6\cdot10^{-6}$ for $t\le1.5$. Gaps are lvl0 Galerkin values (inverse iteration); by the Ritz property they are upper bounds for the true gaps. No determinant sign change anywhere (as Prop. 34.3.3 requires).

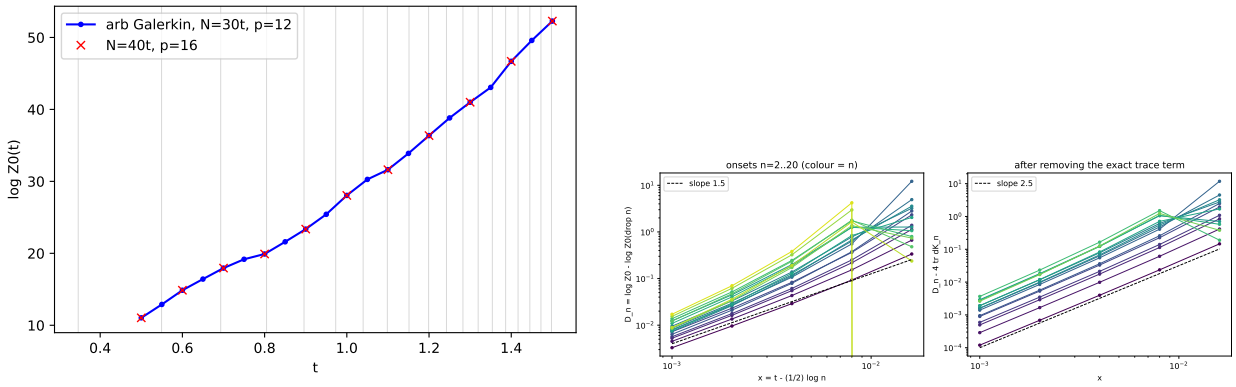

FIGURE 34.4. Left: $\log Z_0(t)$ to $t=1.5$ (Arb Galerkin, two resolutions; grey lines at $\frac12\log n$). Right: onsets D_n, $n=2..20$, and D_n-T_n. [NUM]

34.8.3. Onsets for all $n\le20$: the trace law. Let $D_n(x)=\log Z_0-\log Z_0^{(\text{drop }n)}$ at $t=t_n+x$, $t_n=\frac12\log n$, where "drop n" removes the single atom n from K.

PROPOSITION 34.8.1 (exact first-order term; [THM], elementary). *The trace part of D_n is $T_n(x):=4\big(\mathrm{tr}_{\mathrm{exact}}-\mathrm{tr}_{\mathrm{exact}}^{(drop\ n)}\big)=2\,b_n\,G_1(2x)$, where $b_n=c_\nu(n)n^{-1/2}$ and $G_1(v)=\int_0^v G$. For $(\frac12,3)$, $G(v)=4\sqrt2\,\pi\,v^{1/2}(1+O(v))$, so $T_n(x)=\frac{64\pi}{3}\,b_n\,x^{3/2}(1+O(x))$.*

PROOF. $\mathrm{tr}\,\mathcal K[t]=\int_{-t}^{t}K(2y)\,dy=\frac12\int_0^{2t}K$, and the atom n contributes $b_nG(v-\log n)$. For the constant: the Laplace transform of G is $(\gamma(p)/\gamma(p+1))^3=\big(\frac{p-1}{p}\sqrt\pi\,\Gamma(\frac p2+1)/\Gamma(\frac p2+\frac32)\big)^3\sim(2\pi)^{3/2}p^{-3/2}$, and Watson's lemma (in its Tauberian direction for this explicit, asymptotically expandable transform) gives $G(v)\sim(2\pi)^{3/2}v^{1/2}/\Gamma(\frac32)=4\sqrt2\pi v^{1/2}$. Numerical check: the table gives $g(0)=17.7715317526335$, and $4\sqrt2\pi=17.7715317526335$. □

Data ([NUM], $x=10^{-3},\dots,1.6\cdot10^{-2}$, all $n=2,\dots,20$; `l34/arb_onset_table.json`, `figs/lemma34_hp_onsets.pdf`). (i) $D_n>0$ for all n, prime powers and composites alike. (ii) T_n/D_n at $x=10^{-3}$ is 0.96 ($n=2$), 0.90 ($n=5$), 0.80 ($n=10$), 0.66 ($n=20$). (iii) The residual D_n-T_n has local exponent 2.52–2.70 at the smallest x (where it is positive), i.e. it is $O(x^{\nu\omega+1})$

at the resolved scales, with a coefficient that grows with n (from $3.8 \cdot 10^3$ at $n = 2$ to $1.8 \cdot 10^5$ at $n = 20$). (iv) lvl1 vs lvl0: $|\Delta D|/D \le 1.1 \cdot 10^{-4}$ ($n = 2, 4, 8, 16$). (v) Points with x beyond the drop-n break (below) are meaningless for D_n ($\log Z_0^{(\text{drop } n)}$ has a logarithmic singularity there); this explains the erratic last slopes for $n \ge 8$ in the table.

CONJECTURE 34.8.2 (revised Conj. 34.5.2; [CONJ]). $D_n(x) = T_n(x) + O_n(x^{\nu\omega+1})$, so every integer n produces a cusp $\log Z_0 \ni \frac{64\pi}{3} b_n (t - t_n)_+^{3/2}$, with $b_n > 0$; the O_n constant grows with n, so the pure $x^{3/2}$ regime shrinks as t grows.

This corrects v1's mechanism (V2): v1 claimed a "naive" bounded-resolvent estimate gives exponent $\nu\omega + 1$ and attributed the observed $\nu\omega$ to edge singularities of the resolvent. In fact the exponent $\nu\omega$ is already in the elementary trace term (Prop. 34.8.1); no edge-resolvent mechanism is needed. The coefficient $\propto b_n$ in v1's conjecture was right. v1's statement that a local prime-side description is "not visible beyond $n \le 3$" is withdrawn: it is visible for all $n \le 20$.

34.8.4. Drop-n breaks: every single atom is necessary. For each n we locate the first $x > 0$ at which the drop-n operator on $(-t, t)$, $t = t_n + x$, has $\det(1 - A) \le 0$ or $\det(1 + A) \le 0$ (12-point scan of $(0, 0.98 \cdot \frac{1}{2} \log \frac{n+1}{n})$, then 8 bisections, lvl0; `num/lemma34_arb_dropbreak*.py`). Both bracket ends were re-checked at lvl1 with rigorous Arb determinant signs (`num/lemma34_arb_dropcheck.py`, `l34/arb_dropcheck.jsonl`, `l34/arb_dropbreak_p.jsonl`): in every case lvl1 is unbroken at x_{lo} and broken at x_{hi}.

n	t_n	x-bracket	$t_{\text{break}} \le$	side	L37 cascade P	L37 t_{break} (diff)
2	0.34657	[0.05193, 0.05200]	0.39857	−1	1 (Γ only)	0.3981 (−0.0005)
3	0.54931	[0.03460, 0.03464]	0.58395	−1	2	0.5844 (+0.0004)
4	0.69315	[0.02595, 0.02598]	0.71913	−1	–	–
5	0.80472	[0.02402, 0.02405]	0.82877	−1	3	0.8319 (+0.0031)
6	0.89588	[0.02028, 0.02031]	0.91619	−1	–	–
7	0.97296	[0.01911, 0.01913]	0.99208	−1	5	0.9981 (+0.0060)
8	1.03972	[0.01601, 0.01603]	1.05575	−1	–	–
9	1.09861	[0.01516, 0.01518]	1.11379	−1	–	–
10	1.15129	[0.01406, 0.01408]	1.16537	−1	–	–
11	1.19895	[0.01441, 0.01442]	1.21337	−1	7	1.2207 (+0.0073)
12	1.24245	[0.01177, 0.01178]	1.25424	−1	–	–
13	1.28247	[0.01309, 0.01310]	1.29557	−1	11	1.3057 (+0.0101)
16	1.38629	[0.01049, 0.01050]	1.39680	−1	–	–
17	1.41661	[0.01111, 0.01112]	1.42773	−1	13	1.4382 (+0.0105)
19	1.47222	[0.01050, 0.01051]	1.48272	−1	17	1.4932 (+0.0105)
20	1.49787	[0.00858, 0.00858]	1.50645	−1	–	–
23	1.56775	[0.00936, 0.00937]	1.57712	−1	19	1.5882 (+0.0111)

TABLE 34.3. Drop-n breaks, $(\frac{1}{2}, 3)$, [NUM] (upper bounds Ritz-sound, see §34.8.1). "side −1": $\det(1 + \mathcal{K}[t]) = 0$, a short in Lemma 37's dictionary. Last column: Lemma 37, Table 1 ($h = 1/800$) and its difference from our upper bound.

PROPOSITION 34.8.3 (drop-p = cascade; [THM], elementary). *If $n = p^+$ is prime and P is the preceding prime, the drop-p^+ kernel and Lemma 37's P-truncated kernel agree on $[0, \log(p^+ + 1))$. Hence on $t \in (t_{p^+}, \frac{1}{2}\log(p^+ + 1))$ the two lines coincide, and a drop-p^+ break in that window is the break of the P-truncated line.*

PROOF. Every $m \le p^+$ other than p^+ is P-smooth, and the P-truncated kernel keeps exactly the P-smooth atoms. All breaks in Table 34.3 occur before $\frac{1}{2}\log(n + 1)$. □

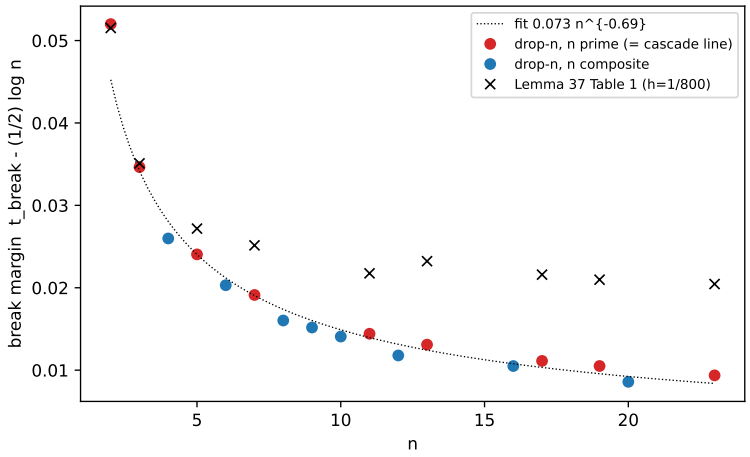

FIGURE 34.5. Drop-n break margins (dots; red = prime n, i.e. cascade lines) vs Lemma 37 Table 1 (crosses), with the empirical fit. [NUM]

Reading. (1) Every computed drop-n line breaks, shortly after t_n and before t_{n+1}, always as a short (-1). This holds for composite n (4, 6, 8, 9, 10, 12, 16, 20) too: removing a single non-prime-power coefficient such as $c_\nu(6)$ breaks the line. Empirically, positivity of the line is *tight in every coefficient*. This is an observation for $n \le 23$, not a theorem. (2) For primes these are the cascade margins of Lemma 37, now at high precision: 0.0520 ($n = 2$), 0.0346, 0.0240, 0.0191, 0.0144, 0.0131, 0.0111, 0.0105, 0.0094 ($n = 23$). Lemma 37's values agree for $t \lesssim 0.6$ but are too large by 0.003–0.011 for $t \gtrsim 0.8$, more than its stated discretisation error ($\le 1.5 \cdot 10^{-3}$). The double-precision $h = 1/800$ scheme cannot resolve gaps below $\sim 10^{-8}$ (Table 34.2), so its Ritz values sit too far inside $(-1, 1)$ and its breaks come late. This is a correction to Lemma 37's Table 1, recorded here (V7); Lemma 37 itself is not edited. Its conclusions (all breaks are shorts; every truncated line survives past the next prime's arrival; margins decrease) stand, and the decrease is faster than Table 1 showed. (3) Over $n \le 23$ the margin fits $x_{\text{break}} \approx 0.073\, n^{-0.69}$ (all 17 values; primes alone $0.076\, n^{-0.68}$, rms log-residual 0.05). This is an empirical fit on 17 points, not a law. Among primes no dependence on the preceding prime gap (raw or Cramér-normalised by $\log p/p$) is detectable at this sample size. Primes have slightly larger margins than neighbouring composites (11 vs 10, 12; 13 vs 12; 17 vs 16). [CONJ] (Lemma 37): margins $\to 0$; the data are consistent with that but cannot establish it.

34.8.5. The archimedean break as $\omega \downarrow 0$ (v1 Question 4). With $\beta = \nu\omega$ fixed, the archimedean symbol tends to $\exp(-2\beta(\gamma'/\gamma)(\frac{1}{2} + p))$, the archimedean part of Lemma 37's exponential line Θ_c with $c = \beta$ (`num/lemma34_q4.py`, `l34/q4.jsonl`; arb Galerkin at two resolutions $(N, p) = (8, 10), (14, 12)$, agreeing to $\le 3 \cdot 10^{-11}$; [NUM]):

β	ω values	t_*^{arch}	limit $\omega \to 0$	limit $-\frac{1}{2}\log 2$
1.5	$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, 0.1, 0.05$	.398570, .399177, .399330, .399348, .399373	.399381	.05281
1.2	$0.1, 0.05$	.390772, .390796	.390804	.04423
1.05, 1.1	–	–	.386900, .388169	.04033, .04160
2.0, 3.0	–	–	.415421, .451701	.06885, .10513

The approach to the limit is $O(\omega^2)$ (differences from the limit $8.1 \cdot 10^{-4}$ at $\omega = \frac{1}{2}$, $2.0 \cdot 10^{-4}$ at $\frac{1}{4}$, $5.1 \cdot 10^{-5}$ at $\frac{1}{8}$: ratio 4.0 per halving), matching Lemma 37 Thm. 4.1. All breaks are shorts. The margin $t_*^{\text{arch}} - \frac{1}{2}\log 2$ stays positive in every case and increases with β. *Structural reason:*

Lemma 37 Thm. 7.4 (prime arrival) proves $t_*^{\mathrm{arch}} > \frac{1}{2}\log 2$ unconditionally for $\omega \geq \frac{1}{2}$, since the arch line coincides with the full line on $[0, \frac{1}{2}\log 2]$. For $\omega < \frac{1}{2}$ and for the limit it is numerical only (Lemma 37 roadmap M5 asks for an interval certificate). The arb value 0.398570 for $(\frac{1}{2}, 3)$ sharpens v1's bound ≤ 0.4000 and agrees with the drop-2 break in Table 34.3 (≤ 0.39857; independent kernel tables), as Prop. 34.8.3 predicts.

34.8.6. $A = 2t$ **(v1 Question 3), reconciled with Lemma 37.** Lemma 37, Prop. 5.1: $A = 2t$ holds exactly as an *information horizon* (both sides see $\Lambda(n)$ for $n < e^A$, plus Γ), but the functionals differ. Our data refine this without contradiction. The line *does* have a genuine first-order onset at every integer, composites included ($D_6, D_{10}, D_{12}, \cdots > 0$ with the exact trace law, weight $b_n > 0$). At non-prime-powers the *weight* carries no new arithmetic information, since c_ν is multiplicative, but the line still responds to it, and removing it breaks the line (Table 34.3). So "onset at $\frac{1}{2}\log n$ for all n" (Thm. 34.5.1) and "composite onsets are echoes" (Lemma 37) are both correct: the first is about the line's singularity structure, the second about information content. Lemma 37's prediction of *no* kink of $\lambda^*(e^A)$ at $A = \log 6$ concerns the Weil-form functional and is not tested here. The v1 Remark, "same edge atoms seen through two functionals", is therefore true for prime powers only, and only in the information sense [ANALOGY].

34.8.7. Reconciliation with Lemma 36 v2, Thm. 7.9. Lemma 36 v2, Thm. 7.9 proves $n_\pm(R[t]) = \kappa/2$ *exactly* for every $t > 0$ and removes hypothesis (M) from Thm. 34.6.4(iv): eventually exactly $\kappa/2$ eigenvalues lie above 1 and exactly $\kappa/2$ below -1, and they tend to $\pm\infty$. Accordingly, Thm. 34.6.4(ii) holds with equality, (iv) holds without (M), and the Corollary after it holds without "modulo (M)" (see the corrected statements, marked v2). The planted-zero counts 1/1 and 2/2 (§34.6.1) are exactly these eventual counts. Our proofs of (i)–(iii) are unchanged.

Question 34.8.4 (v2, new). The drop-n table suggests that every positive coefficient is *individually* necessary, with margin $\approx 0.07\, n^{-0.7}$. Is there an inequality of the form "margin $\geq f(b_n, \mathrm{gap})$" (Lemma 37's Osgood route M4) that the data can guide? Should we extend to $n \leq 60$ ($t \approx 2$, needs a longer G table and $\sim$1000-bit runs)?

Question 34.8.5 (v2, new). Should Lemma 37's Table 1 and Lemma 39's margin fits be re-run with the Arb solver (V7)?

34.9. Status

Claim	Tag	Where
E, $\mathcal{K}[t]$, $H = \mathrm{diag}(Z_0^{-1}, Z_0)$, locality	[CITED]	[257] Thm. 2.1–2.4, Prop. 5.1
$E^{1/2,\nu} \in \overline{HB}$, line exists $\forall t$	[THM]	Prop. 34.3.3
line/string form, $\mathfrak{m} = Z_0^{-2} > 0$	[THM]	Thm. 34.3.4
prime-side kernel, $c_\nu > 0$	[THM]	Prop. 34.3.5
$\log Z_0$ to 4–5 digits, $t \le 0.55$	[NUM]	Table 34.1
not converged $t \ge 0.6$	[NUM]	Fig. 34.1
onsets at $\frac{1}{2}\log n$, all n	[THM]	Thm. 34.5.1
onset exponent $\approx \nu\omega$	[NUM], [CONJ]	Conj. 34.5.2
arch-only active, breaks $t \approx 0.39$	[THM] ($\Theta_\infty(0)$), [NUM]	§34.5
$\kappa = \nu N_+(\frac{1}{2}+\omega)$	[THM] (hyp. B, ND)	Thm. 34.6.2
escape, $\le \kappa/2$ per side	[THM] ((iv) needs M)	Thm. 34.6.4
planted 1/1, 2/2	[NUM]	§34.6.1
RH $\iff$ $\kappa \equiv 0$ $\iff$ Hilbert $\iff$ no break	[RESTATEMENT]	§34.6
"RH $\iff$ $\mathfrak{m} \ge 0$"	false as stated	§34.6, [96]
KW continuation past t_*	[OPEN]	§34.6
(v2) $\log Z_0$ to ∼6 digits, $t \le 1.5$	[NUM]	Table 34.2
(v2) onset trace law $T_n = 2b_nG_1(2x)$	[THM]	Prop. 34.8.1
(v2) trace term dominates, $n \le 20$	[NUM], [CONJ]	Conj. 34.8.2
(v2) every drop-n line breaks ($n \le 23$), shorts	[NUM]	Table 34.3
(v2) drop-p = cascade line	[THM]	Prop. 34.8.3
(v2) $t_*^{\mathrm{arch}} - \frac{1}{2}\log 2 > 0$, $\omega \downarrow 0$	[THM] ($\omega \ge \frac{1}{2}$, L37), [NUM]	§34.8
(v2) exact escape counts, no (M)	[THM] (Lemma 36 v2)	§34.8
G5 supplied?	**No**	§34.7

Files: `num/lemma34_{kernel,tab,tab2,galerkin,checks,laplace,onset,trace,figs}.py`; data `num/l34/`; figures `figs/lemma34_{spectrum,profile,planted}.pdf`; refs `lit/lemma34/verified_refs.md` resume log `num/lemma34_progress.md`. (v2) scripts `num/lemma34_arb.py`, `arb_run`, `arb_analyze`, `arb_dropbreak`, `arb_dropbreak_ext`, `arb_dropbreak_p`, `arb_dropcheck`, `q4`, `hp`, `gcheb_hp`, `v2_tables`; data `num/l34/arb_runs.jsonl`, `arb_dropbreak*.jsonl`, `arb_dropcheck.jsonl`, `q4.jsonl`, `arb_onset_table.json`, `v2_summary.json`; figures `figs/lemma34_hp_onsets.pdf`, `hp_profile.pdf`, `dropbreak.pdf`.

CHAPTER 35

Lemma 35: The Post–Widder sign check of S_+ from the prime side. An exact zero-free formula for $(-1)^n S_+^{(n)}$, a blindness theorem below order $4.7 \cdot 10^{12}$, and why complete monotonicity is domination, not termwise positivity

Standalone note `lemma35.tex`, October 21, 2026.

Summary. Tags. [THM] proved here; [CITED] checked at the source (`lit/lemma35/verified_refs.md`); [CITED, secondary] checked only through a named secondary source; [DEF] definition; [RESTATEMENT] equivalent to RH, no progress; [ANALOGY] analogy; [OPEN] open; [NUM] numerical (`num/lemma35_*.py`); [ESTIMATE] heuristic estimate, not a theorem.

Statement of intellectual honesty. RH is not proved here, and nothing here makes progress on it. G5 (Lemma 32: complete monotonicity of S_+ from the prime side) is *not* supplied. The numerics check an explicit formula against Platt's zeros; they are not evidence for RH beyond what the verified height $3 \cdot 10^{12}$ already gives. Section 35.4 shows that no finite-order sign check up to order $4.7 \cdot 10^{12}$ could have failed.

Findings.

(1) *Exact prime-side formula* (Thm. 35.2.1, [THM]). For every $c > \frac{1}{2}$, $n \geq 0$, $\tau > 0$: $(-1)^n S_+^{(n)}(\tau) = \mathcal{S}_n + \mathcal{A}_n + \mathcal{P}_n$. Here $\mathcal{S}_n$ is an integral of ξ'/ξ over the real segment $[\frac{1}{2}, \frac{1}{2} + c]$, $\mathcal{A}_n$ is a Γ-factor integral on $\operatorname{Re} s = \frac{1}{2} + c$, and $\mathcal{P}_n = -\frac{1}{\pi} \sum_m \Lambda(m) m^{-1/2} \operatorname{Re}\big[\Gamma(n+1, c\log m - ic\tau)/(\tau + i\log m)^{n+1}\big]$. The formula has an explicit tail bound. It uses no zeros. It is a computable form of Cramér's continuation of V (poles at $i\tau = \pm \log m$, consistent with [**135**] [CITED]).

(2) *Numerical agreement* ([NUM]). In all 74 cases ($0 \leq n \leq 2000$, $0.01 \leq \tau \leq 100$, Tables 35.1–35.2), the prime side agrees with the zero side (Platt zeros, high precision) within the rigorous truncation bound. Where that bound is small, the agreement is 22–27 significant digits. In 73 of the 74 cases the prime side alone fixes the sign (relative tail bound < 1). The exception is $n = 20$, $\tau = 20/3$, $c = 4$, where the tail bound is $3.6 \cdot 10^{11}$ times the value; with $c = 8$ the bound is $1.4 \cdot 10^{-3}$. The Post–Widder values (Table 35.3) agree to $\sim 10^{-15}$, the precision at which they were stored, except at $T = 10$, $n = 100$, where the difference $2.4 \cdot 10^{-12}$ is inside the tail bound $3.6 \cdot 10^{-11}$. An uncontrolled truncation can produce a *spurious* sign violation: $n = 0$, $\tau = 3$, $c = 4$ gives $-1.7 \cdot 10^{-18}$, but the tail bound there is $7.7 \cdot 10^{-17}$ and the true value is $+3.84 \cdot 10^{-19}$.

(3) *What order n probes* (Prop. 35.3.1, [THM] + [CITED, secondary]). The Post–Widder transform $\mathrm{PW}_n(T)$ at $\tau = n/T$ sees zeros of height $T \pm T/\sqrt{n}$ through a Gamma kernel. Resolving individual zeros at height T takes $n \gtrsim (T \log T)^2$. On the prime side it takes prime powers up to $\log m \approx \sqrt{2L}\sqrt{n}/T$ for L nats of accuracy, an uncertainty relation $\Delta T \cdot \Delta \log m \approx \sqrt{2L}$ ([ESTIMATE]).

(4) *Blindness theorem* (Thm. 35.4.1, [THM]). Since all zeros with $0 < t \leq 3 \cdot 10^{12}$ are on the line [**228**], $(-1)^n S_+^{(n)}(\tau) > 0$ for *all* $\tau > 0$ and all $0 \leq n \leq \lfloor \frac{\pi}{2} \cdot 3 \cdot 10^{12} \rfloor = 4\,712\,388\,980\,384$. Equivalently, a sign violation at order n forces an off-line zero

of height $< 2n/\pi$. So a finite-order Post–Widder check, prime-side or not, is an RH-verification method at height $\approx 2n/\pi$, and a weak one.

(5) *Detection threshold* (§35.5, [NUM] on a model + [ESTIMATE]). In a model (true zeros plus one planted pair $t_0 \pm i\delta$), the first violation occurs at $n_{\mathrm{det}} = \kappa\, t_0^2/(\delta g)$ with $\kappa \in [2.9, 4.2]$ over 16 cases ($t_0 \le 300$). Here g is the larger neighbouring gap, and the shift is $|\tau - n/t_0| \approx (0.5\text{–}0.6)\pi/\delta$. This is 10–400 times the half-turn order n^* of Lemma 33. Extrapolated to $t_0 = 3 \cdot 10^{12}$ with δ at the MTY24 bound, $n_{\mathrm{det}} \approx 2\text{–}3 \cdot 10^{26}$ ([ESTIMATE]).

(6) *Bernstein and domination* (§35.6, [THM] + [RESTATEMENT]). The prime-side inequality implying complete monotonicity for all n is $D_n(\tau) \ge -\mathcal{A}_n^\theta(\tau)$ for all n, τ. Here $\mathcal{A}_n^\theta = \int t^n e^{-\tau t}\theta'(t)\, dt/\pi$ is the archimedean part, negative for windows below $T \approx 6.29$ (Lemma 17's t_θ). The remainder D_n is a sum of prime atoms with oscillating signs $-\cos((n+1)\arctan(\log m/\tau))$. At $n = 1000$, $T = 1$ the archimedean part is -0.322 (normalized) and the total is 10^{-4553}: the prime atoms cancel the ghost tower to 4553 digits. Termwise positivity is false in every term family. The positive measures one can write down for free (dN, dN_0) are blind to off-line zeros: S_+ CM $\iff$ $S_+ = \mathcal{L}[dN]$ $\iff$ $S_+(\tau_0) \ge \mathcal{L}[dN](\tau_0)$ for one $\tau_0 \in (0, 4\pi)$ ([RESTATEMENT]). The content is an *identity*, not an inequality.

The new statements (Thms. 35.2.1, 35.4.1; Props. 35.3.1, 35.4.4, 35.4.5, 35.6.1) are elementary. We claim no priority.

35.1. Setting and references

Notation follows `NOTATION.md` and Lemmas 32–33. $\rho = \frac{1}{2} + i\gamma_\rho$, $\gamma_\rho = t_\rho - i\delta_\rho$, $|\delta_\rho| < \frac{1}{2}$, multiplicity m_ρ. For $\operatorname{Im} z > 0$, $V(z) = \sum_{t_\rho > 0} m_\rho e^{i\gamma_\rho z}$, and $S_+(\tau) = V(i\tau) = \sum_{t_\rho>0} m_\rho e^{-\tau\gamma_\rho}$ for $\operatorname{Re}\tau > 0$ [DEF]. Lemma 32, Thm. 7.2: RH $\iff$ S_+ is completely monotone (CM) on $(0,\infty)$ [RESTATEMENT]. Lemma 33, Prop. (half-turn): an off-line pair contributes $2m|\gamma|^n e^{-\tau t}\cos(n\theta - \tau\delta)$ to $(-1)^n S_+^{(n)}(\tau)$, with $\theta = \arctan(\delta/t)$.

Reference check (task item (a)).

- Cramér 1919 [**77**]: correct. Illies's introduction (read in full, `lit/lemma35/ill02.pdf`) reports that Cramér's function is $\sum_{\operatorname{Im}\rho>0} e^{\rho t} = e^{t/2}V(t)$, and that $V(t) - \frac{1}{2\pi i}\frac{\log t}{1-e^{-t}}$ continues meromorphically to $\mathbb{C}$ with simple poles [CITED, secondary].
- *Guinand 1948* [**119**] is the summation-formula paper (PLMS 50). The functional equation for V quoted by Illies, $e^{-t/2}(V(t)-1) + e^{t/2}(V(e^{\pi i}t) - 1) = -e^{-t/2}/(1-e^{-2t})$, is from [**120**], PLMS 51 (Crossref 1949, cited by Illies as 1950) [CITED, secondary]. We cite both and use only the latter's statement.
- *Illies 2001* should read 2002: [**135**], Acta Arith. 105 (2002), 103–118 [CITED]. The 2001 paper in this circle is Jorgenson–Lang [**141**], JNT 86 (bibliographic).

We do not use the Guinand equation below. Thm. 35.2.1 is derived from scratch, so the numerics depend only on Davenport-level facts [**80**].

35.2. The exact prime-side formula

Put $p = s - \frac{1}{2}$ and $g(p) = \frac{\xi'}{\xi}(\frac{1}{2} + p)$. Then g is odd, real on $\mathbb{R}$, regular on $\mathbb{R}$ (no real zeros of ξ), and $g(0) = 0$. Its poles are $p = i\gamma_\rho$, with residue m_ρ. From [**80**, Ch. 12],

$$g(p) = A(\tfrac{1}{2} + p) + \frac{\zeta'}{\zeta}(\tfrac{1}{2} + p), \qquad A(s) = \frac{1}{s} + \frac{1}{s-1} - \frac{\log\pi}{2} + \frac{1}{2}\psi(\tfrac{s}{2}). \tag{35.2.1}$$

THEOREM 35.2.1 ([THM]). *Let $c > \frac{1}{2}$, $n \in \mathbb{Z}_{\ge 0}$, $\tau > 0$. Then $(-1)^n S_+^{(n)}(\tau) = \mathcal{S}_n + \mathcal{A}_n + \mathcal{P}_n$ with*

$$\mathcal{S}_n = \frac{(-1)^n}{\pi}\int_0^c g(x)\,x^n \sin(\tau x + \tfrac{n\pi}{2})\,dx, \qquad \mathcal{A}_n = \frac{1}{\pi}\operatorname{Re}\Big[e^{i\tau c}\int_0^\infty A(\tfrac12 + c + it)(t - ic)^n e^{-\tau t}dt\Big],$$
$$\mathcal{P}_n = -\frac{1}{\pi}\sum_{m\ge 2}\frac{\Lambda(m)}{\sqrt{m}}\operatorname{Re}\frac{\Gamma(n+1, z_m)}{w_m^{n+1}}, \qquad w_m = \tau + i\log m, \quad z_m = c\log m - ic\tau.$$

The sum converges absolutely. Its tail over $m > M$ is bounded by $\frac{1}{\pi}\sum_{m>M}\Lambda(m)m^{-1/2}|w_m|^{-n-1}B_n(z_m)$ with $B_n(z) = \min\{n! + |z|^{n+1}/(n+1),\ e^{|z|-\operatorname{Re} z}\Gamma(n+1,|z|)\}$, and for $|z| > n$ also $|\Gamma(n+1,z)| \le e^{-\operatorname{Re} z}|z|^n/(1 - n/|z|)$. All three terms are zero-free: they use ξ'/ξ on $[\frac12, \frac12 + c]$, Γ'/Γ and the pole on $\operatorname{Re} s = \frac12 + c$, and the Dirichlet series of ζ'/ζ on $\operatorname{Re} s = \frac12 + c > 1$.

PROOF. The residue of $g(p)e^{i\tau p}$ at $p = i\gamma_\rho$ is $m_\rho e^{-\tau\gamma_\rho}$. Integrate over the rectangle $[-c, c]\times[0, T]$ in the p-plane, with T a good height [**80**, Ch. 17]. On such lines $\zeta'/\zeta(\sigma + iT) = O(\log^2 T)$ for $-1 \le \sigma \le 2$; outside that strip use the Dirichlet series and the functional equation. With the factor $e^{-\tau T}$, the top edge tends to 0. The rectangle contains exactly the zeros with $0 < t_\rho < T$ (because $|\delta_\rho| < \frac12 < c$), and no pole lies on the real axis.

On the bottom edge, oddness and reality of g give $\frac{1}{2\pi i}\int_{-c}^{c} g(x)e^{i\tau x}dx = \frac{1}{\pi}\int_0^c g(x)\sin\tau x\,dx$. On the side edges $p = \pm c + it$, $g(-c + it) = -\overline{g(c + it)}$, so the two side edges add to $\frac{1}{\pi}\operatorname{Re}[e^{i\tau c}\int_0^T g(c + it)e^{-\tau t}dt]$. Applying $(-1)^n\partial_\tau^n$ multiplies the integrand by $(-ip)^n$, which is $(t - ic)^n$ on the right edge. Differentiation under the integral is legitimate because $|g(c + it)| = O(\log t)$ and $e^{-\tau t}$ decays locally uniformly in τ. This gives $\mathcal{S}_n$, and the A-part of the right edge gives $\mathcal{A}_n$.

For the ζ'/ζ-part, $\zeta'/\zeta(\frac12 + c + it) = -\sum\Lambda(m)m^{-1/2-c}m^{-it}$, absolutely convergent, so sum and integral commute. Each term is $\int_0^\infty (t - ic)^n e^{-w_m t}dt$. Substituting $v = w_m(t - ic)$ and rotating the ray (the integrand is entire and decays in $\operatorname{Re} v > 0$) gives $e^{-icw_m}\Gamma(n + 1, -icw_m)/w_m^{n+1}$, and $-icw_m = z_m$. Finally $e^{i\tau c}e^{-icw_m}m^{-c} = 1$.

For the bounds ($\operatorname{Re} z_m = c\log m > 0$): (1) $\Gamma(n+1, z) = n! - \int_0^z v^n e^{-v}dv$, and on the segment $v = sz$, $|v^n e^{-v}| \le |z|^n s^n$, so $|\Gamma(n+1, z)| \le n! + |z|^{n+1}/(n+1)$. (2) On the horizontal ray $v = z + u$, $u \ge 0$, $|v| \le |z| + u$ and $|e^{-v}| = e^{-\operatorname{Re} z - u}$, so $|\Gamma(n+1, z)| \le e^{-\operatorname{Re} z}\int_0^\infty (|z| + u)^n e^{-u}du = e^{|z|-\operatorname{Re} z}\Gamma(n+1, |z|)$. (3) $\Gamma(n + 1, x) \le x^n e^{-x}/(1 - n/x)$ for $x > n$ (compare with a geometric series), which combined with (2) gives the third bound; note $e^{-\operatorname{Re} z_m} = m^{-c}$. The tail over $m > M$ is compared with an integral, using $\Lambda(m) \le \log m$ and the prime-power count. Monotonicity of the comparison function is checked on a grid in the code. □

REMARK 35.2.2 (structure). (i) As a function of τ, the m-th prime term has its singularities at $\tau = \mp i\log m$, i.e. at $z = i\tau = \pm\log m$. These are the pole locations of Cramér's continuation as reported by Illies (§10: ± logarithms of prime powers) [CITED]. The log-term of Cramér's theorem is hidden in $\mathcal{S} + \mathcal{A}$. (ii) As $n \to \infty$ at fixed τ, $\Gamma(n + 1, z_m) \to n!$, so $\mathcal{P}_n \sim -\frac{n!}{\pi}\sum\Lambda(m)m^{-1/2}\operatorname{Re} w_m^{-n-1}$, independent of c. Relative to the scale $n!/\tau^{n+1}$, the m-th atom carries the factor $(1 + \log^2 m/\tau^2)^{-(n+1)/2}$ and the phase $-(n + 1)\arctan(\log m/\tau)$. (iii) Choice of c. Large c damps the prime sum (m^{-c} through z_m) but makes $\mathcal{S}, \mathcal{A}$ large and cancelling, so it needs more digits. It is good for small n. For large n, $c \approx 0.75$ is best. We take $c = 4, 6, 8$ for $n \le 20$ and $c = 0.75$ beyond; agreement between two values of c is an internal check.

35.2.1. Numerical agreement. `num/lemma35_core.py` evaluates the three terms with `mpmath` (50–80 digits). $\mathcal{S}$ uses Gauss–Legendre with breakpoints at $\frac12$ and every $\approx 1.5/\tau$. It is checked against tanh–sinh with 8/16/32 panels, which agree to 10^{-42}. g is checked against numerical differentiation of $\log\xi$ ($< 10^{-42}$). $\mathcal{A}$ uses tanh–sinh with breakpoints at $n/\tau \pm k\sqrt{n+1}/\tau$, checked with two breakpoint sets (10^{-43}). $\mathcal{P}$ uses exact incomplete Γ. The zero side uses 107 614

Platt zeros $\le 8\cdot 10^4$ (ordinates to 2^{-101}), summed in high precision near the peak of $t^n e^{-\tau t}$. The tail beyond $8\cdot 10^4$ is bounded with HSW's $N(T)$ bound [**127**, Cor. 1.2].

One bug is worth recording. A first version stored $\Lambda(m) = \log p$ in double precision. That produced a c-dependent absolute error of $\sim 10^{-20}$ at $\tau \ge 1$, larger than the claimed tail bounds. With $\log p$ in working precision, the discrepancy disappeared ($n=0$, $\tau=2$, $c=8,10$: 22 digits).

Table 35.1: Prime side (Thm. 35.2.1) versus zero side (107 614 Platt zeros, high precision, plus HSW tail), small orders. M: primes powers $\le M$ used. "rel. tail bound is the rigorous bound on the omitted prime powers $m > M$ divided by the value. [NUM]

n	τ	c	M	$(-1)^n S_+^{(n)}(\tau)$ (prime side)	rel. diff. to zeros	rel. tail bound
0	0.05	4	100000	2.68601504852	$1.8\cdot 10^{-21}$	$1.1\cdot 10^{-19}$
0	0.05	6	100000	2.68601504852	$3.2\cdot 10^{-26}$	$6.8\cdot 10^{-30}$
0	0.2	4	100000	0.0856674423422	$2.1\cdot 10^{-19}$	$3.4\cdot 10^{-18}$
0	0.2	6	100000	0.0856674423422	$2.8\cdot 10^{-27}$	$2.1\cdot 10^{-28}$
0	1	4	100000	$7.27475121688\cdot 10^{-7}$	$2.7\cdot 10^{-14}$	$3.9\cdot 10^{-13}$
0	1	6	100000	$7.27475121688\cdot 10^{-7}$	$4.1\cdot 10^{-25}$	$2.5\cdot 10^{-23}$
0	2	4	100000	$5.28121936158\cdot 10^{-13}$	$4.3\cdot 10^{-8}$	$5.4\cdot 10^{-7}$
0	2	6	100000	$5.28121913338\cdot 10^{-13}$	$1.1\cdot 10^{-18}$	$3.4\cdot 10^{-17}$
0	3	4	100000	$3.76440965435\cdot 10^{-19}$	0.019	0.73
0	3	6	100000	$3.83796140971\cdot 10^{-19}$	$2.2\cdot 10^{-12}$	$4.6\cdot 10^{-11}$
1	0.3333	4	100000	0.1540785148	$1.3\cdot 10^{-19}$	$7.6\cdot 10^{-18}$
1	0.3333	8	100000	0.1540785148	$2.1\cdot 10^{-25}$	$7.0\cdot 10^{-38}$
1	0.1	2	100000	14.8759916731	$7.6\cdot 10^{-11}$	$9.4\cdot 10^{-10}$
1	0.1	4	100000	14.8759916743	$6.1\cdot 10^{-21}$	$7.9\cdot 10^{-20}$
1	0.06667	2	100000	47.055020925	$2.4\cdot 10^{-11}$	$3.0\cdot 10^{-10}$
1	0.06667	4	100000	47.0550209261	$2.0\cdot 10^{-21}$	$2.5\cdot 10^{-20}$
1	0.04	2	100000	180.177161704	$6.4\cdot 10^{-12}$	$7.8\cdot 10^{-11}$
1	0.04	4	100000	180.177161705	$5.4\cdot 10^{-22}$	$6.5\cdot 10^{-21}$
1	0.01	2	100000	5077.90415391	$2.3\cdot 10^{-13}$	$2.7\cdot 10^{-12}$
1	0.01	4	100000	5077.90415391	$2.0\cdot 10^{-23}$	$2.3\cdot 10^{-22}$
2	0.6667	4	100000	0.0165490315979	$9.9\cdot 10^{-18}$	$2.9\cdot 10^{-16}$
2	0.6667	8	100000	0.0165490315979	$2.2\cdot 10^{-25}$	$5.3\cdot 10^{-36}$
2	0.2	2	100000	28.2473154641	$3.4\cdot 10^{-11}$	$1.0\cdot 10^{-9}$
2	0.2	4	100000	28.2473154651	$1.0\cdot 10^{-20}$	$1.7\cdot 10^{-19}$
2	0.1333	2	100000	148.055952036	$4.4\cdot 10^{-12}$	$2.0\cdot 10^{-10}$
2	0.1333	4	100000	148.055952037	$1.4\cdot 10^{-21}$	$3.2\cdot 10^{-20}$
2	0.08	2	100000	1001.41041046	$4.0\cdot 10^{-13}$	$2.9\cdot 10^{-11}$
2	0.08	4	100000	1001.41041046	$1.3\cdot 10^{-22}$	$4.8\cdot 10^{-21}$
2	0.02	2	100000	$1.19243568602\cdot 10^{5}$	$8.4\cdot 10^{-16}$	$2.4\cdot 10^{-13}$
2	0.02	4	100000	$1.19243568602\cdot 10^{5}$	$6.2\cdot 10^{-25}$	$4.0\cdot 10^{-23}$
5	1.667	4	100000	$3.31447246945\cdot 10^{-5}$	$7.1\cdot 10^{-13}$	$9.8\cdot 10^{-12}$
5	1.667	8	100000	$3.31447246945\cdot 10^{-5}$	$1.2\cdot 10^{-25}$	$1.4\cdot 10^{-30}$
5	0.5	2	100000	638.984275176	$1.7\cdot 10^{-11}$	$4.2\cdot 10^{-10}$
5	0.5	4	100000	638.984275187	$2.0\cdot 10^{-20}$	$5.1\cdot 10^{-19}$
5	0.3333	2	100000	$1.33762181402\cdot 10^{4}$	$1.3\cdot 10^{-12}$	$2.0\cdot 10^{-11}$
5	0.3333	4	100000	$1.33762181402\cdot 10^{4}$	$4.2\cdot 10^{-22}$	$2.5\cdot 10^{-20}$
5	0.2	2	100000	$4.40931335362\cdot 10^{5}$	$4.5\cdot 10^{-14}$	$6.1\cdot 10^{-13}$
5	0.2	4	100000	$4.40931335362\cdot 10^{5}$	$4.3\cdot 10^{-23}$	$7.4\cdot 10^{-22}$
5	0.05	2	100000	$3.5006563815\cdot 10^{9}$	$6.2\cdot 10^{-18}$	$7.6\cdot 10^{-17}$
5	0.05	4	100000	$3.5006563815\cdot 10^{9}$	$2.3\cdot 10^{-26}$	$9.4\cdot 10^{-26}$
10	3.333	4	100000	$1.09837972135\cdot 10^{-9}$	$2.5\cdot 10^{-5}$	$3.3\cdot 10^{-4}$
10	3.333	8	100000	$1.09840686722\cdot 10^{-9}$	$1.1\cdot 10^{-22}$	$1.4\cdot 10^{-21}$
10	1	2	100000	$2.45202153856\cdot 10^{5}$	$3.2\cdot 10^{-12}$	$4.6\cdot 10^{-11}$
10	1	4	100000	$2.45202153857\cdot 10^{5}$	$1.1\cdot 10^{-19}$	$1.5\cdot 10^{-18}$
10	0.6667	2	100000	$4.66296550162\cdot 10^{7}$	$1.9\cdot 10^{-14}$	$2.4\cdot 10^{-13}$
10	0.6667	4	100000	$4.66296550162\cdot 10^{7}$	$2.7\cdot 10^{-22}$	$8.1\cdot 10^{-21}$

Table 35.1: Prime side (Thm. 35.2.1) versus zero side (107 614 Platt zeros, high precision, plus HSW tail), small orders. M: primes powers $\le M$ used. "rel. tail bound is the rigorous bound on the omitted prime powers $m > M$ divided by the value. [NUM]

n	τ	c	M	$(-1)^n S_+^{(n)}(\tau)$ (prime side)	rel. diff. to zeros	rel. tail bound
10	0.4	2	100000	$1.96885070357 \cdot 10^{10}$	$3.5 \cdot 10^{-17}$	$5.7 \cdot 10^{-16}$
10	0.4	4	100000	$1.96885070357 \cdot 10^{10}$	$1.5 \cdot 10^{-24}$	$1.9 \cdot 10^{-23}$
10	0.1	2	100000	$1.62662009533 \cdot 10^{17}$	$1.3 \cdot 10^{-24}$	$6.9 \cdot 10^{-23}$
10	0.1	4	100000	$1.62662009533 \cdot 10^{17}$	$7.3 \cdot 10^{-26}$	$2.3 \cdot 10^{-30}$
20	6.667	4	100000	$2.47502074485 \cdot 10^{-8}$	$2.1 \cdot 10^{10}$	$3.6 \cdot 10^{11}$
20	6.667	8	100000	$1.2064260688 \cdot 10^{-18}$	$5.9 \cdot 10^{-5}$	$1.4 \cdot 10^{-3}$
20	2	2	100000	$5.36737280716 \cdot 10^{10}$	$3.7 \cdot 10^{-14}$	$4.8 \cdot 10^{-13}$
20	2	4	100000	$5.36737280716 \cdot 10^{10}$	$7.4 \cdot 10^{-19}$	$9.8 \cdot 10^{-18}$
20	1.333	2	100000	$8.84326362227 \cdot 10^{14}$	$3.2 \cdot 10^{-19}$	$3.0 \cdot 10^{-17}$
20	1.333	4	100000	$8.84326362227 \cdot 10^{14}$	$3.5 \cdot 10^{-23}$	$6.0 \cdot 10^{-22}$
20	0.8	2	100000	$5.91002256479 \cdot 10^{19}$	$3.6 \cdot 10^{-23}$	$4.5 \cdot 10^{-22}$
20	0.8	4	100000	$5.91002256479 \cdot 10^{19}$	$8.1 \cdot 10^{-26}$	$9.0 \cdot 10^{-27}$
20	0.2	2	100000	$5.15516734394 \cdot 10^{32}$	$7.4 \cdot 10^{-26}$	$5.2 \cdot 10^{-35}$
20	0.2	4	100000	$5.15516734394 \cdot 10^{32}$	$7.4 \cdot 10^{-26}$	$1.0 \cdot 10^{-39}$

Table 35.2: As Table 35.1, orders $n \ge 50$ at $\tau = n/T$, $c = 0.75$; M chosen by the uncertainty relation (35.3.1). [NUM]

n	τ	c	M	$(-1)^n S_+^{(n)}(\tau)$ (prime side)	rel. diff. to zeros	rel. tail bound
50	2.5	0.75	200000	$2.76456954663 \cdot 10^{43}$	$5.9 \cdot 10^{-26}$	$7.7 \cdot 10^{-33}$
50	2.5	1	200000	$2.76456954663 \cdot 10^{43}$	$5.9 \cdot 10^{-26}$	$7.7 \cdot 10^{-33}$
100	5	0.75	3000	$4.49616148662 \cdot 10^{86}$	$9.5 \cdot 10^{-27}$	$1.2 \cdot 10^{-26}$
100	2	0.75	3000	$1.21928967459 \cdot 10^{127}$	$1.3 \cdot 10^{-25}$	$2.6 \cdot 10^{-61}$
100	0.3333	0.75	3000	$8.8895972301 \cdot 10^{205}$	$6.1 \cdot 10^{-27}$	$6.4 \cdot 10^{-139}$
100	0.1	0.75	3000	$7.53787088276 \cdot 10^{258}$	$1.2 \cdot 10^{-26}$	$8.2 \cdot 10^{-192}$
500	25	0.75	200000	$1.24031251952 \cdot 10^{433}$	$1.5 \cdot 10^{-22}$	$5.5 \cdot 10^{-21}$
500	10	0.75	3000	$4.60759167782 \cdot 10^{632}$	$4.3 \cdot 10^{-26}$	$2.0 \cdot 10^{-53}$
500	1.667	0.75	3000	$5.36251559257 \cdot 10^{1022}$	$3.5 \cdot 10^{-26}$	$3.1 \cdot 10^{-346}$
500	0.5	0.75	3000	$6.44673238865 \cdot 10^{1284}$	$5.5 \cdot 10^{-26}$	$3.9 \cdot 10^{-604}$
2000	100	0.75	200000	$2.3666510106 \cdot 10^{1732}$	$3.3 \cdot 10^{-5}$	$4.7 \cdot 10^{-3}$
2000	40	0.75	64748	$2.71108817388 \cdot 10^{2529}$	$2.8 \cdot 10^{-26}$	$1.3 \cdot 10^{-30}$
2000	6.667	0.75	3000	$4.65968933109 \cdot 10^{4086}$	$2.1 \cdot 10^{-26}$	$1.6 \cdot 10^{-388}$
2000	2	0.75	3000	$1.16538196699 \cdot 10^{5133}$	$8.0 \cdot 10^{-28}$	$1.4 \cdot 10^{-1232}$

Reading the tables [NUM]. Each row is consistent: $|\text{prime} - \text{zero}| \le$ (prime tail bound) + (zero tail bound) + printing precision (25 digits). The relative difference is the actual agreement. Some rows have a relative tail bound larger than the difference; there the prime side alone does *not* certify the value, and agreement with the zeros is then a check of the code, not a certification. Where the relative tail bound is < 1 and the value is positive, the prime side alone certifies the sign. That holds in 73 of 74 rows (the exception is $n = 20$, $\tau = 6.667$, $c = 4$). But it is a sign at a single point (n, τ), where Thm. 35.4.1 already guarantees positivity. The row $n = 2000$, $\tau = 100$ ($T = 20$) shows the uncertainty relation (35.3.1): with $M = 2 \cdot 10^5$ the relative tail is still $4.7 \cdot 10^{-3}$. Nothing in these tables is interval arithmetic. The bounds are analytic, evaluated in 50+ digit floating point.

Two warnings [NUM]. (1) $n = 0$, $\tau = 3$, $c = 4$, $M = 2 \cdot 10^4$ gives $-1.71 \cdot 10^{-18}$, with tail bound $7.7 \cdot 10^{-17}$. With $c = 6$ it gives $3.83796134 \cdot 10^{-19}$ (tail $1.2 \cdot 10^{-25}$), and the zeros give $3.837961\ldots \cdot 10^{-19}$. An apparent violation is only meaningful inside the bound. (2) $n = 0$, $\tau = 0.05$: $S_+ = 2.686015048515766$, against Lemma 32's small-τ asymptotic 2.72323. The residual $-0.0372 = -0.74\tau$ matches Lemma 32's observed -0.72τ.

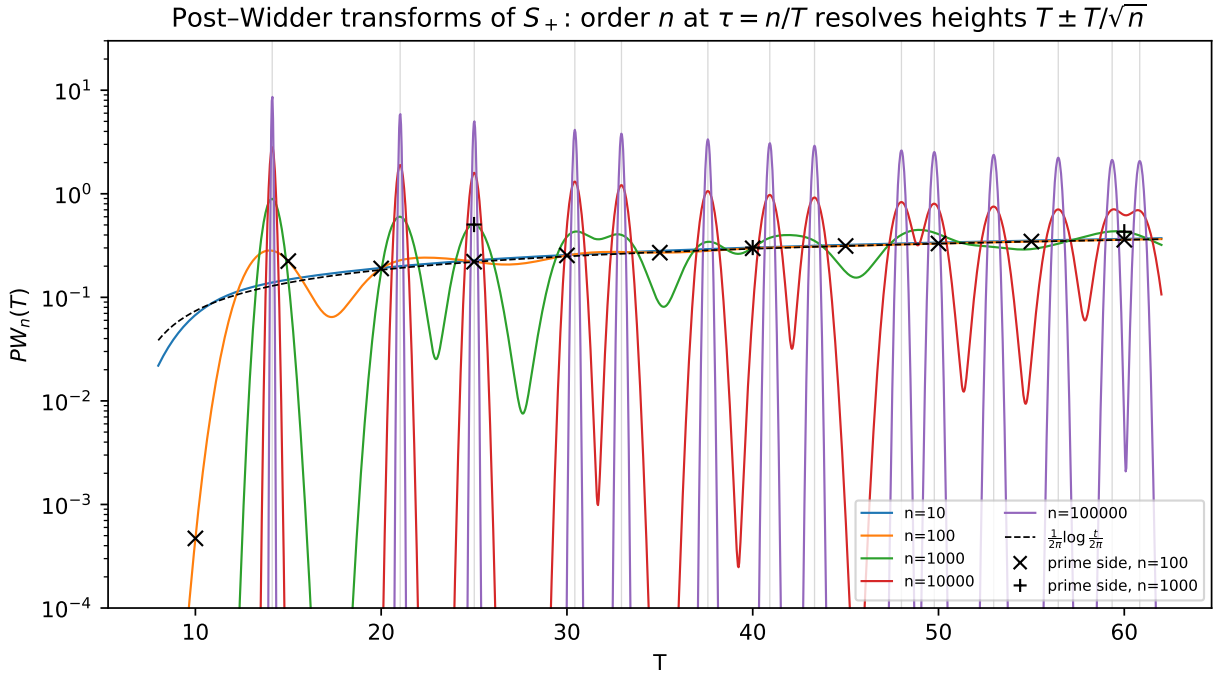

FIGURE 35.1. $\mathrm{PW}_n(T)$ for $n = 10, \ldots, 10^5$ from Platt zeros (grey lines: zeros), the smooth density $\frac{1}{2\pi}\log\frac{t}{2\pi}$, and prime-side values (Thm. 35.2.1, Table 35.3). Order 10^5 resolves the zeros below 62, while $n = 10$ sees only the smooth density. [NUM]

35.3. Post–Widder: which heights order n probes

PROPOSITION 35.3.1 ([THM]; Post's formula [CITED, secondary]). *Define* $\mathrm{PW}_n(T) = \frac{1}{n!}(n/T)^{n+1}(-1)^n S_+^{(n)}(n/T)$ *[DEF]. Unconditionally,*

$$\mathrm{PW}_n(T) = \frac{1}{T}\sum_{t_\rho>0} m_\rho\, k_n(\gamma_\rho/T), \qquad k_n(u) = \frac{n^{n+1}}{n!}u^n e^{-nu},$$

with k_n continued analytically. If RH holds, k_n is evaluated at real u, where it is the Gamma$(n+1, n)$ density: mean $(n+1)/n$, variance $(n+1)/n^2$. So $\mathrm{PW}_n(T)$ is the zero density at T smoothed over a window $T \pm T/\sqrt{n}$. Then $\mathrm{PW}_n(T) \to N'(T)$ in the distributional sense, which is Post's inversion formula [**232**] *(statement via Wikipedia, [CITED, secondary]). By Williamson's theorem* [**294**] *(via Fasshauer's notes, [CITED, secondary]) and its Post–Widder form* [**186**] *(via Wikipedia, [CITED, secondary]), S_+ is N-monotone iff it has a representation $\int(1-\tau x)_+^{N-1}d\mu(x)$ with $\mu \geq 0$, and PW_n is the natural density candidate.*

PROOF. Termwise differentiation of the absolutely convergent series S_+ gives $(-1)^n S_+^{(n)}(\tau) = \sum m\gamma^n e^{-\tau\gamma}$. Substitute $\tau = n/T$. The Gamma moments are standard. □

Resolution and the prime-side cost [ESTIMATE]. To separate zeros at height T (mean spacing $2\pi/\log(T/2\pi)$), the window $T/\sqrt{n}$ must be below the spacing: $n \gtrsim (T\log(T/2\pi)/2\pi)^2$. Fig. 35.1 shows this at $n = 10^4$–10^5 for $T \leq 62$. On the prime side, by Rem. 35.2.2(ii) the m-th atom is suppressed by $\exp(-n\log^2 m/2\tau^2)$ when $\log m \ll \tau$. To reach relative accuracy e^{-L} one needs

$$\log M \approx \tau\sqrt{2L/n} = \sqrt{2L}\,\frac{\sqrt{n}}{T}, \qquad \text{i.e.} \qquad \Delta T \cdot \Delta \log m \approx \sqrt{2L}, \quad \Delta T = T/\sqrt{n}. \tag{35.3.1}$$

n	T	$\mathrm{PW}_n(T)$ prime side	rel. diff.	rel. tail
100	10	$4.7032904737 \cdot 10^{-4}$	$2.4 \cdot 10^{-12}$	$3.6 \cdot 10^{-11}$
100	15	0.223952915108	$1.8 \cdot 10^{-15}$	$6.7 \cdot 10^{-25}$
100	20	0.19002410006	$2.6 \cdot 10^{-16}$	$1.7 \cdot 10^{-34}$
100	25	0.220802355277	$1.4 \cdot 10^{-15}$	$8.2 \cdot 10^{-43}$
100	30	0.254003945903	$1.5 \cdot 10^{-15}$	$5.4 \cdot 10^{-50}$
100	35	0.269994128399	$1.1 \cdot 10^{-15}$	$3.1 \cdot 10^{-56}$
100	40	0.297534748907	$4.5 \cdot 10^{-16}$	$9.0 \cdot 10^{-62}$
100	45	0.313219125209	$9.5 \cdot 10^{-16}$	$1.0 \cdot 10^{-66}$
100	50	0.331232393919	$6.8 \cdot 10^{-16}$	$3.6 \cdot 10^{-71}$
100	55	0.345971087368	$1.0 \cdot 10^{-15}$	$3.1 \cdot 10^{-75}$
100	60	0.359945849385	$6.7 \cdot 10^{-16}$	$5.7 \cdot 10^{-79}$
1000	25	0.504537231711	$4.9 \cdot 10^{-16}$	$2.4 \cdot 10^{-17}$
1000	40	0.302381952732	$8.8 \cdot 10^{-16}$	$1.3 \cdot 10^{-44}$
1000	60	0.430280995482	$3.8 \cdot 10^{-16}$	$6.6 \cdot 10^{-92}$

TABLE 35.3. Post–Widder transform from the prime side ($c = 0.75$, $M = 3 \cdot 10^4$ for $n = 100$, $2 \cdot 10^5$ for $n = 1000$) against the zeros. Prime totals are stored to 15 digits, so the relative difference has a floor near 10^{-15}. [NUM]

Resolving single zeros at height T ($\Delta T \approx 2\pi/\log T$) costs $M \approx \exp(\sqrt{2L}\log T/2\pi) = T^{\sqrt{2L}/2\pi}$. For windows below t_1 (small T, large n), M is astronomically large, so those windows are not computable this way. They are, however, exactly where the archimedean term is negative (§35.6). This is the familiar explicit-formula duality, not a new obstruction. The large-table choice $M = \min(2 \cdot 10^5, \max(3000, e^{1.15\sqrt{116n}/T}))$ implements (35.3.1) with $L = 58$.

35.4. Blindness below order $\frac{\pi}{2} \cdot 3 \cdot 10^{12}$

THEOREM 35.4.1 ([THM]). *Let $H \geq 3 \cdot 10^{12}$ and suppose all zeros with $0 < t_\rho \leq H$ have $\delta_\rho = 0$. This holds for $H = 3 \cdot 10^{12}$ by [228] [CITED]. Then $(-1)^n S_+^{(n)}(\tau) > 0$ for all $\tau > 0$ and all integers $0 \leq n \leq \frac{\pi}{2}H$. For $H = 3 \cdot 10^{12}$, $n_{\max} = 4\,712\,388\,980\,384$.*

PROOF. On-line terms are $mt^n e^{-\tau t} > 0$. An off-line pair has $t > H$ and $0 < \delta < \frac{1}{2}$, so $0 < \theta < \delta/t < 1/(2H)$. Its contribution is $2m|\gamma|^n e^{-\tau t}\cos(n\theta - \tau\delta)$.

Case 1: $\tau \leq \pi$. Here $n\theta < \frac{\pi}{2}H \cdot \frac{1}{2H} < \frac{\pi}{2}$ and $0 < \tau\delta < \frac{\pi}{2}$, so the phase lies in $(-\frac{\pi}{2}, \frac{\pi}{2})$ and every term is positive.

Case 2: $\tau > \pi$. Write $f(t) = t^n e^{-\tau t}$, so $(\log f)' = n/t - \tau$. For $t \geq H$ this is $\leq \frac{\pi}{2} - \pi = -\frac{\pi}{2}$, and for $t \geq H - 20$ it is $\leq \frac{\pi}{2}\frac{H}{H-20} - \pi < 0$. Hence f is decreasing on $[H-20, \infty)$, and $f(t) \leq f(H)e^{-\frac{\pi}{2}(t-H)}$ for $t \geq H$. So (i) the on-line zeros in $(H-20, H]$ contribute at least $f(H)\,[N(H) - N(H-20)]$; (ii) the off-line terms, using $|\gamma|^n \leq t^n(1 + \frac{1}{4t^2})^{n/2} \leq t^n e^{n/8H^2}$ and $f(t) \leq f(H)e^{-\frac{\pi}{2}(t-H)}$, are at least $-f(H)e^{n/8H^2}\int_0^\infty \frac{\pi}{2}e^{-\pi u/2}(N(H+u) - N(H))\,du$ in total (Stieltjes integration by parts). With $|N(T) - \frac{T}{2\pi}\log\frac{T}{2\pi e}| \leq R(T) = 0.1038\log T + 0.2573\log\log T + 9.3675$ [127, Cor. 1.2] [CITED], (i) is $\geq f(H) \cdot 59.17$ and (ii) is $\geq -f(H) \cdot 29.15$ at $H = 3 \cdot 10^{12}$ (`num/lemma35_blind.json`; $e^{n/8H^2} < 1 + 10^{-12}$). The margin 30.02 is positive. Its derivative in $\log H$ is about $\frac{20}{2\pi} - \frac{1}{\pi^2} - 4(0.1038) - O(1/\log H) > 0$, so the margin stays positive for all $H \geq 3 \cdot 10^{12}$. □

REMARK 35.4.2. (a) A window of length 10 instead of 20 fails: $16.37 < 29.15$. Trudgian's 2014 constants were not used, because HSW report errors in them. (b) The proof with window $H/4$ instead of 20 works numerically on the grid $H \in \{300, 500, \ldots, 10^4\}$ (margin 2.5 at 300) and fails at $H \leq 200$ [NUM], not claimed as a theorem. (c) *Contrapositive.* If $(-1)^n S_+^{(n)}(\tau) < 0$ for some τ, then

the lowest off-line zero has height $H^* < 2n/\pi$. So order n checks RH up to height $\approx 2n/\pi$, while a single zero computation checks the line at height T with work polynomial in log-size. Prime-side Post–Widder is not a competitive verification method [ESTIMATE].

COROLLARY 35.4.3 ([THM] + [CITED, secondary]). *S_+ is N-monotone for $N \le 4.7\cdot 10^{12}$, hence has a Williamson representation of that order with a positive measure* [**294**]. *Explicitly, Mattner's form* [**186**] *(via Wikipedia, [CITED, secondary]) gives $S_+(\tau) = \int_0^\infty (1-\tau x/n)_+^{n-1}\mathrm{PW}_n(x)\,dx$, with PW_n as in Prop. 35.3.1. By Thm. 35.4.1, $\mathrm{PW}_n(x)\,dx \ge 0$ is such a measure for every $n \le 4\,712\,388\,980\,384$, and it is computable from the prime side by Thm. 35.2.1.*

PROPOSITION 35.4.4 (necessary condition, [THM]). *If $(-1)^n S_+^{(n)}(\tau) < 0$, some off-line zero has $n\theta_\rho \ge \frac{\pi}{2}$ or $\tau\delta_\rho \ge \frac{\pi}{2}$. In particular, for $\tau \le \pi$ (where $\tau\delta < \frac{\pi}{2}$) a violation needs $n \ge \pi/(2\theta) > \pi t_0/(2\delta) > \pi H \approx 9.4\cdot 10^{12}$ (Lemma 33's half-turn order at $\tau = 0$).*

PROOF. Otherwise every phase is in $(-\frac{\pi}{2}, \frac{\pi}{2})$, as in Case 1. □

PROPOSITION 35.4.5 (phase cancellation at the focus, [THM]). *The pair weight $|\gamma|^n e^{-\tau t}$ peaks at $\tau = n/t_0$, and there the phase is $n\theta - n\delta/t_0 = -n(\tan\theta - \theta) = -\frac{n}{3}\theta^3 + O(n\theta^5)$. So, at its own focus, a pair turns negative only for $n > \frac{3\pi}{2}(t_0/\delta)^3$, about 10^{39} for $t_0 = 3\cdot 10^{12}$, $\delta = \frac{1}{2}$. Earlier violations must come from τ shifted off the focus by $s \sim \pi/\delta$, where the pair's phase is $\approx -s\delta$ but its weight relative to on-line zeros at $t_0 + \Delta$ changes by $e^{s\Delta}$. That competition is §35.5.*

PROOF. $\tan\theta = \delta/t_0$, so $n\theta - \tau\delta = n(\theta - \tan\theta)$. □

35.5. Detection threshold: which (δ, γ_0) order n reveals

Thm. 35.4.1 and Prop. 35.4.5 are rigorous bounds. The actual detection order is a competition. An off-line pair at $t_0 \pm i\delta$, evaluated at $\tau = n/t_0 + s$, has phase $\approx -s\delta$ and weight $e^{-st_0}\cdot$(Gaussian peak). Its neighbours at $t_0+\Delta$ have weights $\exp(-n\Delta^2/2t_0^2 - s\Delta)$. Taking $s = \pm\pi/\delta$ to make the pair negative, the pair beats the neighbour on the far side when $n\Delta^2/2t_0^2 \gtrsim \pi\Delta/\delta$. That gives

$$n_{\mathrm{det}} \approx \kappa\,\frac{t_0^2}{\delta g}, \qquad g = \text{larger gap to a neighbouring zero}, \quad \kappa = O(1) \qquad \text{[ESTIMATE]}. \tag{35.5.1}$$

Model test [NUM] (`num/lemma35_planted.py`). We took the Platt zeros (all on the line), planted one pair $t_0 \pm i\delta$, and scanned $s \in [-4\pi/\delta, 4\pi/\delta]$ (801 points) over a 4% geometric grid of n. This is a *model*, not ζ: no zero of ζ was moved. Over $t_0 \in \{30, 50, 100, 300\}$ and $\delta \in \{0.05, 0.1, 0.2, 0.4\}$, the first violation has $\kappa = n_{\mathrm{det}}\delta g/t_0^2 \in [2.90, 4.13]$. The shift is $|s|\delta/\pi \in [0.51, 0.63]$, and the ratio to the half-turn order n^* is 11–390 (Fig. 35.2). Examples: $(t_0, \delta) = (30, 0.4)$, $n_{\mathrm{det}} = 1529$, $n^* = 118$; $(300, 0.05)$, $n_{\mathrm{det}} = 3.70\cdot 10^6$, $n^* = 9425$. The $1/\delta$ scaling holds within the 4% grid.

Consistency with known zero-free information [ESTIMATE]. Any off-line zero has $t_0 > 3\cdot 10^{12}$ [**228**] and $\delta < \frac{1}{2} - 1/(5.558691\log t_0)$ [**200**], which is 0.4937 at $t_0 = 3\cdot 10^{12}$. The mean gap there is $2\pi/\log(t_0/2\pi) = 0.234$. With (35.5.1) and $\kappa \in [2.9, 4.2]$, $n_{\mathrm{det}} \approx 2.3$–$3.3\cdot 10^{26}$, far above the rigorous floor $4.7\cdot 10^{12}$. By (35.3.1), the prime-side cost at that order and height, at 1% accuracy ($L \approx 5$), is $\log M \approx \sqrt{10}\cdot 1.6\cdot 10^{13}/(3\cdot 10^{12}) \approx 17$, about $10^{7.4}$ prime powers per τ point. The incomplete Γ's have order 10^{26}, so uniform asymptotics would be needed. Even then, each evaluation tests only the zeros within $T/\sqrt{n} \approx 0.2$ of one height t_0. A direct zero computation does that more cheaply. Covering heights beyond $3\cdot 10^{12}$ this way is not a sensible verification method. The extrapolation from $t_0 \le 300$ to $3\cdot 10^{12}$ is itself unproven.

35.6. Bernstein: what inequality would give all n, and why it is domination

35.6.1. The free positive measures, and their blindness. Let dN count all zeros with $t_\rho > 0$ by height (an off-line pair counts twice at t_0), and dN_0 only the on-line ones [DEF]. Both

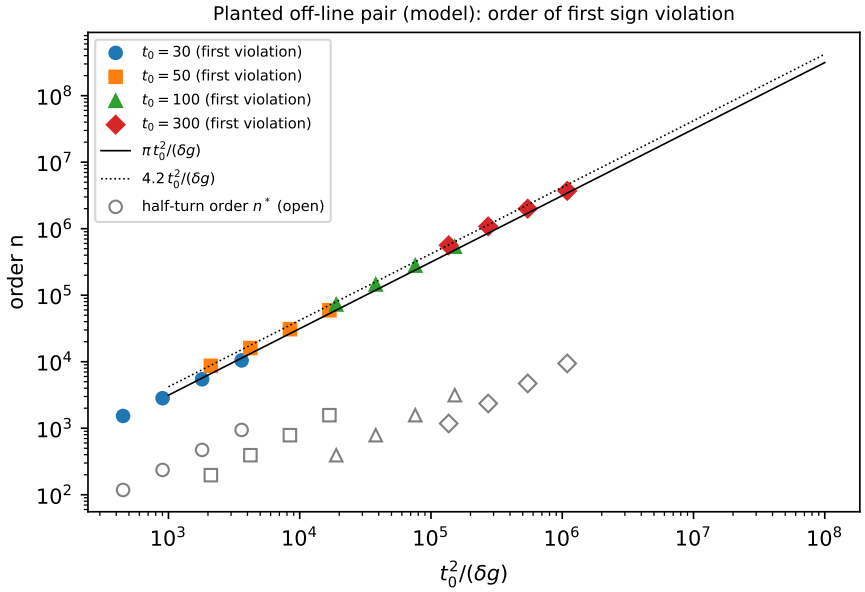

FIGURE 35.2. Planted-pair model: order of the first sign violation (filled) against $t_0^2/(\delta g)$, the lines π and 4.2 times it, and Lemma 33's half-turn order n^* (open). [NUM]

are positive. Both are zero-free computable: $N(T)$ by the argument principle [**80**, Ch. 15], and $dN_0 = \frac{1}{\pi}[\theta'(t) + \lim_{a\downarrow 0} \operatorname{Re} \frac{\zeta'}{\zeta}(\frac{1}{2}+a+it)]dt$ (Lemma 33, Thm. (boundary), with $\operatorname{Re} A(\frac{1}{2}+it) = \theta'(t)$). Unconditionally,

$$S_+ = \mathcal{L}[dN_0] + \sum_{\text{off pairs}} 2m\, e^{-\tau t}\cos\tau\delta, \qquad S_+ = \mathcal{L}[dN] - 2\sum_{\text{off pairs}} m\, e^{-\tau t}(1-\cos\tau\delta). \tag{35.6.1}$$

PROPOSITION 35.6.1 ([THM] + [RESTATEMENT]). *(a) RH $\iff$ $S_+(\tau_0) \geq \mathcal{L}[dN](\tau_0)$ for one (equivalently every) $\tau_0 \in (0, 4\pi)$. (b) If S_+ is CM, its Bernstein measure $\mu \geq 0$ satisfies $\mu = dN_0 = dN$. Hence "$(-1)^n S_+^{(n)} \geq 0$ for all n" is equivalent to the* identity $S_+ \equiv \mathcal{L}[dN]$, *i.e. to the vanishing of the off-line sum. The inequality form carries no extra content: positivity of a representing measure is free once the measure is identified, and the identification is the whole problem.*

PROOF. (a) For $0 < \tau\delta < 2\pi$, which holds when $\tau < 4\pi$, each $1-\cos\tau\delta > 0$, so the off-line sum in (35.6.1) is zero iff there are no off-line zeros. (b) This is the argument of Lemma 32, Thm. 7.2, (c)⇒(a). A positive Bernstein measure gives $|S_+(a-ib)| \leq S_+(a)$ on vertical lines. An off-line pair contributes terms of modulus $e^{-at}e^{\pm b\delta}$ with Bohr frequency t, which cannot be cancelled by the other frequencies, so it is excluded. Then $S_+ = \mathcal{L}[dN_0] = \mathcal{L}[dN]$, and $\mu = dN$ by uniqueness of Laplace–Stieltjes transforms. □

35.6.2. The canonical decomposition and its signs. Define $D_n := (-1)^n S_+^{(n)} - \mathcal{A}_n^\theta$ [DEF], where

$$\mathcal{A}_n^\theta(\tau) = \frac{1}{\pi}\int_0^\infty t^n e^{-\tau t}\theta'(t)\,dt, \qquad \frac{\theta'(t)}{\pi} = \frac{1}{2\pi}\Big[-\gamma - \log\pi + \sum_{k\geq 0}\Big(\frac{1}{k+1} - \frac{2(2k+\frac{1}{2})}{(2k+\frac{1}{2})^2+t^2}\Big)\Big].$$

- The Lorentzians are the minus sign of Lemma 32's archimedean ghost tower $k_\infty(u) = \sum_k e^{-(2k+1/2)|u|}$ (its Fourier transform). $\theta' < 0$ on $(0, t_\theta)$ with $t_\theta = 6.28983598884$, and $\theta'(0) = \frac{1}{2}\psi(\frac{1}{4}) - \frac{1}{2}\log\pi = -2.68609$ (Lemma 17; [NUM] recomputed).

- [NUM] (`num/lemma35_arch.json`): normalized by $n!/\tau^{n+1}$ at $\tau = n/T$, $\mathcal{A}_n^\theta < 0$ exactly for $T \lesssim 6.3$, i.e. $T \in \{1,2,4\}$ for $n = 1, 10$ and $\{1,2,4,6\}$ for $n = 100, 1000$. At $(n,T) = (1000,1)$, $\mathcal{A}^\theta = -0.3223$ while the total is $1.1 \cdot 10^{-4553}$. At $(100,6)$, -0.00673 against $1.4 \cdot 10^{-22}$. So D_n must cancel the archimedean part to thousands of digits.
- D_n is zero-free by Thm. 35.2.1. It contains $\mathcal{S}_n$, the prime atoms, and $\mathcal{A}_n - \mathcal{A}_n^\theta$. The prime atoms have signs $-\cos((n+1)\arctan(\log m/\tau))$ (Rem. 35.2.2(ii)), which oscillate in m and n. D_n itself changes sign (normalized): -0.0243 at $(n,T) = (10,8)$, -0.0141 at $(10,10)$, $-1.5 \cdot 10^{-5}$ at $(10,30)$, -0.0391 at $(100,8)$, but $+0.155$ at $(100,14)$.

PROPOSITION 35.6.2 ([RESTATEMENT]). *RH* $\iff$ $D_n(\tau) \geq -\mathcal{A}_n^\theta(\tau)$ *for all* $n \geq 0$, $\tau > 0$.

This is the "prime-side inequality" asked for in item (c). In it, $\mathcal{A}_n^\theta$ and D_n are zero-free, and neither side has a sign. It is domination: prime atoms (both signs) plus the segment must beat a negative ghost tower at low heights, and at high heights only fluctuations are allowed. There is no termwise positivity:

(i) the ghost-tower Lorentzians enter with $-$;
(ii) prime atoms enter with $-\Lambda(m)m^{-1/2}$ times an oscillating phase;
(iii) the segment term $\mathcal{S}_n$ has the oscillating factor $\sin(\tau x + n\pi/2)$.

Moreover, by Prop. 35.6.1(b) and Thm. 35.4.1, any inequality true at all orders $\leq 4.7 \cdot 10^{12}$ is already implied by known zero verification. The content sits at orders $\gtrsim 10^{26}$ (§35.5), where the inequality is a statement about individual zeros at heights $> 3 \cdot 10^{12}$.

Circularity flag. The decomposition is zero-free, but every route we see to proving $D_n \geq -\mathcal{A}_n^\theta$ at the relevant orders passes through the zeros (D_n is a smoothed $S(T)$-type fluctuation). There is no explicit formula for non-holomorphic weights such as $e^{-\tau \operatorname{Im}\rho}$, which would be needed to read $\mathcal{L}[dN] - S_+$ directly from primes. **G5 is not supplied.**

35.7. Open questions

QUESTION 35.7.1 ([OPEN]). *Quantitative finite-order converse.* Thm. 35.4.1 shows N-monotonicity for $N \leq \frac{\pi}{2}H$. Is there a converse: N-monotonicity of $S_+ \Rightarrow$ no off-line zero below $c\sqrt{N\delta g}$ (cf. (35.5.1))? What do the Williamson measures μ_N (density PW_N) look like near an off-line pair?

QUESTION 35.7.2 ([OPEN]). *Complex* τ. Prop. 35.4.5 shows that the real-τ test suffers a cubic phase cancellation at the focus. Widder/Pick-type tests in complex τ (Lemma 33's half-turn, $\operatorname{Re}\tau > 0$) rotate the phase directly. Is there a complex-τ positivity test with $n_{\det}$ linear in t_0/δ?

QUESTION 35.7.3 ([OPEN]). *A Littlewood-lemma identity for the defect.* $\mathcal{L}[dN] - S_+ = 2\sum m e^{-\tau t}(1 - \cos\tau\delta) \geq 0$ is a positive quantity measuring off-line zeros (Prop. 35.6.1(a)). Littlewood's lemma turns $\sum(\beta - \sigma_0)$ into a contour integral of $\log|\zeta|$. Is there an analogue expressing this defect through $\log|\zeta|$ on vertical lines, i.e. an explicit formula for the non-holomorphic weight $e^{-\tau t}(1 - \cos\tau\delta)$?

QUESTION 35.7.4 ([OPEN]). *Link to the Krein-string option.* The Post–Widder densities PW_n are Gamma-smoothed zero densities, and the Krein string of $E_{1/2}$ (Lemma 33) encodes the same spectral measure under RH. Does the string density at length ℓ correspond to a definite (n,T) window, and does the uncertainty relation (35.3.1) have a string-side meaning?

35.8. Ledger

Theorems (proved here): Thm. 35.2.1 (prime-side formula + tail bounds); Prop. 35.3.1 (kernel identity); Thm. 35.4.1 (blindness, $n \leq 4\,712\,388\,980\,384$); Props. 35.4.4, 35.4.5; Prop. 35.6.1(a) as a two-sided equivalence. **Restatements:** CM of S_+; $S_+ \geq \mathcal{L}[dN]$; $D_n \geq -\mathcal{A}_n^\theta$. **Numerics:** Tables 35.1–35.3; Fig. 35.1; planted-pair model (Fig. 35.2); archimedean signs. **Estimates**

(heuristic): (35.3.1), (35.5.1), $n_{\mathrm{det}} \approx 2\text{–}3 \cdot 10^{26}$ at the verified height. **Cited:** [**77**, **120**, **135**] (Cramér continuation, via Illies), [**228**, **200**, **127**], [**232**, **294**, **186**] (secondary). **Not supplied:** G5.

CHAPTER 36

Lemma 36: The positivity ledger

Every positivity at the obstruction, what each one supplies, and the single statement that is missing

Standalone note `lemma36.tex`, October 21, 2026; v2 October 22, 2026.

Summary. **Tags.** [THM] proved here or in the cited TAA lemma; [CITED] checked at the source (`lit/lemma36/verified_refs.md` and the files it points to); [CITED, secondary] checked only through a named secondary source; [RESTATEMENT] equivalent to RH (or to a named open rung), no progress; [ANALOGY] analogy only; [OPEN] open; [PARTIAL] unconditional but partial; [NUM] numerical (`num/lemma36_checks.{py,json,out}`; v2: `num/lemma36_sommerfeld_*.py`); [DEF] definition; [BIB] bibliographic record only, content stated as standard background.

This is a synthesis. It collects every positivity notion that TAA has met at the obstruction (Lemmas 1, 10–33) into one ledger (§36.2) and one implication diagram (§36.3). It then carries out J. W.'s Kramers–Kronig thread precisely (§36.4) and adds two sections that J. W. asked for: Hessian, Morse and Pontryagin index (§36.5), and the pole as an "edge" mode (§36.6). The ledger ends with one gap sentence per element (§36.8), the collapsed missing statement (§36.9), and questions (§36.10).

Main points.

(1) *Where free positivity ends.* Every positivity that holds unconditionally lives in one of four places: (i) on $\sigma > 1$ or in a zero-free region (the Euler product), (ii) on the boundary line, where off-line zeros are invisible, (iii) on finite pieces (windows $A \le 1.6$, heights $|t| \le 3 \cdot 10^{12}$), or (iv) on structures that are line-blind (arrow-of-time positivity, KMS of $\xi(1+it)$, exponential convexity). G5 begins where a positivity has to see the whole interior $\frac{1}{2} < \sigma \le 1$ at all heights. A single parameter $h = \sigma - \frac{1}{2}$ runs through the passivity, de Branges and damping faces: free for $h \ge \frac{1}{2}$, open for uniform $h < \frac{1}{2}$ (rung P3), RH at $h = 0$ (Thm. 36.3.1).

(2) *Kramers–Kronig, done precisely* (§36.4). Dispersion relations (causality) are free. They become positivity only with a positive spectral measure *and* stability (no poles in the right half-plane), and for ζ stability *is* RH. New here: an exact *absorption–phase sum rule* (Thm. 36.4.3, [THM]): the integrated absorption on $\sigma = \sigma_0$ falls short of the line strength $\pi N(T)$ by exactly $2\pi N(\sigma_0, T)$, a deficit of *definite sign*. This answers Lemma 33's question on a gain–loss sum rule. It is the differentiated form of Littlewood's lemma, the engine of zero-density theory, and it does not prove RH.

(3) *Index* (§36.5). The zero-free form $\mathcal{L} = \mathcal{P} - \mathcal{Q}$ has positive index $1 + N_+$. Each off-line quadruple raises it by exactly 2, i.e. by 1 per mirror pair (Lemma 1(vi), Lemma 15, [THM]). RH $\iff$ exactly one exceptional direction: a Hodge-index or Lorentzian shape ([RESTATEMENT]). The Hamiltonian–Krein count $k_r + 2k_c + 2k_i^- = n(L) - n(D)$ of Kapitula–Kevrekidis–Sandstede ([CITED]) has the same bookkeeping: on-line zeros are stable modes of positive signature, off-line quadruples are complex quadruplets, and $k_r = k_i^- = 0$ for ζ ([THM]). The Hamiltonian reading is an [ANALOGY].

(4) *Edge* (§36.6). The pole is "edge" in three precise senses ([THM]/[CITED]): cohomological degree (H^0, H^2 versus H^1), the boundary terms $f(0), \hat{f}(0)$ of Poisson summation (Riemann's and Tate's proofs), and the edge $|\operatorname{Im} z| = \frac{1}{2}$ of the spectral strip, with non-normalizable eigenfunctions $e^{\pm u/2}$. "Edge state" in the bulk–boundary (topological) sense is an [ANALOGY]: no index theorem is supplied.

(5) *The missing statement, collapsed* (§36.9). All the RH-equivalent rows are faces of one sentence, "$\nu = \sum_{t_\rho>0} m_\rho \delta_{\gamma_\rho}$ is a positive measure on $(0,\infty)$". Its sharpest *zero-free* form is the domination inequality $\mathcal{L}(F,F) \le \mathcal{P}(F,F)$ for all F. In words, the single exceptional (pole) direction dominates the archimedean ghost tower plus the prime atoms. Equivalent sharp forms: a reflection "elsewhere", prime-side complete monotonicity, and a passive realization from the Euler product.

(6) *Integrality* (v2, §36.7). The deficit is a winding number, $D = \Delta_{\partial Q} \arg \xi = 2\pi N(\sigma_0, T)$ ([THM]), so no deficit lies in $(0, 2\pi)$. Through Lemma 34 it is an exact count: $\frac{\nu}{\pi} D(\frac{1}{2} + \omega, \infty)$ eigenvalues of the Hankel sections eventually escape $[-1, 1]$, half through $+1$ and half through -1 (Thm. 36.7.9, [THM] modulo Lemma 34's sketched contour shift). Sommerfeld/Maslov is an [ANALOGY]: $\frac{7}{8}$ is not an integer Maslov index (it corresponds to $\mu = -\frac{3}{2}$ or $\frac{5}{2}$), and its $-\frac{1}{8}$ is a Stirling phase. "RH = vanishing of a topological index" is [RESTATEMENT].

Nothing here proves RH. RH is not proven. Every row tagged [RESTATEMENT] is a reformulation. The new theorems (Thms. 36.3.1, 36.4.3, 36.5.4; Props. 36.4.1, 36.6.1; v2: Thms. 36.7.1, 36.7.9, 36.7.15) are elementary. We claim no priority for them.

Revision note (v2, October 22, 2026; approved by J. W. [t131u]). *Added:* §36.7, "Integrality: the deficit as a winding number (Sommerfeld/Maslov)". It contains: the integrality theorem with boundary conventions (Thm. 36.7.1, Cor. 36.7.2); the exact split of the "+1" (Prop. 36.7.4); the Bohr–Sommerfeld/Maslov analogy with the arithmetic of $\frac{7}{8}$ (Props. 36.7.6, 36.7.7; Table 36.2); a new *exact escape-count* theorem for Lemma 34's Hankel sections (Thm. 36.7.9, Cor. 36.7.10); truncated-line lower bounds (Prop. 36.7.13); homotopy invariance and a scored deformation program (Thm. 36.7.15, Table 36.3); new questions (Q. 36.10.6–36.10.9). *Corrected or refined:* items (e)–(h) of §36.11. The v1 text is otherwise unchanged (backup `archive/lemma36.before-sommerfeld-20261022-070804.{tex,pdf}`). Its mentions of Lemmas 34 and 35 as "in preparation" are superseded: both are complete.

36.1. Conventions

Notation follows NOTATION.md. Zeros: $\rho = \frac{1}{2} + i\gamma_\rho$, $\gamma_\rho = t_\rho - i\delta_\rho$, $\beta_\rho = \frac{1}{2} + \delta_\rho$, with multiplicity m_ρ. $\Theta = \sup \beta_\rho$, and $N_+ = \#\{\rho : \beta_\rho > \frac{1}{2}\}$ counted over *all* zeros, both signs of t (Lemma 1). $\mathcal{Q}(F,G) = \sum_\rho m_\rho \hat{F}(\gamma_\rho)\overline{\hat{G}(\bar{\gamma}_\rho)}$ is the Weil form. $\mathcal{P}(F,G) = a(F)\overline{b(G)} + b(F)\overline{a(G)}$, with $a(F) = \int F e^{-u/2}$ and $b(F) = \int F e^{u/2}$, is the pole form, and $\mathcal{L} = \mathcal{P} - \mathcal{Q}$ is zero-free (Lemma 15, eq. (1)). $\mathsf{Z}(p) = \frac{\xi'}{\xi}(\frac{1}{2} + p)$ is the impedance (Lemma 33). $S_+(\tau) = \sum_{t_\rho>0} m_\rho e^{-\tau\gamma_\rho}$ (Lemma 32). P0–P4 is the passivity ladder of Lemma 25, Prop. 1.8. Lemmas 34 (Krein string of $E_{1/2}$) and 35 (prime-side Post–Widder) were in preparation when v1 was written; both are now complete, and §36.7 uses Lemma 34 and Lemma 37.

36.2. The master table

Columns: the precise statement; its status for ζ; the *cost* (what has to be supplied to get it); where TAA treats it; and the label. "Free" means unconditional.

notion	precise statement	status for ζ	cost	TAA	label
1. Spectrum condition / causality / analyticity	(a) $\mathsf{Z}(p) = \int_0^\infty e^{-pu}\,\mathcal{W}_+(u)\,du$ on $\operatorname{Re} p > \frac{1}{2}$ (support causality). (b) KK: Re and Im of boundary values are a Hilbert pair (Titchmarsh). (c) Lemma 32's V is holomorphic on $\operatorname{Im} z > 0$ (spectrum condition). (d) Stability: Z holomorphic on $\operatorname{Re} p > 0$.	(a)–(c) unconditional. (d) $\iff$ RH, since the poles of Z in $\operatorname{Re} p > 0$ are exactly the zeros with $\delta > 0$.	(a)–(c): Euler product + Γ. (d): G5.	L33 Thm 3.2, Rem 3.3; L32 §7	[THM]; (d) [RESTATEMENT]
2. Passivity / positive-real	$\operatorname{Re}\mathsf{Z}(p) > 0$ on $\operatorname{Re} p > \theta - \frac{1}{2}$ $\iff$ all $\beta \le \theta$ (Lagarias, Hinkkanen). Foster: under RH $\mathsf{Z} = \sum 2mp/(p^2+t^2)$, reactive on the line. Ladder P0 ($0 \le \beta \le 1$), P1 ($\beta < 1$), P2 (dlVP region), P3 ($\Theta < 1$), P4 ($\delta \equiv 0$).	P0–P2 unconditional: PR on $\sigma > 1$; PR outside the shadow half-discs (so for $\lvert t\rvert \le 3 \cdot 10^{12} - \frac{1}{2}$ and right of the MTY curve). P3 [OPEN]. P4 = PR on $\sigma > \frac{1}{2}$ $\iff$ RH. Boundary absorption $\pi \sum_{\delta=0} m\delta_t \ge 0$ is free.	PR on all of $\sigma > \frac{1}{2}$: G5.	L25 Prop 1.8; L33 §§3–5, Thms 4.1, 4.2, 4.4, 5.1	[CITED], [THM]; P4 [RESTATEMENT]
3. Hilbert-space positivity	$\mathcal{Q} \ge 0$ $\iff$ unitary Hilbert realization of the scaling flow $\iff$ bounded matrix coefficients. Negative index $\kappa(\mathcal{Q}) = N_+$.	$\iff$ RH. Unconditionally $\mathcal{Q}$ is a Krein (Pontryagin if $N_+ < \infty$) form with J_{FE} as fundamental symmetry.	G5.	L1 Thm 2.1; L30	[THM], [RESTATEMENT]
4. OS reflection positivity	Half-line RP is vacuous ($\mathcal{Q}$ translation invariant). Parity identity $\mathcal{Q}(F \pm R_a F) = 2(\mathcal{Q} \pm \mathrm{OS}_a)$. RH $\iff$ $\lvert\mathrm{OS}_a(F)\rvert \le \mathcal{Q}(F)$. On $u > 0$: $\mathcal{W} = e^{u/2} - \sum_{k\ge1} e^{-(2k+\frac{1}{2})u} - \sum \Lambda(n) n^{-1/2} \delta_{\log n}$.	$\lvert\mathrm{OS}_a\rvert \le \mathcal{Q}$ $\iff$ RH. $\mathrm{OS}_a \ge 0$ is *false*. The tachyon $e^{u/2}$ is RP^+, the Pauli–Villars tower RP^-, the primes non-Laplace atoms. Free: spectrum condition and contraction of the OS semigroup.	Domination of tower + atoms by the tachyon: G5.	L32 Thms 5.4, 5.5, 7.2	[THM], [RESTATEMENT]
5. KMS / modular	Tomita–Takesaki: $\langle \Omega, xJxJ\Omega\rangle = \lVert\Delta^{1/4} x^* \Omega\rVert^2 \ge 0$. Bisognano–Wichmann: modular group = boosts (spacetime side). Thm 6.5: $\xi(1+it)$ is a KMS_1 two-point function, $d\mu = 2\Phi(\lambda/2) e^{\lambda/2} d\lambda$, $\Phi > 0$.	Thm 6.5 is *unconditional*, hence not G5. BC KMS positivity is prime-side ($\sigma > 1$). A flow-invariant positive half exists $\iff$ RH. Modular realization $\Rightarrow$ RH. $\langle \xi, J\xi\rangle \ge 0$ as stated earlier is false.	State positivity on the zero side: G5 (= OS4 + OS5).	L31 Thm 6.5, §§3, 6, 7	[THM], [CITED], [RESTATEMENT]
6. Complete monotonicity / Bernstein–Widder	S_+ CM on $(0,\infty)$ $\iff$ Laplace transform of a positive measure (Bernstein); Widder for two-sided. Half-turn: $d/d\tau = -\gamma$ is an exact half-turn $\iff$ γ real.	RH $\iff$ S_+ CM $\iff$ OS/Widder RP (Thm 7.2). Free: $\xi(\frac{1}{2}+\tau)$ is exponentially convex. Prime-side CM: Lemma 35 (in preparation).	Prime-side CM: G5.	L32 Thm 7.2; L33 Thm 7.1, Prop 7.2	[CITED], [THM], [RESTATEMENT]
7. Hermite–Biehler / de Branges	$E = \Xi + i\Xi'$ HB $\iff$ RH. $E_h(z) = \xi(\frac{1}{2} + h - iz)$ is HB for $h \ge \Theta - \frac{1}{2}$, and the threshold is exact. Lagarias: HB for $h \ge \frac{1}{2}$ unconditionally.	$h \ge \frac{1}{2}$ free. Some $h < \frac{1}{2}$ $\iff$ P3 [OPEN]. $h = 0$ $\iff$ RH. De Branges' *stronger* positivity conditions fail for ζ (Conrey–Li).	$h \downarrow 0$: G5.	L33 Thms 6.1, 6.2; [La05, CL00]	[THM], [CITED], [RESTATEMENT]
8. Krein string	A string $m(x) \ge 0$ (equivalently a canonical system with $H \ge 0$) whose spectral data are the zeros.	Positive string exists $\iff$ RH (Kac–Krein, secondary). String of $E_{1/2}$: Lemma 34 (in preparation).	Construct it from the Euler product: G5.	L33 Prop 6.3	[CITED, secondary], [RESTATEMENT]

notion	precise statement	status for ζ	cost	TAA	label
9. Hodge–Riemann / polarization	Two halves: integrality (Bohr–Sommerfeld, Kostant–Souriau) and positivity (Kähler, Riemann form > 0). For curves, Hodge index on $C \times C$ gives RH (Weil).	Integrality holds at finite places and fails at ∞ ($\mathrm{Pic}(\overline{\mathrm{Spec}\,\mathbb{Z}}) \cong \mathbb{R}$). The positive half is G5. Integrality does not imply positivity ($E_1 \oplus -E_2$).	Positive half: G5.	L32 §8; L27; L15	[CITED], [ANALOGY] (for ζ)
10. Weil positivity	$\mathcal{Q}(F,F) \geq 0$ for all $F \in C_c^\infty(\mathbb{R})$. Equivalently $\mathcal{W}$ is positive definite; equivalently Li's $\lambda_n \geq 0$.	$\Longleftrightarrow$ RH. Windows: positive definite on every window of length ≤ 1.6 (certified, L16; agrees with Zhu). Small support: Yoshida.	All windows: G5.	L1, L10, L16, L17	[THM], [CITED], [RESTATEMENT]
11. Hessian / Morse / Pontryagin index	$n_+(\mathcal{L}\|_V)$ over finite-dimensional V has supremum $1 + N_+$. Each off-line quadruple gives $+2$. Window Morse index $= \kappa_A$. KKS count $k_r + 2k_c + 2k_i^- = n(L) - n(D)$.	$n_+ \geq 1$ is free (the pole plane is timelike). Index $= 1 \Longleftrightarrow$ RH. Lorentzian (Brändén–Huh) shape of window Grams: numerical, RH-equivalent on all windows.	Index = 1: G5.	§36.5; L15, L17, L27	[THM], [CITED], [RESTATEMENT], [ANALOGY]
12. Toeplitz / Levinson certificates	Certified lower bounds for $\lambda^*(e^A) = \min$ of $\mathcal{Q}$ on window A (Arb ball arithmetic).	[PARTIAL]. Certified: $\lambda^*(e^{1.6}) \geq 9.2792 \cdot 10^{-18}$ (L16); Toeplitz pipeline: prime-only $A = 1.6$: $\geq 1.4646 \cdot 10^{-17}$; $A = 1.5$: prime-only $\geq 3.2380 \cdot 10^{-15}$, hybrid $\geq 2.4463 \cdot 10^{-15}$. The hybrid fails at $A = 1.6$ and 2.4 (a minorant failing, *not* $\mathcal{Q} < 0$). Separate: *middle-band sampling bounds* certified to $A = 2.8$; these are *not* window positivity.	Cost doubly exponential in A. Cannot reach RH.	L10, L16, L24	[THM] (CA), [NUM]
13. Sum rule (new)	$\int_0^T \mathrm{Re}\,\mathsf{Z}(a+it)\,dt = \pi[N(T) - 2N(\frac{1}{2} + a, T)] + \int_{1/2}^{1/2+a} \mathrm{Im}\,\frac{\xi'}{\xi}(\sigma + iT)\,d\sigma$.	Identity unconditional. Deficit ≥ 0 always. Deficit $= 0$ for all $a, T \Longleftrightarrow$ RH.	Prime-side evaluation for $\sigma_0 < 1$: G5 (zero-density route).	§36.4, Thm 36.4.3	[THM], [NUM], [RESTATEMENT]

36.3. The implication diagram and the spine

THEOREM 36.3.1 (the passivity spine; [THM] assembled from Lemmas 18, 25, 33 and [La99, La05).] *For $h \in (0, \frac{1}{2}]$ the following are equivalent:*

(i) every zero has $\beta \le \frac{1}{2} + h$;
(ii) $\operatorname{Re} \frac{\xi'}{\xi}(s) > 0$ on $\sigma > \frac{1}{2} + h$ (positive-real at depth h);
(iii) $E_{h'}$ is Hermite–Biehler for every $h' > h$.

Under (i), the damped Weil forms of Lemma 18 at $\sigma = \frac{1}{2} + h'$, $h' > h$, are positive. At $h = \frac{1}{2}$ all three hold unconditionally (P0). Every $h < \frac{1}{2}$ is rung P3. As $h \downarrow 0$ the statement becomes RH (P4).

PROOF. (i) $\iff$ (ii) is Lagarias' Thm. 1.1 with $\theta = \frac{1}{2}+h$ [**168**] ([CITED]). (i)$\Rightarrow$(iii) is Lemma 33, Thm. 6.2(a): $E_{h'}$ is HB once $h' \ge \Theta - \frac{1}{2}$. (iii)$\Rightarrow$(i): if a zero has $\beta > \frac{1}{2}+h$, then for $h < h' < \beta - \frac{1}{2}$ the function $E_{h'}$ has a zero in $\operatorname{Im} z > 0$, so it is not HB (Lemma 33, Thm. 6.2, exactness). The damped statement is Lemma 18, Thm. 2.1 (for $\sigma \ge \sup \beta$ the damped form equals Lagarias' form W^σ with weight $2 \operatorname{Re} \frac{\xi'}{\xi}(\sigma + it) > 0$). P0 is Lemma 25, Prop. 1.8; at $h = \frac{1}{2}$, (iii) is [**169**, Lemma 6.1]. □

Figure 36.1 records the implications. Solid arrows are theorems. Double arrows are equivalences. The crossed dashed arrow stands for the *proved* non-implications listed in the caption, each with a counterexample in the TAA lemma named.

REMARK 36.3.2 (reading the boundary). The line "G5 begins" separates two kinds of positivity. Below it, the positivity holds for every Dirichlet series with an Euler product and a functional equation, *or* it is blind to off-line zeros. Above it, the positivity must see the whole strip. The yellow rung P3 is the cheapest statement that crosses the line. It is open, and it is far weaker than RH.

36.4. The Kramers–Kronig thread, done precisely

36.4.1. Three statements that are easy to conflate.

PROPOSITION 36.4.1 (the KK ladder; [THM] from Lemma 33 and standard facts). *Consider $\mathsf{Z}(p) = \frac{\xi'}{\xi}(\frac{1}{2} + p)$.*

(a) Support causality *(free). Z is the Laplace transform of Weil's distribution restricted to $u > 0$ (Lemma 33, Thm. 3.2), so dispersion relations hold on every vertical line $\operatorname{Re} p = c > \frac{1}{2}$: $\operatorname{Re} \mathsf{Z}(c + i\cdot)$ and $\operatorname{Im} \mathsf{Z}(c + i\cdot)$ (after subtracting the $e^{u/2}$ term) are a Hilbert pair* [**268**, Thm. 95] *([CITED, secondary]).*

(b) Positive spectral measure *(free on the boundary). The distributional boundary value of $\operatorname{Re} \mathsf{Z}(a + it)$ as $a \downarrow 0$ is $\pi \sum_{\delta_\rho = 0} m_\rho \, \delta_{t_\rho} \ge 0$ (Lemma 33, Thm. 5.1).*

(c) Stability *($\iff$ RH). Z is holomorphic on $\operatorname{Re} p > 0$ iff no zero has $\delta > 0$.*

(d) Passivity = *(b) + (c) + growth. A function holomorphic on the right half-plane with non-negative real part is a Herglotz–Nevanlinna function, and has a Poisson–Herglotz representation by a positive boundary measure* [**291**] *([CITED, secondary]). Conversely, (c) together with (b) gives $\operatorname{Re} \mathsf{Z} > 0$ on $\operatorname{Re} p > 0$. Under (c), Z is the Foster sum $\sum 2mp/(p^2 + t^2)$, whose real part is the Poisson extension of the measure in (b).*

PROOF. (a), (b) are Lemma 33. (c): the poles of Z are $p = \rho - \frac{1}{2} = \delta_\rho + it_\rho$. (d): under (c) the Foster form holds (Lemma 33, Thm. 5.3), and each term $2mp/(p^2 + t^2)$ has positive real part on $\operatorname{Re} p > 0$. □

REMARK 36.4.2 (what turns a dispersion relation into positivity). A dispersion relation is a statement about analyticity in a region. It says nothing about signs. To get positivity one needs

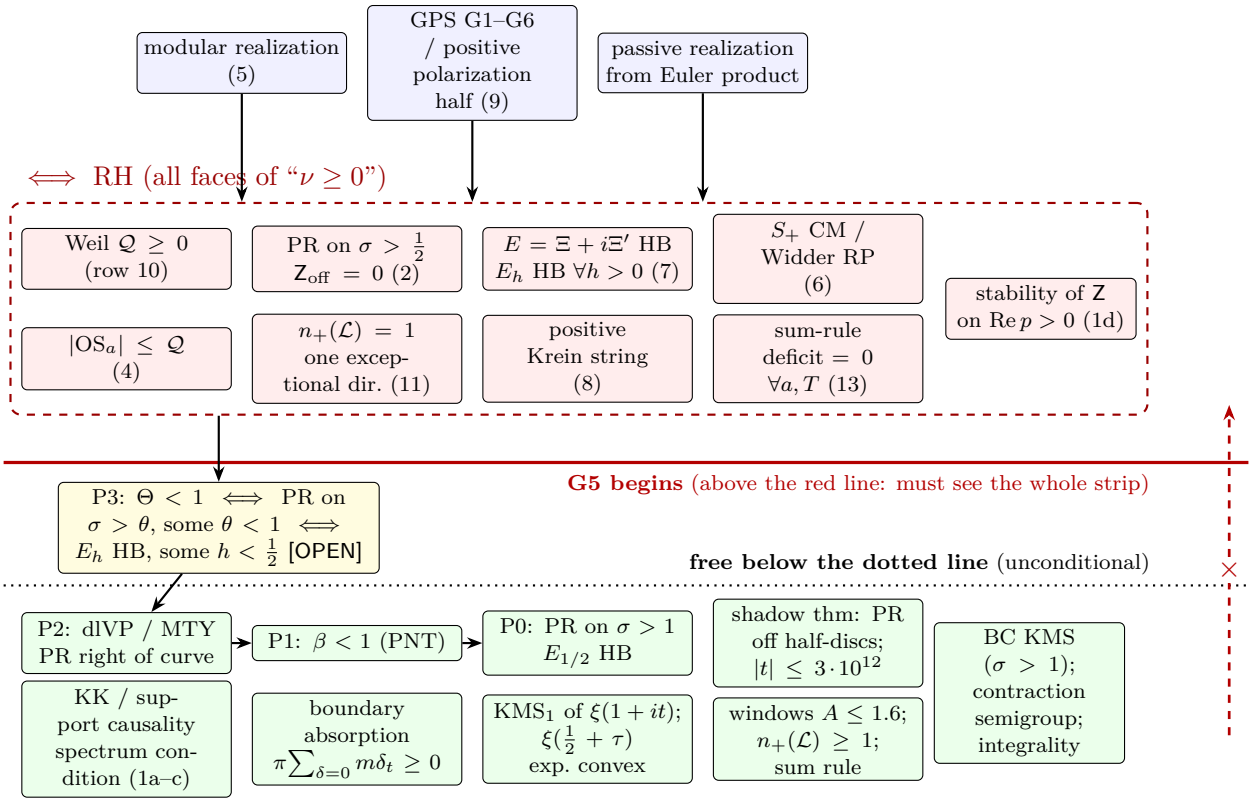

FIGURE 36.1. Implication diagram. Green: unconditional (free). Yellow: open intermediate rung. Red box: mutually equivalent to RH. Blue: sufficient structures (each $\Rightarrow$ RH; the converse is open or sketched). The crossed arrow summarizes proved non-implications: Causality does not give passivity (gain Lorentzian, L33 Thm 2.1). The boundary spectrum is blind to off-line zeros (L33 Thm 4.1). KMS of $\xi(1+it)$ is unconditional, so it cannot imply RH (L31 Thm 6.5). Short windows do not propagate (exact A_{pair}, L29 Thm 3.1). Prime-side and arrow-of-time positivity hold for $\zeta(s)\zeta(s+a)$, which has off-line zeros (L31 Prop 2.2).

two extra inputs: a *positive spectral measure* on the boundary (here free) and *no singularities* in the passive half-plane (here RH). Sum rules, i.e. moments of the absorption, add nothing new unless they compare the absorption at depth a with a strength computed independently. Theorem 36.4.3 is such a comparison.

36.4.2. The pole is a gain line; the absorption sees only on-line zeros. Lemma 33, Thm. 2.1: near $x = 0$, $\frac{1}{s-1} = -\frac{2}{1-2ix}$ on the line. This is a Lorentzian of HWHM $\frac{1}{2}$ and *negative* strength -2: a gain line ([THM]). In ξ it is cancelled by the factor $s-1$, and in Z it reappears as the pole term $e^{u/2}$ of $\mathcal{W}_+$. The boundary absorption spectrum (Prop. 36.4.1(b)) contains only on-line zeros. An off-line pair contributes $O(a)$ uniformly as $a \downarrow 0$.

36.4.3. A sum rule in which off-line zeros enter with a definite sign. Let $N(T) = \#\{\rho : 0 < t_\rho \leq T\}$ and $N(\sigma_0, T) = \#\{\rho : \beta_\rho > \sigma_0,\ 0 < t_\rho \leq T\}$, with multiplicity.

THEOREM 36.4.3 (absorption–phase sum rule; [THM]). *Let $\sigma_0 > \frac{1}{2}$ and $T > 0$. Assume ξ has no zero on the segment $[\frac{1}{2} + iT, \sigma_0 + iT]$ or on $\{\sigma_0 + it : 0 \leq t \leq T\}$. Then*

$$\mathcal{I}(\sigma_0, T) := \int_0^T \mathrm{Re}\, \frac{\xi'}{\xi}(\sigma_0 + it)\, dt = \pi\big[N(T) - 2N(\sigma_0, T)\big] + \int_{1/2}^{\sigma_0} \mathrm{Im}\, \frac{\xi'}{\xi}(\sigma + iT)\, d\sigma.$$

PROOF. Let $\phi(\sigma,t)$ be a continuous branch of $\arg\xi(\sigma+it)$ on the closed rectangle $R=[1-\sigma_0,\sigma_0]\times[0,T]$ minus the zeros, with $\phi=0$ on the real segment (there $\xi>0$). Since $\frac{d}{dt}\log\xi(\sigma_0+it)=i\frac{\xi'}{\xi}$, we have $\mathcal{I}(\sigma_0,T)=\phi(\sigma_0,T)$. Write $g=\phi(\sigma_0,T)$ and $D=\int_{1/2}^{\sigma_0}\operatorname{Im}\frac{\xi'}{\xi}(\sigma+iT)\,d\sigma$, which is the change of $\arg\xi$ from $\frac12+iT$ to σ_0+iT.

The zeros inside R are those with $0<t\le T$ and $1-\sigma_0<\beta<\sigma_0$. By the functional equation they number $N(T)-2N(\sigma_0,T)$, since the zeros with $\beta>\sigma_0$ and their mirrors $\beta<1-\sigma_0$ are excluded. The hypotheses keep the boundary of R zero-free: none on the right edge or on the top-right half by assumption, none on the mirrored left edge and top-left half by symmetry, none on the real segment since $\xi>0$ there. By the argument principle, $2\pi(N(T)-2N(\sigma_0,T))$ equals the total change of $\arg\xi$ around ∂R, counterclockwise.

The reflection identity $\xi(1-\bar s)=\overline{\xi(s)}$ reduces this change to the right half. Going up the right edge contributes g. Going down the left edge contributes g as well, because the values there are the conjugates of those on the right. The top edge from σ_0+iT to $1-\sigma_0+iT$ contributes $-2D$, because its two halves contribute equally: conjugate values traversed in the opposite direction. The bottom edge contributes 0. Hence $2\pi(N(T)-2N(\sigma_0,T))=2g-2D$, which is the claim. □

COROLLARY 36.4.4 (definite-sign deficit; [THM], [RESTATEMENT]). *With $a=\sigma_0-\frac12$,*

$$\pi N(T)+\int_{1/2}^{\sigma_0}\operatorname{Im}\frac{\xi'}{\xi}(\sigma+iT)\,d\sigma-\mathcal{I}(\sigma_0,T)=2\pi N(\sigma_0,T)\ \ge 0.$$

*The deficit vanishes for all admissible (σ_0,T) iff RH. By Platt–Trudgian [**228**] it vanishes for $T\le 3\cdot10^{12}$.*

REMARK 36.4.5 (reading, and why this is not a proof; [ANALOGY] for the optics). *Optics.* $\pi N(T)$ is the total line strength. It is computable from the phase on the critical line: $\pi N(T)=\vartheta(T)+\pi+\pi S(T)$ (Riemann–von Mangoldt). The left side $\mathcal{I}$ is the integrated absorption at depth a. An on-line zero, or a zero with $|\delta|<a$, is a loss line of integrated strength $+\pi$. A zero with $\delta>a$ is a gain line of strength $-\pi$, and its mirror ($\delta<-a$) is a loss line $+\pi$. So the pair is invisible in the integrated absorption, but it shows up as a deficit of exactly 2π against the phase-counted strength. This is the precise form of J. W.'s anomalous-dispersion picture: gain lines are where absorption and phase disagree. The optical vocabulary is an analogy ([ANALOGY]). The identity is a theorem.

Literature. Integrating Theorem 36.4.3 in σ_0 gives the rectangle formula of [**269**, §9.9, (9.9.1)], Littlewood's lemma, $\int F\,ds=-2\pi i\int\nu(\sigma,T)\,d\sigma$ ([CITED], local scan). The identity is classical in substance; we claim only the packaging.

Why it does not prove RH. To use the corollary one needs $\mathcal{I}(\sigma_0,T)$ to within 2π, at every T, from zero-free data. For $\sigma_0<1$ the Euler product does not converge. Mean-value estimates for $\log|\zeta|$ bound $\int N(\sigma,T)\,d\sigma$ (zero-density theorems, [**269**, §9.19]), but they cannot force an integer count to vanish (Lemma 29, obstruction O3). Lemma 33's question on a "Bode/Fano sum rule with a definite sign" is therefore answered *yes, and it is Littlewood's lemma.* Its use leads to zero-density theory, not to G5.

REMARK 36.4.6 (numerical check; [NUM]). `num/lemma36_checks.py` (mpmath) evaluates both sides of Thm. 36.4.3. For $(\sigma_0,T)=(0.6,40)$: $N=6$, $\mathcal{I}=19.010866$, edge 0.161310. For $(0.75,30)$: $N=3$, $\mathcal{I}=10.168057$, edge 0.743279. The residual is $<10^{-14}$ in both cases (zero deficit, as Platt–Trudgian predict). On a finite zero model with a planted pair $\delta=0.2$ at $t=27.5$, the deficit is exactly 2π at $a=0.1$ and 0 at $a=0.3$ (residual $\le 7\cdot10^{-15}$).

36.5. Hessian, Morse index and Pontryagin index

36.5.1. Lorentzian / Hodge-index shape. A real symmetric form of signature $(1,n-1)$ is a Pontryagin Π_1 form. The Hodge index theorem says the intersection form on $\mathrm{NS}(S)_{\mathbb R}$ of a

surface has exactly one positive eigenvalue [126, V, Thm. 1.9] ([CITED], carried over from Lemmas 13, 20). Brändén–Huh [46, Def. 2.1, Lemma 2.5]: a quadratic form with nonnegative coefficients is Lorentzian iff its Hessian has at most one positive eigenvalue (with M-convex support), iff it is real stable ([CITED], via tropical note). Weil proved RH for curves over $\mathbb{F}_q$ [285] through the Castelnuovo–Severi inequality on $C \times C$, which follows from Hodge index [193, Thm. 1.5], [126, V, Ex. 1.9–1.10]. Grothendieck gave the Hodge-index proof [118] ([CITED], bibliographic).

PROPOSITION 36.5.1 (one exceptional direction; [THM] = Lemma 15 Thm. 1.3 with signs made explicit). Space. *Complex-valued $F \in C_c^\infty(\mathbb{R})$; V ranges over finite-dimensional subspaces.* Signs. *$\mathcal{Q} = \mathcal{P} - \mathcal{L}$, where the pole form $\mathcal{P}$ has signature $(1,1)$ on the pole plane $(a(F), b(F))$. On primitive F ($a = b = 0$), $\mathcal{L} = -\mathcal{Q}$. The ample-like (timelike) direction is positive for $\mathcal{L}$; with $\mathcal{Q}$-signs it is a negative direction of $-\mathcal{P}$.*

(a) $\sup_V n_+(\mathcal{L}|_V) = 1 + N_+$, *and $n_+ \geq 1$ unconditionally.*

(b) RH $\iff$ $n_+(\mathcal{L}|_V) \leq 1$ for all V $\iff$ $\mathcal{L} \leq 0$ on primitive vectors (a Hodge-index shape) ([RESTATEMENT]).

(c) Index per zero. *N_+ counts zeros with $\beta > \frac{1}{2}$ over both signs of t. A quadruple $\{\rho, \bar\rho, 1-\rho, 1-\bar\rho\}$ with $\delta \neq 0$ contains exactly two such zeros, ρ and $\bar\rho$. So it raises $\kappa(\mathcal{Q})$ and $n_+(\mathcal{L})$ by exactly 2, i.e. by 1 per mirror pair $\{\rho, 1-\bar\rho\}$. Restricted to* real *F the index is the same, 1 in each parity sector (Lemma 11, Prop. 4.2, with Bombieri's eventual equality).*

PROOF. (a), (b) are Lemma 15, Thm. 1.3, whose upper bound uses $\kappa(\mathcal{Q}) = N_+$ (Lemma 1(vi)). (c): $\kappa(\mathcal{Q}) = N_+$ by Lemma 1(vi). In Lemma 30's spectral Albert step each mirror pair spans a hyperbolic plane of signature $(1,1)$ for the Krein form. For real F the form is the real part of a Hermitian form invariant under complex conjugation of the zero set, so the real and complex indices agree. `num/toy_offline.py` finds $2k$ negative eigenvalues for k planted quadruples (Lemma 1 numerics), and Lemma 15 finds $n_+ = 3$ for one planted quadruple. □

REMARK 36.5.2 (Morse index). Lemma 17's window variational principle (Thm. 3.1, Rem. 3.2) (Kato closed form, min–max) gives: the number of negative stationary values of the window action equals κ_A, the Morse index of the window Hessian. It is nondecreasing in A with supremum N_+. RH $\iff$ the first conjugate length A_1^* is infinite. The Jacobi-field picture is an analogy with a precise half (Lemma 17).

36.5.2. The Hamiltonian–Krein index theorem.

THEOREM 36.5.3 (Kapitula–Kevrekidis–Sandstede [153, 152]; [CITED]). *Let J be bounded, skew-symmetric and invertible, and L self-adjoint, under the hypotheses of* [153, §2]. *Let k_r be the number of positive real eigenvalues of JL, k_c the number of quadruplets $\{\pm\lambda, \pm\bar\lambda\}$ with $\lambda \notin \mathbb{R} \cup i\mathbb{R}$, and k_i^- the number of negative squares of L on the purely imaginary eigenspaces (with $\operatorname{Im}\lambda > 0$; Def. 2.4). If the symmetry matrix D of (3.1) is nondegenerate,*

$$k_r + 2k_c + 2k_i^- = n(L) - n(D).$$

Eigenvalues leave the imaginary axis only through collisions of opposite Krein signature, or from an embedded negative-signature eigenvalue (§2.2 there). The 2005 item is the authors' addendum. Pelinovsky [223] gives sharp bounds by the inertia law for the coupled NLS case ([CITED], abstract).

THEOREM 36.5.4 (the zero count in KKS bookkeeping; [THM]). *In the zero space $\mathcal{H}_\mathcal{Z}$ with Krein form $[x,y] = \langle J_{\mathrm{FE}} x, y\rangle$ (negative index N_+), the generator $\boldsymbol{\gamma}$ is J_{FE}-self-adjoint with eigenvalues γ_ρ (Lemma 25, Thm. 1.2). Put $\lambda_\rho = i\gamma_\rho$. Then:*

(a) the spectrum $\{\gamma_\rho\}$ is invariant under $\gamma \mapsto -\gamma$ and $\gamma \mapsto \bar\gamma$, so $\{\lambda_\rho\}$ has the KKS symmetry $\{\pm\lambda, \pm\bar\lambda\}$;

(b) $k_r = 0$: a real λ needs $t_\rho = 0$, i.e. a real zero in $(0,1)$, and $\zeta < 0$ on $(0,1)$;

(c) $k_i^- = 0$*: on-line zeros have positive weight* $m_\rho \geq 1$ *(Lemma 17, Thm. 5.3, "no Ostrogradsky ghosts");*

(d) *each off-line quadruple of zeros is exactly one KKS quadruplet. The count reads* $2k_c = N_+ = \kappa$*, in agreement with Prop. 36.5.1(c).*

PROOF. (a) is the functional equation together with reality. (b) is classical: $\zeta(\sigma) < 0$ for $0 < \sigma < 1$ follows from $(1 - 2^{1-\sigma})\zeta(\sigma) = \sum(-1)^{n-1}n^{-\sigma} > 0$. (c) m_ρ is a multiplicity. (d) Count the zeros with $\beta > \frac{1}{2}$ per quadruple, which is 2. □

REMARK 36.5.5 (what is analogy; relation to Lemma 27). The shared theorem is the *Pontryagin inertia count* for a self-adjoint operator in a Π_κ space (Lemma 19, Remark (c), with IKL 1982, Bognár 1974, Pontryagin 1944 cited there). KKS is its Hamiltonian instance, and Theorem 36.5.4 is its ζ instance. Calling on-line zeros "stable modes" and off-line quadruples "unstable quadruplets" of a Hamiltonian PDE is an [ANALOGY]. No symplectic J and energy L have been built from the explicit formula with $n(L) = N_+$ and JL having spectrum $\{i\gamma_\rho\}$. Lemma 27 matches the KKS mechanism in finite windows. Ghost *birth* ($\kappa_A : 0 \to 1$) passes an eigenvalue through ∞ and returns it with negative type (the analogue of an embedded negative-signature eigenvalue). The *boost onset* is a genuine opposite-signature Krein collision, and $\kappa_A = n_{\mathbb{C}} + n_-$ is the finite Pontryagin count. We do not assert a factor-by-factor dictionary between $(n_{\mathbb{C}}, n_-)$, which lives in the variable γ^2, and (k_c, k_i^-). The index of $E_{1/2}$ and its string is the subject of Lemma 34 (in preparation).

36.6. The exceptional direction as an edge mode

PROPOSITION 36.6.1 (three precise senses of "edge"; [THM]/[CITED]). *(1)* Maximum modulus and the $+1$. *On a compact connected Riemann surface X, a holomorphic function attains* $\max|f|$*, so it is constant: $H^0(X, \mathcal{O}) = \mathbb{C}$. In Riemann–Roch $\ell(D) - \ell(K - D) = \deg D + 1 - g$* [**126**]*, the "+1" is $h^0(\mathcal{O})$. Over a function field, Riemann–Roch is adelic Poisson summation applied to the indicator of $\mathcal{O}_{\mathbb{A}}(D)$* ([**231**, §5.6], *[CITED, secondary]). Over $\mathbb{Q}$, van der Geer–Schoof define $h^0(D) = \log\sum_{f\in I} e^{-\pi\|f\|_D^2}$, and Poisson summation gives $h^0(D) - h^0(\kappa - D) = \deg D - \frac{1}{2}\log|\Delta|$* [**275**, Prop. 1] *([CITED], arXiv text). The $f = 0$ term (the constant section) is always present.*

(2) The poles are the boundary terms. *Riemann's theta proof* [**269**, §2.6] *([CITED], local scan; [NUM] check): with $\psi(x) = \sum_{n\geq 1} e^{-\pi n^2 x}$,*

$$\pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s) = \frac{1}{s(s-1)} + \int_1^\infty (x^{s/2-1} + x^{-s/2-1/2})\psi(x)\,dx.$$

The integral is entire. The term $\frac{1}{s(s-1)} = \frac{1}{s-1} - \frac{1}{s}$ comes from splitting $\int_0^\infty$ at $x = 1$. One part is the $n = 0$ term of $\theta = 1 + 2\psi$ (the value $f(0)$). The other is the $\sqrt{x}$ produced by $\theta(1/x) = \sqrt{x}\,\theta(x)$ (the value $\hat{f}(0)$). In Tate's thesis the same split at $|x| = 1$ gives simple poles at $s = 0, 1$ with residues $-\kappa f(0)$ and $\kappa\hat{f}(0)$, and $\zeta(f, c) = \zeta(\hat{f}, \hat{c})$ [**265**], [**156**, Thm. 6.6.9], [**233**, Ch. 7] *([CITED, secondary]). The poles come from the trivial character and the constant mode, and from nothing else.*

(3) Cohomological degree (curves). *$Z(C, T) = P(C, T)/((1-T)(1-qT))$, with P of degree $2g$ carrying the zeros, "using Riemann–Roch"* [**193**, §1]*. In the Lefschetz form, (coincidences) $= \mathrm{Tr}(X|H^0) - \mathrm{Tr}(X|H^1) + \mathrm{Tr}(X|H^2)$, with the H^0, H^2 terms equal to the fibre intersection numbers $d_2(X), d_1(X)$* [**193**, eq. (2)]*. The pole factors come from H^0 and H^2, and the zeros from H^1. On $C \times C$ the hyperbolic plane $\langle H, V\rangle$ of fibre classes is the exceptional (non-negative) part of Hodge index, and the RH-carrying part is its orthogonal complement (Lemma 27, citing* [**193**]*).*

(4) Spectral edge and non-normalizability (ζ, [THM]). *The pole directions are the functionals $a(F) = \int Fe^{-u/2}$ and $b(F) = \int Fe^{u/2}$. They are eigenfunctionals of the scaling flow with*

real exponents $e^{\mp t/2}$ (Lemma 15, Thm. 1.2(b)), i.e. at $z = \pm\frac{i}{2}$, the edge *of the strip $|\operatorname{Im} z| \le \frac{1}{2}$ that contains every γ_ρ (P1 gives $|\operatorname{Im}\gamma_\rho| < \frac{1}{2}$ strictly). Since $e^{\pm u/2} \notin L^2(\mathbb{R})$, neither functional is L^2-bounded. The modes live at the ends $u \to \pm\infty$, i.e. $x = e^u \to 0, \infty$ in $\mathbb{R}_{>0} \cong$ the connected part of the idele class group of $\mathbb{Q}$. In Lemma 32's kernel the term $e^{u/2}$ on $u > 0$ is this non-normalizable tachyon.*

PROOF. (1) The maximum principle, then the cited sources. (2) The identity is checked by `lemma36_checks.py` at $s = 2$, $0.3 + 5i$, $\frac{1}{2} + 14i$ (residuals in `lemma36_checks.json`). The decomposition is Riemann's: write $\int_0^1 = \int_1^\infty$ after $x \mapsto 1/x$, use $\theta(1/x) = \sqrt{x}\theta(x)$, and integrate the leftover $\int_1^\infty x^{-s/2-1}dx$ and $\int_1^\infty x^{(s-1)/2-1}\,dx$-type terms. (3) Cited. (4) Direct. □

REMARK 36.6.2 (what this explains, and what it does not). Prop. 36.6.1 is why the one exceptional direction of §36.5 is "allowed". It is the constant / trivial-character mode, it sits at the spectral edge, and it is not in the Hilbert space. So it can carry positive $\mathcal{L}$-norm without being a ghost. All of this is [THM]/[CITED]. Calling it an *edge state* in the topological, bulk–boundary sense would require an index theorem identifying a bulk invariant with the number of boundary modes. None is supplied, so that reading is an [ANALOGY]. Nor does the edge picture say why the *bulk* (H^1, the zeros) should be definite. That is G5.

36.7. Integrality: the deficit as a winding number (Sommerfeld/Maslov)

Origin. J. W. asked [t130u] whether the fact that the missing absorption of Cor. 36.4.4 comes in whole multiples of 2π is related to Sommerfeld. He then asked [t131u] for the connections to be made concrete. This section separates what is a theorem from what is an analogy. In short: the integrality is a theorem, and it is topological in the precise sense that the deficit *is* a winding number (§36.7.1). The Bohr–Sommerfeld/Maslov reading is an analogy with one precise half (§36.7.2). Through Lemma 34 the integer becomes an exact count of escaped eigenvalues of the Hankel sections (§36.7.3). "RH as the vanishing of a topological index" is a restatement; making it a method is the open program of §36.7.4.

36.7.1. The integrality theorem. For $\sigma_0 > \frac{1}{2}$ and $T > 0$ let $Q = Q(\sigma_0, T) = [\sigma_0, 2] \times [0, T]$, with boundary ∂Q oriented counterclockwise, and

$$D(\sigma_0, T) := \pi N(T) + \int_{1/2}^{\sigma_0} \operatorname{Im} \frac{\xi'}{\xi}(\sigma + iT)\, d\sigma - \mathcal{I}(\sigma_0, T)$$

(the left side of Cor. 36.4.4). We call (σ_0, T) *admissible* [DEF] if ξ has no zero on the vertical segment $\{\sigma_0 + it : 0 \le t \le T\}$ or on the horizontal segment $[\frac{1}{2} + iT, 2 + iT]$. Non-admissible pairs form a closed set with empty interior: a countable union of vertical lines $\sigma_0 = \beta_\rho$ and horizontal lines $T = t_\rho$, locally finite.

THEOREM 36.7.1 (integrality of the deficit; [THM], unconditional). *(a)* Winding form. *If (σ_0, T) is admissible, then $\xi \neq 0$ on ∂Q and*

$$D(\sigma_0, T) = \Delta_{\partial Q} \arg \xi = \frac{1}{i} \oint_{\partial Q} \frac{\xi'}{\xi}(s)\, ds = 2\pi N(\sigma_0, T) \in 2\pi\mathbb{Z}_{\ge 0}.$$

(b) Multiplicity and jumps. *$N(\sigma_0, T)$ counts zeros with multiplicity. On each connected component of the admissible set D is constant. Crossing $\sigma_0 = \beta$ from right to left (decreasing σ_0), D jumps up by $2\pi \sum m_\rho$, the sum running over the zeros with $\beta_\rho = \beta$ and $0 < t_\rho \le T$. Crossing $T = t$ upward, D jumps up by $2\pi \sum m_\rho$ over the zeros with $t_\rho = t$ and $\beta_\rho > \sigma_0$. So D is nonincreasing in σ_0 and nondecreasing in T.*

(c) Boundary conventions. *At non-admissible points, define D by the right-continuous extension $D(\sigma_0, T) := \lim_{\varepsilon \downarrow 0} D(\sigma_0 + \varepsilon, T + \varepsilon)$. This equals $2\pi N(\sigma_0, T)$ with Lemma 36's convention ($\beta > \sigma_0$ strict, $0 < t \le T$) and is still in $2\pi\mathbb{Z}_{\ge 0}$. The* principal-value *winding*

number, by contrast, gives a zero of multiplicity m on a smooth edge of ∂Q weight $m/2$, and one at the corner σ_0+iT weight $m/4$. It lies in $\frac{2\pi}{4}\mathbb{Z}_{\geq 0}$ and is not integral in units of 2π. Integrality is a statement about the generic (admissible) set.

(d) Index per quadruple. *An off-line quadruple $\{\rho,\bar\rho,1-\rho,1-\bar\rho\}$ ($\delta_\rho>0$, multiplicity m) with $\beta_\rho>\sigma_0$ and $t_\rho\leq T$ contributes exactly $2\pi m$ to $D(\sigma_0,T)$: only ρ has both $\beta>\sigma_0$ and $t>0$. Hence, with $D(\sigma_0,\infty):=\lim_{T\to\infty}D(\sigma_0,T)\in 2\pi\mathbb{Z}_{\geq 0}\cup\{\infty\}$,*

$$N_+=\frac{1}{\pi}\lim_{\sigma_0\downarrow 1/2}D(\sigma_0,\infty),$$

which is the "+2 per quadruple" of Prop. 36.5.1(c), divided by the 2π quantum. Under Lemma 34's hypotheses (B), (ND), its Krein index is $\kappa(\omega)=\nu N_+(\frac12+\omega)=\frac{\nu}{\pi}D(\frac12+\omega,\infty)$.

Proof. (a) $\xi>0$ on the real axis; ξ has no zeros with $\beta\geq 1$ (so none on $\sigma=2$); the other two edges of ∂Q are zero-free by admissibility. The zeros inside Q are those with $\sigma_0<\beta<2$ and $0<t<T$, i.e. the zeros counted by $N(\sigma_0,T)$ (none have $t=T$ or $\beta=\sigma_0$, by admissibility). The argument principle gives $\Delta_{\partial Q}\arg\xi=2\pi N(\sigma_0,T)$. Admissibility includes the hypotheses of Thm. 36.4.3, so Cor. 36.4.4 gives $D=2\pi N(\sigma_0,T)$. (b) $N(\sigma_0,T)$ is an integer-valued step function of (σ_0,T). The zeros are locally finite, so it changes only on the non-admissible lines, by the stated amounts. (c) For small $\varepsilon>0$ the point $(\sigma_0+\varepsilon,T+\varepsilon)$ is admissible, and $N(\sigma_0+\varepsilon,T+\varepsilon)=N(\sigma_0,T)$, since no zero has $\sigma_0<\beta\leq\sigma_0+\varepsilon$ or $T<t\leq T+\varepsilon$ once ε is small. For the principal value, indent ∂Q around the boundary zero by an arc of radius r excluding it. The indented contour has winding N° (the interior count). The excluded arc subtends the interior angle π (edge) or $\frac{\pi}{2}$ (corner) and contributes $m\pi$ or $m\frac{\pi}{2}$ to $\Delta\arg$ as $r\to 0$. (d) Of the four members only ρ and $\bar\rho$ have $\beta>\frac12$, and of these only ρ has $t>0$. N_+ counts both ρ and $\bar\rho$, so $N_+=2\lim_{\sigma_0\downarrow 1/2}N(\sigma_0,\infty)$. The formula for κ is Lemma 34, Thm. 5.2 (there: $\kappa(\omega)=\nu N_+(\frac12+\omega)$ with $N_+(\sigma)$ counting $\beta>\sigma$, both signs of t). □

Corollary 36.7.2 (quantization gap; [THM]). *For admissible (σ_0,T), $D(\sigma_0,T)\notin(0,2\pi)$. Hence any computation (certified numerics, an inequality, an asymptotic) that places $D(\sigma_0,T)$ in an interval of length $<2\pi$ determines $N(\sigma_0,T)$ exactly, and $D<2\pi$ already implies $D=0$.*

Remark 36.7.3 (what integrality cannot do; [THM] for the facts, [RESTATEMENT] for the last line). Integrality counts *all* zeros inside the contour with weight 1. An on-line zero inside a contour contributes the same 2π as an off-line one: in Thm. 36.4.3's rectangle $[1-\sigma_0,\sigma_0]\times[0,T]$ the integer is $N(T)-2N(\sigma_0,T)$, which counts both kinds. Only the *placement* of the contour (right of $\sigma_0>\frac12$) separates them, and the integer is then exactly the unknown count. Integrality uses only that ξ is holomorphic, and $\xi>0$ on $\mathbb{R}$. It holds verbatim for the Davenport–Heilbronn-type function of [**269**, §10.25] ([CITED], local scan), which has a functional equation of the same shape, no Euler product, infinitely many zeros in $\sigma>1$ (proved there by Rouché's theorem), and a computed zero in the strip off the line (see also [**79**]). So no argument that uses only integrality, the functional equation and analyticity can prove RH. *Restatement:*

$$\text{RH}\iff D(\sigma_0,\infty)=0\text{ for every }\sigma_0>\tfrac12\iff N_+=0.$$

By Platt–Trudgian [**228**], $D(\sigma_0,T)=0$ for all $\sigma_0>\frac12$ and $T\leq 3\cdot 10^{12}$.

Proposition 36.7.4 (where the "+1" comes from; [THM], refines Lemma 14). *For $T>0$ not an ordinate, $\pi N(T)=\theta(T)+\Delta\arg[s(s-1)]+\pi S(T)$, with Δ taken along $2\to 2+iT\to\frac12+iT$, and exactly*

$$\Delta\arg[s(s-1)]=\arg(\tfrac12+iT)+\arg(-\tfrac12+iT)=\arctan 2T+(\pi-\arctan 2T)=\pi.$$

So the "+1" in $N(T)=\frac{1}{\pi}\theta(T)+1+S(T)$ comes jointly from the two factors s and $s-1$ of $\xi=\frac12 s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$. These are the factors that remove the poles of $\Lambda=\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ at

$s=0$ (from Γ) and $s=1$ (from ζ), i.e. the two pole functionals $a(F), b(F)$ of §36.1. Their shares are $\frac{1}{\pi}\arctan 2T$ and $1-\frac{1}{\pi}\arctan 2T$, which tend to $\frac{1}{2}$ each. Lemma 14 attributed the $+1$ to "the pole of ζ at $s=1$"; that is half the story for finite T and exactly half asymptotically.

PROOF. Half of the argument principle on $[-1,2]\times[0,T]$: by $\xi(1-\bar{s})=\overline{\xi(s)}$, the change of $\arg\xi$ along $2\to 2+iT\to\frac{1}{2}+iT$ is $\pi N(T)$. Split $\arg\xi$ into the four factors. $\arg\pi^{-s/2}\Gamma(\frac{s}{2})$ contributes $\theta(T)$: the right edge and top edge give $\arg\Gamma(\frac{1}{4}+\frac{iT}{2})-\frac{T}{2}\log\pi$ by continuity from $s=2$, where the value is real. $\arg\zeta$ contributes $\pi S(T)$ by definition. For $s(s-1)$: s moves from 2 to $\frac{1}{2}+iT$ in the closed first quadrant, and $s-1$ moves from 1 to $-\frac{1}{2}+iT$ in the closed upper half-plane, so principal arguments are continuous along the path. [NUM]: $\Delta\arg[s(s-1)]-\pi=0$ to 15 digits at $T=10,100,1000$, and $\theta(T)-\frac{T}{2}\log\frac{T}{2\pi}+\frac{T}{2}+\frac{\pi}{8}-\frac{1}{48T}=O(T^{-3})$ (`num/lemma36_sommerfeld_winding.out`). □

REMARK 36.7.5 (numerical checks; [NUM]). `num/lemma36_sommerfeld_winding.py` (mpmath, 20 digits, Gauss–Legendre quadrature of ξ'/ξ around ∂Q, with ζ'/ζ from `mp.zeta(s,derivative=1)`) gives $W=\frac{1}{2\pi i}\oint_{\partial Q}\xi'/\xi$: $|W|\le 2\cdot 10^{-5}$ on $Q(0.51,50)$ (quadrature error grows as the contour approaches the on-line zeros), and $W=0$ to working precision on $Q(0.6,40)$ and $Q(0.75,100)$. On $[-1,2]\times[0,T]$, $W=3$ and 10 for $T=30,50$, matching `mp.nzeros`. With a planted quadruple ($f=\xi\cdot(s-r)(s-\bar{r})(s-1+r)(s-1+\bar{r})$, $r=\beta+27.5i$), $W(Q(0.6,40))$ and $D/2\pi$ are 0 (to $\le 4\cdot 10^{-5}$) for $\beta\in\{0.5,0.56,0.59\}$ and 1 (to $\le 10^{-9}$) for $\beta\in\{0.62,0.71,0.8\}$. For $\beta=0.596,0.602$, at distance 0.004 from the contour, the default panels give non-integers (-0.017, 1.148). These are quadrature errors: with refined panels near $t=27.5$ the values are 0 and 1 to $4\cdot 10^{-5}$ (run `refine`). So the integer stays fixed as β moves, and it jumps only when β crosses σ_0. Values are in the `.out` file.

36.7.2. The Bohr–Sommerfeld/Maslov analogy. Background ([BIB]: Crossref records only, content standard). For a one-degree-of-freedom system, the corrected Bohr–Sommerfeld (EBK) rule is $\oint p\,dq=2\pi\hbar(n+\frac{\mu}{4})$, $n\ge 0$, where the Maslov index $\mu\in\mathbb{Z}$ counts caustic (turning-point) crossings: $\mu=2$ for the oscillator. This is due to Keller [**157**] and was put in topological form by Arnol'd [**10**].

PROPOSITION 36.7.6 (smoothed count of a Bohr–Sommerfeld spectrum; [THM], elementary). *If the levels satisfy $A(E_n)=2\pi\hbar(n+\frac{\mu}{4})$, $n=0,1,\dots$, with A increasing, then with $x=A(E)/2\pi\hbar$ the counting function is $N(E)=\lfloor x-\frac{\mu}{4}\rfloor+1$, whose zero-mean smooth part is $\bar{N}=x-\frac{\mu}{4}+\frac{1}{2}$.*

PROOF. $\lfloor y\rfloor=y-\frac{1}{2}-(\{y\}-\frac{1}{2})$, and $\{y\}-\frac{1}{2}$ has mean 0. [NUM]: `num/lemma36_sommerfeld_maslov.out`. □

Riemann side ([THM]). $N(T)=\frac{1}{\pi}\theta(T)+1+S(T)$ exactly, where S has mean 0 ($\int_0^T S=O(\log T)$, Littlewood; Lemma 14), and $\theta(T)=\frac{T}{2}\log\frac{T}{2\pi}-\frac{T}{2}-\frac{\pi}{8}+\frac{1}{48T}+O(T^{-3})$ (Lemma 14, Prop. 36.7.4). Berry–Keating write $\langle N(t)\rangle=\frac{\theta(t)}{\pi}+1=\frac{t}{2\pi}\log\frac{t}{2\pi e}+\frac{7}{8}+O(\frac{1}{t})$ [**30**, eq. (2.3)] ([CITED], full text). In [**30**, §6] ([CITED]) they take the classical $H_{\mathrm{cl}}=XP$, count the area under $E=XP$ in Planck cells $2\pi\hbar=2\pi$ "with a Maslov index correction given by $\alpha/4\pi$, where α is the angle turned through along the orbit", and regularize by truncating X and P. They report that the result is "precisely the asymptotics of the smoothed counting function for the Riemann zeros … including the term 7/8". They do not state the value of α. They stress that the XP motion is unbounded, and that closing the phase space is "a central unsolved problem". Separately, they list that the Maslov phases of the conjectured Riemann dynamics' periodic orbits "are all π" [**30**, §6(f)]. Sierra–Rodríguez-Laguna [**245**] state the same count with the $\frac{7}{8}$ attributed to a Maslov phase ([CITED, secondary], carried from Lemma 13's refs).

PROPOSITION 36.7.7 (the arithmetic of $\frac{7}{8}$; [THM]). *Match $\bar{N}$ of Prop. 36.7.6 to $N(T)-S(T)$ under the three natural choices of "action" x:*

action x	*constant c to match*	$\mu = 4(\frac{1}{2} - c)$
$\frac{1}{\pi}\theta(T)$ *(exact phase)*	1	-2 *(integer)*
$\frac{T}{2\pi}\log\frac{T}{2\pi e}$ *(Weyl term only)*	$\frac{7}{8}$	$-\frac{3}{2}$ *(not an integer)*
$\frac{1}{2\pi}\cdot$*(cutoff area* $E\log\frac{E}{l_x l_p} - E + l_x l_p$*)*, $l_x l_p = 2\pi$	$-\frac{1}{8}$	$\frac{5}{2}$ *(not an integer)*

In Berry–Keating's normalization $\langle N\rangle = A/2\pi + \alpha/4\pi$, the cutoff-area reading needs $\alpha/4\pi = -\frac{1}{8}$, i.e. $\alpha = -\frac{\pi}{2}$.

PROOF. Direct arithmetic. The cutoff area was checked by quadrature (`lemma36_sommerfeld_maslov.out`: $A/2\pi - [\frac{E}{2\pi}\log\frac{E}{2\pi e} + 1] = O(10^{-30})$ at $E = 50$). The $-\frac{1}{8}$ is $\frac{1}{\pi}\operatorname{Im}[(-\frac{1}{4})\log(iT/2)]$, a Stirling term of $\Gamma(\frac{1}{4} + \frac{iT}{2})$ (Lemma 14). □

REMARK 36.7.8 (verdict; [ANALOGY]). The *precise half*: in both settings an exact phase identity, "phase integral = $2\pi\times$integer + boundary phase", counts states. On the Riemann side this is Thm. 36.7.1 and Prop. 36.7.4. The boundary phase is the exact π of $s(s-1)$, i.e. of the two edge poles. In the exact-phase reading it is formally an integer Maslov-type term ($\mu = -2$). The *analogy half*: (i) $\frac{7}{8}$ is not a Maslov index in the standard 1-dof convention. It corresponds to $\mu = -\frac{3}{2}$ or $\frac{5}{2}$, not integers. Its fractional part $-\frac{1}{8}$ is a Stirling phase of $\Gamma(\frac{1}{4} + \frac{iT}{2})$, $\frac{1}{4}\cdot\frac{\pi}{2}$ divided by π, not a count of caustics. (ii) The XP orbits are not closed, so there is no $\oint p\,dq$ and no Maslov class in the sense of [**10**]. BK's α is not computed, and the cutoffs fix the $+1$ by hand (Lemma 14, Heuristic 1.2). (iii) $D = 2\pi N(\sigma_0, T)$ counts *zeros off to the right of a contour*, while Bohr–Sommerfeld counts *states below an energy*. The common structure is the argument principle (a winding number), not quantization. Calling the deficit a "Sommerfeld quantum" is therefore [ANALOGY]. Calling it a winding number is [THM].

statement	status	where
$D(\sigma_0, T) = 2\pi N(\sigma_0, T)$, a winding number	[THM]	Thm. 36.7.1(a)
no deficit in $(0, 2\pi)$	[THM]	Cor. 36.7.2
$+2$ index per quadruple $= \frac{1}{\pi}\times$ one 2π quantum	[THM]	Thm. 36.7.1(d)
RH $\iff$ $D(\sigma_0, \infty) = 0$ for all $\sigma_0 > \frac{1}{2}$	[RESTATEMENT]	Rem. 36.7.3
the $+1$ is $\frac{1}{\pi}\Delta\arg[s(s-1)]$, exactly π	[THM]	Prop. 36.7.4
$\frac{7}{8} = 1 - \frac{1}{8}$, $-\frac{1}{8}$ from Stirling	[THM]	Lemma 14, Prop. 36.7.7
$\frac{7}{8}$ as a Maslov index	[ANALOGY] (non-integer μ)	Prop. 36.7.7, Rem. 36.7.8
XP area = smooth count (with cutoffs)	[ANALOGY] ([CITED] for BK's statement)	[**30**, §6]
deficit = "Sommerfeld quantum"	[ANALOGY]	Rem. 36.7.8
escape count $= \kappa = \frac{\nu}{\pi}D(\frac{1}{2} + \omega, \infty)$	[THM] (under (B))	Thm. 36.7.9, Cor. 36.7.10
escape count = −spectral flow	[THM] (definitional)	Prop. 36.7.12
an APS/Chern index theorem for RH	[ANALOGY]/[OPEN]	§36.7.4

TABLE 36.2. Theorem versus analogy in this section.

36.7.3. Connection to Lemmas 34 and 37: the integer as a count of line breaks. *Setting.* Lemma 34, Thm. 5.4 ([THM] there, with the contour shift in (i) sketched): $\Theta = E^{\#}/E$ is unimodular on $\mathbb{R}$ and real-symmetric, $\Theta(-\bar{z}) = \overline{\Theta(z)}$, with finitely many poles in $\mathbb{C}_+$ (hypothesis (B)). None is on $\mathbb{R}$ or on $i\mathbb{R}$. $\mathcal{K}[t] = U[t] + R[t]$ with $\|U[t]\| \le 1$, and $R[t]$ is the Hankel operator with kernel $R(x+y)$, $R(x) = \sum_j P_j(x)e^{-iz_j x}$, the residue sum over the $\mathbb{C}_+$ poles. Let κ_Θ be the total order of these poles. Under (ND) there is no cancellation, and $\kappa_\Theta = \kappa = \nu N_+(\frac{1}{2} + \omega)$. Lemma 34 proves $n_\pm(R[t]) \le \kappa_\Theta/2$, at most $\kappa_\Theta/2$ eigenvalues of $\mathcal{K}[t]$ above 1 and at most $\kappa_\Theta/2$ below -1, and, under its hypothesis (M), at least one exit each way.

THEOREM 36.7.9 (exact escape count; [THM], new; removes (M)). *Let Θ be meromorphic with $|\Theta| \le 1$ on $\mathbb{R}$, $\Theta(-\bar z) = \overline{\Theta(z)}$, and finitely many poles in $\mathbb{C}_+$ of total order κ_Θ, none on $\mathbb{R} \cup i\mathbb{R}$. Assume the decomposition $\mathcal{K}[t] = U[t] + R[t]$, $\|U[t]\| \le 1$, of Lemma 34, Thm. 5.4(i). This covers Lemma 34's E_ω under (B), modulo the sketched contour shift there, and every rational Θ of this kind, where the shift is elementary. The identity $\Theta(-z) = 1/\Theta(z)$ is not used. Then:*

(i) $n_+(R[t]) = n_-(R[t]) = \kappa_\Theta/2$ for every *$t > 0$;*

(ii) there is $t_* < \infty$ such that for all $t > t_*$, $\mathcal{K}[t]$ has exactly $\kappa_\Theta/2$ eigenvalues in $(1, \infty)$ and exactly $\kappa_\Theta/2$ in $(-\infty, -1)$, counted with multiplicity. Moreover the k-th largest and k-th smallest of them tend to $\pm\infty$ for $k \le \kappa_\Theta/2$.

PROOF. *Structure of R.* Group the $\mathbb{C}_+$ poles into pairs $\{w, -\bar w\}$ (real symmetry; $w \notin i\mathbb{R}$), with representative $w = a + ib$, $a > 0$, $b > 0$, and order m. The pair contributes $2\,\mathrm{Re}[P(x)e^{-iwx}]$ with $P(x) = \sum_{n<m} p_n x^n$ and $p_{m-1} \ne 0$ (the leading Laurent coefficient of a pole of exact order m). With $\zeta_k(f) = \int_{-t}^{t} y^k e^{-iwy} f(y)\,dy$ for real f, expand $(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$:

$$\langle R_w[t]f, f\rangle = 2\,\mathrm{Re}\,\textstyle\sum_{k,l} H_{kl}\zeta_k\zeta_l, \qquad H_{kl} = \binom{k+l}{k} p_{k+l}\ (k+l \le m-1),\ H_{kl} = 0\ (k+l > m-1).$$

H is complex symmetric and anti-triangular, with antidiagonal entries $\binom{m-1}{k} p_{m-1} \ne 0$, so it is invertible. *Signature of one block.* By Takagi's factorization $H = V^{\mathsf T}\Sigma V$, with V unitary and $\Sigma > 0$ diagonal (Takagi factorization, [**133**]). In the coordinates $\eta = V\zeta$ the form is $2\sum_j \sigma_j\,\mathrm{Re}\,\eta_j^2 = 2\sum_j \sigma_j(u_j^2 - v_j^2)$, so $q_w = 2\,\mathrm{Re}(\zeta^{\mathsf T} H\zeta)$ on $\mathbb{C}^m \cong \mathbb{R}^{2m}$ has signature exactly (m, m). *Surjectivity.* The real functions $\mathrm{Re}(y^k e^{-iw_j y})$, $\mathrm{Im}(y^k e^{-iw_j y})$ (j over pairs, $k < m_j$) are linearly independent over $\mathbb{R}$ on any interval: their complex span is that of $y^k e^{-iw_j y}$ and $y^k e^{i\bar w_j y}$, which are exponential monomials with pairwise distinct exponents ($-iw_j$ and $i\bar w_j$ differ since $a_j \ne 0$, and distinct pairs have distinct w_j). So $f \mapsto (\mathrm{Re}\,\zeta, \mathrm{Im}\,\zeta) \in \mathbb{R}^{\kappa_\Theta}$ is onto on $L^2(-t,t)$ for every $t > 0$, and $R[t] = A_t^* S A_t$, with $S = \bigoplus_j q_{w_j}$ nondegenerate of signature $(\kappa_\Theta/2, \kappa_\Theta/2)$ and A_t onto. Hence $n_\pm(R[t]) = n_\pm(S) = \kappa_\Theta/2$, which is (i). *Growth.* The nonzero eigenvalues of $A_t^* S A_t$ are those of $G_t^{1/2} S G_t^{1/2}$, where $G_t = A_t A_t^*$ is the Gram matrix of the κ_Θ real functions φ_i on $(-t,t)$. For $k \le n_+(S)$ Courant–Fischer, applied with $u = G_t^{1/2} v$ on the span of the top k eigenvectors of S, gives $\lambda_k(G_t^{1/2} S G_t^{1/2}) \ge \lambda_k(S)\,\lambda_{\min}(G_t)$, and similarly at the bottom. G_t is Loewner-increasing in t. We claim $\lambda_{\min}(G_t) \to \infty$. Otherwise $c_t^{\mathsf T} G_t c_t \le M$ for unit vectors c_t. A limit point c then has $c^{\mathsf T} G_s c \le M$ for all s (use $G_s \le G_t$ for $s \le t$), i.e. $g = \sum c_i\varphi_i \in L^2(0,\infty)$. But $g \ne 0$ is a real exponential polynomial. Let b be its largest growth rate and d its top degree at that rate. Then $g(y) = y^d e^{by}(h(y) + o(1))$, where $h(y) = \sum_j \mathrm{Re}(q_j e^{-ia_j y})$ is a nonzero trigonometric polynomial with distinct frequencies $a_j > 0$, hence of positive mean square $\frac12 \sum |q_j|^2$. So $\int_Y^{Y+L} g^2 \to \infty$, a contradiction. *Exits.* By Weyl's inequality and $\|U[t]\| \le 1$, $\lambda_k(\mathcal{K}[t]) \ge \lambda_k(R[t]) - 1 \to \infty$ for $k \le \kappa_\Theta/2$, and symmetrically at the bottom. Lemma 34's bound caps the counts at $\kappa_\Theta/2$. Its proof: on the span of eigenvectors with eigenvalue > 1, $\langle R[t]f, f\rangle = \langle \mathcal{K}[t]f, f\rangle - \langle U[t]f, f\rangle > 0$. This gives (ii). □

COROLLARY 36.7.10 (the 2π quantum counts line breaks; [THM] under (B), (ND)). *For admissible (ω, ν) under (B), (ND), the number of eigenvalues of $\mathcal{K}[t]$ that eventually lie outside $[-1, 1]$ is*

$$\kappa = \nu N_+(\tfrac12 + \omega) = \frac{\nu}{\pi}\, D(\tfrac12 + \omega, \infty),$$

half of them opens (> 1) and half shorts (< -1). If $\kappa_\Theta = 0$ then $R = 0$ and $\|\mathcal{K}[t]\| \le 1$ for all t. So, under (B), the line of Lemma 37 is passive iff $\kappa_\Theta = 0$.

PROOF. Thm. 36.7.9, Thm. 36.7.1(d), and Lemma 34, Thm. 5.4(i). □

REMARK 36.7.11 ("breaks are shorts" is a first-exit statement; [THM] + [NUM]). Lemma 37 records that every break computed in TAA is a short, i.e. exits through -1. Corollary 36.7.10 shows this cannot persist for lines with $\kappa_\Theta > 0$: eventually exactly $\kappa_\Theta/2$ opens occur as well. The

sign of the *first* exit is not topological. In the toy test below with $\nu = 2, 3$ the first exit is an open. (Lemma 37's data concern the archimedean and truncated lines, which are not in this class; see Prop. 36.7.13.)

PROPOSITION 36.7.12 (escape count as spectral flow; [THM], definitional). *$t \mapsto \mathcal{K}[t]$ is norm-continuous with compact values (Lemma 34), so $1 \mp \mathcal{K}[t]$ are paths of self-adjoint Fredholm operators. With the spectral flow of Phillips [226], for t with $\pm 1 \notin \sigma(\mathcal{K}[t])$:*

$$\#\{\lambda > 1\}(\mathcal{K}[t]) = -\mathrm{sf}\big(1 - \mathcal{K}[s]\big)_{s\in[0,t]}, \qquad \#\{\lambda < -1\}(\mathcal{K}[t]) = -\mathrm{sf}\big(1 + \mathcal{K}[s]\big)_{s\in[0,t]}.$$

PROOF. For paths in $1 + (\text{compact})$ the spectral flow is the net number of eigenvalues crossing 0 upward. This equals $n_-(\text{start}) - n_-(\text{end})$ when the endpoints are invertible, because negative spectral subspaces are finite-dimensional. Here $\mathcal{K}[0] = 0$, so $n_-(1 \mp \mathcal{K}[0]) = 0$. □

Thm. 36.7.9 thus says that the eventual spectral flow of each of $1 \mp \mathcal{K}[t]$ is $-\kappa_\Theta/2$, and Cor. 36.7.10 ties it to the 2π quantum. Reading this as an Atiyah–Patodi–Singer index [17, 18], with κ as a bulk invariant and $D/2\pi$ as a boundary η-type term, is [ANALOGY]. No elliptic operator or boundary condition is supplied.

PROPOSITION 36.7.13 (truncated lines, $\omega < \frac{1}{2}$: lower bounds; [THM], numerical prediction untested). *Let $\omega \in (0, \frac{1}{2})$, $\nu > 1/\omega$, $P < \infty$. By Lemma 37, Thm. 7.4(c), Θ_P has one $\mathbb{C}_+$ pole, at $s_0 = 1 - \omega$ on the imaginary z-axis, of order ν. The truncated θ_p have their poles at $\mathrm{Re}\, s = \omega < \frac{1}{2}$. Assume the contour shift of Lemma 34, Thm. 5.4(i) for Θ_P. It holds by the same proof, since $M_P := \sup_{\mathbb{R}} |\Theta_P| < \infty$ and Θ_P decays like $|s|^{-\nu\omega}$ vertically. Then $\mathcal{K}_P[t] = U_P[t] + R_P[t]$ with $\|U_P\| \le M_P$ and $R_P(x) = p(x)e^{(1/2-\omega)x}$, p real of degree exactly $\nu - 1$. The real Hankel form has signature $(\frac{\nu}{2}, \frac{\nu}{2})$ for ν even, and $(\frac{\nu+1}{2}, \frac{\nu-1}{2})$ or $(\frac{\nu-1}{2}, \frac{\nu+1}{2})$ for ν odd according to the sign of the leading coefficient. For all large t, $\mathcal{K}_P[t]$ has* at least *n_+ eigenvalues > 1 and* at least *n_- eigenvalues < -1. Since $M_P > 1$ (e.g. $\Theta_\infty(0) = 1336.5$ at $(0.3, 4)$, Lemma 34), no upper bound follows.* Prediction: *the truncated lines at $(0.3, 4)$ and $(0.2, 6)$ eventually show at least 2, resp. 3, opens, in addition to the shorts seen in Lemma 37. This has* not *been computed.*

PROOF. As in Thm. 36.7.9, with one real block. H is real, symmetric and anti-triangular with nonzero antidiagonal. The path $H_s = H_{\text{anti}} + s(H - H_{\text{anti}})$, $s \in [0, 1]$, has constant nonzero determinant, so the signature is that of the antidiagonal part. That part splits into 2×2 blocks $\left(\begin{smallmatrix} 0 & a \\ a & 0 \end{smallmatrix}\right)$ of signature $(1, 1)$, plus a middle entry $\binom{\nu-1}{(\nu-1)/2} p_{\nu-1}$ when ν is odd. Use Weyl with $\|U_P\| \le M_P$ in place of 1. □

REMARK 36.7.14 (toy test of Thm. 36.7.9 and Prop. 36.7.13; [NUM]). `num/lemma36_sommerfeld_escape.py`: Lemma 34's kernel convention $K(x) = -i \sum \mathrm{Res}\,[\Theta(z)e^{-ixz}]$ $(x > 0)$. Laurent coefficients come from 96-point trapezoid rules on circles (mpmath, 25 digits; $\max |\mathrm{Im}\, K| \le 2 \cdot 10^{-25}$), followed by Gauss–Legendre Nyström with $n = 120$ and 200 nodes. (a) Sanity: $\Theta = (\frac{i}{z+i})^2$ (no $\mathbb{C}_+$ pole) gives all eigenvalues in $(-1, 1)$ for $t \le 6$. (b) $\Theta = (\frac{i}{z+i})^2 \beta_w(z)^{-\nu}$, $w = 1.5 + 0.4i$, β_w the Blaschke factor of the pair $\{w, -\bar{w}\}$, $\kappa = 2\nu$. Inertia of $R[t]$: (ν, ν) at $t = 0.5, 1, 3$ for $\nu = 1, 2, 3$. Exits: $(1, 1)$ from $t = 2$ $(\nu = 1)$; $(1, 0), (1, 1), (2, 1), (2, 2)$ at $t = 1, 2, 4, 6$ $(\nu = 2$, stable to $t = 12)$; $(1, 0), (1, 1), (2, 2), (3, 3)$ at $t = 1, 2, 4, 8$ $(\nu = 3$, stable to $t = 12)$. Both resolutions agree. (c) Imaginary-axis pole, $\Theta = (\frac{i}{z+i})^2 (\frac{z+0.4i}{z-0.4i})^\nu$. Inertia $(1, 0), (1, 1), (2, 1), (2, 2)$ for $\nu = 1, 2, 3, 4$ (leading coefficient > 0). Eventual exits are $(1, 0), (1, 1), (2, 1), (2, 2)$ by $t = 3, 3, 6, 10$, as predicted; here $\|U\| \le 1$, so the counts are exact. One discrepancy is a resolution artifact: at $\nu = 1$, $t = 18$, the $n = 120$ run shows a spurious eigenvalue -1.006, absent at $n = 200$. The lower edge of U approaches -1.

36.7.4. Open program: RH as the vanishing of a topological index.

THEOREM 36.7.15 (homotopy invariance; [THM], Rouché/Hurwitz). *Let $\bar{Q}$ be a closed rectangle, F_λ ($\lambda \in [0,1]$) holomorphic near $\bar{Q}$, jointly continuous in (λ, s), and $F_\lambda \neq 0$ on ∂Q for all λ. Then $W(F_\lambda; Q) = \frac{1}{2\pi i} \oint_{\partial Q} F_\lambda'/F_\lambda$ is independent of λ. For the half-strip $Q_\infty(\sigma_0) = [\sigma_0, 2] \times [0, \infty)$ the same holds if, in addition, there is a T_0 such that no F_λ vanishes in $[\sigma_0, 2] \times [T_0, \infty)$ (uniform tail control). Without tail control, zeros can enter from infinity.*

PROOF. F_λ'/F_λ is continuous on $[0,1] \times \partial Q$ (Cauchy estimates give continuity of F_λ'), so W is continuous and integer-valued, hence constant. For the half-strip, apply this to $Q(\sigma_0, T_0)$. □

Restatement ([RESTATEMENT]). With $\iota(\sigma_0) := D(\sigma_0, \infty)/2\pi$, RH $\iff$ $\iota(\sigma_0) = 0$ for all $\sigma_0 > \frac{1}{2}$. A *method* would be a family F_λ from some F_0 with known index 0 to $F_1 = \xi$, such that no zero crosses $\operatorname{Re} s = \sigma_0$ for any $\sigma_0 > \frac{1}{2}$, with tail control. The candidates are below. Scores are 0.6·significance +0.4·probability, each on 0–10. Significance is the value of the stated target; probability is our estimate that the target is reachable with known tools. Both are judgments ([OPEN]), not computations.

(D1) **de Bruijn–Newman heat flow** ([CITED]: [229] full text, §1, Thm. 1.2, Thm. 3.2; [236] abstract and introduction; [82, 209] [CITED, secondary] via these). $H_t(z) = \int_0^\infty e^{tu^2} \Phi(u) \cos zu \, du$ with $H_0(z) = \frac{1}{8}\xi(\frac{1}{2} + \frac{iz}{2})$. de Bruijn: if the zeros of H_{t_0} lie in $|\operatorname{Im} z| \le y_0$, then for $t > t_0$ those of H_t lie in $|\operatorname{Im} z| \le \max(y_0^2 - 2(t - t_0), 0)^{1/2}$ ([82, Thm. 13] as stated in [229, Thm. 3.2]). So H_t has index 0 for $t \ge \frac{1}{2}$, and there is a de Bruijn–Newman constant Λ with H_t having only real zeros iff $t \ge \Lambda$. Known: $0 \le \Lambda \le 0.22$ [236, 229], and RH $\iff$ $\Lambda \le 0$, i.e. $\Lambda = 0$. In index language, $\iota_t := \#\{\text{zeros of } H_t \text{ with } \operatorname{Im} z > 0\}$ is 0 for $t \ge \Lambda$, and RH says $\iota_0 = 0$. Polymath's barrier criterion [229, Thm. 1.2] is, in substance, an instance of Thm. 36.7.15: a zero-free barrier region over a time interval prevents zeros from moving across it, and numerics control the region below the barrier. *Why it does not reach RH:* Rodgers–Tao [236] show $\Lambda \ge 0$, so $t = 0$ is the endpoint. There is no slack: any index-transport argument down to $t = 0$ must be sharp. Partial crossing control would come from Lemma 33's shadow theorem only up to height $3 \cdot 10^{12}$, which is exactly the range where barrier numerics already work.

(D2) **ω-family / depth** ([THM] for the facts). For fixed ν, $\omega \mapsto \kappa(\omega) = \frac{\nu}{\pi} D(\frac{1}{2} + \omega, \infty)$ is nonincreasing and right-continuous, and it is 0 for $\omega \ge \frac{1}{2}$ (no zeros with $\beta \ge 1$). Lowering ω is a deformation of the *contour*, not of the function, so homotopy invariance is trivial and crossings are exactly zeros with $\beta = \frac{1}{2} + \omega$. Lemma 37's depth theorem is this statement. Partial crossing control: Lemma 33, Cor. 4.3(a) gives κ restricted to $|t| \le 3 \cdot 10^{12}$ equal to 0 for every $\omega > 0$, and the Mossinghoff–Trudgian–Yang zero-free region used there gives $N(\sigma_0, T) = 0$ for $\sigma_0 \ge 1 - 1/(5.558691 \log T)$, $T \ge 2$. Neither is uniform in height. A uniform $\kappa(\omega_0) = 0$ for some fixed $\omega_0 < \frac{1}{2}$ is a quasi-RH (rung P3).

(D3) **P-truncated Euler products** ([THM], Lemma 37 + Prop. 36.7.13). For $\omega < \frac{1}{2}$ every Θ_P has index $\ge \nu$ (the archimedean pole at $s = 1 - \omega$), while Θ_∞ has index $\kappa = \nu N_+(\frac{1}{2} + \omega)$. The pole is cancelled only by the pole of $\zeta(s + \omega)$, a property of the *full* product (Lemma 37, remark after Thm. 7.4). So the index is *not* continuous as $P \to \infty$: there is no tail control, and the deformation fails Thm. 36.7.15's hypotheses at $P = \infty$. Partial sums $\sum_{n \le N} n^{-s}$ fail more drastically: they have zeros in $\sigma > 1$ for large N. Montgomery [197] ([CITED, secondary], via [195]) shows that for $0 < c < \frac{4}{\pi} - 1$ there are zeros in $\sigma > 1 + c\frac{\log\log N}{\log N}$, so zeros cross every line $\sigma_0 < 1+$.

(D4) **Exponential line Θ_c** (Lemma 37, Thm. 4.1). $\Theta_c = \exp(-2c\,\xi'/\xi)$ has *essential* singularities at the off-line zeros, not poles, so Thm. 36.7.9 does not apply and there is no finite index. It is the $\omega \to 0$ limit of (D2) with $\nu\omega = c$, in which $\kappa = \nu N_+ \to \infty$ unless $N_+ = 0$. As an index problem it is all-or-nothing.

(D5) **Archimedean-only line** Θ_∞ (Lemma 34). It is not unimodular on $\mathbb{R}$ (Lemma 34: $\Theta_\infty(0) = 1336.5$ at $(0.3, 4)$), so it is not in the class of Thm. 36.7.9. As a base point F_0 it has nonzero index for $\omega < \frac{1}{2}$ (the pole at $s = 1 - \omega$). It is a wrong starting point, not a candidate.

(D6) **Spectral flow/APS** (Prop. 36.7.12). The escape count *is* a spectral flow ([THM]). An index *theorem*, i.e. a formula for sf in terms of data computable without zeros, would be a method; none is known ([ANALOGY]/[OPEN]). Any such formula must fail for the Davenport–Heilbronn-type function (Rem. 36.7.3), so it must use the Euler product.

target	sig.	prob.	score	status
(D1) index transport $t : \frac{1}{2} \to 0$ (=RH)	10	0.3	6.1	[RESTATEMENT], [OPEN]
(D1′) Polymath barrier as Thm. 36.7.15, plus shadow input	3	9	5.4	[PARTIAL], routine
(D2) uniform $\kappa(\omega_0) = 0$, some $\omega_0 < \frac{1}{2}$	9	0.5	5.6	[OPEN] (P3)
(D3) compute opens on truncated lines (Prop. 36.7.13)	4	8	5.6	[NUM] pending
(D3′) index continuity as $P \to \infty$	8	0.5	5.0	fails as stated
(D4) finite index for Θ_c	3	1	2.2	fails (essential sing.)
(D6) APS-type formula for sf without zeros	9	0.5	5.6	[ANALOGY], [OPEN]

TABLE 36.3. Deformation candidates. Score $= 0.6\,\text{sig} + 0.4\,\text{prob}$. The high-significance rows score high only through significance; their probabilities are near zero.

Recommendation. The cheap and informative item is (D3). Run Lemma 37's truncated-line code past the first short and look for the predicted opens. That tests Prop. 36.7.13 and the claim that the sign of the first exit is not topological. (D1′) is a clean write-up exercise. Neither moves RH.

36.8. Gap sentences: what each element supplies and what remains

(1) Causality/KK supplies dispersion relations and the Laplace representation of Z by $\mathcal{W}_+$ for free. What remains is stability on $\operatorname{Re} p > 0$, which is RH.

(2) Passivity supplies PR on $\sigma > 1$, right of the MTY curve, and outside the shadow half-discs (all $|t| \le 3 \cdot 10^{12}$). What remains is PR on the rest of $\sigma > \frac{1}{2}$, i.e. $\mathsf{Z}_{\text{off}} = 0$.

(3) Hilbert positivity supplies a Krein space with fundamental symmetry J_{FE} and index N_+. What remains is $N_+ = 0$.

(4) OS supplies the spectrum condition, a contraction semigroup, and an RP^+ tachyon. What remains is for the tachyon to dominate the Pauli–Villars tower plus prime atoms: $|\text{OS}_a| \le \mathcal{Q}$.

(5) KMS/modular supplies KMS_1 positivity of $\xi(1 + it)$ and prime-side BC positivity. What remains is a state positivity on the zero side, i.e. a flow-invariant positive half (OS4 + OS5).

(6) Complete monotonicity supplies exponential convexity of $\xi(\frac{1}{2} + \tau)$. What remains is CM of S_+, best obtained prime-side (Lemma 35, in preparation).

(7) Hermite–Biehler supplies E_h for $h \ge \frac{1}{2}$. What remains is to lower h to 0; even $h < \frac{1}{2}$ is P3. De Branges' stronger conditions are known to fail [76].

(8) Krein string supplies (Lemma 34, in preparation) the string of $E_{1/2}$. What remains is a positive string for E_0 built without the zeros.

(9) Hodge–Riemann supplies the integrality half at finite places. What remains is the positive half, which has no carrier at ∞.

(10) Weil supplies certified positivity on windows ≤ 1.6. What remains is all windows; they do not propagate.

(11) Index supplies $n_+(\mathcal{L}) \ge 1$ and the exact count $1 + N_+$. What remains is index exactly 1.

(12) Certificates supply $\lambda^* > 0$ at $A = 1.5, 1.6$ and mid-band sampling to $A = 2.8$. What remains is $A \to \infty$, at doubly exponential cost.

(13) Sum rule supplies a definite-sign deficit identity. What remains is to evaluate the absorption at $\sigma_0 < 1$ prime-side to within 2π.

(14) Edge supplies the reason the pole is the one allowed exceptional mode. What remains is definiteness of the bulk (H^1).

(15) Integrality (v2) supplies the 2π quantum, a quantization gap, homotopy invariance, and an exact count of line breaks. What remains is an index formula, or a crossing-free deformation, that sees the Euler product. Integrality alone is line-blind (Rem. 36.7.3).

36.9. The collapsed missing statement (G5)

All rows tagged [RESTATEMENT] are faces of one sentence (Lemma 33, §7):

$$\boxed{\nu = \textstyle\sum_{t_\rho > 0} m_\rho\, \delta_{\gamma_\rho} \text{ is a positive measure on } (0, \infty).}$$

As stated it mentions the zeros, so by G6 (non-circularity, Lemma 30) it is the *target*, not a method. Its sharpest zero-free forms are:

(G5a) **Domination** ([RESTATEMENT], Lemma 15 eq. (1)). For all $F \in C_c^\infty(\mathbb{R})$ with $H = F * F^*$,

$$H(0)\log\pi - \frac{1}{2\pi}\int_{\mathbb{R}} \hat{H}(r)\,\mathrm{Re}\,\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{ir}{2}\Big)dr + \sum_{n\geq 2}\frac{\Lambda(n)}{\sqrt{n}}\big(H(\log n) + H(-\log n)\big) \;\leq\; \hat{H}(\tfrac{i}{2}) + \hat{H}(-\tfrac{i}{2}).$$

The archimedean ghost tower plus the prime atoms never beat the single exceptional (pole) direction. Equivalently, $n_+(\mathcal{L}|_V) \leq 1$ for every V, or $|\mathrm{OS}_a| \leq \mathcal{Q}$.

(G5b) **Reflection elsewhere.** A pair (θ, E_+), other than the mirror and other than a half-line, whose OS quotient is Weil's Hilbert space and whose RP is proved from the Euler product (Lemma 32, §6).

(G5c) **Prime-side CM.** $(-1)^n S_+^{(n)}(\tau) \geq 0$ for all n and $\tau > 0$, proved from the Cramér/Guinand prime-side expression of S_+ (Post–Widder: Lemma 35, in preparation).

(G5d) **Passive realization.** A Krein string $m \geq 0$, or a Darlington/Cauer network with nonnegative elements, built from ζ's Euler product and Γ-factor, whose impedance is Z (Lemma 34, in preparation).

(G5a) is the only form written entirely in known data. (G5b)–(G5d) are *mechanisms*: each would prove (G5a).

36.10. Open questions

Question 36.10.1 (which face is the official target). Volume 2 should name one sharp G5. Options: (G5a) domination (the physicist's "the pole beats tower + atoms"), or (G5d) passive realization. Which matches TAA's physical intuition best?

Question 36.10.2 (edge atoms). The hybrid certificate fails at $A = 1.6$, and the failure is attributed to the atom $\log 5 \in (A, A + \varepsilon]$ (Lemma 24 log). Does the margin $\lambda^*(e^A)$ show a kink or dip each time a new prime power enters ($A = \log n$)? This is computable with existing code (Lemma 11 Galerkin). It would show the domination (G5a) becoming tight exactly at new atoms.

Question 36.10.3 (a smoothed sum rule). Theorem 36.4.3 with a sharp cutoff at $t = T$ produces the edge term. With a positive smooth weight $w(t)$ in place of $1_{[0,T]}$, is there a version whose deficit is still ≥ 0 and whose left side has a usable prime-side approximation at $\sigma_0 < 1$? Any such version is a zero-density tool. The physics question is whether some "optical theorem" fixes the total strength independently of ϑ.

QUESTION 36.10.4 (physical-state condition). Gupta's indefinite-metric QED [121] keeps positivity on a subspace cut out by a constraint and then quotients by null states. Lemma 32's OS quotient already deletes the off-line zeros in finite models. Is there a *zero-free* constraint (a "physical-state condition") whose solution space carries $\mathcal{Q} \geq 0$? This is (G5b) in other words, labeled [ANALOGY] until a constraint is written down.

QUESTION 36.10.5 (KKS with an energy from the explicit formula). Can one write a Hamiltonian system JL on a space built from $\mathcal{W}$ with $n(L) = N_+$ and spectrum $\{i\gamma_\rho\}$? If yes, Theorem 36.5.4 becomes a theorem about that system, and "RH = spectral stability" stops being an analogy. It would still be a restatement.

QUESTION 36.10.6 (v2: test the opens). Prop. 36.7.13 predicts that the truncated lines at $(0.3, 4)$ and $(0.2, 6)$ eventually have at least 2, resp. 3, eigenvalues above $+1$ (opens), in addition to the shorts of Lemma 37. Shall we run Lemma 37's truncated-line code past the first exit to test this? It is cheap and would show that "breaks are shorts" is a first-exit fact, not a structural one.

QUESTION 36.10.7 (v2: which "action"). Prop. 36.7.7 shows that the Maslov reading of the Riemann constant depends on what is called the action. Only the exact phase $\theta(T)$ gives an integer ($\mu = -2$, i.e. the $+1$ of $s(s-1)$). Do you want the TAA dictionary to fix $\theta(T)$ as "the action"? Then the $\frac{7}{8}$ is retired as a Maslov quantity, and $-\frac{1}{8}$ is recorded as a Stirling phase.

QUESTION 36.10.8 (v2: half a quantum per edge pole). The $+1$ splits as $\frac{1}{\pi}\arctan 2T$ (factor s) plus $1 - \frac{1}{\pi}\arctan 2T$ (factor $s-1$), i.e. $\frac{1}{2} + \frac{1}{2}$ asymptotically (Prop. 36.7.4). Is there a physical boundary condition (a hard wall has reflection phase π) that you would attach to each of the two edge poles? This is an [ANALOGY] until someone writes the boundary-value problem down.

QUESTION 36.10.9 (v2: priority among deformations). Table 36.3 ranks the deformations. Should Volume 2 record "RH as vanishing of an index" only as a restatement ([RESTATEMENT]), or should one of (D1′)/(D3) become a lemma?

36.11. Corrections to earlier claims

(a) "$A \leq 2.8$ certified" (Lemma 24 thread) refers to the *middle-band sampling inequality* ($L_{\min} = 0.450$, $c_{\mathrm{mid}} = 0.4256$, $c_{\mathrm{out}} = 0.00183$ at $A = 2.8$, $\theta = 0.05$). It is *not* Weil positivity on a window of length 2.8. Certified window positivity stands at $A = 1.6$.

(b) The hybrid Galerkin *minorant* is not positive at $A = 1.6$ and 2.4. This is a failure of the lower-bound form, not evidence that $\mathcal{Q} < 0$.

(c) Lemma 31's byline reads "TAA working notes". The project rule is "J. W. McGreevy; formalization by Grok Bot". (Not edited here.)

(d) Lemma 33, Question "Bode/Fano sum rule with definite sign": answered by Thm. 36.4.3. It exists, and it is Littlewood's lemma, so it is classical.

(e) (v2) Lemma 14, proof of Prop. 1.1: "the $+1$ comes from the pole of ζ at $s = 1$". Refined: it is $\frac{1}{\pi}\Delta \arg[s(s-1)] = 1$ exactly, shared between the factors s (pole of Λ at 0) and $s-1$ (pole at 1), asymptotically $\frac{1}{2}$ each (Prop. 36.7.4).

(f) (v2) The "Maslov $-\frac{1}{8}$" (Lemma 14, Heuristic 1.2; [245]) is not an integer Maslov index in the standard convention ($\mu = \frac{5}{2}$ with the cutoff area, $-\frac{3}{2}$ for the whole $\frac{7}{8}$). It is a Stirling phase (Prop. 36.7.7). Berry–Keating themselves leave α unspecified [30, §6].

(g) (v2) Lemma 34, Thm. 5.4: (ii) holds with *equality*, $n_\pm(R[t]) = \kappa/2$ for every $t > 0$, and (iv) holds *without* (M), with exact eventual counts $\kappa/2$ each way (Thm. 36.7.9).

(h) (v2) Lemma 37, "every break computed in TAA is a short": true as recorded[1], but only for first exits. For $\kappa_\Theta > 0$ opens must eventually occur (Cor. 36.7.10), and for the truncated lines with ν even at least $\nu/2$ opens are forced (Prop. 36.7.13).

[1] **[Vol. 3 editorial note]** Chapter 37 (v2) records the same correction as its item K9.

CHAPTER 37

Lemma 37 (v3): G5 as a passive line — the official target

Statement, equivalences, the exponential line, physical dictionary, proof strategies, and the finite Euler cascade

Standalone note `lemma37.tex`, October 22, 2026.

Summary. **Statement of intellectual honesty.** RH is not proven, here or anywhere in TAA. Every claim carries a tag: [THM] proved here (or an elementary consequence of cited results, proof given); [CITED] checked at the source (`lit/lemma37/verified_refs.md`); [CITED, secondary] checked only through a named secondary source; [BIB] bibliographic record only, content stated as standard background and not checked at the source; [RESTATEMENT] equivalent to RH (no progress); [ANALOGY] analogy only; [NUM] numerical (`num/lemma37_*.py`, data `num/l37/`); [CONJ] conjecture; [OPEN] open. Lemma 34's data and code are used read-only.

What J. W. decided. The transmission-line form (Suzuki's det-one canonical system, equivalently a lossless line or Krein string) is the official face of the missing axiom G5 (Lemma 30). This lemma fixes the statement and examines it. *v2:* J. W. (1) *accepts* the corrected official statement: $\|\mathcal{K}^{\omega,\nu}[t]\| < 1$ for every t, equivalently $\sup_t \leq 1$ (not "$\sup_t < 1$", which is false in the passive case); (2) makes the *exponential line* the headline form.

Results.

(1) *Passivity criterion* (Thm. 37.2.2, [THM], from Nehari's theorem). For every symbol in Suzuki's class, $\sup_{t\geq 0}\|\mathcal{K}[t]\| = \|\Theta\|_{H^\infty(\mathbb{C}_+)} \in [0,\infty]$ exactly. Each finite segment is passive ($\|\mathcal{K}[t]\| \leq 1$) iff some Schur function reproduces the first $2t$ of the arithmetic impulse response. *Correction* (Cor. 37.2.4): the formulation "$\sup_t\|\mathcal{K}[t]\| < 1$" in Lemma 34 §(d) and in the task is *false* even where the line is passive. At $\omega = \frac{1}{2}$ (unconditionally) and under RH for every ω, $\sup_t\|\mathcal{K}[t]\| = 1$, approached but never attained. The correct target is "$\|\mathcal{K}[t]\| < 1$ for every t", which is the same as $\sup_t \leq 1$.

(2) *Official statement and equivalence* (Def. 37.3.1, Thm. 37.3.2, [THM]; the RH clause is [RESTATEMENT]). $\mathrm{G5}_{\mathrm{line}}$: for all $\omega \in (0,\frac{1}{2})$ and $\nu > 1/\omega$, the line $L^{\omega,\nu}$ never breaks. Unconditionally, for each depth $h \in [0,\frac{1}{2})$: all zeros have $\beta \leq \frac{1}{2}+h \iff$ the lines with $\omega > h$ never break $\iff$ they are passive along any *one* sequence $\omega_n \downarrow h$. At $h = 0$ this is $\mathrm{G5}_{\mathrm{line}} \iff$ RH. Suzuki's endpoint condition $J \to 0$ is *not needed* for this direction. Passivity at depth ω does not depend on ν. It is the inequality $|\xi(s-\omega)| \leq |\xi(s+\omega)|$ on $\mathrm{Re}\, s > \frac{1}{2}$, a step-$2\omega$ version of Sondow–Dumitrescu monotonicity.

(3) *Boundary member* (Thm. 37.4.1, [THM]/[RESTATEMENT]). As $\omega \to 0$ with $\nu\omega = c$ fixed, $\Theta^{\omega,\nu} \to \exp(-2c\,\xi'/\xi)$ (error $O(\omega^2)$, [NUM]). The *exponential line* with this symbol is passive for one $c > 0$ iff RH. It is Lagarias' positive-real condition exponentiated, since $|e^{-2cZ}| \leq 1 \iff \mathrm{Re}\, Z \geq 0$. Its data on $[0,t]$ are exactly Weil's distribution on $[0,2t)$.

(4) *Lemma 24 window vs line* (Prop. 37.5.1, [THM]). $A = 2t$ holds precisely as an *information horizon*: both see exactly $\{\Lambda(n) : n < e^A\}$ and Γ. The functionals differ, though. The line has onsets at *every* integer n, but the onsets at non-prime-powers are multiple-scattering echoes of the prime-power atoms. The Weil window has atoms only at prime powers.

(5) *The finite Euler cascade* (Thm. 37.7.4, §37.7, J. W.'s strategy (a)). [THM]: every truncated line (archimedean section plus primes $\le P$) breaks at a finite $t_*(P)$ when $\omega \le \frac{1}{2}$. It survives *exactly* until the next prime p^+ arrives, $t_*(P) > \frac{1}{2}\log p^+$, unconditionally for $\omega \ge \frac{1}{2}$. [RESTATEMENT]: for every ω, "the next prime always arrives in time" (for all P) $\iff$ the line never breaks. So $\mathrm{G5}_{\mathrm{line}}$ is a countable list of finite-prime statements. [NUM] (v2, corrected; Tables 37.1, 37.5): margins $t_*(P) - \frac{1}{2}\log p^+$ decrease steadily, $0.052 \to 0.0094$ for $P \le 19$ at $(\frac{1}{2}, 3)$ (v1's $0.052 \to 0.020$ was too large, K8). On the resolution-converged range ($P \le 47$ and $P = 67$) they follow $\approx 0.0552\,(p^+)^{-0.56}$; to $P = 127$ we have upper estimates only (≤ 0.00520). *v3:* at $P = 97$ the $r36p14$ margin 0.0043502 is 7.3% below the $r30p12$ value 0.0046952, so $r30$ is *not* converged there; whether the decay slows beyond $p^+ \approx 70$ is *open* (M6.2–M6.3). [THM] (v2, Thm. 37.10.1): the margin is that of the *full* line with the single atom p^+ removed, so it depends on p^+ alone. [NUM]: no dependence on the prime gap, raw or Cramér-normalized; Lemma 39's mean-energy pattern is falsified. [THM]: without the archimedean section the line is broken from $t = 0^+$, and no single Euler factor is passive. The local sections are *stable but active*.

(6) *Strategies* (§37.11): assessed for circularity and plausibility; roadmap (§37.12); questions (§37.13).

Nothing here supplies G5.

Corrections/updates in v3 (October 22, 2026)

v2 is archived as `archive/lemma37.v2-final-before-v3-20261022-1251.{tex,pdf}`. v3 changes no theorem; it records one finished run and the data that completed after v2 was compiled. All numbers are read from `num/l37/m6_*.jsonl`; tables, macros and figures were regenerated by `num/run_l37_pipeline.sh`.

(v3.1) $r36p14$ *at* $P = 97$, $(\frac{1}{2}, 3)$, *finished* ([NUM]). Break $t_* = 2.3119105$ (bracket width $7 \cdot 10^{-7}$), margin 0.0043502. The $r24, r30, r36$ margins at $P = 97$ are 0.0054979, 0.0046952, 0.0043502: $r36$ lies $3.45 \cdot 10^{-4}$ (7.3%) below $r30$. At $P = 67$ the same two resolutions differ by $7.3 \cdot 10^{-7}$. *So* $r30$ *is not converged at* $P = 97$, and the resolution-converged window stays $P \le 47$ and $P = 67$ ($t \le 2.14$). The $r36$ value at $P = 97$ is itself an unconfirmed upper estimate (Ritz sense).

(v3.2) *The decay-slowing question is open* (corrects the v2 reading of (M6.3)). v2 said that *if* $r30$ were converged at $p^+ = 101$, the decay would slow beyond $q \approx 70$ ($m\sqrt{q}$ rising from 0.0433 to 0.0472). That premise is false. With $r36$, $m\sqrt{q} \le 0.0437$ at $q = 101$ (vs. 0.0433 at $q = 71$), and the $r36$ margin is 4.2% above the power-law prediction 0.00417 of the fit on the converged window. Since the $r24 \to r30 \to r36$ steps shrink only by a factor 0.43, the converged value may lie above or below the prediction; we do not know which. An $r42p16$ run at $P = 97$ (and $P = 67$ as a control) is queued[1] (§37.10, (M6.2)).

(v3.3) *Data finished after v2 was compiled, now included* ([NUM]). Exponential line $\Theta_{1.5}$: $r24p10$ now to $P = 127$; $r30p12$ at $P = 7, 19, 31, 47, 67, 97$ agrees with $r24p10$ to $\le 2.2 \cdot 10^{-6}$ (up to $t = 2.32$). $(1.5, 1)$: $r30p12$ now also at $P = 47, 67, 97$, still differing from $r24$ by 3.0–$3.2 \cdot 10^{-4}$ (not converged, as in (M6.6)). The regenerated gap tests and fits change only in the second digit and *no verdict changes* (no gap dependence; L39 pattern falsified, its $P = 97$ ratio now 2.6).

(v3.4) *Minor numerical correction.* v2 quoted the $P = 67$ agreement of $r36$ and $r30$ as $8 \cdot 10^{-7}$; the jsonl values give $7.3 \cdot 10^{-7}$ (v2 subtracted rounded values).

[1] **[Vol. 3 editorial note]** Run r42 was still running when this volume was built (the $P = 97$ run started on 22 October 2026; $P = 67$ follows). Its result is placeholder P1 of Appendix B: **[pending: P1,** $P = 97$] ($P = 97$) and **[pending: P1,** $P = 67$] ($P = 67$). No value is filled in until the run finishes.

Revision note (v2, October 22, 2026)

v1 is archived as `archive/lemma37.before-v2-20261021-222246.{tex,pdf}`. Changes: (i) J. W.'s two decisions on v1's Questions 1–2 are recorded (in the abstract, and STATE §46 addendum). (ii) The *exponential line* $\Theta_c = \exp(-2c\,\xi'/\xi)$ is now the *headline* form of $\mathrm{G5}_{\mathrm{line}}$ (Def. 37.4.3); the Suzuki family is its regularization. (iii) New numerics, all [NUM], and the headline definition (Def. 37.4.3, Prop. 37.4.4): M7, the exponential line (§37.8); M6, high-precision cascade margins to $P = 127$ with error bars, the gap-dependence question and the Lemma 39 model test (§37.10); D3, the lines past their first exit, opens and shorts (§37.9). M6 and the M7 breaks at $t > 0.8$ use a ball-arithmetic solver (`python-flint arb`, 480 bits), independent of but cross-checked against Lemma 34 v2's. (iv) A new theorem (Thm. 37.10.1, [THM]): each cascade margin depends on the next prime p^+ only, not on P or on the gap below p^+. (v) *Corrections* K8–K13 (§37.14): v1's Table 1 margins were too large for $t \gtrsim 0.8$ (Lemma 34 V7); "every break is a short" holds for *first* exits only; the $(0.3, 4)/(0.2, 6)$ agreement was overstated; Γ in Remark 37.7.7 is $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$; Lemma 39's margin pattern fails on the corrected data. Sections 1–6 are otherwise unchanged from v1. The abstract's item (5) reflects the corrected numerics.

37.1. Conventions and inputs

$s = \frac12 - iz$, $p = s - \frac12 = -iz$, so $\mathbb{C}_+ = \{\operatorname{Im} z > 0\} = \{\operatorname{Re} s > \frac12\}$. Zeros $\rho = \beta + i\gamma$ with multiplicity m_ρ; $\beta^* = \sup \beta_\rho$. (Lemma 36's Θ for $\sup\beta$ is renamed β^* here, because Θ is Suzuki's symbol.) $\xi(s) = \frac12 s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and $\gamma(s) = \xi(s)/\zeta(s) = (s-1)\pi^{-s/2}\Gamma(\frac{s}{2}+1)$, as in Lemma 34.

Suzuki's line (Lemma 34 §2; [257], [CITED]). For $\omega > 0$, $\nu \in \mathbb{Z}_{>0}$, $\nu\omega > 1$ ("admissible"):

$$\Theta^{\omega,\nu}(z) = \Big(\frac{\xi(\frac12 - \omega - iz)}{\xi(\frac12 + \omega - iz)}\Big)^\nu = \Big(\frac{\xi(s-\omega)}{\xi(s+\omega)}\Big)^\nu, \qquad K(x) = \frac{1}{2\pi}\int_{\operatorname{Im} z = v} \Theta^{\omega,\nu}(z) e^{-ixz}\,dz,$$

so that $\Theta^{\omega,\nu}(z) = \int_0^\infty K(x)e^{ixz}dx$ for $\operatorname{Im} z > v_0$. Here $K = 0$ on $x < 0$ and $K = \sum_{n \le e^x} c_\nu(n) n^{-1/2} G(x - \log n)$ with $c_\nu > 0$ (Lemma 34, Prop. 2.3). The operator $\mathcal{K}[t]f(x) = \int_{-t}^t K(x+y)f(y)\,dy$ acts on $L^2(-t,t)$, and $Z_0(t) = (\det(1+\mathcal{K}[t])/\det(1-\mathcal{K}[t]))^2$ gives $H = \operatorname{diag}(Z_0^{-1}, Z_0)$. We use: Thm. 2.1 (Hilbert–Schmidt, continuous kernel), Prop. 5.1 ($E \in \overline{HB} \Rightarrow \|\mathcal{K}[t]\| < 1\ \forall t$), Prop. 2.2, Thm. 2.4, the locality remark ($\mathcal{K}[t]$ depends on $K|_{[0,2t)}$), and the bound (4.3) ($\Theta^{\omega,\nu}$ bounded on $\operatorname{Im} z \ge v_0$).

Definition 37.1.1 (symbol class $\mathfrak{S}$; [DEF]). $\Theta \in \mathfrak{S}$ if Θ is holomorphic and bounded on $\{\operatorname{Im} z > v_0\}$ for some $v_0 \ge 0$, and its kernel K (defined as above, as a distribution) vanishes on $(-\infty, 0)$ and gives bounded operators $\mathcal{K}[t]$. A *break* at t means $\|\mathcal{K}[t]\| \ge 1$; the break length is $t_* = \inf\{t : \|\mathcal{K}[t]\| \ge 1\} \in (0, \infty]$.

Suzuki's symbols, the archimedean-only symbol $\Theta_\infty = (\gamma(s-\omega)/\gamma(s+\omega))^\nu$, its Euler truncations (§37.7), and the exponential symbol of §37.4 are all in $\mathfrak{S}$, by Stirling and the finite Euler factors being bounded on $\operatorname{Re} s \ge \frac12 + v_0$ ([THM], elementary). For $\nu\omega > 1$ (resp. $c > 1$) the kernels are continuous and $\mathcal{K}[t]$ is compact. Then a break at t_* is an eigenvalue ± 1, i.e. $\det(1 \mp \mathcal{K}[t_*]) = 0$, and $\|\mathcal{K}[t]\|$ is continuous in t.

Lemma 37.1.2 (monotone sections; [THM]). *For $s < t$, $P_s\mathcal{K}[t]P_s = \mathcal{K}[s]$, where P_s is restriction to $(-s, s)$. Hence $t \mapsto \|\mathcal{K}[t]\|$ and $t \mapsto \lambda_{\max}(\mathcal{K}[t])$ are nondecreasing, and $\lambda_{\min}(\mathcal{K}[t])$ is nonincreasing. Once the line has broken it stays broken.*

Proof. The kernel of $P_s\mathcal{K}[t]P_s$ is $K(x+y)$ on $(-s,s)^2$. Compression does not increase the norm, and the max/min of a Rayleigh quotient over a subspace is bounded by those over the whole space. □

37.2. The passivity criterion (Nehari)

THEOREM 37.2.1 (Nehari, half-line form; [224, Ch. 1, §8, Thm. 8.1], statement checked in [55]; [CITED, secondary]). *Let f be a distribution on $\mathbb{R}_+$. Then $\Gamma_f g(a) = \int_0^\infty f(a+b)g(b)\,db$ is bounded on $L^2(\mathbb{R}_+)$ iff $f = \hat{b}|_{\mathbb{R}_+}$ for some $b \in L^\infty(\mathbb{R})$, and b can be chosen with $\|b\|_\infty = \|\Gamma_f\|$.*

We also use the standard Hardy-space fact: $b \in L^\infty(\mathbb{R})$ has inverse Fourier transform supported in $[0,\infty)$ iff b is the boundary function of some $\psi \in H^\infty(\mathbb{C}_+)$, and then $\|\psi\|_{H^\infty(\mathbb{C}_+)} = \|b\|_\infty$, with $\psi(z) = \langle \check{b}, e^{ixz}\rangle$ for $\operatorname{Im} z > 0$ ([224, Ch. 1], [219, Ch. 4]; [BIB], standard).

THEOREM 37.2.2 (passivity criterion; [THM]). *Let $\Theta \in \mathfrak{S}$ with kernel K.*

(i) (finite horizon) $\|\mathcal{K}[t]\| = \min\{\|\psi\|_{H^\infty(\mathbb{C}_+)} :\ \psi \in H^\infty(\mathbb{C}_+),\ \check{\psi} = K \text{ on } (-\infty, 2t)\}$.

(ii) (ultimate gain) $\sup_{t\ge 0}\|\mathcal{K}[t]\| = \|\Theta\|_{H^\infty(\mathbb{C}_+)}$. The right side is the sup over $\mathbb{C}_+$ of the holomorphic continuation of Θ if that continuation exists and is bounded, and $+\infty$ otherwise.

PROOF. *Reduction.* $J_t f(a) = f(t-a)$ is unitary $L^2(-\infty,t) \to L^2(\mathbb{R}_+)$, and $\mathcal{K}[t]$ annihilates $L^2(-\infty,-t)$ (Lemma 34, §2). Moreover $J_t\mathcal{K}[t]J_t^{-1} = \Gamma_{k_t}$ with $k_t(a) = K(2t-a)$, $a>0$.

(i) "$\le$": if $\psi \in H^\infty(\mathbb{C}_+)$ and $\check{\psi} = K$ on $(-\infty,2t)$, then $b(\lambda) = e^{2it\lambda}\psi(\lambda)$, up to the normalization of the Fourier transform and the reflection $\lambda \mapsto -\lambda$, satisfies $\hat{b} = k_t$ on $\mathbb{R}_+$. The easy half of Thm. 37.2.1, which is Plancherel and Cauchy–Schwarz, gives $\|\Gamma_{k_t}\| \le \|b\|_\infty = \|\psi\|$. "$\ge$": Thm. 37.2.1 gives b with $\|b\|_\infty = \|\mathcal{K}[t]\|$ and $\hat{b} = k_t$ on $\mathbb{R}_+$. Undo the reflection and translation: the distribution $K_t^*(x) := \hat{b}(2t-x)$ has bounded Fourier transform of sup-norm $\|b\|_\infty$, and $K_t^* = K$ on $(-\infty,2t)$. In particular $K_t^* = 0$ on $(-\infty,0)$, so by the Hardy-space fact its Fourier transform is the boundary function of some $\psi_t \in H^\infty(\mathbb{C}_+)$ with $\|\psi_t\| = \|\mathcal{K}[t]\|$, attaining the minimum.

(ii) "$\le$": if Θ continues to $\Theta^\sharp \in H^\infty(\mathbb{C}_+)$, the distributions $\check{\Theta}^\sharp$ and K are both supported in $[0,\infty)$ and have the same Laplace transform on $\operatorname{Im} z > v_0$, so they coincide. Then (i) with $\psi = \Theta^\sharp$ gives $\|\mathcal{K}[t]\| \le \|\Theta^\sharp\|$ for all t. "$\ge$": assume $M = \sup_t\|\mathcal{K}[t]\| < \infty$ and take ψ_t from (i), so $\|\psi_t\| \le M$. Fix $v_1 > v_0$ and z_0 with $\operatorname{Im} z_0 > v_1$. The distribution $D_t = K - \check{\psi}_t$ is supported in $[2t,\infty)$. Also $e^{-v_1 x}D_t$ has Fourier transform $\Theta(\cdot + iv_1) - \psi_t(\cdot + iv_1)$, bounded by $C_1 = \sup_{\operatorname{Im} z = v_1}|\Theta| + M$. Let χ be smooth, $=1$ on $[-\tfrac12,\infty)$ and $=0$ on $(-\infty,-1]$, and put $g_t(x) = \chi(x-2t)e^{ix(z_0 - iv_1)}$. This is a Schwartz function (it decays like $e^{-(\operatorname{Im} z_0 - v_1)x}$ with all derivatives), and $g_t = e^{2it(z_0-iv_1)}g_0(\cdot - 2t)$. Hence

$$|\Theta(z_0) - \psi_t(z_0)| = |\langle e^{-v_1 x}D_t, g_t\rangle| \le \tfrac{1}{2\pi}C_1\|\hat{g}_0\|_{L^1}\, e^{-2t(\operatorname{Im} z_0 - v_1)} \longrightarrow 0.$$

The family $\{\psi_t\}$ is bounded by M on $\mathbb{C}_+$. By Vitali, a subsequence converges locally uniformly on $\mathbb{C}_+$ to some ψ with $\|\psi\| \le M$, and $\psi = \Theta$ on $\operatorname{Im} z > v_1$. So Θ continues to $\mathbb{C}_+$ with $\|\Theta\|_{H^\infty(\mathbb{C}_+)} \le M$. □

COROLLARY 37.2.3 (passive $\iff$ Schur; [THM]). *For $\Theta \in \mathfrak{S}$ with compact sections: $\|\mathcal{K}[t]\| < 1$ for all t $\Rightarrow$ $\sup_t\|\mathcal{K}[t]\| \le 1$ $\iff$ Θ is a Schur function on $\mathbb{C}_+$ (holomorphic, $|\Theta| \le 1$). For Suzuki's symbols with $E^{\omega,\nu} \in \overline{HB}$, Schur $\Rightarrow$ $\|\mathcal{K}[t]\| < 1$ for all t (Prop. 5.1 of [257]), so all three are equivalent.*

COROLLARY 37.2.4 (the uniform strict form is false; [THM]; correction to Lemma 34 §(d)). *If Θ is a nonconstant inner function (Schur, $|\Theta| = 1$ a.e. on $\mathbb{R}$), then $\sup_t\|\mathcal{K}[t]\| = 1$, but $\|\mathcal{K}[t]\| < 1$ for each t when $E \in \overline{HB}$. This applies to $\Theta^{1/2,\nu}$ unconditionally (Lemma 34, Prop. 2.4), and to every admissible $\Theta^{\omega,\nu}$ under RH. So "$\sup_t\|\mathcal{K}^{\omega,\nu}[t]\| < 1$" never holds in the passive case. Lemma 34's numerics agree: the top gap $1 - \lambda_{\max}$ collapses ($8.6\cdot10^{-2}$ at $t = 0.25$, $2.2\cdot10^{-4}$ at 0.5; Lemma 34, Table 1, [NUM]).*

PROOF. $\|\Theta\|_{H^\infty(\mathbb{C}_+)} = \operatorname{ess\,sup}_{\mathbb{R}}|\Theta| = 1$; apply Thm. 37.2.2(ii). □

COROLLARY 37.2.5 (ultimate gain of the archimedean line; [THM] + [NUM]). *For $\omega < \frac{1}{2}$, Θ_∞ has a pole of order ν at $z = i(\frac{1}{2} - \omega)$, i.e. at $s = 1 - \omega$, coming from the factor $(s - 1 + \omega)^{-\nu}$ of $\gamma(s + \omega)^{-\nu}$. So $\sup_t \|\mathcal{K}_\infty[t]\| = \infty$. For $\omega = \frac{1}{2}$ the pole sits at $z = 0 \in \mathbb{R}$, and again the sup is ∞. For $\omega > \frac{1}{2}$, $\Theta_\infty \in H^\infty(\mathbb{C}_+)$, and*

$$\sup_t \|\mathcal{K}_\infty[t]\| = \sup_{u \in \mathbb{R}} \Big| \frac{\zeta(\frac{1}{2} + \omega - iu)}{\zeta(\frac{1}{2} - \omega - iu)} \Big|^\nu,$$

using the modulus identity of Lemma 34 §4. At $(1.5, 1)$ a scan of $u \in [0, 60]$ (`num/l37/archsup.json`) gives the maximum $2\pi^2 = 19.739\ldots$ at $u = 0$, where $\Theta_\infty(0) = \zeta(2)/\zeta(-1) = -2\pi^2$ [NUM]. The Ritz extremes of $\mathcal{K}_\infty[t]$ climb toward it: $\lambda_{\min} = -1.54, -5.47, -12.34, -17.23$ at $t = 0.5, 1, 2, 4$, stable to 10^{-5} under $h = \frac{1}{100} \to \frac{1}{400}$ (`num/l37/gain_arch15.json`, [NUM]). The negative sign of $\Theta_\infty(0)$ matches Lemma 34's observed exit through -1 [ANALOGY] (heuristic).

37.3. The official statement and its equivalence with RH

DEFINITION 37.3.1 ($\mathrm{G5}_{\mathrm{line}}$, the official G5; [DEF]). For admissible (ω, ν) call $L^{\omega,\nu}$ *passive* if $\|\mathcal{K}^{\omega,\nu}[t]\| < 1$ for every $t \geq 0$. Equivalently, $\det(1 \pm \mathcal{K}^{\omega,\nu}[t]) \neq 0$ for all t, or $Z_0^{\omega,\nu}$ is defined, finite and positive on $[0, \infty)$; the equivalence uses compactness, continuity and Lemma 37.1.2.

$$\boxed{\mathrm{G5}_{\mathrm{line}}: \quad \text{for every } \omega \in (0, \tfrac{1}{2}) \text{ and every integer } \nu > 1/\omega,\ L^{\omega,\nu} \text{ is passive,}}$$

proved from the Euler product and the Γ-factor, without the zeros (axiom G6 of Lemma 30). The endpoint-augmented form $\mathrm{G5}^+_{\mathrm{line}}$ adds Suzuki's condition $J(t; z, z) \to 0$ $(t \to \infty,\ z \in \mathbb{C}_+)$ along some sequence $\omega_n \downarrow 0$.

THEOREM 37.3.2 (depth theorem; [THM]). *Fix $h \in [0, \frac{1}{2})$. The following are equivalent:*

(a) every zero has $\beta \leq \frac{1}{2} + h$;
(b) $L^{\omega,\nu}$ is passive for every admissible (ω, ν) with $\omega > h$;
(c) there is a sequence $\omega_n \downarrow h$, $\omega_n > h$, with admissible ν_n, such that $\sup_t \|\mathcal{K}^{\omega_n,\nu_n}[t]\| \leq 1$ for every n.

Moreover, for each admissible (ω, ν), $\sup_t \|\mathcal{K}^{\omega,\nu}[t]\| \leq 1$ iff

$$|\xi(s - \omega)| \leq |\xi(s + \omega)| \qquad (\mathrm{Re}\, s > \tfrac{1}{2}), \tag{37.3.1}$$

a condition independent of ν. Equivalently, $|\xi(x + i\tau)| \leq |\xi(x + 2\omega + i\tau)|$ for all $x > \frac{1}{2} - \omega$ and all τ.

PROOF. The criterion: by Cor. 37.2.3, sup ≤ 1 iff $\Theta^{\omega,\nu} = (\xi(s - \omega)/\xi(s + \omega))^\nu$ is Schur on $\mathrm{Re}\, s > \frac{1}{2}$. Schur means: holomorphic with $|\cdot| \leq 1$. A ν-th power is Schur iff its base is, since poles and moduli > 1 are preserved. Its continuation is the meromorphic function itself, so this is (37.3.1) (a pole would violate the bound). The second form substitutes $s = x + \omega + i\tau$ and uses nothing else.

(a)⇒(b): for $\omega > h \geq \beta^* - \frac{1}{2}$, $E_\omega(z) = \xi(\frac{1}{2} + \omega - iz)$ is HB by Lemma 33, Thm. 6.2(a). It has no real zeros, because a real zero means a zero on $\mathrm{Re}\, s = \frac{1}{2} + \omega$, and $\beta \leq \frac{1}{2} + h < \frac{1}{2} + \omega$. Powers preserve both properties, so $E^{\omega,\nu} \in \overline{HB}$, and Suzuki's Prop. 5.1 gives $\|\mathcal{K}[t]\| < 1$ for all t. (b)⇒(c) is trivial (take $\nu_n = \lceil 1/\omega_n \rceil + 1$).

(c)⇒(a): suppose ρ has $\beta = \frac{1}{2} + \delta$ with $\delta > h$. For n large, $\omega_n < \delta$, and $s_n = \rho - \omega_n$ has $\mathrm{Re}\, s_n = \frac{1}{2} + \delta - \omega_n > \frac{1}{2}$. At s_n the denominator $\xi(s + \omega_n)$ vanishes to order m_ρ. By (37.3.1) Θ^{ω_n,ν_n} is bounded, so the numerator must vanish there to at least that order: $\xi(\rho - 2\omega_n) = 0$. These are zeros at the common height $\mathrm{Im}\, \rho$ with real parts $\beta - 2\omega_n \in (\beta - 2\omega_1, \beta)$, pairwise distinct. Infinitely many zeros of the entire function $\xi \not\equiv 0$ in a compact set is impossible. □

COROLLARY 37.3.3 ($\mathrm{G5}_{\mathrm{line}} \iff$ RH; [THM] for the logic, [RESTATEMENT] in content). $\mathrm{G5}_{\mathrm{line}} \iff \mathrm{G5}^{+}_{\mathrm{line}} \iff$ *RH* $\iff$ *passivity along* one *sequence* $\omega_n \downarrow 0$. *The endpoint condition is not needed to reach RH: Thm. 37.3.2(c) at* $h = 0$ *suffices. It is implied by RH via Suzuki's Thm. 2.4 (*$\Rightarrow$ *direction).*

REMARK 37.3.4 (what this does and does not add; [THM]/[ANALOGY]). (1) Thm. 37.3.2 slightly strengthens Lemma 33, Thm. 6.2(b) and Lemma 36's spine (Thm. 3.1). One sequence $\omega_n \downarrow h$, not all $h' > h$, suffices, and it is phrased in line language. (2) For a *single* ω, passivity allows "ladders": it forces every zero with $\beta > \frac{1}{2} + \omega$ to be accompanied by a zero at $\rho - 2\omega$ of at least the same multiplicity, so iterating gives a horizontal chain with step 2ω ending in $|\beta - \frac{1}{2}| \le \omega$. This is why a sequence or interval of ω is needed. (3) Condition (37.3.1) is a discrete-step form of the Sondow–Dumitrescu monotonicity: $|\xi|$ increases strictly along horizontal half-lines in zero-free right half-planes, and RH $\iff$ it increases on $\sigma > \frac{1}{2}$ [**249**] ([CITED], abstract). As $\omega \to 0$, (37.3.1) for all small ω becomes that monotonicity. So $\mathrm{G5}_{\mathrm{line}}$ is a physical dressing of a known RH-equivalent. Its *value* lies elsewhere: in the causal filtration by t with finite prime data (§37.7), and in the layer-peeling dictionary (§37.6). (4) P3 in line form: some interval of ω below $\frac{1}{2}$ with passive lines $\iff$ P3 (Lemma 25) [RESTATEMENT].

37.3.1. Relations to the other faces.

- **Lagarias PR / Lemma 33.** $\mathrm{G5}_{\mathrm{line}}$ at depth h is the spine of Lemma 36, Thm. 3.1, (i) $\iff$ (ii) $\iff$ (iii), now with a fourth equivalent (iv): passive lines for $\omega \downarrow h$. The Cayley transform $Z_\omega = (1 - \Theta^{\omega,1})/(1 + \Theta^{\omega,1}) = \frac{\xi(s+\omega) - \xi(s-\omega)}{\xi(s+\omega) + \xi(s-\omega)}$ is positive-real on $\mathrm{Re}\, s > \frac{1}{2}$ iff the line at depth ω is passive (Cayley maps Schur to Carathéodory, [THM]). Moreover $Z_\omega/\omega \to \xi'/\xi$ as $\omega \to 0$, the impedance of Lemma 33.
- **(G5a) domination, Lemma 32 OS form.** Weil positivity is "$|\mathrm{OS}_a(F)| \le \mathcal{Q}(F)$" (Lemma 32, Thm. 4.2): a *Hankel* form of W dominated by the *Toeplitz* form of W. Segment passivity is $|\langle \mathcal{K}[t] f, \bar{f} \rangle| \le \|f\|^2$: a Hankel form of K dominated by the identity. Both are "reflection bounded by transmission" [ANALOGY]. For the exponential line the two kernels are linked exactly: $K_c = \exp_*(-2cW_+)$ (convolution exponential, Thm. 37.4.1). Both are $\iff$ RH. No implication between their *finite-horizon* versions is known [OPEN].
- **Lemma 24 certificates.** See Prop. 37.5.1.
- **Lemma 34 escape theorem.** Under (B), (ND), (M), $\kappa > 0 \Rightarrow$ break. Thm. 37.3.2 gives the converse direction without those hypotheses (no break along $\omega_n \downarrow 0 \Rightarrow$ RH).

37.4. The boundary member: the exponential line

THEOREM 37.4.1 (exponential line; [THM], last clause [RESTATEMENT]). *For* $c > 0$ *put* $\Theta_c(s) = \exp(-2c\,\xi'(s)/\xi(s))$, *holomorphic on* $\mathbb{C}$ *minus the zeros.*

(i) $\Theta_c \in \mathfrak{S}$: $|\Theta_c| < 1$ *on* $\mathrm{Re}\, s > 1$ *(since* $\mathrm{Re}\,\xi'/\xi > 0$ *there, Lemma 33/P0), and* $|\Theta_c| = 1$ *on* $\mathrm{Re}\, s = \frac{1}{2}$ *off the zeros, unconditionally (there* $\mathrm{Re}\,\xi'/\xi = 0$, *because* $\xi(\frac{1}{2} + i\tau)$ *is real).*

(ii) $\Theta_c = \exp(-2c\gamma'/\gamma) \cdot \exp(2c \sum_n \Lambda(n) n^{-s})$, *with* $\exp(2c \sum \Lambda(n) n^{-s}) = \sum_n a_c(n) n^{-s}$, $a_c(1) = 1$, *and* $a_c(n) \log n = \sum_{d \mid n, d > 1} 2c\Lambda(d) \log d$ $a_c(n/d) > 0$ *for all* $n \ge 1$. *So* $K_c = \sum_n a_c(n) n^{-1/2} G_c(x - \log n)$, *with* $G_c \sim C x^{c-1}/\Gamma(c)$ *at* 0^+ *(continuous for* $c > 1$*).*

(iii) $\Theta^{\omega, c/\omega} \to \Theta_c$ *locally uniformly on* $\mathrm{Re}\, s > 1$ *as* $\omega \to 0$ *through* $c/\omega \in \mathbb{Z}$, *with error* $O(\omega^2)$. *[NUM]: relative errors* $2.3{\cdot}10^{-5}, 1.0{\cdot}10^{-5}, 2.6{\cdot}10^{-6}, 6.5{\cdot}10^{-7}$ *at* $s = 2$, $\omega = 0.3, 0.2, 0.1, 0.05$*; ratio* ≈ 4 *per halving (*`num/l37/explimit.json`*). Truncated lines at* $(0.3, 4)$ *and* $(0.2, 6)$ *(both* $c = 1.2$*) agree to* $\le 3 \cdot 10^{-4}$ *in all Ritz extremes (§37.7).* v2: *their breaks differ from the exponential line's by* $\approx 3.2 \cdot 10^{-3} \omega^2$ *(§37.8, (M7.2)).*

(iv) $\sup_t \|\mathcal{K}_c[t]\| \le 1 \iff$ *RH, for any* single $c > 0$.

(v) $\mathcal{K}_c[t]$ *is a function of* $W_+|_{[0,2t)}$, *the restriction of Weil's distribution (Lemma 32), via* $K_c = \exp_*(-2cW_+) = \sum_k (-2c)^k W_+^{*k}/k!$.

PROOF. (i) is in the statement. (ii): $\xi'/\xi = \gamma'/\gamma + \zeta'/\zeta$ and $-\zeta'/\zeta = \sum \Lambda(n) n^{-s}$. The recursion comes from differentiating $F = e^g$ ($F' = g'F$) termwise as Dirichlet series. Positivity follows by induction. The behaviour at 0^+ comes from $\gamma'/\gamma(s) \sim \frac{1}{2}\log s$ (Stirling), so $\exp(-2c\gamma'/\gamma) \asymp s^{-c}$. (iii): $\nu[\log \xi(s-\omega) - \log \xi(s+\omega)] = -2\nu\omega\, \xi'/\xi(s) + O(\nu\omega^3)$. (iv) "$\Leftarrow$": under RH, $\operatorname{Re} \xi'/\xi(s) = \sum_\rho \operatorname{Re} \frac{1}{s-\rho} > 0$ for $\operatorname{Re} s > \frac{1}{2}$ (Hadamard; [**168**, Thm. 1.1] with $\theta = \frac{1}{2}$), so Θ_c is Schur; apply Thm. 37.2.2(ii). "$\Rightarrow$": Thm. 37.2.2(ii) gives a bounded holomorphic continuation to $\operatorname{Re} s > \frac{1}{2}$. This continuation is Θ_c itself on the connected set $\{\operatorname{Re} s > \frac{1}{2}\} \setminus \{\rho\}$. If $\beta_\rho > \frac{1}{2}$, then $-2c\,\xi'/\xi(s) = -2cm_\rho/(s-\rho) + O(1)$, which tends to $+\infty$ as $s = \rho - \varepsilon$, $\varepsilon \downarrow 0$. So $|\Theta_c|$ is unbounded, a contradiction. (v): $\xi'/\xi(\frac{1}{2}+p) = \int e^{-pu} W_+(u)\,du$ (Lemma 33, Thm. 3.2), and convolution powers of distributions supported in $[0,\infty)$ restrict to $[0,X)$ through their restrictions to $[0,X)$. □

REMARK 37.4.2 (reading; [ANALOGY] for the physics, [THM] for the algebra). $\Theta_c = e^{-2cZ}$ with $Z = \xi'/\xi(\frac{1}{2}+p)$ has the form of the transmission factor $e^{-\Gamma\ell}$ of a uniform slab of length $\ell = c$ with propagation constant $\Gamma = 2Z$. Passivity is "$\operatorname{Re}\Gamma \geq 0$, no gain", and $\Theta_{c_1+c_2} = \Theta_{c_1}\Theta_{c_2}$ is the cascade of slabs. The exponential line is the cleanest single-object statement of $\mathrm{G5}_{\mathrm{line}}$, with no limit $\omega \to 0$ left in it. Whether a det-one canonical system with symbol Θ_c exists (Suzuki's construction requires $\Theta = E^\#/E$ with E entire, and Θ_c is not meromorphic) is [OPEN]. Its Hankel passivity (Thm. 37.4.1(iv)) does not need it.

DEFINITION 37.4.3 (headline form of $\mathrm{G5}_{\mathrm{line}}$ (v2, J. W.'s decision); [DEF]). For $c > 0$ call the exponential line L_c *passive* if $\|\mathcal{K}_c[t]\| < 1$ for every $t \geq 0$, where $\mathcal{K}_c[t]$ is the Hankel section with kernel K_c of Thm. 37.4.1(ii).

$$\boxed{\mathrm{G5}_{\exp}: \quad \text{for some } c > 0,\ L_c \text{ is passive,}}$$

proved from the Euler product and the Γ*-factor, without the zeros.* The weak form $\mathrm{G5}^{\leq}_{\exp}$ asks only $\sup_t \|\mathcal{K}_c[t]\| \leq 1$.

PROPOSITION 37.4.4 (status of the headline form; [THM], content [RESTATEMENT]). *(a)* $\mathrm{G5}_{\exp} \Rightarrow \mathrm{G5}^{\leq}_{\exp} \iff$ *RH, and in* $\mathrm{G5}^{\leq}_{\exp}$ *"some c" and "every c" are equivalent. (b) RH* $\Rightarrow \sup_t \|\mathcal{K}_c[t]\| \leq 1$ *for every c. Whether RH also gives the strict pointwise bound* $\|\mathcal{K}_c[t]\| < 1$ *at each finite t is [OPEN] for* Θ_c*: for the Suzuki family the step is Suzuki's Prop. 5.1 (it needs* $\Theta = E^\#/E$ *with* $E \in \overline{HB}$*), and* Θ_c *is not of that form (it is not meromorphic). (c) For the Suzuki family, Def. 37.3.1 (pointwise) and the* sup ≤ 1 *form coincide, so J. W.'s accepted statement and the headline form differ only by this open step at the boundary member.*

PROOF. (a) Pointwise < 1 gives sup ≤ 1. $\sup_t \|\mathcal{K}_c[t]\| \leq 1 \iff \Theta_c$ Schur on $\mathbb{C}_+$ (Thm. 37.2.2) $\iff$ RH (Thm. 37.4.1(iv)); the last equivalence holds for each single c, hence "some" $\iff$ "every". (b) is (a) read backwards. (c) is Def. 37.3.1 with Cor. 37.2.3. □

Numerics (§37.8, (M7.3), [NUM]): no Ritz value of $\mathcal{K}_{1.5}[t]$ leaves $(-1,1)$ for $t \leq 2.45$, consistent with the strict form.

37.5. The Lemma 24 window and the line: is $A = 2t$?

PROPOSITION 37.5.1 (horizon matching; [THM]). *(a) Lemma 24's window margin* $\lambda^*(e^A)$ *(Weil's form on F supported in an interval of length A) is a function of* $W|_{(-A,A)}$*. Since W is even, this means* $W_+|_{[0,A)}$*, i.e. of* Γ *and* $\{\Lambda(n) : n < e^A\}$*. (b)* $\mathcal{K}^{\omega,\nu}[t]$ *is a function of G and* $\{c_\nu(n) : n < e^{2t}\}$*. Since* c_ν *is multiplicative and determined by the primes below n, this is again a function of* Γ *and* $\{\Lambda(n) : n < e^{2t}\}$*. (c)* $\mathcal{K}_c[t]$ *is a function of* $W_+|_{[0,2t)}$ *(Thm. 37.4.1(v)). Hence* $A = 2t$ *holds exactly as an* information horizon*. Physically it is the round trip: a pulse at unit speed probes depth t by time 2t [ANALOGY]. (d) The* functionals *differ.* λ^* *can change character only at* $A = \log n$ *with* $\Lambda(n) \neq 0$ *(prime powers). The line has onsets at* every *n (Lemma 34,*

Thm. 4.1), since $c_\nu(n) > 0$ and $a_c(n) > 0$ for all n. The onset at a non-prime-power n carries no new data: $c_\nu(n)$ and $a_c(n)$ are polynomials in the prime-power data below n. These onsets are multiple-scattering echoes. *For example, the $n = 6$ onset of the exponential line has weight $a_c(6) = a_c(2)a_c(3) = (2c)^2 \log 2 \log 3$ (multiplicativity), built entirely from the $\log 2$ and $\log 3$ reflectors.*

PROOF. (a) $\mathcal{Q}(F) = \iint W(x-y)F(x)\bar{F}(y)$ and $x - y \in (-A, A)$. (b) is locality (Su21) plus Lemma 34, Prop. 2.3. (d) uses the Dirichlet-exponential recursion of Thm. 37.4.1(ii). For $\zeta(s-\omega)/\zeta(s+\omega) = \exp(\sum \frac{\Lambda(n)}{\log n}(n^\omega - n^{-\omega})n^{-s})$ the same holds. □

Answer to Lemmas 34/36 (kinks at $A = \log n$). The prime-power atoms are common to both functionals (same horizon, same data). The line's onsets at composite non-prime-powers ($n = 6, 10, 12, \dots$) have no counterpart in λ^*. This difference is testable with existing code: no kink of $\lambda^*(e^A)$ is predicted at $A = \log 6$. Whether the prime-power kinks of λ^* and the line onsets have related *shapes* is [OPEN].

37.6. Physical dictionary, done rigorously

Each row gives the exact mathematics (tag) and the engineering name. The names are standard in inverse scattering [**50**, **95**, **109**] ([BIB]); using them for ζ is [ANALOGY].

object	exact statement	name
$\Theta(z)$, $\operatorname{Re} s > \frac{1}{2}$	Schur iff passive (Cor. 37.2.3) [THM]	scattering / Schur function
$K(x)$, $x \geq 0$	$\Theta = \int_0^\infty K e^{ixz}$; causal ($K = 0$, $x < 0$) [CITED]	reflection impulse response
$\mathcal{K}[t]$	Hankel of $K\|_{[0,2t)}$; depends on nothing else [CITED]	GL/Krein/Marchenko operator
$\|\mathcal{K}[t]\| \leq 1$	some passive ψ matches K on $[0, 2t)$ (Thm. 37.2.2(i)) [THM]	record so far is consistent with a passive medium ("no gain")
$\sup_t \|\mathcal{K}[t]\|$	$= \|\Theta\|_{H^\infty(\mathbb{C}_+)}$ (Thm. 37.2.2(ii)) [THM]	ultimate gain
$Z_0(t)$	$(\det(1+\mathcal{K}[t])/\det(1-\mathcal{K}[t]))^2$ [CITED]	characteristic impedance at depth t
$r(t) = \frac{1}{2}(\log Z_0)'$	$\log Z_0(t) = 2\int_0^t r$ (Lemma 34, Thm. 2.5) [THM]	local reflection coefficient
break, $\det(1-\mathcal{K}[t]) = 0$	eigenvalue $+1$; $Z_0 \to \infty$ [THM]	open circuit, total reflection $+1$
break, $\det(1+\mathcal{K}[t]) = 0$	eigenvalue -1; $Z_0 \to 0$ [THM]	short circuit, total reflection -1
$1 - U[t]^2 = B_t^* B_t$	Prop. 37.6.1 [THM]	energy not reflected = leakage

Every first exit computed in TAA is a short (exit through -1): Lemma 34's archimedean exits, all truncated lines of §37.7, and all M6/M7 lines [NUM]. *v2 correction (K9):* this concerns *first* exits only. Past the first exit, opens appear at once and then alternate with shorts (§37.9). Lemma 36's toy lines with $\nu = 2, 3$ even exit through $+1$ first.

Layer peeling. In the discrete setting, Schur's algorithm [**239**] peels a Schur function into parameters $|\gamma_k| \leq 1$. Its Toeplitz counterpart is the Szegő/Levinson recursion [**260**, **177**, **247**], in which a Toeplitz matrix is positive definite iff all Verblunsky coefficients satisfy $|\alpha_k| < 1$ ([BIB], standard). Its continuous analogues are Krein systems [**92**] and Gelfand–Levitan [**109**] ([BIB]). In a Goupillaud (equal-travel-time) discretization of our line, the layer reflection coefficients are $(Z_{0,k+1} - Z_{0,k})/(Z_{0,k+1} + Z_{0,k})$. These lie in $(-1, 1)$ *automatically* whenever $Z_{0,k} \in (0, \infty)$ [THM]. So the "$|r_k| < 1$" inequality is not an extra condition: it is "no break" again (§37.11(c)).

PROPOSITION 37.6.1 (energy identity; [THM]). *Let Θ be unimodular on $\mathbb{R}$ with $\Theta(-\lambda) = \overline{\Theta(\lambda)}$, which holds for $\Theta^{\omega,\nu}$ unconditionally by the functional equation. Let $U = \mathcal{F}^{-1}\Theta\mathcal{F} \circ (f \mapsto f(-\cdot))$, built from the boundary values. Then U is a self-adjoint unitary, so $U^2 = 1$. With $P = P_{(-\infty,t)}$, $Q = 1 - P$ and $U[t] = PUP$,*

$$1 - U[t]^2 = (QUP)^*(QUP) \ \geq 0.$$

So $\|U[t]f\| = \|f\|$ iff Uf is supported in $(-\infty, t)$: a break of the boundary part is a trapped state. *If Θ is inner (passive case), K is the kernel of U and $\mathcal{K}[t] = U[t]$. Otherwise $\mathcal{K}[t] = U[t] + R[t]$ with the finite-rank pole part of Lemma 34, Thm. 5.3, under (B).*

PROOF. $P = PU^2P$, so $P - PUPUP = PU(1-P)UP$. □

37.7. Strategy (a): per-prime sections and the finite Euler cascade

This section makes J. W.'s heuristic precise: each prime is a "peak" (fast rise, slow tail), small primes set a baseline, and larger integers are superpositions via unique factorization.

37.7.1. A prime as a reflector.

PROPOSITION 37.7.1 (the local Euler section; [THM]). *(1)* Feedback loop. $(1 - p^{-s})^{-1} = \sum_{k\geq 0} p^{-ks}$ *is the transfer function of a single feedback loop with round-trip delay* $\log p$ *(in $u = \log x$) and loop gain $p^{-\sigma}$. On the critical line the loop gain has modulus $p^{-1/2} < 1$, so the loop is* stable*: the echo train $\sum_k p^{-k/2}\delta_{k\log p}$ converges. In $-\zeta'/\zeta$ the train is $\log p \sum_k p^{-k/2}\delta_{k\log p}$, which are Lemma 32's atoms $\Lambda(n)n^{-1/2}\delta_{\log n}$. In the linear variable x the train sits at $x = p^k$, crowded near 1 with a long tail. In $u = \log x$ with the half-density $x^{-1/2} = e^{-u/2}$ it is* uniformly spaced with a pure exponential envelope $e^{-u/2}$*, so the asymmetry J. W. sees in linear scale is the exponential map and disappears. (2)* Stable is not passive *(correction to the heuristic). $|(1 - p^{-1/2-i\tau})^{-1}|$ ranges over $[(1 + p^{-1/2})^{-1}, (1 - p^{-1/2})^{-1}]$, which exceeds 1 near the resonances $\tau \in \frac{2\pi}{\log p}\mathbb{Z}$. In Suzuki's symbol the local factor $\theta_p(s) = (\frac{1-p^{-s-\omega}}{1-p^{-s+\omega}})^{\nu}$ satisfies $\theta_p(s) > 1$ for real $s > \max(\omega, \frac{1}{2})$, a point of $\mathbb{C}_+$. In the exponential symbol, $\theta_{c,p} = \exp(2c \log p \frac{p^{-s}}{1-p^{-s}}) > 1$ for real $s > 0$.* No Euler factor is a Schur function*: every local section is active somewhere in $\mathbb{C}_+$. (3)* Products of passive sections are passive *([THM], trivial): $|\Theta_1\Theta_2| \leq 1$. For inner factors the model spaces cascade, $H^2 \ominus \Theta_1\Theta_2 H^2 = (H^2 \ominus \Theta_1 H^2) \oplus \Theta_1(H^2 \ominus \Theta_2 H^2)$. For transfer matrices, $M_i^* J M_i \leq J \Rightarrow (M_1M_2)^* J (M_1M_2) \leq J$. But by (2) the Euler factorization supplies no passive factors to cascade.*

PROPOSITION 37.7.2 (Tate's local integral; how p factors into all integers; [THM], [265] [CITED, secondary]). *With multiplicative Haar measure on $\mathbb{Q}_p^\times$ normalized by $\mathrm{vol}(\mathbb{Z}_p^\times) = 1$,*

$$\int_{\mathbb{Z}_p \setminus \{0\}} |x|_p^s \, d^\times x = \sum_{k\geq 0} p^{-ks} = (1 - p^{-s})^{-1}.$$

Equivalently, with additive Haar measure dx ($\mathrm{vol}\,\mathbb{Z}_p = 1$), $\int_{\mathbb{Z}_p} |x|_p^{s-1} dx = (1-p^{-1})(1-p^{-s})^{-1}$. The natural density of $\{n : v_p(n) = k\}$ is $p^{-k}(1 - p^{-1}) = \mathrm{vol}(p^k\mathbb{Z}_p^\times)$. So the distribution of $v_p(n)$ over the integers is *p-adic Haar measure, and the Euler factor is its $|\cdot|_p^s$-moment. This is the precise form of "how p factors into all integers". By CRT the v_p for distinct p are independent in density, which is unique factorization in measure form.*

PROOF. $\mathbb{Z}_p \setminus \{0\} = \bigsqcup_k p^k\mathbb{Z}_p^\times$; $|x|_p = p^{-k}$ on the k-th piece. The density statement is $\#\{n \leq N : p^k \| n\} = N(p^{-k} - p^{-k-1}) + O(1)$. □

PROPOSITION 37.7.3 (causality and the thin smooth set; [THM]). *(i) The line on $[0, t]$ sees exactly the integers $n < e^{2t}$ (Lemma 34, Thm. 4.1). Every $n < 13$ is 11-smooth, so* the primes ≤ 11 fix the line exactly on $[0, \frac{1}{2}\log 13] = [0, 1.2825]$. *(J. W.'s $\frac{1}{2}\log 12$ should be $\frac{1}{2}\log 13$, because*

12 *is* 11*-smooth.) In general, primes* $\le P$ *fix the line exactly up to* $\frac{1}{2}\log p^+$*, where* p^+ *is the least prime* $> P$*. (ii) For fixed* y*,* $\Psi(x,y) = \#\{n \le x : \ p \mid n \Rightarrow p \le y\} = \frac{(\log x)^{\pi(y)}}{\pi(y)!\prod_{p\le y}\log p}(1+O(1/\log x))$*. For* $y = 11$ *this is* $\asymp (\log x)^5$*. So 11-smooth numbers have density 0: small primes set the line exactly on a fixed initial segment but cannot set the pattern globally.*

PROOF. (ii) $\Psi(x,y)$ counts lattice points $k \in \mathbb{Z}_{\ge 0}^{\pi(y)}$ with $\sum k_p \log p \le \log x$, i.e. in a simplex of volume $(\log x)^{\pi(y)}/(\pi(y)!\prod \log p)$. The boundary contributes $O((\log x)^{\pi(y)-1})$. □

37.7.2. The truncated lines: the next prime must arrive in time. For $P \ge 1$ let $\Theta_P = \Theta_\infty \prod_{p\le P}\theta_p$ ($P = 1$: archimedean only), with kernel $K_P = \sum_{n\ P\text{-smooth}} c_\nu(n) n^{-1/2} G(x - \log n)$, and let $t_*(P) \in (0,\infty]$ be its break length.

THEOREM 37.7.4 (prime arrival; [THM], (d) is [RESTATEMENT]). *Let* (ω,ν) *be admissible.*

(a) $K_P = K$ *on* $[0, \log p^+)$*, so* $\mathcal{K}_P[t] = \mathcal{K}[t]$ *for* $t \le \frac{1}{2}\log p^+$*.*

(b) If $L^{\omega,\nu}$ *is passive (unconditionally for* $\omega \ge \frac{1}{2}$*; under RH for all* ω*), then* $t_*(P) > \frac{1}{2}\log p^+$ *for every* P*. In particular the archimedean line survives past* $\frac{1}{2}\log 2 = 0.3466$*. This answers Lemma 34's question "is* $t_*^{\mathrm{arch}} - \frac{1}{2}\log 2 > 0$ *structural?": yes.*

(c) If $\omega \le \frac{1}{2}$*, every truncated line breaks:* $t_*(P) < \infty$*. The reason is that* Θ_P *keeps the archimedean pole at* $s = 1-\omega$*, since finite Euler products do not vanish there. So* $\Theta_P \notin H^\infty(\mathbb{C}_+)$*, and Thm. 37.2.2(ii) and Lemma 37.1.2 apply.*

(d) (cascade form of G5) For each admissible (ω,ν)*:* $L^{\omega,\nu}$ *is passive* $\iff$ $t_*(P) > \frac{1}{2}\log p^+(P)$ *for every* P*. Hence* $\mathrm{G5}_{\mathrm{line}} \iff$ *for all* $\omega \in (0,\frac{1}{2})$*,* $\nu > 1/\omega$ *and primes* P*, the* P*-truncated line survives until* p^+ *arrives.*

(e) (no archimedean section) The pure Euler cascade $\prod_{p\le P}\theta_p$ *(kernel* $\sum c_\nu(n) n^{-1/2}\delta_{\log n}$*) is broken from* $t = 0^+$*. For* $t < \frac{1}{2}\log 2$ *its section is the reflection* $f \mapsto f(-\cdot)$ *of* $L^2(-t,t)$*, with eigenvalues exactly* ± 1*. Its symbol tends to* 1 *at high frequency, so the input is totally reflecting. The* Γ*-factor is what makes the input non-reflecting:* $\Theta_\infty \to 0$ *like* $|s|^{-\nu\omega}$*.*

PROOF. (a) Every $n < p^+$ is P-smooth, and c_ν is multiplicative, so the Dirichlet coefficients of $\prod_{p\le P}\theta_p$ are c_ν restricted to P-smooth n. Use locality. (b) By (a), Lemma 37.1.2, and $\|\mathcal{K}[t]\| < 1$ at $t = \frac{1}{2}\log p^+$; strictness follows from continuity. (c) At $s_0 = 1-\omega$, $\gamma(s+\omega)$ has a simple zero, $\gamma(s_0-\omega) = \gamma(1-2\omega) \ne 0$ for $\omega \in (0,\frac{1}{2})$, and $\theta_p(s_0) = \left(\frac{1-p^{-1}}{1-p^{2\omega-1}}\right)^\nu \in (0,\infty)$. So Θ_P has a pole of order ν at s_0, with $\operatorname{Re} s_0 > \frac{1}{2}$ (for $\omega = \frac{1}{2}$, $s_0 = \frac{1}{2}$ on the boundary, and Θ_P is unbounded near it). Hence $\sup_t \|\mathcal{K}_P[t]\| = \infty$. By continuity and monotonicity there is a first t with norm 1. (d) Every t satisfies $t \le \frac{1}{2}\log p^+(P)$ for P large, so apply (a). (e) $c_\nu(1) = 1$, and the next atom is at $\log 2$. Use Thm. 37.2.2(ii) and Prop. 37.7.1(2). □

REMARK 37.7.5 (where the archimedean section is rescued; [THM]). For $\omega < \frac{1}{2}$ the active part of the archimedean section is one pole, at $s = 1-\omega$, of order ν. It is the image of the zero of γ at $s = 1$, i.e. of the factor $(s-1)$ that cancels the pole of ζ: the Riemann–Roch "+1"/tachyon direction of Lemmas 32 and 36. The full product $\zeta(s-\omega)/\zeta(s+\omega)$ has a zero exactly there, coming from the pole of $\zeta(s+\omega)$ at $s+\omega = 1$, i.e. from the divergence of $\sum_n n^{-1}$, equivalently from $\sum_p \log p\, p^{-s} \sim \frac{1}{s-1}$. *No finite set of primes supplies this zero* (Thm. 37.7.4(c)). It is a property of the mean density of primes. So the rescue of the archimedean section is global in p: it is the pole of ζ. G5 then says the *fluctuations* of the primes create no new active poles in $\operatorname{Re} s > \frac{1}{2}$.

37.7.3. Numerics of the cascade ([NUM]; corrected in v2). v1 method (superseded for $t \gtrsim 0.8$**).** v1 used Lemma 34's Galerkin scheme with cell indicators of width $h = \frac{1}{800}$ in double precision (`num/lemma37_trunc.py`). Lemma 34 v2 (V7, STATE §43 addendum) showed that this solver puts breaks *late* by 0.003–0.011 for $t \gtrsim 0.8$. There the full line's gaps $1 \pm \lambda$ fall below

double-precision resolution $(6.3\cdot10^{-3}, 3.7\cdot10^{-5}, 2.6\cdot10^{-7}, 9.1\cdot10^{-11}$ at $t = 0.5, 0.7, 0.9, 1.1$ for $(\frac{1}{2}, 3)$). **v2 method.** Legendre Galerkin in ball arithmetic (480 bits, 95-digit kernel tables), with the first exit found by a forward scan of the signs of $\det(1 \pm A)$ (step 10^{-3}–$1.5\cdot10^{-3}$) and then bisection to 10^{-6}. A scan is needed, not sign bisection: each simple crossing flips one determinant's sign, so bisection on signs can jump over a pair of crossings (observed at $(1.5, 1)$). The method is the same in kind as Lemma 34 v2's solver, and the values agree with Lemma 34 V7's ($\frac{1}{2}$, 3: $0.0240, 0.0191, 0.0144, 0.0131, 0.0111, 0.0105, 0.0094$ for $P = 3, \ldots, 19$).

P	p^+	$\frac{1}{2}\log p^+$	$(\frac{1}{2},3)$ v2	v1	$(0.3,4)$ v2	v1	$(1.5,1)$ v2	v1
arch	2	0.3466	0.05200	0.0516	0.04394	0.0441	0.04583	0.0453
2	3	0.5493	0.03462	0.0351	0.02865	0.0289	0.01963	0.0201
3	5	0.8047	0.02405	0.0272	0.01952	0.0222	0.00768	0.0134
5	7	0.9730	0.01912	0.0252	0.01561	0.0202	0.00428	0.0102
7	11	1.1989	0.01441	0.0218	$0.01318\pm5.9\cdot10^{-6}$	0.0179	$0.00199\pm3.9\cdot10^{-4}$	0.008
11	13	1.2825	0.01309	0.0232	—	0.0195	0.00266 ± 10^{-3}	0.007
13	17	1.4166	0.01112	0.0216	—	0.0179	$0.00222\pm1.1\cdot10^{-3}$	0.0054
17	19	1.4722	0.01050	0.0209	—	—	$0.00215\pm9.8\cdot10^{-4}$	—
19	23	1.5677	0.00937	0.0204	—	—	$0.00125\pm2.7\cdot10^{-4}$	—

TABLE 37.1. **Corrected Table 1 (v2)**, [NUM]. Margins $t_*(P) - \frac{1}{2}\log p^+$. "v2": ball-arithmetic solver at the highest available resolution ($r24p10$, or $r30p12$ when present). The error is the difference to the next-lower resolution plus the bisection width, shown only when $\geq 5\cdot10^{-6}$. $*$: one resolution only. d: double precision $r45p14$ ($t < 0.8$, where double is reliable). "v1": v1's values ($h = \frac{1}{800}$), *too large* for $t \gtrsim 0.8$ (correction K8, citing Lemma 34 V7). Every exit is a short. Data `num/l37/m6_*.jsonl`, `m7_double.json`; table `num/lemma37_v2_tables.py`.

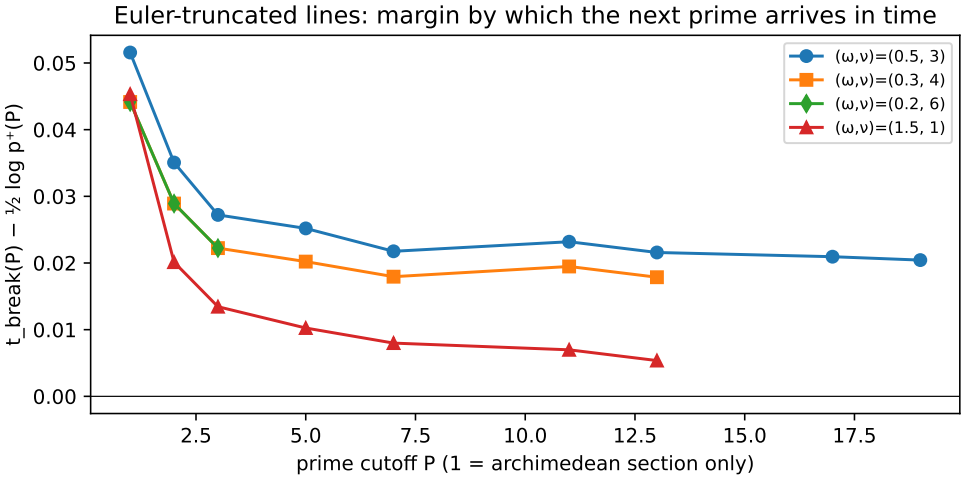

FIGURE 37.1. (v1 figure, retained for the record; uses the v1 margins, which are too large for $t \gtrsim 0.8$.) Margin versus cutoff P. Corrected data: Fig. 37.5. [NUM]

Observations (v2). (1) Every truncated line breaks, and *every one* survives past the arrival of the next prime, as Thm. 37.7.4 requires for $\omega \geq \frac{1}{2}$. At $\omega < \frac{1}{2}$ this is consistent with RH and tests nothing new. (2) Corrected margins shrink steadily: at $(\frac{1}{2}, 3)$ they go $0.052, 0.035, 0.024, 0.019, 0.014, 0.013, 0.011, 0.0105, 0.0094$ for $P = 1, \ldots, 19$. v1's apparent plateau at ≈ 0.02 was an artefact. At $(1.5, 1)$ the margins are 4–5 times smaller, $0.0458 \to 0.0019$. v1's

erratic values there came from double precision and multiple crossings. (3) At $\nu\omega = 1.2$, $(0.3, 4)$ and $(0.2, 6)$ do *not* agree to 10^{-4} (v1's claim, K10). They differ by $1.6 \cdot 10^{-4}$ (archimedean) up to $4.2 \cdot 10^{-4}$ ($P = 7$), exactly as $O(\omega^2)$ predicts (§37.8, (M7.2)). (4) M6 (§37.10) extends this table to $P = 127$.

Conjecture 37.7.6 ([CONJ], from v1 Table 1; status in v2: see §37.10). At fixed admissible (ω, ν) with $\omega \le \frac{1}{2}$, $\inf_P(t_*(P) - \frac{1}{2}\log p^+(P)) = 0$: the margins tend to 0 along a subsequence of P (large prime gaps would help). Under RH they stay positive for every P. *v2:* Thm. 37.10.1 removes the "large gaps would help" mechanism. The corrected data (§37.10) show margins decreasing along the *whole* sequence, smoothly in p^+, with no detectable gap dependence; whether they tend to 0 is not decided by data to $P = 127$ (the decay slows, (M6.3)). At $\omega = \frac{1}{2}$ they are positive for every P unconditionally (Thm. 37.7.4(b)), so the conjecture there is only $\inf = 0$.

Remark 37.7.7 (Iwasawa theory and finite cascades; [ANALOGY], citations [BIB]). J. W. suggests that Iwasawa theory bears on whether finite cascades stay passive. What exists: Iwasawa's analogy compares the action of $\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)) \cong 1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$ (p odd; v2 correction K11, flagged by Lemma 38; equivalently $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})^+/\mathbb{Q}(\mu_p)^+)$) on the inverse limit X_∞ of p-parts of class groups in the tower $\mathbb{Q}(\mu_{p^n})$ with Frobenius acting on the Tate module of a curve's Jacobian. The characteristic power series of X_∞ plays the role of the numerator $P(T)$ of a curve's zeta function [**139, 140**]. The Main Conjecture, proved for $\mathbb{Q}$ by Mazur–Wiles [**188**], identifies that characteristic power series with the p-adic L-function, up to units. It is a *spectral interpretation of the zeros of p-adic L-functions*, the closest existing realization of a Hilbert–Pólya-type statement [**280**] (content stated as standard background, not checked at the sources). It differs from our setting in two ways. (i) It is p-adic: there is no archimedean positivity, unitarity or Schur bound, so nothing in it can yield $|\Theta| \le 1$. (ii) It runs in the *level* direction ($\mathbb{Q}(\mu_{p^n})$, $n \to \infty$, fixed p), whereas cascade passivity runs in the *prime-cutoff* direction ($P \to \infty$, archimedean). The Ferrero–Washington theorem $\mu_p = 0$ [**100**] (no runaway p-power growth of class numbers in the tower) is at most a "no runaway gain" *analogue* of passivity. No precise statement linking a $\mathbb{Z}_p$-tower to $t_*(P)$ is known to us; the link is [ANALOGY].

37.8. M7: the exponential line, numerically (v2)

Method ([NUM]; `num/lemma37_gtab.py`, `lemma37_gtab_hp.py`, `lemma37_hpline.py`, `lemma37_arbline.py`). The archimedean kernel $G_c = \mathcal{L}^{-1}[\exp(-2c\,\gamma'/\gamma(\frac{1}{2} + p))]$, with $\gamma'/\gamma(s) = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\psi(\frac{s}{2} + 1)$, is tabulated as $G_c(v) = v^{c-1}g(v)$ with g piecewise Chebyshev. Two table sets are used: double, validation error $\le 5 \cdot 10^{-15}$; and 95-digit Talbot inversion, $M = 140$, validation error $\le 3.5 \cdot 10^{-90}$. The Dirichlet coefficients come from Thm. 37.4.1(ii) ($a_c(n)\log n = \sum_{d|n} 2c\Lambda(d)\log d\, a_c(n/d)$). The Hankel section is discretized by orthonormal Legendre polynomials of degree $< p$ on $N = \lceil rt \rceil + 1$ equal cells, written "$rXpY$". Matrix entries are exact moment sums of the tables: Gauss–Legendre for regular pieces, Gauss–Jacobi with weight $(\tau - \tau_0)^{c-1}$ at the singular corner. This is a compression of $\mathcal{K}_c[t]$, so its eigenvalues are *inner* (Ritz) bounds. A Ritz value outside $[-1, 1]$ certifies a break of the line, up to the table error. The computed break times are therefore *upper* estimates of t_*; under refinement they are expected (not guaranteed: the meshes are not nested) to decrease. Validation: the drop-7 break at $(\frac{1}{2}, 3)$ gives 0.0191158 at $r45p14$, inside Lemma 34 v2's ball-arithmetic bracket $[0.0191052, 0.0191265]$. The exponential-line archimedean breaks at $c = 1.5$ and $c = 1.2$ agree with Lemma 34's `q4` values (0.3993808826, 0.3908043279) to all printed digits.

Truncated exponential lines (Table 37.2; data `num/l37/m7_double.json`). $\Theta_{c,P} = \exp(-2c\gamma'/\gamma)\prod_{p \le P}\exp(2c\log p\,\frac{p^{-s}}{1-p^{-s}})$, for $P = 1$ (archimedean, "Γ-only") and $P = 2, 3, 5, 7$.

Findings ([NUM]).

symbol	arch ($P=1$)	$P=2$	$P=3$	$P=5$	$P=7$
$c=0.5$	0.37561 (0.02904)	0.56598 (0.01667)	0.81381 (0.00909)†	0.98075 (0.00779)†	1.20490 (0.00596)$^{\dagger\pm 2.0\cdot 10^{-3}}$
$c=0.75$	0.38006 (0.03349)	0.56980 (0.02049)	0.81741 (0.01269)†	0.98336 (0.01041)†	1.20739 (0.00845)†
$c=1$	0.38566 (0.03909)	0.57427 (0.02497)	0.82142 (0.01670)†	0.98635 (0.01339)†	1.21030 (0.01136)†
$c=1.2$	0.39080 (0.04423)	0.57825 (0.02895)	0.82486 (0.02014)†	0.98898 (0.01602)†	1.21285 (0.01390)†
$c=1.5$	0.39938 (0.05281)	0.58475 (0.03545)	0.83035 (0.02563)†	0.99332 (0.02036)†	1.21696 (0.01801)†
$c=2$	0.41542 (0.06885)	0.59674 (0.04743)	0.84024 (0.03552)†	1.00161 (0.02865)†	1.22447 (0.02552)†
$c=3$	0.45170 (0.10513)	0.62406 (0.07475)	0.86257 (0.05785)†	1.02202 (0.04907)†	1.24191 (0.04296)†
$(0.1,12)$	0.39077 (0.04420)	0.57822 (0.02891)	0.82480 (0.02008)†	0.98893 (0.01598)†	1.21278 (0.01383)†
$(0.2,6)$	0.39067 (0.04410)	0.57812 (0.02881)	0.82460 (0.01988)†	0.98879 (0.01584)†	1.21255 (0.01360)†
$(0.3,4)$	0.39051 (0.04394)	0.57796 (0.02865)	0.82424 (0.01952)†	0.98856 (0.01561)†	1.21213 (0.01318)†
$(\frac{1}{2},3)$	0.39857 (0.05200)	0.58393 (0.03462)	0.82877 (0.02405)†	0.99207 (0.01912)†	1.21336 (0.01441)†
$(1.5,1)$	0.39240 (0.04583)	0.56893 (0.01963)	double fails‡	double fails‡	double fails‡

TABLE 37.2. M7, [NUM]. Break $t_*(P)$ and margin $t_*(P)-\frac{1}{2}\log p^+$ (in parentheses), double precision at $r45p14$. Resolution differences $|r45-r30|\le 4\cdot 10^{-7}$ except where marked $\pm$. $\dagger$: $t>0.8$, where Lemma 34 V7 found the old double-precision solver unreliable. Our Legendre solver agrees with the arb solver there to $\le 10^{-6}$ (Table 37.3); the arb values are authoritative. $\ddagger$: at $(1.5,1)$ the double-precision gaps collapse by $t\approx 0.8$, and the exits are found only by the arb solver (Table 37.5). Every exit listed is through -1 (a short). All exits lie *after* $\frac{1}{2}\log p^+$.

(M7.1) *Every truncated exponential line survives the next prime's arrival*, for all seven $c\in[0.5,3]$ and $P\le 7$. The margins increase monotonically with c: 0.029 at $c=0.5$ to 0.105 at $c=3$ (archimedean), and 0.006 to 0.043 at $P=7$. This is consistent with $\Theta_{c_1+c_2}=\Theta_{c_1}\Theta_{c_2}$: a longer slab is more lossy away from the boundary. The ordering in P is the same at every c. At $c=0.5$ ($G_c\sim x^{-1/2}$, not continuous) convergence is poor ($2\cdot 10^{-3}$ at $P=7$), and $c\le\frac{1}{2}$ is outside the compact-section regime of §37.1.

(M7.2) *$c\leftrightarrow\nu\omega$ is exact to $O(\omega^2)$, with a universal constant at $P=1$* (`num/lemma37_m7_cnu.py`, `l37/m7_cnu.json`). The archimedean break of the Suzuki line at (ω,ν) lies *below* that of the exponential line at $c=\nu\omega$ by $(3.24\pm 0.01)\cdot 10^{-3}\,\omega^2$ for $(0.1,12)$, $(0.2,6)$, $(0.3,4)$ ($c=1.2$) and $(\frac{1}{2},3)$ ($c=1.5$), and by $3.10\cdot 10^{-3}\omega^2$ for $(1.5,1)$. At $P=2,3,5,7$ the coefficient varies between $3.3\cdot 10^{-3}$ and $8\cdot 10^{-3}$ (and $1.4\cdot 10^{-2}$ for $(\frac{1}{2},3)$ at $P=7$), but at fixed P it is stable in ω to within $\approx 10\%$. The ω^2 law is what Thm. 37.4.1(iii) predicts ($\nu[\log\xi(s-\omega)-\log\xi(s+\omega)]=-2c\,\xi'/\xi-\frac{c\omega^2}{3}(\xi'/\xi)''+O(\omega^4)$). The constant is an observation.

(M7.3) *The full exponential line* (`num/lemma37_m7_full.py`, `l37/m7full_{exp_c1.5,w0.5_nu3}.jsonl`, arb). On the grid $t=0.10,0.15,\dots,2.45$, at $r15p8$ and $r24p10$, *both* $\det(1\pm\mathcal{K}_{1.5}[t])$ are positive, with certified sign: the balls exclude 0, and $\log_{10}|\det(1+\cdot)|$ exceeds $\log_{10}$ of its ball radius by ≥ 92 at every grid point. The same holds for the $(\frac{1}{2},3)$ line. No Ritz value leaves $(-1,1)$ up to $t=2.45$, i.e. up to $n<e^{4.9}\approx 134$. Caveats: (a) Ritz values are inner bounds, so this is *consistent with* passivity, not a proof of it. (b) A sign check on a grid detects an even number of crossings between grid points only through the Ritz count, which stays zero by continuity of the extreme Ritz values; we did not certify that. (c) The tables are numerical (Talbot, 10^{-90}), not interval-certified. The gaps shrink super-exponentially: $\log_{10}\det(1+\mathcal{K}_{1.5}[t])=-14.3,-41.1,-84.9,-140.7$ at $t=1,1.5,2,2.45$, and $\log_{10}\det(1-\cdot)$ is 6–16 lower. $r15$ and $r24$ agree to ≤ 0.2 in $\log_{10}\det$ up to $t=2.0$, and diverge beyond (*convergence is lost at $t\approx 2$* at $r24p10$). For $(\frac{1}{2},3)$ the dets are smaller (-263 at $t=2.45$), and the resolutions agree only to $t\approx 1.5$. Double precision resolves the full lines (both gaps $>10^{-12}$, resolutions consistent) only up to $t=0.90$ ($c=0.5$), 0.95 ($c=0.75$), 1.05 ($c=1$), 1.10 ($c=1.2$), 1.15 ($c=1.5$), 1.20 ($c=2$), 1.25 ($c=3$).

symbol	P	double $r45p14$	arb $r24p10$	$\lvert\text{arb } r24 - r15\rvert$
$c = 0.75$	3	0.01269	0.01269	$6.8 \cdot 10^{-6}$
$c = 0.75$	5	0.01041	0.01041	$9.8 \cdot 10^{-6}$
$c = 0.75$	7	0.00845	0.00845	$3.2 \cdot 10^{-5}$
$c = 1$	3	0.01670	0.01670	$2.9 \cdot 10^{-6}$
$c = 1$	5	0.01339	0.01339	$2.9 \cdot 10^{-6}$
$c = 1$	7	0.01136	0.01136	$1.2 \cdot 10^{-5}$
$c = 1.2$	3	0.02014	0.02014	$9.8 \cdot 10^{-7}$
$c = 1.2$	5	0.01602	0.01602	$2.0 \cdot 10^{-6}$
$c = 1.2$	7	0.01390	0.01390	$4.9 \cdot 10^{-6}$
$c = 1.5$	3	0.02563	0.02563	—
$c = 1.5$	5	0.02036	0.02036	—
$c = 1.5$	7	0.01801	0.01801	—
$c = 2$	3	0.03552	0.03552	0
$c = 2$	5	0.02865	0.02865	0
$c = 2$	7	0.02552	0.02552	0
$c = 3$	3	0.05785	0.05785	0
$c = 3$	5	0.04907	0.04907	0
$c = 3$	7	0.04296	0.04296	0
$(0.3, 4)$	3	0.01952	0.01952	$9.8 \cdot 10^{-7}$
$(0.3, 4)$	5	0.01561	0.01561	$2.0 \cdot 10^{-6}$
$(0.3, 4)$	7	0.01318	0.01318	$4.9 \cdot 10^{-6}$
$(\frac{1}{2}, 3)$	3	0.02405	0.02405	$8.5 \cdot 10^{-7}$
$(\frac{1}{2}, 3)$	5	0.01912	0.01912	$8.5 \cdot 10^{-7}$
$(\frac{1}{2}, 3)$	7	0.01441	0.01441	$8.5 \cdot 10^{-7}$
$(1.5, 1)$	3	—	0.00768	0
$(1.5, 1)$	5	—	0.00428	$7.1 \cdot 10^{-4}$
$(1.5, 1)$	7	—	0.00288	$1.2 \cdot 10^{-3}$

TABLE 37.3. M7 re-check of the $t > 0.8$ entries of Table 37.2 by the ball-arithmetic solver (480 bits, 95-digit tables), [NUM]. Data `num/l37/m6_{tag}_r{15p8,24p10}_s0.001.jsonl`, script `num/lemma37_m6.py`. Rows appear as the runs complete.

(M7.4) *Passivity range, summarized.* Numerically ([NUM], Ritz sense): the full exponential line at $c = 1.5$ shows no break on $[0, 2.45]$, with the deepest resolution-converged point at $t \approx 2.0$. Theoretically: Θ_c is passive for all t iff RH (Thm. 37.4.1(iv)), and $\|\mathcal{K}_c[t]\| < 1$ is unconditional only for $t < t_1(c)$ from the Schur/HS test (§37.11(e2)). There is no theorem in between.

37.9. D3: past the first exit — opens and shorts (v2)

Question (Lemma 36 v2, Prop. 7.13, Thm. 7.9, Cor. 7.10, read-only). For $\omega < \frac{1}{2}$ the truncated line Θ_P has one $\mathbb{C}_+$ pole, of order ν, and $\mathcal{K}_P[t] = U_P[t] + R_P[t]$ with R_P of signature $(\frac{\nu}{2}, \frac{\nu}{2})$ for even ν. Lemma 36 predicts *at least* 2 opens at $(0.3, 4)$ and at least 3 at $(0.2, 6)$, eventually.

Method ([NUM]; `num/lemma37_d3.py`, data `num/l37/d3_{tag}_P{P}.json`). Double-precision Legendre Galerkin at $r15p8$ and $r24p10$ on $t \in [0.3, T]$, step 0.01. At each t we count Ritz values $> 1 + \varepsilon$ (opens) and $< -1 - \varepsilon$ (shorts), $\varepsilon = 10^{-9}$ and 10^{-6}. Ritz values are inner bounds, so each count is a *lower bound* for the true count, up to table and rounding error. Values exceeding 1 by $\geq 10^{-6}$ are far above both.

(D3.1) *The prediction holds, with room* ([NUM]). The archimedean lines at $(0.3, 4)$ and $(0.2, 6)$ reach 6 opens and 6 shorts by $t = 3.45$, identical at both resolutions and both thresholds. Lemma 36 requires ≥ 2 and ≥ 3 opens. The truncated lines $P = 2, 3$ show $10 + 10$ and 15–17 + 15–17.

(D3.2) *First exits are shorts, but the opens follow at once.* In every case the first exit is a short, and the first open follows within 0.02–0.06 in t. Correction (K9): v1's "every break computed

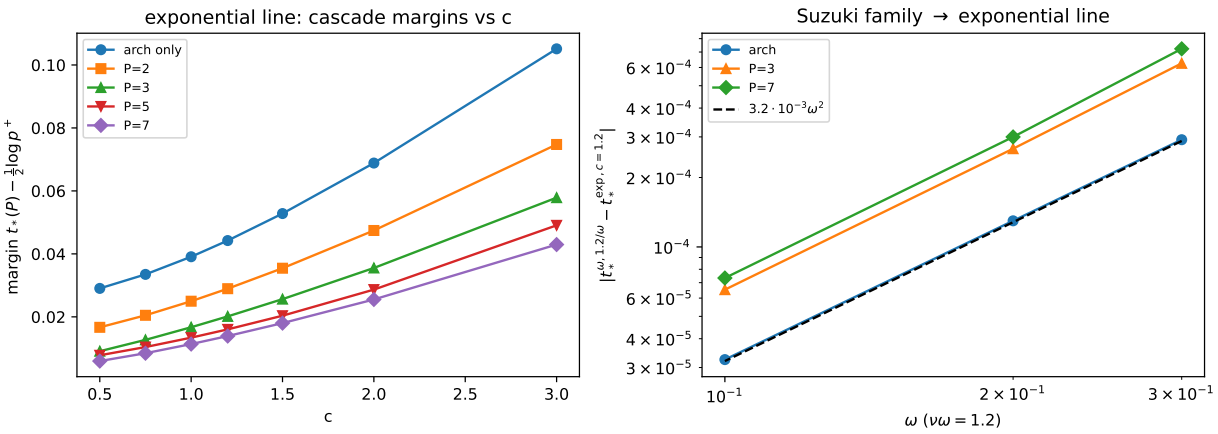

FIGURE 37.2. M7, [NUM]. Left: cascade margins of the exponential line versus c. Right: the Suzuki family approaches the exponential line as ω^2 at fixed $\nu\omega = 1.2$; dashed: $3.2 \cdot 10^{-3}\omega^2$.

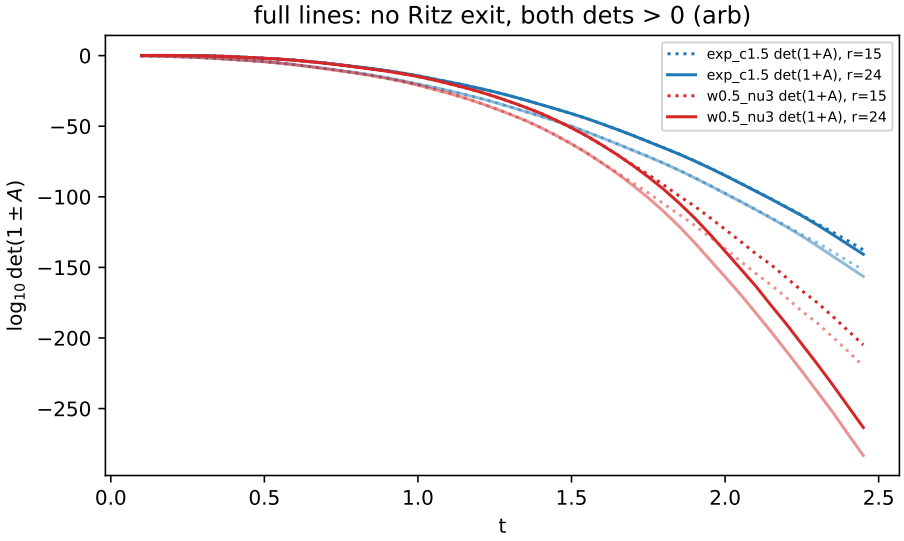

FIGURE 37.3. M7, [NUM]. Full lines in ball arithmetic: $\log_{10}\det(1+\mathcal{K}[t])$ (dark) and $\log_{10}\det(1-\mathcal{K}[t])$ (light) for $\Theta_{1.5}$ and $\Theta^{1/2,3}$ at two resolutions. Both stay positive (no Ritz exit).

in TAA is a short" is a statement about *first* exits only. Lemma 36's toy runs with $\nu = 2, 3$ even have an open first.

(D3.3) *The counts do not saturate.* Opens and shorts alternate and grow roughly linearly in t, with no plateau at $\nu/2$ or any finite number. This is consistent with Lemma 36, Prop. 7.13. There the bound $\|U_P\| \le M_P$ with $M_P = \sup_{\mathbb{R}} |\Theta_P| > 1$ gives only lower bounds; Thm. 7.9's hypothesis $|\Theta| \le 1$ on $\mathbb{R}$ *fails* for truncations, and so does its exact count $\kappa/2$. A Landau–Slepian heuristic ([ANALOGY]) explains the growth. The part of U_P where $|\Theta_P| > 1$ on the boundary contributes about $2t|E|/2\pi$ Ritz values of modulus > 1, where $|E|$ is the measure of that set. For the archimedean line $|\Theta_\infty(\frac{1}{2} + i\tau)| \approx (|\tau|/2\pi)^{-\nu\omega} > 1$ exactly for $|\tau| < 2\pi$, so $|E| \approx 4\pi$ and the rate is ≈ 4 per unit t. We measure $|E| = 12.57$ (all four families),

line	P	t-range	opens	shorts	first short	first open	Landau rate (pred. / obs.)
$c = 1.5$	arch	$[0.30, 2.55]$	4	4	0.40	0.46	4.0 / 3.7
$(0.2, 6)$	arch	$[0.30, 3.45]$	6	6	0.40	0.44	4.0 / 3.9
$(0.2, 6)$	2	$[0.30, 3.45]$	10	10	0.58	0.62	9.7 / 7.0
$(0.2, 6)$	3	$[0.30, 3.45]$	15/16	15/16	0.83	0.85	15.6 / 12.2
$(0.3, 4)$	arch	$[0.30, 3.45]$	6	6	0.40	0.44	4.0 / 3.9
$(0.3, 4)$	2	$[0.30, 3.45]$	10	10	0.58	0.62	11.1 / 7.0
$(0.3, 4)$	3	$[0.30, 3.45]$	15/17	15/17	0.83	0.85	16.9 / 13.0
$(\frac{1}{2}, 3)$	arch	$[0.30, 2.55]$	4	4	0.40	0.45	4.0 / 3.7

TABLE 37.4. D3, [NUM]. Counts at the end of the range ($\varepsilon = 10^{-9}$; identical at $\varepsilon = 10^{-6}$). "a/b" means $r15p8/r24p10$ when they differ. "First short/open" are grid times (step 0.01). Landau rate: predicted $|E|/\pi$ with $E = \{\tau : |\Theta_P(\frac{1}{2} + i\tau)| > 1\}$ (`num/lemma37_landau.py`, `l37/landau.json`; heuristic), observed = total exits / (range after the first exit).

predicted rate 4.0, observed 3.7–3.9. For $P = 2, 3$ the heuristic over-predicts (11.1, 16.9 vs. 7.0, 13.0 at $(0.3, 4)$); the asymptotic regime is not reached.

(D3.4) $\omega = \frac{1}{2}$ *and* $c = 1.5$ (outside Prop. 7.13): the archimedean lines show $4 + 4$ by $t = 2.55$, also unbounded growth. The $(\frac{1}{2}, 3)$ pole sits on the boundary.

(D3.5) *Convergence evidence.* The archimedean counts are identical at $r15p8$ and $r24p10$, and so are their exit times to the grid step, except one short at $(0.2, 6)$ (2.23 vs. 2.21). For $P = 3$ the finer resolution finds 1–2 *more* exits late in the range, as inner bounds should. The counts are numerical lower bounds, not certified values.

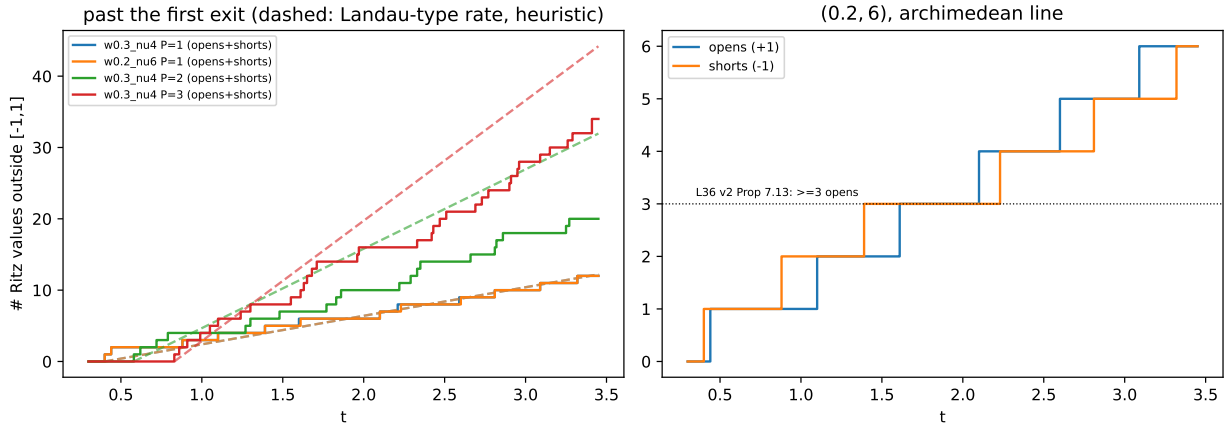

FIGURE 37.4. D3, [NUM]. Left: number of Ritz values outside $[-1, 1]$ versus t (dashed: Landau-type rate, heuristic). Right: opens and shorts of the $(0.2, 6)$ archimedean line; the dotted line is Lemma 36's lower bound of 3 opens.

37.10. M6: high-precision cascade margins to $P = 127$ (v2)

Method ([NUM]; `num/lemma37_m6.py`, solver `num/lemma37_arbline.py`). Ball-arithmetic Legendre Galerkin (480 bits) on the 95-digit tables of §37.8, at three resolutions $r15p8$, $r24p10$, $r30p12$, plus $r36p14$ at selected points ($N = \mathrm{round}(r\, t_{\mathrm{arr}}) + 1$ cells, degree $< p$). Starting at the arrival time $t_{\mathrm{arr}} = \frac{1}{2} \log p^+$ (before which the truncated line equals the full line, Thm. 37.7.4(a)), the signs of $\det(1 \pm A)$ are scanned forward with step 10^{-3} or $1.5 \cdot 10^{-3}$, and the first sign change is bisected to width $\leq 2 \cdot 10^{-6}$. The determinant signs at the bracket ends are rigorous for the computed matrix.

The matrix itself rests on numerical (Talbot) tables, so the outputs are *Ritz-sound upper estimates* of $t_*(P)$, not interval-certified values (the same label as Lemma 34 v2, V4). **Error bars**: the difference between the highest and the next-lower resolution, plus the bracket width. Where only $r24p10$ and $r15p8$ exist, this difference is *conservative*: at $(\frac{1}{2},3)$, $P=31$, $|r24-r15|=1.1\cdot10^{-3}$ while $|r30-r24|<10^{-6}$. **Retracted internal run**: a first M6 pass that bisected on determinant signs without a scan gave spurious margins (≈ 0.05) for $P\geq 67$, because a pair of crossings leaves both signs unchanged. Those values were discarded (`num/l37/old_m6_bisect/`) and are not used anywhere.

THEOREM 37.10.1 (single-atom reduction; [THM]). *Let P be 1 or a prime, $p^+<p^{++}$ the next two primes, and let $L^{(\hat{q})}$ be the* full *line (Suzuki symbol $\Theta^{\omega,\nu}$, or Θ_c) with the single atom at $\log q$ removed from its kernel. Then on $[0,\frac{1}{2}\log p^{++})$ the P-truncated line coincides with $L^{\widehat{(p^+)}}$:*

$$\mathcal{K}_P[t]=\mathcal{K}^{\widehat{(p^+)}}[t]\qquad(0\leq t<\tfrac{1}{2}\log p^{++}).$$

Hence, if either line breaks before $\frac{1}{2}\log p^{++}$, both break at the same time, and the cascade margin is a function of p^+ alone:

$$m(P)=\mu(p^+),\qquad \mu(q):=t_*(L^{(\hat{q})})-\tfrac{1}{2}\log q.$$

The previous prime P, and so the gap p^+-P below p^+, does not enter the definition of μ.

PROOF. The coefficients are multiplicative: $\Theta^{\omega,\nu}$ and Θ_c are Euler products over p times the archimedean factor, so the P-truncated symbol has the same Dirichlet coefficients as the full one at P-smooth n and zero elsewhere. The kernel section $\mathcal{K}[t]$ sees only the atoms with $\log n<2t$, i.e. $n<e^{2t}<p^{++}$. An integer $n<p^{++}$ that is not P-smooth has a prime factor $q\geq p^+$. Then $q<p^{++}$ forces $q=p^+$, and $n=p^+m$ with $m<p^{++}/p^+<2$ (Bertrand: $p^{++}<2p^+$), so $n=p^+$. So on this range the two kernels differ by exactly the atom at $\log p^+$. The archimedean part is common to both. Equal sections give equal break times. □

REMARK 37.10.2 (what the theorem changes; [THM]/[NUM]). (1) This is Lemma 34 v2, Prop. 8.3 (drop-p^+ line = P-truncated line on $(t_{p^+},\frac{1}{2}\log(p^++1))$), extended to $[0,\frac{1}{2}\log p^{++})$ by Bertrand's postulate. Table 37.11 checks it numerically: Lemma 34's independent drop-n brackets and our M6 margins agree for all nine primes $n\leq 23$: every M6 margin lies inside Lemma 34's bracket. Every M6 break satisfies the hypothesis $t_*(P)<\frac{1}{2}\log p^{++}$ (checked in `num/l37/m6_analysis.json`). (2) *The cascade question is a question about the full line.* "The next prime arrives in time" becomes: *removing the newest prime atom from the full line breaks it within $\mu(q)$ of that prime's arrival.* The margin is the full line's tolerance to losing one prime. Lemma 34 v2's drop-n data show that composites behave the same way (removing $n=4,6,8,\ldots$ also breaks the line), with prime margins slightly larger than those of neighbouring composites. (3) *Gap dependence.* v1's conjecture suggested that large prime gaps might help the margins reach 0. Under Thm. 37.10.1 the gap below p^+ is not an input. It could only matter through the way $\mu(q)$ itself depends on q. The numerical test below finds no such dependence. (4) *Not linear response in the atom weight* ([THM] for the weights, [NUM] for the margins). The removed atom has weight $b_q=c_\nu(q)q^{-1/2}=\nu(q^\omega-q^{-\omega})q^{-1/2}$ (Suzuki; $=3(1-q^{-1})$ at $(\frac{1}{2},3)$, $=q-q^{-2}$ at $(1.5,1)$), and $b_q=2c\log q\cdot q^{-1/2}$ for Θ_c. These are *flat*, *growing* and *slowly decreasing* in q, respectively, while all three margin sequences decrease (Table 37.5). So $\mu(q)$ is not proportional to the size of the missing atom. This also disposes of Lemma 39's "missing atom" candidate in its linear form.

Findings ([NUM] unless marked).

(M6.1) *Every truncated line survives the next prime's arrival*, for all $P\leq 127$ at $(\frac{1}{2},3)$, $P\leq 127$ for $\Theta_{1.5}$ and $P\leq 127$ at $(1.5,1)$. For $\omega\geq\frac{1}{2}$ this is required by Thm. 37.7.4(b), so it checks the code. For $\Theta_{1.5}$ it is consistent with RH and tests nothing new. Every first exit is a short.

P	p^+	$\frac{1}{2}\log p^+$	$(\frac{1}{2},3)$	$c=1.5$ (exp. line)	$(1.5,1)$ (upper est.)
arch	2	0.3466	$0.05200_{\pm 9e-7}$	$0.05281_{\pm 1e-6}$	$0.04583_{\pm 1e-6}$
2	3	0.5493	$0.03462_{\pm 9e-7}$	$0.03545_{\pm 1e-6}$	$0.01963_{\pm 1e-6}$
3	5	0.8047	$0.02405_{\pm 2e-6}$	$0.02563_{\pm 1e-6}$	$0.00768_{\pm 1e-6}$
5	7	0.9730	$0.01912_{\pm 2e-6}$	$0.02036_{\pm 1e-6}$	$0.00428^{c}_{\pm 1e-6}$
7	11	1.1989	$0.01441^{c}_{\pm 7e-7}$	$0.01801^{c}_{\pm 7e-7}$	$0.00199^{\alpha}_{\pm 4e-4}$
11	13	1.2825	$0.01309_{\pm 3e-6}$	$0.01643_{\pm 2e-6}$	$0.00266_{\pm 1e-3}$
13	17	1.4166	$0.01112_{\pm 4e-6}$	$0.01554_{\pm 4e-6}$	$0.00222_{\pm 1e-3}$
17	19	1.4722	$0.01050_{\pm 4e-6}$	$0.01573_{\pm 5e-6}$	$0.00215_{\pm 1e-3}$
19	23	1.5677	$0.00937^{c}_{\pm 1e-6}$	$0.01406^{c}_{\pm 7e-7}$	$0.00125^{\alpha}_{\pm 3e-4}$
23	29	1.6836	$0.00836^{\flat}_{\pm 6e-4}$	$0.01371^{\flat}_{\pm 8e-6}$	$0.00173^{\flat}_{\pm 9e-4}$
29	31	1.7170	$0.00812^{\flat}_{\pm 8e-4}$	$0.01336^{\flat}_{\pm 8e-6}$	$0.00167^{\flat}_{\pm 9e-4}$
31	37	1.8055	$0.00728^{c}_{\pm 7e-7}$	$0.01353^{c}_{\pm 1e-6}$	$0.00117^{c}_{\pm 4e-4}$
37	41	1.8568	$0.00698^{\flat}_{\pm 1e-3}$	0.01306^{*}	0.00152^{*}
41	43	1.8806	$0.00673^{\flat}_{\pm 2e-3}$	0.01263^{*}	0.00149^{*}
43	47	1.9251	$0.00638^{\flat}_{\pm 2e-3}$	0.01265^{*}	0.00145^{*}
47	53	1.9851	$0.00606^{c}_{\pm 4e-5}$	$0.01211^{c}_{\pm 2e-6}$	$0.00102^{c}_{\pm 3e-4}$
53	59	2.0388	$0.00622^{\flat}_{\pm 2e-3}$	0.01181^{*}	0.00131^{*}
59	61	2.0554	0.00619^{*}	0.01196^{*}	0.00131^{*}
61	67	2.1023	0.00585^{*}	0.01205^{*}	0.00127^{*}
67	71	2.1313	$0.00513^{\alpha}_{\pm 1e-6}$	$0.01200^{c}_{\pm 3e-6}$	$0.00093^{c}_{\pm 3e-4}$
71	73	2.1452	0.00580^{*}	0.01179^{*}	0.00123^{*}
73	79	2.1847	0.00585^{*}	0.01196^{*}	0.00120^{*}
79	83	2.2094	0.00567^{*}	0.01188^{*}	0.00118^{*}
83	89	2.2443	0.00554^{*}	0.01139^{*}	0.00115^{*}
89	97	2.2874	0.00562^{*}	0.01103^{*}	0.00112^{*}
97	101	2.3076	$0.00435^{\alpha}_{\pm 3e-4}$	$0.01086^{c}_{\pm 3e-6}$	$0.00081^{c}_{\pm 3e-4}$
101	103	2.3174	0.00547^{*}	0.01088^{*}	0.00108^{*}
103	107	2.3364	0.00541^{*}	0.01090^{*}	0.00108^{*}
107	109	2.3457	0.00546^{*}	0.01076^{*}	0.00108^{*}
109	113	2.3637	0.00549^{*}	0.01071^{*}	0.00104^{*}
113	127	2.4221	0.00531^{*}	0.01091^{*}	0.00102^{*}
127	131	2.4376	0.00520^{*}	0.01081^{*}	0.00102^{*}

TABLE 37.5. M6, [NUM]. Cascade margins $m(P) = t_*(P) - \frac{1}{2}\log p^+$ in ball arithmetic. Subscript: error bar = difference to the next-lower resolution plus bracket width. No mark: $r24p10$ vs. $r15p8$; ^{c}: $r30p12$ vs. $r24p10$; $^{\alpha}$: $r36p14$ vs. $r30p12$; $^{\flat}$: $r24p10$ vs. $r15p8$ at $t > 1.68$, where $r15p8$ is itself unconverged (conservative, see text); *: one resolution only. All values are Ritz-sound *upper* estimates of $t_*(P) - \frac{1}{2}\log p^+$ (a Ritz value is an inner bound, so the true break is no later; the meshes are not nested, so finer resolution is expected, not guaranteed, to lower them). $(\frac{1}{2},3)$ with $P \geq 53$ and all entries with $t_* > 2.0$ lie outside the resolution-converged window and may be 10–20% high (M6.2). Every exit is a short (-1), and every break precedes $\frac{1}{2}\log p^{++}$. $(1.5,1)$ is not converged (M6.6). Data `num/l37/m6_{tag}_r{15p8,24p10,30p12,36p14}_s{0.001,0.0015}.jsonl`; table `num/lemma37_v2_m6tab.py`.

(M6.2) *Convergence.* At $(\frac{1}{2},3)$, $r30p12$ agrees with $r24p10$ to $\leq 4\cdot 10^{-5}$ at $P = 7, 19, 31, 47$ (all $t_* \leq 2.0$). Beyond $t \approx 2.0$ it does not: the $r24, r30, r36$ margins are $0.00580, 0.00514, 0.00513$ at $P = 67$ ($t \approx 2.13$; $r24$ high by 13%) and $0.00550, 0.00470, 0.00435$ at $P = 97$ ($t \approx 2.31$). At $P = 67$, however, $r36p14$ confirms $r30p12$ to $7.3\cdot 10^{-7}$: $r24$ is 13% high there, $r30$ is converged. *At $P = 97$ it is not* (v3): $r36$ is $3.45\cdot 10^{-4}$ (7.3%) below $r30$. We call a margin *resolution-converged* if $t_* \leq 2.0$ (where $r30$ confirms $r24$), or if its two finest resolutions, both $\geq r30p12$, agree to $\leq 10^{-5}$; so far this is $P \leq 47$ and $P = 67$ ($t \leq 2.14$). All other $(\frac{1}{2},3)$ entries with $P \geq 53$ are *upper estimates* that may be 10–20% high; at $P = 97$ the best value is the $r36$ margin 0.0043502, unconfirmed (an $r42p16$ run at $P = 97$, with $P = 67$ as

a control, is queued). The $r15$-based error bars ($^\flat$) overstate the error for $P \le 47$, where $r30$ confirms $r24$. They understate nothing, because every entry is an upper bound in the Ritz sense. For $\Theta_{1.5}$ the line is easier to resolve: $r15$ and $r24$ agree to $\le 10^{-5}$ up to $P = 47$, and (v3) $r30$ confirms $r24$ to $\le 2.2 \cdot 10^{-6}$ at $P = 7, 19, 31, 47, 67, 97$.

(M6.3) *Margin decay: a clean power law on the converged window* (corrects v1, K12; [NUM]). On the converged window at $(\frac{1}{2}, 3)$ ($5 \le P \le 67$), $m \approx 0.0552\,(p^+)^{-0.56}$. On the $r24$ part alone ($5 \le P \le 47$) $\beta = 0.56 \pm 0.01$ ($n = 13$) and $\beta = 0.54 \pm 0.02$ on $23 \le P \le 47$ ($n = 7$). Equivalently, $m\sqrt{p^+}$ stays in $[0.0433, 0.0458]$ for $17 \le p^+ \le 71$ (converged points) (Table 37.8, Table 37.9). The fit predicts $m = 0.00508$, 0.00417, 0.00361 at $p^+ = 71, 101, 131$. At $p^+ = 71$ the converged value ($r36$) is 0.005135, within 1% of the prediction (the fit includes this point; without it the prediction is 0.00507). At $p^+ = 101$ the $r30$ value 0.00470 is 13% above the prediction, and v2 noted that *if* $r30$ were converged there the decay would slow ($m\sqrt{q}$ from 0.0433 at $q = 71$ to 0.0472 at $q = 101$). *v3:* it is not converged. The $r36$ value is 0.0043502, 4.2% above the prediction, so $m\sqrt{q} \le 0.0437$ at $q = 101$. The successive differences $r24 \to r30 \to r36$ are $8.03 \cdot 10^{-4}$ and $3.45 \cdot 10^{-4}$ (ratio 0.43). A geometric extrapolation of them (heuristic, not a bound; the meshes are not nested) would give ≈ 0.0041, slightly *below* the prediction. *Whether the decay slows beyond $q \approx 70$ is therefore open*; an $r42p16$ run at $P = 97$ is queued. At $p^+ = 131$ only $r24$ exists (0.00520, an upper estimate). A fit over the whole $r24$ range would give a much smaller local exponent (0.28 on $P \ge 23$); that flattening is a resolution artefact of the unconverged $r24$ entries and is not used. In integers, the survival window $\Delta n = p^+(e^{2m} - 1) \approx 2mp^+$ grows like $\sqrt{p^+}$ but stays below one integer on the converged window (max 0.73): for every converged $q \le 71$ the line without its newest prime shorts before $q + 1$, as in Lemma 34 v2's drop-n data ($n \le 23$). None of the fitted laws is a candidate asymptotic. A fitted $a + b/\log p^+$ has $a < 0$ and would reach $m = 0$ at finite p^+, contradicting Thm. 37.7.4(b) (every margin at $\omega = \frac{1}{2}$ is positive), so it can only be a local description. (Table 37.6 lists all fits on the converged window.)

(M6.4) *Gap dependence: none detected, raw or Cramér-normalized.* Detrend the margins by $a + b/\log p^+$ and correlate the residuals with the raw rapidity gap $\frac{1}{2}\log(p^+/P)$, with the Cramér-normalized gap $(p^+ - P)/\log P$, and with the normalized *next* gap $(p^{++} - p^+)/\log p^+$. On the converged window ($n = 15$ at $(\frac{1}{2}, 3)$; $n = 15$ for $\Theta_{1.5}$; Table 37.8), Pearson $r = +0.44, +0.18, -0.01$ with permutation $p = 0.10, 0.52, 0.96$ at $(\frac{1}{2}, 3)$, and $p = 0.99, 0.54, 0.08$ for $\Theta_{1.5}$. On the full uniform $r24p10$ series (including the unconverged entries; Table 37.7) the smallest of 9 linear-detrend p-values is 0.04 (at the unconverged $(1.5, 1)$ line), and a quadratic-in-$\log p^+$ detrend gives $p \ge 0.35$; with 9 tests at the 5% level about 0.45 false positives are expected. *Verdict: no gap dependence at the resolution of the data.* This is what Thm. 37.10.1 leads one to expect: P is not an input to the margin, so the gap below p^+ could act only through $\mu(p^+)$ itself, and in the data μ varies smoothly with p^+. J. W.'s normalizations (Table 37.9, converged window): at $(\frac{1}{2}, 3)$ the flattest are the power-normalized margin (flat by construction), $\Delta n/\log p^+$ (the expected number of primes in the survival window, mean ≈ 0.15, CV 0.10, no significant slope) and $m\sqrt{p^+}$ (CV 0.09, small negative slope). These two descriptions differ in law ($\Delta n/\log q$ constant means $m \propto \log q/q$; the power law means $m \propto q^{-0.56}$) but agree on $q \le 53$, where $\sqrt{q}/\log q$ varies little. Converged data at $q \gtrsim 100$ would separate them; the $r36$ value at $q = 101$ (0.00435, an unconfirmed upper estimate; v3) lies above both (≈ 0.0042 and ≈ 0.0034).

(M6.5) *Lemma 39's mean-energy pattern fails* (Lemma 39, §5, R2; Table 37.10). Lemma 39 proposed $m(P) \approx b \log p^+/(\sqrt{p^+} - 1)$ with $b = 0.0226$, fitted to v1's (too large) margins, and predicted $m(31), m(47), m(97), m(113) \approx 0.016, 0.014, 0.0115, 0.011$. The corrected values are $0.0073, 0.0061, 0.0044, 0.0053$, smaller by factors $2.2, 2.3, 2.6$. Since our values are upper

estimates, the true discrepancy can only be larger. Refitting b on the corrected $P \le 19$ data ($b = 0.0166$) still over-predicts: $0.0118, 0.0105, 0.0085$. Out of sample it is the worst of the six one- and two-parameter models (Table 37.6). Lemma 39's trend alternative $(0.019, 0.019, 0.018)$ fails too, by a factor of about 3. *Verdict: the pattern is falsified.* It was post hoc and low-confidence, and its only support came from the biased v1 data. Lemma 39 is not edited; the falsification is recorded here (K13) and in STATE.

(M6.6) $(1.5, 1)$ *is not converged.* At $P = 7$ the four resolutions $r15, r24, r30, r36$ give $0.00409, 0.00288, 0.00238, 0.00199$, and at $P = 19$ they give $0.00284, 0.00193, 0.00152, 0.00125$. The values fall roughly linearly in $1/r$, and a linear extrapolation in $1/r$ (heuristic, not a bound) would put the $P = 7$ margin at 1–$2 \cdot 10^{-4}$. Thm. 37.7.4(b) still guarantees $m > 0$. So at $\omega = 1.5$ the cascade is passive by a margin that is unknown but smaller than the listed upper estimates. A non-smooth eigenvector is the likely cause; the atom weights $b_q \approx q$ grow there (Remark 37.10.2(4)), but we have not verified this. v1's $(1.5, 1)$ column and this column are both upper estimates.

(M6.7) *The headline line.* For $\Theta_{1.5}$ the margins ($2 \le P \le 127$) are 2–150% larger than at $(\frac{1}{2}, 3)$, the ratio growing with P, and they decay more slowly ($\beta = 0.20 \pm 0.01$ on $5 \le P \le 127$). Larger margins on the exponential line agree in sign with (M7.2): there the Suzuki break lies below the exponential one, by an $O(\omega^2)$ amount whose coefficient depends on P.

model	line	coef. (all $P \ge 2$)	rms (all)	rms test ($P > 19$; train $P \le 19$)	n
$a + b/\log p^+$	$(\frac{1}{2}, 3)$	-0.004861, 0.04477	$5.7 \cdot 10^{-4}$	$9.3 \cdot 10^{-4}$	16
$b\,(p^+)^{-1/2}$	$(\frac{1}{2}, 3)$	0.05165	$1.6 \cdot 10^{-3}$	$1.3 \cdot 10^{-3}$	16
$a + b\,(p^+)^{-1/2}$	$(\frac{1}{2}, 3)$	-0.00314, 0.06191	$8.2 \cdot 10^{-4}$	$1.9 \cdot 10^{-3}$	16
$b/\log p^+$	$(\frac{1}{2}, 3)$	0.03405	$2.1 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	16
L39: $b \log p^+/(\sqrt{p^+} - 1)$	$(\frac{1}{2}, 3)$	0.01482	$4.2 \cdot 10^{-3}$	$4.5 \cdot 10^{-3}$	16
$b\,(p^+)^{-\beta}$	$(\frac{1}{2}, 3)$	$b = 0.0615,\ \beta = 0.591 \pm 0.009$	$6.7 \cdot 10^{-4}$	$5.7 \cdot 10^{-4}$	16
$a + b/\log p^+$	$c = 1.5$	0.003355, 0.03502	$4.0 \cdot 10^{-4}$	$4.6 \cdot 10^{-4}$	17
$b\,(p^+)^{-1/2}$	$c = 1.5$	0.06398	$2.7 \cdot 10^{-3}$	$3.8 \cdot 10^{-3}$	17
$a + b\,(p^+)^{-1/2}$	$c = 1.5$	0.005146, 0.04689	$1.3 \cdot 10^{-3}$	$2.8 \cdot 10^{-3}$	17
$b/\log p^+$	$c = 1.5$	0.04256	$1.5 \cdot 10^{-3}$	$1.9 \cdot 10^{-3}$	17
L39: $b \log p^+/(\sqrt{p^+} - 1)$	$c = 1.5$	0.01909	$2.1 \cdot 10^{-3}$	$7.0 \cdot 10^{-4}$	17
$b\,(p^+)^{-\beta}$	$c = 1.5$	$b = 0.0398,\ \beta = 0.304 \pm 0.023$	$1.9 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$	17
$a + b/\log p^+$	$(1.5, 1)$	-0.004398, 0.02204	$1.2 \cdot 10^{-3}$	$3.4 \cdot 10^{-3}$	31
$b\,(p^+)^{-1/2}$	$(1.5, 1)$	0.01652	$2.2 \cdot 10^{-3}$	$1.2 \cdot 10^{-3}$	31
$a + b\,(p^+)^{-1/2}$	$(1.5, 1)$	-0.002354, 0.02599	$1.8 \cdot 10^{-3}$	$5.5 \cdot 10^{-3}$	31
$b/\log p^+$	$(1.5, 1)$	0.01019	$2.3 \cdot 10^{-3}$	$2.0 \cdot 10^{-3}$	31
L39: $b \log p^+/(\sqrt{p^+} - 1)$	$(1.5, 1)$	0.004005	$2.8 \cdot 10^{-3}$	$2.1 \cdot 10^{-3}$	31
$b\,(p^+)^{-\beta}$	$(1.5, 1)$	$b = 0.0145,\ \beta = 0.594 \pm 0.053$	$2.2 \cdot 10^{-3}$	$8.3 \cdot 10^{-4}$	31

TABLE 37.6. Fits of the margins on the resolution-converged window ($P \ge 2$, $t_* \le 2.0$, best resolution per point; $(1.5, 1)$: all points, none converged), [NUM]. "rms test": trained on $P \le 19$, evaluated on $P > 19$. `num/lemma37_v2_tables2.py`.

37.11. Proof strategies, assessed

(a) Euler factorization into local lossless sections. *Status:* the naive form is *refuted* (Prop. 37.7.1(2)): no Euler factor is Schur, and the archimedean section alone is active (Cor. 37.2.5). Passivity of products of passive factors is trivial, but there are no passive factors. The rescue of the archimedean section is global in p (the pole of ζ). Any regrouping of local factors with local archimedean counterterms that made each section Schur, *with a product convergent on* $\operatorname{Re} s > \frac{1}{2}$, would prove RH directly. Convergence of Euler-type products in the strip is itself of RH depth.

line (r24p10 only)	n	detrend	raw gap r (perm. p)	Cramér gap r (perm. p)	Cramér next gap r (perm. p)
$(\frac{1}{2},3)$	31	$a+b/\log p^+$	+0.21 (0.27)	+0.12 (0.54)	+0.04 (0.84)
$(\frac{1}{2},3)$	31	quadr. in $\log p^+$ ($\log m$)	+0.08 (0.67)	+0.05 (0.78)	+0.01 (0.98)
$c=1.5$	31	$a+b/\log p^+$	−0.04 (0.84)	+0.15 (0.41)	−0.24 (0.18)
$c=1.5$	31	quadr. in $\log p^+$ ($\log m$)	−0.06 (0.76)	+0.07 (0.72)	−0.15 (0.42)
$(1.5,1)$	31	$a+b/\log p^+$	−0.38 (0.04)	−0.11 (0.55)	−0.03 (0.88)
$(1.5,1)$	31	quadr. in $\log p^+$ ($\log m$)	−0.17 (0.35)	−0.13 (0.50)	−0.07 (0.71)

TABLE 37.7. Gap dependence of the detrended margins, uniform $r24p10$ series over the *whole* range (entries with $t_* > 2.0$ unconverged), [NUM]. Permutation p-values from 4000 shuffles. `num/lemma37_v2_m6tab.py`, `l37/m6_uniform.json`.

line	n	P range	β ($b\,(p^+)^{-\beta}$)	β on $P \geq 23$	raw gap r (p)	Cramér gap r (p)	Cramér next r (p)
$(\frac{1}{2},3)$	15	2–47	0.56 ± 0.01	0.54 ± 0.02 (7)	+0.44 (0.10)	+0.18 (0.52)	−0.01 (0.96)
$c=1.5$	15	2–47	0.24 ± 0.01	0.19 ± 0.03 (7)	+0.00 (0.99)	+0.17 (0.54)	−0.46 (0.08)

TABLE 37.8. Converged window ($r24p10$, $t_* \leq 2.0$, $2 \leq P \leq 47$), [NUM]: power-law exponent of the margin and gap tests of the $a+b/\log p^+$ residuals (Pearson r, permutation p from 4000 shuffles). `num/lemma37_v2_conv.py`, `l37/m6_conv.json`.

line	normalized margin	mean	CV	slope vs. $\log p^+$ (rel., p)
$(\frac{1}{2},3)$	m (rapidity)	0.01196	0.64	−0.69 (2.4e-08)
$(\frac{1}{2},3)$	m/g, $g=\frac{1}{2}\log(p^+/P)$	0.1379	0.44	+0.10 (0.44)
$(\frac{1}{2},3)$	$\Delta n = p^+(e^{2m}-1)$ (integers)	0.4564	0.32	+0.36 (6.1e-10)
$(\frac{1}{2},3)$	$\Delta n/\log p^+$ (expected primes)	0.1508	0.10	−0.01 (0.83)
$(\frac{1}{2},3)$	$m\log p^+$	0.02987	0.17	−0.19 (1.4e-11)
$(\frac{1}{2},3)$	$m\sqrt{p^+}$	0.0469	0.09	−0.10 (3.3e-07)
$(\frac{1}{2},3)$	$m\sqrt{p^+}/\log p^+$ (L39)	0.01835	0.60	−0.62 (2.5e-06)
$(\frac{1}{2},3)$	$m\,(p^+)^{\beta}$ (fitted β)	0.06157	0.03	−0.00 (0.96)
$c=1.5$	m (rapidity)	0.01618	0.37	−0.36 (6.6e-07)
$c=1.5$	m/g, $g=\frac{1}{2}\log(p^+/P)$	0.2419	0.59	+0.41 (0.0047)
$c=1.5$	$\Delta n = p^+(e^{2m}-1)$ (integers)	0.8648	0.61	+0.60 (5.1e-07)
$c=1.5$	$\Delta n/\log p^+$ (expected primes)	0.2547	0.35	+0.31 (2.1e-05)
$c=1.5$	$m\log p^+$	0.04556	0.08	+0.08 (2.5e-09)
$c=1.5$	$m\sqrt{p^+}$	0.07492	0.20	+0.20 (1.1e-06)
$c=1.5$	$m\sqrt{p^+}/\log p^+$ (L39)	0.02574	0.32	−0.25 (0.0011)
$c=1.5$	$m\,(p^+)^{\beta}$ (fitted β)	0.03997	0.08	−0.00 (0.89)
$(1.5,1)$	m (rapidity)	0.002249	1.52	−1.10 (6.9e-06)
$(1.5,1)$	m/g, $g=\frac{1}{2}\log(p^+/P)$	0.043	0.67	+0.13 (0.29)
$(1.5,1)$	$\Delta n = p^+(e^{2m}-1)$ (integers)	0.1482	0.43	+0.36 (3.3e-08)
$(1.5,1)$	$\Delta n/\log p^+$ (expected primes)	0.03978	0.41	−0.02 (0.79)
$(1.5,1)$	$m\log p^+$	0.006022	0.54	−0.39 (6.7e-06)
$(1.5,1)$	$m\sqrt{p^+}$	0.01077	0.43	−0.20 (0.011)
$(1.5,1)$	$m\sqrt{p^+}/\log p^+$ (L39)	0.003767	1.38	−0.92 (7.2e-05)
$(1.5,1)$	$m\,(p^+)^{\beta}$ (fitted β)	0.01514	0.33	−0.06 (0.32)

TABLE 37.9. J. W.'s normalizations of the margin (converged window $t_* \leq 2.0$, best resolution per point; $(1.5,1)$ unconverged), [NUM]: mean, coefficient of variation, and the relative slope against $\log p^+$ with its p-value. "Flat" would mean CV small and slope ≈ 0. The power-normalized row uses the in-sample β and is flat by construction. `l37/m6_analysis.json`.

Circularity: none formally; the risk is that any convergent regrouping presupposes zero-free information. *Plausibility:* low as a proof route, high as a diagnostic: the cascade form (Thm. 37.7.4(d)) turns $\mathrm{G5}_{\mathrm{line}}$ into countably many finite-prime statements.

P	p^+	L39 pred. ($b = 0.0226$)	L39 trend pred.	L39 refit ($b = 0.0166$, $P \le 19$)	power law ($P \le 19$)	corrected $m(P)$
31	37	0.016	0.019	0.0118	0.0068	$0.0073 \pm 7e-07$
47	53	0.014	0.019	0.0105	0.0054	$0.0061 \pm 4e-05$
97	101	0.0115	0.018	0.0085	0.0036	0.0044 ± 0.0003
113	127	0.011	0.018	0.0078	0.0031	0.0053

TABLE 37.10. Test of Lemma 39's prediction (R2), [NUM]. "L39 pred." and "L39 trend pred." are copied from Lemma 39, §5. "Refit" uses the same model with b fitted on the corrected $P \le 19$ margins; "power law" is $b\,(p^+)^{-\beta}$ fitted on $P \le 19$. The corrected values are upper estimates.

n	prime?	L34 drop-n margin (Arb bracket)	M6 $m(P)$, $p^+ = n$	difference	side
2	yes	[0.05193, 0.05200]	0.05200	+3.1e-05	-1
3	yes	[0.03460, 0.03464]	0.03462	-1.8e-06	-1
4	no	[0.02595, 0.02598]	—	—	-1
5	yes	[0.02402, 0.02405]	0.02405	+1.4e-05	-1
6	no	[0.02028, 0.02031]	—	—	-1
7	yes	[0.01911, 0.01913]	0.01912	+3.5e-08	-1
8	no	[0.01601, 0.01603]	—	—	-1
9	no	[0.01516, 0.01518]	—	—	-1
10	no	[0.01406, 0.01408]	—	—	-1
11	yes	[0.01441, 0.01442]	0.01441	+6.0e-07	-1
12	no	[0.01177, 0.01178]	—	—	-1
13	yes	[0.01309, 0.01310]	0.01309	+1.1e-06	-1
16	no	[0.01049, 0.01050]	—	—	-1
17	yes	[0.01111, 0.01112]	0.01112	-4.6e-07	-1
19	yes	[0.01050, 0.01051]	0.01050	-1.4e-06	-1
20	no	[0.00858, 0.00858]	—	—	-1
23	yes	[0.00936, 0.00937]	0.00937	+2.6e-06	-1

TABLE 37.11. Numerical check of Thm. 37.10.1: Lemma 34 v2's drop-n margins (independent Arb solver, read-only from `num/l34/v2_summary.json`) against M6 margins with $p^+ = n$, [NUM]. Composites are listed for comparison: removing them also shorts the line.

(b) Monotonicity / comparison (de Branges ordering, Krein comparison). The de Branges ordering theorem orders the spaces $\mathcal{H}(E_t)$ of one chain [**81, 234**] ([BIB]). It compares nested spaces, not different symbols, so it does not move ω. The natural comparison in ω is the cascade identity [THM]

$$\Theta^{\omega_1+\omega_2,1}(s) = \Theta^{\omega_1,1}(s-\omega_2)\,\Theta^{\omega_2,1}(s+\omega_1).$$

The second factor is a lossy, robustly passive section if the depth-ω_2 line is passive. The first is the depth-ω_1 symbol evaluated *deeper* in the strip, on $\operatorname{Re} s > \frac{1}{2} - \omega_2$, which is uncontrolled. So deforming ω downward from $\frac{1}{2}$ needs at every step exactly the information of P3. Upward monotonicity (passive at $\omega \Rightarrow$ passive at $\omega' > \omega$) holds when the passivity at ω involves no ladder cancellations (Remark 37.3.4(2) and Lemma 33, Thm. 6.2(a)); in general it is [OPEN]. *Plausibility:* low; it is P3 in disguise.

(c) Schur algorithm / Verblunsky coefficients. In the continuous det-one gauge every real $r \in L^1_{\mathrm{loc}}$ gives a passive line. The "Schur parameter" inequality degenerates into finiteness of $\int r$, and in a Goupillaud discretization $|r_k| < 1$ is automatic for $Z_{0,k} \in (0, \infty)$ (§37.6). So (c) restates "no break" and adds no strength. Szegő- and Baxter-type theorems [**26, 247**] relate coefficient decay to an absolutely continuous spectral measure with a Szegő weight. Our spectral data are discrete (the zeros), so the Szegő condition fails and those theorems do not apply. The real use of (c) is *numerical*: an $O(N^2)$ layer-peeling evaluation of Z_0 to large t. *Plausibility as proof:* very low; as a tool: good.

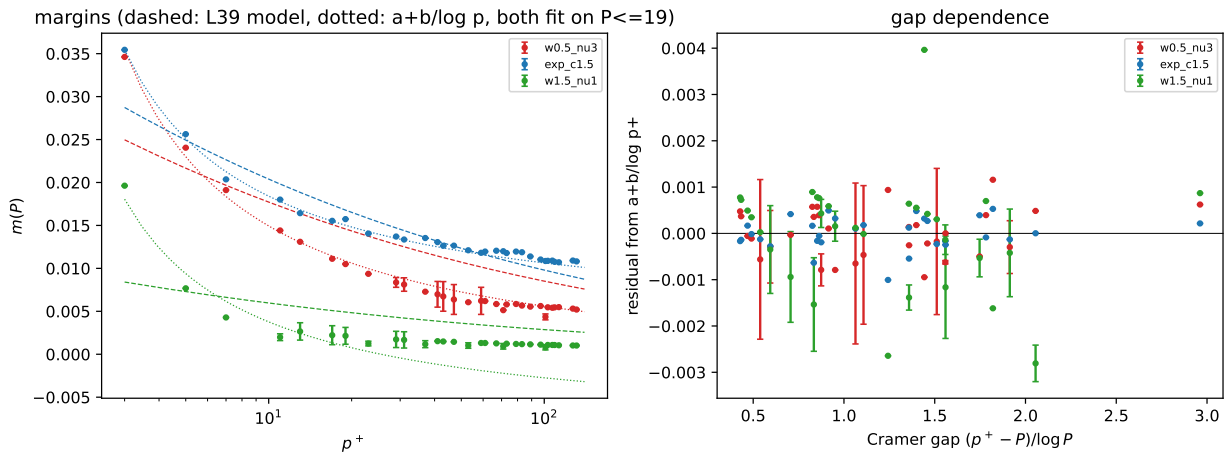

FIGURE 37.5. M6, [NUM]. Left: corrected margins versus p^+ (log scale), with Lemma 39's model (dashed) and $a + b/\log p^+$ (dotted), both fitted on $P \le 19$. Right: residuals from $a + b/\log p^+$ versus the Cramér-normalized gap.

(d) Energy / Lyapunov functional along t. Prop. 37.6.1 identifies the defect $1 - U[t]^2$ as leakage. The target form is an a priori inequality for the gap $g(t) = 1 - \|\mathcal{K}[t]\|$: *if* $g' \ge -F(g)$ with $\int_{0^+} dg/F(g) = \infty$ (Osgood), *then* $g > 0$ for all t ([THM], ODE comparison). The data demand super-exponential collapse ($g = 8.6 \cdot 10^{-2}$, $2.2 \cdot 10^{-4}$ at $t = 0.25, 0.5$), so $F(g) \approx Cg \log(1/g)$ is the scale. No such inequality has been derived, and $\log Z_0$ is not monotone in general (r can change sign). The collapse is reminiscent of Slepian concentration for time- and band-limiting (Lemma 22's prolate) [ANALOGY]. *Circularity:* an a priori bound must use the Euler product, or it would hold for $\zeta(s)\zeta(s+a)$-type symbols with off-line zeros (Lemma 31, Prop. 2.2). *Plausibility:* very low, but this is the form a proof would take.

(e) What is unconditional now.

(e1) $\omega \ge \frac{1}{2}$: passive for all t (Lemma 34, Prop. 2.4; Su21), with $\sup_t \|\mathcal{K}[t]\| = 1$ exactly (Cor. 37.2.4) [THM].

(e2) All admissible (ω, ν), short horizon. The Schur test $\|\mathcal{K}[t]\| \le \int_0^{2t} |K|$ and the Hilbert–Schmidt bound $\|\mathcal{K}[t]\|^2 \le \int_0^{2t} (2t - x)K(x)^2 dx$ give $\|\mathcal{K}[t]\| < 1$ for $t < t_1$, with $t_1 = 0.224$ $(\frac{1}{2}, 3)$, 0.190 $(0.3, 4)$, 0.190 $(0.2, 6)$, 0.224 $(1.5, 1)$. These are computed from Lemma 34's G tables (`num/l37/schurtest.json`): [THM] for the bounds, [NUM] for the numbers (Talbot tables, not interval-certified). This is the strongest explicit no-break horizon we can state for $\omega < \frac{1}{2}$. It is short of $\frac{1}{2}\log 2$ and says nothing about RH: small t only probes the archimedean factor.

(e3) Prime arrival at $\omega \ge \frac{1}{2}$: $t_*(P) > \frac{1}{2}\log p^+$ for all P (Thm. 37.7.4(b)) [THM].

(e4) Depth: passivity for ω in a sequence $\downarrow h$ is *equivalent* to $\beta^* \le \frac{1}{2} + h$. Only $h = \frac{1}{2}$ is known; any $h < \frac{1}{2}$ is P3 [RESTATEMENT].

(e5) *Not available:* a blindness theorem for the line, i.e. "off-line zeros above height H cannot break the line before $t(H)$", the analogue of Lemma 35, Thm. 4.1. Lemma 34 §3(4) asserts such a scaling informally. We flag it as *unproved*: the near-unit gaps ($\le 10^{-7}$ at $t \ge 0.7$) make perturbative arguments fragile [OPEN].

37.12. Roadmap

Score = 0.6·significance+0.4·probability, both in $[0, 1]$ and estimated. Because the formula weights significance, RH-depth items score well despite tiny probability; we list both numbers so this is visible.

	milestone	sig.	prob.	score
M1	Passivity criterion $\sup_t \|\mathcal{K}[t]\| = \|\Theta\|_{H^\infty(\mathbb{C}_+)}$; depth theorem; correction of "sup < 1" (*done here*)	0.7	1.0	0.82
M2	Prime arrival theorem, cascade form of G5, cascade numerics to $P = 19$ (*done here*)	0.5	1.0	0.70
M3	Prove $t_*(P) > \frac{1}{2}\log p^+$ for all P at one $\omega < \frac{1}{2}$ (= no zeros with $\beta > \frac{1}{2} + \omega$ up to ladders)	1.0	0.01	0.60
M4	A priori gap inequality $g' \geq -F(g)$ (Osgood) from the Euler product	0.8	0.03	0.49
M5	Interval-certified no-break up to $t = \frac{1}{2}\log 2$ at $\omega < \frac{1}{2}$ (rigorous G via ball arithmetic + rigorous norm upper bounds)	0.25	0.7	0.43
M6	High-precision cascade margins to $P \sim 10^2$ ($t \sim 2.3$); gap test (*done in v2*: §37.10; no gap dependence)	0.3	1.0	0.58
M7	Exponential-line numerics: direct kernel K_c, cascade, comparison with $\omega \to 0$ (*done in v2*: §37.8)	0.25	1.0	0.55
M10	(v2) Asymptotics of the single-atom margin $\mu(q)$ (Thm. 37.10.1); a lower bound $\mu(q) > 0$ for all q at one $\omega < \frac{1}{2}$ is M3	0.5	0.1	0.34
M11	(v2) Strict pointwise bound for Θ_c under RH (Prop. 37.4.4(b))	0.3	0.4	0.34
M8	Upward monotonicity in ω without the no-ladder hypothesis	0.3	0.5	0.38
M9	Blindness theorem for the line (analogue of Lemma 35, Thm. 4.1)	0.4	0.3	0.36

By score (v2): M1, M2, M6, M7 (done), M3, M4, M5, M8, M9, M10, M11. Practically, the next *attemptable* items are M5, M11 and the M6 refinements of Question 37.13.4. M3 and M4 are where G5 lives; Thm. 37.10.1 restates M3 as positivity of the single-atom margins $\mu(q)$.

37.13. Open questions

v1 Questions 1, 2 and 6 were answered (corrected $\mathrm{G5}_{\mathrm{line}}$ accepted; exponential line as headline; M6, M7 approved). v1 Questions 4 (arch rescue vs. small-primes baseline) and 5 (Iwasawa tower in the prime-cutoff direction) remain open as stated in v1. v2 questions:

Question 37.13.1 (gap dependence). The margin is the full line's tolerance to losing its newest prime (Thm. 37.10.1), and the data show no gap dependence, raw or Cramér-normalized ((M6.4)). Does your peak/tail intuition predict how $\mu(q)$ depends on q itself? Observed on the converged window ($q \leq 71$): $\mu(q) \approx 0.0552\, q^{-0.56}$ at $(\frac{1}{2}, 3)$, i.e. $\mu(q)\sqrt{q} \approx 0.044$, but only $\approx q^{-0.24}$ on the exponential line $c = 1.5$; not proportional to the removed atom's weight (Remark 37.10.2(4)). Does the $q^{-1/2}$ scale, which matches the loop gain $q^{-1/2}$ of the local Euler section on the critical line (Prop. 37.7.1; a coincidence of exponents, [ANALOGY], not a derivation), fit your picture?

Question 37.13.2 (Lemma 39). Lemma 39's mean-energy pattern is falsified ((M6.5)). Retire it, and record the falsification in a Lemma 39 erratum (Lemma 39 itself not edited)?

Question 37.13.3 (headline form). Accept $\mathrm{G5}_{\mathrm{exp}}$ (Def. 37.4.3) with the open step of Prop. 37.4.4(b)? That step is whether RH gives the strict bound $\|\mathcal{K}_c[t]\| < 1$ at each t for the non-meromorphic Θ_c. The weak form $\sup_t \leq 1$ is RH-equivalent unconditionally.

Question 37.13.4 (next computation). Which is next? (v3: an $r42p16$ run of $(\frac{1}{2}, 3)$ at $P = 97$ and $P = 67$ is queued behind the Lemma 24 runs.) (a) M6 at $r36p14$ (or finer) for all $P \geq 53$, to extend the converged window beyond $t = 2.0$ ((M6.2)) and separate the two laws of (M6.4). (b)

The exponential line to $P = 127$ at $r30p12$. (c) $(1.5, 1)$ at $r45p16$ to settle its convergence ((M6.6)). (d) Composite drop-n to $n \le 60$, the matching Lemma 34 question.

Question 37.13.5 (past the first exit). D3 shows unbounded open/short counts for truncated lines, with a Landau-type rate ((D3.3)). Should the Landau–Slepian heuristic be turned into a lemma, the truncated counterpart of Lemma 36's exact $\kappa/2$?

37.14. Corrections

(K1) Lemma 34 §(d) and the task statement: "$\sup_t \|\mathcal{K}^{\omega,\nu}[t]\| < 1$" is false whenever the line is passive and Θ is nonconstant inner (Cor. 37.2.4). The correct form is pointwise < 1, equivalently $\sup \le 1$. Suzuki's $J \to 0$ is not needed for ⇒RH.

(K2) J. W.'s heuristic: local Euler sections are *stable* on the critical line but not *passive* (Prop. 37.7.1(2)).

(K3) $\frac{1}{2}\log 12 \to \frac{1}{2}\log 13 = 1.2825$ for the exact reach of the primes ≤ 11 (Prop. 37.7.3).

(K4) Lemma 34's question on $t_*^{\mathrm{arch}} - \frac{1}{2}\log 2 > 0$: answered, structural (Thm. 37.7.4(b)).

(K5) Lemma 34 Remark on $A = 2t$: true as an information horizon; the atoms agree only at prime powers, and composite onsets are echoes (Prop. 37.5.1).

(K6) Lemma 34 §3(4), "off-line zeros at height $\gtrsim 10^{12}$ act only at $t \gtrsim \frac{\nu}{2}\log 10^{12}$": unproved heuristic (§37.11(e5)).

(K7) Lemma 33, Thm. 6.2(b): one sequence $h' \downarrow h_1$ suffices (Thm. 37.3.2).

v2 corrections (errors in v1 found during the v2 work; v1 is archived unchanged).

(K8) *v1 Table 1 margins were too large for $t \gtrsim 0.8$* (found by Lemma 34 v2, correction V7). At $(\frac{1}{2}, 3)$ the v1 breaks were late by 0.003–0.011, against a stated h-stability of $\le 1.5 \cdot 10^{-3}$. Corrected margins (Table 37.1): $P = 3$: 0.0240 (v1 0.0272); 5: 0.0191 (0.0252); 7: 0.0144 (0.0218); 11: 0.0131 (0.0232); 13: 0.0111 (0.0216); 17: 0.0105 (0.0209); 19: 0.0094 (0.0204). $P = 1, 2$ agree to $\le 5 \cdot 10^{-4}$. At $(1.5, 1)$ v1's values were too large by factors of 1.7–3. Cause: double precision cannot resolve gaps $1 \pm \lambda$ below $\sim 10^{-12}$, reached by $t \approx 1.1$ (§37.7.3).

(K9) "Every break computed in TAA is a short" holds for *first* exits only. Past the first exit, opens appear within 0.02–0.06 and then alternate with shorts (§37.9).

(K10) v1 said $(0.3, 4)$ and $(0.2, 6)$ give identical exits to 10^{-4}. They differ by $1.6 \cdot 10^{-4}$ to $4.2 \cdot 10^{-4}$, as the $O(\omega^2)$ law predicts ((M7.2)).

(K11) Remark 37.7.7: Iwasawa's Γ is $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)) \cong 1 + p\mathbb{Z}_p$ (flagged by Lemma 38), not the full $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times$.

(K12) v1's summary and Question 3 described the margins as "about 0.02", i.e. "about 4% in n", with an apparent plateau. The plateau was an artefact of K8: the corrected margins keep decreasing (§37.10). v1's conjecture mechanism "large prime gaps would help" is void: the margin depends on p^+ only (Thm. 37.10.1).

(K13) *For Lemma 39 §5 (not edited here):* its margin fits used v1's Table 1. Its post-hoc pattern $m(P) \approx b \log p^+/(\sqrt{p^+} - 1)$ and the prediction $m(97) \approx 0.0115$ fail on the corrected data (§37.10, (M6.5)).

v3 corrections (v2 is archived unchanged).

(K14) *$r30p12$ is not converged at $P = 97$* at $(\frac{1}{2}, 3)$: $r36p14$ gives 0.0043502 vs. 0.0046952 (−7.3%). v2's conditional "the decay slows beyond $q \approx 70$" rested on the $r30$ value; the question is open ((M6.2), (M6.3)).

(K15) The $r36$–$r30$ agreement at $P = 67$ is $7.3 \cdot 10^{-7}$, not $8 \cdot 10^{-7}$.

37.15. Status

claim	tag	where
$\sup_t \|\mathcal{K}[t]\| = \|\Theta\|_{H^\infty(\mathbb{C}_+)}$; finite-horizon CF form	[THM] (Nehari [CITED, secondary])	Thm. 37.2.2
"$\sup_t < 1$" false in passive case	[THM]	Cor. 37.2.4
arch ultimate gain $2\pi^2$ at $(1.5, 1)$; ∞ for $\omega \le \frac{1}{2}$	[THM], [NUM]	Cor. 37.2.5
depth theorem; $G5_{\text{line}} \iff$ RH; ν-independence	[THM], [RESTATEMENT]	Thm. 37.3.2, Cor. 37.3.3
exponential line $\iff$ RH for one c; limit $O(\omega^2)$	[THM], [RESTATEMENT], [NUM]	Thm. 37.4.1
$A = 2t$ horizon; composite onsets = echoes	[THM]	Prop. 37.5.1
energy identity	[THM]	Prop. 37.6.1
local sections stable, not passive; Tate; smooth numbers	[THM]	Props. 37.7.1–37.7.3
prime arrival; cascade form of G5	[THM], [RESTATEMENT]	Thm. 37.7.4
cascade margins $P \le 19$ (v1 values too large, K8; corrected)	[NUM]	Table 37.1
headline form $G5_{\text{exp}}$; weak form $\iff$ RH; strict form under RH open	[DEF], [THM], [OPEN]	Def. 37.4.3, Prop. 37.4.4
single-atom reduction $m(P) = \mu(p^+)$	[THM]	Thm. 37.10.1
M6 margins to $P = 127$; no gap dependence; L39 pattern falsified	[NUM]	§37.10
(v3) $r30$ unconverged at $P = 97$; whether the margin decay slows beyond $p^+ \approx 70$	[NUM], [OPEN]	(M6.2), (M6.3)
M7 exponential line; $c \leftrightarrow \nu\omega$ at $O(\omega^2)$; full line no exit to $t = 2.45$	[NUM]	§37.8
D3 opens/shorts past first exit; first exits shorts only	[NUM], [ANALOGY] (Landau)	§37.9
margins $\to 0$ along a subsequence	[CONJ]	§37.7.3
Iwasawa link	[ANALOGY]	Rem. 37.7.7
G5 supplied?	**No**	

Files: `num/lemma37_{trunc,gain,archsup,schurtest,explimit,table}.py`; v2: `num/lemma37_{gtab, gtab_hp,hpline,arbline,m6,m6_analysis,m6_probe,m7_double,m7_cnu,m7_full,d3, landau,v2_tables,v2_tables2,v2_m6tab,v2_macros,v2_figs}.py`, data `num/l37/m6_*`, `m7*`, `d3_*`, `gtabhp_*`, log `num/lemma37_progress.md`; data `num/l37/`; figure `figs/lemma37_cascade.{pdf, png}`; references `lit/lemma37/verified_refs.md`, `crossref.txt`, `cp_nehari.{pdf,txt}`.

CHAPTER 38

Lemma 38: Rapidity and the Iwasawa parameter — one idele class group

Galois symmetry, Bisognano–Wichmann rapidity, Bost–Connes, and where the dictionary breaks

Standalone note `lemma38.tex`, October 21, 2026.

Abstract and verdict

Tags. [THM] proved here; [CITED] checked at the source (`lit/lemma38/verified_refs.md`); [CITED, secondary] checked only through a named secondary source; [BIB] bibliographic data only (Crossref), content not read; [DEF] definition; [ANALOGY] analogy only; [OBSERVATION] a coincidence or correspondence noticed but not proved to be more; [NUM] numerical check (`num/lemma38_checks.py`, `.json`, `.out`); [RESTATEMENT] equivalent restatement of RH/GRH; [CONJ] conjecture.

Question (J. W.). Is Iwasawa's "group element acting on a module" the same structure as rapidity acting through the modular (Bisognano–Wichmann) flow on Rindler = Mellin modes (Lemma 31)? And what, precisely, is a Galois symmetry, and how does it translate into rapidity?

Answer. Both groups sit inside *one* group, the idele class group $C_{\mathbb{Q}} \cong \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^{\times}$ (Thm. 38.2.1), but in *different factors*, and they are seen by *different number systems*.

(1) There is exactly one global rapidity: $\eta_\infty = \log|\cdot|$, the logarithm of the idelic norm. It is the archimedean scaling/boost of Lemma 31, and every local valuation feeds into it, the prime p with step $\log p$, the Euler-factor delay of Lemma 37 (Prop. 38.2.2). Galois symmetry is the compact complement $C_{\mathbb{Q}}^1 \cong \widehat{\mathbb{Z}}^{\times} \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, which commutes with rapidity. Iwasawa's $\Gamma \cong 1 + p\mathbb{Z}_p$ lies in this compact Galois factor (§38.1). **So the chat claim "the Iwasawa group is the p-adic rapidity group" is false inside** $C_{\mathbb{Q}}$ (Correction 38.2.6).

(2) It becomes true in a precise p-adic sense. Each Frobenius idele class f_n has an archimedean rapidity $-\log n$ and, for $p \nmid n$, a p-adic rapidity $-\log_p n$ (Prop. 38.2.4). Complex characters see only the first, and p-adic characters see the second. The Mellin variable $x^s = e^{s\eta_\infty}$ and the Iwasawa variable $T = \kappa(\gamma)^s - 1 = \exp_p(s \log_p \kappa(\gamma)) - 1$ are the same construction, the exponent dual to a logarithm (Thm. 38.3.1). But the complex unitary dual of Γ is the *discrete* group μ_{p^∞}, so Γ has no "critical line" and no modular flow. No nontrivial continuous homomorphism exists between $\mathbb{R}$ and $\mathbb{Z}_p$ in either direction (Prop. 38.2.5).

(3) *Strongest rigorous bridge: Bost–Connes* (§38.5). The BC time evolution is the action *dual* to rapidity. The energies are the rapidities $\log n$, and $(\beta, t) = (\mathrm{Re}\, s, -\mathrm{Im}\, s)$ (Prop. 38.5.3). $\widehat{\mathbb{Z}}^{\times} = \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ acts by symmetries commuting with it [CITED]. The Galois *sector* χ carries the order parameter $\tau(\bar\chi)L(\beta,\chi)/\zeta(\beta)$ (Thm. 38.5.4, [THM] + [NUM]). This vanishes as $\beta \downarrow 1$ *because* of the pole of ζ and Dirichlet's $L(1,\chi) \neq 0$. At integer $\beta = n$ and along the p-cyclotomic tower, these order parameters are, up to explicit archimedean factors, the Kubota–Leopoldt values $L_p(1-n, \bar\chi\omega^n)$ that Iwasawa's characteristic series

interpolates (Thm. 38.5.6). Connes–Consani construct a p-adic BC system whose partition function is $L_p(\beta,1)$ and whose "Hamiltonian" is the Iwasawa logarithm, extended to Iwasawa's disc [CITED].

(4) *Where it breaks* (§38.7). $\mathbb{C}_p$ is not ordered, so there are no states, no Hilbert space and no positivity. Iwasawa's λ counts *all* p-adic zeros, and no "off-line" notion exists, so there is no analogue of our index $\kappa = N_+$. A "p-adic modular theory" keeps only the algebraic KMS relation.

(5) *What transfers* (§38.8). (a) Lemma 31 Thm. 6.5 ($\xi(1+it)$ is KMS_1) is one member of a family. $\xi(c+it)$ is KMS_{2c-1} for every real c (Prop. 38.8.1, [NUM]). Its "$\beta=1$" is a choice of reference line, which coincides with the BC critical point because both sit at the pole $s=1$. This is a genuine but shallow relation, not a shared mechanism. (b) The modular conjugation J of a BC KMS state swaps the Galois sectors, $JE_\chi\Omega = \chi(-1)E_{\bar\chi}\Omega$. In the same way, the reality symmetry $\rho \mapsto \bar\rho$ of Lemma 31 swaps the zero sets of $L(s,\chi)$ and $L(s,\bar\chi)$, while the mirror $\rho \mapsto 1-\bar\rho$ stays inside each sector. Complex conjugation *as a Galois element* ($-1 \in \widehat{\mathbb{Z}}^\times$) acts inside each sector, by the parity $\chi(-1)$ (Prop. 38.8.3). (c) There is one Lemma 37 line per Galois sector [CONJ]. The BC ordered phase $\beta > 1$ is the region $\omega > \frac12$ where Lemma 37's lines are unconditionally passive, and $\beta = 1$ is where the archimedean-only gain blows up ([OBSERVATION]; the common cause, the pole, is [THM]).

Nothing here proves RH or GRH. No positivity crosses from the p-adic side.

38.1. Conventions and what a Galois symmetry is

Notation follows Lemmas 1, 4, 31, 37. We write $\rho = \frac12 + i\gamma_\rho$ for a zero, $\mathcal{Q}$ for Weil's form, $U_u = e^{iu\gamma}$ for the scaling flow on $\mathcal{H}_\mathcal{Z}$, and p for an *odd* prime throughout. ω is the Teichmüller character, $\langle a\rangle = a/\omega(a) \in 1+p\mathbb{Z}_p$ for $a \in \mathbb{Z}_p^\times$, and $\log_p$ is the Iwasawa logarithm ($\log_p p = 0$). Γ, $\Lambda_{\mathrm{Iw}} = \mathbb{Z}_p[[T]]$, X_∞, V_k, W_{ω^k} are as in Lemma 4.

Definition 38.1.1 (Galois symmetry; [DEF]). Let $K/\mathbb{Q}$ be a field extension. A *Galois symmetry* of K is a field automorphism $\sigma : K \to K$ fixing $\mathbb{Q}$ pointwise. It preserves every polynomial identity with rational coefficients, so it permutes the roots of each rational polynomial and cannot be detected by any rational measurement. For $K = \mathbb{Q}(\mu_N)$, generated by $\zeta_N = e^{2\pi i/N}$, σ is fixed by $\sigma(\zeta_N) = \zeta_N^a$ with $\gcd(a,N)=1$, and $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N)^\times$. Passing to all roots of unity, $\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\mu_\infty)$ (Kronecker–Weber), gives the *cyclotomic character*

$$\kappa : \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) \xrightarrow{\sim} \varprojlim (\mathbb{Z}/N)^\times = \widehat{\mathbb{Z}}^\times = \textstyle\prod_\ell \mathbb{Z}_\ell^\times, \qquad \sigma(\zeta) = \zeta^{\kappa(\sigma)} \quad (\zeta \in \mu_\infty).$$

Plain language. A Galois symmetry is a relabelling of the roots of unity, $\zeta \mapsto \zeta^a$, that no rational equation can tell apart. Complex conjugation is the element $a=-1$: rational algebra cannot distinguish i from $-i$. The p-power part is $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times = \mu_{p-1} \times (1+p\mathbb{Z}_p)$, and Iwasawa's group is

$$\Gamma = \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p)) \xrightarrow{\kappa} 1+p\mathbb{Z}_p \xrightarrow{\log_p} p\mathbb{Z}_p \cong \mathbb{Z}_p.$$

So $\gamma \in \Gamma$ acts on p-power roots of unity by the *p-adic power* $\zeta \mapsto \zeta^{\kappa(\gamma)}$. This is a "scaling" of the exponent, which is what made the rapidity comparison tempting. §38.2 locates that scaling exactly.

Remark 38.1.2 (physics reading; [ANALOGY]). In QFT language, a Galois symmetry is an *internal, compact, discrete-spectrum* symmetry: it relabels sectors and commutes with the dynamics. That is the role of global $U(1)$ or $SO(3)$ in a system whose time evolution is a boost or scaling. The Fock/Bargmann $SO(4)$ versus $SO(3,1)$ contrast of Lemma 31 §4 (compact rotation symmetry versus non-compact boost) is the nearest picture already in our notes. §38.5 shows that this reading

is *literally* true in the Bost–Connes system, where Galois acts by symmetries commuting with time evolution [CITED]. Outside BC it is an analogy.

38.2. One idele class group

Let $\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \prod'_{\ell} \mathbb{Q}_{\ell}$ and let $\mathbb{A}_{\mathbb{Q}}^{\times}$ be the ideles, with norm $|x| = \prod_v |x_v|_v$ and $C_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$ (Definition 5).

THEOREM 38.2.1 ([THM]). *The map $\mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^{\times} \to C_{\mathbb{Q}}$, $(r, (u_\ell)) \mapsto [(r, (u_\ell)_\ell)]$, is an isomorphism of topological groups. Under it, $|\cdot|$ is the projection onto $\mathbb{R}_{>0}$, so $C_{\mathbb{Q}}^1 := \ker|\cdot| \cong \widehat{\mathbb{Z}}^{\times}$ is compact.*

PROOF. *Surjective.* For $x \in \mathbb{A}_{\mathbb{Q}}^{\times}$ put $q = \operatorname{sgn}(x_\infty) \prod_\ell \ell^{v_\ell(x_\ell)} \in \mathbb{Q}^{\times}$, a finite product because $x_\ell \in \mathbb{Z}_\ell^{\times}$ for almost all ℓ. Then $q^{-1}x$ has positive archimedean component and $v_\ell(q^{-1}x_\ell) = 0$ for all ℓ. *Injective.* If $q \in \mathbb{Q}^{\times}$ is positive and lies in $\mathbb{Z}_\ell^{\times}$ for every ℓ, then $q = 1$. *Topology.* $\mathbb{Q}^{\times}$ is discrete in $\mathbb{A}_{\mathbb{Q}}^{\times}$, $\mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^{\times}$ is open in $\mathbb{A}_{\mathbb{Q}}^{\times}$, and the restricted map is a continuous bijective homomorphism onto the quotient that is open. *Norm.* For (r, u) we have $|u_\ell|_\ell = 1$, so $|(r, u)| = r$. □

PROPOSITION 38.2.2 (where the local groups land; [THM]). *Fix a prime p and write $\mathbb{Q}_p^{\times} = p^{\mathbb{Z}} \times \mathbb{Z}_p^{\times}$, with $\mathbb{Z}_p^{\times} = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$. Embed $\mathbb{Q}_p^{\times} \hookrightarrow \mathbb{A}_{\mathbb{Q}}^{\times}$ at the place p (other components 1). Under Thm. 38.2.1:*

(a) a unit $u \in \mathbb{Z}_p^{\times}$ maps to $(1, u)$, with u in the p-component of $\widehat{\mathbb{Z}}^{\times}$, inside the compact Galois factor*;*

(b) the uniformizer p maps to $(p^{-1}, (p^{-1})_{\ell \neq p} \times 1_p)$. Its rapidity is $\eta_\infty = \log|\cdot| = -\log p$.

So the local "valuation direction" at p (step $\log p$) is part of the one global rapidity*, and the local units are part of Galois. In particular $\Gamma \cong 1 + p\mathbb{Z}_p$ lies in $C_{\mathbb{Q}}^1$.*

PROOF. (a) is immediate. (b): the idele with p at the place p is equivalent modulo $\mathbb{Q}^{\times}$ to its product with p^{-1}, which is p^{-1} at ∞ and at every $\ell \neq p$, and 1 at p. □

Reading. The product formula $\prod_v |q|_v = 1$ is what glues the local valuation steps $\log p$ into the archimedean rapidity. This is why the delays $\log p$ of Lemma 37's Euler sections are measured in the same unit as the Rindler rapidity of Lemma 31.

PROPOSITION 38.2.3 (class field theory, normalization made explicit; [THM] from [BIB]). *Normalize the reciprocity map $\mathrm{rec} : C_{\mathbb{Q}} \to \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ so that the class of a uniformizer at ℓ acts on $\mathbb{Q}(\mu_N)$, $\ell \nmid N$, as the arithmetic Frobenius $\zeta \mapsto \zeta^{\ell}$* [**208**, Ch. VI] *[BIB]. Then* rec *is trivial on $\mathbb{R}_{>0}$, and for $u \in \widehat{\mathbb{Z}}^{\times}$*

$$\kappa(\mathrm{rec}(u)) = u^{-1}.$$

Hence rec *restricts to an isomorphism $\widehat{\mathbb{Z}}^{\times} \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$, and after restriction to $\mathbb{Q}(\mu_{p^\infty})$ the subgroup $1 + p\mathbb{Z}_p$ (the p-component of $\widehat{\mathbb{Z}}^{\times}$) maps isomorphically onto Γ.*

PROOF. rec kills the connected component $\mathbb{R}_{>0}$. Fix N and a prime $\ell \nmid N$. By Prop. 38.2.2(b) the uniformizer class at ℓ is $(\ell^{-1}, u^{(\ell)})$ with $u^{(\ell)}_{\ell'} = \ell^{-1}$ for $\ell' \neq \ell$. On $\mathbb{Q}(\mu_N)$ only the components at $\ell' \mid N$ act, because local units at unramified places act trivially. So $\mathrm{rec}(u^{(\ell)})|_{\mathbb{Q}(\mu_N)}$ is $\zeta \mapsto \zeta^{\ell}$, while $u^{(\ell)} \equiv \ell^{-1} \pmod N$. By Dirichlet's theorem such ℓ fill $(\mathbb{Z}/N)^{\times}$, so $\mathrm{rec}(u)$ acts as $\zeta \mapsto \zeta^{u^{-1}}$ for every u, by continuity. □

The inverse is a convention artefact. It reappears as $\theta_u(e(y)) = e(u^{-1}y)$ in [**44**, §5, (9)] and as $\theta^{-1}(\gamma)$ in [**74**, (3.137)] [CITED]. We track it and nothing below depends on it.

PROPOSITION 38.2.4 (the two rapidities of a Frobenius class; [THM]). *For $n \geq 1$ let $f_n \in C_{\mathbb{Q}}$ be the class of the idele $(\ell^{v_\ell(n)})_\ell$ (components 1 at ∞ and at $\ell \nmid n$). Then $f_n = (n^{-1}, (n_{(\ell)}^{-1})_\ell)$, where $n_{(\ell)} = n\,\ell^{-v_\ell(n)}$. Consequently:*

(a) *archimedean rapidity:* $\eta_\infty(f_n) = -\log n$, *and* $|f_n|^s = n^{-s}$ *is the Dirichlet-series term;*
(b) *for* $p \nmid n$, *the* p*-component is* $n^{-1} \in \mathbb{Z}_p^\times$, *and its* p-adic rapidity $\eta_p(f_n) := \log_p\langle n^{-1}\rangle = -\log_p n \in p\mathbb{Z}_p$.

Proof. Multiply the idele by $n^{-1} \in \mathbb{Q}^\times$ and use $\log_p \omega(\cdot) = 0$. □

The integers are the "particles" common to both analyses. Complex analysis reads each f_n through $\log n$, p-adic analysis through $\log_p n$. $\log_p$ is a homomorphism on $\mathbb{Z}_p^\times$, and $\langle n\rangle = (1+p)^{i_p(n)}$ with $i_p(n) = \log_p n/\log_p(1+p) \in \mathbb{Z}_p$. Both are checked mod p^{10} for $p = 5, 7, 37$ (144 pairs, 20 values each; [NUM]).

Proposition 38.2.5 (no continuous bridge between $\mathbb{R}$ and $\mathbb{Z}_p$; [THM]). *Every continuous homomorphism* $\mathbb{R} \to \mathbb{Z}_p$ *or* $\mathbb{Z}_p \to \mathbb{R}$ *is trivial.*

Proof. The image of connected $\mathbb{R}$ in totally disconnected $\mathbb{Z}_p$ is a point. The image of compact $\mathbb{Z}_p$ in $\mathbb{R}$ is a compact subgroup, hence $\{0\}$. □

Correction 38.2.6 (to the chat answer of [t126u] and to item (1) of the task brief). "The Iwasawa group is the p-adic rapidity group" is **false as a statement inside** $C_\mathbb{Q}$. By Thm. 38.2.1 and Prop. 38.2.2, $\Gamma \subset 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times \subset \widehat{\mathbb{Z}}^\times = C_\mathbb{Q}^1$ lies in the compact Galois factor, which commutes with the archimedean rapidity $\log|\cdot|$. The local analogue of rapidity at p is the valuation direction $p^\mathbb{Z}$ (step $\log p$), and globally it is *identified* with archimedean rapidity (Prop. 38.2.2(b)). By Prop. 38.2.5, Γ can neither be nor contain a modular one-parameter group Δ^{it}. The statement is **true only** p**-adically**: through $\log_p : 1 + p\mathbb{Z}_p \cong p\mathbb{Z}_p$ and the characters $\gamma \mapsto \kappa(\gamma)^s$, Γ plays the role that $\mathbb{R}_{>0}$ plays for the Mellin transform (Thm. 38.3.1). The dictionary table of §38.6 is corrected accordingly.

38.3. Matching the variables: Mellin x^s versus Iwasawa $T = \kappa(\gamma)^s - 1$

Theorem 38.3.1 (characters of the two rapidity groups; [THM]). *(a)* Archimedean. *The continuous homomorphisms* $\mathbb{R}_{>0} \to \mathbb{C}^\times$ *are* $x \mapsto x^s = e^{s\eta_\infty}$, $s \in \mathbb{C}$, *with* $\eta_\infty = \log x$. *The unitary ones are those with* $\operatorname{Re} s = 0$, *or* $\operatorname{Re} s = \frac{1}{2}$ *after the half-density twist* $x^{1/2}$ *of Lemma 1. They form the continuum* $\mathbb{R}$*: the critical line.*

(b) Γ into $\mathbb{C}^\times$. *Every continuous homomorphism* $\psi : \Gamma \to \mathbb{C}^\times$ *has finite image in* μ_{p^∞}. *So the complex (unitary) dual of* Γ *is the* discrete *group* $\mathbb{Q}_p/\mathbb{Z}_p \cong \mu_{p^\infty}$*: these are the Dirichlet characters of* p*-power order and conductor* p^k, $k \geq 2$, *together with* 1.

(c) Γ into $\mathbb{C}_p^\times$. *Fix a topological generator* γ_0 *and put* $u = \kappa(\gamma_0) \in 1 + p\mathbb{Z}_p$. *The continuous homomorphisms* $\Gamma \to \mathbb{C}_p^\times$ *are* $\gamma_0 \mapsto 1 + T$ *with* $|T|_p < 1$. *Evaluating* $f \in \Lambda_{\mathrm{Iw}} = \mathbb{Z}_p[[T]]$ *at such* T *is evaluating the measure* f *on that character.*

(d) The p-adic Mellin line. *For* $|s|_p < p^{1-1/(p-1)}$ *the character* $\gamma_0^a \mapsto \kappa(\gamma_0^a)^s := \exp_p(as\log_p u)$ *is defined. It corresponds to* $T = u^s - 1$. *With* $\eta_p(\gamma) := \log_p \kappa(\gamma) \in p\mathbb{Z}_p$,

$$\boxed{x^s = \exp(s\,\eta_\infty) \quad \longleftrightarrow \quad \kappa(\gamma)^s = \exp_p(s\,\eta_p(\gamma)), \qquad T = \kappa(\gamma_0)^s - 1}$$

Both are "exponent times logarithm" on a one-dimensional rapidity group: $\mathbb{R}$ *via* log, *and* $p\mathbb{Z}_p$ *via* $\log_p$. *The torsion points* $T = \zeta - 1$, $\zeta \in \mu_{p^\infty}$, *of (c) are exactly the complex-unitary characters of (b).*

Proof. (a) Standard: a continuous homomorphism $\mathbb{R} \to \mathbb{C}^\times$ is $\eta \mapsto e^{s\eta}$. (b) The image is a compact subgroup of $\mathbb{C}^\times$, hence lies in S^1. Some neighbourhood of 1 in S^1 contains no nontrivial subgroup, and its preimage is an open subgroup of $\Gamma \cong \mathbb{Z}_p$, so it contains some $p^k\mathbb{Z}_p$. Thus ψ factors through $\mathbb{Z}/p^k$. (c) $a \mapsto t^a$ extends continuously from $\mathbb{Z}$ to $\mathbb{Z}_p$ iff $t^{p^k} \to 1$. If $|t-1|_p < 1$ this holds by the binomial theorem. If $|t|_p \neq 1$ it fails. If $|t|_p = 1$ and $\bar{t} \neq 1$ in $\bar{\mathbb{F}}_p^\times$, then $\bar{t}^{p^k} \neq 1$ for all

k because Frobenius is bijective on $\bar{\mathbb{F}}_p^\times$, so it fails again. (d) $|\log_p u|_p \le p^{-1}$ and $\exp_p$ converges on $|x|_p < p^{-1/(p-1)}$. This is the "extended s-disc" D_p of [**67**, (91), (97)–(98)] [CITED]. □

COROLLARY 38.3.2 (the Iwasawa evaluation is a p-adic Mellin transform; [THM] from [CITED]). *By Lemma 4 (K1) ([**60**, Thm. 2.4]; Kubota–Leopoldt, Iwasawa), $L_p(s,\omega^k) = W_{\omega^k}(u^s-1)$. For $k=0$ the same form, $L_p(\beta,1) = g((1+p)^\beta-1)/(1-(1+p)^{1-\beta})$ with $\frac{1}{2}g \in \mathbb{Z}_p[[T]]^\times$, is quoted from Washington's Thm. 7.10 in* [**67**, (134)] *[CITED, secondary]. So the characteristic series is a function on the space of p-adic characters of Γ (Thm. 38.3.1(c)). The Kubota–Leopoldt function is its restriction to the p-adic Mellin line (d), exactly as $\zeta(s)$ is the Mellin transform restricted to $x \mapsto x^s$. Connes–Consani extend from D_p to the whole disc $M = \{\lambda : |\lambda-1|_p < 1\}$ via $\lambda \mapsto \ell(\lambda) = \log_p \lambda / \log_p(1+q)$, which is surjective with kernel μ_{p^∞}* [**67**, §7.5, (142)–(146)] *[CITED]. Their M is Iwasawa's variable $1+T$.*

REMARK 38.3.3 (twisted interpolation; flagged). Iwasawa's series also interpolate twists by the finite-order characters ψ of (b), via evaluation at $T = \zeta_\psi u^s - 1$ (Washington Thm. 7.10 in general). We could not check the general twisted statement at the source, so **nothing below uses it**. Thm. 38.5.6 uses only the classical values $L(1-n,\chi)$ and the defining interpolation property recorded by Greenberg [**114**, (6)–(7)] [CITED, secondary].

38.4. The spectral sides: Iwasawa, Connes, Deninger

PROPOSITION 38.4.1 (Iwasawa side; from Lemma 4 (D4), [THM] from [CITED]). *For even $k \not\equiv 0 \pmod{p-1}$ with $\mu(W_{\omega^k}) = 0$, the generator γ_0 acts on the λ-dimensional $\mathbb{Q}_p$-space $V_k = X_\infty^{(1-k)} \otimes \mathbb{Q}_p$ with $\det(T-(\gamma_0-1) \mid V_k) =$ the distinguished polynomial of W_{ω^k}. The main conjecture (Mazur–Wiles [**188**]; also [**293**]) is cited via* [**60**, Thm. 3.5]. *The eigenvalues of γ_0 are $1+T_0$ with $T_0 \in \mathfrak{m}_{\mathbb{C}_p}$. They are* non-torsion *unless $T_0 = 0$, so they are never values of a complex unitary character (Thm. 38.3.1(b)).*

THEOREM 38.4.2 (where Γ acts in Connes' spectral realization; [THM] from [CITED]). *Let $\mathcal{H}$ be Connes' space with the $C_{\mathbb{Q}}$-representation W* [**75**, (16)–(26)]. *With $K = C_{\mathbb{Q}}^1 \cong \widehat{\mathbb{Z}}^\times$ (Thm. 38.2.1), $\mathcal{H} = \bigoplus_{\chi \in \widehat{K}} \mathcal{H}_\chi$* [**75**, (23)]. *For $\delta > 1$, the generator D_χ of $\mathbb{R}_{>0}$ on $\mathcal{H}_\chi$ has discrete spectrum equal to the imaginary parts of the* critical *zeros of $L(\tilde{\chi}, \frac{1}{2}+\cdot)$* [**75**, Thm. 1] *[CITED]. Here $\widehat{\widehat{\mathbb{Z}}^\times}$ is the group of primitive Dirichlet characters. Then:*

- *(a) the archimedean rapidity group $\mathbb{R}_{>0}$ supplies the spectral parameter, and the Galois factor $\widehat{\mathbb{Z}}^\times$ supplies the sector label χ;*
- *(b) $\Gamma \cong 1+p\mathbb{Z}_p \subset \widehat{\mathbb{Z}}^\times$ acts on $\mathcal{H}_\chi$ by the scalar $\chi|_{1+p\mathbb{Z}_p}$, of finite order (Thm. 38.3.1(b)). It commutes with every D_χ, and its isotypic component of type ψ is $\bigoplus_{\chi|_\Gamma=\psi} \mathcal{H}_\chi$;*
- *(c) no vector of $\mathcal{H}$ is a Γ-eigenvector with Iwasawa's eigenvalue $1+T_0$, $T_0 \neq 0$.*

PROOF. (a) is (23)–(26) of [**75**]. (b) $W|_K$ is unitary and diagonal on the $\mathcal{H}_\chi$. (c) follows from (b) and Prop. 38.4.1. □

Same group, two spectral roles. In Connes' complex unitary picture, Γ only *labels sectors*, and rapidity carries the zeros. In Iwasawa's p-adic picture, Γ itself carries the zeros. This does not contradict anything: the two are representations over different fields, $\mathbb{C}$ and $\mathbb{C}_p$, and Thm. 38.3.1(b)–(c) is exactly the difference. It is the precise content of J. W.'s "confluence": *the same compact group Γ, read complexly, is Galois (sector labels); read p-adically, it is a spectral flow.*

REMARK 38.4.3 (Deninger's program: the existing literature closest to J. W.'s idea; [CITED]). (i)
Local factors as regularized determinants (proved): $\zeta_{\mathfrak{p}}(s) = \det_\infty(\frac{1}{2\pi}(s-\Theta) \mid R_{\mathfrak{p}})^{-1}$ for every place, $\Theta = d/dy$ the generator of an $\mathbb{R}$-action [**87**, Prop. 3.1], originally [**89**].

(ii) *Global formula* (conjectural): $\hat{\zeta}_K(s) = \prod_i \det_\infty(\frac{1}{2\pi}(s-\Theta) \mid H^i_{\mathrm{dyn}})^{(-1)^{i+1}}$ [87, (3)], introduced there as "suggests by analogy that a formula of the following type might hold". A Hodge $*$ commuting with Θ would give RH [91, §3].

(iii) *Iwasawa*: "The analogue of the Alexander polynomial of a knot turns out to be the Iwasawa zeta function" (Mazur's analogy), with primes as knots and periodic orbits of length $\log N\mathfrak{p}$ [87, p. 37]; see also [86] [BIB].

(iv) *Galois commuting with the flow*: in [88, §7, (56)–(61)], for an arithmetic scheme over $\operatorname{spec}\mathbb{Z}$, the space $\tilde{X} = \check{X}(\mathbb{C}) \times \mathbb{R}_{>0}$ maps $\mathbb{Q}_{>0}$-equivariantly to $\mathbb{A}_f \times \mathbb{R}_{>0}$. Here $\sigma \in G$ acts on $\widehat{\mathbb{Z}}$ "by multiplication with $\kappa(\sigma) \in \widehat{\mathbb{Z}}^\times$" (the cyclotomic character), and G acts on the first factor only, so it commutes with the $\mathbb{R}_{>0}$-factor. Deninger then asks whether the Bost–Connes/Connes point of view applies there [CITED]. This is our Thm. 38.2.1 picture in his framework: Galois sits on the profinite coordinate and rapidity on $\mathbb{R}_{>0}$.

So the chain *Iwasawa module* $\leftrightarrow$ *Alexander module* $\leftrightarrow$ *monodromy of a flow on a foliated space* $\leftrightarrow$ *scaling flow* is in the literature as a program. Its last link, the existence of the space, is open. We found no paper that identifies Iwasawa's Γ with the flow itself. In (iv) the flow and Galois are the two commuting factors, as in Thm. 38.2.1.

Conjecture 38.4.4 (archimedean "main conjecture" shape; [CONJ], Deninger's, restated). There is a $C_\mathbb{Q}$-equivariant "H^1" with $\widehat{\mathbb{Z}}^\times$-sectors H^1_χ and an $\mathbb{R}_{>0}$-generator Θ_χ, such that the completed $L(s,\chi)$ equals $\det_\infty(\frac{1}{2\pi}(s-\Theta_\chi) \mid H^1_\chi)$. In this picture Iwasawa's $\Gamma \subset \widehat{\mathbb{Z}}^\times$ acts by finite-order scalars, consistently with Thm. 38.4.2.

Shape match and honest gaps. Mazur–Wiles: "characteristic ideal of γ on a finitely generated torsion Λ_{Iw}-module $=$ p-adic L" (Lemma 4 (K4)). Conj. 38.4.4: "regularized characteristic determinant of Θ on $H^1 = \xi$". There are three gaps. (i) Λ_{Iw} is Noetherian and $\dim V_k = \lambda < \infty$, while H^1 is infinite-dimensional and needs zeta-regularization. (ii) The Mazur–Wiles proof runs through Eisenstein congruences and Galois representations (Lemma 4 (D1)–(D2)), and there is no archimedean analogue. (iii) Positivity (Hodge $*$) is extra input in Deninger's program, and is G5 (Lemma 30). *Circularity flag:* a Θ whose spectrum is the zeros, built from the zeros, is the GNS circularity of Lemma 31 §3.

38.5. Bost–Connes: the rigorous meeting point of modular theory and Galois

Theorem 38.5.1 (the BC system; [CITED]). *Let $\mathcal{A}_{\mathrm{BC}} = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times$ $(= C(\widehat{\mathbb{Z}}) \rtimes \mathbb{N}^\times)$, generated by isometries μ_n and unitaries $e(r)$, $r \in \mathbb{Q}/\mathbb{Z}$ [74, Prop. 3.23, 3.25], [44, §§3–4].*

(i) Time evolution: $\sigma_t(\mu_n) = n^{it}\mu_n$ and $\sigma_t(e(r)) = e(r)$ [74, (3.60)], [44, §5 (2)]. Equivalently, $\sigma_t(f)(L,L') = |L/L'|^{it} f(L,L')$, the ratio of covolumes of commensurable $\mathbb{Q}$-lattices [74, (3.58)–(3.59), Lemma 3.24].

(ii) In the representation π_ρ on $\ell^2(\mathbb{N})$ we have $H\varepsilon_m = (\log m)\varepsilon_m$ and $Z(\beta) = \operatorname{Tr} e^{-\beta H} = \zeta(\beta)$, with the pole at $\beta = 1$ "where the phase transition occurs" [74, (3.140)–(3.142)].

(iii) $0 < \beta \le 1$: there is a unique KMS$_\beta$ state [74, Thm. 3.32], of type III$_1$ [44, Thm. 5(a)].

(iv) $1 < \beta \le \infty$: the extremal states are $\varphi_{\beta,\rho}(e(r)) = \zeta(\beta)^{-1}\mathrm{Li}_\beta(\rho(\zeta_r))$, indexed by $\rho \in \widehat{\mathbb{Z}}^\times$. $\widehat{\mathbb{Z}}^\times$ acts by automorphisms of $(\mathcal{A}_{\mathrm{BC}}, \sigma_t)$, freely and transitively on these states [74, Thm. 3.32, (3.135)], [44, Thm. 25].

(v) $\beta = \infty$: $\varphi(\mathcal{A}_{1,\mathbb{Q}}) \subset \mathbb{Q}^{\mathrm{cycl}}$, and $\gamma\varphi(f) = \varphi(\theta^{-1}(\gamma)f)$ for $\gamma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{cycl}}/\mathbb{Q})$ [74, (3.136)–(3.137)]. In BC's own form, $\varphi_{\infty,\alpha}(\sum x_j Y_j) = \alpha(\sum x_j \exp 2\pi i y_j)$ [44, proof of Thm. 25, (6)–(7)].

(vi) The symmetry group $W = \widehat{\mathbb{Z}}^\times$ (BC's "idele class group W") acts with $\theta_u(\mu_n) = \mu_n$ and $\theta_u(e(y)) = e(u^{-1}y)$. It commutes with σ_t, and its fixed-point algebra is $C^*(\mathbb{N}^\times)$ [44, Prop. 21, Lemma 22].

(vii) $C^*(\mathbb{N}^\times) = \bigotimes_p \mathcal{T}_p$ *(Toeplitz algebras) with* $\sigma_t = \bigotimes_p \sigma_{t,p}$. *For every* $\beta > 0$ *there is a unique KMS*$_\beta$ *state* $\bigotimes_p \varphi_{\beta,p}$, *and the eigenvalue list of* $\varphi_{\beta,p}$ *is* $\{(1-p^{-\beta})p^{-n\beta}\}_{n\geq 0}$. *The product is type* I_∞ *for* $\beta > 1$ *and* III_1 *for* $\beta \leq 1$ [**44**, Props. 7–8].

(viii) For $0 < \beta \leq 1$, *every KMS*$_\beta$ *state vanishes on the spectral subspaces* $\mathcal{A}_{\mathrm{BC}\chi} = \{x : \theta_u(x) = \chi(u)x\}$, $\chi \neq 1$ [**44**, Lemma 27(c)].

PROPOSITION 38.5.2 (modular flow of a KMS$_\beta$ state; [THM]). *Let* φ *be a faithful KMS*$_\beta$ *state of* $(\mathcal{A}_{\mathrm{BC}}, \sigma_t)$, *with convention* [**74**, Def. 3.6]. *Then its modular group is* $\sigma_t^\varphi = \sigma_{\beta t}$ *in the convention* $\sigma_t^\varphi = \Delta^{-it} \cdot \Delta^{it}$ *of* [**74**, (4.125)], *and* $\sigma_{-\beta t}$ *in the convention* $\Delta^{it} \cdot \Delta^{-it}$. *For the type I states of (iv),* $\Delta k = e^{-\beta H} k e^{\beta H}$ *on Hilbert–Schmidt operators, so* $\log \Delta$ *has spectrum* $\beta \log \mathbb{Q}_{>0}$ *(Lemma 31 Prop. 3.9(a)). At* $\beta = 1$ *the state induced by Haar measure on* $\widehat{\mathbb{Z}}$ *has modular group equal to the BC time evolution* [**74**, Lemma 4.49] *[CITED].*

PROOF. φ is KMS$_1$ for $t \mapsto \sigma_{\beta t}$. The modular group is the unique flow for which φ is KMS$_1$ (Takesaki; [**74**, Lemma 4.47]). The type I formula is the computation in Lemma 31 Prop. 3.9. □

PROPOSITION 38.5.3 (BC time is dual to rapidity; [THM]). *For* $\beta > 1$, $\varphi_\beta(e^{itH})$, *i.e.* $\mathrm{Tr}(e^{-\beta H}e^{itH})/\zeta(\beta)$, *equals* $\zeta(\beta - it)/\zeta(\beta)$. *Hence*

$$(\textit{inverse temperature, time}) = (\beta, t) = (\mathrm{Re}\, s, -\mathrm{Im}\, s), \qquad \textit{energies } = \log n = -\eta_\infty(f_n) \textit{ (Prop. 38.2.4).}$$

In BC the rapidities are the spectrum *of the modular generator, and the modular* parameter *is conjugate to them. In Bisognano–Wichmann (Lemma 31 §6(2)) the modular parameter* is *the rapidity (*Δ^{it} = *boost by* $\mp 2\pi t$*) and the spectrum is the Rindler frequency. The two are Fourier-dual. This is Lemma 31 Prop. 3.10 ("the positivities are Fourier-dual") at the level of the flows. Connes' adele class space, "the type II dual of the BC-system"* [**67**, §1] *[CITED], is the side on which* $\mathbb{R}_{>0}$ *acts geometrically, so the rapidity is the flow parameter and the spectrum is the zeros (Thm. 38.4.2).*

THEOREM 38.5.4 (Galois sectors and their order parameters; [THM] from [CITED], [NUM]). *Let* χ *be a primitive Dirichlet character mod* N, *let* $\varphi_\beta = \varphi_{\beta,1}$ *(standard embedding), and put*

$$E_\chi := \sum_{a \in (\mathbb{Z}/N)^\times} \bar{\chi}(a)\, e(a/N) \in C^*(\mathbb{Q}/\mathbb{Z}).$$

(a) $\theta_u(E_\chi) = \bar{\chi}(u)E_\chi$, *so* E_χ *lies in the Galois sector* $\mathcal{A}_{\mathrm{BC}\bar{\chi}}$, *an isotypic component of* $\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$.

(b) For $\beta > 1$: $\varphi_\beta(E_\chi) = \tau(\bar{\chi})\, L(\beta, \chi)/\zeta(\beta)$, *where* τ *is the Gauss sum.*

(c) For $0 < \beta \leq 1$ *and* $\chi \neq 1$: $\varphi_\beta(E_\chi) = 0$ *(Thm. 38.5.1(viii)).*

(d) As $\beta \downarrow 1$, $\varphi_\beta(E_\chi) = (\beta - 1)\,\tau(\bar{\chi})L(1,\chi) + O((\beta-1)^2)$ *for* $\chi \neq 1$, *and* $L(1,\chi) \neq 0$ *(Dirichlet). So the Galois order parameter vanishes* linearly, *and it vanishes* because *the trivial sector's normalization* $\zeta(\beta)$ *has a pole at* 1 *while* $L(s,\chi)$, $\chi \neq 1$, *has none.*

PROOF. (a) $\theta_u(E_\chi) = \sum_a \bar{\chi}(a) e(u^{-1}a/N) = \sum_b \bar{\chi}(ub) e(b/N)$. (b) By Thm. 38.5.1(iv), $\varphi_\beta(e(a/N)) = \zeta(\beta)^{-1} \sum_k k^{-\beta} e^{2\pi i a k/N}$. For primitive $\bar{\chi}$ and every k, $\sum_a \bar{\chi}(a) e^{2\pi i a k/N} = \chi(k)\tau(\bar{\chi})$. (d) $\zeta(\beta)^{-1} = (\beta - 1) + O((\beta-1)^2)$. □

[NUM]: (b) was checked against direct polylogarithm sums for all primitive χ mod $5, 7, 9, 25, 27$ at $\beta \in \{1.2, 2, 2.5, 3\}$ (160 cases, max relative error 9×10^{-40}).

REMARK 38.5.5 (physics reading; [ANALOGY]). (d) is the statement "spontaneous symmetry breaking with order parameters $\langle E_\chi \rangle_\beta$, critical point at the pole, mean-field-like linear onset". The critical exponent 1 is the order of the pole. The analytic *continuation* of $\langle E_\chi \rangle_\beta$ into complex β meets the zeros of $L(\cdot, \chi)$ (Fisher-type zeros, Lemma 31 Prop. 3.9(c)). There is no Gibbs state there, so no positivity attaches to them.

THEOREM 38.5.6 (BC Galois sectors at $\beta = n$ are Kubota–Leopoldt data; [THM] from [CITED], [CITED, secondary], [NUM]). *Let χ be primitive mod $N = p^k$ ($k \geq 1$), with parity $a \in \{0,1\}$, $\chi(-1) = (-1)^a$, and let $n \geq 2$ with $n \equiv a \pmod 2$. Then*

$$\varphi_n(E_\chi) = \frac{(-1)^{(n-a)/2}\chi(-1)\,N\,(2\pi/N)^n\,n}{2\,i^a\,n!\,\zeta(n)}\,L(1-n,\bar\chi), \qquad L(1-n,\bar\chi) = -\frac{B_{n,\bar\chi}}{n} = L_p(1-n,\bar\chi\omega^n).$$

So, up to the archimedean factor shown and the trivial-sector normalization $\zeta(n)$, the Galois-sector order parameters of BC along the p-cyclotomic tower, at integer inverse temperature, are the values that the Kubota–Leopoldt function interpolates. These are the values of the Iwasawa characteristic series (Cor. 38.3.2, Prop. 38.4.1).

PROOF. Combine Thm. 38.5.4(b), the functional equation $L(n,\chi) = (-1)^{1+(n-a)/2}\frac{\tau(\chi)}{2i^a}(\frac{2\pi}{N})^n\frac{B_{n,\bar\chi}}{n!}$ (classical; [NUM] below), $\tau(\chi)\tau(\bar\chi) = \chi(-1)N$, and $L(1-n,\psi) = -B_{n,\psi}/n$. Then use the interpolation property $L_p(1-n,\psi) = (1-\psi\omega^{-n}(p)p^{n-1})L(1-n,\psi\omega^{-n})$ [**114**, (6)–(7)] [CITED, secondary] with $\psi = \bar\chi\omega^n$. The Euler factor is 1 because $p \mid N$ gives $\bar\chi(p) = 0$. □

[NUM]: the FE formula holds in 69 cases and the identity $L(1-n,\chi) = -B_{n,\chi}/n$ in 168 cases (mod $5,7,9,25$, $n \leq 6$). The displayed identity for $\varphi_n(E_\chi)$ holds in 120 cases (mod $5,7,9,25,27$, $n \leq 7$). All relative errors are $\leq 9 \times 10^{-40}$.

REMARK 38.5.7 (what Thm. 38.5.6 does and does not say). It is a *rigorous meeting point*: one object (the BC equilibrium state), one symmetry (Galois), and the sector data are Iwasawa's interpolation data. The temperature n and the tame part of the sector are locked ($\bar\chi\omega^n$ depends on n mod $(p-1)$), which is the Kummer-congruence structure. It does *not* say that Γ acts as time. Time is σ_t, which is Galois-invariant. It also gives no information on complex zeros.

PROPOSITION 38.5.8 (finite cascades have no phase transition; [THM] from [CITED], [NUM]). *For finite P, the free subsystem $\bigotimes_{p\leq P}\mathcal{T}_p$ with $\bigotimes_{p\leq P}\sigma_{t,p}$ has a unique KMS_β state for every $\beta > 0$, of type I. Its partition function $\prod_{p\leq P}(1-p^{-\beta})^{-1}$ is holomorphic on $\mathrm{Re}\,\beta > 0$. The singularity at $\beta = 1$ and the type change appear only for $P = \infty$. [NUM]: $\prod_{p\leq X}(1-1/p)^{-1} =$ 8.311, 12.351, 16.424, 20.512 versus $e^\gamma \log X =$ 8.202, 12.303, 16.404, 20.505 for $X = 10^2,\ldots,10^5$ (Mertens [**192**] [BIB]).*

PROOF. Uniqueness of each local KMS state is [**44**, Prop. 8(a), proof]. A finite tensor product of normal states of type I factors is type I. □

Link to Lemma 37 [OBSERVATION]. Lemma 37 Thm. 7.4 and Cor. 2.5 found that the archimedean rescue of the line is the pole of ζ, global in p, which no finite set of primes supplies. Prop. 38.5.8 is the thermodynamic face of the same fact: each prime's local system is regular at every temperature, and criticality is collective. The two statements are separately theorems. That they are "the same phenomenon" is [ANALOGY]. Galois acts trivially on this cascade (Thm. 38.5.1(vi)), so *the Euler cascade is the Galois-invariant sector.*

REMARK 38.5.9 (the p-adic BC system of Connes–Consani; [CITED]). [**67**] construct, for each p, p-adic representations of the integral BC system "that are the p-adic analogues of the complex, extremal KMS_∞ states". They also build automorphisms $\sigma^{(\beta)}$ using "the Iwasawa logarithm as a substitute for the complex Hamiltonian H" (§7.1): $\sigma[\lambda](\tilde\mu_a e(\gamma)\mu_b^*) = \omega(b/a)\lambda^{ip(b/a)}\tilde\mu_a e(\gamma)\mu_b^*$ (147). The partition function is $Z(\beta) = L_p(\beta,1)$ (133), and the theory extends to the Iwasawa disc M (§7.5). So, *in the literature, Iwasawa's logarithm* $\log_p n = -\eta_p(f_n)$ *replaces the BC energy* $\log n = -\eta_\infty(f_n)$. That is Prop. 38.2.4 realized as dynamics, and it is the precise sense in which Γ is a "p-adic rapidity". Two features matter for us. (1) The "KMS" condition is the algebraic identity $\varphi(x\sigma(y)) = \varphi(yx)$ (87), with no positivity. (2) The p-adic functionals are invariant under $e(\gamma) \mapsto e(-\gamma)$, i.e. complex conjugation $-1 \in \widehat{\mathbb{Z}}^\times$, "a consequence of the vanishing of the p-adic

L-functions associated to odd Dirichlet characters" (§1). In the complex case that symmetry is broken for $\beta > 1$.

38.6. The corrected dictionary

The steering correction asked for the table to be redone. The archimedean rapidity and the Galois factor are the two *commuting* factors of $C_{\mathbb{Q}}$. The Iwasawa column is the p-adic reading of the Galois factor's pro-p part.

	Rapidity (archimedean, $\mathbb{R}_{>0}$)	**Galois** (compact, $\widehat{\mathbb{Z}}^\times$; complex characters)	**Iwasawa** ($\Gamma \subset \mathbb{Z}_p^*$; p-adic characters)
group in $C_{\mathbb{Q}}$	$C_{\mathbb{Q}}/C_{\mathbb{Q}}^1 = \mathbb{R}_{>0}$ (Thm. 38.2.1)	$C_{\mathbb{Q}}^1 = \widehat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$	$\Gamma = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^* \subset \widehat{\mathbb{Z}}^\times$
logarithm	$\eta_\infty = \log\lvert\cdot\rvert$; $\eta_\infty(f_n) = -\log n$	none: compact (angles)	$\log_p \kappa$; $\eta_p(f_n) = -\log_p n$ (Prop. 38.2.4)
at the prime p	$p^{\mathbb{Z}} \subset \mathbb{Q}_p^*$, step $\log p$ (Prop. 38.2.2)	$\mathbb{Z}_p^*$, the "angular" part of $\mathbb{Q}_p^*$	$1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^*$
characters	$x^s = e^{s\eta_\infty}$, $s \in \mathbb{C}$	Dirichlet χ, finite order	$\gamma \mapsto \kappa(\gamma)^s = 1 + T$, $\lvert T\rvert_p < 1$
unitary dual	$\mathbb{R}$ (the line $\mathrm{Re}\, s = c$)	discrete $\widehat{\widehat{\mathbb{Z}}^\times}$	complex: discrete μ_{p^∞}; p-adic: no unitarity
modular role	BW: flow parameter (Lemma 31 §6); BC: dual to energy (Prop. 38.5.3)	BC symmetry commuting with σ_t; labels sectors (Thm. 38.5.4)	CC14 p-adic "time" $\sigma^{(\beta)}$ via $\log_p$ (Rem. 38.5.9); algebraic KMS only
spectral object	zeros of ζ (Connes D; Deninger Θ, conj.)	zeros of $L(s,\chi)$, sector χ (Thm. 38.4.2)	γ_0 on X_∞: eigenvalues $1 + T_0$, T_0 zeros of W_{ω^k} (Prop. 38.4.1)
generating function	$\zeta(s)$, Ξ	$L(s,\chi)$; BC: $\tau(\bar\chi)L(\beta,\chi)/\zeta(\beta)$	$W_{\omega^k}(T)$; $L_p(s,\omega^k)$
positivity	yes (Hilbert space, KMS states, Weil)	yes (each sector a Hilbert space)	none: $\mathbb{C}_p$ is not ordered
counting index	$\kappa = N_+$ (off-line zeros), open	GRH: $\kappa_\chi = 0$, open	λ (all zeros), $\mu = 0$ (Ferrero–Washington); no "off-line"
proved link	—	BC: Galois commutes with time [CITED]	p-adic BC partition function $= L_p(\beta, 1)$ [CITED]; Thm. 38.5.6 [THM]

Reading the table. Rows 1–3 are [THM]. In the Rapidity column, "BW" is Lemma 31 §6 [CITED], and the Connes and Deninger entries are as in §38.4. The arrow "Galois → Iwasawa" changes the *coefficient field* of characters ($\mathbb{C} \to \mathbb{C}_p$), not the group. That change is what turns an angle into a logarithm.

38.7. Where the dictionary breaks

(B1) *No order, hence no states* [THM]. A state requires $\varphi(x^*x) \geq 0$, and $\mathbb{C}_p$ (or $\mathbb{Q}_p$, for any p) carries no ordering compatible with its field structure, because -1 is a sum of squares in $\mathbb{Q}_p$. So GNS, Hilbert spaces, Tomita's J and Δ, and every positivity used in Lemmas 31–37 (detailed balance $d\mu \geq 0$, the de Branges/Schur bound, Weil positivity) have no p-adic counterpart. Connes–Consani's p-adic "KMS" is the bare identity $\varphi(x\sigma(y)) = \varphi(yx)$ [**67**, (87)]. This is the key break: RH-type statements in our framework *are* positivity statements (Lemma 31 Thm. 3.7, Lemma 37 Thm. 3.2), and positivity does not transfer.

(B2) λ *is not* κ [THM]. Iwasawa's λ is the number of zeros of W_{ω^k} in the open unit disc, i.e. the dimension of V_k, with every zero counted. There is no distinguished "line" inside $\mathfrak{m}_{\mathbb{C}_p}$, so there is no analogue of "off-line". Our $\kappa = N_+$ (Lemma 34) counts *deviations from a line*. The

p-adic main conjecture is an equality of characteristic ideals. It is a theorem, and it implies nothing about positions.

(B3) *The groups do not talk* [THM]. Prop. 38.2.5: there is no continuous homomorphism $\mathbb{R} \to \mathbb{Z}_p$ or $\mathbb{Z}_p \to \mathbb{R}$ except 0. So "the Iwasawa generator is a rapidity" cannot be a group homomorphism. The shared structure is only the exponent-dual-to-a-logarithm pattern (Thm. 38.3.1). That pattern is a parallel, not a map. It is [ANALOGY] when used as an identification.

(B4) *Different spectral roles* [THM]. In Connes' realization $\Gamma \subset \widehat{\mathbb{Z}}^\times$ *labels sectors*: it is a symmetry, and its eigenvalues are roots of unity. In Iwasawa theory γ_0 *is* the operator whose eigenvalues encode the zeros. The analogue of Iwasawa's γ_0 on the archimedean side is the scaling generator D, not any element of $\widehat{\mathbb{Z}}^\times$.

(B5) *What a "p-adic modular theory" would need* [OBSERVATION]. It would need (i) a p-adic one-parameter group: CC14 has one, $\lambda^{i_p(\cdot)}$; (ii) a KMS-type relation: algebraic, present; (iii) a substitute for positivity that can force zeros onto a locus. For (iii) there is nothing in the literature we found. A "p-adic Tomita theory" in the sense of a modular operator on a p-adic Banach space with a polar decomposition $S = J\Delta^{1/2}$ would need square roots and adjoints that p-adic Banach spaces lack in general. We record this as an obstruction, not a theorem.

38.8. What transfers

PROPOSITION 38.8.1 (Lemma 31 Thm. 6.5 is one member of a family; [THM], [NUM]). *For real c let $F_c(t) = \xi(c + it)$. In the notation of Lemma 31 Thm. 6.5, $F_c(t) = \int e^{i\lambda t} d\mu_c(\lambda)$ with $d\mu_c = 2\Phi(\lambda/2)e^{(c-\frac{1}{2})\lambda}d\lambda \geq 0$. It satisfies $d\mu_c(-\lambda) = e^{-(2c-1)\lambda}d\mu_c(\lambda)$, i.e. F_c is a KMS$_\beta$ two-point function with $\beta = 2c - 1$. The KMS relation $F_c(t + i\beta) = F_c(-t)$ is, for every c, the functional equation $\xi(s) = \xi(1 - s)$. β is the width of the strip between the line $\operatorname{Re} s = c$ and its mirror $\operatorname{Re} s = 1 - c$. $c = \frac{1}{2}$ (the critical line) is $\beta = 0$, a tracial two-point function, with Ξ real and even.*

PROOF. Evenness of Φ, exactly as in Lemma 31 Thm. 6.5(b)–(c), with $e^{\lambda/2}$ replaced by $e^{(c-\frac{1}{2})\lambda}$. □

[NUM]: F_c was compared with the integral for $c \in \{0.75, 1, 1.5\}$, $t \in \{0.3, 0.7, 3\}$, together with the relation $F_c(t + i\beta) = F_c(-t)$ (18 cases, max relative error 2.8×10^{-26}).

CORRECTION 38.8.2 (answer to steering question (4); [THM] for the facts, [OBSERVATION] for the reading). *Is "KMS$_1$" in Lemma 31 Thm. 6.5 the BC critical temperature?* It is related, but genuinely and shallowly, not through a shared mechanism. (i) BC's β is $\operatorname{Re} s$ in $\zeta(\beta - it)$ (Prop. 38.5.3). Thm. 6.5's β is $2c - 1$, a strip width, and any β is available (Prop. 38.8.1). (ii) At $c = 1$ the strip is the critical strip $0 \leq \operatorname{Re} s \leq 1$, whose right edge $\operatorname{Re} s = 1$ is the BC critical line. $\xi(1 + it) = \frac{1}{2}(1 + it)(it)\pi^{-(1+it)/2}\Gamma(\frac{1+it}{2})\zeta(1 + it)$ contains the boundary value of the BC partition function at $\beta = 1$, with the pole cancelled by the factor $s - 1$. That is the genuine link. (iii) There is no KMS$_1$ *state* of BC whose two-point function is $\xi(1 + it)$: the BC KMS$_1$ state is of type III$_1$ and unique, and the Gibbs trace diverges at $\beta = 1$. So the two "$\beta = 1$" coincide because *the pole at $s = 1$ and the functional-equation centre $\frac{1}{2}$ fix both*: the pole-to-mirror-pole distance is 1. It is not a coincidence, and not an identity of structures. Lemma 31's own remark that Thm. 6.5 is "a reading of the functional equation" stands.

PROPOSITION 38.8.3 (modular conjugation, reality and Galois sectors; [THM]). *(a) Let $\beta > 1$, let $\varphi = \varphi_{\beta,1}$ and let $(\pi, \mathcal{H}, \Omega)$ be its GNS triple, realized on Hilbert–Schmidt operators with $Jk = k^*$ (Lemma 31 Prop. 3.9(a)). Then $JE_\chi\Omega = \chi(-1)E_{\bar\chi}\Omega$. So J* swaps the Galois sectors $\chi \leftrightarrow \bar\chi$ *and, unlike the BC symmetries θ_u, it does not commute with the $\widehat{\mathbb{Z}}^\times$-grading. (b) Let $Z(\chi)$ be the multiset of non-trivial zeros of $L(s, \chi)$ and $\Lambda(s, \chi)$ the completed L-function. Then $\overline{\Lambda(\bar s, \chi)} = \Lambda(s, \bar\chi)$ and $\Lambda(s, \chi) = \epsilon(\chi)\Lambda(1 - s, \bar\chi)$ with $|\epsilon(\chi)| = 1$. Hence $\rho \mapsto \bar\rho$ maps $Z(\chi) \to Z(\bar\chi)$, $\rho \mapsto 1 - \rho$ maps $Z(\chi) \to Z(\bar\chi)$, and the mirror $\rho \mapsto 1 - \bar\rho$ maps $Z(\chi) \to Z(\chi)$. (c) The Galois element $-1 \in \widehat{\mathbb{Z}}^\times$*

(complex conjugation in $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$*) acts on the sector* χ *by the scalar* $\bar\chi(-1)=\chi(-1)$*, the parity, and does not move sectors. What moves* χ *to* $\bar\chi$ *is the complex conjugation of the* values *of* χ*, implemented in BC by the antilinear* $*$ *and hence by* J*.*

Proof. (a) $\sigma_t(E_\chi)=E_\chi$, so $E_\chi\Omega$ is fixed by Δ^{it} and $\Delta^{1/2}E_\chi\Omega=E_\chi\Omega$. With $S=J\Delta^{1/2}$ and $SE_\chi\Omega=E_\chi^*\Omega$ we get $JE_\chi\Omega=E_\chi^*\Omega$, and $E_\chi^*=\sum_a\chi(a)e(-a/N)=\chi(-1)E_{\bar\chi}$. (b) This is the standard functional equation for primitive χ. (c) $\theta_{-1}(E_\chi)=\bar\chi(-1)E_\chi$ by Thm. 38.5.4(a). □

Remark 38.8.4. For $\chi=1$, (b) is Lemma 31 Prop. 3.4 (the Klein four-group). For $\chi\neq1$ the Klein group breaks into a sector-preserving $\mathbb{Z}/2$ (the mirror) and a sector-swapping pair. Real χ (quadratic) keep the whole group. The earlier draft sentence "reality *is* Galois complex conjugation" is wrong as stated and is replaced by (c).

Conjecture 38.8.5 (one Lemma 37 line per Galois sector; [CONJ], its RH content [RESTATEMENT]). For primitive χ and $\omega>0$ put $E_{\chi,\omega}(z)=\Lambda(\frac12+\omega-iz,\chi)$. By Prop. 38.8.3(b), for real z, $E^{\#}_{\chi,\omega}(z)=\Lambda(\frac12+\omega+iz,\bar\chi)=\overline{E_{\chi,\omega}(z)}$, so $\Theta_\chi=E^{\#}_{\chi,\omega}/E_{\chi,\omega}$ is unimodular on $\mathbb{R}$ ([THM]). Conjecture: Lemma 37's construction (exponential line, passivity at ω, depth theorem) carries over sector by sector. Then "all sector lines passive at every $\omega>0$" is GRH for χ, by the argument of Lemma 37 Cor. 3.3 ([RESTATEMENT]). *Not checked*: Hermite–Biehler for $E_{\chi,\omega}$ with the right ω-range for $L(s,\chi)$, and any χ-version of the Suzuki-type kernels Lemma 37 uses. Also note that $E^{\#}_{\chi,\omega}$ lies in the $\bar\chi$ sector, so the natural object is the *pair* $(\chi,\bar\chi)$, a 2×2 line for complex χ. That is (a) of Prop. 38.8.3 in the line picture [OBSERVATION].

Proposition 38.8.6 (thresholds; facts [THM]/[CITED], matching [OBSERVATION]). *Write the Lemma 37 line at depth* ω *via* $s=\frac12+\omega$*, so* $\beta_{\mathrm{BC}}=\mathrm{Re}\,s=\frac12+\omega$*. (i) BC's ordered phase* $\beta>1$ *is* $\omega>\frac12$*. There Lemma 37 proves passivity unconditionally (STATE §46(e)), because* ζ *has an absolutely convergent Euler product there. That is* the same reason *BC has a Gibbs state:* $\zeta(\beta)<\infty$*. This common cause is [THM]. (ii) BC's critical point* $\beta=1$ *is* $\omega=\frac12$*. There Lemma 37 Cor. 2.5 finds the archimedean-only gain becomes infinite, and the pole edge mode* $e^{u/2}$ *appears. Both are the pole of* ζ *at* $s=1$*, as is the vanishing of the order parameters (Thm. 38.5.4(d)). (iii) Lemma 37's archimedean break* $t^*\approx0.39$ *concerns the transverse (horizon) variable, not* $\mathrm{Re}\,s$*, and corresponds to nothing in BC.*

Answer to J. W.'s question (5): symmetry breaking at $\beta=1$ corresponds to the *pole edge state*, not to the archimedean break. The identification "SSB = edge mode" is [OBSERVATION]: same location, same cause, no proved functorial map.

Remark 38.8.7 (a "main conjecture" shape via Deninger's determinants; [CONJ]). Conjecture 38.4.4 and Prop. 38.8.3 combine into the expected shape $\Lambda(s,\chi)\doteq\det_\infty\left(\frac{1}{2\pi}(s-\Theta)\mid H^1_{\mathrm{dyn}}\right)_\chi$, a χ-isotypic piece under a Galois action commuting with Θ [88, §7]. This parallels the Iwasawa main conjecture's ω^k-isotypic pieces V_k. The parallel is *exactly the grading*: isotypic components of the compact factor. The *operators* differ: Θ is the infinitesimal rapidity, while γ_0 is a compact-group element read p-adically. Circularity flag: as in §38.4.

38.9. Open questions

(Q1) Do you accept the corrected dictionary (§38.6), in which Iwasawa's γ_0 is "Galois read p-adically" rather than "p-adic rapidity"? The rapidity-like reading survives only as Prop. 38.2.4 and in CC14's $\sigma^{(\beta)}$.

(Q2) Should we develop Conj. 38.8.5, a GRH family of lines with $(\chi,\bar\chi)$ pairing, as a future lemma? The first step is Hermite–Biehler for $\Lambda(\frac12+\omega-iz,\chi)$.

(Q3) The BC order parameters continue analytically to complex β and meet zeros of $L(\cdot,\chi)$ (Fisher zeros). Is a "Lee–Yang for Galois sectors" worth a lemma, knowing that no Gibbs state exists there?

(Q4) Should TAA adopt Deninger's language ($\mathbb{R}_{>0}$-system, Galois on the $\widehat{\mathbb{Z}}$-coordinate, Θ on H^1) as the official frame for Lemmas 31–38? It already contains Thm. 38.2.1's split [**88**, §7].

(Q5) CC14's p-adic states keep the conjugation symmetry that the complex states break. Is this the p-adic face of Prop. 38.8.3? ([OBSERVATION].)

(Q6) Lemma 37 asked for a "tower in the prime-cutoff direction". The Galois object is $\prod_{p\in S}\mathbb{Z}_p^*$, the Galois group of the S-ramified cyclotomic field, with multi-variable Iwasawa theory [**107**] [BIB]. The BC object is $\bigotimes_{p\in S}\mathcal{T}_p$ (Prop. 38.5.8). Which one do you mean?

38.10. Literature, citation corrections, summary

Search report. We searched (web, Crossref, and the sources read) for work identifying an Iwasawa $\mathbb{Z}_p$-action with a Tomita–Takesaki or Bost–Connes time evolution. *We found none that makes this identification.* The nearest works are: BC95 and CM08 (Galois as symmetry commuting with time); [**67**] (p-adic BC with Iwasawa log and L_p); [**88**] (Galois via κ commuting with the $\mathbb{R}_{>0}$-flow); [**185**], [**298**], [**65**] (endomotives and arithmetic models of BC; bibliographic, [BIB]); [**71**] (archimedean Weil positivity; title and venue verified, content not used); [**198**] (knots and primes, [BIB]). An unrefereed 2026 Zenodo deposit (doi:10.5281/zenodo.22373693) was seen only as a search snippet claiming "no p-adic KMS states". It is *consulted, not cited.*

Citation corrections and flags.

(i) *BC95 Thm. 5(a)* states uniqueness for $0 < \beta < 1$. Uniqueness at $\beta = 1$ is in CM08 Thm. 3.32 ($0 < \beta \le 1$). Cite CM08 for "$\beta \le 1$".

(ii) *"σ_t is the $\mathbb{R}_{>0}$ scaling action"* (brief) is refined: σ_t is the covolume-ratio phase, the action *dual* to scaling (Prop. 38.5.3). "Modular flow $= \sigma_{-\beta t}$" is convention-dependent; in CM08's convention it is $\sigma_{\beta t}$ (Prop. 38.5.2).

(iii) *Chat claim "Iwasawa generator = p-adic rapidity"*: corrected (Correction 38.2.6).

(iv) *"Reality = Galois complex conjugation"*: corrected (Prop. 38.8.3(c)).

(v) *Washington Thm. 7.10* is used only as quoted in [**67**, (134)] ([CITED, secondary]). The Washington copy available to us had only the front matter.

(vi) *Greenberg (6)–(7)* was checked from a search-result rendering of the article, because Project Euclid blocked the fetch ([CITED, secondary]).

(vii) *Lemma 4's byline* reads "TAA working notes (J. W. McGreevy; formalization by Grok Bot)", which differs from the standard byline. Flagged, not edited.

(viii) *Lemma 37 Rem. 7.6* calls[1] Γ "$\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})^+/\mathbb{Q})$-type". Precisely, $\Gamma \cong \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$, and also $\cong \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})^+/\mathbb{Q}(\mu_p)^+)$. Minor, flagged, not edited.

(ix) *CC21* title verified as "Weil positivity and trace formula the archimedean place", Selecta 27 (2021).

Summary of labels. [THM]: Thm. 38.2.1, Props. 38.2.2–38.2.5, Thm. 38.3.1, Props. 38.5.2, 38.5.3, Thms. 38.5.4, 38.5.6, Props. 38.5.8, 38.8.1, 38.8.3. [CITED]: Thm. 38.5.1, Rem. 38.5.9, Thm. 38.4.2. [CONJ]: Conjs. 38.4.4, 38.8.5. [OBSERVATION]: Correction 38.8.2 reading, Prop. 38.8.6 matching. [ANALOGY]: the QFT readings. **RH and GRH remain unproved; nothing here bears on them beyond restatement.**

[1] **[Vol. 3 editorial note]** Already fixed: Lemma 37 v2 (correction K11) gives the precise group, and Remark 37.7.7 as printed in Chapter 37 carries it.

CHAPTER 39

Lemma 39: The prime clock

Scaling time, the Bost–Connes Galois tower, relative entropy, and causal polarization

Standalone note `lemma39.tex`, October 22, 2026.

Summary. Statement of intellectual honesty. RH is not proven, here or anywhere in TAA, and this lemma does not supply the missing axiom G5. Every claim carries a tag: [THM] proved here (or an elementary consequence of cited results, proof given); [CITED] checked at the source (`lit/lemma39/verified_refs.md`); [CITED, secondary] checked only through a named secondary source; [BIB] bibliographic record only (content standard, not read at the source); [RESTATEMENT] equivalent to RH or to a known statement (no progress); [ANALOGY] analogy only; [NUM] numerical (`num/lemma39_*.py`, data `num/l39/`); [CONJ] conjecture; [OPEN] open. Lemma 37's data (`num/l37/`) are used read-only.

Summary

Question (J. W.). Is the "gap in time" between prime arrivals on the Lemma 37 line a proper-time, scale-free quantity, and can positivity (G5) be read as an entropic statement, phrased in Galois-symmetry terms?

Results.

(1) *Prime clock* (§39.2, [THM]). The line coordinate $t = \frac{1}{2}\log x$ is the parameter of the scaling group $\mathbb{R}_{>0}$; arrival gaps $\frac{1}{2}\log(p^+/P)$ are Haar-invariant. An off-line displacement δ is one exponential rate per unit log-scale, the same for every prime; $\delta = 0$ is exactly simultaneous unitarity of all the characters $p^{\mathbb{Z}} \ni p^k \mapsto p^{-ks}p^{k/2}$. *Correction* (Rem. 39.2.4): in the Bost–Connes (BC) system the log-scale axis is the *energy* axis ($H\varepsilon_n = \log n\, \varepsilon_n$), and the scaling action of $\mathbb{R}_{>0}$ is the *dual action* of the BC time evolution (Takesaki duality), not the modular flow itself. "Line time = modular time" is therefore an analogy; the theorem-level statement is that each arrival gap is an eigenvalue of the modular Hamiltonian of the next Galois level at $\beta = \frac{1}{2}$ (Prop. 39.2.5).

(2) *Galois tower* (§39.3). [CITED]: the Toeplitz part of BC factorizes over primes, $C^*(\mathbb{N}^\times) = \bigotimes_p \mathcal{T}_p$, with KMS states $\bigotimes_p \varphi_{\beta,p}$ (BC95 Props. 7–8). [THM]: the full BC algebra is a tensor product over primes *twisted by Galois*: the endomorphism implemented by μ_q is the local shift at q composed with the Galois element $g_q \in \widehat{\mathbb{Z}}^*$, $(g_q)_p = q$ $(p \neq q)$ (Prop. 39.3.2). The P-truncated system has Galois group $\prod_{p\le P}\mathbb{Z}_p^* = \mathrm{Gal}(\mathbb{Q}(\mu_{P^\infty})/\mathbb{Q})$ and partition function $\prod_{p\le P}(1-p^{-\beta})^{-1}$, exactly Lemma 37's truncated Euler product; it has *no* phase transition (Prop. 39.3.4). The BC transition at $\beta = 1$ and Lemma 37's archimedean rescue are the *same* fact: the pole of ζ at 1, global in p. [THM]: BC's high-temperature state restricted to level P is the Galois (Haar) average of the truncated Gibbs states (Prop. 39.3.5; checked to 10^{-40}).

(3) *Entropy* (§39.4). Corrected statement: relative entropy is *invariant* under the state's own modular flow and monotone under restriction/CP maps (Uhlmann) and along half-sided modular inclusions. [THM]: per-prime closed forms; $S_P(1) = \log P + \log\log P + o(1)$ (the Euler constants cancel; Thm. 39.4.2); relative entropies between two temperatures diverge iff $\min(\beta,\beta') \le 1$

(Prop. 39.4.3). [RESTATEMENT]: the Suzuki symbol on the real axis is a free-energy difference, and the exponential line is $\exp(2c \cdot \text{BC mean energy})$ (Prop. 39.4.4).

(4) *Margin↔entropy verdict* (§39.5, [NUM]). **No relation is established.** Every decreasing function of p^+, including trivial controls such as $1/\log p^+$, fits Lemma 37's margins to within the grid error ($\pm 1.6\times 10^{-3}$). After detrending, the residuals are at noise level. One post-hoc pattern ([CONJ], low confidence) is that $m(P) \approx b \log p^+/(\sqrt{p^+}-1)$, i.e. proportional to the BC mean energy of the next prime at $\beta = \frac{1}{2}$, with max relative error 8–9% for $\omega \le \frac{1}{2}$; it fails at $\omega = 1.5$. It makes a falsifiable prediction for the running $P \approx 100$ computation: $m(97) \approx 0.0115$ versus 0.018 for the trend model, at $(\frac{1}{2}, 3)$.

(5) *Galois ↔ rapidity* (§39.6, [THM]). In $C_{\mathbb{Q}} = \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^*$ the uniformizer $p \in \mathbb{Q}_p^*$ maps to (p^{-1}, g_p^{-1}). One step of p-adic rapidity is a scaling by $\log p$ *together with* the Frobenius at p in the Galois factor; the units $\mathbb{Z}_p^*$ (and Iwasawa's $\Gamma = 1 + p\mathbb{Z}_p$) lie entirely in the compact factor. A Galois-equivariant entropy source of G5 is assessed and faces four obstacles, including circularity (§39.6).

(6) *Algebras, causality, polarization* (§39.7, per J. W.). (a) [THM]: by Gelfand–Pontryagin duality the P-truncation is the quotient $\widehat{\mathbb{Z}} \to \prod_{p\le P} \mathbb{Z}_p$ (and $C_{\mathbb{Q}} \to C_{\mathbb{Q}}/\prod_{p>P} \mathbb{Z}_p^*$). (b) [CITED]: Borchers/Wiesbrock half-sided modular inclusions; [THM] the causality (Paley–Wiener/Titchmarsh/KK) reading of a positive generator. This is spectrum-condition positivity, which comes free and is not the positivity RH needs. [CITED] the BC Hecke algebra comes from the $ax+b$ pair $P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+$. [THM]: *no* finite Galois level, at any β, carries a half-sided modular structure, and the prime tower is a split (tensor) inclusion, the opposite of half-sided; a half-sided structure could only appear in the infinite-prime limit at $\beta \le 1$ ([OPEN]). (c) [ANALOGY]: Hodge–Riemann polarization of H^1 vs. the Hardy split and the one-particle complex structure. RH would need a *causal polarization* of Deninger's H^1 whose positivity is not defined through the zeros.

39.1. Conventions and inputs

$s = \sigma + i\tau$; zeros $\rho = \frac{1}{2} + \delta + i\gamma$. p^+ is the least prime $> P$; $\mathbb{N}_P$ is the multiplicative monoid of P-smooth positive integers; $\zeta_P(s) = \prod_{p\le P}(1-p^{-s})^{-1} = \sum_{n\in\mathbb{N}_P} n^{-s}$, absolutely convergent for $\operatorname{Re} s > 0$. From Lemma 34 (Suzuki's line, [**257**]) we use: $E = \xi(\frac{1}{2} + \omega - iz)^\nu$, kernel $K = \sum_n c_\nu(n) n^{-1/2} G(x - \log n)$, Hankel sections $\mathcal{K}[t]$ on $(-t, t)$ seeing $K|_{[0,2t)}$; so integer n switches on at $t_n = \frac{1}{2}\log n$. From Lemma 37: the P-truncated symbol $\Theta_P = \Theta_\infty \prod_{p\le P} \theta_p$, break length $t_*(P)$, margin $m(P) = t_*(P) - \frac{1}{2}\log p^+$ (Table 1 there), the prime-arrival theorem (Thm. 7.4) and Cor. 2.5 (the truncated line breaks for $\omega \le \frac{1}{2}$ because the archimedean pole is not cancelled). From Lemma 31: the BC facts and the theorem that no Borchers half-sided translations exist when $\log\Delta$ has pure point spectrum (Lemma 31, §6(5)). From Lemma 38 (in preparation by another worker; not edited here): $C_{\mathbb{Q}} = \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^*$ and the correction that Iwasawa's Γ lies in the compact factor.

39.2. The prime clock

THEOREM 39.2.1 (scale-free arrival gaps; [THM]). *(a) $\tau : (\mathbb{R}_{>0}, \cdot) \to (\mathbb{R}, +)$, $x \mapsto \frac{1}{2}\log x$, is a group isomorphism carrying Haar measure dx/x to $2\,dt$. The onset of n on the line is $\tau(n)$. The arrival gap between consecutive primes $P < p^+$ is $\tau(p^+) - \tau(P) = \frac{1}{2}\log(p^+/P)$. It is unchanged by every rescaling $x \mapsto \lambda x$, and it is the Haar length of $[P, p^+]$ (divided by 2).*

(b) Write $u = \log x$. For $s = \frac{1}{2} + \delta + i\gamma$, $x^{-s} = x^{-1/2} e^{-\delta u} e^{-i\gamma u}$, so $|p^{-s}| = p^{-1/2} e^{-\delta \log p}$. Hence δ is a single exponential rate per unit log-scale, the same at every prime and at every x.

(c) For each prime p the map $\chi_{s,p} : p^{\mathbb{Z}} \to \mathbb{C}^\times$, $p^k \mapsto p^{k/2} p^{-ks}$, is a character of the p-adic rapidity lattice $p^{\mathbb{Z}} \subset \mathbb{R}_{>0}$ (step $\log p$ in u). It is unitary iff $\delta = 0$, and then for all p simultaneously. Equivalently, $x \mapsto x^{1/2-s}$ is a unitary character of $\mathbb{R}_{>0}$ iff $\operatorname{Re} s = \frac{1}{2}$.

PROOF. (a),(b) are immediate. (c): $|\chi_{s,p}(p^k)| = p^{-k\delta}$, which is 1 for all k iff $\delta = 0$; the condition does not depend on p. □

REMARK 39.2.2 (what (c) does and does not say; [RESTATEMENT]). (c) is a statement about characters, true for every s. It becomes an RH statement only once the zeros are realized as the characters that occur in a unitary representation; that realization is G5 again (Lemma 30). So "the critical line is simultaneous unitarity of the scaling action at every prime" is an exact *reformulation*, not a mechanism.

THEOREM 39.2.3 (Bisognano–Wichmann; [CITED, secondary] via CR94 §4.3, Borchers ESI446). *For a Wightman field theory, the vacuum and the algebra of a Rindler wedge W satisfy $\Delta_W^{it} = U(\Lambda_W(-2\pi t))$, where $\Lambda_W(\eta)$ is the boost fixing W with rapidity η* [**33**, **34**, **72**]. *Modular time t is rapidity divided by -2π.*

What is proved and what is analogy. Thm. 39.2.3 needs a Hilbert space, a wedge algebra, a vacuum and the Poincaré group. For ζ none of these is given; the scaling group $\mathbb{R}_{>0}$ plays the role of the boost group only formally. Lemma 31 (§6) proved that the BW match is exact on the *spacetime* side (Rindler modes = Mellin modes; off-line = complex Rindler frequency), and that the Hilbert-space side is G5. So "prime-clock time = modular time" is [ANALOGY]. The next remark sharpens this and corrects the wording used in the conversation.

REMARK 39.2.4 (correction: scaling is *dual* to BC time; [CITED] CM08 Ch. 4 Def. 4.60, [THM] for the reading). In BC, $\sigma_\tau(\mu_n) = n^{i\tau}\mu_n$, $H\varepsilon_n = (\log n)\varepsilon_n$ [**44**, Prop. 7], [**74**, (3.141)]. The position $\log n$ of n on the log-scale is an *energy*, not a time. Connes–Marcolli's dual system $(\hat{A} = A \rtimes_\sigma \mathbb{R}, \theta)$ carries the scaling action $\theta_\lambda(\int x(t)U_t dt) = \int \lambda^{it} x(t) U_t dt$ of $\mathbb{R}_+^*$ [**74**, Lemma 4.59, Def. 4.60], the Takesaki dual action [**262**]. It is this action, combined with the Galois action, whose spectrum gives the zeros in their spectral realization [**74**, Intro., "third step"; §4.3]. Thus, in BC terms, the prime clock $u = \log x$ is the variable *Fourier dual* to modular time, which is the reason BW cannot be transplanted literally.

PROPOSITION 39.2.5 (arrival gaps are modular eigenvalues; [THM]). *Let $\rho_\beta = e^{-\beta H}/\zeta_{P'}(\beta)$ be the Gibbs state on $\ell^2(\mathbb{N}_{P'})$ (§39.3). The modular operator of its GNS representation satisfies $\log\Delta = \log\rho_\beta \otimes 1 - 1 \otimes \log\rho_\beta^T$, with pure point spectrum $\{\beta\log(m/n) : m, n \in \mathbb{N}_{P'}\}$. In particular, for $P' = p^+$ and $\beta = \frac{1}{2}$, the arrival gap $\frac{1}{2}\log(p^+/P)$ is the eigenvalue of $-\log\Delta$ on $\varepsilon_{p^+} \otimes \overline{\varepsilon_P}$, up to sign convention.*

PROOF. Type I Gibbs states have $\Delta = \rho \otimes \rho^{-1}$ in the standard form on Hilbert–Schmidt operators (CR94 (39)–(44), [CITED] in Lemma 31). Both P and p^+ lie in $\mathbb{N}_{p^+}$. □

The choice $\beta = \frac{1}{2}$ only matches Suzuki's normalization $t = \frac{1}{2}\log x$. That it is also the critical abscissa is suggestive, but no deeper meaning is claimed ([ANALOGY]).

39.3. The Galois tower

39.3.1. BC facts used.

THEOREM 39.3.1 ([CITED] BC95 Thm. 5, Props. 7, 8, 21; CM08 Thm. 3.32, (3.73)–(3.76)). *(a) The BC algebra is generated by isometries μ_n ($n \in \mathbb{N}^\times$) and unitaries $e(r)$ ($r \in \mathbb{Q}/\mathbb{Z}$), with time evolution $\sigma_t(\mu_n) = n^{it}\mu_n$, $\sigma_t(e(r)) = e(r)$. The group $W = \widehat{\mathbb{Z}}^*$ acts by symmetries, $g(\mu_n) = \mu_n$, $g(e(r)) = e(gr)$, commuting with σ_t, and the fixed-point algebra is $C^*(\mathbb{N}^*)$ generated by the μ_n* [**44**, Prop. 21], [**74**, (3.75)]. *(b) $C^*(\mathbb{N}^*) = \bigotimes_p \mathcal{T}_p$ (Toeplitz algebras, $\mathcal{T}_p = C^*(\mu_p)$), $\sigma_t = \bigotimes_p \sigma_{t,p}$* [**44**, Prop. 7]. *(c) For every $\beta > 0$ there is a unique KMS$_\beta$ state on $(C^*(\mathbb{N}^*), \sigma)$, namely $\bigotimes_p \varphi_{\beta,p}$ with eigenvalue list $\{(1-p^{-\beta})p^{-n\beta}\}_{n\geq 0}$. It is of type I_∞ for $\beta > 1$ and of type III_1 for $\beta \leq 1$* [**44**, Prop. 8]. *(d) On the full system: for $0 < \beta \leq 1$ the KMS$_\beta$ state is unique, with $\varphi_\beta(e(a/b)) = b^{-\beta}\prod_{p|b}\frac{1-p^{\beta-1}}{1-p^{-1}}$.*

For $\beta > 1$ the extremal KMS$_\beta$ states are indexed by $\rho \in \widehat{\mathbb{Z}}^*$, the action of $\widehat{\mathbb{Z}}^*$ on them is free and transitive, and the partition function is $\zeta(\beta)$ [74, Thm. 3.32, (3.138), (3.142)], [44, Thm. 5]. *(e) BC95 eq. (7): $f\mu_n = \mu_n f_n$ with $f_n(b) = f(nb)$ for $f \in C^*(\mathbb{Q}/\mathbb{Z}) = C(\widehat{\mathbb{Z}})$. Eq. (8): if f is localized in a finite set F of primes and $q \notin F$, then $f\mu_q = \mu_q \theta_{g_q}(f)$, where $(g_q)_p = q$ for $p \neq q$ and $(g_q)_q = 1$.*

Proposition 39.3.2 (BC is a Galois-twisted tensor product over primes; [THM] from Thm. 39.3.1(e)). *Identify $C^*(\mathbb{Q}/\mathbb{Z}) = C(\widehat{\mathbb{Z}}) = \bigotimes_p C(\mathbb{Z}_p)$. For a prime q, $\operatorname{Ad}\mu_q^*$ restricted to $C(\widehat{\mathbb{Z}})$ is $f \mapsto f(q\,\cdot)$. Multiplication by q on $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ splits as*

$$q \cdot (x_p)_p = \big(\underbrace{qx_q}_{\text{local shift at } q} \quad , \quad \underbrace{(qx_p)_{p\neq q}}_{\text{Galois element } g_q} \big),$$

so $\operatorname{Ad}\mu_q^* = \alpha_q^{\mathrm{loc}} \circ \theta_{g_q} = \theta_{g_q} \circ \alpha_q^{\mathrm{loc}}$. Here α_q^{loc} acts only on the $\mathbb{Z}_q$ tensor factor and θ_{g_q} is the Galois symmetry by $g_q \in \widehat{\mathbb{Z}}^*$. Consequently: (i) the BC system would be the tensor product $\bigotimes_p (C(\mathbb{Z}_p) \rtimes p^{\mathbb{N}})$ of local systems if *the Galois cocycle $q \mapsto g_q$ were removed; (ii) on the Galois-invariant subalgebra the twist acts trivially, and there the factorization of Thm. 39.3.1(b) is exact.* Galois symmetry is precisely what couples different primes in BC.

Proof. From $\mu_q^*\mu_q = 1$ and (e), $\mu_q^* f \mu_q = f_q$. The splitting of $q\cdot$ is componentwise; on $\mathbb{Z}_p$ with $p \neq q$, q is a unit, so the action is that of $g_q \in \widehat{\mathbb{Z}}^*$. Galois-invariant f satisfy $\theta_{g_q} f = f$. □

39.3.2. The P-truncated system.

Definition 39.3.3 ([DEF]). $\mathcal{A}_{\leq P}$ is the C*-subalgebra generated by $e(r)$ with P-smooth denominator and by μ_n, $n \in \mathbb{N}_P$. It is σ- and $\widehat{\mathbb{Z}}^*$-invariant, since $\mu_n e(r) \mu_n^* = \frac{1}{n}\sum_{ns=r} e(s)$ keeps P-smooth denominators. Its symmetry group acts through $G_{\leq P} := \prod_{p\leq P} \mathbb{Z}_p^*$.

Proposition 39.3.4 (truncated BC; [THM]). *(a) $G_{\leq P} \cong \varprojlim_{n\in\mathbb{N}_P} (\mathbb{Z}/n)^* \cong \operatorname{Gal}(\mathbb{Q}(\mu_{P^\infty})/\mathbb{Q})$, where $\mathbb{Q}(\mu_{P^\infty})$ is generated by all roots of unity of P-smooth order. The Galois levels form a tower $\cdots \to G_{\leq p^+} \to G_{\leq P} \to \cdots$ of quotients of $\widehat{\mathbb{Z}}^*$. (b) For every embedding $\rho \in G_{\leq P}$ the representation π_ρ on $\ell^2(\mathbb{N}_P)$ (the restriction of CM08 (3.140)) has $H\varepsilon_m = \log m\, \varepsilon_m$ and $\operatorname{Tr} e^{-\beta H} = \zeta_P(\beta) = \prod_{p\leq P}(1-p^{-\beta})^{-1} < \infty$ for* every *$\beta > 0$. Hence the Gibbs states $\varphi^P_{\beta,\rho} = \zeta_P(\beta)^{-1}\operatorname{Tr}(\pi_\rho(\cdot)e^{-\beta H})$ are KMS$_\beta$ for all $\beta > 0$. $G_{\leq P}$ permutes them transitively, and their common restriction to $C^*(\mathbb{N}_P)$ is $\bigotimes_{p\leq P}\varphi_{\beta,p}$. (c) The partition function of level P is Lemma 37's truncated Euler product, free of poles on $\operatorname{Re}\beta > 0$. No finite level has the BC phase transition: the pole at $\beta = 1$ appears only as $P \to \infty$, through the divergence of $\sum_p p^{-1}$. (d) The analytic object behind (c) is the same one behind Lemma 37 Cor. 2.5 and Thm. 7.4 ("$\omega \leq \frac{1}{2}$: every truncated line breaks because the archimedean pole is uncancelled"). In both cases the missing ingredient is the pole of ζ at 1, which is global in p. [RESTATEMENT] as an identification of the two phenomena; the identity of the function ζ_P is [THM].*

Proof. (a) Cyclotomic theory: $\operatorname{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n)^*$ compatibly in n; take the limit over $n \in \mathbb{N}_P$ and use CRT. (b) Gibbs states with trace-class $e^{-\beta H}$ are KMS$_\beta$ (standard, [47]). $g \in G_{\leq P}$ maps π_ρ to $\pi_{g\rho}$, and $\pi_\rho(\mu_n)$ does not depend on ρ. (c) ζ_P is a finite product of functions holomorphic on $\operatorname{Re}\beta > 0$. □

Not claimed: that $\{\varphi^P_{\beta,\rho}\}$ exhausts the extremal KMS$_\beta$ states of $\mathcal{A}_{\leq P}$, or that the $\varphi^P_{\beta,\rho}$ are pairwise distinct ([OPEN]; this is expected by analogy with BC95 Thm. 5(b)).

Proposition 39.3.5 (the BC high-temperature state, seen at level P, is a Galois average; [THM], [NUM]). *For $0 < \beta \leq 1$ and P-smooth b, $\varphi_\beta(e(a/b)) = \int_{G_{\leq P}} \varphi^P_{\beta,\rho}(e(a/b))\, d\rho$ (Haar probability). The right-hand side is defined, and the identity holds as an algebraic identity, for every $\beta > 0$.*

PROOF. $\int \rho(\zeta_b^{am})d\rho$ is the average of a primitive d-th root of unity, $d = b/(b,m)$, which equals $\mu(d)/\varphi(d)$. The sum $\zeta_P(\beta)^{-1}\sum_{m\in\mathbb{N}_P} m^{-\beta}\mu(d)/\varphi(d)$ factorizes over $p^k\|b$. With $x = p^{-\beta}$ and $v_p(m) = j$ geometrically distributed, the local factor is $x^k-(1-x)x^{k-1}/(p-1) = x^{k-1}(px-1)/(p-1)$. CM08's local factor is $p^{-k\beta}(1-p^{\beta-1})/(1-p^{-1}) = x^{k-1}(px-1)/(p-1)$. [NUM]: 98 cases ($\beta \in [0.3, 3]$, $b \le 1024$), max relative error 1.6×10^{-40} (`num/l39/checks.json`). □

Reading (Galois terms). Below $\beta = 1$ BC is in its symmetric phase, and level by level that phase is exactly the Galois average of the symmetry-broken Gibbs states that exist at every finite level. Symmetry breaking is *present* at every finite level and at all temperatures; it is *restored* only by the infinitude of primes when $\beta \le 1$. This parallels Lemma 37: every finite cascade is "active", and only the global pole repairs it.

39.4. Entropy

39.4.1. Correct statements.

THEOREM 39.4.1 ([CITED] Uhl77 abstract; [BIB] Ara76/77, Lin75, Ume62; [THM] for (b)). *(a) (Uhlmann) Relative entropy, defined for positive linear forms on a unital $*$-algebra (agreeing with Araki's for normal states), is jointly convex and* decreases *under identity-preserving completely positive maps* [271]. *In particular $S_{\rm rel}(\psi|_{\mathcal{B}}\|\varphi|_{\mathcal{B}}) \le S_{\rm rel}(\psi\|\varphi)$ for every unital subalgebra $\mathcal{B}$ (restriction is unital CP). (b) For any automorphism α, $S_{\rm rel}(\psi\circ\alpha\|\varphi\circ\alpha) = S_{\rm rel}(\psi\|\varphi)$ (apply (a) to α and α^{-1}). Since $\varphi\circ\sigma_t^{\varphi} = \varphi$, we get $S_{\rm rel}(\psi\circ\sigma_t^{\varphi}\|\varphi) = S_{\rm rel}(\psi\|\varphi)$:* relative entropy is invariant along the state's own modular flow*. This corrects the conversational phrase "changes monotonically along modular time". (c) Monotonicity in a geometric parameter does exist for half-sided modular inclusions: for $\mathcal{A}_B \subset \mathcal{A}_A$ an HSMI, $b \mapsto S_{\rm rel}(\psi\|\Omega;\mathcal{A}_{A_b})$ is monotonically decreasing, and convex under finite averaged null energy (QNEC)* [57, §2, Lemma 1, Thm. 2, Cor. 2] *([CITED], arXiv full text).*

39.4.2. Per-prime thermodynamics. For one prime at inverse temperature β, write $x = p^{-\beta}$. Then ([THM], geometric law)

$$\log z_p = -\log(1-x), \quad e_p = \frac{x\log p}{1-x} = \frac{\log p}{p^\beta - 1}, \quad s_p = \log z_p + \beta e_p, \quad \mathrm{Var}_p = \frac{x\log^2 p}{(1-x)^2},$$

and $d_p(\beta\|\beta') := S_{\rm rel}(\varphi_{\beta,p}\|\varphi_{\beta',p}) = \log\frac{1-x}{1-y} + \frac{x}{1-x}\log\frac{x}{y}$ with $y = p^{-\beta'}$. All level-P quantities are sums over $p \le P$ (product states). [NUM]: for $\beta = 2, 3$ the sum to 2×10^5 matches $\log\zeta(\beta)-\beta\zeta'/\zeta(\beta)$ to 10^{-5} and 4×10^{-11}.

THEOREM 39.4.2 (entropy of level P at the critical temperature; [THM]). $S_P(1) = \sum_{p\le P} s_p(1) = \log P + \log\log P + o(1)$.

PROOF. $s_p(1) = -\log(1-p^{-1})+\frac{\log p}{p-1}$. Mertens' third theorem [192] gives $\sum_{p\le P} -\log(1-1/p) = \log\log P + \gamma + o(1)$. Next, $\sum_{p\le P}\frac{\log p}{p-1} = \sum_{n\le P}\frac{\Lambda(n)}{n} + \sum_{p\le P}\sum_{k:p^k>P}\frac{\log p}{p^k}$. The double sum is $\le 2\sum_{p\le\sqrt{P}}\frac{\log p}{P}+2\sum_{\sqrt{P}<p}\frac{\log p}{p^2} = o(1)$. Finally $\sum_{n\le P}\Lambda(n)/n = \log P-\gamma+o(1)$. To see this, partial summation gives $\psi(P)/P + \int_1^P \psi(u)u^{-2}du = \log P + 1 + \int_1^P R(u)u^{-2}du + o(1)$ with $R = \psi - u = O(u/\log^2 u)$ (de la Vallée Poussin), so the integral converges. Its value is $-1-\gamma$, by comparing the Laurent expansion $-\zeta'/\zeta(s) = \frac{1}{s-1}-\gamma+O(s-1)$ with $s\int_1^\infty \psi(u)u^{-s-1}du = \frac{s}{s-1}+s\int_1^\infty R(u)u^{-s-1}du$ as $s \downarrow 1$. The two γ's cancel. [NUM]: the remainder is 0.308, 0.100, 0.037, 0.014, 0.0042, 7.0×10^{-4}, 2.0×10^{-4} at $P = 10^1, \ldots, 10^7$. □

So at $\beta = 1$ each new Galois level adds entropy $s_{p^+}(1) \approx (\log p^+ + 1)/p^+$, and the total diverges like the logarithm of the cutoff, a "log-UV" divergence. For $\beta < 1$, PNT gives $S_P(\beta) \sim \frac{\beta}{1-\beta}P^{1-\beta}$ ([THM], partial summation); for $\beta > 1$ the sum converges.

PROPOSITION 39.4.3 (two temperatures; [THM]). *For $\beta \neq \beta'$, $\sum_p d_p(\beta\|\beta') < \infty$ iff $\min(\beta,\beta') > 1$. Hence the level-P relative entropies $D_P(\beta\|\beta') = \sum_{p\le P} d_p$ increase to $+\infty$ whenever $\min(\beta,\beta') \le 1$. By Kakutani's dichotomy [147] ([BIB]), the corresponding infinite product measures (diagonal laws of $\bigotimes \varphi_{\beta,p}$) are then mutually singular.*

PROOF. $d_p = (y-x) + x(\beta'-\beta)\log p + O(x^2\log p + y^2)$. If $\beta < \beta'$ the term $x(\beta'-\beta)\log p$ dominates, so the sum diverges iff $\sum p^{-\beta}\log p = \infty$, iff $\beta \le 1$. If $\beta > \beta'$ then $y/x \to \infty$ and $d_p \sim y$, which diverges iff $\beta' \le 1$. For Kakutani, the Hellinger defect per prime is comparable to $p^{-\min(\beta,\beta')}$ up to logarithms, with the same dichotomy. □

The increments $D_{p^+} - D_P = d_{p^+} \ge 0$ are Uhlmann monotonicity along the Galois tower, here an *equality of additivity*: in a product tower, monotonicity carries no information beyond the local positivity $d_p \ge 0$.

PROPOSITION 39.4.4 (the line symbol is thermodynamic on the real axis; [THM] identity, [RESTATEMENT] content). *Let φ_β be the level-P Gibbs state, $Z_P = \zeta_P$, $E_P(\beta) = -\zeta_P'/\zeta_P(\beta) = \sum_{p\le P} e_p(\beta)$. For real $s = \beta$ with $\beta - \omega > 0$:*

$$\log \frac{\zeta_P(\beta-\omega)}{\zeta_P(\beta+\omega)} = S_{\mathrm{rel}}(\varphi_{\beta+\omega}\|\varphi_{\beta-\omega}) + 2\omega\, E_P(\beta+\omega),$$

so the prime part of $\log \Theta_P^{\omega,\nu}(\beta)$ is ν times a relative entropy plus an energy term. For the exponential line, $\Theta_c = \exp(-2c\,\xi'/\xi)$ truncated at P is $\exp(-2c\gamma'/\gamma(\beta))\cdot\exp(2cE_P(\beta))$: exponentiated BC mean energy. For complex s, $\zeta_P(\beta+i\tau)/\zeta_P(\beta) = \varphi_\beta(e^{-i\tau H})$ is the Gibbs characteristic function of the energy.

PROOF. $S_{\mathrm{rel}}(\varphi_a\|\varphi_b) = (b-a)E(a) + \log Z(b) - \log Z(a)$ for Gibbs states of one Hamiltonian; take $a = \beta+\omega$, $b = \beta-\omega$. □

All of these are positive, prime-side, Euler-product quantities. Passivity of the line (Lemma 37 Thm. 3.2: $|\xi(s-\omega)| \le |\xi(s+\omega)|$ on $\mathrm{Re}\, s > \frac12$) needs them on $\frac12 < \mathrm{Re}\, s \le 1+\omega$, *outside* the Gibbs region. That is where the archimedean factor and the zeros enter, and where no entropy interpretation is available.

39.5. The key test: margin vs. entropy differences between Galois levels

Data (read-only): Lemma 37 margins $m(P)$ at $h = 1/800$ for $(\omega,\nu) = (\frac12, 3)$ ($P = 2,\dots,19$; 8 points), $(0.3, 4)$ ($P \le 13$; 6 points; identical to $(0.2, 6)$), $(1.5, 1)$ (6 points). The grid error, estimated from $h = 1/1600$ reruns, is $\le 1.6\times10^{-3}$. **Candidates** $X(p^+)$: entropy increments $s_{p^+}(\beta)$, $\beta \in \{\frac12, 1, \frac32\}$; energy increments $e_{p^+}(\beta)$; two-temperature relative-entropy increments $d_{p^+}(\beta_- + 2\omega\|\beta_-)$ and Jeffreys increments, $\beta_- \in \{\frac12, 1\}$ (the two temperatures the symbol compares, Prop. 39.4.4); the cumulative $S_P(1)$; the gap $\frac12\log(p^+/P)$ and the Cramér-normalized gap; the first-order missing atom $\log p^+/\sqrt{p^+}$; and the *controls* $1/\log p^+$, $p^{+-1/2}$. Fits: affine, leave-one-out, proportional; residuals after detrending by $a + b/\log p^+$ (`num/lemma39_entropy.py`, `lemma39_checks.py`; `num/l39/`).

Verdict ([NUM]; honest).

(i) *No relation is established.* Any slowly decreasing function of p^+ fits, and the trivial controls fit as well as the entropy quantities. After removing a two-parameter trend, the residuals (4–7×10^{-4}) are below the grid error, so residual correlations (Table data in `entropy.json`) carry no information. With 6–8 points per setting the data cannot discriminate between "entropy increment" and "any monotone function".

(ii) The *gap* (raw or Cramér-normalized) and the *first-order missing atom* do *not* explain the margin ($R^2 \le 0.56$). This confirms the Lemma 37 report: the margin hardly notices the gap.

	$(\frac{1}{2},3)$	$(0.3,4)$	$(1.5,1)$
affine R^2: $s_{p^+}(1)$ / $d_{p^+}(\beta_-{=}\frac{1}{2})$ / $S_P(1)$	.97 / .87 / .91	.95 / .79 / .88	.998 / .99 / .99
affine R^2: control $1/\log p^+$ / control $p^{+-1/2}$	.98 / .95	.96 / .91	.99 / .99
affine R^2: gap / normalized gap / missing atom	.30 / .02 / .10	.11 / .18 / .56	.28 / .07 / .39
rms residual after detrending by $a + b/\log p^+$	5.8×10^{-4}	7.1×10^{-4}	3.8×10^{-4}
proportional $m = b\, e_{p^+}(\frac{1}{2})$: max rel. err.	8%	9%	64%
proportional $m = b\, s_{p^+}(1)$: max rel. err.	56%	50%	10%

TABLE 39.1. Margin vs. Galois-level quantities ([NUM]). Grid error $\approx 1.6 \times 10^{-3}$.

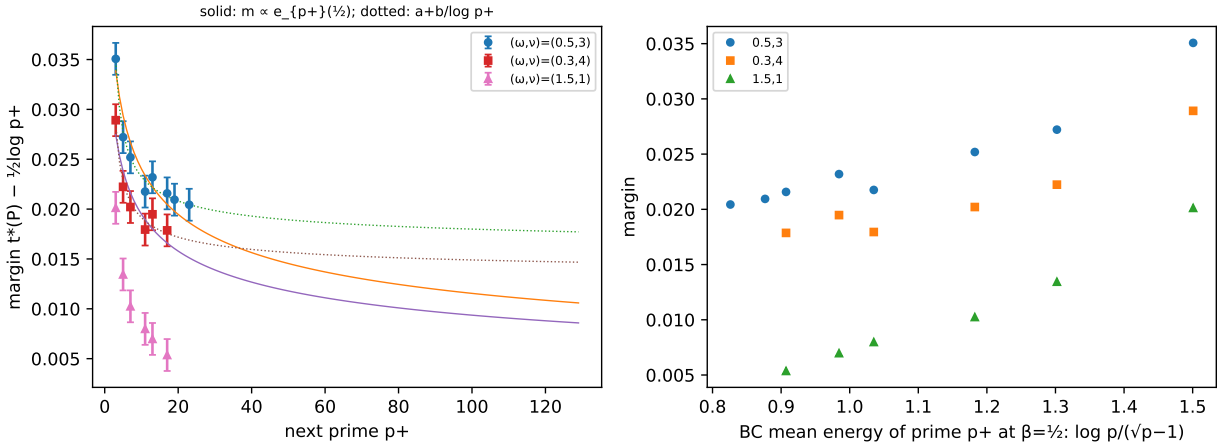

FIGURE 39.1. Left: margins with $\pm 1.6 \times 10^{-3}$ bars; solid lines show $m \propto e_{p^+}(\frac{1}{2})$, dotted lines the trend $a + b/\log p^+$. Right: margin against the BC mean energy of the next prime at $\beta = \frac{1}{2}$.

(iii) No two-temperature relative-entropy increment is singled out; Uhlmann monotonicity along this product tower is just local additivity (Prop. 39.4.3), so it cannot carry the information the margin would need.

(iv) Post-hoc pattern ([CONJ], low confidence, chosen after looking at 8 candidates $\times$ 3 settings): for $\omega \le \frac{1}{2}$,

$$m(P) \approx b_{\omega,\nu}\, e_{p^+}(\tfrac{1}{2}) = b_{\omega,\nu} \frac{\log p^+}{\sqrt{p^+ - 1}} = b_{\omega,\nu} \sum_{k \ge 1} \Lambda(p^{+k}) p^{+-k/2},$$

with $b = 0.0226$ at $(\frac{1}{2},3)$ and 0.0183 at $\nu\omega = 1.2$. This is the total critical-line weight of the missing prime and all of its powers, and equally the BC mean energy of the new Galois level at $\beta = \frac{1}{2}$. It fails at $\omega = 1.5$, where $s_{p^+}(1)$ fits instead. *Falsifiable prediction*[1] for the running $P \approx 100$ computation (Lemma 37 milestone M6) at $(\frac{1}{2},3)$: $m(31), m(47), m(97), m(113) \approx 0.016, 0.014, 0.0115, 0.011$ under this pattern, versus $0.019, 0.019, 0.018, 0.018$ under the trend model. The two differ by $> 4\times$ the grid error at $P = 97$. If the pattern holds, $m \to 0$ like $\log p/\sqrt{p}$, in line with Lemma 37's conjecture that margins tend to 0.

[1] **[Vol. 3 editorial note]** This prediction was tested and failed. Lemma 37 v2 (corrections K12, K13; (M6.5)) and Lemma 34 v2 (V7) show that the margins fitted here are the v1 values of Lemma 37's Table 1, which were too large for $t \gtrsim 0.8$. On the corrected data the pattern fails. See Appendix A, item D2.

39.6. Galois ↔ rapidity

THEOREM 39.6.1 (where a p-adic step lands in $C_{\mathbb{Q}}$; [THM] elementary, [BIB] Neu99 for the Artin map). *Use $C_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^* \cong \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^*$, choosing in each class the representative with positive archimedean component and unit finite components* [**74**, (3.76), (3.134)]. *The local group $\mathbb{Q}_p^* = p^{\mathbb{Z}} \times \mathbb{Z}_p^*$ embeds by*

$$u \in \mathbb{Z}_p^* \mapsto (1;\ u \text{ at } p), \qquad p \mapsto (p^{-1};\ g_p^{-1}), \quad (g_p)_q = p\ (q \neq p),\ (g_p)_p = 1.$$

So a unit step of p-adic rapidity (the uniformizer, $\log p$ in u) is a scaling by p^{-1} in $\mathbb{R}_{>0}$ together with *the Galois element g_p^{-1}. Under the Artin map $g_p^{\pm 1}$ (the sign depends on normalization) is the Frobenius at p on $\mathbb{Q}(\mu_n)$, $p \nmid n$. The g_p are exactly BC95's elements (Thm. 39.3.1(e)) that twist the tensor product in Prop. 39.3.2.*

PROOF. The idele $(1, \dots, p_{(p)}, \dots, 1)$ times $p^{-1} \in \mathbb{Q}^*$ is $(p_{(\infty)}^{-1}, (p^{-1})_{q\neq p}, 1_{(p)})$, whose finite part is a unit idele. □

Consequences. (1) The compact factor $\widehat{\mathbb{Z}}^* = \prod_p \mathbb{Z}_p^*$ contains each local unit group, and inside $\mathbb{Z}_p^* = \mu_{p-1} \times (1 + p\mathbb{Z}_p)$ it contains Iwasawa's $\Gamma = 1 + p\mathbb{Z}_p \cong \mathbb{Z}_p$ (p odd). This is consistent with Lemma 38's correction: *Iwasawa theory lives in the Galois factor, not in the rapidity factor* ([THM]). (2) The rapidity factor $\mathbb{R}_{>0}$ sees the primes only through the discrete lattices $p^{\mathbb{Z}}$ (steps $\log p$), and each step drags a Frobenius along. "Galois symmetry ↔ rapidity" is therefore not an identification of two groups: they are commuting *different* factors, *coupled* by the global field $\mathbb{Q}^*$ through $p \mapsto (p^{-1}, g_p^{-1})$ ([THM]). (3) BC time acts through the characters $n \mapsto n^{it}$ of the positive rationals (the norm side), Galois acts through $\widehat{\mathbb{Z}}^*$, and the two commute (Thm. 39.3.1(a)).

Can a Galois-equivariant entropy functional be a source of G5? Candidate. A functional $\mathcal{S}$ on states of BC (or of its levels), $\widehat{\mathbb{Z}}^*$-equivariant, whose positivity or monotonicity along the Galois tower is a theorem, and which equals or dominates the G5 quantity (Lemma 37's $1 - \|\mathcal{K}[t]\|$, or Lemma 36's $\mathcal{P} - \mathcal{L}$). **Obstacles.**

(O1) *Freeness* (Lemma 31 §2, [THM] there). All BC thermodynamic positivities also hold for $\zeta(s)\zeta(s+a)$, which has zeros on two lines. An entropy built only from BC states is line-blind unless it uses extra structure.

(O2) *Galois invariance kills the twist* ([THM]). Relative entropy is automorphism-invariant (Thm. 39.4.1(b)), so a Galois-equivariant entropy is constant on Galois orbits and depends only on Galois-invariant data. By Prop. 39.3.2, those data are the untwisted product $\bigotimes \mathcal{T}_p$, i.e. the Euler product alone, and they see no zeros (Prop. 39.4.4).

(O3) *Product towers make monotonicity trivial* (Prop. 39.4.3). Increments are local and ≥ 0 automatically, so they can never fail, while G5 can fail (it is equivalent to RH). An equivalence would require a non-product state, and the only known ones are defined from the zeros or from ξ on $\operatorname{Re} s < 1$.

(O4) *Circularity.* Defining the state or the subalgebras through ξ (e.g. GNS of the Weil form, Lemma 31 G6) makes the positivity of the entropy equal to G5 by construction ([RESTATEMENT]). Positivity does not cross Fourier duality (Lemma 31, [THM]), so prime-side entropy positivity does not transfer to the zero side.

Score (0.6 sig+0.4 prob): significance 0.9, probability 0.03, so 0.55. The value is carried by significance; on current evidence, success is unlikely.

39.7. Algebras, causality, polarization (per J. W.)

39.7.1. (a) Subalgebras, conditional expectations, quotients.

THEOREM 39.7.1 ([CITED] Uhl77; [BIB] Tak72, Gelfand–Naimark; [THM] for the identification). *(i) Restriction to a subalgebra and φ-preserving conditional expectations are unital CP maps, so relative entropy decreases under them (Thm. 39.4.1(a)). A φ-preserving normal conditional expectation onto a von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ exists iff $\mathcal{N}$ is globally invariant under σ^{φ} (Takesaki [**261**], [BIB]). For the product KMS state on $\bigotimes_p \mathcal{T}_p$ the slice map $E_P = \mathrm{id}_{\le P} \otimes \varphi_{\beta,>P}$ is such an expectation onto level P ([THM], product state). (ii) Gelfand duality: unital C^*-subalgebras $C(Y) \subset C(X)$ correspond to continuous surjections $X \to Y$. For the group C^*-algebra of a discrete abelian group, $C^*(H) \cong C(\hat{H})$ and an inclusion $H \subset H'$ dualizes to the restriction (quotient) $\hat{H}' \to \hat{H}$. (iii)* Identification *([THM]). The commutative part of level P is $C^*(\oplus_{p\le P}\mathbb{Q}_p/\mathbb{Z}_p) \subset C^*(\mathbb{Q}/\mathbb{Z})$. Since $\mathbb{Q}/\mathbb{Z} = \oplus_p \mathbb{Q}_p/\mathbb{Z}_p$ and $\widehat{\mathbb{Q}_p/\mathbb{Z}_p} = \mathbb{Z}_p$, it is $C(\prod_{p\le P}\mathbb{Z}_p) \subset C(\widehat{\mathbb{Z}})$, dual to the quotient*

$$\widehat{\mathbb{Z}} = \textstyle\prod_p \mathbb{Z}_p \;\twoheadrightarrow\; \prod_{p\le P}\mathbb{Z}_p \qquad (\textit{forget the } p\textit{-adic coordinates, } p > P).$$

On symmetries, the truncation is $\widehat{\mathbb{Z}}^* \twoheadrightarrow G_{\le P}$. In idele-class terms it is $C_{\mathbb{Q}} = \mathbb{R}_{>0} \times \widehat{\mathbb{Z}}^* \twoheadrightarrow \mathbb{R}_{>0} \times G_{\le P} = C_{\mathbb{Q}}/\prod_{p>P}\mathbb{Z}_p^*$, the quotient by a compact *subgroup. The rapidity factor is untouched; what is lost is Galois information at the primes $> P$. The semigroup part is cut from $\mathbb{N}^\times$ to $\mathbb{N}_P$ (the lattices $p^{\mathbb{Z}}$, $p \le P$), which is not a quotient but a sub-monoid. Hence the Lemma 37 truncation is* a quotient on the Galois side and a restriction on the rapidity side.

39.7.2. (b) Half-sided modular inclusions and causality.

THEOREM 39.7.2 (Borchers; Wiesbrock; [CITED] abstracts of Bor92/Wie93, statements as in CF20 Def. 1 and (12)). *(i) [**40**] Let $\mathcal{M}$ have cyclic separating Ω, and let $U(a) = e^{iaP}$, $P \ge 0$, with $U(a)\Omega = \Omega$ and $U(a)\mathcal{M}U(-a) \subset \mathcal{M}$ for $a \ge 0$. Then $\Delta^{it}U(a)\Delta^{-it} = U(e^{-2\pi t}a)$ and $JU(a)J = U(-a)$. (ii) [**289**] Conversely, if $\mathcal{N} \subset \mathcal{M}$ have common cyclic separating Ω and $\Delta_{\mathcal{M}}^{-it}\mathcal{N}\Delta_{\mathcal{M}}^{it} \subset \mathcal{N}$ for $t \ge 0$ (half-sided modular inclusion), then $P = \frac{1}{2\pi}(\log\Delta_{\mathcal{N}} - \log\Delta_{\mathcal{M}}) \ge 0$ generates such translations, and $\mathcal{N} = U(1)\mathcal{M}U(-1)$ (with the normalization of CF20 (9)).*

THEOREM 39.7.3 (causality = positive generator; [THM] from the spectral theorem and Paley–Wiener [**218**], Titchmarsh [**268**] ([CITED, secondary])). *For a strongly continuous unitary group $U(a) = e^{iaP}$ the following are equivalent: (1) $P \ge 0$; (2) for all ϕ, ψ, $a \mapsto \langle\phi, U(a)\psi\rangle$ extends to a bounded holomorphic function on $\mathrm{Im}\, a > 0$, continuous up to $\mathbb{R}$; (3) every spectral measure $\mu_{\phi,\psi}$ is supported on $[0,\infty)$. When $\mu_{\psi,\psi}$ has an L^2 density, (2) is Titchmarsh's theorem: the boundary value is in H^2_+, and its real and imaginary parts are Hilbert-transform (Kramers–Kronig) pairs.*

PROOF. (1)⇔(3) is the spectral theorem. (3)⇒(2): $\int_0^\infty e^{ia\lambda}d\mu(\lambda)$ is holomorphic and bounded by $\|\mu\|$ on $\mathrm{Im}\, a \ge 0$. (2)⇒(3): a finite measure whose Fourier transform extends boundedly to the upper half-plane is supported on $[0,\infty)$ (F. and M. Riesz/Paley–Wiener type argument; for L^2 densities this is Titchmarsh's Thm. 95, secondary). □

Caveat ([THM] via Lemmas 31, 32). Thm. 39.7.3 is *spectrum-condition* positivity, the arrow-of-time kind. It is free and line-blind (Lemma 31 §2: finite symmetric models with ghosts are strictly passive), and it is *not* the Hilbert-space (OS/reflection) positivity that RH needs. Lemma 32 showed that G5 is (OS4)+(OS5) and that Weil's form lacks a half-space. Kramers–Kronig gives causality, not unitarity (Lemma 36 Prop. 4.1: support causality and positivity of the boundary measure are free; stability ⇔ RH).

PROPOSITION 39.7.4 (the $ax+b$ structure of BC; [CITED] BC95 Thm. 5, CM08 (3.61)–(3.62), (3.71)). *The BC Hecke algebra is $\mathcal{H}(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$, where P is the $ax+b$ group $\{\left(\begin{smallmatrix}1 & b\\ 0 & a\end{smallmatrix}\right)\}$ and $+$ means $a > 0$. The inclusion $P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+$ is almost normal (finite orbits), and the time evolution is $\sigma_t(f)(\gamma) = (L(\gamma)/R(\gamma))^{-it}f(\gamma)$ with L, R the left/right orbit sizes. It is the modular group of φ_1, a KMS$_1$ state.*

HSMIs are also built on $ax+b$ (Δ^{it} dilates the translations, Thm. 39.7.2). In BC the roles are *transposed*: σ_t *fixes* the translations $e(r)$ and rescales the dilation isometries μ_n (Lemma 31 §6(5), [THM] there).

THEOREM 39.7.5 (no half-sided structure at any finite Galois level; [THM]). *(i) For every $P<\infty$ and $\beta>0$, every Gibbs state $\varphi^P_{\beta,\rho}$ and the Galois-averaged state $\int\varphi^P_{\beta,\rho}d\rho$ (which is BC's φ_β restricted to $\mathcal{A}_{\le P}$ when $\beta\le 1$, Prop. 39.3.5) have modular operators with pure point spectrum contained in $\beta\log\mathbb{Q}_{>0}$ (P-smooth ratios). Hence, by Lemma 31 §6(5), they admit no nontrivial Borchers translations. (ii) The prime tower is not a half-sided modular inclusion. In the GNS space of the product state $\bigotimes_p\varphi_{\beta,p}$ on $\bigotimes_p\mathcal{T}_p$ (any β), let $\mathcal{M}_P$ be the weak closure of level P. Then $\overline{\mathcal{M}_P\Omega}=\mathcal{H}_{\le P}\otimes\Omega_{>P}\neq\mathcal{H}$, so Ω is not cyclic for $\mathcal{M}_P$. Moreover $\Delta^{it}_{\mathcal{M}_{P'}}\mathcal{M}_P\Delta^{-it}_{\mathcal{M}_{P'}}=\mathcal{M}_P$ for* all *$t\in\mathbb{R}$. The inclusion is split and modular-invariant in both directions, the opposite of half-sided.*

PROOF. (i) Prop. 39.2.5 for each ρ; the averaged state is a direct integral of these, with the same eigenvalue set. (ii) The modular group of a product state is the product of modular groups, which preserves each tensor factor. □

QUESTION 39.7.6 ([OPEN]). Can a half-sided modular structure arise only in the infinite-prime limit at $\beta\le 1$, where the KMS state is of type III_1 and the modular spectrum can be continuous? Or in Connes' adele class space, from the scaling flow together with some additive translation (Lemma 31 §5: the additive "$+b$" does not descend)? If it did, Borchers' theorem would supply a positive generator P with $\Delta^{it}P\Delta^{-it}=e^{-2\pi t}P$, a canonical "null energy" of arithmetic. Even then the positivity obtained would be of spectrum-condition type (caveat above), and making it the G5 positivity would require an extra OS-type step. Score: significance 0.7, probability 0.15, so 0.48.

39.7.3. (c) Polarization: Hodge–Riemann vs. Hardy ([ANALOGY]). Geometry ([BIB] Griffiths–Harris; [CITED] in earlier lemmas: Hart V Thm. 1.9, Ex. 1.9–1.10, MilAV Thm. I.14.3). For a compact Riemann surface C of genus g: $H^1(C,\mathbb{C})=H^{1,0}\oplus H^{0,1}$, each of dimension g (half the Betti number $2g$), $H^{0,1}=\overline{H^{1,0}}$, and $i\int_C\omega\wedge\bar\omega>0$ for $0\neq\omega\in H^{1,0}$ (Hodge–Riemann). The complex structure J ($=i$ on $H^{1,0}$, $-i$ on $H^{0,1}$) is a *polarization*: the cup product $\langle a,Jb\rangle$ is positive definite on $H^1(C,\mathbb{R})$. Over $\mathbb{F}_q$, Weil's RH comes from the positivity of the Rosati involution and Castelnuovo–Severi, i.e. the Hodge index on $C\times C$ (Lemmas 13, 27, 30).

Analysis ([THM], Paley–Wiener). $L^2(\mathbb{R})=H^2_+\oplus H^2_-$, with Hilbert transform $\mathcal{H}$ ($\mathcal{H}^2=-1$). On real L^2, $J=\mathcal{H}$ is a complex structure whose $+i$ eigenspace is the positive-frequency (causal) part. In QFT, a one-particle structure is a choice of such J compatible with the symplectic form and with positive energy; it fixes the vacuum ([BIB], standard; cf. [47]).

Dictionary ([ANALOGY]). $H^{1,0}\leftrightarrow H^2_+$ (positive frequency), $H^{0,1}\leftrightarrow H^2_-$, cup product $\leftrightarrow$ symplectic form, Hodge–Riemann $\leftrightarrow$ positivity of $\omega(\cdot,J\cdot)$, choice of complex structure $\leftrightarrow$ choice of vacuum. Both splittings are "half and half", and in both the *split itself is free*; the content is the *positivity* of the form it induces.

What RH would need ([RESTATEMENT] plus [OPEN]). In Deninger's picture [91, 87] ([CITED] in Lemma 30), H^1 is infinite-dimensional, the flow generator Θ has the zeros as spectrum, and a Hodge $*$ with $\Theta=\frac12+A$, A skew, would give RH. A *causal polarization* would be a splitting $H^1=H^1_+\oplus H^1_-$ by the sign of iA (the Hardy split of the flow), together with a cup product $\cup$ such that $\langle a,Jb\rangle=\cup(a,Jb)$ is positive definite. Translated: (1) J must come from the causal structure of the scaling flow, not from the zeros; (2) the form must come from a cup product, i.e. from geometry or the explicit formula without reference to the zeros; (3) positivity of $\cup(\cdot,J\cdot)$ is then exactly G5 (Lemma 30: positive involution). Without (1)–(2) the construction is circular: if J is defined as "sign(γ) on zero modes" it presupposes real γ, i.e. RH, because off-line modes have complex γ and no sign. We claim no implication in either direction between "a causal polarization exists" and G5 beyond (3).

Score (causal polarization of H^1): significance 0.9, probability 0.05, so 0.56. The object H^1 is itself conjectural.

39.8. Roadmap

id	item	sig	prob	score
R4	causal polarization of Deninger's H^1 (§39.7(c))	.9	.05	.56
R1	Galois-equivariant entropy source of G5 (§39.6)	.9	.03	.55
R5	Galois-twisted cascade: χ-sectors of $\mathcal{A}_{\leq P}$ as Dirichlet-L truncated lines	.5	.6	.54
R3	half-sided structure in the infinite-prime, $\beta \leq 1$ limit (Question in §39.7)	.7	.15	.48
R2	test $m(P) \propto e_{p^+}(\frac{1}{2})$ against M6 data, $P \approx 100$	.45	.5	.47

R2 is the cheapest, and it is decisive for the only pattern seen. R5 is concrete: each nontrivial character of $G_{\leq P}$ gives an isotypic sector whose partition function is a truncated Dirichlet L-function (Lemma 38 checks, `bc_isotypic`), so Lemma 37's cascade can be run in each Galois sector.

39.9. Corrections recorded

(K1) "Relative entropy changes monotonically along modular time" (conversation) is false. It is invariant along the state's own modular flow (Thm. 39.4.1(b)); monotonicity holds under restriction/CP maps and along HSMI translations.

(K2) "Line time = modular time" is an analogy. In BC the log-scale is the energy axis, and scaling is the dual action of the modular flow (Rem. 39.2.4).

(K3) "The BC system is the tensor product of per-prime systems" holds only for the Galois-invariant part $C^*(\mathbb{N}^*)$ (BC95 Prop. 7). The full system is a Galois-twisted product (Prop. 39.3.2).

(K4) The margin is not shown to be a relative-entropy difference (§39.5).

39.10. Open questions

(Q1) The Galois factor and the rapidity factor commute but are coupled by $p \mapsto (p^{-1}, g_p^{-1})$ (Thm. 39.6.1): each prime-step of rapidity carries a Frobenius. Is *this* coupling the "translation between Galois symmetry and rapidity" you are after, rather than an identification of the two?

(Q2) Should R2 (testing $m(P) \propto \log p^+/(\sqrt{p^+}-1)$ on the $P \approx 100$ run) take priority? It is cheap and would either kill or promote the only pattern found.

(Q3) R5: run Lemma 37's cascade in each Galois (Dirichlet-character) sector of the truncated BC system? This is the most concrete Galois-framed computation available.

(Q4) For the polarization program (R4), is the right target a cup product on a Deninger-type H^1, or would you prefer the more concrete half-sided-inclusion question (R3) in BC at $\beta \leq 1$?

(Q5) Is the reading "symmetry is broken at every finite Galois level and restored only by infinitely many primes" (Prop. 39.3.5) the thermal counterpart of your intuition that small primes set a baseline that no finite set can complete?

CHAPTER 40

Lemma 40: Polarization as Galois-twisted positivity

Hodge–Riemann, Rosati, Krein unitarity, and one theorem behind Weil's curve proof, the L_4/Routh kernel and the ζ ghost

Standalone note `lemma40.tex`, October 22, 2026.

Summary. Statement of intellectual honesty. RH is not proved here or anywhere in TAA, and nothing in this lemma supplies the missing axiom G5. Every claim carries a tag: [THM] proved here, or an elementary consequence of cited results with the proof given; [CITED] checked at the source (`lit/lemma40/verified_refs.md`); [CITED, secondary] checked only through a named secondary source; [BIB] bibliographic record only, content stated as standard background; [RESTATEMENT] equivalent to RH, so no progress; [ANALOGY] analogy only; [NUM] numerical (`num/lemma40_*.py`, data `num/l40/`); [CONJ] conjecture; [OPEN] open. Lemmas 27, 30, 34, 36–39 are used read-only.

The question. J. W. singled out the Hodge split $H^1 = H^{1,0} \oplus H^{0,1}$ (two halves of dimension g, "half Betti") with the Riemann bilinear positivity $i \int \omega \wedge \bar{\omega} > 0$ as the most concise object attached to the positivity question. He asked for its connection to Galois symmetry (Lemma 38) and to the Krein arena (Lemmas 27, 34, 36). His steering [t142u] fixed the central theme: *Krein unitarity unifies Weil's curve proof, the L_4/Routh kernel and the ζ ghost.*

Results.

(1) *The Hodge split is a Krein space* (Thm. 40.2.1, [THM]). The form $h(\alpha, \beta) = i \int \alpha \wedge \bar{\beta}$ on $H^1(X, \mathbb{C})$ has signature (g, g). The Hodge decomposition is a fundamental decomposition, and the archimedean conjugation $c_\infty \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is an antiunitary map exchanging the positive and negative halves. The Hodge–Riemann positive form is $h(\mathfrak{J}\cdot, \cdot)$, where $\mathfrak{J}$ is the fundamental symmetry ($\mathfrak{J} = -iC$, with C the Weil operator). So "half Betti plus Riemann positivity" is literally "a Krein space of signature (g, g) plus a chosen fundamental symmetry".

(2) *Weil's proof as Krein unitarity* (§40.3; [CITED] Milne, [THM] reformulation). Rosati positivity makes $F/\sqrt{q}$ unitary for a positive Hermitian form, which forces $|\text{eigenvalues}| = \sqrt{q}$. The ℓ-adic Weil pairing alone gives only a symplectic similitude, so only the functional equation $\lambda \mapsto q/\lambda$ follows. Explicitly, for an elliptic curve the Rosati Gram matrix on $\mathbb{Q}[\pi]$ is $\left(\begin{smallmatrix} 2 & a \\ a & 2p \end{smallmatrix}\right)$, and positivity is *exactly* Hasse's bound ([THM], [NUM]). The intersection-theoretic proof (Castelnuovo–Severi from Hodge index on $C \times C$) is the char-p source of the same positivity, where no Hodge split exists.

(3) *Krein unitaries* (§40.4, [THM] with full proofs in finite dimensions). The spectrum is symmetric under $\lambda \mapsto 1/\bar{\lambda}$. Off-circle root spaces are neutral. Each off-circle pair spans a *hyperbolic plane*. In Π_κ at most κ eigenvalues lie outside the circle. Definite-type eigenvalues cannot leave the circle (Krein–Gelfand–Lidskii), and in a continuous family starting from a definite form *no ghost can be born before the form degenerates.*

(4) *Central theorem: the Pick kernel of ξ'/ξ is the zero-space Krein form* (Thm. 40.5.2, [THM], new to TAA). For $F(p) = \xi'/\xi(\frac{1}{2}+p)$,

$$\frac{F(z)+\overline{F(w)}}{z+\bar{w}} = \sum_{\rho} m_\rho \frac{1}{z-(\rho-\frac{1}{2})} \overline{\frac{1}{w-(\rho^*-\frac{1}{2})}}, \qquad \rho^* = 1-\bar{\rho},$$

absolutely convergent. So every Pick matrix equals $\Gamma M \Gamma^*$, where M is the Krein Gram $[e_\rho, e_\sigma] = m_\rho \delta_{\sigma,\rho^*}$ of Lemmas 27/30. Each off-line mirror pair is a hyperbolic plane $\left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$, and $\sup(\#\text{negative eigenvalues}) = N_+^{\mathrm{dist}}$, the number of *distinct* zeros with $\beta > \frac{1}{2}$. Lagarias' criterion and the Krein count become one statement. [NUM]: the kernel identity holds to within its tail estimate, and a planted quadruple gives exactly 2 negative eigenvalues; its visibility decays steeply with distance.

(5) *Unification* (§40.5). Weil: the form is definite, so no hyperbolic plane can exist (= Lemma 30 Thm. 2.2). L_4: the form is indefinite, and stability is lost at Routh's value by an opposite-signature collision ([ANALOGY], with a shared theorem). ζ: the on-line zeros have definite type, so a ghost needs a *degeneration* first. The chain "ghost ⇔ Krein-form degeneracy ⇔ line break ⇔ 2π jump" is proved at the level of *depth thresholds* (Thm. 40.5.5: all equal $\beta^* - \frac{1}{2}$) and within finite models. It is *not* proved as an identity of individual events.

(6) *Bridges and dictionary* (§§40.7–40.8). Causality alone does not give Hilbert positivity (Gupta–Bleuler). Positive-real is equivalent to a PSD Pick kernel, which gives a Hilbert space; Krein–Langer N_κ counts the failures. Three conjugations are kept distinct, following Lemma 38 Prop. 8.3. The *Rosati role is played by the mirror* $\sharp$, $\rho \mapsto 1-\bar{\rho}$, not by c_∞ and not by $-1 \in \hat{\mathbb{Z}}^\times$. The involution already exists (Lemma 30), so what is missing is the *non-degeneracy/positivity of the arithmetic Krein form for a geometric reason*: G5 in Krein form (Def. 40.5.6, [RESTATEMENT] as stated, [OPEN] as a mechanism).

(7) *Kepler, regularization, modular* (§40.6). Lagrange's Gram determinant of (r, v) is $|L|^2$: constant, zero only for collision orbits, and never indefinite. Levi-Civita is a $\{\pm 1\}$-Galois cover, and collision is a sheet change ([THM]). The pencil eigenvalue $\mu = \tau^2$ lives on the same kind of cover, with deck map = FE and branch points $0, \infty$ ([THM]). Identifying the two sign flips is [ANALOGY]. The Fock/Killing form is $8E \cdot I_3$ on Runge–Lenz directions. Separation criterion (Thm. 40.6.8, [THM]): the stages separate iff $v^* K v \neq 0$ on the degenerating vector. A Kepler/Fock-type two-mode pencil merges them, and a ζ-type one separates them (gap $\sim \kappa^2/4c^2$, [NUM]). J always acts as c_∞, never as $\sharp$ ([THM]). Covariant natural-cone positivity is only Bochner positivity, so equating it with Weil's form is RH. The missing step is stated (Def. 40.6.12, [OPEN]).

(8) *Bridge table Weil → OS* (§40.9): four rungs, labelled. Rung 4, an algebra/cone giving Weil's form without zeros, is open and is G5.

40.1. Conventions; three conjugations

Zeros of ξ are $\rho = \beta + i\gamma$, with multiplicity m_ρ. Put $a_\rho = \rho - \frac{1}{2}$, $\beta^* = \sup \beta$, and $N_+ = \sum_{\beta > 1/2} m_\rho$. Write $N_+^{\mathrm{dist}} = \#\{\rho : \beta > \frac{1}{2}\}$ for the count of *distinct* zeros. The mirror is $\rho^* = 1 - \bar{\rho}$, so $a_{\rho^*} = -\bar{a}_\rho$. $F(p) = \frac{\xi'}{\xi}(\frac{1}{2}+p)$ on $\operatorname{Re} p > 0$ (Lemma 33). A *Krein space* is a vector space with a nondegenerate Hermitian form $[\cdot,\cdot]$ admitting a decomposition $\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-$ into orthogonal positive and negative definite (complete) parts. The fundamental symmetry is $\mathfrak{J} = \pm 1$ on $\mathcal{K}_\pm$. Π_κ means $\dim \mathcal{K}_- = \kappa < \infty$ (Pontryagin space) [**230**, **36**, **137**, **20**] [BIB]. An operator U is *Krein-unitary* if it is invertible and $[Ux, Uy] = [x, y]$. In finite dimensions, with $[x, y] = y^* G x$, this reads $U^* G U = G$.

Three conjugations (Lemma 38 Prop. 8.3, [THM] there; Lemma 31 Prop. 3.4). They must not be conflated.

	curve side	ζ / BC side
$c_\infty \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ (coefficients)	$H^{1,0} \leftrightarrow H^{0,1}$; $\overline{H^1(X,\mathbb{C})}$	reality $\rho \mapsto \bar\rho$ (pairs $\frac12 + i\gamma \leftrightarrow \frac12 - i\gamma$); implemented by modular J_{mod}; swaps sectors $\chi \leftrightarrow \bar\chi$ (L38 Prop. 8.3(a,b))
$-1 \in \hat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$	(an automorphism commuting with Frobenius; labels isotypic pieces, [ANALOGY])	acts on sector χ by the scalar $\chi(-1)$; moves nothing (L38 Prop. 8.3(c))
Rosati $\dagger$ / its antilinear extension	$\alpha^\dagger = \lambda^{-1}\alpha^\vee\lambda$; $\pi^\dagger = q/\pi$	$\sharp$: $f^\sharp(x) = x^{-1}\overline{f(1/x)}$ (L30); on the spectrum the mirror $\rho \mapsto 1-\bar\rho$; sector-internal (L38 Prop. 8.3(b))

The earlier chat sentence "complex conjugation pairs the zero at $\frac12 + i\gamma$ with the one at $\frac12 - i\gamma$ (Galois side)" is corrected as follows. This pairing is *archimedean* conjugation c_∞, which is reality, implemented by J_{mod}. It is not the Galois element $-1 \in \hat{\mathbb{Z}}^\times$ (correction (E1), §40.12).

40.2. The Hodge split is a Krein space

Let X be a compact Riemann surface of genus g. Then $H^1(X,\mathbb{C}) = H^1(X,\mathbb{R}) \otimes \mathbb{C}$, and c_∞ acts on the second factor. Hodge theory gives $H^1(X,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$ with $H^{1,0} = H^0(X,\Omega^1)$ and $H^{0,1} = c_\infty(H^{1,0})$, each of dimension g [**117**, **277**] [BIB]. Write $Q(\alpha,\beta) = \int_X \alpha\wedge\beta$ for the cup product, which is the intersection pairing on $H^1(X,\mathbb{Z})$ [BIB].

THEOREM 40.2.1 (Hodge split as fundamental decomposition; [THM] from the standard facts above). *Put $h(\alpha,\beta) = i\,Q(\alpha, c_\infty\beta) = i\int_X \alpha\wedge\bar\beta$.*

- *(a) h is Hermitian and nondegenerate on $H^1(X,\mathbb{C})$, with $h>0$ on $H^{1,0}\setminus 0$, $h<0$ on $H^{0,1}\setminus 0$, and $H^{1,0} \perp_h H^{0,1}$. So $(H^1(X,\mathbb{C}),h)$ is a Krein space of signature (g,g), and the Hodge decomposition is a fundamental decomposition.*
- *(b) c_∞ is antiunitary-up-to-sign, $h(\bar\alpha,\bar\beta) = -\overline{h(\alpha,\beta)}$. It exchanges the positive and negative halves.*
- *(c) $H^{1,0}$ and $H^{0,1}$ are Q-Lagrangian, i.e. $Q|_{H^{1,0}\times H^{1,0}} = 0$.*
- *(d) With the fundamental symmetry $\mathfrak{J} = \pm 1$ on $H^{1,0}/H^{0,1}$ and the Weil operator $C = i^{p-q}$ ($C = i\mathfrak{J}$), the form $\langle\alpha,\beta\rangle := h(\mathfrak{J}\alpha,\beta) = Q(C\alpha,\bar\beta)$ is positive definite. This is the Hodge–Riemann form, i.e. the polarization.*

PROOF. For $\omega = f\,dz$ locally, $\omega\wedge\bar\omega = |f|^2 dz\wedge d\bar z = -2i|f|^2 dx\wedge dy$, so $h(\omega,\omega) = 2\int|f|^2 > 0$. For $\bar\omega \in H^{0,1}$, $h(\bar\omega,\bar\omega) = i\int\bar\omega\wedge\omega = -h(\omega,\omega) < 0$. A product of two $(1,0)$-forms is a $(2,0)$-form, which vanishes on a curve. This gives (c) and the orthogonality $h(\omega,\bar\eta) = iQ(\omega,\eta) = 0$. Hermitian symmetry follows from $\overline{\alpha\wedge\bar\beta} = \bar\alpha\wedge\beta = -\beta\wedge\bar\alpha$ for 1-forms. (b) is the same computation. (d): on $H^{1,0}$, $Q(C\omega,\bar\omega) = iQ(\omega,\bar\omega) = h(\omega,\omega)$. On $H^{0,1}$, $Q(C\bar\omega,\omega) = -iQ(\bar\omega,\omega) = -h(\bar\omega,\bar\omega) > 0$. The cross terms vanish by (c). □

REMARK 40.2.2 (reading for J. W.; [THM]). "Half Betti" is the statement that a maximal positive subspace of the Krein space (H^1,h) has dimension exactly half. "Riemann positivity" is the statement that the holomorphic half *is* such a maximal positive subspace and is also Q-Lagrangian. The Galois element c_∞ does not create positivity. It is the map that *relates* the two halves, and it turns the bilinear Q into the Hermitian h. Positivity is the extra choice of $\mathfrak{J}$. In Lemma 39 §7(c) the same C is called the complex structure "J"; here it is $C = i\mathfrak{J}$, to avoid a clash with the modular J_{mod}. That lemma's Hardy dictionary $H^{1,0} \leftrightarrow H^2_+$ is [ANALOGY] and is not repeated here.

PROPOSITION 40.2.3 (polarization of the Jacobian; [CITED] MilAV Lemma 2.4, Thm. 2.8; MilpRH p. 19). *Let $\mathrm{Jac}(X) = H^{0,1}/H^1(X,\mathbb{Z})$ (equivalently $H^0(\Omega^1)^\vee/H_1(X,\mathbb{Z})$). The intersection form E on the lattice $H_1(X,\mathbb{Z})$ is a Riemann form: $E(iv,iw) = E(v,w)$, and $H(v,w) = E(iv,w)+$*

$iE(v,w)$ is positive definite [**194**, Lemma 2.4 and the definition following it]. *Polarizable tori are exactly the abelian varieties* [**194**, Thm. 2.8]. *For $\alpha \in \mathrm{End}^0(\mathrm{Jac})$, the Rosati involution is the H-adjoint, $H(\alpha x, y) = H(x, \alpha^\dagger y)$. Hence $\mathrm{Tr}(\alpha\alpha^\dagger) > 0$ for $\alpha \neq 0$, since $\alpha\alpha^\dagger$ is a nonzero positive operator for a positive definite form* [**193**, p. 19].

Translation ([THM]). Over $\mathbb{C}$, Rosati positivity is Thm. 40.2.1(d) transported to the Jacobian. A polarization is a choice of fundamental symmetry that turns the integral symplectic form into a positive Hermitian one. Lemma 32 §(polarization) separates the two halves of a Riemann form: integrality, which is free at the finite places, and positivity, which is G5. This lemma concerns only the positive half.

40.3. Weil's proof for curves over $\mathbb{F}_q$: Rosati positivity as unitarity

Let $C/\mathbb{F}_q$ be smooth projective of genus g, $A = \mathrm{Jac}(C)$, $\pi \in \mathrm{End}(A)$ the Frobenius, $\dagger$ the Rosati involution of the canonical polarization, and $P(T) = \det(T - \pi \mid V_\ell A)$.

Theorem 40.3.1 (Weil 1948; [CITED] MilpRH Thm. 1.27, Cor. 1.29, Thm. 1.30, Summary 1.31). *(i) $\dagger$ is positive: $\mathrm{Tr}_{E/\mathbb{Q}}(\alpha\alpha^\dagger) > 0$ for $0 \neq \alpha \in E = \mathrm{End}^0(A)$. Indeed $\mathrm{Tr}(\alpha\alpha^\dagger) = \frac{2g}{(D^g)}(D^{g-1} \cdot \alpha^{-1}D)$ for the ample D of the polarization (Lang), and the right side is > 0 by dimension theory. (ii) $\pi\pi^\dagger = q$, proved via the Weil pairing: $e(\alpha x, y) = e(x, \alpha^\dagger y)$ and $e(\pi x, \pi y) = q\, e(x,y)$. (iii) Hence, for every $\mathbb{Q}[\pi] \to \mathbb{C}$, $\rho(\pi^\dagger) = \overline{\rho(\pi)}$ and $|\rho(\pi)| = q^{1/2}$. Since $P(T)$ has integer coefficients and $|C(\mathbb{F}_{q^n})| = q^n + 1 - \sum \alpha_i^n$, this is RH for C* [**193**]. *Weil's sources are* [**285, 287**] *[BIB], and the modern reference is* [**202**] *[BIB].*

Proof sketch of (iii) (as in [**193**, Cor. 1.29]). $\mathbb{Q}[\pi]$ is $\dagger$-stable, and it is semisimple by (i) ([**193**, Cor. 1.28]). So $\mathbb{R} \otimes \mathbb{Q}[\pi] \cong \mathbb{R}^r \times \mathbb{C}^s$. The only involution with $\mathrm{Tr}(\beta\beta^\dagger) \geq 0$ is the identity on the real factors and complex conjugation on the complex factors. Then $q = \rho(\pi\pi^\dagger) = |\rho(\pi)|^2$.

Proposition 40.3.2 (the same proof as Krein/Hilbert unitarity; [THM]). *On $V = \mathbb{Q}[\pi] \otimes \mathbb{C}$ put $\langle x, y\rangle = \mathrm{Tr}(x\,\bar{y}^\dagger)$, where $^-$ is c_∞ on the coefficient and $\dagger$ is extended $\mathbb{C}$-linearly. Then $\langle\cdot,\cdot\rangle$ is Hermitian. Left multiplication L_π satisfies $\langle \pi x, \pi y\rangle = q\langle x, y\rangle$, i.e. $U = L_\pi/\sqrt{q}$ is unitary for this form. The eigenvalues of L_π are the $\rho(\pi)$. If the form is positive definite, which is (i), then the $\rho(\pi)$ have modulus $\sqrt{q}$, since unitaries on a Hilbert space have unimodular spectrum. If the form were merely nondegenerate and indefinite of signature (r_+, r_-), then U would be Krein-unitary, and §40.4 applies: at most r_- eigenvalues off $|\lambda| = 1$ on each side.*

Proof. $\langle \pi x, \pi y\rangle = \mathrm{Tr}(\pi x \bar{y}^\dagger \pi^\dagger) = \mathrm{Tr}(x\bar{y}^\dagger \pi^\dagger \pi) = q\langle x, y\rangle$, using that $\mathbb{Q}[\pi]$ is commutative and $\pi^\dagger\pi = q$. Hermitian symmetry holds because $\dagger$ fixes the trace, $\mathrm{Tr}(z^\dagger) = \mathrm{Tr}(z)$. If $\pi x = \lambda x$, then $q\langle x,x\rangle = |\lambda|^2\langle x,x\rangle$. □

Example 40.3.3 (elliptic curves: Rosati positivity = Hasse; [THM], [NUM]). For $g = 1$, with $\pi^2 - a\pi + p = 0$ and basis $\{1, \pi\}$: $\mathrm{Tr}(1) = 2$, $\mathrm{Tr}(\pi^\dagger) = \mathrm{Tr}(p/\pi) = a$, and $\mathrm{Tr}(\pi\pi^\dagger) = 2p$. The Rosati Gram matrix is $\left(\begin{smallmatrix} 2 & a \\ a & 2p \end{smallmatrix}\right)$, positive definite $\iff$ $a^2 < 4p$. That is Hasse's bound, *exactly.* [NUM] (`lemma40_checks.py`, B1): $y^2 = x^3 + x + 1$ ($p = 101$, $a = -3$, $\det = 395$); $x^3 + 2x + 3$ (1009, −58, 672); $x^3 - x$ (1013, 46, 1936); $x^3 + 7$ (10007, 0, 40028); all positive. Genus 2 (B2): for $y^2 = x^5 + x + 1$ over $\mathbb{F}_{11}, \mathbb{F}_{31}, \mathbb{F}_{101}$ and $y^2 = x^5 + 2x^2 + 3$ over $\mathbb{F}_{53}$, the polynomial P was reconstructed from point counts over $\mathbb{F}_p, \mathbb{F}_{p^2}$. All roots have $|\alpha|/\sqrt{p} = 1$ to 10^{-15}, and the 4×4 Rosati Gram matrix $\left(\sum_i \alpha_i^j (p/\alpha_i)^k\right)_{j,k \leq 3}$ is positive definite. An artificial Weil-type polynomial with one off-circle quadruple, $\sqrt{31}\{u, 1/\bar{u}, \bar{u}, 1/u\}$ with $u = 1.3e^{0.8i}$, gives signature $(2,2)$ (B3), matching Lemma 30 Thm. 2.2.

Remark 40.3.4 (ℓ-adic versus endomorphism-algebra setting; [THM]). On $V_\ell A$ the Weil pairing is a nondegenerate alternating $\mathbb{Q}_\ell$-form, and π is a *symplectic similitude*, $e(\pi x, \pi y) = q\, e(x,y)$

[193, Thm. 1.30, Ex. 3.8]. That alone forces only $\operatorname{spec}(\pi) = q/\operatorname{spec}(\pi)$, the functional equation $P(T) = q^{-g}T^{2g}P(q/T)$, because $\mathbb{Q}_\ell$ is not ordered and has no positivity (compare Lemma 38 (B1)). Positivity lives on the $\mathbb{Q}$-algebra E, where $\operatorname{Tr}(\alpha\alpha^\dagger) \in \mathbb{Q}_{>0}$. Then c_∞ enters only to define Hermitian forms after $\otimes\mathbb{C}$. *So in Weil's proof the positivity is a statement over the ordered field $\mathbb{Q}$, coming from ampleness (an intersection number). The Galois conjugation c_∞ only converts it into unitarity.*

THEOREM 40.3.5 (intersection-theoretic proof; [CITED] MilpRH Thm. 1.5, Cor. 1.6, Ex. 1.7 and p. 10; Hart V Ex. 1.9–1.10). *On $V = C \times C$ over $\bar{\mathbb{F}}_q$ the intersection form on $N(V)$ has index 1 (Hodge index). Hence $D^2 \le 2d_1d_2$ (Castelnuovo–Severi), with $\operatorname{def}(D) = 2d_1d_2 - D^2 \ge 0$ and $|D \cdot D' - d_1d_2' - d_2d_1'| \le (\operatorname{def} D \operatorname{def} D')^{1/2}$. With $\operatorname{def}(\Delta) = 2g$ and $\operatorname{def}(\Gamma_\pi) = 2gq$ this gives $|\Gamma_\pi \cdot \Delta - q - 1| \le 2g\sqrt{q}$. Applied over all $\mathbb{F}_{q^n}$, this is RH.*

REMARK 40.3.6 (no Hodge split in characteristic p; the target $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$; [THM]/[CITED]/[OPEN]). Over $\mathbb{F}_q$ there is no $H^{1,0} \oplus H^{0,1}$ on ℓ-adic H^1, and no c_∞ acting on it compatibly with Frobenius. Positivity comes from (a) ampleness, i.e. positivity of an intersection number (Thm. 40.3.1(i)), or (b) the Hodge *index* theorem on the surface $C \times C$ (Thm. 40.3.5), which is a signature statement $(1, \rho - 1)$ about a $\mathbb{Q}$-valued intersection form on divisors. Lemma 27 records the cited identity $D^2 = -\operatorname{Tr}(\alpha\alpha^\dagger \mid V_\ell J)$ on the transcendental part [193, fn. 7, p. 11], so (a) and (b) are the same positivity. For ζ the analogue of (b) is the programme "intersection theory and Riemann–Roch on the square of the arithmetic site" [62], with $s(f, f) = D \bullet D$ [70] ([CITED] in Lemma 30). Its H^1 is "still open" [70]. *The transferable lesson* ([THM] for curves, [OPEN] for ζ): one needs a $\mathbb{Q}$-valued (ordered) form with a Hodge-index signature, and positivity comes from geometry, not from c_∞.

40.4. Krein unitaries: the spectral facts

All statements here are proved in finite dimensions. For Π_κ we use only that eigenvalues off the circle are isolated with finite-dimensional root spaces [137, Ch. II] [BIB].

THEOREM 40.4.1 ([THM]). *Let U be Krein-unitary on $(\mathcal{K}, [\cdot,\cdot])$, $\dim \mathcal{K} < \infty$, with $[x, y] = y^*Gx$.*

(a) $\operatorname{spec}(U)$ is invariant under $\lambda \mapsto 1/\bar\lambda$, with equal algebraic multiplicities.

(b) Let $\mathcal{R}_\Omega$ be the spectral (root) subspace of U for a set Ω of eigenvalues. If $\lambda\bar\mu \ne 1$ for all $\lambda, \mu \in \Omega$, then $\mathcal{R}_\Omega$ is neutral, $[x, y] = 0$ on $\mathcal{R}_\Omega$. In particular $\mathcal{R}_{\{|\lambda|>1\}}$ and $\mathcal{R}_{\{|\lambda|<1\}}$ are neutral, and $\mathcal{R}_\Omega \perp \mathcal{R}_{\Omega'}$ unless Ω' meets $1/\bar\Omega$.

(c) For a simple off-circle eigenvalue λ, eigenvectors x (for λ) and y (for $1/\bar\lambda$) span a hyperbolic plane*: Gram $\left(\begin{smallmatrix} 0 & c \\ \bar c & 0 \end{smallmatrix}\right)$ with $c \ne 0$, signature $(1, 1)$. It contains exactly one positive and one negative direction, $x \pm (c/|c|)\, y$, with $[x + ty, x + ty] = 2\operatorname{Re}(\bar t c)$.*

(d) If $\mathcal{K}$ has κ negative squares, then $\#\{|\lambda| > 1\} = \#\{|\lambda| < 1\} \le \kappa$ (algebraic multiplicities).

PROOF. (a) $U^*GU = G$ gives $U^* = GU^{-1}G^{-1}$, so $\operatorname{spec}(U^*) = \overline{\operatorname{spec} U}$ equals $\operatorname{spec}(U^{-1})$, with multiplicities. (b) Let V be a basis matrix of $\mathcal{R}_\Omega$, $UV = VA$, $\operatorname{spec} A = \Omega$, and $B = V^*GV$. Then $A^*BA = B$, i.e. $B - A^*BA = 0$. The Stein operator $B \mapsto B - A^*BA$ has eigenvalues $1 - \bar\lambda\mu$ $(\lambda, \mu \in \Omega)$, all nonzero, so $B = 0$. The cross statement is the same with two blocks. (c) By (b), x and y are neutral. Since G is nondegenerate and $\mathcal{R}_{\{\lambda\}}$ is orthogonal to every root space except $\mathcal{R}_{\{1/\bar\lambda\}}$, we get $c = [x, y] \ne 0$. (d) $\mathcal{R}_{\{|\lambda|>1\}}$ is neutral. A neutral subspace L meets every subspace of codimension κ on which the form is positive definite, if $\dim L > \kappa$, which is impossible. So $\dim L \le \kappa$. □

THEOREM 40.4.2 (definite type is stable; Krein–Gelfand–Lidskii; [THM] in finite dimensions, cf. [110] [BIB], [191, Thm. 4.7] [CITED] (selfadjoint form, equivalent by Cayley transform)). *Let λ_0, $|\lambda_0| = 1$, be an eigenvalue of a Krein-unitary U_0 whose root space $\mathcal{R}_0$ is* definite *($\pm[x, x] > 0$ on $\mathcal{R}_0 \setminus 0$). Then there is $\varepsilon > 0$ such that every Krein-unitary U (for the same G) with $\|U - U_0\| < \varepsilon$ has all eigenvalues near λ_0 on the unit circle. Consequently an eigenvalue can leave the circle only*

from a root space that is indefinite. *By Thm. 40.4.1(b) the root space of a unimodular eigenvalue is orthogonal to all other root spaces, hence nondegenerate. Indefinite therefore means that it contains vectors of both signs: a collision of opposite Krein signature (or a non-semisimple mixed block).*

PROOF. Let $P(U)$ be the Riesz projection onto eigenvalues inside a small circle around λ_0. It is continuous in U, and so is its range. Definiteness on $\operatorname{ran} P(U_0)$ is an open condition, so the form stays definite on $\operatorname{ran} P(U)$. Any eigenvector there with $|\lambda| \neq 1$ would be neutral by Thm. 40.4.1(b), a contradiction. □

COROLLARY 40.4.3 (no ghost before degeneration; [THM]). *Let $\tau \mapsto (G(\tau), U(\tau))$, $\tau \in [0,1]$, be continuous, with $G(\tau)$ Hermitian, $U(\tau)^* G(\tau) U(\tau) = G(\tau)$ wherever $\det G(\tau) \neq 0$, and $G(0) > 0$. Let $\tau_1 = \inf\{\tau : U(\tau)$ has an eigenvalue off $|\lambda| = 1\}$ and $\tau_0 = \inf\{\tau : \det G(\tau) = 0\}$. Then $\tau_0 \leq \tau_1$. Moreover, between τ_0 and τ_1 the form may already be indefinite with all eigenvalues still on the circle: degeneration is necessary but not sufficient.*

PROOF. On $[0, \tau_0)$ the eigenvalues of $G(\tau)$ are continuous and never 0, so $G(\tau) > 0$ there. Then $U(\tau)$ is unitary for a positive definite form and its spectrum is unimodular. Hence no $\tau < \tau_0$ has an off-circle eigenvalue, i.e. $\tau_1 \geq \tau_0$. For non-sufficiency, take $G = \operatorname{diag}(1, -1)$ and $U = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2})$, $\theta_1 \neq \theta_2$: indefinite, with all eigenvalues on the circle and of definite (opposite) type. □

This is the two-stage mechanism of Lemma 27 Thm. 4.4 in abstract form: (a) degeneration of the form (Sylvester; there, $\det G_A = 0$, with a pencil eigenvalue passing through ∞ and returning with negative type), then (b) an opposite-signature collision, after which the pair leaves as $(\gamma \pm i\delta)^2$.

REMARK 40.4.4 (counting conventions, matched; [THM]). A real Weil-type polynomial with an off-circle root $u\sqrt{q}$ has the quadruple $\sqrt{q}\{u, 1/\bar{u}, \bar{u}, 1/u\}$. That is *two* off-circle pairs $(u, 1/\bar{u})$, $(\bar{u}, 1/u)$, hence two hyperbolic planes and $\kappa \geq 2$. Under $\alpha = q^\rho$ (zeros of $Z(C, q^{-s})$), $|u| > 1 \iff \beta > \frac{1}{2}$. So "$\kappa$ = number of eigenvalues outside the circle" $= N_+$, which is *two per quadruple*. This matches Lemma 36 Prop. 5.1 (index +2 per quadruple) and Lemma 34 ($\kappa = \nu N_+$, so $\kappa = N_+$ at $\nu = 1$). Lemma 30 Thm. 2.2 gives signature $(\#\mathrm{Fix}\,\iota + p,\, p)$, with p the number of 2-cycles of $\iota(\alpha) = q/\bar{\alpha}$: again one negative square per off-circle pair. The Pick count of Thm. 40.5.2 counts *distinct* zeros, N_+^{dist}. It agrees with N_+ for simple zeros and differs for multiple off-line zeros. That is a genuine convention difference, recorded and not resolved.

Numerics ([NUM], `lemma40_checks.py`, A1–A3). (A1) A planted 4×4 Krein-unitary in Π_1 ($G = \operatorname{diag}(1, 1, 1, -1)$), generically conjugated: $\|U^* G U - G\| = 5.6 \times 10^{-16}$. The eigenvalues are $e^{0.4i \pm 0.3}$ (moduli 1.3499, 0.7408) plus two unimodular ones, with $|\lambda_+ \bar{\lambda}_- - 1| = 5.6 \times 10^{-16}$. The off-circle eigenvectors are neutral ($\leq 1.2 \times 10^{-15}$) and the unimodular ones are definite. (A2) A real $O(2,2)$ quadruple: moduli $0.7788^{(2)}, 1.2840^{(2)}$, i.e. $e^{\mp 0.25}$, at angles ± 0.9. (A3) 16000 random Krein-unitaries $\exp(iG^{-1}H)$ in signatures $(3,1), (2,2), (5,3), (6,2)$: the number of eigenvalues outside the circle never exceeds $\kappa = \min(p, q)$ and attains it. Histograms are in `l40/checks.json`; for $(2,2)$ they are $\{0{:}67, 1{:}1629, 2{:}2304\}$.

40.5. Central: Krein unitarity unifies Weil's curve proof, the L_4/Routh kernel and the ζ ghost

The shared theorem is §40.4: Thm. 40.4.1 (symmetry, neutrality, hyperbolic planes, Π_κ count), Thm. 40.4.2 (definite type is stable; exit only through an indefinite root space) and Cor. 40.4.3 (no ghost before the form degenerates). In all three settings below the same three alternatives appear: the form is *definite*, so nothing can leave; or it is *indefinite but fixed*, so leaving requires an opposite-signature collision; or the form *degenerates*, which changes the inertia and is how a definite system acquires its first negative direction.

40.5.1. Weil: definite form, so no hyperbolic plane can exist.

PROPOSITION 40.5.1 ([THM]; Lemma 30 Thm. 2.2 in Krein language). *On $V = \mathbb{Q}[\pi] \otimes \mathbb{C}$ with the Hermitian form of Prop. 40.3.2, the off-circle eigenvalues of $U = L_\pi/\sqrt{q}$ come in pairs $\lambda, 1/\bar{\lambda}$, i.e. 2-cycles of $\iota(\alpha) = q/\bar{\alpha}$, and each pair spans a hyperbolic plane. The signature is $(\#\mathrm{Fix}\,\iota + p,\, p)$, with p the number of pairs (Lemma 30 Thm. 2.2). Rosati positivity (Thm. 40.3.1(i)) says $p = 0$. So the Krein space is a Hilbert space, no hyperbolic plane exists, and RH holds.*

PROOF. Thm. 40.4.1(a)–(c) applied to U, together with Lemma 30 Thm. 2.2 for the signature. □

Structural point ([THM]). Weil needs no deformation and no stability argument. The form is *parameter-free* and definite for an algebraic reason, the identity $\mathrm{Tr}(\alpha\alpha^\dagger) = \frac{2g}{(D^g)}(D^{g-1} \cdot \alpha^{-1}D)$. Neither stage (a) nor stage (b) can occur.

40.5.2. L_4/Routh: indefinite fixed form, so instability by collision. Facts (Lemma 27 Addendum B, [THM]/[NUM] there). At L_4 the energy Hessian has inertia $\mathrm{In}\,S = (2,2)$ (L_4 is a maximum of the effective potential). The fast mode has Krein sign + and the slow mode −, with characteristic polynomial $\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu)$. Linear stability is lost at Routh's value $27\mu(1-\mu) = 1$, $\mu_R = 0.0385209$, where the two frequencies collide and leave the imaginary axis.
Krein reading ([THM] for the linear algebra). The time-t flow e^{tJS} is symplectic, i.e. Krein-unitary for the Hermitian form $[x,y] = i\,y^*\Omega x$ on $\mathbb{C}^4$, which has signature $(2,2)$. The Krein sign of a unimodular Floquet eigenvalue is the sign of the energy on its eigenvector. Since the energy is indefinite, stability is *conditional*: it holds while the two opposite-type modes stay apart (Thm. 40.4.2), and Routh's value is where they collide (stage (b)). The limit $\mu \downarrow 0$ is degenerate: the slow mode is born at frequency 0, which Lemma 27 matched with stage-(a) birth.
Status ([ANALOGY]). The dynamics is unrelated to ζ. The shared content is exactly Thm. 40.4.2 and Cor. 40.4.3, which are theorems in both places.

40.5.3. ζ: the Pick kernel of ξ'/ξ is the zero-space Krein form.

THEOREM 40.5.2 ([THM]; new in TAA). *For $\mathrm{Re}\,z, \mathrm{Re}\,w > 0$ (away from poles) put $g_\rho(z) = \frac{1}{z-a_\rho}$ and $K_F(z,w) = \dfrac{F(z) + \overline{F(w)}}{z + \bar{w}}$. Then*

$$K_F(z,w) = \sum_\rho m_\rho\, g_\rho(z)\, \overline{g_{\rho^*}(w)} \qquad \text{(sum over distinct zeros, absolutely convergent).} \tag{40.5.1}$$

Consequently:

- *(a) For points $z_1, \dots, z_n$ the Pick matrix is $P = \Gamma M \Gamma^*$, with $\Gamma_{i\rho} = g_\rho(z_i)$ and $M_{\rho\sigma} = m_\rho \delta_{\sigma,\rho^*}$. The matrix M is the Krein Gram $[e_\rho, e_\sigma] = m_\rho \delta_{\sigma,\rho^*}$ of the zero space (Lemma 30 §8, Lemma 27 Prop. 1.1).*
- *(b) An on-line zero contributes the rank-one positive term $m_\rho g_\rho \bar{g}_\rho$ (definite type). An off-line mirror pair $\{\rho, \rho^*\}$ contributes $h^{\mathrm{T}}\left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)\bar{h}$ with $h = (g_\rho, g_{\rho^*})$, a* hyperbolic plane.
- *(c) $\kappa_{\mathrm{Pick}} := \sup_{n,\{z_i\}} n_-(P) = N_+^{\mathrm{dist}}$, the number of distinct zeros with $\beta > \frac{1}{2}$, whether finite or infinite.*
- *(d) Hence ([RESTATEMENT]) RH $\iff$ $\kappa_{\mathrm{Pick}} = 0$ $\iff$ K_F is a positive kernel $\iff$ F is positive-real on $\mathrm{Re}\,p > 0$. This is Lagarias' criterion [168] ([CITED]) seen as a Krein count.*

PROOF. By Hadamard [80, Ch. 12] (as used in Lemma 33), $\xi'/\xi(s) = \lim_{T\to\infty} \sum_{|\gamma|<T} \frac{1}{s-\rho}$, summed symmetrically: the constant B cancels $\sum 1/\rho$ under pairing $\rho \leftrightarrow \bar{\rho}$. So $F(z) + \overline{F(w)} =$

$\lim_T \sum_{|\gamma|<T}\big[\frac{1}{z-a_\rho}+\frac{1}{\bar w-\bar a_\rho}\big]$. The map $\rho\mapsto\rho^*$ preserves $\{|\gamma|<T\}$ and m_ρ, and $\bar a_{\rho^*}=-a_\rho$. Re-indexing the second sum gives $\sum\big[\frac{1}{z-a_\rho}+\frac{1}{\bar w+a_\rho}\big]$. Termwise,

$$\frac{1}{z+\bar w}\Big(\frac{1}{z-a}+\frac{1}{\bar w+a}\Big)=\frac{1}{(z-a)(\bar w+a)}=g_\rho(z)\,\overline{g_{\rho^*}(w)},$$

because $g_{\rho^*}(w)=1/(w+\bar a_\rho)$ (checked symbolically, `lemma40_pair.py`). Absolute convergence follows from $|g_\rho(z)g_{\rho^*}(w)|\ll_{z,w}\gamma^{-2}$ and $\sum m_\rho\gamma^{-2}<\infty$. (a) is a rewriting. (b): for $\beta=\frac12$, $\rho^*=\rho$. For a pair, the identity with $M=\left(\begin{smallmatrix}0&m\\m&0\end{smallmatrix}\right)$ is verified symbolically (residual 0). (c) Upper bound: $P=\Gamma M_{\mathrm{on}}\Gamma^*+\sum_{\text{pairs}}\Gamma_jM_j\Gamma_j^*$, the first term PSD and each pair term of rank ≤2 with at most one negative eigenvalue. Subadditivity of n_- gives $n_-(P)\le\#\text{pairs}=N_+^{\mathrm{dist}}$. Lower bound: for distinct $a_1,\dots,a_k$ with $\operatorname{Re}a_j>0$, take $z_j=a_j-\varepsilon m_j$. Then $F(z_j)=-1/\varepsilon+O(1)$, so $P_{ij}=-\frac{2}{\varepsilon}\frac{1}{a_i+\bar a_j}+O(1)$. The Cauchy matrix $\big(\frac{1}{a_i+\bar a_j}\big)=\big(\int_0^\infty e^{-a_it}\overline{e^{-a_jt}}dt\big)$ is positive definite for distinct a_j in the right half-plane, so $n_-(P)=k$ for small ε. (d): (c) together with $N_+^{\mathrm{dist}}=0\iff$ RH and the diagonal $K_F(z,z)=\operatorname{Re}F(z)/\operatorname{Re}z$. □

Remark 40.5.3 (what is new, and what is not; [THM]). Lagarias' theorem, the zero-space Krein form (Lemmas 27, 30) and the index count (Lemmas 34, 36) were separate items in TAA. Eq. (40.5.1) is one line, but it identifies them: *the Nevanlinna–Pick kernel of ξ'/ξ is the zero-space Krein Gram, pulled back by the Cauchy feature map $z\mapsto(g_\rho(z))_\rho$.* The mirror $\rho\mapsto\rho^*$ supplies the pairing and c_∞ supplies the bar on w. That is the precise "Galois twist": the Hermitian form is $\sharp\circ c_\infty$, the same structure as $\langle x,y\rangle=\operatorname{Tr}(x\bar y^\dagger)$ in Prop. 40.3.2. We have not found this identity stated in the sources we checked. It is elementary, and we claim no priority.

Corollary 40.5.4 (ghosts need a degeneration; [THM] in finite models). *In any finite zero model (finitely many zeros, Lemma 27 §3) deformed continuously through FE-symmetric configurations, an on-line zero,* simple or double*, has definite Krein type. It can become a ghost (leave the line) only after the zero-space form, or the test-function Gram G_A of Lemma 27, degenerates. This is Cor. 40.4.3, and it matches Lemma 27 Thm. 4.4 numerically ($(\gamma\pm i\delta)^2$ to 6–7 digits).*

40.5.4. Ghost birth, Krein degeneracy, line break, 2π jump: what is proved. For $h\in[0,\frac12)$ put $F_h(p)=\frac{\xi'}{\xi}(\frac12+h+p)$ and define the thresholds

$$\begin{aligned}h_{\mathrm{Pick}}&=\inf\{h: K_{F_h}\text{ is a positive kernel on }\operatorname{Re}p>0\},\\ h_{\mathrm{line}}&=\inf\{h: L^{\omega,\nu}\text{ never breaks for all admissible }\omega>h\}\quad\text{(Lemma 37)},\\ h_{\mathrm{def}}&=\inf\{h:\textstyle\int_0^T\operatorname{Re}\frac{\xi'}{\xi}(\sigma_0+it)dt\text{ has zero deficit }\forall T,\ \forall\sigma_0>\frac12+h\}\quad\text{(Lemma 36 Thm. 4.3)}.\end{aligned}$$

Theorem 40.5.5 (thresholds coincide; [THM] synthesis, [RESTATEMENT] at $h=0$). $h_{\mathrm{Pick}}=h_{\mathrm{line}}=h_{\mathrm{def}}=\max(\beta^*-\frac12,0)$.

Proof. Pick: a positive kernel $\iff\operatorname{Re}F_h\ge0$ on $\operatorname{Re}p>0$ (Prop. 40.7.1), and Lagarias [**168**, Thm. 1.1] gives $\iff$ all $\beta\le\frac12+h$. Line: Lemma 37 Thm. 3.2 (depth theorem). Deficit: Lemma 36 Cor. 4.4, deficit $=2\pi N(\sigma_0,T)$, which vanishes for all T iff no zero has $\beta>\sigma_0$. □

link	status	where
ghost ⇒ prior degeneracy of the Krein form (finite models)	[THM]	Cor. 40.4.3, 40.5.4; L27 Thm. 4.4
degeneracy ⇒ ghost	**false** in general ([THM] counterexample)	Cor. 40.4.3; L27 stage $\delta_b < \delta < \delta_c$
line break at t_* ⟺ degeneracy of $1 \mp \mathcal{K}[t_*]$	[THM] (compact $\mathcal{K}[t]$)	L37 §1, L34 escape thm
some line breaks (depth $\omega > h$) ⟺ ghost with $\beta > \frac{1}{2} + h$	[THM]; [RESTATEMENT] at $h = 0$	L37 Thm. 3.2
ghost ⟺ deficit $2\pi N(\sigma_0, T) > 0$; jump $2\pi m$ per zero	[THM]	L36 Cor. 4.4
ghost ⟺ negative square of K_F; count N_+^{dist}	[THM]	Thm. 40.5.2
de Branges index of the line $= \nu N_+$	[THM] under (B),(ND)	L34 Thm. 5.2
# eventual exits of $\mathcal{K}[t]$ from $[-1, 1] = \nu N_+ = \frac{\nu}{\pi} D(\frac{1}{2} + \omega, \infty)$ (half opens, half shorts)	[THM] under (B),(ND)	L36 Thm. 7.9, Cor. 7.10
each pole pair of Θ gives a Hankel block of signature (m, m) (hyperbolic)	[THM]	L36 Thm. 7.9 proof; cf. Thm. 40.5.2(b)
all depth thresholds equal $\beta^* - \frac{1}{2}$	[THM]	Thm. 40.5.5
same event: the time t_* of each exit determined by the ghost's (δ, γ); the exiting eigenvector ↔ the ghost's hyperbolic plane	[OPEN] / [CONJ]	L34 planted pairs ($t_* \leq 2.0$); L36 Rem. 7.14 toys

Summary. "Ghost birth ⇔ Krein-form degeneracy ⇔ line break ⇔ 2π jump" is a theorem *as an equivalence of existence statements, threshold by threshold in depth.* Under Lemma 34's hypotheses (B), (ND) it is also a theorem *as a count*: eventual line exits $= \nu N_+ = \frac{\nu}{\pi}\times$deficit (Lemma 36 Cor. 7.10), with $N_+^{\mathrm{dist}} = N_+$ Pick negative squares when the zeros are simple. Within finite models, ghost ⇒ degeneracy is a theorem and the converse is false. The identification of the individual events across the three pictures (the line's parameter t, the deficit's (σ_0, T), the deformation parameter of Lemma 27) is *not* proved.

40.5.5. G5 in Krein form.

Definition 40.5.6 ($\mathrm{G5_K}$; [DEF]). There is a geometric reason, built from the Euler product and Γ without the zeros (G6), why the line forms $1 \mp \mathcal{K}[t]^{\omega,\nu}$ *never degenerate* along the depth deformation $\omega \downarrow 0$. The starting point $\omega \geq \frac{1}{2}$ is unconditionally passive (Lemma 37).

Status. As a statement, $\mathrm{G5_K} \iff \mathrm{G5_{line}} \iff$ RH ([RESTATEMENT]; Lemma 37 Thm. 3.2 and Cor. 40.4.3). As a *mechanism* it is [OPEN]. Two theorems constrain the mechanism.

(i) *No spectral-gap proof* ([THM], from Lemma 37 Cor. 2.4, correction K1). In the passive case $\sup_t \|\mathcal{K}[t]\| = 1$, approached and not attained, so $\inf_t(1 - \|\mathcal{K}[t]\|) = 0$. No uniform-gap or continuity argument can show non-degeneracy. The reason must be *structural*, an identity, as in Weil, where $\mathrm{Tr}(\alpha\alpha^\dagger)$ *equals* an intersection number.

(ii) *The zero-space form is circular* ([THM]). M in Thm. 40.5.2 is built from the zeros, and K_F is built from ξ'/ξ, whose positivity is RH. Neither is a G6-admissible input. Only the line forms $1 \mp \mathcal{K}[t]$ and Weil's Q on test functions are zero-free.

What a Rosati-type reason would look like ([OPEN]). A formula $\mathrm{Tr}_Z(f * f^\sharp) = c\cdot$(intersection number of a divisor D_f on a surface-like object with a Hodge-index theorem. This is the Connes–Consani programme ($s(f, f) = D \bullet D$, [70]), Lemma 30 G5, and Remark 40.3.6. In this language the Hodge index on $C \times C$ says that the intersection form on $N(V)$ has exactly *one* positive direction, spent on the ample class $C_1 + C_2$ [193, Thm. 1.5 proof]. Up to sign this is a Π_1 form. Its ζ-side counterpart is Lemma 36's $n_+(\mathcal{L}) = 1$, whose exceptional direction is the pole, the Riemann–Roch "+1" (Lemma 36 §6, Prop. 7.4). The ghosts would be *additional* hyperbolic planes beyond that one direction.

40.6. Kepler, regularization and the two-stage ghost birth

This section answers J. W.'s [t145u, t146u]. It deepens Lemma 31 §4 (KS lift, Kepler algebra, dictionary Prop. 4.4) and does not repeat it. Numerics: `num/lemma40_kepler.py`, output `num/l40/kepler.json`. The separate question of whether the two stages of Lemma 27 can coincide is Lemma 41 (in preparation); it is only referenced here.

40.6.1. (A1) Lagrange's identity: the Gram form of (r, v).

PROPOSITION 40.6.1 ([THM], [NUM]). *For $r, v \in \mathbb{R}^3$, $\det\left(\begin{smallmatrix} |r|^2 & r\cdot v \\ r\cdot v & |v|^2 \end{smallmatrix}\right) = |r|^2|v|^2 - (r\cdot v)^2 = |r\times v|^2$. Along a Kepler orbit (unit mass) this is $|L|^2$, hence:*

- *(a) the Gram determinant is* constant *along each orbit;*
- *(b) it vanishes iff $L = 0$, i.e. iff the orbit is radial (collision/ejection);*
- *(c) at the apsides $r \cdot v = r\dot{r} = 0$, so the Gram matrix is* diagonal *there, and $|r|\,|v| \geq |L|$ with equality exactly at the apsides. So $|r||v|$ is* minimal *at the apsides;*
- *(d) the Gram matrix of two real vectors is positive semidefinite. It can degenerate, but it can never become indefinite.*

PROOF. Expand $|r\times v|^2$. (a) L is conserved. (c) $|r|^2|v|^2 = L^2 + (r\cdot v)^2$. (d) It is a Gram matrix. [NUM]: three orbits (one with $L^2 = 2.5\cdot 10^{-5}$, nearly radial) give $\max|\det - L^2| \leq 1.1\cdot 10^{-15}$ and $|r\cdot v| \leq 4\cdot 10^{-3}$ at the sampled apsides (grid-limited), against $\max|r\cdot v| \approx 0.6$–$1.0$. □

Gentle correction (E8). The apsides are not where the Gram determinant is maximal: it does not vary along an orbit. They are where the frame (r, v) is orthogonal, so the Gram matrix is diagonal and $|r||v|$ is smallest. Among orbits of one energy, L^2 is largest for the circular orbit. What does carry over to the Krein picture is (b)+(d). Kepler's Lagrange form is a *Hilbert* Gram form, so its only failure is *degeneration* (collision). A ghost needs an indefinite form, and the Lagrange form never produces one. In Kepler the sign flip has to come from somewhere else, and regularization supplies it.

40.6.2. (A2) Levi-Civita: a two-sheeted cover with deck group $\{\pm 1\}$.

PROPOSITION 40.6.2 (Levi-Civita regularization; [THM] (classical computation [176]), [NUM]). *In the plane, put $z = u^2$ ($z, u \in \mathbb{C}$) and Sundman time $dt = |z|\,ds$. A Kepler orbit $\ddot{z} = -z/|z|^3$ of energy E becomes the linear oscillator $u'' = \frac{E}{2}u$ (so $\omega^2 = -E/2$, as for KS in Lemma 31 Prop. 4.1). [NUM]: a generic orbit integrated both ways agrees to $1.1\cdot 10^{-11}$.*

THEOREM 40.6.3 (collision = sheet change; [THM]). *$\pi : u \mapsto u^2$ is a two-sheeted cover of $\mathbb{C}$ branched only at 0. Its deck group is $\{\pm 1\} \cong \mathbb{Z}/2$, which is the Galois group of $\mathbb{C}(u)/\mathbb{C}(z)$, $u = \sqrt{z}$. Let $E < 0$ and let $u(s)$ be the regularized collision orbit through $u(0) = 0$. Then $u(s) = a\sin(\omega s)e^{i\theta}$ and:*

- *(a) $u(-s) = -u(s)$, so the post-collision half of the regularized orbit is the deck image of the pre-collision half, run backwards. It passes* smoothly *through the branch point onto the other sheet;*
- *(b) downstairs, $z(-s) = z(s)$ and $\dot{z}(-s) = -\dot{z}(s)$. The particle returns along the same ray with reversed velocity (a bounce).*

More generally, any C^1 curve through 0 with $u'(0) \neq 0$ has its two halves in opposite half-lines $\pm e^{i\theta}\mathbb{R}_+$, and these are exchanged by the deck transformation.

PROOF. The cover and its Galois group are standard. The orbit solves $u'' = -\omega^2 u$ with $u(0) = 0$, so it is odd. Then $z = u^2$ is even, and $\dot{z} = (dz/ds)/(dt/ds) = 2\omega\cot(\omega s)e^{2i\theta}$ is odd. [NUM]: u-oddness and z-evenness hold exactly, and $\dot{z}$ changes sign across $s = 0$ (table in `kepler.json`). □

40.6.3. (A3) The frequency cover of the pencil, and its branch points.

PROPOSITION 40.6.4 ([THM]). *Lemma 27's pencil eigenvalue is a* squared *frequency, $\mu = \tau^2$, where a zero is $\rho = \frac{1}{2} + i\tau$ (on-line: $\tau = \gamma \in \mathbb{R}$). The map $\tau \mapsto \tau^2$ is the same two-sheeted cover $\mathbb{P}^1 \to \mathbb{P}^1$, now with* two *branch points,* 0 *and* ∞*, and deck group* $\{\pm 1\}$*. The Klein four-group of Lemma 31 Prop. 3.4 acts on τ as follows:*

$$\tau \mapsto -\tau \quad (FE,\ \rho \mapsto 1-\rho) = deck; \qquad \tau \mapsto \bar{\tau} \quad (\sharp,\ \rho \mapsto 1-\bar{\rho}); \qquad \tau \mapsto -\bar{\tau} \quad (c_\infty,\ \rho \mapsto \bar{\rho}).$$

So the deck transformation is the FE involution. Complex conjugation of the pencil eigenvalue, $\mu \mapsto \bar{\mu}$, is induced by $\sharp$ (equivalently by c_∞, since the two differ by the deck map). Since ζ has no zeros on $(0,1)$, μ is real iff τ is real iff ρ is on the line. In Prop. 40.8.1 language: μ is real iff the Rosati-type $\sharp$ acts trivially on the spectrum.

PROOF. $\rho = \frac{1}{2} + i\tau$ gives $1-\rho = \frac{1}{2} + i(-\tau)$, $\bar{\rho} = \frac{1}{2} + i(-\bar{\tau})$ and $1-\bar{\rho} = \frac{1}{2} + i\bar{\tau}$. Finally, $\tau^2 \in \mathbb{R}$ iff $\tau \in \mathbb{R} \cup i\mathbb{R}$. □

LEMMA 40.6.5 (type flip through ∞; [THM]). *Let $K(e), G(e)$ be Hermitian and continuous, with $K(e)x = \mu(e)G(e)x$, μ real and $x(e)$ continuous. Then $x^*Kx = \mu\, x^*Gx$. If $x^*Kx \neq 0$ near e_0 and μ passes through $\infty \in \mathbb{P}^1$ at e_0 (that is, $1/\mu$ changes sign through a simple zero), then the Krein type* $\operatorname{sgn} x^*Gx = \operatorname{sgn}(x^*Kx)\operatorname{sgn}(1/\mu)$ *flips. This is Lemma 27's stage (a) (Lemma 27 Lemma 4.1, Thm. 4.4).*

REMARK 40.6.6 (the Kepler–pencil identification: [ANALOGY]; correction E9). *What is a theorem:* (i) LC collision is a passage through the branch point 0 of a $\{\pm 1\}$-cover, ending on the other sheet (Thm. 40.6.3). (ii) Lemma 27's pencil eigenvalue lives on a $\{\pm 1\}$-cover whose deck map is the FE (Prop. 40.6.4). Stage (a) is a passage of μ through the other branch point, ∞, with a Krein-type flip (Lemma 40.6.5). Lemma 27's numerics then have the reborn eigenvalue climb from $-\infty$ through $\mu = 0$ to the collision near γ_1^2. So it visits *both* branch points. *What is analogy:* identifying the Kepler sheet swap with the Krein sign flip. The quantity that reverses in Kepler is a velocity, a vector. In the pencil it is the value x^*Gx of a form. The covers are also different: the LC cover lives in configuration space, and the frequency cover lives in spectral space. The earlier chat claim that the returning branch "*literally*" is the negative-type eigenvalue overstated this. The covering statements are [THM], and the identification is [ANALOGY].

40.6.4. (A4) Fock: the Killing form of the Runge–Lenz algebra degenerates at $E = 0$.

PROPOSITION 40.6.7 ([THM], sympy). *Let $\mathfrak{g}_E = \operatorname{span}\{L_i, A_i\}$ with $[L_i, L_j] = \epsilon_{ijk}L_k$, $[A_i, L_j] = \epsilon_{ijk}A_k$ and $[A_i, A_j] = -2E\epsilon_{ijk}L_k$ (Lemma 31 Prop. 4.3; $m = k = 1$; Jacobi residual 0). Its Killing form is*

$$\kappa_E = \operatorname{diag}(-4, -4, -4,\ 8E, 8E, 8E).$$

The signature is $(0,6)$ for $E < 0$ (compact $\mathfrak{so}(4)$), $(3,3)$ for $E > 0$ ($\mathfrak{so}(3,1)$), and rank 3 at $E = 0$ ($\mathfrak{e}(3)$, not semisimple). So the invariant form on the Runge–Lenz directions is $8E \cdot I_3$. It passes through 0 and *changes sign exactly at $E = 0$, where Fock's rescaling $D = A/\sqrt{2|E|}$ is singular: this is the Inönü–Wigner contraction* [**136**] *[BIB] (see also* [**102, 22, 199**] *[BIB]).*

Structural difference ([THM] for the facts, [ANALOGY] for the comparison). In Kepler, two things happen at the same parameter on the same (Runge–Lenz/radial) mode: the invariant form degenerates ($8E \to 0$, Prop. 40.6.7), and the stiffness changes sign ($\omega^2 = -E/2 \to 0$, so escape). So *degeneration and escape merge at $E = 0$.* In ζ (Lemma 27) they are separated: first $\det G_A = 0$ (stage a), then, strictly later, an opposite-sign collision (stage b), with $\delta_c/\delta_b \to \infty$ in the explicit model (Lemma 27 Thm. 9.5). The next result says exactly which feature decides between the two.

THEOREM 40.6.8 (separation criterion; [THM]). *Let $K(e), G(e)$ be Hermitian $n\times n$, analytic in e. Suppose $G(e_0)$ has a one-dimensional kernel spanned by v, and the pencil $K(e_0)-\mu G(e_0)$ has $n-1$ simple real finite eigenvalues.*

(a) (ζ-type.) If $v^*K(e_0)v\neq 0$, then for $|e-e_0|$ small all *eigenvalues are real. One of them passes through ∞ with a type flip (Lemma 40.6.5). So degeneration (stage a) and escape off the real axis (stage b) are separated: $e_c\neq e_0$.*

(b) (Fock/Kepler-type, 2×2.) Take $G=\mathrm{diag}(1,-e)$ and $K=\left(\begin{smallmatrix}k_1 & c\\ c & \kappa-\nu e\end{smallmatrix}\right)$ with $c\neq 0$ and $\nu\neq k_1$. If $\kappa=0$, then $v=e_2$ has $v^*K(0)v=0$: both forms degenerate on the same vector. Both eigenvalues pass through ∞ together as $e\uparrow 0$ ($\mu\approx\pm c/\sqrt{|e|}$), and for $0<e<4c^2/(\nu-k_1)^2$ they are a complex pair. So stages (a) and (b) merge *at $e=0$. If $\kappa\neq 0$, they separate, with*

$$e_c=\frac{\kappa^2}{4c^2}(1+O(\kappa)).$$

PROOF. (a) In $\nu=1/\mu$, $p(\nu)=\det(\nu K-G)$ is a polynomial with real coefficients (it is real for real ν). By Jacobi's formula, $p'(0)=\mathrm{tr}(\mathrm{adj}(-G)K)$. Since $\mathrm{adj}(-G(e_0))=c_0vv^*$ with $c_0\neq 0$ (the product of the nonzero eigenvalues of $-G$), $p'(0)=c_0v^*Kv\neq 0$, so $\nu=0$ is a simple root. The finite eigenvalues give simple real roots too. Simple real roots of a real polynomial stay real under small real perturbation, because a nonreal root would bring its conjugate and the root count near each simple root is 1. (b) $\det(K-\mu G)=0$ reads $-e\mu^2+((\nu+k_1)e-\kappa)\mu+k_1(\kappa-\nu e)-c^2=0$. For $\kappa=0$ divide by $-e$: $\mu^2-(\nu+k_1)\mu+k_1\nu+c^2/e=0$, with discriminant $(\nu-k_1)^2-4c^2/e$. For $\kappa\neq 0$, the discriminant in e is $D(e)=Ae^2+Be+\kappa^2$ with $A=(\nu-k_1)^2$ and $B=-2\kappa(\nu-k_1)-4c^2$. Its smallest positive root is $\kappa^2/(4c^2)(1+O(\kappa))$. □

[NUM] two-mode pencils ($k_1=4$, $c=3$, $\nu=1$); `K4`, `K5b`				
model	e_b (det $G=0$)	e_c (first complex pair)	type of reborn eigenvalue	verdict
ζ-type ($k_2\equiv 1$)	0	0.0365	-1 ($\mu=-90.5$ at $e=0.01$)	two stages
Fock/Kepler-type ($\kappa=0$)	0	0^+ ($\mu=2.5\pm 30.0i$ at $e=0.01$)	none (both through ∞)	merged
interpolation κ:	$e_c/(\kappa^2/4c^2)=1.0002,\ 1.0017,\ 1.017,\ 1.053,\ 1.212$ for $\kappa=10^{-3},10^{-2},0.1,0.3,1$			

Honest report. The explicit two-mode Fock/Kepler-type pencil *merges* the stages, and the ζ-type pencil *separates* them. The deciding quantity is v^*Kv on the degenerating vector, and the gap closes like $(v^*Kv)^2$. The model was built to exhibit the criterion, so the numerics confirm the theorem and are not independent evidence about ζ. One feature of Lemma 27 the toy does *not* reproduce: in the toy the collision happens at negative μ, while Lemma 27's reborn eigenvalue climbs through $\mu=0$ before colliding near γ_1^2. *Testable consequence* (for Lemma 41, not done here): Lemma 27 conjectures $\delta_b,\delta_c\to 0$ as $A\to\infty$, merging into the canonical degeneration. By Thm. 40.6.8, that merging should go with $v_A^*K_Av_A\to 0$ at δ_b, i.e. a Kepler-type limit, with gap $\propto(v_A^*K_Av_A)^2$.

40.6.5. (B) Regularization → thermality → J → Galois.

PROPOSITION 40.6.9 ([THM], [NUM]). *(a) The metric $d\varrho^2+\varrho^2d\theta^2$ with $\theta\in\mathbb{R}/\beta\mathbb{Z}$ is a flat cone of total angle β. It extends smoothly across $\varrho=0$ iff $\beta=2\pi$ (angle deficit $2\pi-\beta$). (b) Rindler coordinates are $x^0=\varrho\sinh\eta$ and $x^1=\varrho\cosh\eta$. Under $\eta=-i\theta$, $x^0=-ix^4$, they become Euclidean polar coordinates, and boosts become rotations. So regularity at the horizon forces rapidity period 2π in imaginary rapidity. (c) $B(\eta)=\left(\begin{smallmatrix}\cosh\eta & \sinh\eta\\ \sinh\eta & \cosh\eta\end{smallmatrix}\right)$ satisfies $B(i\pi)=-I$ ([NUM] exactly) and $B(i\pi/2)=\left(\begin{smallmatrix}0 & i\\ i & 0\end{smallmatrix}\right)$. Continuing the boost by $i\pi$ gives the half-turn $(x^0,x^1)\mapsto(-x^0,-x^1)$, which maps the right wedge onto its causal complement.*

[CITED, secondary] (as in Lemma 31 §6(2) and Lemma 39 Thm. 2.3, via [72]): by Bisognano–Wichmann [33, 34], the vacuum on the wedge algebra is KMS at $\beta = 2\pi$ for the boosts, $\Delta^{it} = U(B(-2\pi t))$, and $J = \Theta U(R(\pi))$ (CPT times the half-turn). Then $\Delta^{1/2}$ is the boost continued to rapidity $i\pi$, i.e. the half-turn of (c), and $S = J\Delta^{1/2}$. Gibbons–Hawking [111] and Unruh [273] [BIB] give the heuristic "no conical singularity $\Rightarrow \beta = 2\pi$" of (a)–(b). *The heuristic is (a)+(b); the theorem is BW.* [ANALOGY]: the LC deck map $u \mapsto -u = e^{i\pi}u$ and the BW half-turn $B(i\pi) = -I$ are both "-1 reached by a rotation through π". Both resolve a singular point by passing to a branch: upstairs in LC, the commutant in BW. They are not the same map. The LC deck map covers the identity downstairs, while the BW half-turn exchanges wedge and complement.

Natural-cone positivity: pairing with your conjugate ([THM], Lemma 31 Lemma 3.2; cone [CITED] [8, 73, 124] bibliographic). $\langle \Omega, xJxJ\Omega\rangle = \|\Delta^{1/4}x^*\Omega\|^2 \geq 0$, and $j(x) = JxJ \in M'$. The four "pair with your conjugate" positivities, side by side:

positivity	conjugation	action on ζ-spectrum	status
Riemann $i\int\omega\wedge\bar\omega > 0$	c_∞ (Hodge)	$\rho \mapsto \bar\rho$ (twist)	[THM] (Thm. 40.2.1)
Rosati $\mathrm{Tr}(\alpha\alpha^\dagger) > 0$	$\dagger$	$\sharp$: $\rho \mapsto 1-\bar\rho$	[CITED] (Prop. 40.8.1)
natural cone $\langle\Omega, xj(x)\Omega\rangle \geq 0$	J	c_∞ (forced, Prop. 40.6.10)	[THM], free
OS $\langle\theta F, F\rangle \geq 0$ on E_+	θ	Weil's θ = mirror $\sharp$ (L31 Prop. 7.3)	[OPEN] for ζ (L32)

PROPOSITION 40.6.10 (J is always the twist, never the Rosati; [THM]). [1] *Tomita–Takesaki gives $J\Delta^{it}J = \Delta^{it}$ [CITED] ([72] summary; Lemma 31 Def. 3.1). Suppose the modular flow is the scaling flow and e_τ is a mode with $\Delta^{it}e_\tau = e^{i\tau t}e_\tau$ ($\tau \in \mathbb{C}$ allowed in a Krein model). Then $\Delta^{it}Je_\tau = e^{-i\bar\tau t}Je_\tau$. So J acts on exponents by $\tau \mapsto -\bar\tau$, which is c_∞ (Prop. 40.6.4) and never $\sharp$ ($\tau \mapsto \bar\tau$). This matches Lemma 31 Rem. 3.5 ($\varepsilon = +1$) and Lemma 38 Prop. 8.3 (J swaps the χ and $\bar\chi$ sectors, i.e. it is the Galois-conjugate pairing in the* values, *not $-1 \in \hat{\mathbb{Z}}^\times$).*

PROOF. $\Delta^{it}Je_\tau = J\Delta^{it}e_\tau = J(e^{i\tau t}e_\tau) = \overline{e^{i\tau t}}\,Je_\tau$, since J is antilinear. □

40.6.6. (C) What natural-cone positivity gives, and the missing step.

THEOREM 40.6.11 (Bochner reduction; [THM] from Stone/Bochner–Schwartz [BIB]). *Let (M,Ω) be standard, and let $F \mapsto \xi_F := x_F^*\Omega$ be linear and continuous from $C_c^\infty(\mathbb{R})$, with scaling covariance $\Delta^{it}\xi_F = \xi_{T_tF}$. Then $N(F,G) := \langle \xi_F, \Delta^{1/2}\xi_G\rangle$ is a translation-invariant positive semidefinite Hermitian form. Hence $N(F,F) = \int |\widehat F|^2\, d\nu$ for a positive tempered measure ν on $\mathbb{R}$. If $N(F,F) = Q(F,F)$ for all F, then Weil's distribution is positive definite, which is RH (Weil's criterion; Lemma 32 Thm. 5.1).*

PROOF. Covariance gives $N(T_tF, T_tG) = N(F,G)$, because Δ^{it} commutes with $\Delta^{1/2}$ and is unitary. Positivity is $\Delta^{1/2} \geq 0$. A translation-invariant continuous positive form on C_c^∞ is a positive-definite distribution, so Bochner–Schwartz applies. □

What it gives for ζ ([THM], Lemma 31 Thm. 6.5). For the KMS_1 structure of $\xi(1+it)$, natural-cone/midpoint positivity is the positive-definiteness of $\Xi(t) = \xi(\frac12 + it)$, which comes from $\Phi > 0$. It is *unconditional*, and KMS_1 is *equivalent to the FE only.* The measure ν there lives on the log-scale (prime/theta side). Weil needs positivity of the *zero*-side distribution, and the explicit formula is a Fourier duality that does not carry positivity across (Lemma 31 Prop. 3.10, Lemma 38 Prop. 5.3). On the OS side, Weil's form is $\langle\langle F, \theta F\rangle\rangle$ with $\theta = \sharp$ and *no* half-space (Lemma 31

[1] **[Vol. 3 editorial note]** Lemma 42, correction E3: this proposition, (E10) and row 3′ of the bridge table need the scope "when the scaling flow is the modular flow". In the placement (P2) of Lemma 42, J acts as $\sharp$ on translation frequencies.

Prop. 7.3). Half-line restrictions are vacuous by translation invariance (Lemma 32 Thm. 5.1). A θ-invariant nonnegative domain forces $\theta = \mathrm{id}$ (Lemma 32 Prop. 6.1). So Weil's form is a *degenerate* reflection-positivity form.

Why natural-cone positivity has not reached Weil's form ([THM] for the two facts, [RESTATEMENT] for the conclusion). (1) It is positivity for J, and J acts as the *twist* c_∞ (Prop. 40.6.10). The Rosati role is $\sharp$ (Prop. 40.8.1), and $\sharp = c_\infty \circ \mathrm{FE}$ with FE the deck map of the frequency cover (Prop. 40.6.4). Composing with FE does not preserve positivity, because FE is the fundamental symmetry J_{FE} of the hyperbolic planes (Lemma 27 Prop. 1.1), an indefinite isometry. (2) In any covariant realization it is Bochner positivity of a measure ν (Thm. 40.6.11). Weil positivity *is* the statement that the zero-side distribution has such a ν, so asking for $N = Q$ asks for RH.

Definition 40.6.12 (the missing step; [OPEN] = G5). Construct a standard pair (M, Ω), together with $F \mapsto x_F$, *from prime-side/adelic data alone* (no zeros, and not by GNS of Q), such that (i) the modular flow is the scaling flow, (ii) M is noncommutative (Lemma 31 Prop. 3.3), and (iii) $\langle \Omega, x_F J x_F J \Omega \rangle = Q(F, F)$, equivalently ν in Thm. 40.6.11 is the Fourier transform of Weil's distribution. Given (i)–(ii), condition (iii) is all of G5.

Circularity flags (C7–C9). (C7) GNS of Q gives such a pair under RH (Lemma 31 Rem. 3.8, G6), so this is circular as a proof. (C8) Prescribing ν = zero counting measure presupposes real zeros (Hilbert–Pólya in modular dress). (C9) Any cone built from the BC/KMS$_1$ structure yields prime-side ν and hence only the FE. The known candidates fail for stated reasons: BC (dual grading; Lemma 31 §6(5), remark (i)), the adele class space (Lemma 31 Rem. 3.12), and half-sided inclusions at finite prime levels (Lemma 39 Thm. 7.5; Lemma 31 Prop. 6.3).

Routes from this section (score = 0.6 sig+0.4 prob)

	route	sig	prob	score
K1	Test $v_A^* K_A v_A \to 0$ at δ_b as $A \to \infty$ in Lemma 27 window pencils, gap $\propto (v^* K v)^2$ (feeds Lemma 41)	.5	.7	.58
K2	Branch-point bookkeeping (Prop. 40.6.4) as the Vol. 2 convention for stages (a)/(b)	.3	.95	.56
K3	A Moser-type regularization of the scaling flow (on-line $\leftrightarrow$ geodesic flow on a compact space [**199**])	.7	.05	.44
M1	The missing step, Def. 40.6.12	1.0	.03	.61
M2	Sector natural cones with $J : \chi \leftrightarrow \bar{\chi}$ (likely free positivity only)	.3	.8	.50
M3	Horizon regularity for ζ beyond KMS$_1$ (expected: FE only)	.2	.9	.48

40.7. Bridges from causality (KK) positivity to Hilbert positivity

(1) The spectrum condition does not imply Hilbert positivity ([CITED]). In gauge QFT the Gauss law forces a conflict between positivity and locality of the charged-field correlation functions. The Feynman–Gupta–Bleuler quantization keeps local charged fields with a weak Gauss law (a subsidiary condition) at the price of an indefinite metric. The Coulomb gauge keeps positivity at the price of non-locality [**254**, ch. "Quantization of gauge field theories", abstract]. A covariant weakly local A_μ requires unphysical states [**255, 256**]; see also [**121**] [BIB]. In the Morchio–Strocchi framework of indefinite-metric QFT, positivity is replaced by a "Hilbert space structure condition" while locality, covariance and the spectral condition are retained. Interacting models exist in which the spectral condition is proved [**5**, §1 and abstract]. So causality and the spectrum condition, the KK side, are compatible with a Krein space that has negative-norm states. This is the physics counterpart of Lemma 36's "causality is free; stability on $\mathrm{Re}\, p > 0$ is RH". The abstract route from causality to a positive generator, Borchers–Wiesbrock half-sided modular inclusions, is unavailable at every finite Galois level (Lemma 39 Thm. 7.5), and even where it applies it yields spectrum-type positivity only (Lemma 39 §7(b)).

PROPOSITION 40.7.1 (positive-real $\iff$ positive Pick kernel $\iff$ Hilbert space; [THM] from standard facts). *Let F be holomorphic on $\operatorname{Re} z > 0$. Then $\operatorname{Re} F \geq 0$ there iff $K_F(z,w) = \frac{F(z)+\overline{F(w)}}{z+\bar{w}}$ is a positive kernel. In that case K_F is the reproducing kernel of a Hilbert space of functions on the half-plane (Moore–Aronszajn [11], [BIB]), and the Pick matrices are its Gram matrices.*

PROOF. ($\Leftarrow$) $K_F(z,z) = \operatorname{Re} F(z)/\operatorname{Re} z$. ($\Rightarrow$) By the Herglotz–Nevanlinna representation for the right half-plane (standard, via the Cayley transform from the disc; [BIB]), $F(z) = cz + ib + \int_{\mathbb{R}} (\frac{1}{z-it} - \frac{it}{1+t^2}) d\mu(t)$ with $c \geq 0$, $b \in \mathbb{R}$, $\mu \geq 0$ and $\int \frac{d\mu}{1+t^2} < \infty$. The subtracted terms are purely imaginary and cancel in $F(z) + \overline{F(w)}$. Then $K_F = c + \int_{\mathbb{R}} \frac{d\mu(t)}{(z-it)\overline{(w-it)}}$, using the termwise identity of Thm. 40.5.2 with $a = it$. This is a positive combination of rank-one positive kernels. The disc version is Pick's theorem [227] ([BIB]); see [1] [BIB]. □

(2) Krein–Langer. The generalized Nevanlinna class N_κ consists of functions whose Pick kernel has exactly κ negative squares. Its theory (generalized poles of non-positive type, factorization) is [162] [BIB]. We do *not* rely on that theory. Theorem 40.5.2(c) proves the count we need directly for ξ'/ξ at $h = 0$: in half-plane form, $F \in N_{N_+^{\mathrm{dist}}}$, and the "poles of non-positive type" are exactly the off-line zeros with $\beta > \frac{1}{2}$. For $h > 0$ (the shifted F_h) we prove only $\kappa \geq \#\{\beta > \frac{1}{2} + h\}$ (the lower-bound argument) and $\kappa = 0 \iff$ no such zero (Thm. 40.5.5). The exact count for $h > 0$ is roadmap item R1.

(3) Lagarias and the line. RH $\iff \operatorname{Re} \frac{\xi'}{\xi} > 0$ on $\operatorname{Re} s > \frac{1}{2}$ [168] [CITED]. By Prop. 40.7.1 and Thm. 40.5.2 this is $K_F \geq 0$, i.e. $\kappa_{\mathrm{Pick}} = 0$. Lemma 37 Thm. 4.1 exponentiates it: the exponential line $\Theta_c = e^{-2cF}$ is passive (Schur) iff $\operatorname{Re} F \geq 0$. So $\mathrm{G5}_{\mathrm{line}}$ *is* the bridge: it is the one face in which "causal + passive $\Rightarrow$ Hilbert" is stated with a zero-free input, the Euler product kernel of $\mathcal{K}[t]$ (Lemma 34).

(4) The count is a Gram/Hessian signature ([THM]). Pick matrices are Gram matrices $\Gamma M \Gamma^*$, so the number of their negative eigenvalues is bounded by, and in the sup equal to, $n_-(M) = N_+^{\mathrm{dist}}$. Lemma 36 Prop. 5.1 counts the same objects on Weil's quadratic form, $n_+(\mathcal{L}) = 1 + N_+$, with one extra direction, the pole. The Pick form has no pole direction, because ξ has the pole removed. The pole direction reappears in Weil's form on test functions, which is the Hodge-index Π_1 shape of Remark 40.3.6.

(5) Where the negativity is visible ([THM] + [NUM]). $K_F(z,z) = \operatorname{Re} F(z)/\operatorname{Re} z$ is negative only inside Lemma 33's shadow discs $|s - \frac{1}{2} - it_\rho| < |\delta_\rho|$ (Lemma 33 Thm. 4.2). The lower-bound construction places points just *left* of the pole, inside its disc. Numerics C3 below show that point sets outside the discs see a ghost only through off-diagonal terms, with tiny eigenvalues.

40.8. The arithmetic dictionary for ζ, and which conjugation is Rosati

curve over $\mathbb{F}_q$ / over $\mathbb{C}$	ζ	status
H^1, $\dim 2g$	Deninger's conjectural H^1, zeros = spectrum of Θ [91]	[CONJ]
Frobenius π (a Galois element of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$)	scaling flow $\mathbb{R}_{>0}$, generator D; *not* in $\hat{\mathbb{Z}}^\times$ (L38 Thm. 4.2, §7 (B4))	[THM] (that it is not Galois); [ANALOGY] as identification
automorphisms commuting with π	$\hat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$: labels sectors χ, commutes with σ_t (BC; L38 Thm. 5.1)	[CITED]
c_∞ on coefficients; $H^{1,0} \leftrightarrow H^{0,1}$	reality $\rho \mapsto \bar{\rho}$; J_{mod}; $\chi \leftrightarrow \bar{\chi}$ (L31, L38 Prop. 8.3(a))	[THM]
Rosati †, $\pi^\dagger = q/\pi$	$\sharp$, $\delta_\lambda * \delta_\lambda^\sharp = \lambda$ (L30 Prop. 3.2); mirror on spectrum (L30 Prop. 3.1)	[THM]
$\langle x, y\rangle = \mathrm{Tr}(x\bar{y}^\dagger)$ († twisted by c_∞)	$K_F(z,w) = \sum m\, g_\rho(z)\overline{g_{\rho^*}(w)}$ (Thm. 40.5.2)	[THM]
Rosati positivity	$\mathrm{Tr}_Z(f * f^\sharp) \geq 0$ (Weil) $\iff$ $K_F \geq 0$ (Lagarias)	[RESTATEMENT]
off-circle pair $\lambda, 1/\bar{\lambda}$ = hyperbolic plane	off-line mirror pair $\{\rho, \rho^*\}$ = hyperbolic plane	[THM]
Weil pairing: symplectic similitude, gives the FE only	$\xi(s) = \xi(1-s)$; the Krein structure itself (free)	[ANALOGY] (precise)
Hodge split $H^{1,0} \oplus H^{0,1}$	$\gamma > 0$ / $\gamma < 0$; Hardy $H^2_\pm$ (L39 §7(c))	[ANALOGY]
ampleness; Hodge index on $C \times C$	intersection theory on the square of the arithmetic site [62, 70]; $n_+(\mathcal{L}) = 1$ (pole)	[OPEN]

Proposition 40.8.1 (which conjugation plays the Rosati role; [THM] for the matching, [RESTATEMENT] for the last clause). *(a) The Rosati involution is $\mathbb{Q}$-linear, and $\pi^\dagger = q/\pi$. On the spectrum it acts by $\lambda \mapsto q/\lambda$, the functional-equation involution. Its antilinear extension, † composed with c_∞, acts by $\iota(\lambda) = q/\bar{\lambda}$ (Lemma 30 Cor. 2.3). On the ζ side the corresponding objects are $\rho \mapsto 1-\rho$ (FE) and the mirror $\rho \mapsto \rho^* = 1-\bar{\rho}$, the spectral action of $\sharp$.* So the Rosati/polarization role is played by $\sharp$ (mirror). *It is sector-internal (L38 Prop. 8.3(b)), just as † preserves $\mathbb{Q}[\pi]$. (b) c_∞ (reality, J_{mod}) is the* twist. *It turns the bilinear pairing into a Hermitian one in Prop. 40.3.2 and in Eq. (40.5.1), and it swaps sectors $\chi \leftrightarrow \bar{\chi}$. (c) $-1 \in \hat{\mathbb{Z}}^\times$ plays* no *role in the polarization. It acts by the scalar $\chi(-1)$ on each sector (L38 Prop. 8.3(c)), commutes with everything above, and preserves each Krein form sector by sector. (d) Positivity forces † = c_∞ on the spectrum: $\rho(\pi^\dagger) = \overline{\rho(\pi)}$* [193, Cor. 1.29]*, i.e. $\iota = \mathrm{id}$. On the ζ side the same equation reads $\rho^* = \rho$ for all ρ, which is RH (Lemma 30 Thm. 2.2, [RESTATEMENT]).* This is the precise sense of "Galois-twisted positivity": the positive form is Rosati twisted by c_∞, and its positivity says that Rosati and c_∞ agree on the spectrum.

Remark 40.8.2 (refinement of the Hodge analogy; [THM] about our objects). In Thm. 40.2.1, h is *negative* on $H^{0,1} = c_\infty(H^{1,0})$. On the ζ side the Krein form M is *positive* on both $\gamma > 0$ and $\gamma < 0$ on-line zeros. Reality maps it to its complex conjugate, not to its negative. So the correct counterpart of M (and of K_F) is the *polarized* form $h(\mathfrak{J}\cdot, \cdot)$, not h. The "$\gamma > 0$ half $\leftrightarrow H^{1,0}$" reading (chat; L39 §7(c)) is a statement about a choice of complex structure, the Hardy split, and not about the sign of the arithmetic form (correction (E7)).

40.8.1. Circularity assessment.

(C1) $\mathrm{Tr}_Z(f * f^\sharp) \geq 0$: Weil's criterion, [RESTATEMENT] (Lemma 30 Thm. 3.3; CCM Cor. 6.3).

(C2) $K_F \geq 0$ / $\kappa_{\mathrm{Pick}} = 0$: Lagarias, [RESTATEMENT]. Thm. 40.5.2 shows it *is* the zero-space form, built from the zeros. As an input it is circular (G6).
(C3) "A fundamental symmetry $\mathfrak{J}$ on H^1 with $\Theta = \frac{1}{2} + A$ skew" (Deninger): if $\mathfrak{J}$ is defined through the zeros (for example $\operatorname{sign}\gamma$), this presupposes real γ and is circular (L39 §7(c)). Off RH, $\mathfrak{J}$ would have to choose a direction inside each hyperbolic plane, which needs the zeros.
(C4) $\iota = \mathrm{id}$ / $\rho^* = \rho$: [RESTATEMENT].
(C5) $\mathrm{G5_K}$ (Def. 40.5.6): [RESTATEMENT] as stated. The forms $1 \mp \mathcal{K}[t]$ are zero-free (Euler product $+\Gamma$), so the *input* is non-circular. A proof by gap or continuity is impossible (§40.5.5(i)).
(C6) Non-circular candidates: (i) a Rosati-type *identity*, a trace equal to an intersection number with a Hodge-index sign, on the square of the arithmetic or scaling site [**62, 70**]; (ii) a canonical-system realization of the exponential line with positive mass from the Euler product (Lemma 37 Thm. 4.1, open). Both are [OPEN].

40.8.2. Numerics: Pick matrices. [NUM] (`lemma40_checks.py` C1–C4, `lemma40_hp.py`; mpmath, 50 digits, rechecked at 100–160 digits, all digits stable). F is computed as $\frac{1}{s} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\psi(\frac{s}{2}) + \frac{\zeta'}{\zeta}(s)$.

- C1: 24 points $x + iy$, $x \in \{0.05, 0.15, 0.35\}$, $y \in \{0, 4, 9, 14.1347, 17, 21.022, 25.0109, 30.4249\}$. All eigenvalues of the Pick matrix are positive. The smallest is 3.2003×10^{-40} and the largest 475.09, so the matrix is severely ill-conditioned (Cauchy-like). Its positivity is a consequence of known zero verification, not evidence for RH.
- C2 (identity check): $K_F(z, z)$ against $\sum_{|\gamma| \leq \gamma_{300}}$ of Eq. (40.5.1) at $z = 0.1,\ 0.2 + 14i,\ 0.3 + 20i$. The differences are 0.003199, 0.003201, 0.003203, against the tail estimate $2\int_{\gamma_{300}}^{\infty} \frac{\log(t/2\pi)}{2\pi t^2} dt = 0.003206$.
- C3 (planted quadruple, $\xi_{\mathrm{toy}} = \xi \cdot \prod(s - \rho_j)$ over $\{\rho_0, \bar{\rho}_0, \rho_0^*, \bar{\rho}_0^*\}$, $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$; Thm. 40.5.2 applies, so $\kappa_{\mathrm{Pick}} = 2$ exactly). Points near the ghost ($y \in \gamma_0 + \{-6, -2, 0, 2, 6\}$ and $-\gamma_0$) give exactly 2 negative eigenvalues: $-275.5, -274.0$ ($\delta = 0.1, \gamma_0 = 40$), $-91.3, -90.0$ ($\delta = 0.25$), $-273.5, -272.0$ ($\delta = 0.1, \gamma_0 = 60$). The C1 points ($y \leq 30.4$) give *none* for $\delta = 0.1$ and one, -1.4839×10^{-33}, for $\delta = 0.25$, $\gamma_0 = 40$. A mixed set containing $0.2 \pm 40i$ and $0.4 \pm 41i$ sees both for $\delta = 0.25$, whose disc contains $0.2 + 40i$, and none for $\delta = 0.1$, whose points lie right of the pole.
- C4: 25 random sets of 14 points in $(0.02, 0.6) \times (-50, 50)$ for the $\delta = 0.25$ toy. The histogram of n_- is $\{0{:}19, 1{:}6\}$, never above $\kappa_{\mathrm{Pick}} = 2$.

Reading. The Krein count is exact, but its *visibility* from a finite sample decays very steeply outside the shadow discs. This is the Pick-side version of Lemma 35's blindness theorem.

40.9. Bridge table: from Weil's (Rosati) positivity to OS reflection positivity

J. W. [t148u] reads Lemma 40 as the bridge between Weil's curve positivity and the reflection positivity RH would need. Here is that bridge, rung by rung. Every entry carries a label and its source. Lemma 41 (in preparation) treats the coincidence of the two stages and is not duplicated here.

rung	statement	label	where
1	Weil: Rosati positivity $\Rightarrow F/\sqrt{q}$ unitary for a positive Hermitian form $\Rightarrow \lvert\lambda\rvert = \sqrt{q}$, RH for curves. Elliptic case: Rosati Gram $\left(\begin{smallmatrix} 2 & a \\ a & 2p \end{smallmatrix}\right) > 0 \iff$ Hasse.	[CITED] (Weil, Milne), [THM] reformulation, [NUM]	Thm. 40.3.1, Prop. 40.3.2, Ex. 40.3.3
$1'$	Signature $(\#\mathrm{Fix}\,\iota + p,\, p)$; positivity $\iff$ $\iota = \mathrm{id}$; $\iota(\alpha) = q/\bar{\alpha}$	[THM]	Lemma 30 Thm. 2.2, Cor. 2.3
2	Krein unitaries: spectrum symmetric under $\lambda \mapsto 1/\bar{\lambda}$; each off-circle pair spans a hyperbolic plane; its negative half is the ghost; at most κ eigenvalues outside the circle in Π_κ	[THM] (finite dim.)	Thm. 40.4.1, Rem. 40.4.4; Lemma 27 Prop. 1.1
$2'$	Krein–Gelfand–Lidskii: definite type cannot leave the circle; exit only by an opposite-sign collision; no ghost before degeneration	[THM] finite dim.; [CITED] [**191**]	Thm. 40.4.2, Cor. 40.4.3
$2''$	Two stages: (a) $\det G_A = 0$, eigenvalue through ∞ with a type flip; (b) opposite-sign collision $(\gamma \pm i\delta)^2$. Separation $\iff$ $v^*Kv \neq 0$ on the degenerating vector; Kepler/Fock-type merge when $v^*Kv = 0$	[THM] in finite models, [NUM]	Lemma 27 Thm. 4.4, 9.5; Thm. 40.6.8; Lemma 41 (in prep.)
$2'''$	ζ: Pick kernel of ξ'/ξ = zero-space Krein form; thresholds $h_{\mathrm{Pick}} = h_{\mathrm{line}} = h_{\mathrm{def}}$	[THM]	Thm. 40.5.2, Thm. 40.5.5; Lemmas 34, 36, 37
3	Bisognano–Wichmann: wedge vacuum KMS at $\beta = 2\pi$; Δ^{it} = boost; $J = \Theta U(R(\pi))$; $B(i\pi) = -I$	[CITED, secondary] (BW via CR94), [THM] for (c)	Prop. 40.6.9; Lemma 31 §6(2); Lemma 39 Thm. 2.3
$3'$	Natural cone $\langle \Omega, aJaJ\Omega\rangle = \lVert\Delta^{1/4}a^*\Omega\rVert^2 \geq 0$ (free); J acts on the spectrum as c_∞, never as $\sharp$	[THM] (from Tomita [CITED])	Lemma 31 Lemma 3.2; Prop. 40.6.10; Lemma 38 Prop. 8.3
$3''$	OS reflection positivity $\langle \theta F, F\rangle \geq 0$ on $E_+ \Rightarrow$ Hilbert space and positive generator	[CITED] [**217**]	Lemma 32 Prop. 2.2 (OS machine)
$3'''$	What ζ has: $\xi(1+it)$ is $\mathrm{KMS}_1 \iff$ FE only; Ξ positive definite (free); Weil's form is a *degenerate* reflection form ($\theta = \sharp$, no half-space; half-lines vacuous; θ-invariant nonnegative domain $\Rightarrow \theta = \mathrm{id}$); covariant natural-cone positivity is only Bochner positivity	[THM]	Lemma 31 Thm. 6.5, Prop. 7.3; Lemma 32 Thm. 5.1, Prop. 6.1; Thm. 40.6.11
4	**Missing rung:** an algebra and cone, built without zeros, in which $\langle a, Ja\rangle$ (natural cone) or $\langle \theta F, F\rangle$ (OS) *equals* Weil's form. Equivalently, a Rosati-type reason the arithmetic Krein form never degenerates ($\mathrm{G5_K}$), or a passive line ($\mathrm{G5_{line}}$)	[OPEN] = G5; [RESTATEMENT] as stated	Def. 40.6.12, Def. 40.5.6; Lemma 37

Reading. Rungs 1–2 are the *curve* side, all proved: positivity there comes from an *ample class* (Rosati, Hodge index), and its failure would show up as hyperbolic planes. Rungs 3–$3'''$ are the *modular/OS* side: positivity there is *free* (Tomita, OS), but it attaches to $J = c_\infty$ or to a half-space that ζ lacks. The gap between rungs 3 and 1 is a conjugation mismatch: the free positivity is for the twist c_∞, and the needed one is for the Rosati $\sharp = c_\infty \circ \mathrm{FE}$ (Props. 40.8.1, 40.6.4, 40.6.10). Rung 4 is the one piece that would give the $\sharp$-form a Hilbert (ample) source. *Analogy flag:* that Kepler regularization and the BW half-turn "are" the passage between the two sides is [ANALOGY] (Rem. 40.6.6, §40.6(B)). The $\{\pm 1\}$ deck/Galois group of the LC cover is [THM], and identifying its sheet swap with Lemma 27's Krein sign flip is [ANALOGY], *not* literal.

40.10. Roadmap

Score = 0.6·significance+0.4·probability.

	item	sig	prob	score
R1	Exact Krein count for the shifted F_h, $h>0$: $\kappa(F_h)=\#\{\beta>\frac{1}{2}+h\}$ (distinct), via a Blaschke/Krein–Langer factorization; completes Thm. 40.5.5 as a count	.5	.8	.62
R2	Rosati-type *identity* on the ζ side: $\mathrm{Tr}_Z(f*f^\sharp)=$ intersection number with a Hodge-index sign (Connes–Consani square)	1.0	.05	.62
R3	Function-field control: Pick kernel of ξ_C'/ξ_C for curves over $\mathbb{F}_q$; check Eq. (40.5.1), $\kappa=0$ and agreement with the Rosati Gram matrix (Prop. 40.3.2); planted fake roots	.4	.9	.60
R4	Event-level identification: each line exit time and eigenvector versus the ghost's hyperbolic plane (planted zeros, Lemma 34/36 code)	.7	.4	.58
R5	Sector version (L38 Conj. 8.5): Pick kernel of $\Lambda'/\Lambda(s,\chi)$ with the $\chi\leftrightarrow\bar\chi$ pairing; 2×2 hyperbolic structure for complex χ	.5	.7	.58
R6	Dictionary between the line's de Branges–Pontryagin space $\mathcal{P}(E)$ (index νN_+) and the zero-space Krein form M (index N_+^{dist}): explicit isometry, multiplicity convention	.6	.5	.56
R7	Arb-certified Pick positivity on fixed point sets (finite; certifies only known zero data)	.2	.95	.50

Nothing on this list supplies G5 except R2, whose probability is low. Routes K1–K3, M1–M3 from §40.6 are scored there. The top new item is M1 (.61, the missing rung, Def. 40.6.12), and the top feasible one is K1 (.58, feeding Lemma 41).

40.11. Open questions

Q1. Do you accept the three-conjugation assignment (Prop. 40.8.1)? The Rosati role is the mirror $\sharp$, the "Galois twist" is c_∞ (reality, J_{mod}), and $-1\in\hat{\mathbb{Z}}^\times$ plays no role in the polarization.

Q2. Should Thm. 40.5.2 (Pick kernel = zero-space Krein form) be the official "Krein face" of Lagarias in Volume 2?[2] Which multiplicity convention do you want: N_+^{dist} (Pick, distinct zeros) or N_+ (Lemmas 34/36, with multiplicity)?

Q3. Should $\mathrm{G5_K}$ (non-degeneracy along the depth deformation) be recorded as an equivalent face of the official $\mathrm{G5_{line}}$, with $\mathrm{G5_{line}}$ kept primary?

Q4. The no-gap theorem (§40.5.5(i)) says non-degeneracy cannot come from an estimate; it needs an identity, as in Weil. Does this match your intuition about where the "geometric reason" must come from?

Q5. Shall I write a short lemma "where ghosts are visible"? It would combine Lemma 33's shadow discs, Lemma 35's blindness and the Pick sampling of C3.

Q6. L_4: is the Krein reading of Routh's criterion in §40.5.2 the level of detail you want, or should the explicit Krein-signature computation of Lemma 27 Addendum B be redone in this lemma's notation?

Q7. Kepler: do you accept the corrected reading of the apsides (Prop. 40.6.1): the Gram determinant is constant, and at the apsides the Gram matrix is diagonal and $|r||v|$ is minimal?

Q8. Do you accept that the Kepler sheet swap $\leftrightarrow$ Krein type flip is [ANALOGY], with the covering statements as [THM] (Rem. 40.6.6)? And should the branch-point convention (0 and ∞ of the frequency cover, deck = FE) be the Vol. 2 language for stages[3] (a)/(b)?

[2] **[Vol. 3 editorial note]** With the approved two-volume split, the series is Vol. I (Lemmas 1–13), Vol. II (Lemmas 14–29) and Vol. III (Lemmas 30–47, this volume). "Volume 2" in this question meant the next volume of the series; for Lemmas 30–47 that is this volume. See Appendix A, item D1.

[3] **[Vol. 3 editorial note]** See the note to the previous question.

Q9. Bridge table (§40.9): is "free positivity is for c_∞, the needed one is for $\sharp = c_\infty \circ \mathrm{FE}$" the formulation of the gap you want carried into Volume 2?

40.12. Corrections recorded

(E1) Chat: "complex conjugation pairs $\frac{1}{2} + i\gamma$ with $\frac{1}{2} - i\gamma$" under the heading "Galois side". This pairing is archimedean c_∞ (reality, J_{mod}), not the Galois element $-1 \in \hat{\mathbb{Z}}^\times$ (L38 Prop. 8.3).

(E2) Chat: "Krein-unitary eigenvalues off the circle in pairs $\lambda, 1/\bar{\lambda}$ — exactly the shape of an off-line zero quadruple". A pair corresponds to a *mirror pair* $\{\rho, \rho^*\}$, which is half a quadruple. A quadruple is two pairs, hence two hyperbolic planes (Remark 40.4.4).

(E3) Chat: "what's missing is a Rosati-type positive involution on the arithmetic H^1". The involution exists (♯, Lemma 30). What is missing is the positivity, equivalently the non-degeneracy, of the associated form *for a geometric reason* (G5).

(E4) Chat: "in Π_κ at most κ such pairs". This is correct, and sharpened to: at most κ eigenvalues outside the circle with multiplicity (Thm. 40.4.1(d)).

(E5) Chat: "Frobenius F, also a Galois element". This is true for curves. Its ζ analogue, the scaling flow, is *not* a Galois element (L38).

(E6) Naming: Lemma 39 §7(c) writes J for the complex structure C. Here $C = i\mathfrak{J}$, to avoid the modular J_{mod}. This is not an error.

(E7) The "$\gamma > 0 \leftrightarrow H^{1,0}$" analogy is refined (Remark 40.8.2): the arithmetic form corresponds to the *polarized* Hodge form, not to h.

(E8) Steering/chat: the Lagrange Gram determinant is "maximal at the apsides". It is constant along an orbit ($= |L|^2$). At the apsides the Gram matrix is diagonal and $|r||v| = |L|$ is minimal (Prop. 40.6.1).

(E9) Chat: the regularized returning branch is "literally" the negative-type pencil eigenvalue. The $\{\pm 1\}$ deck (Galois) group of the LC cover and the sheet change at collision are [THM] (Thm. 40.6.3). The identification with the Krein type flip is [ANALOGY] (Rem. 40.6.6).

(E10) Natural-cone positivity is *not* a Rosati positivity: J commutes with the modular flow, so it acts on the spectrum as c_∞, never as ♯ (Prop. 40.6.10).

CHAPTER 41

Lemma 41: Do the two stages of ghost birth coincide?

Degeneration, collision, and the detuning that separates them

Standalone note `lemma41.tex`, October 22, 2026.

Summary. **Statement of intellectual honesty.** RH is not proved here, and nothing in this lemma supplies G5. Tags: [THM] proved here (or an elementary consequence of cited results, proof given); [CITED] checked at the source (`lit/lemma41/verified_refs.md`); [BIB] bibliographic only; [TAA] a result of an earlier TAA lemma, used read-only; [RESTATEMENT] restatement; [ANALOGY] analogy; [NUM] numerical, high precision, not interval-certified (`num/lemma41_*.py`, data `num/l41/`); [CONJ] conjecture; [OPEN] open.

J. W. asked whether the two stages of ghost birth found in Lemma 27 can coincide: (a) the Krein space degenerates ($\det \mathsf{G}_A = 0$, an eigenvalue passes through ∞ and returns with negative type), then (b) an opposite-signature collision pushes a pair off the axis. Lemma 40 Thm. 6.8 gives the criterion: the stages separate exactly when $v^*Kv \neq 0$ on the degenerating vector. **Answer.** (1) At the *canonical* level the two stages coincide *exactly* ([THM]). On the pair block of a splitting double zero the Krein space degenerates and a mixed-type double eigenvalue occurs at the same instant $\delta = 0$. These are one event seen in two charts related by the blow-up $g = \delta f'$. The same holds in every finite zero model whose test space separates the zero functionals, and in the de Branges line setting once all boundary conditions are allowed. (2) This coincidence is *not* a constraint on ζ. It holds for every zero configuration with the functional-equation and reality symmetry, including Davenport–Heilbronn-type functions, which have off-line zeros. (3) The separation seen in windows is a regularization effect, and we identify exactly what it measures. It is a *detuning*: in the window it is $v^*\mathsf{K}_A v = R(v)(\Omega_v^2 - \gamma^2)$, the leakage of the ghost mode into the other zeros times the mismatch between their mean squared frequency and γ^2. In the line setting it is the angle between the chosen boundary condition and the resonant one. (4) Central numerical test (prediction K1 of Lemma 40): in Lemma 27's window pencils $v^*\mathsf{K}_A v \to 0$ at birth, like $\varepsilon^{1.01}$ (doubly exponentially in h). The normalized detuning, however, does *not* go to 0: it stays $\geq \gamma_2^2 - \gamma_1^2$ (a theorem under the zero-side representation for $k = 1$), and numerically it is $10^{4.5}$–$10^{5.7}$ and grows with N. Birth-point data predict the collision threshold through the cube-root law $\delta_c \simeq (\kappa_v/2\beta_v)^{1/3}$, to within 0.1 decades in every converged window. So both thresholds go to 0 (they "coincide in the limit" in absolute terms), but $\delta_c/\delta_b \to \infty$: the window limit is a leakage limit, not a resonant (Kepler-type) limit.

0. Verdict

setting	do (a) and (b) coincide?	what separates them / status
canonical pair block (§41.2)	*exactly*, at $\delta = 0$	nothing; Thm. 41.2.1 [THM]
finite zero model, test space separating (§41.3)	*exactly*, $\delta_b = \delta_c = 0^+$	nothing; Thm. 41.3.1 [THM], [NUM] precision scaling
finite zero model, compressed	no, $0 < \delta_b < \delta_c$	compression; [NUM] (4 backgrounds, incl. non-arithmetic)
two-mode normal form (§41.4)	iff $\varepsilon\Delta = 0$	leakage $\varepsilon \times$ detuning Δ; Thm. 41.4.1 [THM]
Lemma 27 window (§41.5)	no; both $\to 0$, ratio $\to \infty$	$v^*\mathsf{K}_A v = R(v)(\Omega_v^2 - \gamma^2) > 0$; Prop. 41.5.1 [THM], [NUM]
de Branges line, fixed boundary condition θ (§41.6)	no: $\omega_b(\theta) = \delta_\rho \lvert\cos(\theta - \theta_*)\rvert$	angle $\theta - \theta_*$; Thm. 41.6.3 [THM] (leading order), [NUM]
de Branges line, all θ	*exactly*: $\sup_\theta \omega_b(\theta) = \omega_a$	nothing; Thm. 41.6.2 [THM] (unconditional for ξ)

(i) *Coincide in the limit vs. coincide exactly.* Exactly: at the canonical level, in separating finite models, and in the line setting over all boundary conditions. In the limit only, and only in absolute terms ($\delta_b, \delta_c \to 0$, $\delta_c - \delta_b \to 0$, $\delta_c/\delta_b \to \infty$): in Lemma 27's windows as $h \to \infty$ ([NUM]; $\delta_b \to 0$ is [TAA] Lemma 27 Thm. 9.3, $\delta_c \to 0$ remains [OPEN]).

(ii) *What structural property forces coincidence?* On the pair block the shifted form $K - cG$ ($c = \gamma^2 - \delta^2$) vanishes on the degenerating vector and couples it to the value direction only at order δ^2, because the functional equation makes the $O(\delta)$ term purely imaginary. This is the "single quadratic form controlling both", the analogue of Kepler's energy (Lemma 40 §6, Fock), and it is generic (Thm. 41.2.1). Any exit from the line goes through a multiple zero (Thm. 41.2.3) and, in the normalized limit frame, through a semisimple double eigenvalue of mixed type.

(iii) *What is the gap?* A detuning between the chosen regularization and the resonant one. In windows the detuning is bounded below by the zero gap $\gamma_2^2 - \gamma_1^2$ (for $k = 1$, Cor. 41.5.2), so the window can never become Kepler-type there.

(iv) *Consequence for ζ.* None that bears on RH. The canonical coincidence is a theorem about all FE-symmetric zero configurations. It is not arithmetic and holds for Davenport–Heilbronn-type functions (§41.7). Correction (C4) records that the chat's "structural constraint" does not survive.

41.1. The thresholds, defined in each setting

Throughout, a double on-line zero at height γ splits as $\gamma \pm i\delta$ (zeros $\rho = \frac{1}{2} \pm \delta + i\gamma$ and mirrors), with multiplicity weight m. The Krein-space objects are those of Lemma 27 ([TAA]): a Hermitian pair (K, G), the Krein type $\operatorname{sgn} x^*Gx$ of a real eigenvalue, $\kappa =$ number of negative eigenvalues of G, $n_{\mathbb{C}} =$ number of non-real pairs, $n_- =$ number of negative-type real eigenvalues, and the count identity $\kappa = n_{\mathbb{C}} + n_-$ (Lemma 27 Prop. 4.2).

Definition 41.1.1 (window pencil; [DEF], Lemma 27). For the window $[-h, h]$, parity p, basis size N, and the zero γ_k, let $\mathsf{G}_A(\delta), \mathsf{K}_A(\delta)$ be the window Grams of Q and of $F \mapsto Q(F', F')$ for the replacement model $Q - T_{\gamma_k} + T_{\gamma_k + i\delta} + T_{\gamma_k - i\delta}$. Put

$$\delta_b(h, N, p, k) = \inf\{\delta > 0 : \kappa(\mathsf{G}_A(\delta)) \geq 1\}, \qquad \delta_c(h, N, p, k) = \inf\{\delta > 0 : n_{\mathbb{C}}(\mathsf{K}_A(\delta), \mathsf{G}_A(\delta)) \geq 1\}.$$

Stage (a) happens at δ_b, stage (b) at δ_c, and $\delta_b \leq \delta_c$ (Lemma 27 Thm. 4.4(b)). The limits of interest are $h \to \infty$ (the window grows, Lemma 27's "$A \to \infty$") and $N \to \infty$ (full window space). Both thresholds *decrease* as h grows. See correction (C1): the limit is $h \to \infty$, not $h \to 0$.

Definition 41.1.2 (finite zero model; [DEF]). Fix M background heights $t_j > 0$ and the pair. In the even sector, with test functions $\phi_1, \dots, \phi_N$, let $G = \sum_j T_{t_j} + T_{\gamma+i\delta} + T_{\gamma-i\delta}$ and $K = \sum_j t_j^2 T_{t_j} + \sum_{\pm}(\gamma \pm i\delta)^2 T_{\gamma\pm i\delta}$ (zero-side forms, $T_w(F,G) = \mathrm{Re}(\widehat{F}(w)\widehat{G}(-w) + \widehat{F}(-w)\widehat{G}(w))$, as in Lemma 27 Prop. 3.2). The *rank* is $M+2$, the number of zero functionals. δ_b, δ_c are defined as in Def. 41.1.1. The model is *separating* if $N = M+2$ and the evaluation matrix is invertible for $\delta > 0$, and *compressed* if $N < M+2$.

Definition 41.1.3 (line setting; [DEF], after Lemma 34). For $\omega > 0$ let $E_\omega(z) = \xi(\frac{1}{2} + \omega - iz)$ (the exponent ν of Lemma 34 changes nothing below), $E^{\#}(z) = \overline{E(\bar{z})}$, $\Theta = E^{\#}/E$, and $A_\theta = \frac{1}{2}(e^{i\theta}E + e^{-i\theta}E^{\#})$, $\theta \in [0, \pi)$. Stage (a) is the *degeneration threshold* $\omega_a = \sup\{\omega : E_\omega \text{ has a zero in } \overline{\mathbb{C}_+}\} = \beta^* - \frac{1}{2}$. There E acquires a real zero, $E, E^{\#}$ share it, and the index $\kappa(\omega)$ jumps ([TAA] Lemma 34 §6, Lemma 40 Thm. 5.5). Stage (b) at boundary condition θ is $\omega_b(\theta) = \sup\{\omega : A_\theta \text{ has a non-real zero}\}$, and $\omega_b^* = \sup_\theta \omega_b(\theta)$. In the Hermite–Biehler case the zeros of A_θ are the spectra of the selfadjoint extensions of multiplication by z in the de Branges space [**81**] ([BIB]). For $\kappa > 0$ we use only the zeros of A_θ and leave the operator interpretation [OPEN]. In Lemma 37's dictionary $\theta = 0, \frac{\pi}{2}$ correspond to the two boundary conditions "open" and "short" ([ANALOGY]).

Definition 41.1.4 (canonical pair block; [DEF]). For real-on-$\mathbb{R}$ transforms ($\widehat{F}(\bar{z}) = \overline{\widehat{F}(z)}$) put $a = \widehat{F}(\gamma + i\delta) = f + ig$, with f, g real linear functionals. The pair contributes $G_{\text{pair}} = T_{\gamma+i\delta} + T_{\gamma-i\delta} = 4\,\mathrm{Re}(a^2)$ and $K_{\text{pair}} = 4\,\mathrm{Re}((\gamma+i\delta)^2 a^2)$ (Lemma 27 Prop. 9.1 normalization, $m = 2$ per pair). The canonical object is the pencil $(K_{\text{pair}}, G_{\text{pair}})$ on the two-dimensional coordinate space of (f, g), or of (f, f') with $f' = \widehat{F}'(\gamma)$.

41.2. The canonical level: the two stages are one event

It is convenient to use $u = \delta^2$ as the parameter. The zero configuration depends only on u: for $u > 0$ it is the off-line pair $\gamma \pm i\sqrt{u}$; at $u = 0$ a double on-line zero; and for $u < 0$ two simple on-line zeros $\gamma \pm s$, $s = \sqrt{-u}$. So $\delta = \sqrt{u}$ is the two-sheeted cover branched at $u = 0$ whose deck map $\delta \mapsto -\delta$ fixes the configuration. This is the cover of Lemma 40 Prop. 6.4, restricted to one pair.

Theorem 41.2.1 (canonical coincidence; [THM]). *Let* $f = \frac{1}{2}(\widehat{F}(\gamma + \sqrt{u}) + \widehat{F}(\gamma - \sqrt{u}))$ *and* $\tilde{g} = \frac{1}{2\sqrt{u}}(\widehat{F}(\gamma + \sqrt{u}) - \widehat{F}(\gamma - \sqrt{u}))$, *with* $\sqrt{u} = i\delta$ *for* $u > 0$. *These are real linear functionals for every real* u, *analytic in* u, *with* $\tilde{g} \to f' = \widehat{F}'(\gamma)$ *as* $u \to 0$. *In the* blown-up chart $(f, \tilde{g})$ *the pair block of Def. 41.1.4 is, exactly and for every real* u,

$$G(u) = 4\begin{pmatrix} 1 & 0 \\ 0 & -u \end{pmatrix}, \qquad K(u) = 4\begin{pmatrix} c & -2\gamma u \\ -2\gamma u & -cu \end{pmatrix}, \qquad c = \gamma^2 - u.$$

(a) *For* $u < 0$, $G > 0$ *and the pencil eigenvalues are* $(\gamma \pm s)^2$, *real and of positive (definite) type. For* $u > 0$, $\kappa(G) = 1$ *and the eigenvalues are* $(\gamma \pm i\delta)^2$, *non-real. Hence* $\delta_b = \delta_c = 0$*: the stages coincide* exactly.

(b) *(Stage (a) at* $u = 0$.*)* $G(0)$ *and* $K(0)$ *are both singular on* $v = (0, 1)$, *with* $v^* K(0) v = 0$. *This is the Fock/Kepler-type, merged case* $\kappa = 0$ *of Lemma 40 Thm. 6.8(b) ([TAA]), with the coupling vanishing linearly in* u *instead of being constant. In the original chart* (f, g), $g = \mathrm{Im}\,\widehat{F}(\gamma + i\delta) = \delta\tilde{g}$, *the coordinate map from test functions drops rank at* $u = 0$*: the Krein space degenerates.*

(c) *(Stage (b) at* $u = 0$.*) In the Krein-normalized chart* (f, g) *(*$u > 0$*) one has* $G = 4\,\mathrm{diag}(1, -1)$ *and* $G^{-1}K = \left(\begin{smallmatrix} c & -2\gamma\delta \\ 2\gamma\delta & c \end{smallmatrix}\right)$. *As* $u \downarrow 0$ *this tends to* $(4\,\mathrm{diag}(1, -1), \gamma^2 I)$*: a semisimple double eigenvalue whose two partial multiplicities have* opposite *signs. That is an eigenvalue of mixed type in the sense of* [**191**, Def. 4.6] *(transferred to selfadjoint pencils as in Lemma 27 Thm. 4.3), i.e. an opposite-signature collision.*

(d) (The controlling form.) $K - cG$ has zero diagonal in both charts. Its only entry is the coupling $-8\gamma u$ in $(f, \tilde{g})$, resp. $-8\gamma\delta$ in (f, g). Equivalently, the $O(\delta)$ part of the pair term is purely imaginary (Lemma 27 Prop. 3.3, Lemma 25). This is the functional equation together with reality.

PROOF. For $u > 0$ write $a = \widehat{F}(\gamma + i\delta) = f + ig$; reality gives $\widehat{F}(\gamma - i\delta) = \bar{a}$, so $f = \operatorname{Re} a$ and $g = \operatorname{Im} a = \delta\tilde{g}$. Then $G_{\text{pair}} = 4\operatorname{Re} a^2 = 4(f^2 - g^2)$ and $K_{\text{pair}} = 4\operatorname{Re}((\gamma+i\delta)a)^2 = 4[(\gamma f - \delta g)^2 - (\delta f + \gamma g)^2] = 4[c(f^2 - g^2) - 4\gamma\delta fg]$. Substituting $g = \delta\tilde{g}$, $\delta^2 = u$ gives the displayed matrices. For $u < 0$ the pair is $T_{\gamma+s} + T_{\gamma-s}$, so $G = 2(\widehat{F}(\gamma+s)^2 + \widehat{F}(\gamma-s)^2) = 4(f^2 + s^2\tilde{g}^2)$, and the same computation with weights $(\gamma \pm s)^2$ gives the same polynomial matrices (all entries are polynomials in u, so the identity holds for all real u). (a) $\det(K - \mu G) = 16[(c-\mu)(-cu+\mu u) - 4\gamma^2u^2] = -16u[(\mu-c)^2 + 4\gamma^2 u]$, whose roots are $\mu = c \pm 2\gamma\sqrt{-u} = (\gamma \pm \sqrt{-u})^2$. For $u < 0$, $G > 0$, so every eigenvector has positive G-norm (positive type). (b), (c), (d) are read off. □

REMARK 41.2.2 (two charts, one event; [THM] for the charts, [ANALOGY] for the Kepler reading). Lemma 27's canonical remark ("the double on-line zero is of definite type; it leaves only because the Krein space degenerates") and Thm. 41.2.1(c) ("it leaves through a mixed-type collision") describe the same instant in two charts, related by the singular rescaling $g = \delta\tilde{g}$. In the fixed chart the Krein space degenerates. In the normalized chart the limiting frame is nondegenerate but of mixed signature. The approach from $u < 0$ is a collision of two eigenvalues of the *same* (positive) type. Krein–Gelfand–Lidskii forbids such a pair from leaving the axis inside a fixed Krein space [**191**, Thm. 4.7(1)], and the pair leaves only because the space degenerates at the very moment of collision. The structural match with Kepler is close (Lemma 40 §6). The parameter u plays the energy E: $u < 0$ gives rotations on the line (bound), $u = 0$ is parabolic, where the invariant form degenerates on the same mode on which the stiffness changes sign (Lemma 40 Prop. 6.7), and $u > 0$ gives a boost of rapidity $\sqrt{u}$ (escape). The blow-up $g = \delta\tilde{g}$ is the analogue of the Levi-Civita square root (Lemma 40 Thm. 6.3). *At the canonical level ζ's zero splitting is Kepler-type: degeneration and escape merge.*

THEOREM 41.2.3 (exits only through multiple zeros; [THM]). *Let F_s ($s \in [0,1]$) be entire functions, continuous in s locally uniformly, not identically 0, whose zero sets are invariant under $\rho \mapsto 1 - \bar{\rho}$. If $\rho_0 = \frac{1}{2} + i\gamma_0$ is a* simple *zero of F_{s_0}, then for s near s_0 the function F_s has exactly one zero near ρ_0, and it lies on the critical line. Consequently, a zero can leave the line at (s_0, ρ_0) only if ρ_0 is a zero of F_{s_0} of multiplicity ≥ 2. In the multiplicity-two case the local pair block is that of Thm. 41.2.1 to leading order, so the exit is the merged event (a)=(b).*

PROOF. Choose a disc D centered at ρ_0, invariant under $\rho \mapsto 1 - \bar{\rho}$ (reflection in the line), containing no other zero of F_{s_0}. By Hurwitz's theorem F_s has exactly one zero in D for s near s_0. The reflection maps zeros in D to zeros in D, so this zero is fixed, i.e. on the line. The last sentence follows by expanding the local factor as in Lemma 27 Prop. 3.3. □

Reading. Thm. 41.2.3 is the zero-side counterpart of definite-type stability [**191**, Thm. 4.7(1)]: a simple on-line zero is a rank-one positive block and cannot leave. It is general (it uses only the reflection symmetry), so it says nothing specific about ζ.

41.3. Finite zero models: coincidence when the test space separates

THEOREM 41.3.1 ([THM]). *In a separating finite zero model (Def. 41.1.2) one has $\delta_b = \delta_c = 0$, i.e. $\kappa = 1$ and the pair of eigenvalues $(\gamma \pm i\delta)^2$ is non-real for every $\delta > 0$.*

PROOF. By Lemma 27 Prop. 3.2 ([TAA]), $G^{-1}K = E^{-1}\operatorname{diag}(w_j^2)E$ with w_j the zeros, so the eigenvalues are exactly t_j^2 and $(\gamma \pm i\delta)^2$. And $G = E^*WE$ with W positive on the background and of signature $(1,1)$ on the pair (Thm. 41.2.1), so $\kappa(G) = 1$ by Sylvester. □

[NUM] (`lemma41_finite.py`, `l41/finite_*.json`). Window $h = 1$, even sector, pair at γ_1, four backgrounds of four zeros: ζ $(\gamma_2, \dots, \gamma_5)$, a lattice $\gamma_1 + 8j$, uniform random in $[18, 45]$ (seed 1), and $\{10, 18, 26, 33\}$ (zeros *below and above* γ_1). In the separating case ($N = 6$) the computed thresholds are pure precision floors: at 60 digits $\log_{10}\delta_b \in [-30.4, -28.8]$ (or below the search floor) and $\log_{10}\delta_c \in [-19.1, -18.4]$; at 120 digits $\log_{10}\delta_b \le -60.3$ and $\log_{10}\delta_c \in [-39.1, -38.5]$. Both scale with the working precision, as they must if the true value is 0. At $\delta = 10^{-3}$ the eigenvalue is $199.7904538 \pm 0.0282694503\,i = (\gamma_1 \pm 10^{-3}i)^2$ to all printed digits. *Compressed* models separate:

background	$N=5$: $\log_{10}\delta_b$	$\log_{10}\delta_c$	$N=4$: $\log_{10}\delta_b$	$\log_{10}\delta_c$
ζ	-1.80	-0.78	-1.77	-0.74
lattice	-1.67	-0.63	-1.29	-0.49
random	-1.74	-0.86	-1.42	-0.52
below+above	-2.10	-1.36	-0.29	-0.16

The separation is a compression phenomenon. It is qualitatively the same for arithmetic and non-arithmetic backgrounds.

41.4. The two-mode normal form: what separates the stages

Add to the canonical block of Thm. 41.2.1 the two things a window does to the degenerating direction v. It gives v a positive background ε (*leakage*: the ghost test function cannot vanish on all other zeros), and it gives that background a squared frequency ω^2, which in general differs from c (*detuning*). With the coupling written as βu^k (canonically $k = 1$, $\beta = 2\gamma$), this is Lemma 27 Def. 9.4 when $k = 1$:

$$G = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon - u \end{pmatrix}, \qquad K = \begin{pmatrix} c & -\beta u^k \\ -\beta u^k & \varepsilon\omega^2 - cu \end{pmatrix}, \qquad c = \gamma^2 - u, \quad \Delta = \omega^2 - c.$$

THEOREM 41.4.1 ([THM]). *Here $v = e_2$, $v^*K(u_b)v = \varepsilon\Delta$ at $u_b = \varepsilon$, and $\det(K - \mu G) = 0$ iff $(\varepsilon - u)\nu^2 - \varepsilon\Delta\nu - \beta^2 u^{2k} = 0$, $\nu = \mu - c$. Therefore:*

(a) $u_b = \varepsilon$ (stage (a)), and a non-real pair exists iff $4\beta^2 u^{2k}(u - \varepsilon) > \varepsilon^2\Delta^2$.

(b) The stages coincide, $u_c = u_b$, iff $\varepsilon\Delta = 0$*: either no leakage ($\varepsilon = 0$, the canonical case) or resonant leakage ($\Delta = 0$). This is Lemma 40 Thm. 6.8 ([TAA]) in the present family.*

(c) If $\kappa := \varepsilon\Delta \neq 0$, the threshold $u_c > u_b$ is unique, and as $\kappa \to 0$ with $\varepsilon \ll u_c$,

$$u_c - \varepsilon = \Big(\frac{\kappa}{2\beta}\Big)^{2/(2k+1)}(1 + o(1)).$$

For $k = 0$ (constant coupling, Lemma 40's Fock pencil) this is $\kappa^2/4\beta^2$, quadratic in κ. For $k = 1$ (the ζ case) it is $(\kappa/2\beta)^{2/3}$, i.e. $\delta_c \simeq (\kappa/2\beta)^{1/3}$.

(d) For $k = 1$: $\delta_c/\delta_b \to 1$ iff $\kappa = o(\beta\varepsilon^{3/2})$, i.e. iff $\Delta = o(\beta\sqrt{\varepsilon})$. If Δ stays bounded below while $\varepsilon \to 0$, then $\delta_b, \delta_c \to 0$ but $\delta_c/\delta_b \simeq (\Delta/2\beta)^{1/3}\varepsilon^{-1/6} \to \infty$.

PROOF. Expand the determinant (it is Lemma 27 Thm. 9.5 with $16\gamma^2\delta^4$ replaced by $4\beta^2u^{2k}$). The discriminant in ν is $\varepsilon^2\Delta^2 + 4\beta^2u^{2k}(\varepsilon - u)$. Its second term is negative and decreasing for $u > \varepsilon$, which gives (a), uniqueness and (b). Solving $4\beta^2u^{2k}(u - \varepsilon) = \kappa^2$ with $u \gg \varepsilon$ gives (c). Then (d) follows by taking ratios with $u_b = \varepsilon$. □

COROLLARY 41.4.2 (the minimal separating ingredient; [THM]). *Start from Lemma 40's Fock/Kepler pencil, which merges. (i) Making the coupling vanish like u (the functional equation) keeps the stages merged; this is the canonical ζ block. (ii) Adding leakage $\varepsilon > 0$ with $\Delta = 0$ still keeps them merged. (iii) Detuning $\Delta \neq 0$ together with leakage separates them.* The minimal separating ingredient is the product $\kappa = \varepsilon\Delta$*: detuned leakage. The functional equation does not create the separation, but it changes the gap law from κ^2 to $\kappa^{2/3}$, so the separation becomes much more robust as $\kappa \to 0$.*

[NUM] (`lemma41_twomode.py`, `l41/twomode.json`; $\beta = 2\gamma_1$). With $\Delta = 0$, $\delta_c/\delta_b = 1$ to 10^{-8} for $\varepsilon = 10^{-4}, \dots, 10^{-32}$. With $\Delta = 242 \approx \gamma_2^2 - \gamma_1^2$, $\delta_c/\delta_b = 7.6,\ 35,\ 754,\ 3.5 \cdot 10^5$ for $\log_{10}\varepsilon = -4, -8, -16, -32$, and the cube-root law of (c) holds to < 0.002 decades for $\varepsilon \le 10^{-8}$. At equal $\kappa = 10^{-8}$ the Kepler-type gap is $u_c - \varepsilon = 10^{-15.5}$, while the ζ-type gap is $10^{-6.5}$ (coupling frozen at its value at $\delta = 0.1$ for the comparison).

41.5. The central test K1 in Lemma 27's window pencils

Lemma 40 (route K1) predicted that if Lemma 27's stages merge as the window grows, then $v_A^* \mathsf{K}_A v_A \to 0$ at δ_b. We compute v_A, the null vector of $\mathsf{G}_A(\delta_b)$, in every window, and decompose $v_A^* \mathsf{K}_A v_A$ exactly.

Proposition 41.5.1 (exact decomposition; [THM]). *Let R, R_K be the background Grams (γ_k removed), $G_{\text{pair}}, K_{\text{pair}}$ the pair Grams, and v with $\mathsf{G}_A(\delta_b)v = 0$, $\|v\| = 1$ (L^2-normalized test functions). Put $R(v) = v^{\mathsf{T}} R v$, $\Omega_v^2 = v^{\mathsf{T}} R_K v / R(v)$ and $c = \gamma_k^2 - \delta_b^2$. Then*

$$k_v := v^{\mathsf{T}} \mathsf{K}_A(\delta_b) v = R(v)\,(\Omega_v^2 - c) + X_v, \qquad X_v = v^{\mathsf{T}}(K_{\text{pair}} - c G_{\text{pair}})v = -16\gamma_k \delta_b\, f_v g_v.$$

If the background admits the zero-side representation $R = \sum_{\rho \ne \rho_k} T_\rho$, $R_K = \sum_{\rho \ne \rho_k} \gamma_\rho^2 T_\rho$ with γ_ρ real (the hypothesis of Lemma 27 Thm. 9.3: Weil's explicit formula, RH for the background zeros), then $T_\rho(v) \ge 0$, so Ω_v^2 is a weighted mean of the background γ_ρ^2. In particular $\Omega_v^2 \ge \gamma_2^2$ for $k = 1$.

Proof. $\mathsf{K}_A = R_K + K_{\text{pair}}$ and $0 = v^{\mathsf{T}} \mathsf{G}_A v = R(v) + G_{\text{pair}}(v)$, so $k_v = R_K(v) - cR(v) + (K_{\text{pair}} - cG_{\text{pair}})(v)$. The last term is Thm. 41.2.1(d). For real on-line γ and real test functions, $T_\gamma(F,F) = 2|\widehat{F}(\gamma)|^2 \ge 0$. □

Corollary 41.5.2 (no Kepler-type window limit at the first zero; [THM] given the hypothesis, plus [NUM]). *For $k = 1$, $\Delta_v := k_v/R(v) \ge \gamma_2^2 - \gamma_1^2 + \delta_b^2 - |X_v|/R(v)$. In every computed window $|X_v|/R(v) \le 0.37$ (largest at $h = 0.5$), and $\le 3 \cdot 10^{-3}$ for $h \ge 0.6$. So $\Delta_v \ge 241$. The normalized detuning never approaches resonance, and in the two-mode reduction (Thm. 41.4.1(d)) $\delta_c/\delta_b \to \infty$ as $\varepsilon \to 0$.*

Numerics ([NUM]; `lemma41_vkv.py`, `l41/vkv_{k1,k2,q240}.json`, `lemma41_analyze.py`). The setup is Lemma 27's: Q100 window matrices (dps 120) for $N = 20, 30, 40$, $h = 0.5, \dots, 1.2$, both parities, $k = 1$ (48 windows) and $N = 30$, $k = 2$ (16). In addition we use the Q240 matrices (dps 200) for $N = 60, 80, 100$, $h = 0.9$–1.2, parity 0. We locate δ_b by a secant iteration on $\lambda_{\min}(\mathsf{G}_A)$, with residual $|\lambda_{\min}| \le 10^{-60}$. The vector v comes from inverse iteration. $\kappa_v = k_v / |v^{\mathsf{T}} \partial_u \mathsf{G}_A v|$ (normalization of Thm. 41.4.1), and $\beta_v = |\phi^{\mathsf{T}} \mathsf{K}_A v| / \delta_b^2$ with ϕ the G_A-normalized value-Christoffel direction ($\phi^{\mathsf{T}} \mathsf{G}_A v = 0$ automatically). We then compute δ_c^{pred} from Thm. 41.4.1(a) with these inputs, and δ_c^{meas} from Lemma 27's bisection data (`lemma27_thresholds_*.json`).

(N1) *K1, literal form: confirmed.* In all $48 + 16 + 12$ windows $k_v > 0$. So the reborn eigenvalue returns from $-\infty$ with negative type (Lemma 40 Lemma 6.5), and $v^* \mathsf{K}_A v \ne 0$: the window is ζ-type (separating) at every finite h, as Lemma 40 Thm. 6.8(a) requires. And $k_v \to 0$ with $\log_{10} k_v = s \log_{10}\varepsilon + \text{const}$: $s = 1.11, 1.08$ ($N = 20$), $1.02, 1.03$ ($N = 30$), $1.014, 1.009$ ($N = 40$), 1.001 (Q240, $N = 100$). Hence $k_v \propto \varepsilon^{1.00 \pm 0.01}$, i.e. $\ln(1/k_v) \approx 11.8\, d$ with $d = e^{2h}$ (from the birth law $\ln(1/\delta_b) = 5.90\, d - 14.7$, Q240 $N = 100$, $h = 0.9$–1.2; Lemma 27 had $5.83\, d - 14.0$).

(N2) *K1, normalized form: refuted.* $\Delta_v = k_v/R(v)$ does not go to 0. It ranges from $2.9 \cdot 10^4$ ($N = 20$) to $1.96 \cdot 10^6$ ($N = 100$), and $\Omega_v^2 \ge 2.87 \cdot 10^4 \gg \gamma_2^2 = 441.9$ in every window, consistent with Prop. 41.5.1. At fixed h it grows with N ($\propto N^{1.5\text{–}3.4}$ on $N = 20$–40; $\propto N^{1.2\text{–}1.8}$ on $N = 60$–100). The ghost vector leaks into ever higher zeros as the basis grows. So the window limit is a *leakage* limit ($\varepsilon \to 0$), not a *resonance* limit ($\Delta \to 0$): it is not Kepler-type.

h	N	$\log_{10}\varepsilon$	$\log_{10}\delta_b$	$\log_{10}k_v$	Δ_v	$\lvert X_v\rvert/R(v)$	$\beta_v/2\gamma_1$	$\log_{10}\delta_c^{\text{meas}}$	$\log_{10}\delta_c^{\text{pred}}$
0.5	40	−2.80	−1.40	−0.94	$1.20\cdot10^{5}$	$3.5\cdot10^{-2}$	57.2	−0.36	−0.41
0.6	40	−5.27	−2.63	−3.63	$1.44\cdot10^{5}$	$1.3\cdot10^{-4}$	33.2	−1.12	−1.13
0.7	40	−8.63	−4.31	−7.16	$1.61\cdot10^{5}$	$6.3\cdot10^{-8}$	24.6	−2.18	−2.19
0.8	40	−12.85	−6.43	−11.50	$1.83\cdot10^{5}$	$4.0\cdot10^{-12}$	14.3	−3.53	−3.50
0.9	40	−18.30	−9.15	−16.98	$2.35\cdot10^{5}$	$1.5\cdot10^{-17}$	14.0	−5.26	−5.28
1.0	40	−24.96	−12.48	−23.64	$3.02\cdot10^{5}$	$3.4\cdot10^{-24}$	10.2	−7.42	−7.41
0.9	100	−18.33	−9.17	−16.32	$1.16\cdot10^{6}$	$1.4\cdot10^{-17}$	67.9	$-5.26^{(49)}$	−5.285
1.0	100	−25.07	−12.54	−23.12	$1.31\cdot10^{6}$	$2.6\cdot10^{-24}$	44.5	$-7.44^{(49)}$	−7.452
1.1	100	−33.47	−16.73	−31.51	$1.60\cdot10^{6}$	$1.1\cdot10^{-32}$	34.0	$\le -9.95^{(49)}$	−10.182
1.2	100	−43.82	−21.91	−41.85	$1.96\cdot10^{6}$	$4.8\cdot10^{-43}$	25.3	$\le -11.87^{(49)}$	−13.562

TABLE 41.1. Birth-point data, $k = 1$, parity 0. Superscript (49): Lemma 27's $N = 49$ value (upper bounds where not converged).

(N3) *Birth-point data predict the collision.* Δ_v and β_v grow together with N, and their ratio, which is what enters $\delta_c^{\text{pred}} = (\kappa_v/2\beta_v)^{1/3}$, converges. For $N = 40$, $h = 0.5$–1.2, both parities, $|\log_{10}\delta_c^{\text{meas}} - \log_{10}\delta_c^{\text{pred}}| \le 0.09$ decades (median 0.01). For $N = 20, 30$ the same holds except at four windows (0.19–0.41 decades) where δ_c^{meas} is not N-converged. For $k = 2$ ($N = 30$) the predictions $-1.39, -2.60, -4.33$ (parity 0, $h = 0.7, 0.8, 0.9$) and -0.98 (parity 1, $h = 0.7$) compare with measured bisections $-1.37, -2.64, -4.29$ and -0.97 (`141/k2dc.json`). Here too $\Omega_v^2 \ge 5.3\cdot10^4 \gg \gamma_2^2$: no resonance, even though a lower zero (γ_1) is now present in the background. With Q240 the prediction is N-stable to ≤ 0.02 decades for $h \le 1.1$ ($N = 60/80/100$: $-10.164, -10.178, -10.182$ at $h = 1.1$) and 0.01 at $h = 1.2$ ($N = 80/100$). This gives *new collision thresholds* $\log_{10}\delta_c(1.1) \approx -10.18\pm0.02$ and $\log_{10}\delta_c(1.2) \approx -13.56 \pm 0.02$ (parity 0), where Lemma 27 had only upper bounds. These are predictions validated where checkable, not bisections. Bracket checks at $10^{\pm0.15}\delta_c^{\text{pred}}$ are in `141/dccheck.json`; see §41.5.1.

(N4) *Rates.* Collision law, refitted ($h = 0.7$–1.0 measured, $N = 40$, plus the two new points): $\ln(1/\delta_c) = 3.77\,d - 10.6$ (max residual 0.27), against Lemma 27's $3.64\,d - 9.8$. Slope ratio $a_b/a_c = 5.90/3.77 = 1.57$ (two-mode model: 3/2). Merging index $M = \kappa_v/(2\beta_v\delta_b^3) \simeq (\delta_c/\delta_b)^3$: $\log_{10}M$ rises from 6.4 ($h = 0.5$) to 20.2 ($h = 1.2$, $N = 40$) and 25.0 ($h = 1.2$, $N = 100$). So $\delta_c/\delta_b \to \infty$ doubly exponentially, while $\delta_c - \delta_b \to 0$.

(N5) *Cross term.* $|X_v|/R(v) \le 0.37$ (at $h = 0.5$, parity 1), and it decays doubly exponentially (like δ_b^2, up to a factor $\le 10^2$), so Cor. 41.5.2 applies to every computed window.

Verdict on K1. Both thresholds go to 0, so the stages "coincide in the limit" in absolute terms ([NUM]; $\delta_c \to 0$ unproved). They never coincide in ratio, and they do not merge in the Kepler sense. In the window the Kepler-type merger happens only at $h = \infty$, through the leakage $\varepsilon \to 0$, which turns the window into the canonical block of Thm. 41.2.1. Detuning never goes to zero.

41.5.1. Additional checks. [NUM] (`lemma41_dccheck.py`, `141/dccheck.json`; Q240, dps 200). Direct eigenvalue checks at $\delta = 10^{\pm0.15}\delta_c^{\text{pred}}$. For $h = 1.1$, $N = 60$: $n_{\mathbb{C}} = 0$ at $10^{-10.31}$ and $n_{\mathbb{C}} = 1$ at $10^{-10.01}$, with $\mu = 199.790454832 \pm 2.6105\cdot10^{-9}i$ (asymptotic $2\gamma_1\delta = 2.73\cdot10^{-9}$). So the new threshold is bracketed within ±0.15 decades of the prediction by an actual eigenvalue computation. The same holds for $h = 1.1$, $N = 80$ ($n_{\mathbb{C}} = 0$ at $10^{-10.33}$, $n_{\mathbb{C}} = 1$ at $10^{-10.03}$, $\operatorname{Im}\mu = 2.51\cdot10^{-9}$) and for $h = 1.0$, $N = 60$ ($n_{\mathbb{C}} = 0$ at $10^{-7.60}$, $n_{\mathbb{C}} = 1$ at $10^{-7.30}$, $\operatorname{Im}\mu = 1.32\cdot10^{-6}$; Lemma 27's bisections give -7.42 ($N = 40$) and -7.44 ($N = 49$)). In all three windows the collision threshold therefore lies within ±0.15 decades of δ_c^{pred}. That gives $\log_{10}\delta_c(1.1) \in [-10.31, -10.01]$ ($N = 60$) and $[-10.33, -10.03]$ ($N = 80$), both strictly below Lemma 27's upper bound -9.95.

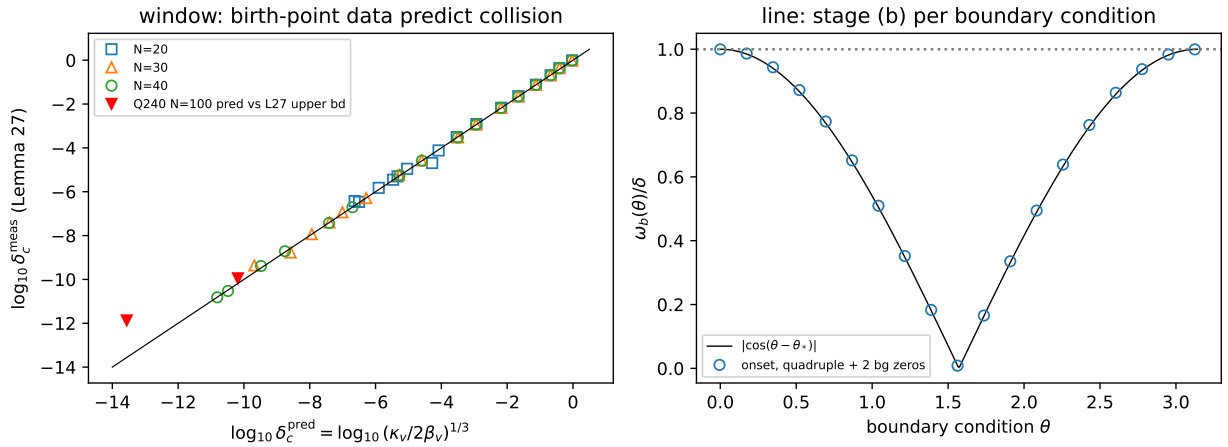

FIGURE 41.1. Left: measured collision thresholds (Lemma 27 bisections) against the two-mode prediction from birth-point data, $k = 1$, both parities; red: Q240 predictions at $h = 1.1, 1.2$ against Lemma 27's upper bounds. Right: line setting, per-boundary-condition onset of stage (b) against Thm. 41.6.3.

41.6. The line setting: boundary conditions and an exact coincidence

THEOREM 41.6.1 (Hermite–Biehler ⇒ no stage (b); [THM]). *If E is entire with $|E^{\#}(z)| < |E(z)|$ on $\mathbb{C}_+$, then every A_θ has only real zeros.*

PROOF. $A_\theta^{\#} = A_\theta$, so the zeros of A_θ are symmetric under $z \mapsto \bar{z}$. At a zero $z \in \mathbb{C}_+$ we would have $|E(z)| = |E^{\#}(z)|$, which contradicts the hypothesis. □

THEOREM 41.6.2 (coincidence over all boundary conditions; [THM]). *Let E be entire, not identically zero, with a zero $z_0 \in \mathbb{C}_+$, and suppose $|\Theta(z_1)| < 1$ at some $z_1 \in \mathbb{C}_+$. Then some A_θ has a non-real zero. For $E_\omega(z) = \xi(\frac{1}{2} + \omega - iz)$ this holds for every $\omega \in (0, \omega_a)$, unconditionally, and therefore*

$$\omega_b^* = \sup_\theta \omega_b(\theta) = \omega_a = \beta^* - \tfrac{1}{2}.$$

PROOF. If $E^{\#}(z_0) = 0$, then $A_\theta(z_0) = 0$ for every θ. Otherwise $\Theta = E^{\#}/E$ is meromorphic on $\mathbb{C}_+$ with a pole at z_0. Join z_1 to z_0 by a path in $\mathbb{C}_+$ and stop at the first point where $|\Theta| \geq 1$. Such a point exists and comes before any pole is reached, since $|\Theta| \to \infty$ at poles. By continuity $|\Theta(z_*)| = 1$, say $\Theta(z_*) = e^{i\alpha}$, and then $A_\theta(z_*) = 0$ for $e^{2i\theta} = -e^{i\alpha}$. For ξ: E_ω has a zero in $\mathbb{C}_+$ iff some $\delta_\rho > \omega$ (zeros at $z = \mp\gamma_\rho + i(\delta_\rho - \omega)$). Also $\Theta(iy) = \xi(\frac{1}{2} - \omega + y)/\xi(\frac{1}{2} + \omega + y)$ by the functional equation, which is < 1 for $y \geq \frac{1}{2} + \omega$ because $\xi > 0$ is increasing on $[1, \infty)$: $\xi'/\xi(\sigma) = \sum_\rho \mathrm{Re}(\sigma - \rho)^{-1} > 0$ for $\sigma \geq 1$, by the Hadamard product with the standard real-part normalization [**80**] ([BIB]). So $\omega_b^* \geq \omega_a$. Conversely, for $\omega > \omega_a$, E_ω is Hermite–Biehler (Lemma 36 Thm. 3.1, Lemma 40 Thm. 5.5; [TAA]), and Thm. 41.6.1 gives $\omega_b(\theta) \leq \omega_a$ for every θ. □

So the line setting has a fourth threshold, $h_{\mathrm{ext}} := \omega_b^*$, equal to Lemma 40's $h_{\mathrm{Pick}} = h_{\mathrm{line}} = h_{\mathrm{def}}$. It is not new information: it is $\beta^* - \frac{1}{2}$ again, and RH $\Leftrightarrow \omega_b^* = 0$ is a [RESTATEMENT].

THEOREM 41.6.3 (per-boundary-condition law; [THM] at leading order, error term sketched). *Let $\rho = \frac{1}{2} + \delta + i\gamma$ be a simple off-line zero with δ small compared with its distance to the other zeros, and write $\eta = \delta - \omega$. Near $z = \gamma$, $E_\omega(z) = (z - \gamma - i\eta)(z - \gamma + i(2\delta - \eta))U(z)$ with U analytic and nonvanishing. Put $\theta_* = -\arg U(\gamma) \bmod \pi$. Then the two zeros of A_θ near γ are non-real iff*

$$\omega < \delta\,|\cos(\theta - \theta_*)| \qquad (\textit{up to } O(\delta^2 \sup |U'/U|)).$$

Hence $\omega_b(\theta) = \delta|\cos(\theta-\theta_*)|$: the resonant boundary condition θ_* (where A_{θ_*} has a double zero at the degeneration) gives exact coincidence, and the gap $\omega_a - \omega_b(\theta) = \delta(1-|\cos(\theta-\theta_*)|) \approx \frac{\delta}{2}(\theta-\theta_*)^2$ is quadratic in the detuning angle. The fraction of boundary conditions in stage (b) at depth ω is $\frac{2}{\pi}\arccos(\omega/\delta)$.

PROOF. Freeze $U = |U|e^{-i\theta_*}$ and put $w = z-\gamma$, $\varphi = \theta-\theta_*$, $b = 2\delta-\eta$. Then $2A_\theta/|U| = e^{i\varphi}(w-i\eta)(w+ib)+e^{-i\varphi}(w+i\eta)(w-ib) = 2[\cos\varphi\,(w^2+\eta b)-\sin\varphi\,(b-\eta)w]$. This has non-real roots iff $\sin^2\varphi\,(b-\eta)^2 < 4\eta b\cos^2\varphi$. With $b-\eta = 2(\delta-\eta)$ and $\eta b = \delta^2-(\delta-\eta)^2$ this is $(\delta-\eta)^2 < \delta^2\cos^2\varphi$, i.e. $\omega < \delta|\cos\varphi|$. *Error term (sketch):* A_θ is real entire, so the two zeros near γ (Rouché, for δ small) are either real or conjugate. Replacing $U(z)$ by $U(\gamma)$ on $|w| \lesssim 2\delta$ perturbs the coefficients by $O(\delta\sup|U'/U|)$ relative to their size. That moves the boundary of the region by $O(\delta^2\sup|U'/U|)$ in ω wherever the discriminant crosses zero transversally. □

[NUM] (`lemma41_lineq.py`, `l41/lineq*.json`; non-arithmetic polynomial controls $p(s) = ((s-\delta)^2+\gamma^2)((s+\delta)^2+\gamma^2)\prod_j(s^2+t_j^2)$). For four configurations (with 0–4 on-line background zeros) no A_θ has non-real zeros for $\omega/\delta \geq 1.01$, and some A_θ does for every $\omega/\delta \leq 0.9999$. The single exception is one case at $\omega/\delta = 1.0001$ (degree 12, $\gamma = 30$), which is attributed to root-finding tolerance. For $(\delta,\gamma) = (0.1, 14)$ with two background zeros, the per-θ onset matches $\delta|\cos(\theta-\theta_*)|$ with best-fit $\theta_* = -2\cdot10^{-4}$, max error 0.0052δ over 19 angles (grid $5\cdot10^{-4}\delta$). The stage-(b) fraction matches $\frac{2}{\pi}\arccos(\omega/\delta)$: 0.094/0.090, 0.293/0.287, 0.514/0.506, 0.669/0.667, 0.812/0.806, 0.934/0.936 at $\omega/\delta = 0.99, 0.9, 0.7, 0.5, 0.3, 0.1$ (181 angles).

REMARK 41.6.4 (boundary conditions as regularizations; [ANALOGY]). A fixed θ is a choice of selfadjoint extension, i.e. a boundary condition at the end of the line. In Lemma 37's dictionary, $\theta = 0$ and $\theta = \frac{\pi}{2}$ are the two break types "open" and "short", and Lemma 36 Thm. 7.9 says both eventually occur, $\kappa/2$ each ([TAA]). Whether the open/short selection is the θ_*-detuning of Thm. 41.6.3 is [OPEN].

41.7. Consequences: what the gap measures, and what it does not say about ζ

(1) Coincidence is generic, not arithmetic ([THM]). Thms. 41.2.1, 41.2.3 and 41.3.1 use only the reflection symmetry $\rho \mapsto 1-\bar\rho$ together with reality. Thm. 41.6.2 uses only a zero in $\mathbb{C}_+$ and one point with $|\Theta| < 1$. All of them apply verbatim to zero configurations with genuine off-line zeros, such as the Davenport–Heilbronn function [**79**], [**269**, §10.25] (it satisfies a Riemann-type functional equation and has zeros with $\sigma > 1$; TAA usage as in Lemma 36 Rem. 7.3). For the DH line, Thm. 41.6.2 needs the monotonicity input; we have not verified it ([OPEN]). So *exact canonical coincidence cannot be a constraint that distinguishes* ζ, and it gives no route to RH. The chat's "structural constraint on ζ" (correction C4) becomes a structural fact about the reflection symmetry.

(2) What the window gap measures ([THM] for the normal form, [NUM] for the window). By Thm. 41.4.1 and Prop. 41.5.1,

$$\delta_c^3 \simeq \frac{\kappa_v}{2\beta_v} = \frac{R(v)\,(\Omega_v^2-\gamma^2)+X_v}{2\beta_v\,|v^{\mathsf T}\partial_u \mathsf{G}_A v|}.$$

This is leakage times detuning. The leakage $R(v) \approx \varepsilon\,|v^{\mathsf T}\partial_u \mathsf{G}_A v|$ is Lemma 27's derivative Christoffel function, the window's ability to isolate $\widehat{F}'(\gamma)$ from all other zeros (the hole effect of Lemmas 11–12). The detuning $\Omega_v^2-\gamma^2$ is how far the zeros that the ghost leaks into sit, in mean squared frequency, from γ^2. The gap is therefore a *spectral-resolution* quantity of the regularization. It depends on ζ only through the local zero configuration (e.g. the lower bound $\gamma_2^2-\gamma_1^2$), and not through the Euler product.

(3) Non-ζ controls. We ran the finite models with lattice, random and below-and-above backgrounds (§41.3), and the line setting with polynomial quadruples (§41.6). All show the same

coincidence/separation pattern. A window computation for the Davenport–Heilbronn function would need its explicit-formula window matrices. We did not do it, and expect nothing new from it by (1)–(2).

(4) A detuning principle ([CONJ], unifying the three settings).

setting	regularization	detuning	gap law
window (Lemma 27)	finite window, basis	$\Delta_v = \Omega_v^2 - \gamma^2$ (times leakage ε)	$\delta_c \simeq (\varepsilon\Delta/2\beta)^{1/3}$
line (Lemma 34)	boundary condition θ	$\theta - \theta_*$	$\omega_a - \omega_b \approx \frac{\delta}{2}(\theta - \theta_*)^2$
Fock/Kepler (Lemma 40)	$\kappa = v^* K v$	κ	$e_c \approx \kappa^2/4c^2$
canonical	none	none	0

Conjecture. Every regularization of the canonical (Kepler-type) pair block that adds a positive term on the degenerating direction separates the stages by an amount controlled by its detuning from resonance, with the exponent fixed by the order at which the coupling vanishes (Thm. 41.4.1(c)). This is proved in the two-mode family and in the local line model.

41.8. Next steps (score $= 0.6\cdot$**significance**$+0.4\cdot$**probability**)

	step	sig	prob	score
D1	Prove the detuning principle in general (Conj. in §41.7(4)): a Schur-complement reduction of an N-dimensional pencil to Thm. 41.4.1, with (κ_v, β_v) as the exact invariants. That would turn the ≤ 0.1-decade agreement into a theorem	.5	.6	.54
D2	Line boundary conditions $\leftrightarrow$ opens/shorts: test whether Lemma 36 Thm. 7.9's eventual open/short exits are the $\theta = 0, \frac{\pi}{2}$ cases of Thm. 41.6.3 on planted lines	.5	.5	.50
D3	Arb-certify one window: inertia at $0.99\delta_b, 1.01\delta_b$ and $n_{\mathbb{C}}$ at $10^{\pm 0.15}\delta_c^{\text{pred}}$ for $h = 1$	.3	.8	.50
D4	Engineer a resonant regularization at $k \geq 2$ (constrain the test space so that $\Omega_v^2 \approx \gamma_k^2$), showing that window coincidence can be produced at will, which confirms that it is not arithmetic	.3	.7	.46
D5	Prove $\delta_c \to 0$ for the full window ($\delta_b \to 0$ is Lemma 27 Thm. 9.3) by perturbing from the separating finite model	.4	.5	.44
D6	Davenport–Heilbronn window matrices (explicit formula for DH)	.3	.6	.42
D7	Look for arithmetic in the detuning (Δ_{eff} against zero spacing)	.3	.4	.34

Recommendation: the question J. W. raised is answered (exact coincidence canonically, generic and not arithmetic). The thread should return to the Lemma 40 polarization line, which is the standing priority. D1 and D2 are the only follow-ups with structural content, and D3 is cheap.

41.9. Open questions

Q1. Do you accept the verdict that the two stages coincide *exactly* at the canonical level, that this is a theorem for every reflection-symmetric zero configuration (Davenport–Heilbronn included), and that it is therefore not a constraint on ζ?

Q2. Vocabulary for Volume 2: "degeneration in the fixed chart = mixed-type collision in the blown-up chart" (Remark 41.2.2), with $u = \delta^2$ as the Kepler energy. Is this the reading of your "regularized singular point / returning branch" intuition that you want recorded?

Q3. Should "detuning" (leakage $\times$ mismatch; boundary-condition angle) become the official name for what separates the stages, and the detuning principle a target (D1)?

Q4. Is the open/short boundary-condition link (D2) worth a lemma, given the Lemma 36 escape theorem?

Q5. Close this thread and return to Lemma 40's polarization line?

41.10. Corrections recorded

(C1) *Direction of the limit.* The task phrase "$\delta_b, \delta_c \to 0$ as $h \to 0$" is reversed. Lemma 27's thresholds *decrease as h grows*, and the limit is $h \to \infty$ (window growing; Lemma 27 Addendum A).

(C2) *Lemma 40 K1, sharpened.* "Merging goes with $v_A^* \mathsf{K}_A v_A \to 0$" holds only for the unnormalized quantity, which tends to 0 like $\varepsilon^{1.01}$ because the leakage $R(v)$ does. The normalized detuning $\Delta_v = k_v / R(v)$ does not tend to 0 ($\geq \gamma_2^2 - \gamma_1^2$ for $k = 1$), so the window limit is not Kepler-type. Lemma 40's gap formula $e_c \approx \kappa^2/4c^2$ assumes constant coupling. With the ζ-type coupling $\propto \delta^2$ the law is $\delta_c \simeq (\kappa/2\beta)^{1/3}$ (Thm. 41.4.1(c)). This is a refinement, not an error in Lemma 40.

(C3) *Lemma 27 canonical remark, refined.* "Definite type; leaves only because the Krein space degenerates" is the fixed-chart description. In the Krein-normalized chart the limiting event is a semisimple double eigenvalue of *mixed* type. Both describe one event (Thm. 41.2.1, Remark 41.2.2).

(C4) *Chat claim.* "If the two stages could be shown to coincide, that would be a structural constraint on ζ": they do coincide canonically, but by a theorem valid for all reflection-symmetric configurations. So it is no constraint on ζ.

(C5) *Lemma 27 Thm. 9.5 (model) is an exact special case* ($k = 1$, $\beta = 2\gamma$) of Thm. 41.4.1. No change is needed there.

Lemma 27, 34, 36, 37 and 40 sources were read and not edited.

41.11. Summary of claims

claim	status	where
canonical pair block is Kepler-type in $u = \delta^2$; $\delta_b = \delta_c = 0$; degeneration = mixed-type collision	[THM]	Thm. 41.2.1
exits only through multiple zeros (reflection symmetry)	[THM]	Thm. 41.2.3
separating finite models: $\delta_b = \delta_c = 0$	[THM], [NUM]	Thm. 41.3.1
two-mode classification; coincide iff $\varepsilon\Delta = 0$; gap law $\kappa^{2/(2k+1)}$	[THM]	Thm. 41.4.1
$k_v = R(v)(\Omega_v^2 - c) + X_v$; $\Omega_v^2 \geq \gamma_2^2$ ($k = 1$, zero-side rep.)	[THM]	Prop. 41.5.1
window: $k_v \propto \varepsilon^{1.01}$; $\Delta_v \geq 241$ (Prop.), measured $\geq 2.9 \cdot 10^4$, grows with N; δ_c^{pred} within 0.1 dec.	[NUM]	§41.5
line: $\sup_\theta \omega_b = \omega_a$ (unconditional for ξ)	[THM]	Thm. 41.6.2
line: $\omega_b(\theta) = \delta\lvert\cos(\theta - \theta_*)\rvert$	[THM] (leading order), [NUM]	Thm. 41.6.3
coincidence is not arithmetic (applies to DH-type)	[THM] (DH monotonicity [OPEN])	§41.7
detuning principle	[CONJ]	§41.7(4)
$\delta_c \to 0$ in full windows	[OPEN] ([NUM] evidence)	§41.5
RH	not addressed beyond [RESTATEMENT]	–

CHAPTER 42

Lemma 42: The missing rung

Half-sided modular inclusions, the orthogonality pincer, the hyperbolic dictionary, and Taylor–Wiles modularity at the nexus

Standalone note `lemma42.tex`, October 22, 2026 (v2).

Summary. Statement of intellectual honesty. RH is not proved here, and nothing in this lemma supplies the missing axiom G5. Every claim carries a tag: [THM] proved here (or an elementary consequence of cited results, with the proof given); [CITED] checked at the source (`lit/lemma42/verified_refs.md`); [BIB] bibliographic record verified, content stated as standard background; [RESTATEMENT] equivalent to RH, hence no progress; [ANALOGY] analogy only; [SKETCH] formal computation, domain questions not settled; [NUM] numerical (`num/lemma42_*.py`, data `num/l42/`); [CONJ] conjecture; [OPEN] open. Lemmas 25, 31, 32, 39, 40 are used read-only. Lemma 41 (STATE §50, in preparation by another worker) is not used.

The question. Lemma 40 (§9, rung 4; Def. 6.12) isolated the *missing rung*: an algebra and a state, built without zeros, in which the free positivity $\langle a, Ja\rangle$ equals Weil's form. It also proved that when the scaling flow is the *modular* flow, J acts as the twist c_∞ and the free positivity reduces to Bochner positivity, so matching Weil is RH (L40 Prop. 6.10, Thm. 6.11). The lead hypothesis tested here: *put log-scale on the translation of a half-sided modular inclusion (HSMI) instead; then J reverses it and acts as the Rosati-type mirror $\sharp = c_\infty \circ \mathrm{FE}$.*

Results.

(1) *The geometric half of the lead is true* (Thm. 42.1.2, [THM]): in an HSMI, J acts on translation frequencies by $p \mapsto \bar{p}$, which is $\sharp$-type.

(2) *The positivity half is false, unconditionally.* The free positivity attached to a flow-reversing J is a *Hankel* (Osterwalder–Schrader) form with a completely monotone kernel and an arrow of time (Thm. 42.2.1). Weil's form is *Toeplitz*; no such form equals it (Thm. 42.2.2, prime atoms). Moreover, a Stone–von Neumann argument shows that in every HSMI the translation generator has Lebesgue spectrum off its kernel (Thm. 42.3.1). So the zeros can never be HSMI energies, *even under RH* (Cor. 42.3.2). The lead is not "equivalent to RH"; it is impossible. This generalizes L31 Prop. 6.3 and L39 Thm. 7.5.

(3) *The orthogonality pincer* (Thm. 42.4.4, [THM]): an involution that commutes with the scaling flow is a reflection *in* the scaling geodesic (c_∞; Toeplitz/Bochner; RH needed for the match). One that reverses it is a reflection in an *orthogonal* geodesic ($\sharp$; Hankel; never Weil's form). No single J does both. The corrected target in the second placement is Cramér's Hankel form (L32 Thm. 7.2), whose positivity is RH [RESTATEMENT]. It cannot be realized with positive energy (Thm. 42.3.1), and only the Möbius placement remains ([OPEN], circular construction).

(4) *Hyperbolic dictionary* (§42.4, [THM] for the identities, [ANALOGY] for readings). It covers the Klein four-group $\{1, c_\infty, \sharp, \mathrm{FE}\}$ as reflections and a half-turn; rapidity; Bisognano–Wichmann; KMS at 2π; and the Wick rotation $\Delta^{1/4}$. Every rung of the Lemma 40 bridge table is mapped onto it (§42.5). Rung 1 becomes: Hasse $\iff$ normalized Frobenius is an *elliptic* element of $\mathrm{SL}_2(\mathbb{R})$.

(5) *Taylor–Wiles at the nexus* (§42.6). Hecke's FE *is* the Fricke half-turn, and ♯ is reflection in the geodesic $|z| = N^{-1/2}$, orthogonal to the scaling geodesic ([THM]). The Petersson space with the Fricke conjugation gives $\langle f, J^\varepsilon f\rangle = \varepsilon_f \|f\|^2$: a *root-number form*, indefinite on $S_2(\Gamma_0(37))$, where the negative direction is the $\varepsilon = -1$ form with its forced on-line zero ([THM], [NUM]). Multiplicity one and the absence of a scaling flow give an obstruction parallel to L40 Thm. 6.11 (Thm. 42.6.5). Theta inversion has the orthogonal-reflection geometry and lands in the Hankel/Möbius placement with spectrum $\frac{1}{4} + \mathbb{N}$ (Prop. 42.6.7). Modularity (Wiles, Taylor–Wiles, BCDT) supplies ♯ and local unitarity (Eichler–Shimura plus Weil) for $L(E, s)$, but not global positivity: *the missing rung recurs.*

Version 2 (October 22, 2026). Two sections were added at J. W.'s request; v1 content is unchanged except for cross-references and the lists below. (i) §42.7, *Angle versus rapidity is Hilbert versus Krein*: Prop. 42.7.1 (the real model, and why the Weil pairing gives only the Krein form), Thm. 42.7.2 (invariant Hermitian forms of an elliptic, hyperbolic or parabolic g; the polarization $\omega(\cdot, J_g\cdot)$), Cor. 42.7.3 (one quantity for both stages of ghost birth), Prop. 42.7.5 (arithmetic Krein margin $\geq \sqrt{3}/2\sqrt{p}$; semisimplicity), Rems. 42.7.4, 42.7.6. (ii) §42.8, *The tropical form*: Thms. 42.8.1, 42.8.2, 42.8.4, 42.8.5, Ex. 42.8.6, Rem. 42.8.7, Prop. 42.8.8, Rem. 42.8.9. New numerics: `num/lemma42_v2.py`, `num/l42/v2_tropical_krein.json`. New corrections of chat claims: (E9)–(E12). New questions: Q7–Q9. New references (verified, `lit/lemma42/verified_refs.md`): Maz72, Maz73, BO78, Yao22, Elk87, Sil09, Den05.

42.1. Conventions and the three placements of log-scale

Conventions follow Lemma 40 §1. A zero is $\rho = \frac{1}{2} + i\tau$, with $\tau = \gamma - i\delta$ complex off the line ($\delta = \operatorname{Re}\rho - \frac{1}{2}$). The three conjugations (L38 Prop. 8.3, L40 Prop. 7.1) act as follows:

$$c_\infty :\ \rho \mapsto \bar\rho\ (\tau \mapsto -\bar\tau), \qquad \mathrm{FE} :\ \rho \mapsto 1-\rho\ (\tau \mapsto -\tau), \qquad \sharp = c_\infty \circ \mathrm{FE} :\ \rho \mapsto 1-\bar\rho\ (\tau \mapsto \bar\tau),$$

and $-1 \in \hat{\mathbb{Z}}^\times$ acts as a parity scalar. Weil's form is $Q(F,F) = \iint \overline{F(a)}F(b)\, w(b-a)\, da\, db$ on $F \in C_c^\infty(\mathbb{R})$ (log-scale variable). It is *Toeplitz*: w is an even distribution with prime atoms $-\Lambda(n)n^{-1/2}(\delta_{\log n} + \delta_{-\log n})$, the pole kernel $2\cosh(u/2)$, and the archimedean part (L25 Prop. 6.3; Lemma 17). Under RH, $Q(F,F) = \sum_\rho m_\rho |\widehat{F}(\gamma_\rho)|^2$. Cramér's form (L32 Thm. 7.2(iii)) is the *Hankel* form

$$\mathcal{R}(f) = \iint \overline{f(a)}f(b)\,\mathrm{Sp}(a+b)\, da\, db, \qquad \mathrm{Sp}(t) = \sum_{t_\rho > 0} m_\rho e^{-\gamma_\rho t}, \quad f \in C_c^\infty(0,\infty),$$

and RH $\iff$ $\mathcal{R} \geq 0$ $\iff$ Sp completely monotone (CM) (L32 Thm. 7.2(v), [RESTATEMENT]).

Definition 42.1.1 (three placements; [DEF]). Let (M, Ω) be standard with modular data (Δ, J).

(P1) *Modular placement* (Lemma 40): log-scale translation is the modular flow Δ^{it}.

(P2) *Translation placement* (the lead): log-scale translation is a Borchers translation $U(a) = e^{iaP}$, $P \geq 0$, $U(a)MU(-a) \subset M$ for $a \geq 0$, $U(a)\Omega = \Omega$. Equivalently (Wiesbrock), it comes from an HSMI. Conventions as in L39 Thm. 7.2: $\Delta^{it}U(a)\Delta^{-it} = U(e^{-2\pi t}a)$ and $JU(a)J = U(-a)$ [CITED] (Bor92, Wie93 via CF20).

(P3) *Endomorphism placement*: log-scale translation T_u, $u \geq 0$, is a semigroup of endomorphisms of M reversed by J ($JT_uJ = T_{-u}$), with *no* positivity assumption on its generator. Model: a Möbius-covariant chiral net, $M = \mathcal{A}(-1,1)$, T_u = dilation $x \mapsto e^{-u}x$ (§42.4).

Theorem 42.1.2 (frequency action; [THM]). *Let J be antiunitary and $U(a) = e^{iaP}$ unitary (P self-adjoint; in a Krein model, let e_p be a generalized eigenvector with $p \in \mathbb{C}$).*

(a) If $JU(a)J = U(-a)$, then $JPJ = P$, and J maps the frequency-p mode to the frequency-$\bar p$ mode. This is ♯-type ($\tau \mapsto \bar\tau$).

(b) If $JU(a)J = U(a)$ (e.g. (P1), $U = \Delta^{i\cdot}$), then J maps $p \mapsto -\bar{p}$. This is c_∞-type (L40 Prop. 6.10).

(c) In (P2), the geometric claim of the lead is therefore true*: J acts on translation frequencies as $\sharp$. More generally,* any *antiunitary V with $VMV = M$ and $V\Omega = \Omega$ satisfies $V\Delta^{it}V = \Delta^{-it}$, so it acts as $\sharp$ on the modular frequencies. But such V (e.g. a time reversal preserving the wedge) carries no free positivity, because free positivity needs a map $M \to M'$.*

PROOF. (a) $Je^{iaP}J = e^{-ia\,JPJ}$ by antilinearity; equating with e^{-iaP} gives $JPJ = P$. Then $U(a)Je_p = JU(-a)e_p = J(e^{-iap}e_p) = e^{ia\bar{p}}Je_p$. (b) Same computation with $U(a)$ in place of $U(-a)$. (c) For $x \in M$: $S(VxV\Omega) = (VxV)^*\Omega = Vx^*V\Omega = VSx\Omega$, so $VSV = S$. By uniqueness of the polar decomposition $S = J\Delta^{1/2}$, $V\Delta^{1/2}V = \Delta^{1/2}$, hence $V \log\Delta\, V = \log\Delta$ and, by antilinearity, $V\Delta^{it}V = \Delta^{-it}$. Apply (a) to $U(t) = \Delta^{it}$. [NUM]: C5 in `num/142/checks.json` (symbolic, true). □

42.2. What the free positivity is in the translation placement

THEOREM 42.2.1 (free positivity is Hankel/OS; [THM], Widder [CITED] via NO18 Thm. 4.1.1). *In (P2), let $\psi \in M\Omega$ and, for $F \in C_c^\infty(0,\infty)$, $\xi_F = \int F(a)U(a)\psi\, da$. Then:*

(a) $\Delta^{1/2}U(a) = U(-a)\Delta^{1/2}$ on $M\Omega$ for $a \geq 0$.

(b) $N(F) := \langle \xi_F, \Delta^{1/2}\xi_F\rangle = \iint \overline{F(a)}F(b)\, h(a+b)\, da\, db \geq 0$, with $h(t) = \langle \psi, U(-t)\Delta^{1/2}\psi\rangle$. So N is a Hankel *form with a positive-definite Hankel kernel.*

(c) h is bounded and continuous on $[0,\infty)$, hence (Widder) completely monotone: $h(t) = \int_{[0,\infty)} e^{-\lambda t} d\mu(\lambda)$, $\mu \geq 0$.

(d) (Arrow of time.) For $T_cF = F(\cdot - c)$, $c \mapsto N(T_cF)$ is nonincreasing on $[0,\infty)$.

So in (P2) the natural cone, OS reflection positivity and the spectrum condition are the same positivity.

PROOF. (a) For $x \in M$ and $a \geq 0$, $x_a := U(a)xU(-a) \in M$, so $SU(a)x\Omega = Sx_a\Omega = x_a^*\Omega = U(a)Sx\Omega$. With $S = J\Delta^{1/2}$: $J\Delta^{1/2}U(a) = U(a)J\Delta^{1/2}$, hence $\Delta^{1/2}U(a) = JU(a)J\,\Delta^{1/2} = U(-a)\Delta^{1/2}$. (b) Since $\psi \in M\Omega \subset D(\Delta^{1/2})$, $\langle U(a)\psi, \Delta^{1/2}U(b)\psi\rangle = \langle U(a)\psi, U(-b)\Delta^{1/2}\psi\rangle = h(a+b)$. Positivity: $\Delta^{1/2} \geq 0$. (c) $|h(t)| \leq \|\psi\|\|\Delta^{1/2}\psi\|$. Widder's theorem: a bounded continuous h on $[0,\infty)$ with $h(a+b)$ positive definite is a Laplace transform of a positive measure on $[0,\infty)$ (NO18 Thm. 4.1.1, carried from Lemma 32). (d) $N(T_cF) = \int e^{-2\lambda c}|\mathcal{L}F(\lambda)|^2 d\mu(\lambda)$. □

THEOREM 42.2.2 (Weil no-go for the translation placement; [THM], unconditional). *No form N as in Thm. 42.2.1 (indeed no form $\iint \bar{F}(a)F(b)h(a+b)$ with h a distribution on $(0,\infty)$) equals Weil's form Q on $C_c^\infty(0,\infty)$. Nor is the Hankel form with Weil's kernel,* $\mathrm{RP}_w(f) = \iint \bar{f}(a)f(b)\, w(a+b)$, *positive.*

PROOF. Polarize: $\iint \bar{F}(a)G(b)h(a+b+2c)$ would equal $\iint \bar{F}(a)G(b)w(b-a)$ for all $c \geq 0$ and all $F, G \in C_c^\infty(0,\infty)$. Differentiating in c gives $h' = 0$ on $(0,\infty)$, so $h \equiv h_0$, and then $w(b-a) = h_0$ on all of $\mathbb{R}$ (differences of points of $(0,\infty)$ cover $\mathbb{R}$). But w carries the atom $-\frac{\log 2}{\sqrt{2}}\delta_{\log 2}$ and is not constant. The second claim is L25 Prop. 6.3(b): a bump at $\frac{1}{2}\log 2$ gives $\mathrm{RP}_w \to -\infty$. Structurally, the Hankel operator of an atom δ_{t_0} on $L^2(0,t_0)$ is the flip $f(u) \mapsto f(t_0-u)$, with equal numbers of eigenvalues ± 1 ([NUM] C9: 200/200). □

REMARK 42.2.3 (the right target in (P2)/(P3); corrects L40 Def. 6.12 there). For a flow-reversing J, the natural target is not Q but a *Hankel* form whose positivity is RH. Cramér's $\mathcal{R}$ is exactly such a form (L32 Thm. 7.2(v), [RESTATEMENT]). So the missing rung splits: (P1) $\langle a, Ja\rangle = Q$ (L40 Def. 6.12, Thm. 6.11), versus (P2)/(P3) $\langle a, Ja\rangle = \mathcal{R}$. The next section shows that the second version is impossible in (P2).

42.3. The Stone–von Neumann dichotomy: zeros are never HSMI energies

THEOREM 42.3.1 (spectral dichotomy for Borchers pairs; [THM] from Stone–von Neumann [BIB] vN31, Mac49). *In (P2), let $\mathcal{H}_0 = \ker P$ and $\mathcal{H}_1 = \mathcal{H}_0^\perp$. Then:*

(a) $\mathcal{H}_0, \mathcal{H}_1$ reduce Δ^{it}, U and J. $U = 1$ on $\mathcal{H}_0$, so there J commutes with the translations and acts as in Thm. 42.1.2(b).

(b) On $\mathcal{H}_1$, $K := \log P$ and Δ^{it} satisfy $\Delta^{it} K \Delta^{-it} = K - 2\pi t$. The pair (e^{isK}, Δ^{it}) is a Weyl pair, so $\mathcal{H}_1 \cong L^2(\mathbb{R}) \otimes \mathcal{K}$ with the Schrödinger representation.

(c) Hence $P|_{\mathcal{H}_1}$ and $\log \Delta|_{\mathcal{H}_1}$ have homogeneous Lebesgue spectrum ($(0,\infty)$ and $\mathbb{R}$ respectively). P has no eigenvalue except possibly 0, and the point spectrum of Δ lives in $\mathcal{H}_0$.

PROOF. $\Delta^{it} P \Delta^{-it} = e^{-2\pi t} P$ (differentiate the Borchers relation), so Δ^{it} preserves $\ker P$; $JPJ = P$ (Thm. 42.1.2(a)) does too. On $\mathcal{H}_1$, $P > 0$ and functional calculus gives $\Delta^{it} \log P \, \Delta^{-it} = \log P - 2\pi t$. So $\Delta^{it} e^{isK} \Delta^{-it} = e^{-2\pi i s t} e^{isK}$, the Weyl relations. Stone–von Neumann (vN31; Mackey Mac49) gives a multiple of the Schrödinger pair, in which both generators have Lebesgue spectrum. An eigenvector of P with eigenvalue $p > 0$ would be an eigenvector of K. The same argument is NO18 Thm. 4.4.1's proof (Lax–Phillips), carried. □

COROLLARY 42.3.2 (the lead fails, unconditionally; [THM]). *(i) Lemma 31 Prop. 6.3 (discrete modular spectrum $\Rightarrow$ $P = 0$) and Lemma 39 Thm. 7.5(i) are special cases of Thm. 42.3.1(c). (ii) Even granting RH, the Cramér reconstruction (L32 Thm. 7.2(vi)), $\mathcal{H} \cong \ell^2(\{t_\rho > 0\})$ with $H = \mathrm{diag}(t_\rho)$, cannot be the positive-energy part of any HSMI: H has pure point spectrum, which Thm. 42.3.1(c) forbids. (iii) Consequently $\mathcal{R}$ is not of the form N in Thm. 42.2.1 with μ the zero measure: in (P2), $\mu = \|\phi_0\|^2 \delta_0 +$ (absolutely continuous) (by Prop. 42.3.3). So the lead is not "equivalent to RH"; it is* impossible.

PROPOSITION 42.3.3 (Tomita's $\Delta^{1/4}$ is the Wick rotation; [SKETCH]). *Formally continuing $\Delta^{is} U(a) \Delta^{-is} = U(e^{-2\pi s} a)$ to $s = i/4$ gives $\Delta^{-1/4} U(-t) \Delta^{1/4} = U(it) = e^{-tP}$. Hence $h(t) = \langle \phi, e^{-tP} \phi \rangle$ with $\phi = \Delta^{1/4} \psi$, and μ in Thm. 42.2.1(c) is the spectral measure of P in ϕ. In (P2), $\Delta^{1/4}$ rotates log-scale translation ($\log|x|$) into Euclidean time ($\arg x$), matching L32 Cor. 7.4. Explicit model ([THM], [NUM] C10): on $L^2(\mathbb{R}_+, d\lambda)$, $P = \lambda$, $(\Delta^{is} f)(\lambda) = e^{-\pi s} f(e^{-2\pi s} \lambda)$ is unitary and covariant.*

REMARK 42.3.4 (sign/polarization obstruction and Connes; [THM] facts, [ANALOGY] reading). (1) Positive energy makes the spectrum one-sided, while the zero measure is c_∞-symmetric ($\gamma \mapsto -\gamma$). Only the $\gamma > 0$ half can be a spectrum condition, exactly as in Cramér and in the polarized Hodge form (L40 Rem. 7.2; $H^{1,0}$, [ANALOGY]). "Half-Weil positivity" $\iff$ RH is a [RESTATEMENT]. (2) The scaling action on $L^2(C_\mathbb{Q})$ has Lebesgue spectrum. In Connes' framework the zeros appear as *absorption* spectrum or as resonances (Con99; CCM09; Pavlov–Faddeev PF75, Lax–Phillips LP76, carried), never as eigenvalues of a unitary group, and NO18 Rem. 4.4.4 says Euclidean realizations are multiples of the translation representation up to fixed points. Thm. 42.3.1 is the HSMI form of the same fact. It *refines* L39 R3: an infinite-limit HSMI over the prime tower would have Lebesgue spectrum, and zeros could enter only as absorption.

42.4. The hyperbolic dictionary and the orthogonality pincer

42.4.1. Möbius reflections, rapidity and orthogonality. On $\mathbb{P}^1(\mathbb{R})$ put $r_0(x) = -x$ (fixed points $0, \infty$), $r_1(x) = 1/x$ (fixed points ± 1), and $S = r_0 r_1 = r_1 r_0 : x \mapsto -1/x$. These form a Klein four-group [THM] ([NUM] C1). In the Cayley/Doppler coordinate $y = \frac{1+x}{1-x}$:

(i) r_0 becomes $y \mapsto 1/y$ and r_1 becomes $y \mapsto -y$ (C2). For a velocity v with $x = v$, $y = \frac{1+v}{1-v} = e^{2\eta}$ (rapidity η), so r_0 is *rapidity reversal* $\eta \mapsto -\eta$. The map $v \mapsto 1/v$ is $y \mapsto -y$, and $(1, v) \perp (v, 1)$ in Minkowski signature: r_1 is *hyperbolic orthogonality* (rest frame $\leftrightarrow$ its simultaneity line) (C3) [THM].

(ii) The dilation flow fixing $0, \infty$ (the null directions) acts on y by $e^{2\pi t}$. The BGL93 double-cone modular flow on a light ray is velocity addition $\tanh(\pi t) \oplus v$, which is the dilation $y \mapsto e^{2\pi t} y$ (C2) [THM]; BGL93 [CITED] for the flow formula.
(iii) $y = x_+/x_-$ is the light-cone ratio. So the fixed points of scaling are the null rays, $y = 1$ is the rest frame and $y = -1$ is its orthogonal (C3).

Mellin form ([THM], [NUM] C6). On $L^2(\mathbb{R}_+, dx)$, $(Rf)(x) = x^{-1} f(1/x)$ is unitary with $\widetilde{Rf}(s) = \tilde{f}(1-s)$. So R realizes FE, and $c_\infty \circ R$ realizes $\sharp : s \mapsto 1 - \bar{s}$, which fixes the critical line *pointwise*. "A weight-h representation $\leftrightarrow$ the line $\operatorname{Re} s = h$" is [ANALOGY].

42.4.2. $\Delta^{1/4}$: from dilation to rotation.

PROPOSITION 42.4.1 ([THM], [NUM] `num/l42/sl2.json`). *In $\mathfrak{sl}_2(\mathbb{R})$ take $D = \frac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (dilation, fixing $0, \infty$), $B = \frac{1}{2}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (fixing ± 1) and $L = \frac{1}{2}\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ (rotation; $e^{\pi L}$ acts as $S : x \mapsto -1/x$). Then $\operatorname{Ad}(e^{zB})D = \frac{1}{2}\left(\begin{smallmatrix} \cosh z & -\sinh z \\ \sinh z & -\cosh z \end{smallmatrix}\right)$, and at $z = i\pi/2$ this is $-iL$ [THM]. The flow generated by B is, up to normalization, the modular group of the interval $(-1, 1)$ in a Möbius-covariant net (Hislop–Longo HL82 [BIB]; BGL93 Thm. 2.3(ii) [CITED]).* Sketch *([SKETCH]): formally continuing that modular group to imaginary parameter, $\Delta_{(-1,1)}^{\pm 1/4}$ acts as $\operatorname{Ad}(e^{\pm i\frac{\pi}{2}B})$ (sign fixed by the normalization), so it turns the dilation semigroup into a rotation semigroup. The OS Hamiltonian of the dilations in (P3) is then the conformal Hamiltonian L_0, whose spectrum in a positive-energy representation of lowest weight h is $h + \mathbb{N}$ ([BIB], standard representation theory of the universal cover of $\mathrm{PSL}_2(\mathbb{R})$).*

REMARK 42.4.2 ((P3) survives where (P2) does not; [THM] contrast, [ANALOGY] reading). In (P2) the OS Hamiltonian is P, with Lebesgue spectrum (Thm. 42.3.1). In (P3) it is L_0, with *discrete* spectrum. So (P3) is the only placement that could carry Cramér's $\{t_\rho\}$. The Möbius model gives the arithmetic progression $h + \mathbb{N}$; deforming L_0 to $\operatorname{diag}(t_\rho)$ without zeros as input is open ([OPEN]), and doing it with zeros as input is circular (C7 of L40). The proposal $H = xp$ of Berry–Keating (BK99) is the dilation D; the observation that $\Delta^{1/4}$ rotates D into the oscillator L_0 is [ANALOGY] only. By CC14 Thm. 2.6 ([CITED]), the arithmetic Frobenius Fr_λ corresponds to the idèle-class scaling itself. So any "Frobenius flow" for ζ is the scaling flow, and this collapses back to (P1) or (P3); it is not a new placement.

42.4.3. The pincer.

THEOREM 42.4.3 (1+1 positive-energy pincer; [THM], [NUM] C4). *Let U be a positive-energy unitary representation of the 1+1 Poincaré group ($H \geq |P_z|$). Then:*

(a) An antiunitary implementing parity $P : (t, z) \mapsto (t, -z)$, or a unitary implementing time reversal T, forces $H = 0$. Hence the geometric wedge-swapping antiunitaries are of PT type.
(b) PT commutes with boosts, so it acts on boost (modular) frequencies as c_∞ (Thm. 42.1.2(b)). It reverses translations, so it acts on translation frequencies as $\sharp$ (Thm. 42.1.2(a)), and those have Lebesgue spectrum (Thm. 42.3.1).
(c) The boost-reversing antiunitary T preserves the wedge W_R, so it acts as $\sharp$ on boost frequencies but carries no free positivity (Thm. 42.1.2(c)).

By Bisognano–Wichmann (BW75/76; CR94; L39 Thm. 2.3, [CITED] secondary), $J_{W_R} = PT$ (up to an internal symmetry) and $\Delta_{W_R}^{it} =$ boost by rapidity $-2\pi t$, KMS at $\beta = 2\pi$. So J is c_∞ for boosts and $\sharp$ for translations, never both on the same flow.

PROOF. (a) If V is antiunitary with $VU(a)V^{-1} = U(Pa)$, then $Ve^{i(a^0 H - a^1 P_z)}V^{-1} = e^{-i(a^0 VHV^{-1} - a^1 VP_zV^{-1})}$ must equal $e^{i(a^0 H + a^1 P_z)}$. So $VHV^{-1} = -H$, and $\sigma(H) \subset [0, \infty) \cap (-\sigma(H)) = \{0\}$. A unitary T gives $VHV^{-1} = -H$ directly. P maps W_R to W_L, and so does PT;

T preserves W_R (C4). (b), (c): C4 checks the conjugation relations (P and T reverse the boost, PT commutes with it); then apply Thm. 42.1.2. □

THEOREM 42.4.4 (orthogonality pincer; [THM], assembling Thms. 42.1.2–42.3.1 and L40 Thm. 6.11). *Let J be a modular conjugation and T_u the log-scale flow, realized in (M, Ω).*

(i) If J commutes with T_u (geometrically: J is a reflection in *the scaling geodesic), then J acts as c_∞. The free positivity is translation invariant, i.e. Bochner positivity, and equality with Q is RH (L40 Thm. 6.11, [RESTATEMENT]).*

(ii) If J reverses T_u (a reflection in a geodesic orthogonal *to the scaling geodesic), then J acts as $\sharp$. The free positivity is Hankel and never equals Q (Thm. 42.2.2). The appropriate target is Cramér's $\mathcal{R}$ ([RESTATEMENT]). With positive energy (P2) it cannot be realized (Thm. 42.3.1). Without it (P3) it is open, and circular if built from zeros.*

So the lead's dream, "$\sharp$-action plus free positivity equal to Weil's Toeplitz form", is excluded in every placement. What remains is (ii) in (P3), with target $\mathcal{R}$.

PROOF. (i) is Thm. 42.1.2(b) plus L40 Thm. 6.11. (ii) is Thm. 42.1.2(a) for the frequency action, Thm. 42.2.1(a)–(b) for the Hankel shape (its proof uses only $JT_uJ = T_{-u}$, $T_uMT_{-u} \subset M$ for $u \geq 0$ and $T_u\Omega = \Omega$, so it holds in (P3) too), Thm. 42.2.2 for the inequality, and Thm. 42.3.1 for (P2). □

42.5. The Lemma 40 bridge table in the hyperbolic picture

Each rung of L40 §9 is restated geometrically. In the picture, "elliptic/rotation/Euclidean" means preserving a positive-definite form, and "hyperbolic/boost/Lorentzian" means preserving only an indefinite form with null eigenvectors.

rung	hyperbolic reading	label / source
1	Hasse $\iff$ the normalized Frobenius $g_p = \left(\begin{smallmatrix} 0 & -1 \\ 1 & \tau_p \end{smallmatrix}\right)$, $\tau_p = a_p/\sqrt{p}$, is *elliptic* in $\mathrm{SL}_2(\mathbb{R})$. Its invariant symmetric form $\left(\begin{smallmatrix} 2 & \tau \\ \tau & 2 \end{smallmatrix}\right)$ ($\det = 4-\tau^2$) is p^{-1}-rescaled from L40's Rosati Gram $\left(\begin{smallmatrix} 2 & a \\ a & 2p \end{smallmatrix}\right)$	[THM]; [NUM] C7 (4 curves, all elliptic); Thm. 42.6.8
1′	Each non-fixed pair of the involution ι is a hyperbolic plane, i.e. a boost direction; $\iota = \mathrm{id}$ means no boosts	[THM] (L30 Thm. 2.2), [ANALOGY] "boost"
2	Off-circle Krein pair $\lambda, 1/\bar{\lambda} = e^{\pm\eta}$: a boost with rapidity η and null eigenvectors. Example: fake $p = 11$, $a = 7$, $\eta = 0.331021434564477$	[THM]; [NUM] C7
2′	KGL: elliptic $\to$ hyperbolic only through the parabolic point $\lvert\tau\rvert = 2$, a Jordan block (null collision); the two eigenlines of an elliptic g have opposite Krein signs under $i\omega$ (Hodge split)	[THM]; [NUM] C8
2″	Two stages of ghost birth = passage through ∞ of the frequency cover, then collision. Lemma 41: coincidence of the stages is *generic* (reflection + reality; also Davenport–Heilbronn), so it gives *no* ζ constraint	[THM] (L41, §50); no structural claim made here
2‴	Mirror pair Gram $\left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$: a hyperbolic plane with null basis (light cone); on-line zero = positive (timelike) line	[THM] (L40 Thm. 5.2)
3	BW: modular flow = boost, rapidity $2\pi t$; Euclidean angle period 2π = KMS at $\beta = 2\pi$	[CITED] secondary (L39 Thm. 2.3)
3′	Natural cone: $J = PT$ preserves rapidity, i.e. reflection *in* the scaling geodesic, so c_∞	[THM] (L40 Prop. 6.10; Thm. 42.4.3)
3″	OS: reflection in the *orthogonal* geodesic; Hankel; $\Delta^{1/4}$ is the Wick rotation (dilation $\to$ rotation)	[THM]/[SKETCH] (Thm. 42.2.1, Props. 42.3.3, 42.4.1)
3‴	Weil's form is Toeplitz (invariant under the whole translation group along the geodesic), so it cannot see the half-space cut out by the orthogonal geodesic	[THM] (Thm. 42.2.2; L32 Thm. 5.1)
4	Missing rung = a J that is the orthogonal reflection ($\sharp$) *and* gives Toeplitz positivity: excluded (Thm. 42.4.4). Corrected rung: an (P3) structure with $\langle a, Ja\rangle = \mathcal{R}$; equivalently, every arithmetic Frobenius lies in the Euclidean section (§42.6.4)	[THM] (exclusion), [OPEN]/[RESTATEMENT] (corrected rung)

42.6. Taylor–Wiles modularity at the nexus of the missing rung

This section was added at J. W.'s request. Throughout, $f = \sum a_n q^n \in S_k(\Gamma_0(N))$ is a newform with k even and real a_n; $(f|_k W_N)(z) = N^{k/2}(Nz)^{-k} f(-1/(Nz))$; $f|_k W_N = \eta_f f$ with $\eta_f = \pm 1$ (Atkin–Lehner AL70 [BIB]); $\Lambda(f,s) = N^{s/2}\int_0^\infty f(iy) y^{s-1} dy = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$.

42.6.1. Hecke's functional equation is the Fricke half-turn.

THEOREM 42.6.1 ([THM] for "⇒"; converse [BIB] Hec36, Wei67). *(a) $f|_k W_N = \eta f$ implies $\Lambda(f,s) = \varepsilon\Lambda(f,k-s)$ with $\varepsilon = i^k\eta$ (for $k = 2$, $\varepsilon = -\eta$). (b) Put $g(u) = e^{ku/2} f(ie^u/\sqrt{N})$. Then $g(-u) = \varepsilon g(u)$ and $\Lambda(f, \frac{k}{2} + i\tau) = \int_{\mathbb{R}} g(u) e^{i\tau u} du$. (c) Conversely, Hecke: the FE (with standard growth) for a Dirichlet series implies invariance of the corresponding q-series under $z \mapsto z+1$ and W_N. Invariance under all of $\Gamma_0(N)$ needs the FEs of twists (Weil's converse theorem).*

PROOF. At $z = iy$: $f(i/(Ny)) = \eta N^{k/2} i^k y^k f(iy)$, since $(Niy)^{-k} = N^{-k} i^{-k} y^{-k}$ and $i^{-k} = i^k$ for k even. Substitute $y = e^u/\sqrt{N}$: $f(ie^{-u}/\sqrt{N}) = \varepsilon e^{ku} f(ie^u/\sqrt{N})$, so $g(-u) = \varepsilon g(u)$. Also

$N^{s/2}y^s = e^{us}$ and $dy/y = du$, so $\Lambda(f,s) = \int g(u)e^{(s-k/2)u}du$. Evenness/oddness of g gives $\Lambda(\frac{k}{2} - i\tau) = \varepsilon\Lambda(\frac{k}{2} + i\tau)$, i.e. $\Lambda(s) = \varepsilon\Lambda(k-s)$. □

PROPOSITION 42.6.2 (the Klein four-group in $\mathbb{H}$; [THM]). *Let $\ell_0 = i\mathbb{R}_+$ (the scaling geodesic: $z \mapsto \lambda z$ translates along it, and $u = \log(\sqrt{N}y)$ is arclength from $z_0 = i/\sqrt{N}$) and let $\ell_1 = \{|z| = N^{-1/2}\}$. Then:*

- *(a) $c_\infty \leftrightarrow \sigma_0 : z \mapsto -\bar{z}$, the reflection in ℓ_0: σ_0 commutes with the scaling and sends $f \mapsto f^\rho$ (conjugate coefficients).*
- *(b) FE $\leftrightarrow w_N : z \mapsto -1/(Nz)$, the half-turn about z_0: it reverses the scaling ($w_N(\lambda z) = \lambda^{-1}w_N(z)$) and acts on ℓ_0 as $u \mapsto -u$.*
- *(c) $\sharp \leftrightarrow \sigma_1 = w_N \circ \sigma_0 : z \mapsto 1/(N\bar{z})$, the reflection in ℓ_1. The geodesic ℓ_1 is* orthogonal *to ℓ_0 at z_0.*

So the Hecke FE inside $\sharp = c_\infty \circ$ FE is a hyperbolic isometry, and $\{1, \sigma_0, \sigma_1, w_N\}$ realizes $\{1, c_\infty, \sharp, \mathrm{FE}\}$. This is the J-geometry of the interval $I = (-N^{-1/2}, N^{-1/2})$, i.e. placement (P3), not (P2): there is no positive generator, since the dilation generator has spectrum $\mathbb{R}$. (Also: $ie^u/\sqrt{N}$ stays in $\mathbb{H}$ for $|\operatorname{Im} u| < \pi/2$, a strip of width π; any KMS reading of it is [ANALOGY].)

PROOF. Direct: $w_N(z_0) = z_0$; $w_N(iy) = i/(Ny)$; σ_1 fixes $|z|^2 = 1/N$ pointwise; ℓ_0 meets the circle ℓ_1 (centred on $\mathbb{R}$) orthogonally. □

42.6.2. Petersson with the Fricke conjugation: a root-number form.

THEOREM 42.6.3 (Petersson $\sharp$-pairing; [THM] given AL70 [BIB]). *On $S_k(\Gamma_0(N))$, k even, put $J_F f := f^\rho|_k W_N$ and $J^\varepsilon := i^k J_F$. Then J_F and J^ε are antiunitary involutions for the Petersson product (positive definite), they commute with the Hecke operators T_n, $(n,N) = 1$, and for every newform with real coefficients*

$$\langle f, J^\varepsilon f\rangle_{\mathrm{Pet}} = \varepsilon_f \, \|f\|^2_{\mathrm{Pet}}.$$

The Hermitian form $B(f,g) = \langle f, J^\varepsilon g\rangle_{\mathrm{Pet}}$ on the span of newforms is diagonal in the newform basis with signs ε_f. It is indefinite *as soon as both root numbers occur.*

PROOF. W_N normalizes $\Gamma_0(N)$, so $|_k W_N$ is Petersson-unitary, and $(f|W_N)|W_N = (-1)^k f = f$ [BIB]. The map ρ is antiunitary, and $(f|W_N)^\rho = f^\rho|W_N$ for k even (direct computation with $\overline{(-N\bar{z})^{-k}} = (-Nz)^{-k}$). Hence $J_F^2 = 1$. Commutation with T_n is AL70 [BIB]. For a newform with real coefficients, $J_F f = \eta_f f$, so $J^\varepsilon f = i^k\eta_f f = \varepsilon_f f$ (Thm. 42.6.1). Newforms are Petersson-orthogonal (Hecke operators are self-adjoint with distinct eigenvalue systems) [BIB]. □

EXAMPLE 42.6.4 ($S_2(\Gamma_0(37))$, signature $(1,1)$; [NUM] `num/142/modular.json`). $\dim S_2(\Gamma_0(37)) = 2$, with the newforms of 37a ($y^2 + y = x^3 - x$) and 37b ($y^2 + y = x^3 + x^2 - 23x - 50$). Building f from point counts and testing all four pairs (a_{37}, η), the Fricke relation $f(i/(Ny)) = -\eta N y^2 f(iy)$ holds for exactly one pair (relative error $\sim 10^{-40}$; the other pairs fail at $O(1)$, or at $\sim 10^{-8}$ when only a_N is wrong): 37a: $a_{37} = -1$, $\eta = +1$, $\varepsilon = -1$, $L(f,1) = 0$; 37b: $a_{37} = +1$, $\eta = -1$, $\varepsilon = +1$, $L(f,1) = 0.725681061936153$. Control 11a: $\eta = -1$, $\varepsilon = +1$, relative error $4.3 \cdot 10^{-41}$, $L(f,1) = 0.253841860855911$. In all three, $a_N = -\eta$. So B has signature $(1,1)$ on $S_2(\Gamma_0(37))$, and its negative direction is the $\varepsilon = -1$ form, whose central zero is forced *on* the line. The indefiniteness records the parity of the FE, not a ghost. (GRH for these L-functions is believed; it is not proved, see §42.6.5.)

THEOREM 42.6.5 (obstructions for the Petersson/Fricke candidate; [THM]). *Let $\mathcal{S}$ be any finite sum of spaces $S_k(\Gamma_0(N))$ with the Fricke conjugation.*

- *(a) (Finite rank / multiplicity one.) Any antiunitary commuting with all T_n, $(n,N) = 1$, preserves each newform eigenline (strong multiplicity one, AL70 [BIB]) and acts on it by a phase. Every form $F \mapsto \langle \xi_F, V\xi_F\rangle$ with $\xi_F \in \mathcal{S}$ has rank $\le \dim \mathcal{S} < \infty$. Weil's*

form for $L(f,s)$ has infinite rank (its Gram matrix on the functionals $F \mapsto \widehat{F}(\gamma_\rho)$ is the nondegenerate zero-space Krein form of L40 Thm. 5.2, with infinitely many distinct zeros). So no such form equals Weil's form.

(b) (No scaling flow.) $\mathrm{diag}(\lambda^{1/2}, \lambda^{-1/2})$ *normalizes* $\Gamma_0(N)$ *iff* $\lambda = 1$. *So scaling does not act on $\mathcal{S}$. It enters only through the Hecke integral (Thm. 42.6.1(b)), where any positivity is positivity of the profile g on the log-scale side: the Thm. 6.11 (L40) pattern, FE only.*

So the Petersson space with Fricke as conjugation yields a positive Hilbert space and a $\sharp$-type involution, but the pairing is the root-number form of Thm. 42.6.3, not Weil's form.

PROOF. (a) Each newform $\langle\cdot,\rangle_{\mathrm{Pet}}$-line is the joint eigenspace of the T_n (multiplicity one), and an antiunitary commuting with self-adjoint T_n with real eigenvalues preserves their eigenspaces. The rank bound is linear algebra. (b) For $\gamma = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$, $\delta\gamma\delta^{-1} = \left(\begin{smallmatrix}1&\lambda\\0&1\end{smallmatrix}\right)$, which lies in $\Gamma_0(N)$ only for $\lambda \in \mathbb{Z}$. Applying the same to the inverse dilation gives $\lambda^{-1} \in \mathbb{Z}$, so $\lambda = 1$ ($\lambda > 0$). □

REMARK 42.6.6 (profile positivity; [NUM], [THM] conditional). [NUM]: $g > 0$ on the grid $u \in [0,6]$ (step 0.05) for 11a, 37a, 37b. If $g \geq 0$ on $\mathbb{R}$ and $\varepsilon = +1$, then $\tau \mapsto \Lambda(f, \frac{k}{2} + i\tau)$ is a positive-definite function, and $|\Lambda(f, \frac{k}{2} + i\tau)| \leq \Lambda(f, \frac{k}{2})$ [THM] (Bochner, trivial direction). This is positivity on the *log-scale* side. Like Ξ's positive-definiteness for ζ (L31 Thm. 6.5), it says nothing about the location of zeros.

42.6.3. Theta inversion for ζ.

PROPOSITION 42.6.7 ([THM] for (a)–(b); [BIB] for (c)). *(a) $\theta(1/y) = \sqrt{y}\,\theta(y)$ for $\theta(y) = \sum_n e^{-\pi n^2 y}$ (Poisson), and ξ's FE follows by Mellin transform (Riemann; Tate67 for the adelic form, carried). (b) On $L^2(\mathbb{R})$, with $D_\lambda f(x) = \lambda^{1/2} f(\lambda x)$ and the Fourier transform $\mathcal{F}$: $\mathcal{F} D_\lambda \mathcal{F}^{-1} = D_{1/\lambda}$. So $\mathcal{F}$ reverses the dilation flow, and the antiunitary involution $c \circ \mathcal{F}$ (c = complex conjugation; $(c\mathcal{F})^2 = 1$) acts on dilation frequencies as $\sharp$ (Thm. 42.1.2(a)). This is the orthogonal-reflection (r_1-type) geometry. (c) In the even metaplectic representation, $\mathcal{F}$ is a rotation by $\pi/2$ generated by the oscillator, and the lowest weight is $\frac{1}{4}$ (spectrum $\frac{1}{4} + \mathbb{N}$ of L_0) [BIB].*

Verdict for theta ([THM] for the shape, [ANALOGY] for the identification). As a candidate J, $c \circ \mathcal{F}$ is in placement (P3). By Thm. 42.4.4(ii) its free positivity, if a standard-subspace structure realizes it (not checked here, [OPEN]), is Hankel, with OS spectrum $\frac{1}{4} + \mathbb{N}$ (oscillator), not $\{t_\rho\}$. The arithmetic enters only through the comb $\sum_n \delta_n$, which is $\mathcal{F}$-invariant and not normalizable. ζ is a Mellin matrix coefficient of a distributional vector, which is Connes' absorption picture (Rem. 42.3.4). Theta inversion supplies $\sharp$ but not positivity.

42.6.4. Hecke eigenvalue = Frobenius trace: elliptic versus hyperbolic. For $E/\mathbb{Q}$ of conductor N with newform f_E (modularity: Wiles Wil95 and Taylor–Wiles TW95 for semistable E; Breuil–Conrad–Diamond–Taylor BCDT01 in general, [BIB]), and $p \nmid N$:

$$a_p(f_E) = p + 1 - \#E(\mathbb{F}_p) = \mathrm{tr}(\mathrm{Frob}_p \mid T_\ell E)$$

(Eichler–Shimura; DS05 [BIB]). The first equality is the modularity statement, the second is Lefschetz [BIB]. [NUM]: a_p from point counts equals the q-coefficients for $p \in \{3,5,7\}$ (and 11 for 37a/b) and reproduces the Fricke relations above to 10^{-40}. Normalize $g_p = \left(\begin{smallmatrix}0&-1\\1&\tau_p\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, $\tau_p = a_p/\sqrt{p}$, which has the characteristic polynomial of $\mathrm{Frob}_p/\sqrt{p}$.

THEOREM 42.6.8 (trace trichotomy; [THM]). *For $g \in \mathrm{SL}_2(\mathbb{R})$, $g \neq \pm 1$:*

(a) $|\mathrm{tr}\, g| < 2 \iff$ g is elliptic, i.e. conjugate to a rotation $R(\theta)$, $\mathrm{tr}\, g = 2\cos\theta$. This holds iff g preserves a positive-definite quadratic form, iff g lies in a compact subgroup (a conjugate of $\mathrm{SO}(2)$).

(b) $|\mathrm{tr}\, g| = 2 \iff$ g is parabolic, a Jordan block (null collision; C8).

(c) $|\operatorname{tr} g| > 2 \iff$ *g is hyperbolic, conjugate to* $\pm\operatorname{diag}(e^{\eta}, e^{-\eta})$ *(a boost, rapidity* $\eta > 0$*),* $\operatorname{tr} g = \pm 2\cosh\eta$*. Its eigenvectors are null for every invariant symmetric form, and every invariant form is indefinite.*

Hence Hasse/Weil $|a_p| \le 2\sqrt{p} \iff$ *g_p is elliptic or* ± 1*, with* $a_p = 2\sqrt{p}\cos\theta_p$*. A violation would mean g_p hyperbolic, with* $a_p = \pm 2\sqrt{p}\cosh\eta_p$*. For $E/\mathbb{Q}$ (weight 2) the bound is Hasse's theorem (= Weil, CITED via L40); in weight k it is Deligne (Del74, [BIB]), with $\sqrt{p}$ replaced by $p^{(k-1)/2}$.*

Proof. The characteristic polynomial is $\lambda^2 - t\lambda + 1$, with roots on the unit circle (nonreal, conjugate) iff $|t| < 2$, a double root ± 1 iff $|t| = 2$, and real roots $e^{\pm\eta}$ iff $|t| > 2$. If g is elliptic, averaging any positive form over the compact closure of $\langle g\rangle$ (which is $\cong \mathrm{SO}(2)$ after conjugation) gives an invariant positive form. Conversely, an invariant positive form makes g orthogonal for it, so g is in a conjugate of $\mathrm{SO}(2)$. For (c), $q(v) = q(gv) = e^{2\eta}q(v)$ on an eigenvector forces $q(v) = 0$, and a form with two independent null vectors is indefinite or zero. The invariant symmetric form of the companion matrix is $\left(\begin{smallmatrix} 2 & \tau \\ \tau & 2 \end{smallmatrix}\right)$, with $\det = 4 - \tau^2$ (C7). □

Proposition 42.6.9 (Rindler: how far the analogy goes; [THM] for (a)–(b), [ANALOGY] for (c)). *(a) The boost generator* $K = \operatorname{diag}(1, -1)$ *and* iL'*, with* $L' = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$*, are conjugate in* $\mathfrak{sl}_2(\mathbb{C})$ *(both have eigenvalues* ± 1*). So* $e^{\eta K}$ *and* $R(\theta) = e^{\theta L'}$ *are the real points of one complex one-parameter group, with* $\eta = i\theta$ *up to conjugation: the Wick rotation. (b)* $R(\theta)$ *is* 2π*-periodic and* $e^{\eta K}$ *is not periodic. In the Rindler wedge, regularity of the Euclidean continuation at the origin (no conical singularity) forces the angle to have period* 2π*, which is KMS at* $\beta = 2\pi$ *(Unruh Unr76, Gibbons–Hawking GH77, BW; carried from L40, [BIB]/[CITED] secondary). (c)* What transfers*: "Euclidean section" = elements lying in a compact subgroup = an invariant positive form exists (Thm. 42.6.8(a)); "Lorentzian" = an indefinite invariant form with null eigenvectors = a Krein off-circle pair* $\lambda, 1/\bar{\lambda}$ *(L40 §4).* What does not transfer*: (1) Wick rotation changes the group element* $(\eta \to i\theta)$*, whereas g_p is fixed by arithmetic. Nothing rotates a_p, so ellipticity is not obtained by continuation. (2) Regularity fixes* one *period,* 2π*, while the Frobenius angles* θ_p *vary and are equidistributed (Sato–Tate, below). The thermal condition is about the* flow*, not about the individual angles. (3) For* Frob_p*, ellipticity is forced by a* positive invariant form *(Rosati/Hodge index), not by a regularity or periodicity condition. So "Frobenius is Euclidean because of horizon regularity" is not a mechanism. "Frobenius is Euclidean because it preserves a positive form" is the theorem.*

Remark 42.6.10 (the same dichotomy elsewhere; [THM] dictionary, [BIB] facts).

object	Euclidean (angle)	Lorentzian (boost)
$\mathrm{Frob}_p/\sqrt{p}$	$a_p = 2\sqrt{p}\cos\theta_p$ (proved: Hasse/Deligne)	$a_p = \pm 2\sqrt{p}\cosh\eta$ (excluded)
Krein unitary (L40 §4)	definite type, on the circle	pair $\lambda, 1/\bar{\lambda} = e^{\pm\eta}$, ghost
ζ zero	$\frac{1}{2} + i\gamma$: $e^{i\tau u}$ pure phase	$\frac{1}{2} + \delta + i\gamma$: $e^{i\gamma u}e^{\delta u}$, boost component δ
Selberg, $\Gamma\backslash\mathbb{H}$	$\lambda = \frac{1}{4} + r^2$, $r \in \mathbb{R}$ (principal series)	$\lambda \in (0, \frac{1}{4})$, $r \in i\mathbb{R}$ (complementary series, exceptional)

Precision (important; corrects the naive reading). In the automorphic setting, complementary-series components are *unitary*, but not tempered. "Ramanujan/Selberg" is *temperedness*, not unitarity. Selberg's conjecture $\lambda_1 \ge \frac{1}{4}$ for congruence groups is open; Selberg proved $\lambda_1 \ge \frac{3}{16}$ (Sel65, [BIB]). So in the Selberg row the boost is a non-tempered *unitary* representation, while in the Krein row it is a non-unitary (indefinite) one. The matrix dichotomy of Thm. 42.6.8 is about a single element and its invariant forms. "Failure $\iff$ hyperbolic" is exact for g_p, and only [ANALOGY] for representations.

Proposition 42.6.11 (Sato–Tate; [CITED] BLGHT11 abstract, [NUM]). *For non-CM newforms of weight* ≥ 2*, and hence for non-CM* $E/\mathbb{Q}$*, the angles* $\theta_p \in [0, \pi]$ *are equidistributed*

for $\frac{2}{\pi}\sin^2\theta\,d\theta$*, the pushforward of Haar measure on* $\mathrm{SU}(2)$ *to conjugacy classes (Barnet-Lamb–Geraghty–Harris–Taylor, BLGHT11; they also note that it can now be proved for any elliptic curve over a totally real field). Earlier cases: Clozel–Harris–Taylor, Taylor, Harris–Shepherd-Barron–Taylor (CHT08, Tay08, HSBT10, [BIB]). [NUM] (*`num/l42/satotate.json`*, 2260 primes* < 20000 *each): all* g_p *elliptic (*$\max|a_p|/2\sqrt{p} = 0.99402, 0.996484, 0.994456$ *for 11a, 37a, 37b). The moments of* $2\cos\theta_p$ *are* $(0.9846, 1.9449, 4.7954)$*,* $(0.9785, 1.9172, 4.7406)$*,* $(1.0014, 1.9864, 4.9246)$ *against the Haar values* $(1, 2, 5)$*. Sato–Tate is a statement* inside *the Euclidean section: it presupposes ellipticity.*

Definition 42.6.12 (RH as "every arithmetic Frobenius is Euclidean"; [DEF], [RESTATEMENT]). Call a flow or element *Euclidean* if it preserves a positive-definite Hermitian form on the relevant (zero/cohomology) space, equivalently lies in a compact (unitary) group.

(i) Curves over $\mathbb{F}_q$: Frobenius on H^1 is Euclidean (Weil, via Rosati: L40 rung 1). [CITED].
(ii) $L(E,s)$ locally: every g_p is Euclidean (Hasse; $\Rightarrow$ local factors' zeros on $\mathrm{Re}\,s = 1$, the local RH). [CITED]/[THM].
(iii) ζ: the role of Frobenius is played by the scaling flow $\lambda \in \mathbb{R}_+^*$ on the adele class space (Connes; CC14 Thm. 2.6: $\mathrm{Fr}_\lambda \leftrightarrow$ idèle-class scaling, [CITED]). Its spectral parameters are the τ_ρ. "Euclidean" means that the scaling flow is unitary for a positive form on the zero space, i.e. Weil positivity (L25 Rem. 6.2(ii); L19 Thm. 3.2(iii)) $\iff$ RH [RESTATEMENT]. For $L(E,s)$ the same statement is GRH, open.

What would force ellipticity: in (i) it is an invariant positive form from an *ample class* (Rosati/Hodge index), i.e. compactness *by construction*. For ζ the analogue is a positive invariant form built without zeros, which is exactly the missing rung. By the pincer (Thm. 42.4.4), free positivity gives such a form only for c_∞-type J (Bochner, needing RH) or in Hankel shape (Cramér target, [RESTATEMENT]). A single scaling "element" has no trace criterion: there is no finite-dimensional $\mathrm{tr}\,g_\lambda$ to bound, and the Connes trace formula is a distributional identity whose positivity is again Weil's ([THM] observation; Con99, carried from L30).

42.6.5. The honest limit: the missing rung recurs.

(1) *What modularity gives* ([BIB], with the [THM] geometry above): $L(E,s) = L(f_E,s)$ has analytic continuation and the FE, i.e. the involution $\sharp$ (Fricke half-turn and reflection in ℓ_1). The Hecke algebra is self-adjoint for a *positive* Petersson product, so the a_n are real (positivity $\Rightarrow$ self-adjoint $\Rightarrow$ real spectrum). That is not Ramanujan.
(2) *What gives local unitarity*: Eichler–Shimura transports T_p to $\mathrm{Frob}_p + p\,\mathrm{Frob}_p^{-1}$ on $X_0(N)$ mod p [BIB], where Weil/Rosati gives $|a_p| \le 2\sqrt{p}$ (Deligne in higher weight). Each g_p is elliptic: rung 1 holds at every prime.
(3) *What is missing*: global unitarity (GRH for $L(E,s)$) is *not* implied by local unitarity at all p plus the FE plus Petersson positivity. These are exactly the ingredients ζ already has (Euler product with trivially unitary local factors, FE via theta, Ξ positive definite), and they do not give RH. So the missing rung recurs for $L(E,s)$, with the same shape (Thms. 42.6.5, 42.4.4). [THM] for the comparison of ingredients; the non-implication is not a theorem, only the absence of any known proof ([OPEN]).
(4) *Why Taylor–Wiles is nonetheless at the nexus* ([ANALOGY]): $R = T$ identifies a Galois deformation ring (where Frobenius lives) with a Hecke algebra (operators self-adjoint for Petersson). It is the one place where a *Galois* object is proved to act through *Hilbert-space* operators. But the positivity it imports is Petersson's, which acts on the finite-dimensional Hecke side (root numbers, Thm. 42.6.3), not on the zero side.

42.7. Angle versus rapidity is Hilbert versus Krein (the 2×2 case)

Added in v2 at J. W.'s request ("is this rapidity vs angle related to the concept of an inner product on a Hilbert or Krein space?"). The answer is yes, and for a single normalized Frobenius it is a theorem. Notation: $E/\mathbb{F}_p$ an elliptic curve, $a=a_p$, $\pi=\mathrm{Frob}_p$, $\pi^2-a\pi+p=0$, $\tau=a/\sqrt{p}$, $g=g_p=\left(\begin{smallmatrix}0&-1\\1&\tau\end{smallmatrix}\right)$ as in §42.6.4. On $\mathbb{C}^2$ put $\omega(x,y)=x_1y_2-x_2y_1$ and

$$H(x,y)=i\,\omega(\bar{x},y),\qquad H(x,x)=-2\,\mathrm{Im}(\bar{x}_1x_2).$$

42.7.1. Where the real model lives (a precision).

Proposition 42.7.1 ([THM]; [BIB] Sil09 Ch. III for the Weil pairing). *(a)* The Weil pairing gives only the Krein form. *On $T_\ell E$ the Weil pairing is alternating and $e_\ell(\pi x,\pi y)=e_\ell(x,y)^{\deg\pi}=e_\ell(x,y)^p$ [BIB]. In rank 2 this carries no information beyond $\det\pi=p$: for* every 2×2 *matrix M and every alternating form, $\omega(Mx,My)=\det M\,\omega(x,y)$. So $\mathrm{Sp}_2=\mathrm{SL}_2$ and $\mathrm{GSp}_2=\mathrm{GL}_2$, and "$g_p$ is symplectic" is automatic once $\det g_p=1$. (The symplectic condition becomes a real constraint only in genus ≥ 2, where $\mathrm{Sp}_{2g}\subsetneq\mathrm{SL}_{2g}$; roadmap S4.)*

(b) Positivity cannot be stated on $T_\ell E$. *$\mathbb{Q}_\ell$ is not ordered and has no complex conjugation, so "positive definite Hermitian" has no meaning on the Tate module. It is meaningful on a real model. The canonical one is $V=\mathbb{Q}(\pi)\otimes_{\mathbb{Q}}\mathbb{R}$ with basis $\{1,\pi/\sqrt{p}\}$ and $g=$ multiplication by $\pi/\sqrt{p}$. In that basis the matrix is exactly the companion matrix g_p ($\pi\cdot 1=\pi$, $\pi\cdot\pi=a\pi-p$), and ω is preserved because $\det g_p=1$. (Assume $\pi\notin\mathbb{Q}$; the case $\pi\in\mathbb{Q}$ is in Prop. 42.7.5(c).)*

(c) The Rosati involution on $\mathbb{Q}(\pi)$ is the nontrivial field automorphism $\sigma:\pi\mapsto p/\pi=a-\pi$ *(L40 §3, [CITED] via Milne), and the Rosati trace form $\langle x,y\rangle=\mathrm{Tr}_{\mathbb{Q}(\pi)/\mathbb{Q}}(x\,\sigma(y))$ has Gram matrix $S_p=\left(\begin{smallmatrix}2&a\\a&2p\end{smallmatrix}\right)$ on $\{1,\pi\}$ (L40 Ex. 3.3, [THM]). Hence:*

- *if $a^2<4p$, then $\mathbb{Q}(\pi)$ is imaginary quadratic, σ is* complex conjugation c_∞ *under either embedding, and $\langle x,x\rangle=2|x|^2>0$: the Rosati form is the pairing of an element with its own Galois conjugate, and it is positive;*
- *if $a^2>4p$ (excluded by Hasse, but well-defined for the fake polynomial), then $\mathbb{Q}(\pi)$ is real quadratic, σ is the Galois conjugation of a* real *field, and $\langle x,x\rangle=2N(x)$ is the norm form, of signature $(1,1)$: a Krein form.*

Proof. (a) $\omega(Mx,My)$ is alternating bilinear in (x,y), and the space of such forms on a 2-dimensional space is 1-dimensional; evaluating on a basis gives the factor $\det M$. (b) The matrix computation is immediate. (c) $\pi\sigma(\pi)=p$ and $\pi+\sigma(\pi)=a$, so σ is the conjugate root. The discriminant a^2-4p decides whether the field is real or imaginary. For $x=u+v\pi$, $\mathrm{Tr}(x\sigma x)=2N(x)=2(u^2+auv+pv^2)$, the quadratic form of S_p, with $\det S_p=4p-a^2$. □

Reading ([THM], and the point J. W. was circling in L40). In the 2×2 model, "Hilbert versus Krein" is literally "imaginary versus real quadratic", i.e. whether the Galois involution that pairs π with p/π is *complex conjugation*. When it is, the class-with-its-own-conjugate pairing is positive. When it is not, the same recipe produces an indefinite norm form. This is the L40 diamond (c_∞ swaps $H^{1,0}\leftrightarrow H^{0,1}$, positivity is $\langle x,c_\infty x\rangle$) made exact at one prime.

42.7.2. The trichotomy for invariant Hermitian forms.

Theorem 42.7.2 (Hilbert versus Krein for $g\in\mathrm{SL}_2(\mathbb{R})$; [THM]). *Let $g\in\mathrm{SL}_2(\mathbb{R})$, $g\neq\pm 1$, $t=\mathrm{tr}\,g$. Then H is g-invariant and nondegenerate of signature $(1,1)$ (a Krein form) for every such g, and every real vector is H-null. Moreover:*

(a) ***Elliptic*** *($|t|<2$, eigenvalues $e^{\pm i\theta}$, $0<\theta<\pi$). The eigenvectors $v,\bar{v}$ satisfy $H(v,\bar{v})=0$ and $H(\bar{v},\bar{v})=-H(v,v)\neq 0$: one eigenline is positive and the other negative (*definite type*). The g-invariant Hermitian forms on $\mathbb{C}^2$ are exactly those that are diagonal in the eigenbasis $(v,\bar{v})$ with real entries (α,β). This is a 2-dimensional real space containing*

the positive-definite cone $\alpha, \beta > 0$; H is the point $\alpha > 0 > \beta$. The operator $J_g = (g - \cos\theta)/\sin\theta$ satisfies $J_g^2 = -1$, commutes with g, preserves ω, and

$$b_g(x, y) := \omega(x, J_g y)$$

is a g-invariant positive-definite *symmetric form ($0 < \theta < \pi$, so $\sin\theta > 0$). For the companion matrix, $b_g = \frac{1}{2\sin\theta}\left(\begin{smallmatrix} 2 & \tau \\ \tau & 2 \end{smallmatrix}\right)$, i.e. $\frac{1}{2\sin\theta}$ times the Rosati Gram S_p rewritten in the basis $\{1, \pi/\sqrt{p}\}$, i.e. Thm. 42.6.8's form, now with its Hodge origin: J_g is a complex structure on $\mathbb{R}^2$ (a weight-1 Hodge structure; L_+, the $e^{i\theta}$-eigenline, is the $+i$-eigenline of J_g, and $L_- = \overline{L_+}$ the $-i$-eigenline), and b_g is its Riemann-bilinear polarization. On $L_\pm$, the Hermitian extension of b_g equals $\pm H$.*

(b) ***Hyperbolic*** *($|t| > 2$, eigenvalues $\pm e^{\pm\eta}$, $\eta > 0$). The eigenvectors are real, hence H-null (light-cone directions). The invariant Hermitian forms are exactly $\left(\begin{smallmatrix} 0 & z \\ \bar{z} & 0 \end{smallmatrix}\right)$ in the eigenbasis. All of them are indefinite or zero, so no Hilbert structure exists. The eigenvalue pair is an off-circle pair $\lambda, 1/\bar{\lambda}$ with $\lambda = \pm e^{\eta}$, the Krein ghost of L40 §4.*

(c) ***Parabolic*** *($|t| = 2$, $g \neq \pm 1$). g is a Jordan block with a single eigenline, which is real and hence H-null. The invariant Hermitian forms are $\left(\begin{smallmatrix} 0 & ib \\ -ib & d \end{smallmatrix}\right)$ ($b, d \in \mathbb{R}$) in a Jordan basis (e_1, e_2) with $ge_1 = \pm e_1$. Such a form is indefinite if $b \neq 0$, and if $b = 0$ it is positive* semi*definite of rank ≤ 1, with kernel the eigenline. So the only invariant "positive" forms are degenerate.*

PROOF. Invariance of H: for real g, $H(gx, gy) = i\omega(g\bar{x}, gy) = \det g \cdot H(x, y)$. Its matrix is $\left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$, with eigenvalues ± 1. For real x, $\bar{x}_1 x_2 \in \mathbb{R}$, so $H(x, x) = 0$. For a sesquilinear invariant h and eigenvectors $gu = \lambda u$, $gw = \mu w$: $h(u, w) = \bar{\lambda}\mu\, h(u, w)$. (a) $\lambda = e^{i\theta}$, $\mu = e^{-i\theta}$ gives $\bar{\lambda}\mu = e^{-2i\theta} \neq 1$, so $h(v, \bar{v}) = 0$. The diagonal entries are free and real. $H(\bar{v}, \bar{v}) = i\omega(v, \bar{v}) = -i\omega(\bar{v}, v) = -H(v, v)$. If $H(v, v) = 0$, then $\omega(\bar{v}, v) = 0$, so $v, \bar{v}$ are dependent and the eigenvalue is real, a contradiction. $g^2 - 2\cos\theta\, g + 1 = 0$ gives $(g - \cos\theta)^2 = -\sin^2\theta$, hence $J_g^2 = -1$. $\det J_g = 1$, so J_g preserves ω. Then $b_g(y, x) = \omega(y, J_g x) = \omega(J_g y, J_g^2 x) = \omega(x, J_g y)$, so b_g is symmetric. It is nonzero on every real $x \neq 0$ ($x, J_g x$ are independent, since J_g has no real eigenvector), hence definite. The companion computation: $J_g = \frac{1}{s}\left(\begin{smallmatrix} -c & -1 \\ 1 & c \end{smallmatrix}\right)$ with $c = \tau/2$, $s = \sin\theta$, and $\Omega J_g = \frac{1}{s}\left(\begin{smallmatrix} 1 & c \\ c & 1 \end{smallmatrix}\right)$. If $gv = e^{i\theta}v$, then $J_g v = \frac{e^{i\theta} - \cos\theta}{\sin\theta} v = iv$, and $J_g \bar{v} = -i\bar{v}$. So on $L_\pm$, $b_g(\bar{x}, x) = \omega(\bar{x}, \pm ix) = \pm H(x, x)$, and for the companion eigenvector $v = (1, -e^{i\theta})$ one has $H(v, v) = 2\sin\theta > 0$. (b) $\lambda = \mu$ real gives $\bar{\lambda}\mu = e^{\pm 2\eta} \neq 1$, which kills the diagonal. The off-diagonal is free. (c) is a direct computation from $ge_1 = \pm e_1$, $ge_2 = \pm(e_1 + e_2)$: $h(e_1, e_1) = 0$, $\operatorname{Re} h(e_1, e_2) = 0$, d free. □

COROLLARY 42.7.3 (one quantity for both stages; [THM]). *In the fixed chart $\{1, \pi/\sqrt{p}\}$ (normalized eigenvector $v = (1, -\lambda)/\sqrt{2}$, $\lambda = e^{i\theta}$):*

$$H(v, v) = \sin\theta = \frac{\sqrt{\det S_p}}{2\sqrt{p}} = \sqrt{1 - \tfrac{a^2}{4p}}, \qquad \frac{b_g}{\|b_g\|} \xrightarrow[\theta \to 0]{} \tfrac{1}{2}\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right),$$

and the kernel of the limit is the parabolic eigenline $(1, -1)$. So as $|a| \uparrow 2\sqrt{p}$, one *number, the Krein norm of the eigenvector in the fixed chart, which is the square root of the Rosati Gram determinant, goes to zero. At the same instant: (a) the Rosati Gram degenerates (rank drops; the normalized polarization becomes a rank-1 semidefinite form with null kernel), and (b) the two opposite-type eigenlines merge into one null line, after which they can leave the circle as $e^{\pm\eta}$.*

PROOF. For $v = (1, -\lambda)/\sqrt{2}$, $H(v, v) = -2\operatorname{Im}(\bar{v}_1 v_2) = -2\operatorname{Im}(-\lambda/2) = \operatorname{Im}\lambda = \sin\theta$, and $\det S_p = 4p - a^2 = 4p\sin^2\theta$. The limit of $b_g/\|b_g\|$ follows from $b_g = \frac{1}{2s}\left(\begin{smallmatrix} 2 & \tau \\ \tau & 2 \end{smallmatrix}\right)$ with $\tau \to 2$. ([NUM]: `v2_tropical_krein.json`, `A_checks`: $|H(v, v)| = 0.968246, 0.661438, 0.099875$ at $\tau = \frac{1}{2}, \frac{3}{2}, 1.99$, equal to $\sin\theta$; $J_g^2 + 1$, $b_g - S/(2s)$ and $g^{\mathsf{T}} b_g g - b_g$ all $\leq 2 \cdot 10^{-39}$; for $\tau = 2.01, 3$ the eigenvector Krein norms are 0 and $\det S < 0$; at $\tau = 2$, $(g - 1) \neq 0$ and $(g - 1)^2 = 0$.) □

REMARK 42.7.4 (relation to Lemma 27's stage (a) and Lemma 41; [THM] for the toy, [ANALOGY] for the transfer). Lemma 27/41 describe ghost birth in two stages: (a) the Krein space degenerates ($\det G = 0$), then (b) an opposite-signature collision pushes a pair off the axis. Lemma 41 Thm. 2.1 and Rem. 2.2 show that at the canonical level these are one event in two charts (fixed chart: degeneration; Krein-normalized chart: mixed-type collision), and Lemma 41 Thm. 4.1 (two-mode normal form) shows they separate iff leakage $\times$ detuning $\varepsilon\Delta \neq 0$. Cor. 42.7.3 is the same structure in the smallest possible case. The fixed chart is $\{1, \pi\}$ with the Rosati Gram S_p, the normalized chart is the Krein form H, and they coincide exactly because a 2×2 block has no other mode to leak into ($\varepsilon = 0$ in Lemma 41's notation). What is a theorem: the matrix family $g(\tau)$, $\tau \in \mathbb{R}$, passes from Hilbert to Krein only through the parabolic point, where stage (a) and stage (b) are simultaneous. What is analogy: identifying this with ghost birth of ζ-zeros. Lemma 41 (C4) already showed that the coincidence is generic, not ζ-specific, and the same holds here: *every* real 2×2 family has it. Moreover the arithmetic Frobenius never moves along τ; it is a fixed point of the family (Prop. 42.7.5).

PROPOSITION 42.7.5 (arithmetic Frobenius stays away from the boundary; [THM], using [BIB] Sil09). *(a) For p prime and $E/\mathbb{F}_p$, g_p is never parabolic, and the Krein margin satisfies*

$$\sin\theta_p = \sqrt{\frac{4p - a_p^2}{4p}} \;\geq\; \frac{\sqrt{3}}{2\sqrt{p}}.$$

(b) Hence the two eigenlines of g_p have Krein norms $\pm\sin\theta_p$ (fixed chart) and are of definite type at every good prime, uniformly up to $O(p^{-1/2})$. (c) Over $\mathbb{F}_q$ with q a square, $a = \pm 2\sqrt{q}$ can occur, but then $\pi = \pm\sqrt{q} \in \mathbb{Z}$ is a scalar: $g = \pm 1$, never a Jordan block. Geometric Frobenius on an elliptic curve is semisimple.

PROOF. (a) $a_p \in \mathbb{Z}$ and $a_p^2 \leq 4p$ (Hasse, Sil09 Thm. V.1.1 [BIB], carried from L19). $a_p^2 = 4p$ is impossible since $4p$ is not a square. If a_p is odd, $4p - a_p^2 \equiv 3 \pmod 4$, so it is ≥ 3. If a_p is even, $4p - a_p^2 = 4(p - (a_p/2)^2) \geq 4$. (c) If $a^2 = 4q$, then $(\pi \mp \sqrt{q})^2 = \pi^2 - a\pi + q = 0$ in $\mathrm{End}(E)$. Since deg is multiplicative and $\deg\phi = 0$ only for $\phi = 0$ (Sil09 Ch. III [BIB]), $\mathrm{End}(E)$ has no nonzero nilpotents, so $\pi = \pm\sqrt{q}$. □

[NUM] (`v2_tropical_krein.json`; 2261 primes < 20000 each, including $p = 2$): for 11a and 37a every g_p has opposite Krein signs on its eigenlines; $\min_p(4p - a_p^2)/p = 0.04770$ (11a), 0.02808 (37a), all positive.

REMARK 42.7.6 (what this does and does not say about the missing rung; [THM]/[ANALOGY]/[OPEN]). (1) [THM]: at a prime, the Krein form H is free (it comes from $\det = 1$, the FE $\lambda \mapsto 1/\lambda$). The Hilbert form is extra, and it is exactly a complex structure J_g commuting with g (a Hodge structure) whose polarization $\omega(\cdot, J_g\cdot)$ is positive. For curves, Rosati supplies it. (2) [RESTATEMENT]/[ANALOGY]: for ζ, the free part is again the Krein structure (FE plus reality, Lemmas 27, 30, 40), and RH is the existence of the extra Hilbert form (Def. 42.6.12). The 2×2 theorem suggests a sharper target: not "a positive form" but "a complex structure J commuting with the scaling flow, compatible with the Krein form, whose polarization is positive". That is Deninger's Hodge $*$ (Den05 §3, carried from L23/L30, [CITED]): $*$ commuting with Θ and $\mathrm{tr}(f \cup *f') > 0$ forces $\Theta = \frac{1}{2} + A$ with A skew. (3) [OPEN]: the pincer (Thm. 42.4.4) says that the free modular J is not such a complex structure. A modular conjugation is *antilinear* with $J^2 = 1$, whereas J_g is linear with $J_g^2 = -1$. Geometrically J_g is the quarter-turn $R(\pi/2)$ of the compact one-parameter group through g (conjugate g to $R(\theta)$; then $J_g \mapsto R(\pi/2)$), and Prop. 42.3.3 identifies $\Delta^{1/4}$ as a quarter-turn (Wick) rotation. Whether a quarter-turn of this kind, commuting with the scaling flow and built without zeros, exists for ζ is the missing rung restated in this language ([OPEN]; not attempted here; the matching of the two quarter-turns is [ANALOGY]).

42.8. The tropical form: Newton polygons, purity, and what tropicalization forgets

Added in v2 at J. W.'s request ("is this fundamentally a tropical type of statement?"). Summary: *the magnitude half of RH is tropical, but no tropical mechanism forces it.* This sharpens Lemma 23 (`tropical_note.pdf`, STATE §32), which found by test that p-adic Newton polygons, archimedean tropical roots and Yu's tropical PSD cone do not separate real curves from fakes. Here we give the 2×2 theorems behind that. Conventions: for $f(x)=\sum c_i x^i$ and a valuation v, the Newton polygon $\mathrm{NP}_v(f)$ is the lower convex hull of $\{(i, v(c_i))\}$. The valuations of the roots are the negatives of its slopes, with multiplicity the horizontal lengths. For $P_p(x)=x^2-a_p x+p$, the points are $(0,1)$, $(1, v(a_p))$, $(2,0)$.

42.8.1. The p-adic polygon.

THEOREM 42.8.1 (p-adic slopes; [THM], with [BIB] Sil09 Ch. V for "ordinary $\iff$ $p\nmid a_p$"). *Let $v=v_p$, normalized by $v(p)=1$, extended to $\bar{\mathbb{Q}}_p$.*

*(a) If $p\nmid a_p$ (*ordinary*), then $\mathrm{NP}_p(P_p)$ has vertices $(0,1),(1,0),(2,0)$, and the roots have valuations $\{0,1\}$: one p-adic unit root $\alpha_u\equiv a_p \pmod p$ (Hensel: $P_p(a_p)=p\equiv 0$, $P_p'(a_p)=a_p\not\equiv 0$), and $\beta=p/\alpha_u$.*

*(b) If $p\mid a_p$ (*supersingular*), then $\mathrm{NP}_p(P_p)$ is the single segment $(0,1)$–$(2,0)$, and both roots have valuation $\frac{1}{2}$. For $p\geq 5$ this means $a_p=0$, since $|a_p|\leq 2\sqrt{p}<p$; equivalently $\theta_p=\pi/2$ exactly. So for $p\geq 5$ the p-adic polygon sees exactly one angle: whether $\theta_p=\pi/2$.*

(c) Symmetry*: the slopes come in pairs $\lambda, 1-\lambda$ because $\alpha\beta=p$. This is the p-adic shadow of the functional equation $\lambda\mapsto q/\lambda$ (L40 §3).*

PROOF. Lower convex hull of three points: if $v(a_p)=0$ the middle point lies below the chord, otherwise on or above it. Hensel gives (a). For (b) with $p\geq 5$: $p\mid a_p$ and $|a_p|<p$ force $a_p=0$, so $a_p=2\sqrt{p}\cos\theta_p$ gives $\cos\theta_p=0$. □

THEOREM 42.8.2 (Newton above Hodge in genus 1: it is integrality; [THM]; general statement [CITED]). *The Hodge polygon of $H^1(E)$ has slope 0 with length $h^{0,1}=\dim H^1(E,\mathcal{O}_E)=1$ and slope 1 with length $h^{1,0}=\dim H^0(E,\Omega^1)=1$, i.e. vertices $(0,1),(1,0),(2,0)$ in our coordinates.* Katz's conjecture*, proved by Mazur under hypotheses (Maz72, Maz73) and by Ogus in general (BO78, Ch. 8), asserts for crystalline cohomology that the Newton polygon lies on or above the Hodge polygon (statement [CITED]: Maz72 pp. 655–657, "first" and "second form of the conjecture"; attribution to Mazur (Maz73) and Ogus (BO78) [CITED] from Yao22 §1.1). For H^1 of an elliptic curve, this is equivalent to $v_p(a_p)\geq 0$, i.e. to $a_p\in\mathbb{Z}_p$, together with the fixed endpoints (*$\deg=2$, $v(\det)=1$*). It is therefore satisfied by* every *integer a, including the fake hyperbolic $P(x)=x^2-7x+11$ ($|a|>2\sqrt{11}$), whose p-adic polygon is identical to that of an ordinary curve.*

PROOF. The Newton polygon has the same endpoints as the Hodge polygon. It lies on or above it iff its middle point $(1, v(a_p))$, or the chord when that is lower, is on or above $(1,0)$. That holds iff $v(a_p)\geq 0$. For the fake polynomial, $v_{11}(7)=0$: ordinary-shaped, with slopes $0,1$ ([NUM], `fake_p11_a7`). □

REMARK 42.8.3 (is Newton-above-Hodge "tropical positivity"? [ANALOGY] with a sharp limit). It is a tropical *inequality* (one polygon above another, i.e. convexity data), and it does encode the Hodge split $H^{1,0}\oplus H^{0,1}$ of the L40 diamond: the unit root corresponds to the slope-0 part $H^1(\mathcal{O})$, and its existence is the Hasse invariant. But it is a *divisibility* statement (Frobenius is divisible by p^i on the i-th Hodge step), not the positivity of a Hermitian form. By Thm. 42.8.2 it holds for fakes. It cannot see the boost $|a|>2\sqrt{p}$, so it is not the Rosati positivity. Calling it "positivity" is [ANALOGY]. The correct statement is: *the p-adic tropical picture contains the Hodge numbers but not the polarization.*

42.8.2. The archimedean polygon: Weil = purity of weight 1.

THEOREM 42.8.4 ([THM] given Hasse/Weil [BIB]). *For $p \nmid N$ and every embedding $\iota : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\log_p|\iota\alpha| = \log_p|\iota\beta| = \frac{1}{2}$. In Deligne's language, H^1 is pure of weight 1 (Del74 [BIB]): the archimedean "Newton polygon" of P_p (slopes $\log_p|\text{root}|$) is a single segment of slope $\frac{1}{2}$. Conversely, if $a^2 > 4p$ and $a = \pm 2\sqrt{p}\cosh\eta$ ($\eta > 0$, the boost of Thm. 42.6.8(c)), then the roots are $\pm\sqrt{p}e^{\pm\eta}$ and the slopes are*

$$\tfrac{1}{2} \pm \frac{\eta}{\ln p}.$$

A boost splits the slope $\frac{1}{2}$ into the broken polygon $\frac{1}{2} \pm \eta/\ln p$. In tropical terms, a ghost is a broken archimedean polygon.

PROOF. $a^2 \le 4p$ makes α, β complex conjugate (or equal, real $\pm\sqrt{p}$) with product p. This holds under every ι, because the conjugates of α are α, β. The converse case: $\alpha\beta = p$, $\alpha + \beta = \pm 2\sqrt{p}\cosh\eta$, so $\{\alpha, \beta\} = \pm\sqrt{p}\,e^{\pm\eta}$, and $\log_p(\sqrt{p}e^{\pm\eta}) = \frac{1}{2} \pm \eta/\ln p$. [NUM]: for 11a, 37a and all 2261 primes < 20000, $\max|\log_p|\text{root}| - \frac{1}{2}| \le 1.2 \cdot 10^{-41}$ (mpmath, 40 digits). For the fake $p = 11$, $a = 7$: $\eta = 0.3310214$, slopes $0.3619533, 0.6380467$, which equal $\frac{1}{2} \mp \eta/\ln 11$ to all printed digits. □

THEOREM 42.8.5 (max-plus tropicalization of the coefficients; [THM]). *Let $\ell = \log_p|a|$ ($\ell = -\infty$ if $a = 0$). The tropical polynomial $\max(1,\ \ell + X,\ 2X)$ of P (archimedean valuation $\log_p|\cdot|$) has tropical roots $\{\frac{1}{2}, \frac{1}{2}\}$ if $|a| \le \sqrt{p}$ and $\{1 - \ell,\ \ell\}$ if $|a| > \sqrt{p}$. Hence:*

- *(a)* Coefficient purity is sufficient, not necessary. *Tropical purity ($|a| \le \sqrt{p}$) implies Weil purity ($|a| \le 2\sqrt{p}$). For $\sqrt{p} < |a_p| \le 2\sqrt{p}$, which occurs for genuine curves, the coefficient polygon shows a* spurious *break of size $\ell - \frac{1}{2} = \log_p(|a_p|/\sqrt{p}) \in (0, \log_p 2]$, although the true polygon is pure.*
- *(b)* Uniform error. *In every case (elliptic or hyperbolic), each tropical root differs from the corresponding true slope by at most $\log_p 2$, i.e. the root moduli are recovered up to a factor 2. In the hyperbolic case the error is exactly $\log_p(1 + e^{-2\eta})$.*
- *(c) So the "factor 2" in Hasse is precisely the gap between the tropical criterion $|a| \le \sqrt{p}$ and the true criterion $|a| \le 2\sqrt{p}$. A coefficient break $> \log_p 2$ certifies a boost; a break in $(0, \log_p 2]$ is undecidable tropically.*

PROOF. The maximum of $1, \ell + X, 2X$ is attained twice at the corners. If $\ell \le \frac{1}{2}$, the line $\ell + X$ never strictly dominates at the corner $X = \frac{1}{2}$ of $\max(1, 2X)$. Otherwise the corners are $1 = \ell + X$ and $\ell + X = 2X$. (a) is then immediate, with $|a| \le 2\sqrt{p}$ giving $\ell - \frac{1}{2} \le \log_p 2$. (b) Elliptic: the true slopes are $\frac{1}{2}$, and $|\ell - \frac{1}{2}| \le \log_p 2$ when $\sqrt{p} < |a| \le 2\sqrt{p}$. Hyperbolic: $\ell = \frac{1}{2} + \log_p(2\cosh\eta)$, and $\log_p(2\cosh\eta) - \eta/\ln p = \log_p(1 + e^{-2\eta}) \in (0, \log_p 2)$. □

[NUM] (`v2_tropical_krein.json`). The fraction of primes < 20000 with a spurious coefficient break ($|a_p| > \sqrt{p}$) is 0.3861 (11a) and 0.3835 (37a). Under Sato–Tate (Prop. 42.6.11) the predicted fraction is $\mu_{\mathrm{ST}}\{|2\cos\theta| > 1\} = \frac{2}{3} - \frac{\sqrt{3}}{2\pi} = 0.3910$ ([THM] integral). The largest spurious break relative to the bound $\log_p 2$ is 0.9913 (11a) and 0.9949 (37a). For the fake $p = 11$, $a = 7$, the coefficient tropical roots are $0.1885, 0.8115$, against the true $0.3620, 0.6380$.

EXAMPLE 42.8.6 (ordinary versus supersingular primes; [NUM]). For $p < 20000$, the supersingular primes are as follows. For 11a: $p = 2$ ($a_2 = -2$) and 19, 29, 199, 569, 809, 1289, 1439, 2539, 3319, 3559, 3919, 5519, 9419, 9539, 9929, 11279, 11549, 13229, 14489, 17239, 18149, 18959, 19319 ($a_p = 0$); 24 of 2261. For 37a: $p = 2$ ($a_2 = -2$), 3 ($a_3 = -3$) and 17, 19, 257, 311, 577, 2161, 2243, 3511, 7951, 10399, 11261, 11579, 15331 ($a_p = 0$); 15 of 2261. All other good primes are ordinary, with slopes $0, 1$. Every supersingular prime has slopes $\frac{1}{2}, \frac{1}{2}$ (Thm. 42.8.1(b)), including $p = 2, 3$, where $a_p \ne 0$ but $p \mid a_p$. (Elkies proved that every $E/\mathbb{Q}$ has infinitely many supersingular primes; Elk87, [BIB].)

42.8.3. What tropicalization forgets: the phase.

REMARK 42.8.7 ([THM] for the split, [BIB] for Sato–Tate). Write $\alpha = \sqrt{p}\, e^{i\theta_p}$. Every valuation, p-adic or archimedean, sees only $|\alpha|_v$. The angle θ_p is invisible to every tropicalization (rotations do not change $\log|\cdot|$). The one exception is the p-adic coincidence $\theta_p = \pi/2 \iff$ supersingular ($p \geq 5$), which reflects the integrality of a_p, not a phase detector. So the arithmetic of a Frobenius factors as

$$\underbrace{\{\log_p |\alpha|_v\}_v}_{\text{tropical: RH is here (all archimedean slopes } \frac{1}{2})} \quad \times \quad \underbrace{\theta_p}_{\text{phase: Sato–Tate is here}} .$$

Sato–Tate (Prop. 42.6.11) is a statement purely about the forgotten coordinate, and it *presupposes* the tropical one: the angle is only defined once the polygon is unbroken. Conversely, the forced *positivity* that makes the polygon unbroken uses the phase (Rosati = pairing with the complex conjugate, Prop. 42.7.1(c)). This is Lemma 23's finding ("positivity enters through a phase") proved here in the 2×2 case. The *statement* is tropical, the *mechanism* is not.

42.8.4. For ζ: RH as purity.

PROPOSITION 42.8.8 ([THM] for (a), [RESTATEMENT] for (b), [ANALOGY]/[CITED] for (c)). *(a)* Function fields. *For a curve $C/\mathbb{F}_q$, $Z(C,T) = \prod_j (1-\alpha_j T)/((1-T)(1-qT))$ with $T = q^{-s}$. So $\zeta_C(s) = 0 \iff q^s = \alpha_j$ for some j, and $\operatorname{Re} s = \log_q |\alpha_j|$.* The slope of a Frobenius eigenvalue is literally the real part of the zero*, and RH for $C \iff$ all archimedean slopes are $\frac{1}{2}$ (purity). (b) ζ. Define the slope of a zero ρ as $\operatorname{Re}\rho$. Then RH $\iff$ "ζ is pure of weight 1" (all slopes $\frac{1}{2}$), and an off-line pair $\rho, 1-\bar{\rho}$ is a broken polygon $\frac{1}{2} \pm \delta$. This is a restatement, not progress ([RESTATEMENT]). (c)* Deninger's framing *(Den05 §3, carried from L23/L30, [CITED]). In a conjectural foliated dynamical system, the zeros are the spectrum of the generator Θ on H^1, and a Hodge $*$ commuting with Θ with positive $\operatorname{tr}(f \cup *f')$ forces $\Theta = \frac{1}{2} + A$ with A skew: purity from a polarization, exactly as in Prop. 42.7.1(c) and Thm. 42.7.2(a). The existence of such a system is open ([OPEN]), and the parallel to the 2×2 case is [ANALOGY].*

REMARK 42.8.9 (the missing rung in tropical language; [OPEN]). For curves, the reason the archimedean polygon cannot break is a non-tropical positivity (Rosati: a complex structure with positive polarization, Thm. 42.7.2(a)). Tropically, the only available constraints are Newton-above-Hodge (integrality, Thm. 42.8.2) and the coefficient polygon (sufficient up to a factor 2, Thm. 42.8.5). Neither forbids the break, and both are satisfied by fakes. So *the missing rung is a global reason the polygon cannot break, and by Thms. 42.8.2 and 42.8.5 and Lemma 23, that reason cannot itself be tropical.* It must use the phase, as J_g does.

42.9. The three threads, in one place

(1) *Positivity.* Spectrum condition = OS/Hankel CM (Thm. 42.2.1); KK/Titchmarsh = holomorphy in $\operatorname{Im} a > 0$ (L39 Thm. 7.3); Pick/Nevanlinna: Cramér's $\mathcal{V}$ is holomorphic in the upper half-plane for free (L32 Thm. 7.2(i)), while the Pick kernel of ξ'/ξ is PSD iff RH (L40 Thm. 5.2, [RESTATEMENT]); KGL: elliptic $\to$ hyperbolic only through a parabolic collision (Thm. 42.6.8(b)); ghosts $\lambda, 1/\bar{\lambda}$ are boosts. In a hypothetical *Krein* HSMI, J would pair complex energies $p \leftrightarrow \bar{p}$, which is exactly the mirror pairing (Thm. 42.1.2(a)): the one structural reason to pursue S1 below.

(2) *Galois.* c_∞ is the reflection in the scaling geodesic, $\sharp$ the reflection in the orthogonal geodesic, FE the half-turn (Prop. 42.6.2); $-1 \in \hat{\mathbb{Z}}^\times$ is the archimedean sign class in $C_{\mathbb{Q}}$ and plays no positivity role (L40 Prop. 7.1). L39's prime levels have discrete modular spectrum, hence no Borchers translations (Cor. 42.3.2(i)). Hodge/Rosati is the ellipticity of Frobenius (Thm. 42.6.8). (v2) At one prime, the Rosati involution on $\mathbb{Q}(\pi)$ *is* complex conjugation exactly in the Hilbert case, and a nontrivial real-field Galois conjugation in the Krein case (Prop. 42.7.1(c)); the Hilbert

form is the polarization of a complex structure J_g (Thm. 42.7.2); tropically, RH is purity and its mechanism uses the phase (Rem. 42.8.9).

(3) *Hyperbolic/thermal.* Rapidity reversal $= r_0$, orthogonality $= r_1$; BW boost with KMS at 2π; $\Delta^{1/4}$ = Wick rotation (Props. 42.3.3, 42.4.1); the Rindler analogy and its limits (Prop. 42.6.9).

42.10. Numerics

All in `num/l42/`. `checks.json` (`lemma42_checks.py`): C1–C5 symbolic identities, all true; C6 Mellin FE, differences $\leq 7.7 \cdot 10^{-47}$; C7 $a_p = -3, 16, 0, -30$ for $p = 101, 997, 10007, 1009$, all elliptic, fake rapidity 0.331021434564477; C8 Krein signs $\mp$ on conjugate eigenlines ($\tau = \frac{1}{2}, \frac{3}{2}, \frac{199}{100}$), parabolic Jordan block; C9 flip 200/200; C10 covariance and unitarity. `sl2.json`: Prop. 42.4.1. `modular.json` (`lemma42_modular.py`, mpmath 40 digits, 900 coefficients): Ex. 42.6.4. `satotate.json` (`lemma42_satotate.py`): Prop. 42.6.11. `v2_tropical_krein.json` (`lemma42_v2.py`, v2): Cor. 42.7.3, Prop. 42.7.5, Thms. 42.8.4, 42.8.5, Ex. 42.8.6 (11a, 37a; all primes < 20000 including $p = 2$, mpmath 40 digits). These numerics illustrate proved identities or known theorems; none bears on RH.

42.11. Circularity and metaphor flags

(F1) Building (P3) with $L_0 = \mathrm{diag}(t_\rho)$ from zeros is circular (L40 C7). (F2) "J is $\sharp$" in (P2) is *true*, but it is not positivity for $\sharp$ in Weil's sense: Hankel $\neq$ Toeplitz. (F3) "Frobenius = Euclidean rotation, ghost = boost" is a theorem for g_p (Thm. 42.6.8), and *metaphor* for Rindler/KMS (Prop. 42.6.9(c)) and for representations (Rem. 42.6.10). (F4) "Taylor–Wiles is at the nexus" is [ANALOGY]. (F5) No structural ζ constraint is claimed from the two-stage coincidence (Lemma 41: it is generic). (F6, v2) The 2×2 "stage (a) = stage (b)" of Cor. 42.7.3 is generic for real 2×2 families, and arithmetic Frobenius never moves along the family (Prop. 42.7.5): [ANALOGY] as a model of ghost birth. (F7, v2) "Newton above Hodge is tropical positivity" is [ANALOGY]. It is integrality and holds for fakes (Thm. 42.8.2). (F8, v2) "RH = purity" for ζ is [RESTATEMENT] (Prop. 42.8.8(b)).

42.12. Roadmap

Score = 0.6 sig+0.4 prob.

	item	sig	prob	score
S1	Krein-HSMI: does Thm. 42.3.1 survive in Krein/Pontryagin spaces? If not, J pairs ghosts as mirror pairs	.7	.45	.60
S2	(P3) Möbius deformation $L_0 \to \{t_\rho\}$ from prime data	1.0	.10	.64
S3	Absorption HSMI on $L^2(C_{\mathbb{Q}})$ (zeros as resonances of a positive-energy structure)	.6	.45	.54
S4	Genus g: ellipticity in $\mathrm{Sp}_{2g}(\mathbb{R})$ (Krein signatures of Frobenius; semisimplicity needed)	.4	.9	.60
S5	Petersson root-number form for general N: signature $= (\#\{\varepsilon = +1\}, \#\{\varepsilon = -1\})$ newforms; tables vs Atkin–Lehner	.3	.9	.54
S6	The pincer (Thm. 42.4.4) as a Vol. 2 standing theorem	.5	.75	.60
S7 (v2)	Genus g version of Thm. 42.7.2: Krein signature (g, g) of $i\omega$, Hodge J commuting with Frobenius, and which Sp_{2g} fakes are Krein-but-not-Hilbert (merges with S4)	.5	.85	.64
S8 (v2)	A quarter-turn J (linear, $J^2 = -1$) commuting with the scaling flow: can any free structure supply it (Rem. 42.7.6(3))?	1.0	.05	.62

S2's high score is driven by significance; its probability is a guess. Nothing here supplies G5.

42.13. Open questions

Q1. Do you accept the verdict: the lead's geometry is true (Thm. 42.1.2), but its positivity is Hankel and the spectrum Lebesgue, so the lead is *impossible*, not RH-equivalent (Cor. 42.3.2)? And the pincer as the official statement of why the rung is missing?

Q2. Should the missing rung be split officially into (P1) "$= Q$" and (P3) "$= \mathcal{R}$" (Rem. 42.2.3)? Is your intuition Euclidean (OS, P3) or Krein (S1)?

Q3. Do you want the hyperbolic y-dictionary (§42.4) and the Klein four-group in $\mathbb{H}$ (Prop. 42.6.2) as Vol. 2 conventions?

Q4. Your idea "RH = every arithmetic Frobenius lies in the Euclidean section" is recorded as Def. 42.6.12 ([RESTATEMENT]). Do you accept the precision that what forces ellipticity is a positive invariant form (compactness), not horizon regularity (Prop. 42.6.9(c)), and that for representations the right notion is temperedness, not unitarity (Rem. 42.6.10)?

Q5. Taylor–Wiles: is the Petersson root-number form (Thm. 42.6.3, signature $(1,1)$ at $N = 37$) worth a Vol. 2 remark[1] as the "$\sharp$-pairing that exists but measures only parity"?

Q6. Should S4 (genus g, Sp_{2g} Krein signatures) be the next feasible lemma, or S1 (Krein-HSMI)?

Q7. (v2) Do you accept the reformulation "at one prime, Hilbert vs Krein = imaginary vs real quadratic $\mathbb{Q}(\pi)$ = whether the Rosati involution is complex conjugation" (Prop. 42.7.1(c)) as the exact form of your "class paired with its own Galois conjugate" at the 2×2 level?

Q8. (v2) Thm. 42.7.2(a) says the extra Hilbert form is a *complex structure* J_g ($J_g^2 = -1$, linear) commuting with Frobenius, not an antilinear involution. Does your intuition for the missing rung point to a linear quarter-turn (Hodge/Deninger $*$) or to an antilinear conjugation (Tomita J)? This decides between S8 and S1.

Q9. (v2) Tropical: is it acceptable to record "the statement of RH is tropical (purity), the mechanism is not (it uses the phase)" as the official Vol. 2 conclusion[2] of the tropical thread (Rem. 42.8.9, with Lemma 23)?

42.14. Corrections recorded

(E1) Chat: "$\langle a, Ja\rangle \geq 0$ in an HSMI would be $\sharp$-type positivity, which is what Weil needs." Half true: the $\sharp$-action holds, but the positivity is Hankel, not Weil's Toeplitz form (Thms. 42.2.1, 42.2.2).

(E2) Chat: "building such an inclusion may be equivalent to RH." False: it is impossible unconditionally (Cor. 42.3.2).

(E3) L40 Prop. 6.10, (E10) and bridge row 3′: add the scope "when the scaling flow is the modular flow". In (P2) J acts as $\sharp$ on translation frequencies. Lemma 40 is not edited.

(E4) L39 R3 refined: an infinite-limit HSMI has Lebesgue spectrum, so zeros can only appear as absorption (Rem. 42.3.4).

(E5) L31 Prop. 6.3 and L39 Thm. 7.5(i) are special cases of Thm. 42.3.1.

(E6) L40 Def. 6.12 has a (P3) version with target $\mathcal{R}$ (Rem. 42.2.3).

(E7) Steering: "failure of Ramanujan $\Longleftrightarrow$ normalized Frobenius hyperbolic". Exact for the single matrix g_p. For automorphic representations, non-tempered complementary series are still unitary, so "boost" there means non-tempered, not non-unitary (Rem. 42.6.10).

(E8) Steering: "Euclidean angle $\theta = i\eta$ with 2π periodicity forced by regularity". True for Rindler/KMS. For Frobenius no periodicity is forced (Sato–Tate angles vary), and ellipticity comes from a positive form (Prop. 42.6.9).

[1] **[Vol. 3 editorial note]** With the approved two-volume split, the series is Vol. I (Lemmas 1–13), Vol. II (Lemmas 14–29) and Vol. III (Lemmas 30–47, this volume). "Volume 2" in this question meant the next volume of the series; for Lemmas 30–47 that is this volume. See Appendix A, item D1.

[2] **[Vol. 3 editorial note]** See the note to the previous question.

(E9) (v2) Chat: "g_p is real 2×2 with determinant 1 *because* the Weil pairing is symplectic". Correct but empty in rank 2: every 2×2 matrix scales an alternating form by its determinant (Prop. 42.7.1(a)). Also, positivity cannot be stated on the Tate module; the real model is $\mathbb{Q}(\pi) \otimes \mathbb{R}$ (Prop. 42.7.1(b)).

(E10) (v2) Chat: "the boundary $|a_p| = 2\sqrt{p}$ is a Jordan block, stage (a) in miniature". True for the matrix family. But it never occurs arithmetically: for p prime $a_p^2 \neq 4p$, the margin is $\geq \sqrt{3}/2\sqrt{p}$, and over square q the boundary case is the scalar ± 1, not a Jordan block (Prop. 42.7.5). The stage-(a)/(b) identification is [ANALOGY] (Rem. 42.7.4).

(E11) (v2) Chat: "Katz–Mazur theorem: Newton above Hodge, a positivity statement written tropically". Attribution: Katz's *conjecture*, proved by Mazur (Maz72, Maz73) under hypotheses and by Ogus in general (BO78). ("Katz–Mazur" usually names their book on moduli of elliptic curves.) Content: in genus 1 it is equivalent to $a_p \in \mathbb{Z}_p$, an integrality, satisfied by fakes; it is not a positivity (Thm. 42.8.2, Rem. 42.8.3). Also the slope labels were swapped: slope 0 comes from $H^{0,1} = H^1(\mathcal{O})$ and slope 1 from $H^{1,0}$.

(E12) (v2) Chat: "the max-plus polynomial recovers purity only up to a bounded error, the factor 2 in Hasse". Correct, and sharpened: coefficient purity is $|a| \leq \sqrt{p}$, strictly stronger than Hasse. About 39% of genuine primes show a spurious break $\leq \log_p 2$, and the error is $\leq \log_p 2$ in all cases (Thm. 42.8.5). Confirmed without change: "Weil means $\log|\alpha| = \frac{1}{2}\log p$ exactly" (for every complex embedding, Thm. 42.8.4) and "a boost splits slope $\frac{1}{2}$ into $\frac{1}{2} \pm \eta/\log p$" (natural log, Thm. 42.8.4).

CHAPTER 43

Lemma 43: Reversible holomorphic flows $\dot{w} = i\Xi(w)$

Fiedler's framework, Broughan's prior art, ♯-reversible fields over $\mathbb{F}_q$, Nash centers and the positivity game

Standalone note `lemma43.tex`, October 22, 2026.

Summary. Statement of intellectual honesty. RH is not proved here. Tags: [THM] proved here (elementary, proof given); [PRIOR ART] known in the literature (Broughan, Broughan–Barnett) and only re-derived here; [CITED] checked at the source (`lit/lemma43/verified_refs.md`); [BIB] bibliographic record verified, content standard; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma43_*.py`, data `num/143/`); [CONJ] conjecture; [OPEN] open.

Summary. We put the holomorphic flow $\dot{w} = f(w)$ of B. Fiedler's 2025 survey on top of the TAA function $\Xi(t) = \xi(\frac{1}{2} + it)$, i.e. $f = i\Xi$. (1) The reversibility of $i\Xi$ under $w \mapsto \bar{w}$ and the classification "simple real zero = center" are *prior art*: Broughan (Nonlinearity 2005) studied exactly this flow (time-reversed: $\dot{s} = \xi(s)$ with $s = \frac{1}{2} + iw$) and Broughan–Barnett (Math. Comp. 2004, 2006) studied the periods. We do not claim novelty there. (2) Several chat claims are corrected: an off-line pair $\rho, \bar{\rho}$ has eigenvalues $\lambda, -\bar{\lambda}$, which are a source/sink pair only if $\operatorname{Re}\lambda \neq 0$. So "RH $\iff$ all centers" is false as stated; Broughan's criterion adds a separatrix condition. (3) New here ([THM]): for an elliptic curve over $\mathbb{F}_q$ the generator of the Riccati flow of $P(T)$ is *exactly* the transpose of the Lemma 42 quarter turn, $M = (\sin\theta\, J_g)^T$, with $\exp((\theta/\sin\theta)M) = g^T$. (4) New ([THM]+[NUM]): a ♯-reversible rational field $f_Z = u\,\Xi_C(u)/((u + u^{-1}) - c)$ for $Z_C(T)$, whose poles $T = 1, 1/q$ form a ♯-pair. Weil curves (11a, 37a at several p including the supersingular $p = 19$; a genus-2 curve over $\mathbb{F}_{11}$) give only centers, and no pole separatrix ends at a Frobenius equilibrium. Fakes have source/sink pairs, and separatrices terminate on them. This is a finite-field analogue of Broughan's band criterion. (5) Periods: no $\mathbb{Z}$-relations (PSLQ), so Lemma 36 integrality is *not* seen; only the Newton field $i\Xi/\Xi'$ has period module $2\pi\mathbb{Z}$. (6) Nash: replicator centers of zero-sum games, KL conservation, and recurrence are cited; the link to $i\Xi$ is [ANALOGY]. (7) Positivity as a zero-sum game: honest value statements, the Bombieri/Yoshida results as verified, naive learners (MMWU), a quantum center theorem with exact hypotheses, and toy numerics in which the DH "ghosts" appear as $\frac{1}{2}\#\{\text{off-line zeros}\}$ negative directions.

43.1. Fiedler's framework (verified, [CITED])

Reference: B. Fiedler, *Real-time blow-up and connection graphs of rational vector fields on the Riemann sphere*, record verified via Crossref: Japan J. Indust. Appl. Math. **42**(4) (2025) 1499–1561, doi 10.1007/s13160-025-00749-8, open access, arXiv 2504.20503. Read in full (HTML).

(1) For $\dot{w} = f(w)$ with f holomorphic and $f(e) = 0$, $f'(e) \neq 0$, the linearization is multiplication by $f'(e)$. The point e is a sink, center or source according as $\operatorname{Re} f'(e) < 0, = 0, > 0$ (Fiedler (1.24)).

(2) A center has minimal period $T = 2\pi i/f'(e)$ (1.37). All nearby orbits have the same period (holomorphic centers are *isochronous*).

(3) The period module is $P = 2\pi i\langle\eta_j\rangle_{\mathbb{Z}}$ with $\eta_j = 1/f'(e_j) = \mathrm{Res}_{e_j}(1/f)$ (1.28)–(1.32). The imaginary-time flow $\dot w = if$ commutes with $\dot w = f$ and defines the iso-periodic foliation (§1.4.2).
(4) Prop. 2.8 concerns *anti*-holomorphic fields $\dot w = \overline{f(w)}$ with $f = -\mathrm{grad}F$. Such a field is the gradient flow of $\mathrm{Re}F$ and the Hamiltonian flow of $\mathrm{Im}F$, as in (2.14).
(5) §3.1 (Masuda): $\dot w = w^2 - 1$. Its flow maps form a one-parameter Möbius subgroup. In real time ±1 are a source and a sink; in imaginary time they are two centers of period $\pm\pi$.

PROPOSITION 43.1.1 ([THM]). *Let $\Phi = \int dw/f$ (multivalued). Then $f = |f|^2\,\overline{\Phi'}$. Thus, after the time change $dt \mapsto |f|^2dt$, $\dot w = f$ is the anti-holomorphic field $\overline{\Phi'}$: the gradient flow of* $\mathrm{Re}\,\Phi$ *and the Hamiltonian flow of* $-\mathrm{Im}\,\Phi$*, up to the sign convention of Prop. 2.8.* $\mathrm{Im}\,\Phi$ *is single-valued near e_j iff* $\mathrm{Re}\,\eta_j = 0$*, i.e. iff e_j is a center. Near a sink or a source every continuous first integral is constant.*

PROOF. $\Phi' = 1/f$, so $\overline{\Phi'} = \bar f/|f|^2$. Going once around e_j changes Φ by $2\pi i\eta_j$, which changes $\mathrm{Im}\Phi$ by $2\pi\,\mathrm{Re}\,\eta_j$. A first integral is constant on orbits; near a sink every orbit accumulates at e_j, so continuity forces it to be constant. □

43.2. Reversibility of $\dot w = i\Xi(w)$: prior art and corrections

$\Xi(w) = \xi(\frac12 + iw)$ is entire, even, and real on $\mathbb{R}$. Let $R(w) = \bar w$. A holomorphic f is *R-reversible* if $f(\bar w) = -\overline{f(w)}$. Then R conjugates the flow to its time reversal (Lamb–Roberts 1998; Devaney 1976 [BIB]).

THEOREM 43.2.1 ([PRIOR ART]; Broughan 2005, re-derived). *Let $f = i\Xi$.*

(a) f is R-reversible.
(b) A simple real zero γ is a center with period $T_\gamma = 2\pi/\Xi'(\gamma)$, whose sign is the Krein sign *of the center. A real zero of multiplicity $m \ge 2$ is degenerate, with $2(m-1)$ elliptic sectors.*
(c) For a non-real simple zero e, $\bar e$ is also a zero, with $f'(\bar e) = -\overline{f'(e)}$: the eigenvalues are $\lambda, -\bar\lambda$. If $\mathrm{Re}\lambda \neq 0$ they are a source/sink pair. If $\Xi'(e) \in \mathbb{R}$ (i.e. $\mathrm{Re}\lambda = 0$) both are centers.
(d) Consecutive simple real zeros have opposite Krein signs, since Ξ' alternates in sign. A pair of real zeros can leave $\mathbb{R}$ under a reversible deformation only after colliding, and the colliding zeros must have opposite signs.

PROOF. (a) $\Xi(\bar w) = \overline{\Xi(w)}$ (real on $\mathbb{R}$), so $i\Xi(\bar w) = -\overline{i\Xi(w)}$. (b) $f'(\gamma) = i\Xi'(\gamma)$ with $\Xi'(\gamma) \in \mathbb{R}^\times$, so $T = 2\pi i/f' = 2\pi/\Xi'$. For multiple zeros, $f \sim ic(w-\gamma)^m$ with c real, and the local phase portrait of z^m has $2(m-1)$ elliptic sectors (Broughan Thm. 3.3 in the preprint numbering, [CITED]). (c) Differentiate $f(\bar w) = -\overline{f(w)}$. (d) Signs alternate at simple zeros of a real function. A real simple zero stays real under small real perturbations (IVT), so it can leave $\mathbb{R}$ only by merging with a neighbour, whose sign is opposite. □

REMARK 43.2.2 (Krein structure, [THM]). For an off-line pair the linearization is $\mathrm{diag}(\lambda, -\bar\lambda)$, which is skew-Hermitian for the form $H = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ (a hyperbolic plane). The multipliers $e^{t\lambda}, e^{-t\bar\lambda}$ are a Krein pair $\mu, 1/\bar\mu$. This is the same hyperbolic-plane signature that carries the DH ghosts in Lemmas 24/42; see §43.8.1.

REMARK 43.2.3 (Broughan's criterion, [CITED]). Broughan's preprint (numbering may differ from the published version) defines bands, the band number, and re-entrant separatrices (Defs. 4.1–4.3). It proves (Thm. 4.8) that band number 0 $\iff$ RH with all zeros simple, and (Thm. 4.9) that RH $\iff$ there are no re-entrant separatrices and every zero is a center or a multiple zero. The separatrix condition is exactly what rules out the accidental off-line centers of Thm. 43.2.1(c). Our flow is Broughan's $\dot s = \xi(s)$ with time reversed.

THEOREM 43.2.4 (Newton field, [THM]). *$\dot w = i\Xi/\Xi'$ has $f'(e) = i/m$ at every zero of multiplicity m, whether on or off the line. Every zero is therefore a center of period $2\pi m$, and the period module is $2\pi\mathbb{Z}$. This field cannot see RH.*

PROOF. If $\Xi = (w-e)^m g$ then $\Xi/\Xi' = (w-e)/m + O((w-e)^2)$. □

43.3. Numerics for ζ, Davenport–Heilbronn, and periods ([NUM])

`num/lemma43_checks.py` (mpmath, 50 digits):

- First 30 zeros of ζ: $\max|\mathrm{Im}\,\Xi'(\gamma)|/|\Xi'(\gamma)| \le 7\times10^{-40}$. The signs of $\Xi'(\gamma_n)$ alternate, starting $-,+$.
- $\log_{10}|T_n|$ for $n = 1..10$: $3.66, 5.55, 6.70, 8.42, 9.19, 10.53, 11.71, 12.40, 13.99, 14.60$. A linear fit of $\log|T_n|$ against γ_n has slope 0.710, against $\pi/4 \approx 0.785$. The residual $\log|T_n| - \frac{\pi}{4}\gamma_n$ is $\approx -1.4\log\gamma_n$ for $n \ge 4$. This is consistent with Broughan–Barnett 2006 ([CITED]: $\log T_n \ge \frac{\pi}{4}\gamma_n + O(\log\gamma_n)$, equality conjectured), allowing for normalization of ξ; ten zeros cannot test it.
- PSLQ on $(T_1,\dots,T_k)$, $k = 2,3,4,6$, coefficients $\le 10^6$: no integer relation found.
- Davenport–Heilbronn: $f = \frac12(1 - i\tan\theta)L(s,\chi_1) + \frac12(1 + i\tan\theta)L(s,\chi_2)$ mod 5, completed by $(5/\pi)^{(s+1)/2}\Gamma(\frac{s+1}{2})$. The functional equation was checked to relative error 1.6×10^{-41} (Titchmarsh §10.25 definition). Off-line zero $\rho = 0.808517182456637+85.6993484853776i$, $|\Xi_{DH}| = 6.7\times10^{-68}$; there $\lambda = 1.2309\times10^{-28} + 3.666\times10^{-29}i$, and at the mirror zero $\lambda' = -\bar\lambda$ to 2.6×10^{-41}. So this pair is a genuine **source/sink pair** with tiny real part. Second pair at $0.650830080609737 + 114.163342730757i$: $\lambda = 1.4348\times10^{-38} + 1.176\times10^{-39}i$. The Newton eigenvalue is i to 10^{-45} (Thm. 43.2.4). Five on-line DH zeros in $[80, 90]$ have alternating Krein signs.

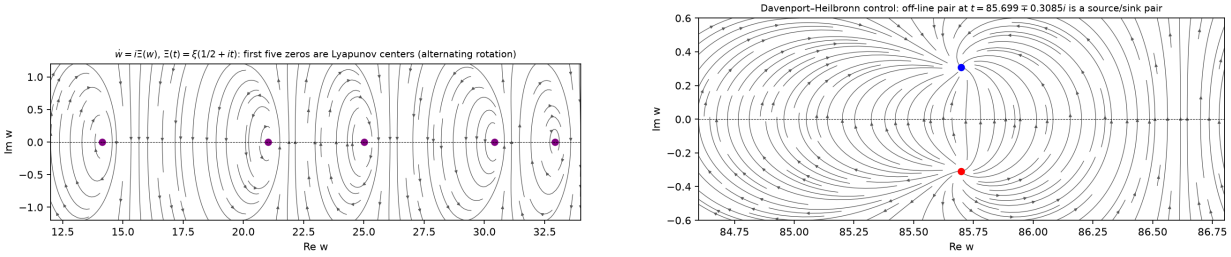

FIGURE 43.1. Left: phase portrait of $\dot w = i\Xi(w)$ near low zeros (normalized field). Right: DH near the off-line pair at height 85.70 (source/sink with $\mathrm{Re}\lambda \sim 10^{-28}$).

REMARK 43.3.1 (Periods vs Lemma 36, [NUM]/[OPEN]). The center periods $2\pi/\Xi'(\gamma)$ grow like $e^{\pi\gamma/4}$ and show no $\mathbb{Z}$-structure. The Lemma 36 integrality is reproduced only by the Newton field (Thm. 43.2.4), where it is automatic and therefore not informative. Chat claim (e) is *not supported*.

43.4. Function fields: Riccati flows and the quarter turn

Let $C/\mathbb{F}_q$ have $P(T) = \prod(1-\alpha_j T)$. Put $T = q^{-1/2}e^{-iw}$, so $\Xi_C(w) = q^{-g/2}T^{-g}P(T)$, which is real and even. The mirror $\sharp$ ($|T| = q^{-1/2}$) becomes $w \mapsto \bar w$. For $g = 1$, $\Xi_C = 2(\cos w - \cos\theta)$ with $a = 2\sqrt{q}\cos\theta$.

THEOREM 43.4.1 ([THM], new identification). *For $g = 1$ the field $f_C(T) = q^{-1/2}(1 - aT + qT^2)$ is a Masuda/Riccati field. Its flow maps form a one-parameter subgroup $\exp(tM) \subset \mathrm{PGL}_2$, where M is the trace-free matrix of the Riccati field. With $J_g = (g - \cos\theta)/\sin\theta$ the Lemma 42 quarter turn and g the normalized Frobenius, $M = (\sin\theta\, J_g)^T$ and $\exp(\frac{\theta}{\sin\theta}M) = g^T$. The two Frobenius*

equilibria are centers with eigenvalues $\mp 2i\sin\theta$ and period $-\pi/\sin\theta$, exactly when the Hasse bound holds ($|\cos\theta|<1$). Otherwise they are a source/sink $\sharp$-pair.

PROOF. For a Riccati field $\dot T=\alpha+\beta T+\gamma T^2$, the flow is the Möbius flow of $M=\left(\begin{smallmatrix}\beta/2 & \alpha\\ -\gamma & -\beta/2\end{smallmatrix}\right)$ up to transpose convention; here $(\alpha,\beta,\gamma)=q^{-1/2}(1,-a,q)$. Since $J_g^2=-1$, $\exp(sJ_g)=\cos s+J_g\sin s$, and at $s=\theta$ this is g. The eigenvalues are $\pm\sqrt{\beta^2-4\alpha\gamma}=\pm\sqrt{a^2/q-4}=\pm 2i\sin\theta$. □

Numerics ([NUM], `num/l43/ff.out`): 11a at $p=2,3,5,7,13,17,19,23,29,31,37$ ($a_p=-2,-1,1,-2,4,-2,0,-1,0,7,3$; $p=19,29$ supersingular). Every root has $|T|\sqrt{q}=1$, Re eigenvalue $\le 3\times10^{-41}$, and both identities hold to $\le 1.8\times10^{-40}$. Fake $q=13,a=8$ (violates Hasse): roots $0.17446, 0.44093$ (a $\sharp$-pair), eigenvalues ∓ 0.96077. Genus 2, $y^2=x^5+x+1$ over $\mathbb{F}_{11}$: $N_1=8,N_2=134$, so $c_1=-4,c_2=14$; all four roots satisfy $|\alpha|=\sqrt{q}$, with center eigenvalues $\pm 2.3687i,\pm 4.0748i$.

43.4.1. The $\sharp$-reversible zeta field (main new computation). Put $u=\sqrt{q}\,T$. Then $\sharp: u\mapsto 1/\bar u$. A holomorphic f is $\sharp$-reversible iff $f(1/\bar u)=\bar u^{-2}\,\overline{f(u)}$.

LEMMA 43.4.2 ([THM]). *$f=u\,h(u)$ with $h(1/\bar u)=\overline{h(u)}$ is $\sharp$-reversible. (Warning: $f=iuh$ is* anti-*reversible; this sign was checked numerically.) On $|u|=1$, $f/u=h$ is real, so the flow crosses the circle orthogonally.*

PROOF. $f(1/\bar u)=\bar u^{-1}h(1/\bar u)=\bar u^{-1}\overline{h(u)}=\bar u^{-2}\overline{uh(u)}$. □

DEFINITION 43.4.3 ([DEF]). $f_Z(u)=u\,\Xi_C(u)\,/\,\big((u+u^{-1})-c\big)$, $c=\sqrt{q}+1/\sqrt{q}$. Here $\Xi_C(u)=u^{-g}P(u/\sqrt{q})\cdot\text{const}$ is $\sharp$-real. The poles are at $u=\sqrt{q},1/\sqrt{q}$, i.e. $T=1/q,1$: the poles of Z_C, forming a $\sharp$-pair. Each (simple) pole has two incoming and two outgoing separatrix directions, alternating (Shapiro–Tahar §6, Prop. 6.1, [CITED]: a pole of order s of R in $-R\partial_z$ has $s+1$ incoming and $s+1$ outgoing directions; here $s=1$). Shapiro–Tahar also show that these directions are the tangent rays of the inflection curve $(R')^{-1}(\mathbb{R}P^1)$ at the pole, and that every bounded component of that curve contains a pole (Prop. 5.4). We did not compute inflection curves of f_Z ([OPEN]).

THEOREM 43.4.4 ([THM], finite-field band criterion, easy half). *Let P have real coefficients, $\deg P=2g$, and functional equation. The following are equivalent: (i) RH for P (all $|\alpha_j|=\sqrt{q}$); (ii) Ξ_C restricted to $|u|=1$ (a real trigonometric polynomial of degree g) has $2g$ zeros counted with multiplicity (equivalently, for simple zeros, $2g$ sign changes). If the zeros are simple, (i) implies that every Frobenius equilibrium of f_Z is a center, and then no separatrix of f_Z ends at a Frobenius equilibrium.*

PROOF. A real trigonometric polynomial of degree g has at most $2g$ zeros on the circle, so (i) $\iff$ (ii) is a count. At a simple zero u_0 on the circle, $f_Z'(u_0)=u_0\Xi_C'(u_0)/(\dots)$ is purely imaginary by the orthogonal-crossing structure. A center is surrounded by periodic orbits, so no trajectory can tend to it. □

CONJECTURE 43.4.5 ([CONJ], finite-field Broughan analogue). If P has a $\sharp$-pair of off-circle roots with $\operatorname{Re}\lambda\neq 0$ (a source/sink pair), then some pole separatrix of f_Z terminates at a Frobenius equilibrium. Accidental off-circle centers ($\operatorname{Re}\lambda=0$) can occur; for those a re-entrant separatrix (Broughan Def. 4.3) is expected to detect the failure.

Numerics ([NUM], `num/l43/sharp.out`, Fig. 43.2) cover 11a at $p=13$ and at the supersingular $p=19$, 37a at $p=13,17$, and $y^2=x^5+x+1$ over $\mathbb{F}_{11}$. In all of them every Frobenius equilibrium is a center, there are $2g$ sign changes, and the pole separatrices either connect the two poles (crossing $|u|=1$ once) or run to $u=0$ or $u=\infty$. In the fakes ($g=1$, $q=13$, $a=8$; $g=2$ with off-circle cos-roots $0.7\pm 0.3i$) there are 0 sign changes, the equilibria are sources and sinks, and pole separatrices *end at them.* The accidental off-circle-center fake was not produced: the root bracketing at $y_0=0.3$ failed. That case is [OPEN].

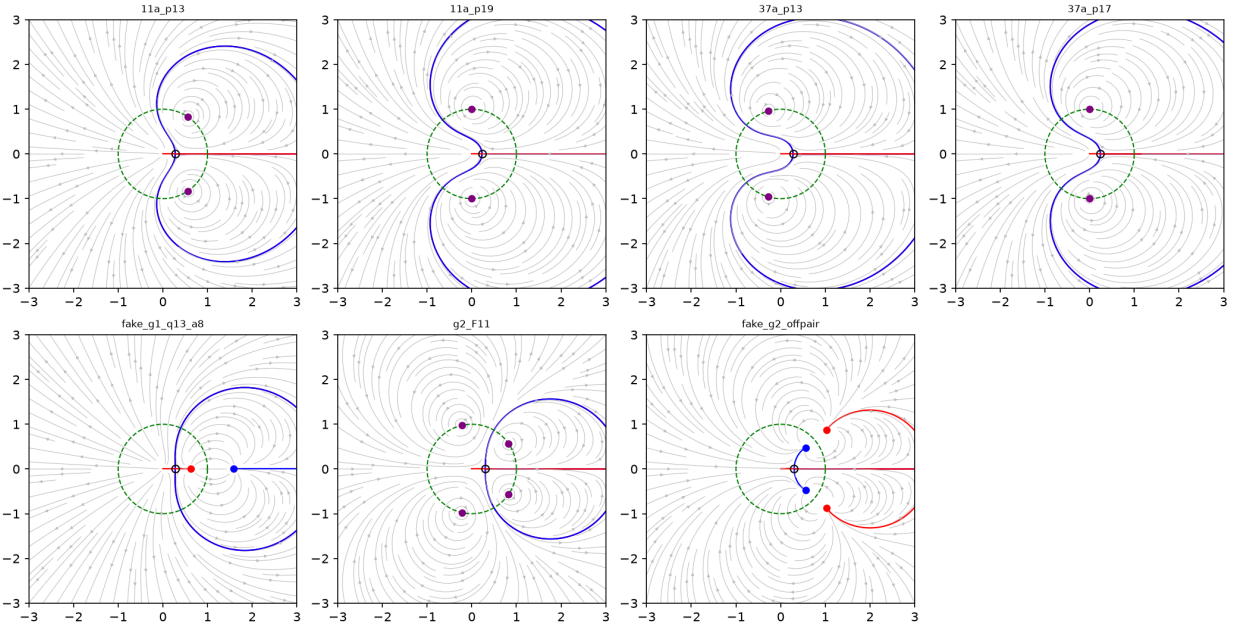

FIGURE 43.2. ♯-reversible fields f_Z. Dashed: $|u| = 1$. Purple: centers; red/blue: sources/sinks; black circle: pole $u = 1/\sqrt{q}$ (the pole $\sqrt{q}$ lies outside the frame). Blue/red curves: pole separatrices.

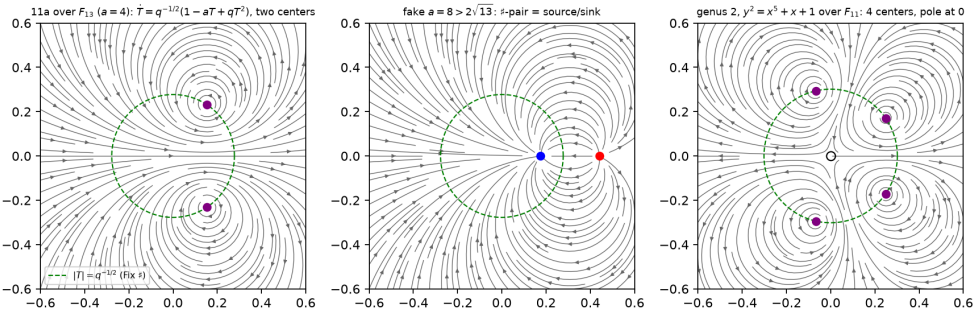

FIGURE 43.3. Riccati flows for 11a (centers) versus the Hasse-violating fake (source/sink).

43.5. First integrals, Prop. 2.8, and the quarter turn

THEOREM 43.5.1 ([THM]). *(a) Near each center γ of $i\Xi$, $H = \operatorname{Re}\int^w ds/\Xi(s)$ is a local first integral. (b) On a horizontal strip containing only simple real zeros, H is a single-valued global first integral iff all residues $1/\Xi'(\gamma_j)$ are real, which is true there. (c) For $f = iP$ with P real-rooted, $H = \sum_j r_j \log|w - \gamma_j|$ with $r_j = 1/P'(\gamma_j) \in \mathbb{R}$ and $\sum r_j = 0$ (for $\deg P \geq 2$). This is the logarithmic potential of a signed, mass-zero measure. Its sign pattern alternates (Krein). (d) For a holomorphic g, the time average of g over a periodic orbit encircling a single center γ equals $g(\gamma)$.*

PROOF. (a)–(b): $dH = \operatorname{Re}(dw/\Xi)$ and $\dot{w}/\Xi = i$, so $\dot{H} = \operatorname{Re} i = 0$; single-valuedness follows from Prop. 43.1.1. (c): partial fractions, using $\sum 1/P'(\gamma_j) = 0$ for $\deg \geq 2$. (d): $\frac{1}{T}\oint g\,dt = \frac{1}{T}\oint g\,dw/f = \frac{2\pi i}{T}\,g(\gamma)/f'(\gamma) = g(\gamma)$. □

REMARK 43.5.2 ([ANALOGY]). H resembles a relative entropy $\sum x^* \log x$ only as an analogy: the weights r_j are signed. Lemma 39 (relative entropy is invariant under its own modular flow) is the same mechanism, "a conserved quantity from an invariant reference state", and nothing more.

REMARK 43.5.3 (correction (d), [THM]). Prop. 2.8 is about *anti*-holomorphic fields, whose linearization anticommutes with i. The complex structure $J = i$ commutes with *every* holomorphic flow, DH included, so it carries no positivity information. The only exact link between a flow generator and a positivity-relevant $J^2 = -1$ is Thm. 43.4.1 ($g = 1$ over $\mathbb{F}_q$). For ζ it is [ANALOGY].

43.6. Connection graphs ([CONJ]/[OPEN])

Fiedler's subject is the connection graph of rational fields. For $f = iP$ with P real with simple real roots, $\mathbb{R}$ is invariant by reversibility (Thm. 43.2.1(a)) and every root is a center; the period annuli of consecutive centers are separated by separatrices of ∞. *Expected* ([CONJ], not proved here): the adjacency graph of the period annuli is a path ordered as on $\mathbb{R}$. Whether the converse holds (path graph $\Rightarrow$ real-rootedness within reversible fields) is [OPEN]; we have no proof and no counterexample. For Ξ itself, Broughan's bands (one on-line zero per band, [CITED]) give the corresponding statement.

43.7. Nash equilibria and replicator centers

Cited facts ([CITED]/[BIB]): replicator dynamics of zero-sum (more generally, rescaled zero-sum) bimatrix games has a conserved $\sum x_i^* \log x_i + \sum y_j^* \log y_j$ at an interior Nash equilibrium (the KL divergence to Nash is constant), interior equilibria are not asymptotically stable, and the system is Hamiltonian (Akin–Losert 1984; Hofbauer 1996; Hofbauer–Sigmund 1998). Time averages of interior orbits converge to the Nash equilibrium (Hofbauer–Sigmund).

PROPOSITION 43.7.1 ([THM], correction to steering). *For two-population 2×2 bimatrix replicator dynamics, the Jacobian at an interior equilibrium has trace* 0. *Interior equilibria are therefore centers (linearly) or saddles,* never *sinks or sources. Sinks and sources arising from "non-zero-sum perturbations" need single-population games, e.g. generalized RPS.*

PROOF. At an interior point, $\partial \dot{x}/\partial x = x(1-x)\partial_x(\text{payoff difference of } x)$, which vanishes because player 1's payoff difference does not depend on x. Likewise for y. □

Numerics ([NUM], `143/games.out`): matching pennies has KL drift $\leq 8 \times 10^{-11}$; periods $6.299, 6.432, 6.743, 7.383, 9.122$ (non-isochronous); time average over one period $= (\frac{1}{2}, \frac{1}{2})$ to 5×10^{-13}. 121 random 2×2 games: $|\mathrm{tr}| \leq 1.6 \times 10^{-16}$, 52 centers, 69 saddles. RPS: $a = b$ gives $\pm 0.57735i$ with KL constant; $(a, b) = (1.2, 1)$ gives $-0.0333 \pm 0.635i$ (sink); $(1, 1.2)$ gives a source.

REMARK 43.7.2 ([ANALOGY], honest). Holomorphic centers are isochronous; replicator centers are not. Hence *no* holomorphic conjugacy exists between a replicator center and a center of $i\Xi$. "Zero of Ξ = Nash equilibrium" is an analogy: both are elliptic equilibria of a reversible or Hamiltonian system, with a conserved log-potential (KL vs. Thm. 43.5.1(c)), and time averages equal the equilibrium (Thm. 43.5.1(d) vs. Hofbauer–Sigmund). As mathematics this is a restatement of standard facts.

43.8. Positivity as a zero-sum game

Value. Let W be the Weil form on a window D_A (test functions supported in an interval of length A), and $\lambda^*(e^A) = \inf_{F \in D_A} Q(F)/\|F\|^2$ (Lemma 24).

PROPOSITION 43.8.1 ([THM]). *For self-adjoint, bounded-below W, $\inf_\rho \mathrm{tr}(W\rho)$ over density matrices equals $\lambda_{\min}(W)$, attained at the bottom eigenvector (the "ghost's" optimal strategy). The number of negative directions equals the Krein negative index. Value* 0 *with a nonzero minimizer*

means the Gram matrix is degenerate (Lemma 42 §7). With a single W*, this is a one-player problem: SDP duality* $\inf \operatorname{tr}(W\rho) = \sup\{t : W - tI \succeq 0\}$ *is trivial.*

Two-player version. The defender mixes over W_k (windows, kernels, or certificates): $y \in \Delta$, $W(y) = \sum y_k W_k$, payoff $\operatorname{tr}(\rho W(y))$, ghost minimizing. *Weil explicit formula reading* ([ANALOGY]/[RESTATEMENT]): archimedean term = defender, prime sum = attacker; "RH $\iff$ value ≥ 0 for every support" is a restatement of Weil's criterion [RESTATEMENT].

Verified small-support results ([CITED], Bombieri 2000 abstract): Bombieri shows the minimum of the Weil functional on the unit ball of $L^2[-t,t]$ is attained. He re-proves Yoshida's theorem that the functional is positive definite for t *sufficiently small*. Once the truncation is big enough, the number of negative eigenvalues is half the number of off-line zeros. Yoshida (ASPM 21, 1992) is cited only as "sufficiently small"; the exact support constant was *not* verified.

Lemma 24 certificates as defender certificates ([THM], exact identification). A Lemma 24 certificate $\lambda^*(e^A) \geq c > 0$ is precisely a dual certificate $W_A - cI \succeq 0$ on D_A, i.e. a defender guarantee of value $\geq c$. Certified values (read-only from `num/lemma24_progress.md`)[1]: $A = 2$, hybrid $\geq 5.038361831 \times 10^{-30}$, prime-only $\geq 4.994651852 \times 10^{-30}$; $A = 1.6$: $\geq 1.464603146 \times 10^{-17}$; $A = 1.55$: $\geq 2.287026830 \times 10^{-16}$; $A = 1.5$: $\geq 2.446287476 \times 10^{-15}$. At $A = 3$ there is *no* window certificate; the $L \geq 0.55$ figure is a mid-band frame bound (Lemma 36 correction (a)), not window positivity. *Missing rung* = a universal defender strategy valid for all A ([OPEN], equivalent to RH [RESTATEMENT]).

Proposition 43.8.2 ([THM], correction). *The symmetric game $x^T W x$ on the simplex is a common-interest (potential) game. Its replicator dynamics is a Shahshahani/Fisher gradient flow ($V = x^T W x$ is monotone), so it has no centers. Numerics: V is monotone and converges to a vertex. Whether some reduction of Weil positivity to a game with Nash* centers *exists is [OPEN].*

43.8.1. Naive learners. (a) The roles are min (ghost) and max (defender). The ghost is the minimizer's limiting strategy. The real presupposition is *complete information*: the learner sees the payoff $\operatorname{tr}(W\rho)$, i.e. it already "knows" W, whose positivity is the question.

(b) No-regret/MWU (Freund–Schapire 1999; Cesa-Bianchi–Lugosi 2006; matrix MWU: Arora–Kale 2007/2016, [BIB]). With cumulative feedback $\sum_s W_s$, the ghost plays the Gibbs state $\rho \propto \exp(-\beta \sum_s W_s)$. For constant W: $\lambda_{\min} = -\lim_{\beta\to\infty} \beta^{-1} \log \operatorname{tr} e^{-\beta W}$, and $\frac{d}{d\beta}\operatorname{tr}(W\rho_\beta) = -\operatorname{Var}_{\rho_\beta}(W) \leq 0$ ([THM]). Cooling is therefore monotone: a gradient flow with no center.

(c)

Theorem 43.8.3 ([THM], adapting MPP18). *In continuous follow-the-regularized-leader with entropic regularizers (continuous MMWU for the ghost, MWU/replicator for the defender), $\dot{Y} = W(y)$, $\rho = e^{-Y}/\operatorname{tr}$, $\dot{z}_k = \operatorname{tr}(\rho W_k)$, $y = \operatorname{softmax}(z)$. If an interior equilibrium exists — $W(y^*) = vI$ (*isotropic defender mix*) and $\operatorname{tr}(\rho^* W_k) = v$ for all k with $\rho^* > 0$ — then $G = S(\rho^*\|\rho) + \operatorname{KL}(y^*\|y)$ is conserved, and the orbits are Poincaré recurrent. If no isotropic mix exists, the ghost's scores grow linearly in some direction, and ρ tends to the boundary (rank-deficient states).*

Proof. $\frac{d}{dt}S(\rho^*\|\rho) = \frac{d}{dt}[\operatorname{tr}\rho^* Y + \log \operatorname{tr} e^{-Y}] = \operatorname{tr}((\rho^* - \rho)W(y))$, and $\frac{d}{dt}\operatorname{KL}(y^*\|y) = -\sum(y_k^* - y_k)\operatorname{tr}(\rho W_k)$. The sum is $\operatorname{tr}(\rho^* W(y)) - \operatorname{tr}(\rho W(y^*)) = \sum_k y_k v - v = 0$. Recurrence: the score dynamics is divergence-free and G is proper, as in Mertikopoulos–Papadimitriou–Piliouras 2018, Thm. 4.2 ([CITED], arXiv 1709.02738). Without an isotropic mix, $\dot{Y} = W(y)$ has a component bounded away from 0 along a fixed direction (separation). This last step is sketched, not proved in full generality. □

[1] **[Vol. 3 editorial note]** These are certificates logged by the Lemma 24 Toeplitz/Levinson pipeline. Volume II lists them in its Appendix on pending results as *logged, write-up pending*, and does not count them as theorems. This volume does not count them either. See Appendix A, item D3.

So the quantum game has a Hamiltonian center *iff* the defender can make W isotropic. For a single Weil Gram ($W \neq vI$) this fails, and the dynamics is the gradient cooling of (b).

(d) Numerics ([NUM], `num/lemma43_learn*.py`, `l43/learn*.json`). The Lemma 24 Gram matrices have $\lambda \sim 10^{-17}$–10^{-30}, below double precision, so we did *not* rerun learners on them; files were read-only and no running job was touched. Instead we built toy *zero-side* Grams, with sine basis on $[0, A]$ and zeros ≤ 120: $W_{jk} = \sum_\rho \mathrm{Re}[\hat{F}_j(t_\rho)\hat{F}_k(-t_\rho) + \hat{F}_k(t_\rho)\hat{F}_j(-t_\rho)]$. For ζ this is PSD *by construction* (circular; it illustrates only).

	$A = 1, J = 40$	$A = 4, J = 150$	$A = 8, J = 300$
ζ (38 zeros) $\lambda_{\min}$	3.9×10^{-7}	≈ 0 (-5×10^{-15})	≈ 0 (rank-deficient)
DH (64 on-line, 2 off-line quadruples) $\lambda_{\min}$	0.0471	-1.570	-42.56
DH #neg	0	3	4

At $A = 8$, DH has $4 = \frac{1}{2} \cdot 8$ negative directions: the off-line zeros $t_\rho, \bar{t}_\rho, -t_\rho, -\bar{t}_\rho$ number 8, matching Bombieri's count statement in this toy. The off-line block alone has signature $(4, 4)$ (hyperbolic planes). At small A the ghosts are masked (the DH toy Gram is PSD at $A = 1$). This is *value discovery*: discrete MMWU at $A = 1$ finds $\mathrm{tr}(W\rho_t) = 0.04705296602$ at $t = 1000$ for DH ($= \lambda_{\min}$), while ζ is still at 3.5×10^{-5} at $t = 4000$ (slow because $\lambda_{\min} \sim 10^{-7}$). Quantum matching pennies, $W(y) = (2y - 1)\sigma_z$ (isotropic at $y^* = \frac{1}{2}$): G drifts by 8.1×10^{-3} over $T = 200$ (explicit Euler, $dt = 10^{-3}$; the drift is integrator error), and the orbit returns within 2.1×10^{-3} (cycling). Adding $0.3\sigma_x$ destroys isotropy: G grows from 0.059 to 17.7, and the ghost converges to a pure state.

(e) [OPEN]: can the payoff rule itself (the Weil form) emerge from learning, e.g. as the unique payoff under which a natural learner has an isotropic interior equilibrium? We have no candidate.

43.9. Corrections to chat claims

(a) "On-line zeros are automatically centers": only *simple* ones are. Multiple zeros give elliptic sectors (Broughan Thm. 3.3).

(b) "Off-line pairs $\lambda, -\bar{\lambda}$ are one source and one sink": only if $\mathrm{Re}\lambda \neq 0$. If $\Xi'(e) \in \mathbb{R}$, both are centers, so "RH $\iff$ all centers" is false. Broughan's Thm. 4.9 adds "no re-entrant separatrices". DH has genuine source/sink pairs with $\mathrm{Re}\lambda \approx 1.2 \times 10^{-28}$.

(c) "Mirror $|T| = q^{-1/2}$ with $\sharp$": correct as a mirror, but the reversible field is $u \cdot h$, not $iu \cdot h$, and orbits cross the circle orthogonally.

(d) "Prop. 2.8 gives a linear $J^2 = -1$ commuting with the flow": Prop. 2.8 is anti-holomorphic. $J = i$ commutes with every holomorphic flow and carries no positivity. The exact link is Thm. 43.4.1 ($g = 1$), and for ζ it is [ANALOGY].

(e) "Center periods relate to L36 integrality": no evidence (PSLQ is null; $\log T \sim \frac{\pi}{4}\gamma$, Broughan–Barnett). Only the Newton field gives $2\pi\mathbb{Z}$, trivially.

(f) Steering: interior equilibria of bimatrix games are never sinks; the single-W positivity game is one-player; $x^T W x$ is common-interest (gradient); a quantum center requires an isotropic $W(y^*) = vI$; $A = 3$ has no window certificate.

43.10. Open questions

(1) Is the $g = 1$ identity $M = (\sin\theta J_g)^T$ the "quarter turn generates Frobenius" statement you want? Should it be pushed to $g \geq 2$ via Sp_{2g}, with one Riccati block per Frobenius angle?

(2) The $\sharp$-reversible zeta field is our finite-field Broughan analogue. Do you want a proof of the conjectured converse (fake $\Rightarrow$ separatrix ends at a Frobenius equilibrium) before going further, or the accidental off-circle-center case first?

(3) The positivity game has a Hamiltonian center only with an isotropic defender mix $W(y^*) = vI$. Is there a natural family of Weil windows whose convex hull contains a multiple of I? That would be a game-theoretic form of the missing rung.
(4) Should the Nash material stay as an [ANALOGY] appendix, given there is no holomorphic conjugacy (isochrony fails)?
(5) Broughan–Barnett periods grow like $e^{\pi\gamma/4}$. Is there any normalization (e.g. dividing by $|\Gamma|$ factors) under which a period lattice could connect to Lemma 36, or should (e) be dropped?

CHAPTER 44

Lemma 44: Arithmetic cavities

Ray-transfer matrices, Gouy phases, finite apertures, and the Planck gas of Frobenius modes

Standalone note `lemma44.tex`, October 22, 2026.

Summary. J.W. asked (t162u) whether the "centers" of Lemma 43 (Fiedler/Broughan), the Nash equilibria, and the aperture discussion of Lemma 21 are one optical object, a focal point; and (steering) whether the cavity picture is a blackbody question. This note makes the optical side exact where it can be made exact and says where it cannot.

What is a theorem. (i) Paraxial optics is $\mathrm{SL}_2(\mathbb{R})$ acting on rays and, by the ABCD law, on the Gaussian-beam parameter $q \in \mathbb{H}$. A resonator is stable iff $|A + D| < 2$ iff its round-trip map has a fixed point $q^* \in \mathbb{H}$ (a self-consistent Gaussian mode) iff q^* is a *center* of the Möbius flow iff there is an invariant positive form; that form is *exactly* the exponent of the Wigner function of the self-consistent beam, $\mathsf{B}_{q^*}(r, \vartheta) = |r - q^*\vartheta|^2 / \operatorname{Im} q^*$. For the normalized Frobenius g_p of Lemma 42 it is *literally* Lemma 42's polarization b_{g_p} (Thm. 44.1.1, Cor. 44.2.3). (ii) In this dictionary: Hasse $\Leftrightarrow$ stable cavity; θ_p = round-trip Gouy phase; Rayleigh range $= \sin\theta_p \geq \sqrt{3}/(2\sqrt{p})$, so the arithmetic mode never loses its focus; *rational Gouy phase (re-entrant cavity)* $\Leftrightarrow$ *supersingular*; $a_p = 0 \Leftrightarrow$ confocal-type quarter turn (Thm. 44.2.4). k round trips give $\operatorname{tr} g_p^k$ (points over $\mathbb{F}_{p^k}$), while Hecke T_{p^k} is the k-photon sector $\operatorname{tr} \operatorname{Sym}^k g_p$. (iii) Weil's form on a window of length A is *exactly* the van Cittert–Zernike mutual intensity, over an aperture of width A, of an incoherent "source" of unit emitters at the zero ordinates. RH $\Leftrightarrow$ the source has nonnegative intensity. An off-line zero is a Deschamps complex source point whose mutual intensity has a negative-norm component (Thm. 44.3.1). (iv) Planck: each Euler factor of ζ is one bosonic mode of energy $\log p$. A degree-2 Euler factor is a two-mode gas twisted by $e^{\pm i\theta_p}$ (Gouy) or $e^{\pm\eta}$ (ghost). Hasse $\Leftrightarrow$ the local Hagedorn point is $\sigma_H = \frac{1}{2} \Leftrightarrow$ the Fisher zeros lie on $\operatorname{Re} s = \frac{1}{2}$ (Lee–Yang form).

What is false or only an analogy. (a) The window A is an aperture width, not a Fresnel number. A Fresnel number needs a second scale, here a frequency band T, and then $c = AT/2$ (Prop. 44.3.4). The Boyd–Gordon prolate modes are the coherent modes of the *uniform-band idealization* of the zero source, not of Weil's form. (b) Fox–Li non-unitarity is not the ghost signature: every truncation is lossy whether or not RH holds. The ghost is *negative source intensity* (Weil) or *gain* (Lemma 37 line), never loss. (c) Depth of focus ($\delta \sim c/A$) does *not* explain Lemma 21's $A_{\mathrm{crit}} = 2.113$, which is δ-independent. It is exactly the law of the *counting-only* detector of Lemma 12 ($\delta A_0 \in [3.71, 8.06]$, certified), i.e. a Rayleigh criterion. With the background known, detection is interferometric: $\delta_{\det}(A) \approx CA^{-3/2}\lambda_0(A)^{0.47}$, from 0.335 at $A = 0.8$ to $5.6 \cdot 10^{-20}$ at $A = 2.4$ for a ghost at γ_1 (new numerics, §44.4). (d) In the Planck reading no real-temperature thermodynamic positivity detects a ghost. Weil factors have *negative* occupation and *negative* heat capacity near their Hagedorn point, while a fake with $a > 2\sqrt{p}$ is a healthy positive gas. Weil constrains where the complex-temperature zeros are; it is not positivity at real temperature. (e) The Berry–Keating hyperbolic orbits and the elliptic per-prime Frobenius do not conflict: the first is the scaling (base) boost of rapidity $\log p$, the second the holonomy on the fibre H^1.

RH is not proved and G5 is not supplied. No statement here is RH-equivalent except where labeled [RESTATEMENT].

Labels. [THM] proved here (or classical with proof); [CITED] cited, and the cited statement was read; [BIB] bibliographic pointer, content standard but not re-read; [RESTATEMENT] restatement of a known equivalence; [ANALOGY] analogy, not mathematics; [OPEN] open; [NUM] numerical; [DEF] definition. Numerics: `num/lemma44_optics.py` (`l44/optics.json`), `num/lemma44_planck.py` (`l44/planck.json`), `num/lemma44_ghost.py`, `_ghost_analyze.py`, `_ghost_mono.py` (`l44/ghost_*.json`). Journal: `num/lemma44_progress.md`. References: `lit/lemma44/verified_refs.md` (every DOI checked against Crossref).

Statement of intellectual honesty. Optics supplies vocabulary and a few genuine identities: the ABCD law, the Gaussian-beam form, van Cittert–Zernike. It supplies no positivity that Weil's form lacks. Every optical statement about primes or zeros is labeled. Where the dictionary is an identity of matrices or kernels it is [THM]; where it reads arithmetic in optical words it is [ANALOGY] or [RESTATEMENT]. Several of my chat claims to J.W. were partly wrong; they are corrected in §44.7.

44.1. Ray-transfer matrices, the q-parameter, and the Gaussian-beam form

44.1.1. Setting. A paraxial ray at a reference plane is $v = (r, \vartheta)^{\mathsf T}$ (height, slope). A lossless first-order system acts by $v \mapsto Mv$ with $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ [**159**, **242**] [BIB]. A Gaussian beam $E(r) = \exp(-ikr^2/2q)$, $k = 2\pi/\lambda$, $\operatorname{Im} q > 0$, has $1/q = 1/R - i\lambda/(\pi w^2)$ and obeys the ABCD law $q' = (Aq+B)/(Cq+D)$ [**159**, eq. (43)] [CITED]. A periodic sequence with element M (an unfolded resonator) has

$$M^n = \frac{1}{\sin\Theta}\begin{pmatrix} A\sin n\Theta - \sin(n-1)\Theta & B\sin n\Theta \\ C\sin n\Theta & D\sin n\Theta - \sin(n-1)\Theta \end{pmatrix}, \qquad \cos\Theta = \tfrac{1}{2}(A+D),$$

and is stable iff $-1 < \frac{1}{2}(A+D) < 1$ [**159**, eqs. (5)–(7)] [CITED]. For two mirrors this reads $0 < g_1 g_2 < 1$ [**159**, eqs. (8),(80)] [CITED]. The self-consistent beam is $1/q = (D-A)/2B - \frac{i}{2B}\sqrt{4-(A+D)^2}$, $w^2 = (2\lambda B/\pi)/\sqrt{4-(A+D)^2}$ [**159**, eqs. (57)–(58)] [CITED], with the sign chosen so that $\operatorname{Im}(1/q) < 0$.

44.1.2. The Gaussian-beam form.

THEOREM 44.1.1 (transport of the beam form). *[THM] For $q \in \mathbb{H}$ put $\mathsf{B}_q(r,\vartheta) = |r - q\vartheta|^2/\operatorname{Im} q$, a positive definite quadratic form of determinant 1 on $\mathbb{R}^2$. For every $M \in \mathrm{SL}_2(\mathbb{R})$ and $q' = (Aq+B)/(Cq+D)$,*

$$r' - q'\vartheta' = \frac{r - q\vartheta}{Cq+D}, \qquad \operatorname{Im} q' = \frac{\operatorname{Im} q}{|Cq+D|^2}, \qquad \textit{hence} \qquad \mathsf{B}_{q'}(Mv) = \mathsf{B}_q(v).$$

Consequently B_q is M-invariant iff q is a fixed point of M in $\mathbb{H}$.

PROOF. $(r'-q'\vartheta')(Cq+D) = (Ar+B\vartheta)(Cq+D) - (Aq+B)(Cr+D\vartheta) = (AD-BC)(r-q\vartheta)$, and $\operatorname{Im}\frac{Aq+B}{Cq+D} = \det M \cdot \operatorname{Im} q/|Cq+D|^2$. Both identities were verified symbolically (`l44/optics.json`, residual 0). If $\mathsf{B}_q \circ M = \mathsf{B}_q$ then $\mathsf{B}_{Mq} = \mathsf{B}_q$, and $q \mapsto \mathsf{B}_q$ is injective: its entries $1/y$, $-x/y$, $|q|^2/y$ determine $q = x + iy$. □

PROPOSITION 44.1.2 (Wigner function of the beam). *[THM] With $W(r,\vartheta) = \int E(r + \frac{s}{2})\overline{E(r-\frac{s}{2})}e^{ik\vartheta s}\,ds$ and $E = \exp(-ikr^2/2q)$, one has $W(r,\vartheta) = W(0,0)\exp(-k\,\mathsf{B}_q(r,\vartheta))$.*

PROOF. Write $1/q = a - ib$ with $b > 0$. The integrand's field product is $\exp(-kb(r^2+s^2/4) - ikars)$. The Gaussian integral in s gives $\exp(-k[br^2 + (\vartheta - ar)^2/b])$, and $br^2 + (\vartheta-ar)^2/b = |\vartheta - r/q|^2/b = |r - q\vartheta|^2/\operatorname{Im} q$ because $\operatorname{Im} q = b|q|^2$. Checked by quadrature at 4 points to 10^{-15} (`optics.json`). With the opposite Fourier sign the slope axis is reflected. □

So Theorem 44.1.1 is the Wigner-function form of the ABCD law: first-order optics transports Wigner functions along rays, $W'(Mv) = W(v)$ (Bastiaans [**25**] [BIB]; Collins integral [**61**] and canonical-operator form [**205**] [BIB]).

COROLLARY 44.1.3 (stability = Lemma 42 Thm 6.8 in optical words). *[THM] For $M \in \mathrm{SL}_2(\mathbb{R})$, $M \neq \pm 1$, the following are equivalent: (a) $|A+D| < 2$ (stable resonator); (b) M has a fixed point $q^* \in \mathbb{H}$ (self-consistent Gaussian mode); (c) M preserves a positive definite form, necessarily a multiple of B_{q^*}; (d) M lies in a compact subgroup (the stabilizer of q^*, $\cong SO(2)$). In that case the eigenvalue on $(q^*,1)^{\mathsf{T}}$ is $Cq^*+D = e^{\pm i\Theta}$ with $\cos\Theta = \frac{1}{2}(A+D)$: the* round-trip Gouy phase. *If $|A+D| = 2$, M is parabolic (Jordan) or ± 1 (every q fixed). If $|A+D| > 2$, M has two real fixed points (geometric spherical-wave modes) and no invariant positive form.*

PROOF. (a)⇔(b): the fixed-point equation $Cq^2+(D-A)q-B = 0$ has discriminant $(A+D)^2-4$. (b)⇒(c) is Theorem 44.1.1; uniqueness up to scale is Lemma 42 Thm. 7.2 (for elliptic M, invariant Hermitian forms are diagonal in the eigenbasis). (c)⇔(d)⇔(a) is Lemma 42 Thm. 6.8. For the eigenvalue: $M(q^*,1)^{\mathsf{T}} = (Aq^*+B, Cq^*+D)^{\mathsf{T}} = (Cq^*+D)(q^*,1)^{\mathsf{T}}$, and $|Cq^*+D|^2 = \operatorname{Im} q^* / \operatorname{Im} q^* = 1$. Random check: for 200 elliptic M, invariance, $|\mathrm{eig}| = 1$ and phase $\pm\Theta$ hold with maximum error $1.6 \cdot 10^{-13}$. □

Transverse modes. The Hermite–Gauss mode of order m in one transverse dimension acquires $e^{-i(m+\frac{1}{2})\Theta}$ per round trip. In Kogelnik–Li's resonance formula this is the term $(m+n+1)\arccos(1-d/R)$ per transit, for two transverse dimensions [**159**, eq. (51)] [CITED]. The $\frac{1}{2}$ is the metaplectic half: $M \mapsto$ (unitary on $L^2(\mathbb{R})$) is a projective representation lifting to the metaplectic cover, and rotation by Θ acts as $e^{-i\Theta(N+\frac{1}{2})}$ on the oscillator [**103, 205**] [BIB]. (Lemma 42 Prop. 6.7 found a similar metaplectic shift on the theta side.)

44.1.3. The center of the Möbius flow.

PROPOSITION 44.1.4 (centers). *[THM] Let $X = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & -\alpha \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{R})$. The Möbius flow of e^{tX} on $\mathbb{H}$ is $\dot q = v(q) = \beta + 2\alpha q - \gamma q^2$, a quadratic rational vector field in Fiedler's class* [**101**]. *If $\det X > 0$ (elliptic), its zero $q^* \in \mathbb{H}$ has $v'(q^*) = \mp 2i\sqrt{\det X}$, which is purely imaginary. So q^* is a* center *(Fiedler's criterion $\operatorname{Re} f'(e) = 0$; Lemma 43 Thm. 2.1) of period $\pi/\sqrt{\det X}$, and e^X has Gouy phase $\Theta = \sqrt{\det X}$. If $\det X < 0$, the zeros are real with $v' = \mp 2\sqrt{-\det X}$: a source and a sink on $\partial\mathbb{H}$. If $\det X = 0 \neq X$, there is a double zero on $\partial\mathbb{H}$.*

PROOF. $\frac{d}{dt}\big|_0 \frac{q+t(\alpha q+\beta)}{1+t(\gamma q-\alpha)} = \beta + 2\alpha q - \gamma q^2$. The zeros are $q = (\alpha \pm i\sqrt{\det X})/\gamma$, and $v'(q) = 2\alpha - 2\gamma q$. □

Optical reading [THM]. Write $q^* = z + iz_R$. Then $z = \operatorname{Re} q^*$ is the distance from the reference plane to the beam waist, and $z_R = \operatorname{Im} q^*$ is the Rayleigh range, half the depth of focus (the confocal parameter is $b = 2z_R$, [**159**, §3]). So J.W.'s "the center is a focal point" is literally right in this setting. The center of the Möbius flow is a *complex focus*: its real part locates the focus and its imaginary part is the depth of focus. A complex source point at q^* radiates exactly this Gaussian beam (Deschamps [**93**] [BIB]).

44.1.4. Marginal cavities.

EXAMPLE 44.1.5 ([THM], `optics.json`). Round-trip matrices, with spacing d and radius R, unfolded as a lens $f = R/2$ after each spacing: planar ($R = \infty$, $d = 1$): $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$, Jordan; concentric ($d = 2R$, $R = 1$): $\left(\begin{smallmatrix} -3 & -4 \\ 4 & 5 \end{smallmatrix}\right)$, $\frac{1}{2}\operatorname{tr} = 1$, Jordan; symmetric confocal ($d = R$): exactly -1, scalar. So the marginal boundary $|A+D| = 2$ holds two different kinds of cavity. In the Jordan kind (planar, concentric), the Gaussian mode ceases to exist. In the scalar kind (confocal), every q is self-consistent and all transverse modes are degenerate. The mode is then selected only by the aperture, which is why Boyd–Gordon's finite-mirror theory is exact there [**45, 9**] [BIB].

44.2. The arithmetic cavity: dictionary, and what it does not say

DEFINITION 44.2.1. [DEF] For an elliptic curve $E/\mathbb{Q}$ and $p \nmid N$, the *arithmetic cavity* at p is the round-trip matrix $g_p = \left(\begin{smallmatrix} 0 & -1 \\ 1 & \tau_p \end{smallmatrix}\right)$, $\tau_p = a_p/\sqrt{p}$. This is Lemma 42's normalization (Thm. 6.8): the companion matrix of $x^2 - \tau_p x + 1$, the normalized Frobenius polynomial. Any $\mathrm{SL}_2(\mathbb{R})$-conjugate describes the same cavity at a different reference plane.

THEOREM 44.2.2 (dictionary, matrix level). *[THM] For g_p: (i) stable $\Leftrightarrow |a_p| < 2\sqrt{p}$, which holds for all p (Hasse [246, Thm. V.1.1] [BIB]); (ii) $q_p^* = -\cos\theta_p + i\sin\theta_p = -e^{-i\theta_p}$ with $2\cos\theta_p = \tau_p$, and the round-trip eigenvalue is $q_p^* + \tau_p = e^{i\theta_p}$, so the Gouy phase is θ_p; (iii) the beam form is $\mathsf{B}_{q_p^*} = \frac{1}{\sin\theta_p}\left(\begin{smallmatrix} 1 & \cos\theta_p \\ \cos\theta_p & 1 \end{smallmatrix}\right) = \frac{1}{2\sin\theta_p}\left(\begin{smallmatrix} 2 & \tau_p \\ \tau_p & 2 \end{smallmatrix}\right)$; (iv) the waist offset is $-\cos\theta_p$, the Rayleigh range $\sin\theta_p$, and the beam width at the reference plane $w^2 = \lambda/(\pi\sin\theta_p)$ (KL66 (58) with $B = -1$); (v) $\sin\theta_p = \sqrt{4p - a_p^2}/(2\sqrt{p}) \geq \sqrt{3}/(2\sqrt{p})$ (Lemma 42 Prop. 7.5). So the arithmetic mode never reaches the marginal (Jordan) boundary, and its Rayleigh range never vanishes.*

PROOF. (ii) solves $q = -1/(q+\tau)$. (iii) is Theorem 44.1.1 at q^*. The rest is substitution. □

COROLLARY 44.2.3 (the Gaussian-beam form is Lemma 42's polarization). *[THM] $\mathsf{B}_{q_p^*} = b_{g_p}$, the positive polarization $\omega(\cdot, J_{g_p}\cdot)$ of Lemma 42 Thm. 7.2. The entrywise difference is $\leq 2.2 \cdot 10^{-16}$ for all 31 primes $p < 60$ of 11a and 37a. The complex structure $J_g = (g - \cos\theta)/\sin\theta$ is the rotation by $\pi/2$ about the complex focus q^*. Lemma 43 Thm. 4.1 ($g^{\mathsf{T}} = \exp((\theta/\sin\theta)M)$, $M = (\sin\theta J_g)^{\mathsf{T}}$) says that the Möbius flow of Prop. 44.1.4 through g_p is the rotation about this focus.*

TABLE 44.1. Arithmetic cavities for 11a and 37a (`144/optics.json`). $q^* = -e^{-i\theta}$; ss = supersingular; period = least N with $g_p^N = 1$.

curve	p	a_p	$4p - a_p^2$	θ_p/π	$\mathrm{Re}\, q^*$	$z_R = \sin\theta_p$	$\frac{\sqrt{3}}{2\sqrt{p}}$	note
11a	2	−2	4	0.750000	0.7071	0.7071	0.6124	ss, period 8
11a	3	−1	11	0.593215	0.2887	0.9574	0.5000	
11a	5	1	19	0.428217	−0.2236	0.9747	0.3873	
11a	7	−2	24	0.623376	0.3780	0.9258	0.3273	
11a	13	4	36	0.312833	−0.5547	0.8321	0.2402	
11a	19	0	76	0.500000	0.0000	1.0000	0.1987	ss, period 4
37a	2	−2	4	0.750000	0.7071	0.7071	0.6124	ss, period 8
37a	3	−3	3	0.833333	0.8660	0.5000	0.5000	ss, period 12; bound attained
37a	5	−2	16	0.647584	0.4472	0.8944	0.3873	
37a	11	−5	19	0.771771	0.7538	0.6571	0.2611	
37a	17	0	68	0.500000	0.0000	1.0000	0.2100	ss, period 4

THEOREM 44.2.4 (rational Gouy phase $\Leftrightarrow$ supersingular). *[THM] Let $p \nmid N$. Then $\theta_p \in \pi\mathbb{Q}$ (the cavity is* re-entrant*, $g_p^n = \pm 1$ for some n: a degenerate cavity in Arnaud's sense [9]) iff E has supersingular reduction at p. The re-entrant cases are exactly: $a_p = 0$ ($\theta = \pi/2$, $g_p^2 = -1$); $p = 2$, $a_2 = \pm 2$ ($\theta = \pi/4, 3\pi/4$, $g^4 = -1$); $p = 3$, $a_3 = \pm 3$ ($\theta = \pi/6, 5\pi/6$, $g^6 = -1$).*

PROOF. If $\theta \in \pi\mathbb{Q}$, then $2\cos 2\theta = \tau^2 - 2 = a^2/p - 2$ is rational and an algebraic integer (a sum of two roots of unity), so it lies in $\{-2, -1, 0, 1, 2\}$, i.e. $a^2 \in \{0, p, 2p, 3p, 4p\}$. For p prime, $a^2 = p$ and $a^2 = 4p$ are impossible, $a^2 = 2p$ forces $p = 2$, $a = \pm 2$, and $a^2 = 3p$ forces $p = 3$, $a = \pm 3$. All of these have $p \mid a_p$, which is supersingularity [246, Exer. 5.10] [BIB]. Conversely, supersingular means $p \mid a_p$, and with $|a_p| < 2\sqrt{p}$ this leaves exactly $a_p = 0$ ($p \geq 5$), $a_2 \in \{0, \pm 2\}$, $a_3 \in \{0, \pm 3\}$.

[NUM]: for all $p < 200$ on 11a and 37a, the re-entrant primes are exactly the supersingular ones (`F_reentrant`). □

REMARK 44.2.5 ([THM] + [ANALOGY]). $a_p = 0$ gives $g_p^2 = -1$. The arithmetic cavity is then a *linear quarter-turn*: the half-trip of a symmetric confocal resonator (a Fourier transform, fractional order 1). Over $\mathbb{F}_{p^2}$ its round trip is the scalar -1, the confocal case of Example 44.1.5 and Lemma 42 Prop. 7.5(c). Lemma 42 Rem. 7.6 (route S8) asked for a linear J with $J^2 = -1$ *commuting with the scaling flow*. Supersingular Frobenius is such a J at one prime, but it commutes with nothing global. It hints at where quarter-turns occur in arithmetic; it does not construct the missing rung [ANALOGY].

THEOREM 44.2.6 (k round trips vs. Hecke T_{p^k}). *[THM] (i)* $\operatorname{tr} g_p^k = 2\cos k\theta_p = p^{-k/2}(p^k + 1 - \#E(\mathbb{F}_{p^k}))$ *(Frobenius of $E/\mathbb{F}_{p^k}$ [246, Thm. V.2.3.1] [BIB]), and $g_p^k = U_{k-1}g_p - U_{k-2}$ (KL66 (5)). (ii)* $\operatorname{tr}\operatorname{Sym}^k g_p = U_k(\cos\theta_p) = \sin((k+1)\theta_p)/\sin\theta_p = p^{-k/2}a_{p^k}$*, the normalized Hecke eigenvalue, by $a_{p^{k+1}} = a_p a_{p^k} - p a_{p^{k-1}}$ [94, Prop. 5.8.5] [BIB]. So "$T_{p^k} = k$ round trips" is false: k round trips give the k-th power sum, and T_{p^k} gives the k-th complete symmetric function. [NUM]: counting $\#E(\mathbb{F}_{p^2})$ directly for $p = 2, 3, 5, 7$ on 11a and 37a gives $\operatorname{tr} F^2 = a_p^2 - 2p$ in all 8 cases. For example, on 37a at $p = 3$: $\#E(\mathbb{F}_9) = 7$, $\operatorname{tr} F^2 = 3$, whereas $a_9 = a_3^2 - 3 = 6$.*

In §44.5 the Hecke side becomes the k-photon sector of a two-mode Bose gas (Thm. 44.5.3).

PROPOSITION 44.2.7 (unstable cavity = boost / Krein ghost). *[THM] Let $|\tau| > 2$, a non-Weil "fake" Frobenius; e.g. $p = 13$, $a = 8$: $\tau = 2.2188$, $\eta = \operatorname{arccosh}(\tau/2) = 0.46360$. Then g has eigenvalues $e^{\pm\eta}$ (0.62902, 1.58978) and two real fixed points $q_\pm = -0.62902, -1.58978$, the source and sink of Prop. 44.1.4 on $\partial\mathbb{H}$. The magnification is $e^\eta = 1.58978$, and both eigenvectors have Krein norm 0 for $H(x,y) = i\omega(\bar{x}, y)$ (`F_fake`). In geometric optics a magnifying unstable resonator with a fixed aperture keeps a fraction $e^{-\eta}$ per round trip in one transverse dimension: the energy is spread over a width larger by e^η (Siegman [243, 244] [BIB]). The ghost is a cavity with* output coupling by magnification*: geometric loss in one mode, geometric gain in its mirror partner.*

REMARK 44.2.8 (Sato–Tate as a distribution of Gouy phases). [CITED]+[NUM]By Barnet-Lamb–Geraghty–Harris–Taylor [23], the Gouy phases of a non-CM curve are equidistributed for $\frac{2}{\pi}\sin^2\theta\, d\theta$. For 11a with $p \le 2 \cdot 10^4$ ($n = 2261$), the moments of $2\cos\theta_p$ of orders $2, 4, 6$ are $0.985, 1.946, 4.797$ (Sato–Tate: $1, 2, 5$). The Kolmogorov–Smirnov distance is 0.0110 (95% critical value 0.0286), and the mean of θ/π is 0.4988. In cavity language, every arithmetic resonator is stable, and their Gouy phases are distributed like the eigenphases of a Haar-random $SU(2)$ matrix. This repeats Lemma 42 Prop. 6.11 over the same range.

What the dictionary does not do. Ellipticity of g_p *is* Hasse; the cavity language adds no proof of it. The positive form is forced by the Rosati involution (Lemma 42 Prop. 7.1), not by any optical principle. For ζ there is no per-prime 2×2 cavity at all, since the degree is 1. Lemma 42 Def. 6.12 already records that ζ's "Frobenius" is the scaling flow [RESTATEMENT].

44.3. Finite apertures: van Cittert–Zernike, Boyd–Gordon, Fox–Li

We use Lemma 21's conventions: $\widehat{F}(z) = \int F(u)e^{izu}du$, $H_F = F * F^*$, $\widehat{H_F}(z) = \widehat{F}(z)\overline{\widehat{F}(\bar{z})}$, $\gamma_\rho = -i(\rho - \frac{1}{2})$, and $Q(F,F) = \sum_\rho \widehat{H_F}(\gamma_\rho)$. A window of length A means $\operatorname{supp} F \subset [-\frac{A}{2}, \frac{A}{2}]$.

THEOREM 44.3.1 (Weil's window form is a mutual intensity). *[THM]+[RESTATEMENT] (i) $Q(F,F) = \iint F(u_1)\overline{F(u_2)}\, J(u_1 - u_2)\, du_1 du_2$ with $J(\Delta) = \sum_\rho e^{i\gamma_\rho\Delta}$, a distribution (symmetric summation). This is the van Cittert–Zernike form [274, 304, 43] [BIB]: J is the mutual intensity across an aperture of the field radiated by an incoherent source whose intensity in the conjugate (far-field) variable t is $I = \sum_\rho \delta_{\gamma_\rho}$, i.e. unit emitters at the zero ordinates. Restricting F to a*

window restricts the mutual intensity to an aperture of width A, which sees separations $|\Delta| < A$. (ii) By the explicit formula, $J = J_\infty - \sum_n \frac{\Lambda(n)}{\sqrt{n}}(\delta_{\log n} + \delta_{-\log n}) + (\text{pole terms})$. In the aperture plane the coherence function is the archimedean term with negative spikes at the prime powers, and an aperture of width A sees only $n < e^A$ (the information horizon of Lemma 37 Prop. 5.1). (iii) [RESTATEMENT] (Weil, Bochner): RH $\Leftrightarrow$ $Q \geq 0$ on every window $\Leftrightarrow$ the zero source is a physical *source: $I \geq 0$, located on the real t-axis. (iv) An off-line quadruple at $\gamma \pm i\delta$ contributes $2e^{i\gamma\Delta}\cosh(\delta\Delta)$. On a window, $\cosh\delta(u_1-u_2) = \cosh\delta u_1 \cosh\delta u_2 - \sinh\delta u_1 \sinh\delta u_2$ has signature $(1,1)$ (Lemma 21 Lemma 1.1). An emitter at the complex position $\gamma + i\delta$ is a Deschamps complex source point* [**93**]*: across the aperture it produces a tilted plane wave with an exponential amplitude ramp, $e^{i\gamma u}e^{-\delta u}$. So the ghost is a* negative-norm component of the mutual intensity*, which no physical (positive-intensity) source produces.*

Proof. (i) $\widehat{H_F}(z) = \iint F(u_1)\overline{F(u_2)}e^{iz(u_1-u_2)}$ by definition, for real or complex z. (ii) is Lemma 21 eq. (1.1) read in the u-variable. (iv) γ_ρ and $\gamma_{1-\bar\rho}$ are $\gamma \mp i\delta$. □

Remark 44.3.2 (lithography). [ANALOGY]In Hopkins' theory of partially coherent imaging [**132**] [BIB] the image is a quadratic form in the mask, with a kernel built from the source intensity and the pupil, and positivity of the image intensity rests on the source intensity being ≥ 0. Lemma 21's lithography analogy is therefore sharper than "resolution". Weil positivity on a window says that the source is physical, tested only through an aperture of width A.

Proposition 44.3.3 (coherent modes). *[THM] The eigenfunctions of the windowed form (the operator $P_A J P_A$ on $L^2[-\frac{A}{2},\frac{A}{2}]$) are the* coherent modes *of the windowed field in Wolf's sense* [**297**] *[BIB], and the eigenvalues are their powers. $\lambda_*(A)$ (Lemmas 11, 22, 24) is the power of the weakest coherent mode. A physical source has all coherent-mode powers ≥ 0. A negative eigenvalue is a negative-power coherent mode, and this is how a ghost first appears (Lemma 27 stage (a), Lemma 41).*

Proposition 44.3.4 (Boyd–Gordon modes and the Fresnel number). *[THM]+[CITED] (i) [CITED]* [**45**]*, content read through KL66 §4 and a secondary text (see verified refs): for a confocal resonator with strip mirrors of half-width a and spacing b, the transverse modes are the angular prolate spheroidal functions $S_{0m}(c, x/a)$, with $c = ka^2/b = 2\pi N$ and $N = a^2/(\lambda b)$ the Fresnel number; for large c they become Hermite–Gaussians. The half-trip is a finite Fourier transform; the round-trip energy fraction of mode m is the Slepian concentration $\lambda_m(c)$* [**248**, **171**]*, and the diffraction loss per transit is $1-|\gamma|^2$* [**159**, eq. (81)] *[CITED]. (ii) [THM]: replace the zero source by a* uniform *source on $|t| \leq T$ ($I = \mathbf{1}_{[-T,T]}$). Then $J(\Delta) = 2\sin(T\Delta)/\Delta$, and $P_A J P_A$ is 2π times the time–band-limiting operator with Slepian parameter $c = AT/2$. Its eigenfunctions are the prolate functions $S_{0m}(c, 2u/A)$. These are Boyd–Gordon's modes with $a \leftrightarrow A/2$ and $c = AT/2$, i.e. Fresnel number $N = AT/4\pi$. (iii) So A plays the role of the aperture (mirror width), and a Fresnel number needs a band T as well. At the resolution band $T = 2\pi e^A$ of Lemma 11, where the zero count $2N(T)$ matches the Landau–Pollak count AT/π up to $O(T)$, one gets $c = \pi A e^A$. (iv) The true windowed Weil form is* not *this operator. The zero source has density $\frac{1}{2\pi}\log\frac{t}{2\pi}$, it is discrete, and J carries the archimedean and prime terms. Lemma 22 found that the counts agree (near-radical dimension $2d - \frac{7}{4} \pm 0.69$, Shannon-type) but the individual eigenvectors differ, and the prolate objects never detect a ghost earlier than full window positivity (Lemma 22, Levels 2–3).*

Proof of (ii). $\int_{-T}^{T} e^{it\Delta}dt = 2\sin(T\Delta)/\Delta$. Rescaling $u = \frac{A}{2}x$ gives the kernel $\frac{\sin c(x-y)}{\pi(x-y)}$ with $c = \frac{A}{2}T$, which is the Slepian–Pollak operator [**248**] with eigenfunctions $S_{0m}(c,\cdot)$. □

Proposition 44.3.5 (what is lost in our setting: Fox–Li vs. the ghost). *[THM]+[RESTATEMENT] (i) Fox–Li* [**105**] *[CITED]: in a finite-mirror resonator the round trip is $K = P_a U P_a$ with U unitary propagation. After many transits the field settles into a self-reproducing distribution that decays at*

a fixed rate. Since $\|P_a\| = 1$, $\|K\| \le 1$, *so every eigenvalue has* $|\gamma| \le 1$ *and loss is generic. (ii) In our setting the analogue of* P_aUP_a *is the compression of the unitary scaling flow* $U(s)$ *(translation in* u*) to the window. Compressions of this type are contractions, and the zeros appear as their absorption spectrum (Lax–Phillips* [**174**]*, Connes* [**75**] *[BIB]; Lemma 42 Rem. 3.4). In the Slepian idealization the loss of coherent mode* m *is* $1 - \lambda_m(c)$*, i.e. leakage out of the band. None of this distinguishes RH from not-RH, since* every *truncation is lossy. (iii) The ghost is not loss. In Weil's form it is negative source intensity (Thm. 44.3.1(iv)). In the Lemma 37 transmission line it is* gain*. By Lemma 37 Thm. 3.2, all zeros have* $\beta \le \frac{1}{2} + h$ *iff the line* $L^{\omega,\nu}$ *is passive for every* $\omega > h$ *(*$\|\mathcal{K}[t]\| \le 1$*, a contraction like Fox–Li's* K*) [RESTATEMENT]. An off-line zero of depth* δ *is a gain coefficient* δ*, and it shows up exactly when the line's own damping* ω *drops below* δ*.*

44.4. Depth of focus and ghost detection: the test

44.4.1. The optical prediction. An off-line zero $\gamma + i\delta$ acts on the aperture as the amplitude ramp $e^{\pm\delta u}$ (Thm. 44.3.1(iv)), whose log-amplitude varies by δA across the aperture. A Rayleigh-type criterion ("detectable once the ramp is of order one") predicts $\delta_{\mathrm{det}} \sim c/A$, a depth-of-focus law. Strictly, δ is an imaginary *lateral* displacement (a complex tilt), not a defocus; defocus would be a quadratic pupil phase. Now suppose the background is known, and only the ghost's deviation from it is measured (interferometric detection). Then by the per-zero form of Lemma 21 the change is $-2|\widehat{S_\delta}|^2 + O(\delta^2)$, quadratic in δ. Detection happens once δ^2 times the ghost curvature exceeds the smallest coherent-mode power $\lambda_*(A)$, so $\delta_{\mathrm{det}} \propto \sqrt{\lambda_*(A)}$ up to powers of A. Lemma 11 Prop. 4.3 makes the lower half rigorous: the ghost is invisible while $4S(\delta, A/2) < \lambda^\circ$, where $S(\delta, h) = \frac{\sinh 2\delta h}{2\delta} - h \approx \frac{2}{3}\delta^2 h^3$ ([THM], cited from Lemma 11).

44.4.2. Three detectors, three laws. (a) Counting-only detector (Lemma 12 Thm. 4.1, certified). Here the background is constrained only by bounds on $N(T)$. The certified detection windows (`num/lemma12_certified.json`, global rows) give

$$\delta \cdot A_0 \in [3.71,\ 8.06] \qquad (\delta \in \{0.5, 0.25, 0.1, 0.05, 0.01\},\ \gamma_0 \in \{\gamma_1, 10^3, 3\cdot 10^{12}, 10^{20}, 10^{100}\}),$$

increasing slowly with $\log\log\gamma_0$ (e.g. at $\delta = 0.01$: 3.71 at γ_1, 4.54 at $3\cdot 10^{12}$, 6.07 at 10^{100}). *This is the depth-of-focus law* $\delta_{\mathrm{det}} = c(\gamma_0)/A$ [NUM]. (STATE §20 quotes $[3.7, 8.4]$; the certified file has maximum 8.06.)

(b) Known-background detector (new, [NUM]). We use the replacement model of the corrected Lemma 11 run, $Q - T_{\gamma_k} + T_{\gamma_k + i\delta} + T_{\gamma_k - i\delta}$, on the Lemma 11 Legendre–Galerkin Grams (50 modes per parity, 800-bit source, `lemma11_Qghost.pkl`, read only). The threshold is found by bisection in $\log\delta$ to relative width 0.2%, with the ghost replacing γ_1 ($k = 1$) or γ_4 ($k = 4$), on 23 windows $A = 0.2, \ldots, 2.4$, both parities, at 90 digits. "Global" is the smaller of the two parity thresholds; it is the even parity in every case.

A	λ_0 (even)	δ_{det}, $k=1$	$\delta_{\mathrm{det}} A$	δ_{det}, $k=4$	ratio to L11 bound ($k=1$)
0.8	$1.81\cdot 10^{-4}$	0.335	0.268	undetected	106
1.0	$9.34\cdot 10^{-7}$	$3.95\cdot 10^{-2}$	$3.95\cdot 10^{-2}$	undetected	557
1.2	$1.61\cdot 10^{-9}$	$2.32\cdot 10^{-3}$	$2.78\cdot 10^{-3}$	undetected	$1.9\cdot 10^{3}$
1.4	$4.18\cdot 10^{-13}$	$4.81\cdot 10^{-5}$	$6.74\cdot 10^{-5}$	$4.49\cdot 10^{-2}$	$5.1\cdot 10^{3}$
1.6	$1.66\cdot 10^{-17}$	$3.69\cdot 10^{-7}$	$5.91\cdot 10^{-7}$	$8.47\cdot 10^{-4}$	$1.1\cdot 10^{4}$
1.8	$4.21\cdot 10^{-23}$	$6.89\cdot 10^{-10}$	$1.24\cdot 10^{-9}$	$3.03\cdot 10^{-6}$	$2.2\cdot 10^{4}$
2.0	$6.27\cdot 10^{-30}$	$3.01\cdot 10^{-13}$	$6.02\cdot 10^{-13}$	$2.19\cdot 10^{-9}$	$3.8\cdot 10^{4}$
2.2	$1.34\cdot 10^{-37}$	$4.90\cdot 10^{-17}$	$1.08\cdot 10^{-16}$	$5.46\cdot 10^{-13}$	$6.4\cdot 10^{4}$
2.4	$1.34\cdot 10^{-43}$	$5.63\cdot 10^{-20}$	$1.35\cdot 10^{-19}$	$1.10\cdot 10^{-15}$	$1.1\cdot 10^{5}$

Below $A = 0.8$ ($k = 1$) and $A = 1.3$ ($k = 4$), no ghost with $\delta \le 0.49$ is detected. *Checks.* $\lambda_0(A = 1.6) = 1.655\cdot 10^{-17}$ agrees with Lemma 11's $\lambda_*(e^{1.6}) = 1.640\cdot 10^{-17}$ to 1%. At $A = 2.4$ the 50-mode value $\lambda_0 = 1.34\cdot 10^{-43}$ is a Ritz upper estimate (Lemma 11 with 120 modes: $6.87\cdot 10^{-49}$), so the

large-A thresholds are upper estimates for this basis. At four spot checks, $\lambda_{\min}(\delta) = \lambda_0(1-(\delta/\delta_{\det})^2)$ to three digits for $\delta/\delta_{\det} \in \{0.3, 0.9, 1.1, 3\}$ (`ghost_mono.json`): the threshold is monotone, and the ghost is an exactly quadratic perturbation at this scale. Fits over the detected range ($k = 1$: $0.8 \le A \le 2.4$, 17 points):

- Power law $\delta_{\det} \propto A^s$: $s = -40$, maximum log-residual 8.5 nats. The $1/A$ law is **refuted** for the known-background detector: $\delta_{\det} A$ falls by 18 orders of magnitude.
- $\log \delta_{\det} = b + e \log \lambda_0 - \frac{3}{2} \log A$: $e = 0.468$ ($k = 4$: 0.437), maximum residual 0.93 nats. So $\delta_{\det} \approx C A^{-3/2} \lambda_0^{0.47}$, close to the interferometric $\sqrt{\lambda_*}$ law, with a slowly growing prefactor: the ratio to the rigorous Lemma 11 lower bound grows from 106 to $1.1 \cdot 10^5$.

Since $\lambda_*(A) = e^{-2\pi^2 e^A (1+o(1))}$ (Zhu's fitted law, Lemma 11), this is the double-logarithmic sensitivity $A_{\det} \approx \log\log(1/\delta) + O(1)$ already seen in Lemmas 11 and 12 ($d_{\text{ghost}} = 2.28 + 0.192 \log(1/\delta)$).

(c) Lemma 21's no-go. $A_{\text{crit}}(H) = 2.113$ is attained with *any* $\delta > 0$ (Lemma 21 Thm. 2.1: adversarial clumping of background zeros on the nodes of $\widehat{F}$, plus one ghost). It is set by the modulation-blindness of counting information, not by δ. A depth-of-focus law $\delta \sim c/A$ cannot produce it.

THEOREM 44.4.1 (verdict on the depth-of-focus conjecture). *[THM]+[NUM] (i) The depth-of-focus (Rayleigh) law $\delta_{\det} = c/A$ is exactly the law of the counting-only detector (Lemma 12: $c \in [3.71, 8.06]$, rising with $\log\log\gamma_0$). (ii) For the known-background (true) Weil form it is false: $\delta_{\det}$ follows $\sqrt{\lambda_*(A)}$ up to powers of A (fit exponent 0.47 / 0.44), so it decreases doubly exponentially in A. (iii) It does not explain Lemma 21's A_{crit}, which is δ-independent.*

REMARK 44.4.2 (where depth of focus does appear). [THM]+[ANALOGY]In the cavity dictionary of §44.2, the depth of focus of the self-consistent mode is $2z_R = 2\sin\theta_p \ge \sqrt{3}/\sqrt{p}$, i.e. twice the Krein margin of Lemma 42 Cor. 7.3. There the parabolic boundary is literally loss of focus ($z_R \to 0$, and the beam width at the reference plane $\to \infty$). So J.W.'s depth-of-focus intuition is exact for the per-prime cavity and wrong for the window.

44.5. Planck and the blackbody: the gas of Frobenius modes

44.5.1. Primon gas.

THEOREM 44.5.1 (Euler factor = one Bose mode). *[THM]+[CITED] (i) $(1-p^{-\beta})^{-1} = \sum_{n\ge 0} e^{-\beta n \log p}$ is the partition function of one bosonic mode of energy $\log p$ ("$\hbar\omega = \log p$"). Its mean occupation is $\langle n\rangle = 1/(p^\beta - 1)$, the Bose–Einstein (Planck) law. (ii) By unique factorization, $\zeta(\beta) = \prod_p (1-p^{-\beta})^{-1} = \operatorname{Tr} e^{-\beta H}$ on the bosonic Fock space over the primes, with $H|n\rangle = \log n|n\rangle$: Julia's primon gas [**143**], the stated motivation of Bost–Connes [**44**] [BIB]. (iii) The number of states of energy $\le E$ is $\lfloor e^E \rfloor$, an exponential (Hagedorn) density of states, so Z diverges for $\beta \le 1$ with a pole at the Hagedorn point $\beta_H = 1$. (iv) For fermionic primons, $\prod_p (1+p^{-\beta}) = \zeta(\beta)/\zeta(2\beta)$, and the graded trace is $\operatorname{Tr}(-1)^F e^{-\beta H} = \prod_p (1-p^{-\beta}) = 1/\zeta(\beta) = \sum \mu(n) n^{-\beta}$. This is Spector's identification of the Möbius function with $(-1)^F$ [**250**] ([CITED], abstract read).*

Rays, modes, quanta. Blackbody physics has three layers. Ray optics and Weyl's law count the modes (Rayleigh–Jeans); each mode has a frequency; Planck quantizes each mode as a Bose oscillator. Our cavity dictionary (§§44.1–44.2) is the *mode* layer: classical paraxial optics, Gouy phases, transverse mode structure. The Euler product is the *Planck* layer: each prime is one mode, quantized. The two layers do not refer to the same modes. In the primon gas the prime *is* the mode (frequency $\log p$); in the cavity dictionary the prime labels a cavity with round trip g_p. They meet in Theorem 44.5.3: for a degree-2 L-function, the Planck layer quantizes the two-dimensional mode space on which the round trip g_p acts.

44.5.2. Bost–Connes.

THEOREM 44.5.2 ([CITED]; read for Lemma 38, abstract re-read here). *Bost–Connes [44] construct a C^*-dynamical system $(C^*_{\mathbb{Q}}, \sigma_t)$ with $\sigma_t(\mu_n) = n^{it}\mu_n$, whose partition function is $\zeta(\beta)$ and whose symmetry group is $\hat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{cycl}}/\mathbb{Q})$, acting by automorphisms that commute with σ_t. There is a phase transition with spontaneous symmetry breaking at $\beta = 1$. For $0 < \beta < 1$ the KMS_β state is unique, hence Galois-invariant (BC95 Thm. 5(a) states the range $0 < \beta < 1$). For $\beta > 1$ the Galois group permutes the extremal KMS_β states freely and transitively. At $\beta = \infty$ the values of extremal ground states on the arithmetic subalgebra generate $\mathbb{Q}^{\mathrm{cycl}}$, and the symmetry intertwines with the Galois action (BC95 Thm. 5; Lemma 38 verified refs).*

KMS = modular condition [BIB]. For a faithful normal state φ, the Tomita–Takesaki modular group σ^φ is characterized by φ satisfying the KMS condition with respect to it (at $\beta = -1$ in Takesaki's convention) [264]. So in the GNS representation of a KMS_β state of Bost–Connes, the modular flow is $\sigma_{-\beta t}$: time evolution is modular flow up to scale. This is the mechanism behind Bisognano–Wichmann in Lemmas 31, 40 and 42 (modular flow = geometric boost), here with the scaling n^{it} as the boost.
It does not contain RH ([THM], from the statements). KMS states exist and are classified for every $\beta > 0$ with no input about zeros, and the partition function is ζ on the half-line $\beta > 1$, where it has no zeros. The zeros enter only through Connes' trace formula on the adele class space, as an absorption spectrum, and RH there is equivalent to positivity of the Weil distribution [75] [BIB] (Lemma 42 Rem. 3.4). My chat claim "Galois acts on low-temperature KMS states" is correct: low temperature means $\beta > 1$, where the symmetry is broken and Galois permutes the extremal states.

44.5.3. The Frobenius-twisted gas. For $E/\mathbb{Q}$ and $p \nmid N$ write $\alpha_p, \beta_p = \sqrt{p}\, e^{\pm i\theta_p}$ (Weil). For a hypothetical non-Weil factor, write $\alpha, \beta = \pm\sqrt{p}\, e^{\pm\eta}$. Put $L_p(s) = ((1-\alpha p^{-s})(1-\beta p^{-s}))^{-1}$ and $x = p^{-s}$.

THEOREM 44.5.3 (Fock form; Hecke = k-photon sector). *[THM] Let $V = \mathbb{C}^2$ and $g \in GL(V)$. Then $\mathrm{Tr}_{\mathrm{Sym}(V)}(g\,x^N) = \det(1-xg)^{-1}$ and $\mathrm{Tr}_{\Lambda(V)}((-1)^F g\,x^N) = \det(1-xg)$. With $g = \sqrt{p}\,g_p$,*

$$L_p(s) = \sum_k p^{k/2}\,\mathrm{tr}(\mathrm{Sym}^k g_p)\,p^{-ks} = \sum_k a_{p^k} p^{-ks}.$$

So the L-factor is a two-mode Bose gas (the two modes span H^1) in which the round trip g_p acts as a twist, and the Hecke eigenvalue a_{p^k} is the twisted trace over the k-photon sector. The Hasse–Weil zeta $Z(E/\mathbb{F}_p, T) = \det(1-Tg|H^1)/((1-T)(1-pT))$ is a supertrace*: H^0 and H^2 are bosonic, and H^1 is fermionic with $(-1)^F$ (Lefschetz), in the spirit of Spector's $(-1)^F$.*

PROOF. Diagonalize g. Sym of a direct sum is the tensor product of single-mode Fock spaces, each contributing $(1-x\lambda)^{-1}$; Λ likewise contributes $(1-x\lambda)$ with the sign. The Hecke identity is Theorem 44.2.6(ii). □

THEOREM 44.5.4 (twist, condensation, Hagedorn point). *[THM] Write $\alpha p^{-s} = \exp(-(s - c_\alpha)\log p)$ with $c_\alpha = \log\alpha/\log p$. Each mode is a Bose mode of energy $\log p$ seen at the shifted inverse temperature $s - c_\alpha$; equivalently, it carries fugacity α per quantum. (i) Weil: $c_{\alpha,\beta} = \frac{1}{2} \pm i\theta_p/\log p$. After the universal shift $s \mapsto s - \frac{1}{2}$ the twist is the rotation $e^{\pm i\theta_p}$: an* imaginary chemical potential*, i.e. a phase per quantum (the Gouy phase). Ghost: $c = \frac{1}{2} \pm \eta/\log p$ when $a > 2\sqrt{p}$ (add $i\pi/\log p$ when $a < -2\sqrt{p}$): a* real *chemical potential, i.e. a boost. (ii) The local Hagedorn point, i.e. the abscissa of convergence of L_p as a Dirichlet series, is $\sigma_H = \log\max(|\alpha|, |\beta|)/\log p$. We have $\sigma_H = \frac{1}{2}$ iff Hasse holds, and a ghost raises it to $\frac{1}{2} + \eta/\log p$ ([NUM]: 0.68074 for $p = 13$, $a = \pm 8$; 0.92757 for $a = 12$). These are the boost slopes of Lemma 42 Thm. 8.4. (iii) (Lee–Yang form.) $1/L_p(s) = 1 - a\,p^{-s} + p^{1-2s}$ vanishes only on $\mathrm{Re}\,s = \frac{1}{2}$ iff $|a| \le 2\sqrt{p}$, which for p prime means $|a| <$*

$2\sqrt{p}$. *So the Fisher zeros of the Frobenius gas (complex-β zeros of the reciprocal partition function) lie on one line iff Hasse holds [RESTATEMENT]. This parallels the Lee–Yang circle theorem for the Ising ferromagnet* [175] *[ANALOGY]; the analogy is only in the conclusion, and no Lee–Yang mechanism is claimed. (iv) (Sign at real temperature.)* $L_p(\sigma) > 0$ *for all real* σ *iff* $a < 2\sqrt{p}$, *and* $L_p(\sigma + i\pi/\log p) > 0$ *for all real* σ *iff* $a > -2\sqrt{p}$. *So Hasse* $\Leftrightarrow$ *the partition function is positive at every real temperature* and *at every* $(-1)^N$*-twisted temperature. Neither half alone suffices.*

PROOF. (ii) The radius of convergence of $\sum(\alpha^k + \beta^k)x^k/k$ is $1/\max|\alpha|$. (iii) The roots in x are $1/\alpha$ and $1/\beta$; they have modulus $p^{-1/2}$ iff $|\alpha| = |\beta| = \sqrt{p}$ iff the discriminant $a^2 - 4p \le 0$. (iv) For $a > 0$, $1 - ax + px^2$ on $x > 0$ has minimum $1 - a^2/4p$ at $x = a/2p$; the twist replaces x by $-x$, i.e. a by $-a$; and $a^2 = 4p$ is impossible for p prime. [NUM]: both one-sided statements hold on 11 test factors (`optics.json` H, `planck.json`). □

THEOREM 44.5.5 (no real-temperature thermodynamic positivity detects a ghost). *[THM]+[NUM] In the physical region* $\sigma > \sigma_H$*: (i) For every Weil factor with* $\theta_p \in (0, \pi)$, *the total occupation* $N(\sigma) = 2\,\mathrm{Re}\,\frac{z}{1-z}$, $z = \alpha p^{-\sigma}$, *tends to* -1 *as* $\sigma \downarrow \frac{1}{2}$. *The energy variance (the heat capacity up to a factor* σ^2*),* $\partial_\sigma^2 \log L_p = 2(\log p)^2\,\mathrm{Re}\,\frac{z}{(1-z)^2}$, *tends to* $-(\log p)^2/(2\sin^2(\theta_p/2)) < 0$. *So every arithmetic Frobenius gas has negative occupation and negative heat capacity near its Hagedorn point ([NUM]: minima of* N *between* -0.91 *and* -0.9996, *and negative variance, for all 7 Weil test factors, including those with* $a_p > 0$*). (ii) A fake with* $a > 2\sqrt{p}$ *is a genuine bosonic gas: all coefficients* $\alpha^k + \beta^k$ *are positive, so* $N > 0$, $\log L_p$ *is convex, and the heat capacity is positive ([NUM]:* $p = 13$, $a = 8, 12$*). (iii) For the full Hasse–Weil zeta in genus 1,* $Z(T) = 1 + \sum_{n\ge1} N_1 \frac{q^n-1}{q-1}T^n$ *with* $N_1 = q + 1 - a$ *(a formal identity, verified symbolically for* $n \le 6$*; for real curves* N_n *counts effective divisors [BIB]). The coefficients are* ≥ 0 *iff* $a \le q + 1$, *so positivity of the full partition function holds for fakes such as* $(q, a) = (13, 8)$ *as well. (iv) KMS positivity of Bost–Connes type is positivity of states, and holds by construction for every* β.

PROOF. (i) $|z| \to 1$ and $z \to e^{i\theta}$. Now $\frac{e^{i\theta}}{1-e^{i\theta}} = -\frac{1}{2} + \frac{i}{2}\cot\frac{\theta}{2}$ and $\frac{e^{i\theta}}{(1-e^{i\theta})^2} = \frac{1}{(e^{-i\theta/2}-e^{i\theta/2})^2} = -\frac{1}{4\sin^2(\theta/2)}$, and the conjugate mode doubles the real part. (ii) $\alpha, \beta > 0$. (iii) Expand $(1 - aT + qT^2)/((1 - T)(1 - qT))$. □

Reading. At real temperature, thermodynamic positivity favors the ghost. A boost is a real chemical potential and gives an honest gas, while a rotation (the Weil case) is an imaginary chemical potential and makes occupations and heat capacities non-positive. The Weil condition says the twist is *unitary*, and its thermal signature is the location of the complex-temperature (Fisher/Lee–Yang) zeros, not positivity. This matches Lemma 42 (RH = unitarity, not positivity of a real quantity) and rules out a "thermodynamic stability" proof of Hasse-type statements.

44.5.4. The Gutzwiller tension.

PROPOSITION 44.5.6 ([THM] computation; [CITED]/[BIB] context). *(i) In Gutzwiller's trace formula* [122] *[BIB], a periodic orbit of period* T *and repetition* m *enters with amplitude* $\propto |\det(M^m - 1)|^{-1/2}$, *where* M *is the transverse monodromy. For hyperbolic* M *(eigenvalues* $e^{\pm\lambda T}$*) this is* $\frac{1}{2\sinh(m\lambda T/2)}$*; for elliptic* M *(eigenvalues* $e^{\pm i\theta}$*) it is* $\frac{1}{2|\sin(m\theta/2)|}$, *resonant at rational* θ/π. *(ii) Berry–Keating* [30] *([CITED], abstract and the counting-fluctuation formula read):* $N_{\mathrm{fl}}(t) = -\frac{1}{\pi}\sum_p\sum_{m\ge1}\frac{1}{m}p^{-m/2}\sin(mt\log p)$, *read as a periodic-orbit sum with primitive periods* $\log p$. *The "Riemann dynamics" should be chaotic, possibly generated by* $H_{\mathrm{cl}} = XP$. *With* $\lambda = 1$ *and* $T = \log p$, $\frac{1}{2\sinh(m\log p/2)} = \frac{p^{-m/2}}{1-p^{-m}}$. *This agrees with the explicit-formula weight* $p^{-m/2}$ *only to leading order in* p^{-m}, *and the oscillatory term has an overall negative sign. Connes reads that sign as absorption rather than emission* [75] *[BIB]. (iii)* XP *generates* $(x, p) \mapsto (e^t x, e^{-t}p)$, *a boost with Lyapunov exponent* 1. *Traversing "orbit* p*" (time* $\log p$*) accumulates rapidity* $\log p$.

THEOREM 44.5.7 (no conflict). *[THM]+[ANALOGY] The hyperbolic monodromy of Berry–Keating and the elliptic per-prime Frobenius are different objects. The first is the linearized scaling flow transverse to an orbit, with rapidity* $\log p$ *per traversal (one unit of rapidity per unit of time, for every p). In Lemma 42 language it is the* base boost*: the scaling flow that Connes–Consani identify with Frobenius for* ζ [69] *(Lemma 42 Def. 6.12). The second acts on the two-dimensional fibre* H^1 *of a curve. For* $L(E,s)$ *both appear: the explicit formula carries* $p^{-m/2}\cdot 2\cos(m\theta_p)\log p$*, a hyperbolic base weight times a fibre character, like a Gutzwiller orbit carrying an internal rotation phase (a Gouy- or Maslov-type factor). A ghost would be a boost in the fibre (*$2\cosh m\eta$*). For* ζ *(degree 1) there is no fibre, so RH for* ζ *is not a per-prime ellipticity statement. This is the recurring gap (Lemma 42 §6.5).*

44.6. Fermat, stationary phase, Nash (flavor only)

This section is [ANALOGY] throughout. Fermat's principle selects rays as *stationary*, not minimal, optical paths, and stationary phase is the asymptotic mechanism. Past a focus a ray stops being a local minimum, and the wave picks up $-\pi/2$ per caustic (a Maslov index). The Gouy phase $\arctan(z/z_R)$ is the smooth version, with total π through a focus (Feng–Winful [99] [BIB]). A Nash equilibrium is also a stationarity condition (no unilateral improvement), and in zero-sum games it is a saddle (minimax), just as Fermat paths through a focus are saddles of the optical length. The shared content is only "saddle, not minimum; index counted by phase jumps". Lemma 43's replicator centers are a third, different object: conservative orbits around an interior equilibrium. No identification is claimed.

44.7. Corrections to chat claims

C1. "Hasse = cavity stability": **correct** as a matrix identity (Thm. 44.2.2(i)). It adds no proof.

C2. "θ_p = Gouy phase": **correct**: it is the round-trip Gouy phase (Thm. 44.2.2(ii)). The transverse mode of order m gets $(m+\frac{1}{2})\theta_p$, not θ_p.

C3. "Gaussian beam form = Lemma 42's invariant form": **correct, exactly**. $\mathsf{B}_{q^*} = b_{g_p}$, and B is the Wigner exponent of the self-consistent beam (Thm. 44.1.1, Prop. 44.1.2, Cor. 44.2.3).

C4. "Boyd–Gordon modes = our Slepian functions, with A as Fresnel number": **half right**. They coincide for the uniform-band idealization, with A as the aperture and Fresnel number $N = AT/4\pi$, which needs a band T. They are not the eigenvectors of Weil's window form (Prop. 44.3.4).

C5. "Fox–Li non-unitarity = window truncation": **structurally right, diagnostically wrong**. Both are compressions, and both are lossy whether or not RH holds. The ghost is negative source intensity (Weil) or gain (Lemma 37), not loss (Prop. 44.3.5).

C6. "Depth of focus may explain Lemma 21's limit": **false**. $A_{\text{crit}} = 2.113$ is δ-independent. The c/A law belongs to the counting-only detector (Lemma 12), and the true known-background threshold scales like $\sqrt{\lambda_*(A)}$ (Thm. 44.4.1). Depth of focus is exact elsewhere: the Rayleigh range $\sin\theta_p$ of the arithmetic cavity.

C7. "Hecke $T_{p^k} = k$ round trips" (task phrasing): **false**. k round trips give $\operatorname{tr} g_p^k = 2\cos k\theta_p$ (points over $\mathbb{F}_{p^k}$); T_{p^k} is $\operatorname{tr}\operatorname{Sym}^k g_p = U_k(\cos\theta_p)$, the k-photon sector (Thms. 44.2.6, 44.5.3).

C8. "Euler factor = Planck mode": **correct** for ζ (Thm. 44.5.1). In degree 2 it is a two-mode gas twisted by g_p.

C9. "Bost–Connes: Galois acts on low-T KMS states": **correct** ($\beta > 1$, extremal states permuted freely and transitively). Uniqueness is stated for $0 < \beta < 1$ in BC95 Thm. 5(a), and RH is not contained.

C10. "The ghost shifts mode energies to $(\frac{1}{2} \pm \eta/\ln p)\log p$": **imprecise**. The mode energy stays $\log p$. The ghost replaces the imaginary chemical potential $\pm i\theta_p$ by a real one, $\pm\eta$, which moves

the condensation (Hagedorn) points to real $s = \frac{1}{2} \pm \eta/\log p$. The numbers $(\frac{1}{2} \pm \eta/\ln p)\log p = \log|\alpha_\pm|$ are log-moduli (Newton slopes times $\log p$), not energies (Thm. 44.5.4).

C11. "Berry–Keating orbits are hyperbolic": **correct as their reading** (XP boost, $\lambda = 1$), with the known mismatch $p^{-m/2}$ vs. $p^{-m/2}/(1-p^{-m})$ and the sign. There is no conflict with elliptic Frobenius (Thm. 44.5.7).

44.8. Status, roadmap, questions

Statement	label	where
ABCD transport of B_q; Wigner form	[THM]	Thm. 44.1.1, Prop. 44.1.2
stable $\Leftrightarrow$ $q^* \in \mathbb{H}$ $\Leftrightarrow$ center $\Leftrightarrow$ positive form; Gouy $= \Theta$	[THM]	Cor. 44.1.3, Prop. 44.1.4
$\mathsf{B}_{q_p^*} = b_{g_p}$ (L42); $z_R = \sin\theta_p \geq \sqrt{3}/2\sqrt{p}$	[THM]	Cor. 44.2.3, Thm. 44.2.2
rational Gouy $\Leftrightarrow$ supersingular; $a_p = 0$ quarter turn	[THM]	Thm. 44.2.4
round trips vs. Hecke; Fock form	[THM]	Thms. 44.2.6, 44.5.3
Weil window = mutual intensity of the zero source	[THM]/[RESTATEMENT]	Thm. 44.3.1
Boyd–Gordon = uniform-band idealization, $N = AT/4\pi$	[THM]/[CITED]	Prop. 44.3.4
ghost = negative intensity / gain, not loss	[THM]/[RESTATEMENT]	Prop. 44.3.5
depth-of-focus law only for the counting-only detector	[NUM]	Thm. 44.4.1
Hasse $\Leftrightarrow$ $\sigma_H = \frac{1}{2}$ $\Leftrightarrow$ Fisher zeros on a line	[THM]/[RESTATEMENT]	Thm. 44.5.4
no real-T positivity detects ghosts	[THM]/[NUM]	Thm. 44.5.5
BK hyperbolic vs. Frobenius elliptic: base vs. fibre	[THM]/[ANALOGY]	Thm. 44.5.7
RH	not addressed	

Roadmap (score = 0.6·significance+0.4·probability, 1–10 scale):

- O1 (6, 6 → 6.0): genus g. Sp_{2g} cavities, with beam forms given by points q^* of the Siegel upper half-space $\mathfrak{H}_g$ (Gaussian beams in g transverse dimensions), and their Krein signatures. Joins Lemma 42 S4/S7 and Lemma 43 Q1.
- O2 (5, 7 → 5.8): the coherent-mode decomposition of the true windowed Weil form vs. prolates at matched c, quantifying how the archimedean and prime terms distort the source (extends Lemma 22).
- O3 (6, 5 → 5.6): a $(-1)^N$-graded (supersymmetric) thermal version of the Lemma 37 line, in which Theorem 44.5.4(iv) has a global analogue. RH-strength if it worked.
- O4 (4, 8 → 5.6): rerun §44.4(b) on the 120-mode Lemma 11 Grams and at larger heights γ_k, to pin down the prefactor of the $\sqrt{\lambda_*}$ law.

QUESTION 44.8.1. Does the van Cittert–Zernike reading (zeros = incoherent emitters, primes = coherence spikes in the aperture plane, ghost = complex source point) match your lithography picture well enough to adopt it as the Volume 2 language for Weil[1] windows?

QUESTION 44.8.2. Theorem 44.2.4 says that a rational Gouy phase is supersingularity, and $a_p = 0$ is a linear quarter-turn. Should route S8 (a linear J with $J^2 = -1$) be pursued from the supersingular side, or stay a remark?

QUESTION 44.8.3. The Planck section shows that thermal positivity at real temperature cannot see ghosts, and that Weil is a Lee–Yang-type statement about complex temperatures. Is that

[1] **[Vol. 3 editorial note]** With the approved two-volume split, the series is Vol. I (Lemmas 1–13), Vol. II (Lemmas 14–29) and Vol. III (Lemmas 30–47, this volume). "Volume 2" in this question meant the next volume of the series; for Lemmas 30–47 that is this volume. See Appendix A, item D1.

consistent with "RH is unitarity" as you mean it, or were you hoping for a positivity statement at real β?

Question 44.8.4. Which detector should set the official ghost-sensitivity scale in Volume 2: the certified counting-only c/A law (Lemma 12), or the known-background $\sqrt{\lambda_*}$ law (numerical, this lemma)?

Question 44.8.5. Should O1 (Siegel-space beams for genus g) come next, ahead of O2–O4?

CHAPTER 45

Lemma 45: Bosons, fermions, spin and ghosts

Two second quantizations of ζ, the supersymmetric primon gas, $SU(2)$ multiplets of Frobenius, and where an off-line zero breaks the Fock metric

Standalone note `lemma45.tex`, October 22, 2026.

Summary. **Statement of intellectual honesty.** RH is not proved here. Tags: [THM] proved here (elementary, proof given); [CITED] checked at the source (`lit/lemma45/verified_refs.md`); [BIB] bibliographic record verified, content standard; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma45_*.py`, data `num/l45/`); [CONJ] conjecture; [OPEN] open. Lemma 44 (cavities and Planck) is cited by its theorem numbers and not redone.

Question (J.W.). If ζ is a Bose gas, is the "infinite-dimensional lattice" of $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$ its extreme UV limit of "bosons in the same state"? And should Pauli exclusion, spin, fine structure and the Zeeman effect matter?

Answer in brief. (1) *Statistics.* Pauli exclusion is a fermion rule; multiple occupation is Bose statistics. Bose condensation is a low-temperature (IR) phenomenon. The UV end of the primon gas is its Hagedorn pole $\beta=1$, and there statistics stop mattering: $Z_{\mathrm{B}}(\beta)=Z_{\mathrm{F}}(\beta)\,\zeta(2\beta)$ exactly, every mode has finite occupation $1/(p-1)$ at $\beta=1$, and the divergence comes from the *number* of modes, not from many bosons in one mode (Prop. 45.1.1). So "UV limit = bosons in the same state" is false as stated. (2) *The lattice.* The infinite-dimensional lattice is exact, but it is the gas's *state space*: Fock basis = the positive cone of $\mathbb{Q}^{\times}_{>0}\cong\bigoplus_p\mathbb{Z}=\operatorname{Div}(\operatorname{Spec}\mathbb{Z})$, with energy = degree. Fermionic states are the vertices of the infinite cube (squarefree n). For a curve, $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$'s counterpart $C\times C$ is the space of *one-particle operators* (correspondences). Its Künneth pieces split exactly into a boson⊗boson hyperbolic plane (the light cone of Lemma 14, carrying the poles) and a fermion-bilinear sector $H^1\otimes H^1$, where Weil's positivity lives (Thm. 45.2.4). (3) *Two second quantizations.* ζ is a Bose gas over *points* (primes), with no fermions at all. "Zeros are fermions" holds in a *different* Fock space, the super-Fock space over *cohomology*. For curves the two agree by the Lefschetz trace formula; this is proved and checked by brute force on 11a over $\mathbb{F}_{p^k}$ (Thm. 45.2.2). For ζ the first exists, and the second is Deninger's conjectural picture. (4) *Spector.* $\mu=(-1)^F$, Pauli = vanishing on non-squarefree n, Witten index $\equiv 1$ = Möbius inversion. RH $\iff$ the even/odd imbalance of the fermionic gas is $O(x^{1/2+\varepsilon})$ (Littlewood; Titchmarsh 14.25(C)), [RESTATEMENT]. Sieve to 10^8: $\max_{x\ge 10^5}|M(x)|/\sqrt{x}=0.4627$. (5) *Spin.* The degree-2 Euler factor is Schwinger's two-boson model of angular momentum: the k-boson sector is spin $k/2$, and $a_{p^k}/p^{k/2}$ is the spin-$k/2$ character at rotation angle $2\theta_p$, i.e. a sum over Zeeman sublevels m ([THM]). Sato–Tate is Haar measure on $SU(2)$ (cited). A ghost makes the "spin" a non-unitary boost with sinh characters, and the Ramanujan bound $|a_{p^k}|\le(k+1)p^{k/2}$ for all k is equivalent to Hasse at p ([THM]). Zeeman and fine structure are [ANALOGY]. (6) *The key test.* An off-line Frobenius pair never violates Pauli *combinatorially*. Real-temperature fermionic positivity, which is what Lemma 44 Thm. 5.4's $(-1)^N$ twist really is, sees real ghosts but misses *every* complex-quadruple ghost in genus 2 (782 of 3591 integer test polynomials at $q=11$). The ghost is a *metric* violation: for every non-degenerate Frobenius-invariant Hermitian metric, the ghost pair state $e\wedge f$

has norm $-|h(e,f)|^2 < 0$ and a unimodular Frobenius eigenvalue. That is a Gupta–Bleuler/Krein ghost. Weil $\iff$ some invariant metric makes the two-fermion sector positive (Thm. 45.6.4). This is a rigorous statement, but it is a Fock-space restatement of "unitary $\iff$ positive invariant form", not a new lever.

45.1. Statistics, the Hagedorn end, and the lattice

45.1.1. Corrections to the setting.

(a) *Pauli exclusion* forbids two identical *fermions* in one state. Multiple occupation is permitted by Bose statistics; Pauli's exclusion principle does not "allow" it. Pauli's 1940 theorem links the two to spin: integer spin goes with Bose, half-integer with Fermi statistics [**221**] [BIB].

(b) *Bose–Einstein condensation* is macroscopic occupation of the ground state at *low* temperature (IR). The primon Hamiltonian $H|n\rangle = \log n\,|n\rangle$ has a non-degenerate ground state $|1\rangle$ and no condensation transition: as $\beta \to \infty$ the Gibbs state simply tends to $|1\rangle$.

(c) The *UV end* of the primon gas is its Hagedorn pole $\beta = 1$ (Lemma 44 Thm. 5.1(iii)). The phase transition with Galois symmetry breaking belongs to the Bost–Connes system: unique KMS state for $0 < \beta < 1$, Galois-permuted extremal states for $\beta > 1$ [**44**] (Lemma 44 Thm. 5.2, [CITED]). For $\beta > 1$ the extremal KMS states are Gibbs states (type I_∞; Lemma 38 verified refs). So the low-temperature phase is an *ordered, symmetry-broken* phase, and "condensate-like" is [ANALOGY] at best: nothing is macroscopically occupied.

45.1.2. At the Hagedorn point statistics do not matter. Write $Z_{\mathrm{B}}(\beta) = \prod_p (1 - p^{-\beta})^{-1} = \zeta(\beta)$ for the bosonic gas and $Z_{\mathrm{F}}(\beta) = \prod_p (1 + p^{-\beta}) = \zeta(\beta)/\zeta(2\beta)$ for the fermionic one (Lemma 44 Thm. 5.1).

PROPOSITION 45.1.1 ([THM]; [NUM]). *(i) Each Bose mode factorizes exactly into a Fermi mode and a Bose "pair" mode of energy* $2\log p$*:* $(1-x)^{-1} = (1+x)(1-x^2)^{-1}$*, i.e.* $n = \varepsilon + 2j$ *with* $\varepsilon \in \{0,1\}$*. Hence* $Z_{\mathrm{B}}(\beta) = Z_{\mathrm{F}}(\beta)\,\zeta(2\beta)$*, and the pair gas* $\zeta(2\beta)$ *has its own Hagedorn point at* $\beta = \frac{1}{2}$*.*

(ii) At $\beta = 1$ *the mean occupation of mode* p *is* $1/(p-1) \le 1$*, and* $\Pr(n_p \ge 2) = p^{-2}$*. The mean energy* $\langle E\rangle_{\mathrm{B}} = -\zeta'(\beta)/\zeta(\beta) \sim 1/(\beta-1)$ *diverges ([NUM]:* 99.42 *at* $\beta = 1.01$*,* 999.42 *at* $\beta = 1.001$*), but*

$$\langle E\rangle_{\mathrm{B}} - \langle E\rangle_{\mathrm{F}} = -2\zeta'(2\beta)/\zeta(2\beta) \xrightarrow{\beta\to 1} -2\zeta'(2)/\zeta(2) = 1.13992\ldots$$

stays finite. So the Hagedorn divergence sits entirely in singly occupied (fermion-like) states, and multiple occupation contributes a finite amount.

PROOF. (i) Multiply over p. (ii) $\langle n_p\rangle = 1/(p^\beta - 1)$; $\Pr(n_p \ge 2) = p^{-2\beta}$; differentiate $\log Z_{\mathrm{B}} - \log Z_{\mathrm{F}} = \log\zeta(2\beta)$. Numbers: `num/l45/spin.json` (mpmath, 30 digits). □

Reading. J.W.'s "extreme UV limit of bosons in the same state" is backwards. The UV divergence comes from the *exponential number of modes* (prime number theorem). The "bosons in the same state" sector is the pair gas $\zeta(2\beta)$, which is harmless at $\beta = 1$. (Its Hagedorn point $\beta = \frac{1}{2}$ coincides numerically with the critical abscissa, but only because $\zeta(2\beta)$ has its pole at $2\beta = 1$. No claim is attached to this.)

45.1.3. The infinite-dimensional lattice is the state space.

PROPOSITION 45.1.2 ([THM]). *(i) The bosonic Fock basis* $\{|n\rangle\}$ *is the positive cone* $\mathbb{N}^{(\mathcal{P})}$ *of the free abelian group* $\mathbb{Q}^\times_{>0} \cong \bigoplus_p \mathbb{Z}$*, which is the (finite part of the) Arakelov divisor group* $\mathrm{Div}(\mathrm{Spec}\,\mathbb{Z})$*. The energy* $\log n$ *is the Arakelov degree of the effective divisor* $\sum k_p[p]$*, so* $\zeta(\beta) = \sum_{D\ge 0} e^{-\beta \deg D}$*. (ii) Fermionic states are the vertices* $\{0,1\}^{(\mathcal{P})}$ *of the infinite cube (squarefree* n*). Points of the lattice off the cube are exactly the multiply occupied states. (iii) Over* $\mathbb{F}_q$ *the same sentence is the*

definition $Z(C,T)=\sum_{D\geq 0}T^{\deg D}=\sum_n \#C^{(n)}(\mathbb{F}_q)T^n$*: the* n*-boson sector is the symmetric power* $C^{(n)}=\mathrm{Sym}^n C$ [**282**] *[BIB].*

Proof. Unique factorization; for (iii), an effective divisor of degree n is a point of $C^{(n)}$. □

So the lattice J.W. describes exists, but it is the lattice of a *single* copy of $\operatorname{Spec}\mathbb{Z}$, read as a gas. The product $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$ plays a different role (§45.2.3).

45.2. Two second quantizations

45.2.1. The super-Fock identity. Let $V=V_0\oplus V_1$ be a finite-dimensional super vector space and F an even endomorphism. Put $\mathbb{S}(V)=\mathrm{Sym}(V_0)\otimes\Lambda(V_1)$, graded by total degree N, with parity $(-1)^{\mathsf{F}}$ equal to the Λ-degree. Let $\mathbb{S}^n$ be the degree-n part.

Theorem 45.2.1 ([THM]).

$$\sum_{n\geq 0}\mathrm{Str}\,(F\,|\,\mathbb{S}^n(V))T^n=\frac{\det(1-TF\,|\,V_1)}{\det(1-TF\,|\,V_0)}=\exp\Big(\sum_{m\geq 1}\mathrm{Str}(F^m|V)\frac{T^m}{m}\Big).$$

Proof. Triangularize F. A bosonic eigenvalue u contributes $\sum_k u^kT^k=(1-uT)^{-1}$. A fermionic eigenvalue a contributes $1-aT$, since Λ stops at degree 1 (Pauli) and the odd state carries the sign. For the exponential form, use $\log\det(1-TF)=-\sum_m\mathrm{Tr}(F^m)T^m/m$. [NUM]: the coefficient identity holds symbolically for $\dim V_0=2$, $\dim V_1=4$, generic eigenvalues, through T^7 (residual 0, `145/fock.json`). □

Newton triangle. For one block, $h_k=\mathrm{Tr}\,\mathrm{Sym}^k$ (bosons), $e_k=\mathrm{Tr}\,\Lambda^k$ (fermions) and $p_k=\mathrm{Tr}\,F^k$ (cycles, i.e. k round trips) satisfy $\sum h_kT^k=\exp(\sum p_kT^k/k)=1/\sum(-1)^ke_kT^k$ (sympy check, `fock.json`). For an elliptic curve, h_k is Hecke's a_{p^k}, p_k gives $\#E(\mathbb{F}_{p^k})$, and $e_1=a_p$, $e_2=p$ (Lemma 44 Thms. 2.6 and 5.3).

45.2.2. Points versus cohomology.

Theorem 45.2.2 ([THM] given Weil's theorem [**282**]; [BIB] Macdonald). *Let* $C/\mathbb{F}_q$ *be a smooth projective curve of genus* g*,* P_d *the number of closed points of degree* d*, and* $N_m=\#C(\mathbb{F}_{q^m})$*.*

- *(A)* Fock space over points. $Z(C,T)=\prod_d(1-T^d)^{-P_d}$ *is the trace of* T^N *on the purely bosonic Fock space with one mode per closed point, of energy* $\deg x$*. Its fermionic counterpart is* $\prod_d(1+T^d)^{P_d}=Z(T)/Z(T^2)$*, which counts squarefree effective divisors.*
- *(B)* Fock space over cohomology. *With* $V_0=H^0\oplus H^2$*,* $V_1=H^1$ *and* F *the geometric Frobenius,* $Z(C,T)=\sum_n\mathrm{Str}(F|\mathbb{S}^nV)T^n=\det(1-TF|H^1)/((1-T)(1-qT))$*. Here there are* 2 *bosonic and* $2g$ *fermionic modes. The poles come from bosons and the zeros from fermions.*
- *(C)* Bridge. *(A)=(B) is equivalent to the Lefschetz formula* $N_m=\mathrm{Str}(F^m|H^*)=1+q^m-\sum_i\alpha_i^m$*. Macdonald's theorem* $H^*(C^{(n)})\cong\mathbb{S}^n(H^*(C))$ [**182**] *[BIB] refines it: the* n*-th coefficient of (B) is the Lefschetz number of Frobenius on the* n*-boson space* $C^{(n)}$ *of Prop. 45.1.2(iii).*

Proof. (A) $\log\prod_d(1-T^d)^{-P_d}=\sum_m\frac{T^m}{m}\sum_{d|m}dP_d=\sum_m N_mT^m/m$, and $(1+x)=(1-x^2)/(1-x)$. (B) Theorem 45.2.1. (C) Compare the exponential forms. □

Remark 45.2.3 (ζ, Deninger; [CITED] for the text, [CONJ] for the content). For ζ the point-Fock space (A) exists: it is the primon gas. The cohomology-Fock space (B) is Deninger's picture. In [**91**, §3, formula (3)] he writes that a formula of the type $\hat\zeta(s)=\prod_{i=0}^2\det_\infty(\frac{1}{2\pi}(s-\Theta)\,|\,H^i)^{(-1)^{i+1}}$ "might hold". It would force $H^0=\mathbb{R}$ with $\Theta=0$, H^1 infinite-dimensional with the nontrivial zeros as spectrum, and $H^2=\mathbb{R}$ with $\Theta=\mathrm{id}$. The consequence $\xi(s)=\prod_\rho\frac{s-\rho}{2\pi}$ (regularized) "turned out to be true" [**89**]. Formula (3) is a regularized *super*determinant, the first-quantized (spectral) form

p	N_k (direct)	P_d	bosonic (A) $= Z$	fermionic $= Z(T)/Z(T^2)$
2	5,5,5,25,25,65	5,0,0,5,4,10	1,5,15,35,75,155,315	1,5,10,10,10,30,80
3	5,15,20,75,275	5,5,5,15,54	1,5,20,65,200,605	1,5,15,40,105,305
5	5,35,140,595	5,15,45,140	1,5,30,155,780	1,5,25,130,625
7	10,60,310,2400	10,25,100,585	1,10,80,570,4000	1,10,70,470,3220

TABLE 45.1. [NUM] Curve 11a, $y^2+y=x^3-x^2-10x-20$, counted directly over $\mathbb{F}_{p^k}$ (our own $\mathbb{F}_{p^k}$ arithmetic; up to $7^4=2401$ elements). In all cases the N_k equal $p^k+1-(\alpha^k+\beta^k)$, the bosonic point-Fock counts equal the coefficients of $Z=(1-a_pT+pT^2)/((1-T)(1-pT))$, and the fermionic counts equal those of $Z(T)/Z(T^2)$. (`num/lemma45_fock.py`, `l45/fock.json`.)

of Theorem 45.2.2(B). So "zeros are fermions, poles are bosons" is exactly Deninger's conjectural cohomology. It is *not* a property of the primon gas, which contains no fermions. The bridge (C) for ζ would be the explicit formula read as a Lefschetz formula, which is Deninger's program and open [OPEN].

45.2.3. $C\times C$ as the space of one-particle operators.

THEOREM 45.2.4 ([THM] + [CITED] [193]; restates Lemma 14). *Let C be a curve over $\mathbb{F}_q$ (or $\mathbb{C}$), $H=[C\times\mathrm{pt}]$, $V=[\mathrm{pt}\times C]$. By Künneth, $H^2(C\times C)=(H^0\otimes H^2)\oplus(H^1\otimes H^1)\oplus(H^2\otimes H^0)$, and $H^1\otimes H^1\cong\mathrm{End}(H^1)$ (via Poincaré duality), so classes act as operators on the one-particle space.*

(i) Boson⊗boson sector = light cone. *H,V span the algebraic part of the two outer summands, with $H^2=V^2=0$ and $H\cdot V=1$: a hyperbolic plane with null basis (Lemma 14(b)).*

(ii) Fermion-bilinear sector. *If $D\cdot H=D\cdot V=0$, the outer Künneth components of D vanish, so $D\in H^1\otimes H^1$. By the Hodge index theorem the intersection form is negative definite on $\langle H,V\rangle^{\perp}\cap NS_{\mathbb{R}}$* [193, Thm. 1.5]*, equivalently the Rosati involution is positive* [193, Thm. 1.27]*. Examples (Lemma 14(g)): $\delta=\Delta-(H+V)$ has $\delta^2=-2g$, and $\gamma=\Gamma_F-(H+qV)$ has $\gamma^2=-2gq$.*

(iii) Lefschetz split. $\Gamma_F\cdot\Delta=\mathrm{Str}(F|H^*)=(1+q)-\mathrm{Tr}(F|H^1)$. *The boson⊗boson part gives $1+q$ (Lemma 14(g)'s "boost" part), and the fermion-bilinear part gives $\delta\cdot\gamma=-\mathrm{Tr}(F|H^1)$. RH for C is $|\delta\cdot\gamma|\le 2g\sqrt{q}$, Cauchy–Schwarz in the negative-definite fermion-bilinear sector.*

PROOF. (i) Standard. (ii) $D\cdot H$ and $D\cdot V$ are the coefficients of the two outer components. Negativity is cited. (iii) Lemma 14(g): $N=q+1+\delta\cdot\gamma$ and $N=q+1-\mathrm{Tr}(F|H^1)$. □

For $\mathrm{Spec}\,\mathbb{Z}\times\mathrm{Spec}\,\mathbb{Z}$ ([RESTATEMENT] of Lemma 15; the rest [ANALOGY]). Lemma 15 Thm. 15.1 splits $\mathcal{L}=P-\mathcal{Q}$, where the pole form P is a hyperbolic plane whose null vectors are the functionals $F\mapsto\widetilde{F}(\pm i/2)$, i.e. the poles $s=1$ (the Hagedorn point) and $s=0$ (its mirror). So the Hagedorn pole and its mirror *are* the boson⊗boson light cone, exactly as in (i). The zero side $\mathcal{Q}$ corresponds to the fermion-bilinear sector (ii), and RH is its definiteness. That is the precise version of "the Hagedorn point is the null part of the light cone". The positivity we lack lives in the two-fermion sector, not in the bosonic gas.

45.3. The supersymmetric primon gas (Spector)

THEOREM 45.3.1 ([THM]; content of [250], [CITED] abstract). *On the fermionic primon Fock space (basis: squarefree n) the grading operator $(-1)^{\mathsf{F}}$ has eigenvalue $\mu(n)$. Extending by 0 to non-squarefree n is Pauli exclusion: those states do not exist. On the supersymmetric Fock space $\mathcal{H}_{\mathrm{B}}\otimes\mathcal{H}_{\mathrm{F}}$ with $H=\log(n_{\mathrm{B}}n_{\mathrm{F}})$, the Witten index* [296] *[BIB] is*

$$\mathrm{Tr}\,(-1)^{\mathsf{F}}e^{-\beta H}=\zeta(\beta)\cdot\frac{1}{\zeta(\beta)}=1\qquad(\beta>1),$$

and the statement that the energy-$\log n$ sector has index $\delta_{n,1}$ is the Möbius identity $\sum_{d|n}\mu(d)=\delta_{n,1}$. Boson–fermion cancellation is exact apart from the vacuum.

PROOF. The states of energy $\log n$ are the pairs $(n/d,d)$ with $d|n$ squarefree, and their index is $\sum_{d|n}\mu(d)$. [NUM]: $\zeta(2)\sum_{n\le 2\cdot10^4}\mu(n)n^{-2}=1.0000001$. □

THEOREM 45.3.2 ([RESTATEMENT]; Littlewood [**180**], Titchmarsh [**269**, Thm. 14.25(B),(C)] [CITED]). *Let $M(x)=\sum_{n\le x}\mu(n)=Q_+(x)-Q_-(x)$, where $Q_\pm$ count squarefree $n\le x$ with $\mu(n)=\pm1$: the even and odd fermion-number states of the fermionic gas with energy $\le\log x$. Then RH $\iff$ $M(x)=O(x^{1/2+\varepsilon})$ for every $\varepsilon>0$ $\iff$ $\sum\mu(n)n^{-s}$ converges for $\operatorname{Re}s>\frac12$. Unconditionally $M(x)\neq o(x^{1/2})$ [**269**, Thm. 14.26(B)], and $|M(x)|<\sqrt{x}$ for all $x>1$ (Mertens) is false* [**213**] *[BIB].*

Reading ([ANALOGY]). The total number of fermionic states is $Q(x)\sim 6x/\pi^2$. RH says the even/odd imbalance is at most of square-root ("random-walk") size, $Q^{1/2+\varepsilon}$. This is *not* the exact supersymmetry of Theorem 45.3.1, which needs the bosons. The fermion gas alone is only "nearly balanced", and how nearly is RH. An off-line zero ρ would make the imbalance grow like $x^{\operatorname{Re}\rho}$.

x	$M(x)$	$M(x)/\sqrt{x}$
10^4	-23	-0.230
10^5	-48	-0.152
10^6	212	0.212
10^7	1037	0.328
10^8	1928	0.193

TABLE 45.2. [NUM] Sieve of μ to $N=10^8$ (`num/lemma45_mertens.py`, `145/mertens.json`; 10 s). The values agree with OEIS A084237. $Q_+(10^8)=30\,397\,311$, $Q_-(10^8)=30\,395\,383$, $Q=60\,792\,694$, versus $6N/\pi^2=60\,792\,710.2$. Over $10^5\le x\le10^8$, $\max|M(x)|/\sqrt{x}=0.4627$, at $x=30\,919\,091$ ($M=-2573$); over $10^3\le x\le10^8$ it is 0.4722, at $x=2803$.

45.4. Spin: Frobenius multiplets

45.4.1. Schwinger bosons.

THEOREM 45.4.1 ([THM]). *Let $E/\mathbb{Q}$, $p\nmid N$, $a_p=2\sqrt{p}\cos\theta_p$, and $g_p\in SU(2)$ with eigenvalues $e^{\pm i\theta_p}$.*

(i) The k-boson sector $\operatorname{Sym}^k\mathbb{C}^2$ of the two-mode gas (Lemma 44 Thm. 5.3) is the spin-$j=k/2$ representation of $SU(2)$: this is the oscillator model of angular momentum, with $J_z=\frac12(n_\alpha-n_\beta)$, so that the sublevel m has $n_\alpha=j+m$, $n_\beta=j-m$.

(ii) $p^{-k/2}a_{p^k}=\chi_{k/2}(g_p)=\sum_{m=-k/2}^{k/2}e^{2im\theta_p}=U_k(\cos\theta_p)$: a sum over the Zeeman sublevels of spin $k/2$, with g_p acting as a rotation by angle $2\theta_p$ about the quantization axis. (The factor 2 is the $SU(2)\to SO(3)$ double cover.)

(iii) The L-function of Sym^kE has local factor $\prod_m(1-e^{2im\theta_p}p^{-s})^{-1}$ (unitary normalization): it is a gas of $k+1$ modes, one per sublevel. Automorphy for all k (non-CM E): Newton–Thorne [**210**] *[BIB].*

PROOF. (i) Sym^k of the defining representation is irreducible of dimension $k+1$, and the weight of $x^{j+m}y^{j-m}$ under $\operatorname{diag}(e^{i\phi},e^{-i\phi})$ is $e^{2im\phi}$. (ii) Lemma 44 Thm. 2.6(ii) gives U_k, and the geometric sum gives $\sin((k+1)\theta)/\sin\theta$. [NUM]: from the q-expansion $q\prod(1-q^n)^2(1-q^{11n})^2$ of 11a to $q^{2^{17}}$, computed independently of the Hecke recursion, the identity holds at all 12 366 prime powers

$p^k \le 2^{17}$ with maximum error $5.8 \cdot 10^{-15}$. The coefficients a_p agree with direct point counts for every $p \le 2 \cdot 10^4$, and multiplicativity holds (`145/spin.json`). □

PROPOSITION 45.4.2 (Sato–Tate = Haar on $SU(2)$; [CITED] [23], [NUM]). *For non-CM E the θ_p are equidistributed for $\frac{2}{\pi}\sin^2\theta\, d\theta$, the image of Haar measure on conjugacy classes of $SU(2)$ (BLGHT, abstract: Sato–Tate "can now be proved for any elliptic curve over a totally real field"; Lemma 42 Prop. 6.11). By Schur orthogonality, $\langle\chi_{k/2}\rangle \to \delta_{k0}$: every non-trivial multiplet averages to zero. [NUM] (11a, the 12 250 good primes below 2^{17}): $\langle\chi_{k/2}\rangle$ for $k = 1, \dots, 8$ is 0.0041, -0.0034, 0.0046, -0.0008, 0.0015, -0.0101, -0.0013, -0.0031, each within 1.2 standard errors (≈ 0.009) of 0.*

45.4.2. The ghost as non-unitary spin.

PROPOSITION 45.4.3 ([THM], [NUM]). *Fix p. If $|a| \le 2\sqrt{p}$ (elliptic, or ± 1), then $|p^{-k/2}a_{p^k}| \le k+1$ for all k. If $a = \pm 2\sqrt{p}\cosh\eta$ with $\eta > 0$ (a boost: Lemma 42 Thm. 6.8; chemical potential $\pm\eta$ in Lemma 44 C10), then $p^{-k/2}a_{p^k} = (\pm 1)^k \sinh((k+1)\eta)/\sinh\eta$, which grows like $e^{k\eta}$. So*

$$\sup_k |a_{p^k}|/((k+1)p^{k/2}) < \infty \iff \sup_k |a_{p^k}|/((k+1)p^{k/2}) \le 1 \iff |a| \le 2\sqrt{p},$$

i.e. the Ramanujan bound in every multiplet is equivalent to Hasse at p. [NUM]: 11a, all $p^k \le 2^{17}$: maximum ratio 0.9967. Fake $p = 13$, $a = 8$ ($\eta = 0.46360$): ratios $1.11, 1.31, 2.09, 7.50, 162.1, 1.39 \cdot 10^5$ at $k = 1, 2, 4, 8, 16, 32$, matching the closed form to 10^{-15} relative.

PROOF. $|\sum_m e^{2im\theta}| \le k+1$; at $\theta = 0$ the sum is $k+1$. In the hyperbolic case the eigenvalues are $\pm e^{\pm\eta}$, and Sym^k has weights $(\pm 1)^k e^{2m\eta}$. □

In representation language, a ghost replaces the compact $SU(2)$ by a hyperbolic element of $SL_2(\mathbb{R})$. Its finite-dimensional "spin" representations are non-unitary for every invariant positive form, which is Lemma 42 Thm. 7.2 promoted to all multiplets by Sym^k. The supersingular case (rational Gouy phase, Lemma 44 Thm. 2.4) makes the multiplet characters periodic in k; for $a_p = 0$ they run $1, 0, -1, 0, \dots$.

45.5. Zeeman splitting and fine structure ([ANALOGY] except where marked)

(1) *Quadratic fields* ([THM] for the identities). $\zeta_K = \zeta \cdot L(\chi_D)$, and the Euler factor at $p \nmid D$ is the two-mode Schwinger gas with $g_p = \mathrm{diag}(1, \chi_D(p))$. For split p, $g_p = I$: all sublevels are in phase, like a degenerate multiplet at zero field. For inert p, $g_p = \mathrm{diag}(1, -1)$: sublevel phases alternate, and the factor is $(1 - p^{-2s})^{-1}$, a single mode of energy $2\log p$. "Splitting" therefore does not lift a degeneracy as a magnetic field does. The Frobenius class sets the phase pattern of the sublevels.

(2) *Artin/Chebotarev* ([THM]/[BIB], standard). For Galois $L/\mathbb{Q}$, $\zeta_L = \prod_\chi L(\chi, s)^{\chi(1)}$: a decomposition into irreducible "multiplets", each appearing with multiplicity equal to its dimension. The local pattern is the Frobenius conjugacy class, and the classes are equidistributed (Chebotarev). [NUM]: for $x^3 - x - 1$ (splitting field S_3, disc. -23) and the 78 495 primes $5 \le p < 10^6$, $p \ne 23$, the frequencies of 3 roots, 1 root and 0 roots are 0.16602, 0.50080, 0.33318, against $\frac{1}{6}, \frac{1}{2}, \frac{1}{3}$. The root counts were cross-checked by brute force for $p < 3000$ (`spin.json`).

(3) *Fine structure* ([ANALOGY], weak). The tower Sym^k, $k = 1, 2, \dots$, refines the single number a_p into finer and finer multiplet data, and Sato–Tate controls all levels at once. Beyond that loose resemblance to successive splittings, we see no mathematical content in "fine structure" here, and none is claimed.

(4) *ζ itself has no spin.* Its Euler factor is a single mode with trivial "Frobenius" 1, i.e. degree 1, $U(1)$ at most (Dirichlet characters supply phases $\chi(p)$). Spin multiplets need degree ≥ 2. This is Lemma 44 Thm. 5.7's base/fibre gap again: RH for ζ is not a per-prime spin statement.

45.6. The key test: does a ghost violate Pauli?

Setting: $V = H^1$ ($\dim V = 2g$), F the Frobenius with reversed characteristic polynomial $P(T) = \det(1 - TF) = \prod(1 - \alpha_i T)$, real, with the functional equation $\alpha \mapsto q/\alpha$. Put $g = F/\sqrt{q}$. A *fake* is such a P with some $|\alpha_i| \neq \sqrt{q}$, i.e. an off-circle ("ghost") pair α, $q/\bar{\alpha}$.

45.6.1. No combinatorial violation.

PROPOSITION 45.6.1 ([THM]). *For every real P of degree $2g$, Weil or fake, $\Lambda^k V = 0$ for $k > 2g$, $\operatorname{Tr} \Lambda^k F = e_k$ is real, $e_{2g-k} = q^{g-k} e_k$, and the fermionic Fock space is 2^{2g}-dimensional. Exclusion holds identically, so a ghost cannot appear as a Pauli violation in the sense of occupation ≥ 2. Coefficient-size ("occupation-magnitude") bounds are also far from enough: in genus 2, of the 3591 integer pairs (a_1, a_2) with $|a_1| \leq 4\sqrt{q}$, $|a_2| \leq 6q$ (the trivial bounds $|e_k| \leq \binom{4}{k} q^{k/2}$), 3190 (88.8%) are fakes at $q = 11$, and 87 388 of 98 253 (88.9%) at $q = 101$ ([NUM], `l45/krein.json`).*

45.6.2. Real-temperature fermionic positivity (Lemma 44's $(-1)^N$ twist).

THEOREM 45.6.2 ([THM]). *For any $g \in GL(V)$,*

$$\operatorname{Tr}_{\mathrm{Sym}V}\left((-1)^N g\, x^N\right) = \det(1 + xg)^{-1} = \left[\operatorname{Tr}_{\Lambda V}\left(g\, x^N\right)\right]^{-1},$$

$$\operatorname{Tr}_{\Lambda V}\left((-1)^N g\, x^N\right) = \det(1 - xg) = \left[\operatorname{Tr}_{\mathrm{Sym}V}\left(g\, x^N\right)\right]^{-1}.$$

So Lemma 44's $(-1)^N$-twisted bosonic temperature is exactly the ungraded *fermion gas $Z_{\mathrm{F}}^{+}(x) = \det(1+xF)$, and the untwisted one is the inverse of the* graded *fermion gas $Z_{\mathrm{F}}^{-}(x) = \det(1-xF) = P(x)$. Lemma 44 Thm. 5.4(iv) therefore reads: in genus 1, Hasse $\iff$ $Z_{\mathrm{F}}^{\pm}(x) > 0$ for all $x > 0$ $\iff$ $P > 0$ on $\mathbb{R}$.*

PROOF. Diagonalize; $(1 + \lambda x)^{-1} = \sum(-\lambda x)^k$; $Z_{\mathrm{F}}^{+}(x) = P(-x)$. □

THEOREM 45.6.3 ([THM] + [NUM]; real-temperature fermionic positivity sees only real ghosts). *$P > 0$ on $\mathbb{R}$ iff every real reciprocal root of P has even multiplicity. Hence:*

- *(i) a* real *ghost pair (simple real $\alpha \neq \pm\sqrt{q}$; chemical potential $\pm\eta$, or $\pm\eta + i\pi$, in Lemma 44's sense) is always detected: P changes sign at $x = 1/\alpha$;*
- *(ii) a* complex *ghost quadruple $\alpha, \bar{\alpha}, q/\alpha, q/\bar{\alpha}$ (complex chemical potential $\eta + i\theta$ with $\theta \notin \{0, \pi\}$) is invisible if the remaining real roots have even multiplicity. Both fermion gases are then positive at every real temperature.*

[NUM] (genus 2, $P = 1 + a_1 T + a_2 T^2 + q a_1 T^3 + q^2 T^4$, full box of Prop. 45.6.1): at $q = 11$ there are 401 Weil, 2408 real-ghost and 782 complex-quadruple polynomials. $P > 0$ on $\mathbb{R}$ for all 401 Weil, none *of the 2408 real-ghost, and* all *782 complex-quadruple ones. At $q = 101$ the numbers are 10 865, 65 724 and 21 664, with the same pattern. Example: $(q, a_1, a_2) = (11, -13, 65)$ has $|\alpha|/\sqrt{q} = 0.7122$ (twice) and 1.4041 (twice), and $\min_{[-3,3]} P = 0.0468 > 0$ (`l45/fermi.json`).*

PROOF. P is real with $P(0) = 1$, and it changes sign exactly at real roots of odd multiplicity. □

So the steering hypothesis ("the $(-1)^N$ twist is the fermionic grading; check whether the fermionic sector is where the ghost becomes visible") gets a sharp, partly negative answer. The twist *is* the fermion gas, but real-temperature positivity of the fermion gas is a genus-1 accident. From genus 2 on it is blind to every ghost carrying a rotation phase. This matches Lemma 44 Thm. 5.5: no real-temperature positivity is the Weil condition.

45.6.3. The metric test: ghosts are Krein (Gupta–Bleuler) states. Let $\mathcal{H}(g) = \{h \text{ Hermitian on } V : h(gx, gy) = h(x, y)\}$, and give $\Lambda^k V$, $\mathrm{Sym}^k V$ the induced forms (Gram determinants and permanents).

THEOREM 45.6.4 ([THM]). *Let g be diagonalizable.*

(a) For $x \in E_\lambda$, $y \in E_\mu$ and $h \in \mathcal{H}(g)$: $(\bar\lambda\mu - 1)\, h(x, y) = 0$.

(b) Ghost pair state. *Let $|\lambda| \neq 1$. Then E_λ is h-neutral for every $h \in \mathcal{H}(g)$. If h is non-degenerate, every $e \in E_\lambda \setminus 0$ has a partner $f \in E_{1/\bar\lambda}$ with $h(e, f) \neq 0$, and*

$$\langle e \wedge f, e \wedge f\rangle_h = -|h(e, f)|^2 < 0, \qquad \Lambda^2 g\,(e \wedge f) = \tfrac{\lambda}{\bar\lambda}\, e \wedge f, \quad \left|\tfrac{\lambda}{\bar\lambda}\right| = 1.$$

The doubly occupied ghost pair is a negative-norm state with an "on-shell" (unimodular) eigenvalue. *The singly occupied states e, f are null. In Sym^2, $e \cdot f$ has norm $+|h(e, f)|^2$, and e^2, f^2 are null.*

(c) Equivalences *(for $\dim V \geq 2$). The following are equivalent: (i) all $|\alpha_i| = \sqrt{q}$; (ii) some $h \in \mathcal{H}(g)$ is positive definite; (iii) some $h \in \mathcal{H}(g)$ induces a positive definite form on $\Lambda^2 V$; (iv) some $h \in \mathcal{H}(g)$ induces a positive definite form on all of ΛV; (v) as (iii) with $\mathrm{Sym}^2 V$.*

(d) Counting. *If h is non-degenerate of signature (n_+, n_-), then $n_-(\Lambda^2 V) = n_-(\mathrm{Sym}^2 V) = n_+ n_-$. Every negative one-particle direction pairs with every positive one: ghost contamination spreads through the two-particle sector of* both *statistics.*

(e) The Krein form is always there. *For real symplectic g (Frobenius preserves the Weil pairing ω up to q: [**193**, Thm. 1.30]), $h_K(x, y) = i\,\omega(\bar x, y) \in \mathcal{H}(g)$ has signature (g, g) for Weil curves and fakes alike. The fermionic Fock space ΛV then has total signature $2^g \sum_j (-1)^j \binom{g}{j} = 0$: it is neutral, a Witten-index-type cancellation.*

PROOF. (a) $h(gx, gy) = \bar\lambda\mu\, h(x, y)$. (b) Neutrality is (a) with $\mu = \lambda$. By (a), $h(e, \cdot)$ vanishes on every E_μ with $\bar\lambda\mu \neq 1$; since $V = \bigoplus E_\mu$ and h is non-degenerate, it cannot vanish on $E_{1/\bar\lambda}$. The norm is the Gram determinant $h(e, e)h(f, f) - |h(e, f)|^2$, and the eigenvalue is $\lambda \cdot \bar\lambda^{-1}$. For Sym^2 use the permanent. (c) (ii)⇒(iv): Gram determinants of a positive form are positive. (iv)⇒(iii) is trivial. (iii)⇒(ii): if h is indefinite non-degenerate, choose h-orthogonal x, y with $h(x, x) > 0 > h(y, y)$; then $\langle x \wedge y\rangle = h(x, x)h(y, y) < 0$. If h is degenerate, take x in the radical and y independent; then $\langle x \wedge y\rangle = 0$. So h is definite, and $\pm h$ is positive and invariant. The same argument works for Sym^2, using $\langle xy\rangle = h(x, x)h(y, y)$ for orthogonal x, y and $\langle x^2\rangle = 2h(x, x)^2 = 0$ on the radical. (ii) $\iff$ (i): g is unitary for h iff its eigenvalues are unimodular, given diagonalizability (average over the eigenbasis). (d) In an h-orthogonal basis the products $x_i \wedge x_j$ (resp. $x_i x_j$) are orthogonal, with sign the product of the signs. (e) $g^{\mathsf T} J g = J$ and g real give $\bar g^{\mathsf T}(iJ)g = iJ$, and iJ has eigenvalues ± 1, each g times. Then $\Lambda(V_+ \oplus V_-) = \Lambda V_+ \otimes \Lambda V_-$, and $\Lambda^j V_-$ has sign $(-1)^j$. □

COROLLARY 45.6.5 (genus 1: the polarization is the pair state; [THM]). *For $g = 1$, $\Lambda^2 H^1$ is the line spanned by the Weil-pairing class θ (the polarization), with $\Lambda^2 F\,\theta = q\,\theta$ for Weil curves and fakes alike. For semisimple F: $|a| < 2\sqrt{q}$ or F scalar $\iff$ some $h \in \mathcal{H}(g)$ has $\langle\theta, \theta\rangle_h > 0$. For a fake, $\langle\theta, \theta\rangle_h = -|h(e, f)|^2 \leq 0$ for all $h \in \mathcal{H}(g)$. The polarization, the "Cooper pair" $\theta = e \wedge f$ of the two one-fermion modes, has negative norm for every invariant metric exactly when RH fails for the curve.*

PROOF. Λ^2 of a 2-dimensional space is det, and $\det F = q$. For elliptic g, $\mathcal{H}(g)$ is the set of real diagonal forms in the eigenbasis (Lemma 42 Thm. 7.2), so $\langle\theta\rangle = h_{11}h_{22}$ can be positive. For a fake, (b) forces the off-diagonal form, which gives $-|c|^2$. □

[NUM] (`num/lemma45_krein.py`, `l45/krein.json`; random invariant forms, 400 samples per case, real symplectic block models with $\eta = 0.4$). Genus 1, 2 Weil: $\dim \mathcal{H}(g) = 2g$; a positive invariant form exists (residual $\leq 2 \cdot 10^{-16}$); random invariant forms of all signatures occur. Fakes

(genus-1 real pair; genus-2 one real pair; genus-2 complex quadruple; genus-3 quadruple + Weil block): no positive invariant form exists (the eigenbasis form has invariance residual 1.34–1.90), all 400 non-degenerate samples are indefinite, all have negative vectors in Λ^2 and Sym^2, and $n_-(\Lambda^2) = n_-(\mathrm{Sym}^2) = n_+n_-$ holds in every sample (asserted in the code). The ghost pair state has $h(e,e), h(f,f) \le 5\cdot 10^{-16}$, $\langle e \wedge f\rangle = -|h(e,f)|^2$ exactly (e.g. -2.0493, -4.9932, -0.5908, -0.8710), and $|\lambda/\bar\lambda| = 1$. Krein form: Λ^2 signatures $(0,1)$, $(2,4)$, $(6,9)$ and total ΛV signature $(2,2)$, $(8,8)$, $(32,32)$ for $g = 1,2,3$.

45.6.4. The ζ version ([RESTATEMENT]) and Gupta–Bleuler ([ANALOGY]). ζ. Lemma 40 Thm. 5.2: the Pick kernel of ξ'/ξ is $\Gamma M \Gamma^*$, where the zero-space Krein Gram M is rank-one positive on each on-line zero and $\left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$ on each mirror pair $\rho, 1-\bar\rho$. Take the span of finitely many zeros as the one-fermion space with metric M. By Theorem 45.6.4(b),(d), each mirror pair contributes a negative two-fermion state $e_\rho \wedge e_{1-\bar\rho}$, and with K simple on-line zeros and N mirror pairs, $n_-(\Lambda^2) = N(K+N)$ ([NUM]: $(K,N) = (5,0), (5,1), (20,1), (20,2)$ give $0, 6, 21, 44$). So RH $\iff$ the fermionic Fock space over every finite set of zeros, with the Pick metric, is positive. This is Lagarias's criterion in Fock language (Lemma 40 Thm. 5.2), a [RESTATEMENT] with no new content.

Gupta–Bleuler ([ANALOGY]). In the covariant quantization of the photon [**121**] [BIB], the scalar and longitudinal modes span an indefinite (hyperbolic) plane with null combinations. The subsidiary condition keeps a positive-semidefinite subspace, and the physical space is its quotient by null vectors (indefinite-metric spaces: [**36**] [BIB]). A Frobenius ghost pair has exactly this structure (Thm. 45.6.4(b)): e, f null, $\langle e, f\rangle \neq 0$. A Gupta–Bleuler-type condition (keep E_λ, discard $E_{1/\bar\lambda}$, then quotient) would remove the ghost *zeros from the spectrum* rather than move them onto the line. Such a mechanism can hide ghosts but cannot prove their absence. What Weil supplies for curves is a positive invariant metric *from geometry* (ampleness, Rosati, Thm. 45.2.4(ii)), so that no subsidiary condition is needed.

45.6.5. Verdict on item (5). *Rigorous statement:* an off-line Frobenius pair is a negative-norm two-fermion state with a unimodular eigenvalue, for every non-degenerate invariant metric (Thm. 45.6.4(b)). Weil holds iff some invariant metric makes the two-fermion sector positive (Thm. 45.6.4(c)). In genus 1 the offending state is the polarization itself (Cor. 45.6.5). *Honest negatives:* (1) There is no combinatorial Pauli violation (Prop. 45.6.1). (2) Real-temperature fermion positivity misses all complex-quadruple ghosts from genus 2 on (Thm. 45.6.3). (3) The metric violation is not specific to fermions: Sym^2 shows the same count (Thm. 45.6.4(d)). (4) The metric statement is the Fock-space form of "unitary $\iff$ invariant positive form" (Lemma 40 §4, Lemma 42 Thm. 7.2). It locates the ghost precisely, in the fermion-bilinear sector of Thm. 45.2.4(ii), but it supplies no positivity. The missing rung is unchanged.

45.7. Corrections to the chat answer

(0a) "Pauli applies to fermions": **correct**. J.W.'s "according to Pauli bosons can share a state" has the right content and the wrong attribution (Bose statistics; spin–statistics [**221**]).

(0b) "BEC is low-T/IR": **correct**. Sharpened: the primon gas has no condensation transition at all, since its ground state $|1\rangle$ is non-degenerate.

(0c) "UV end = Hagedorn pole $\beta = 1$ = light-cone/null part (Lemma 14)": **correct, with two refinements**. At $\beta = 1$ statistics do not matter ($Z_{\mathrm{B}} = Z_{\mathrm{F}}\zeta(2\beta)$; per-mode occupation $1/(p-1)$; Prop. 45.1.1), so the UV end is *not* "bosons in the same state". "Null part" is exact for curves as the boson$\otimes$boson Künneth plane $\langle H, V\rangle$ (Thm. 45.2.4(i)), and for ζ as Lemma 15's pole form, whose two null functionals are the poles $s = 1, 0$ ([RESTATEMENT]).

(0d) "Condensate-like phase = Bost–Connes low-T Galois-broken phase": **analogy only**. It is an ordered phase of Gibbs (type I_∞) KMS states, with nothing macroscopically occupied.

(1) Supertrace for curves: **correct**, with one precision. The inverse denominators are the bosonic Fock traces, and the numerator is the *graded* fermionic trace. More importantly, there are *two* second quantizations (Thm. 45.2.2). "ζ is a Bose gas" refers to points; "zeros are fermions" refers to cohomology. They are equal by Lefschetz/Macdonald (brute force on 11a, Table 45.1). For ζ the cohomological side is Deninger's formula (3), a regularized superdeterminant that he says "might hold". Its ξ-product consequence is proved [**91, 89**].
(2) Spector: **correct** (abstract-level). Littlewood 1912 gives RH $\Rightarrow$ convergence for $\sigma > \frac{1}{2}$. The equivalence is Titchmarsh Thm. 14.25(B),(C). Precision: the *exact* supersymmetry (Witten index 1) is Möbius inversion and needs the bosons. RH is the square-root-size imbalance of the fermion gas *alone*, which is a different statement.
(3) Spin characters: **correct** (Lemma 44 Thms. 2.6, 5.3). Refinements: g_p rotates by angle $2\theta_p$ (double cover). The ghost diagnostic is the Ramanujan bound in all multiplets, which is equivalent to Hasse at p (Prop. 45.4.3). Every finite-dimensional representation of $SL_2(\mathbb{R})$ is non-unitary, so "non-unitary spin" is a statement about the Frobenius element (hyperbolic rather than elliptic), not about the representation.
(4) Zeeman: **analogy**, and partly inverted. Splitting does not lift a degeneracy. The Frobenius class sets the phase pattern of the sublevels: in phase for split primes, alternating for inert ones (§45.5).
(5) Pauli test: **answered** (§45.6). There is no combinatorial violation. The real-temperature fermion test detects only real ghosts in genus ≥ 2. The metric (Krein) violation is rigorous and complete: the ghost pair is a negative-norm two-fermion state for every invariant metric. That statement is a reformulation, not a lever.

45.8. Open questions

Question 45.8.1. Do you accept the reformulation: the "infinite lattice" is the *state space* $\mathrm{Div}^+(\operatorname{Spec}\mathbb{Z})$ of one copy of $\operatorname{Spec}\mathbb{Z}$, while $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ is the space of *one-particle operators*, split into a boson$\otimes$boson light cone (poles, Hagedorn) and a fermion-bilinear sector (zeros, where positivity must live)?

Question 45.8.2. Should "two second quantizations, bridged by Lefschetz" (Thm. 45.2.2) become the official TAA reading of "ζ is a Bose gas and its zeros are fermions"? For ζ the bridge is the explicit formula read as a trace formula, i.e. Deninger's program.

Question 45.8.3. The ghost is a Krein/Gupta–Bleuler state in the two-fermion sector, and in genus 1 it is the polarization θ itself (Cor. 45.6.5). Should the next step look for a "two-fermion Weil form" for ζ, a candidate positive metric on Λ^2 of the zero space built from the prime side, rather than a one-particle one?

Question 45.8.4. Is the Schwinger/Zeeman reading (sublevel $m = \frac{1}{2}(n_\alpha - n_\beta)$, rotation $2\theta_p$, Sato–Tate = Haar) the form of your spin intuition you want to keep, given that ζ itself (degree 1) carries no spin?

Question 45.8.5. Which computation next: (a) the genus-2 Sp_4 Krein signature table for actual curves versus the 782 invisible complex-quadruple fakes (links to Lemma 42 S4/S7), or (b) the Spector gas for $L(E,s)$, i.e. μ_E with $1/L(E,s) = \sum \mu_E(n) n^{-s}$ and $M_E(x)/\sqrt{x}$ on 11a?

CHAPTER 46

Lemma 46: What forces Frobenius to preserve a positive form

Weil, Deligne and Katz–Sarnak over the whole tower of curves; Hilbert versus Krein; the Verschiebung

Standalone note `lemma46.tex`, October 22, 2026.

Summary. Statement of intellectual honesty. RH is not proved here, and no mechanism below is transplanted to ζ. Tags: [THM] proved here (elementary, proof given); [CITED] checked at the source (`lit/lemma46/verified_refs.md`); [BIB] bibliographic record verified, content standard; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma46_*.py`, data `num/l46/`); [CONJ] conjecture; [OPEN] open. Lemmas 23, 30, 40, 42, 45 are cited by number and not redone.

Question (J.W.). Curves over finite fields carry a positive inner product that every Frobenius must preserve (the Hodge index). Taking the entire collection of curves over finite fields out to infinity, what *precisely* demands that Frobenius preserve the positive form? Apply this to the Hilbert/Krein discussion. *Steering:* does $F \circ V = [q]$ make Frobenius a Rosati isometry? What are the global ($F_n V_n = n$) and p-adic (Dieudonné) versions?

Answer in brief. (1) *Algebra gives Krein, geometry gives Hilbert.* Everything in $FV = q$, $V = F^\dagger$, "$F/\sqrt{q}$ is a $\dagger$-isometry" is algebraic. It holds verbatim for an abstract pair (F, V) attached to any fake Weil polynomial. The form it preserves is then indefinite, of signature $(D_{\mathrm{on}} + k,\, k)$, where k is the number of roots outside the circle $|\alpha| = \sqrt{q}$. That is Krein unitarity, with boosts allowed (Thm. 46.6.1). What upgrades Krein to Hilbert is a single inequality, $\mathrm{Tr}(\alpha\alpha^\dagger) > 0$. Weil/Milne obtain it from an *ample divisor* and the fact that a *non-empty intersection count is positive* (Hodge index; Castelnuovo–Severi). Weil's 1948 route obtains it from *the degree of a principal divisor being* 0. Nothing statistical, and nothing about "all curves at once", enters (§46.1). (2) *Exact index count.* For a reciprocal polynomial with k distinct roots outside the circle, the abstract Néron–Severi Gram of $\{C \times \mathrm{pt}, \mathrm{pt} \times C, \Gamma_F, \Gamma_{F^2}, \dots\}$ has $n_+ = 1 + k$ (Thm. 46.1.4). A real ghost pair gives $n_+ = 2$ and a complex quadruple gives $n_+ = 3$. So "ghost $\iff$ index 2" is corrected to "index $= 1 + k$". All 3591 test polynomials at $q = 11$ obey it, including the 782 quadruple ghosts that were invisible to Lemma 45's fermionic test. (3) *Deligne's mechanism is different.* Deligne uses rationality (traces are real, so $\mathrm{Tr}(F|E)^{2m} \geq 0$), a symplectic pairing, big monodromy in a family, Landau's theorem, and the tensor-power trick. Hodge index is not used. In Zeeman language, the positivity is a real-square positivity on the spin-m multiplets. A boosted fibre violates Landau's bound exactly at $m^* = \lceil \ln p/(2\eta) \rceil$ ([NUM]). (4) *Katz–Sarnak* equidistribution (KS distance $0.172 \to 0.00267$ for Legendre, $p = 101 \to 10^6 + 3$) *presupposes* purity. It does not force it. (5) *Real forms.* $\mathrm{Sym}^k g$ preserves a positive form iff g is elliptic. The unique $SL_2(\mathbb{R})$-invariant Hermitian form on Sym^k has signature $(\lceil \frac{k+1}{2} \rceil, \lfloor \frac{k+1}{2} \rfloor)$. The "Zeeman fan" $\frac{1}{2} + 2m\eta/(k \ln p)$ is archimedean, and p-adic slopes do not see it. (6) *Verschiebung.* On the spectrum, V realizes the functional equation $\alpha \mapsto q/\alpha$. The Hermitian Rosati involution is $\sharp = c_\infty \circ \dagger$, so RH $\iff$ "$V = c_\infty$ on the spectrum" $\iff$ $\sharp$ fixes the spectrum ([RESTATEMENT]). Globally, $F_n V_n = n$ for Witt vectors,

$\sigma_n\tilde{\rho}_n = n$ in the integral BC algebra (Connes–Consani, verified), and F_m applied to the Witt vector $Z(C/\mathbb{F}_q)$ *is* base change to $\mathbb{F}_{q^m}$. That makes "the whole tower" a single F-orbit. On $\ell^2(\mathbb{N})$ the normalized maps are isometries, but of weight 0 on the bosonic H^0-type space. Borger proved that no curve of genus ≥ 1 descends to $\mathbb{F}_1$. The resulting H^0/H^1 degree mismatch is formulated as an open problem. Dieudonné's $FV = VF = p$ is integral: fakes have it too, and a "supersingular" fake ($p = a = 13$) is p-adically pure but archimedeanly boosted.

46.1. Weil's mechanism: where the positive form comes from

Throughout, $C/\mathbb{F}_q$ is a smooth projective curve of genus g with Jacobian J and q-Frobenius $\pi = F$. Write $N_n = \#C(\mathbb{F}_{q^n})$ and $s_n = q^n + 1 - N_n = \sum_i \alpha_i^n$ for the reciprocal roots of $P(T) = \det(T - F \mid H^1)$, which satisfies $P(T) = q^{-g}T^{2g}P(q/T)$ (FE).

46.1.1. The cited chain.

THEOREM 46.1.1 (Milne's account of Weil, [CITED] MilpRH Thm. 1.27, Cor. 1.29, Thm. 1.30, Summary 1.31). *Let † be the Rosati involution of* $\mathrm{End}^0(J)$ *for the canonical polarization.*

(a) (Positivity.) For $\alpha \neq 0$, $\mathrm{Tr}(\alpha\alpha^\dagger) = 2g\,(D^{g-1} \cdot \alpha^{-1}D)/(D^g) > 0$, *where* D *is ample. The proof uses only that* D *is ample and that an intersection number of effective cycles meeting properly in a non-empty set is positive.*

(b) (Algebra ⇒ circle, given (a).) If $\alpha\alpha^\dagger = q$, *then* $|\rho(\alpha)| = q^{1/2}$ *for every embedding* $\rho : \mathbb{Q}[\alpha] \to \mathbb{C}$.

(c) (The Frobenius relation.) $\pi^\dagger\pi = q$. *This comes from the Weil pairing, i.e.* $\deg\pi = q^g$ *and* $e(\pi x, \pi y) = e(x, y)^q$.

Weil's original positivity (1948, as reported in MilpRH §1) is $\sigma(\xi \circ \xi') > 0$ for correspondences. Its proof forms $\Phi = \det(\varphi_i(D_j(P)))$ from a basis φ_i of $L(K_C)$ and uses $\deg(\Phi)_0 = \deg(\Phi)_\infty$, i.e. *the degree of a principal divisor is zero.* Castelnuovo (1906), Mattuck–Tate [**187**] and Grothendieck [**118**] give the surface form: Hodge index on $C \times C$. Stepanov [**251**], Schmidt [**238**] and Bombieri [**37**] give a proof that "requires only the Riemann–Roch theorem for curves" (MilpRH, Epilogue) [CITED].

REMARK 46.1.2 (Answer to "what precisely forces it"). The positivity is *local to one curve.* It is forced by (i) an ample class, and (ii) positivity of a non-empty intersection count, or equivalently Weil's zero/pole count of Φ. Item (c) is pure algebra, and (b) is Albert-type linear algebra (Lemma 30 Thm. 2.2). Considering "all curves at once" adds nothing to Weil's argument. The tower over one curve does appear in Stepanov–Bombieri and in Prop. 46.1.6 below, and families appear only in Deligne (§46.3).

46.1.2. Definite forms force the circle; indefinite forms bound the escape.

THEOREM 46.1.3 (Isotropy bound, [THM]). *Let* W *be a finite-dimensional complex space with a non-degenerate Hermitian form* b *of signature* (n_+, n_-), *and let* $\varphi \in GL(W)$ *satisfy* $b(\varphi x, \varphi y) = q\,b(x, y)$ *for some* $q > 0$. *Let* W_λ *denote the generalized eigenspaces.*

(a) If $\lambda\bar{\mu} \neq q$, *then* $b(W_\lambda, W_\mu) = 0$.

(b) $W_> := \bigoplus_{|\lambda|>\sqrt{q}} W_\lambda$ *is totally isotropic, so* $\dim W_> \leq \min(n_+, n_-)$. *The same holds for* $W_<$.

(c) If b *is definite, every eigenvalue has* $|\lambda| = \sqrt{q}$ *and* $\varphi/\sqrt{q}$ *is unitary, hence semisimple.*

The same statements hold for a real symmetric b *on a real* W, *with* $\lambda\mu \neq q$ *in (a).*

PROOF. (a) Let $N = \varphi - \lambda$ on W_λ and $M = \varphi - \mu$ on W_μ, both nilpotent. Expanding $b(\varphi x, \varphi y) = q\,b(x, y)$ gives $(\lambda\bar{\mu} - q)\,b(x, y) = -\lambda b(x, My) - \bar{\mu}b(Nx, y) - b(Nx, My)$. Induct on $\deg_N x + \deg_M y$, where $\deg_N x$ is the least r with $N^r x = 0$. Each term on the right has smaller

total degree, so it vanishes by induction, and $\lambda\bar{\mu} \neq q$ forces $b(x,y)=0$. (b) If $|\lambda|,|\mu| > \sqrt{q}$, then $|\lambda\bar{\mu}| > q$. A totally isotropic subspace of a non-degenerate form of signature (n_+,n_-) has dimension at most $\min(n_+,n_-)$. (c) A definite form has no non-zero isotropic vector, so $W_> = W_< = 0$. The operator $\varphi/\sqrt{q}$ preserves a definite form, so it is unitary and therefore diagonalizable. □

46.1.3. The abstract Néron–Severi Gram and the exact index. Put $H = C \times \mathrm{pt}$, $V_0 = \mathrm{pt} \times C$ and $\Gamma_m = \Gamma_{F^m}$. Then $H \cdot V_0 = 1$, $\Gamma_m \cdot H = q^m$, $\Gamma_m \cdot V_0 = 1$, and

$$\Gamma_m \cdot \Gamma_n = q^{\min(m,n)} N_{|m-n|}, \qquad N_0 := 2-2g,$$

so $\Gamma_F^2 = q(2-2g)$ (Lemma 23; Lemma 45 Thm. 2.4). In terms of $\gamma_m := \Gamma_m - H - q^m V_0$ (the $H^1 \otimes H^1$ part) one has $\gamma_m \cdot H = \gamma_m \cdot V_0 = 0$ and $\gamma_m \cdot \gamma_n = -q^n s_{m-n}$ for $m \geq n$. These formulas make sense for *any* reciprocal integer polynomial. They define an abstract integral Gram matrix $G_n(P)$ on $\mathbb{Z}^{n+2}$, a *lattice* realization of a fake Weil polynomial.

THEOREM 46.1.4 (Index theorem, [THM]). *Let P be monic, reciprocal ($P(T) = q^{-g}T^{2g}P(q/T)$), with real coefficients. Let it have D distinct roots, of which D_{on} lie on $|\alpha| = \sqrt{q}$, and let k be the number of distinct roots with $|\alpha| > \sqrt{q}$, so that $D = D_{\mathrm{on}} + 2k$. For $n \geq D$ the Gram matrix $G_n(P)$ of $\{H, V_0, \Gamma_1, \dots, \Gamma_n\}$ has inertia*

$$(n_+, n_-, n_0) = \big(1+k,\ 1+D_{\mathrm{on}}+k,\ n-D\big).$$

In particular $n_+ = 1$ (Hodge index) $\iff$ $k = 0$ $\iff$ P is a Weil polynomial. A real ghost pair adds 1 to n_+, and a complex quadruple adds 2.

PROOF. $\langle H, V_0\rangle$ is a hyperbolic plane orthogonal to all γ_m, contributing $(1,1)$. For the rest, $\gamma_m \cdot \gamma_n = -\sum_i m_i\, \alpha_i^m (q/\alpha_i)^n$, the sum running over distinct roots with multiplicities m_i. So $\gamma_m \mapsto T^m$ identifies the γ-block with $-\langle\cdot,\cdot\rangle$, where $\langle f, h\rangle = \sum_i m_i f(\alpha_i) h(q/\alpha_i)$ on $A = \mathbb{R}[T]/(P_{\mathrm{red}})$. Since T is invertible in A, the monomials $T, \dots, T^n$ span A for $n \geq D$, which gives the radical dimension $n - D$. Diagonalize $\langle\cdot,\cdot\rangle$ over $\mathbb{R}$ by grouping roots under $\iota : \alpha \mapsto q/\alpha$ and complex conjugation:

- an on-circle root has $q/\alpha = \bar{\alpha}$ and contributes $m_i |f(\alpha_i)|^2$, which is positive (the real span of a conjugate pair gives a positive plane, and $\pm\sqrt{q}$ gives a positive line);
- a real off-circle pair $\{\alpha, q/\alpha\}$ gives $m(f(\alpha)h(q/\alpha) + f(q/\alpha)h(\alpha))$, a hyperbolic plane $(1,1)$;
- a complex quadruple $\{\alpha, \bar{\alpha}, q/\alpha, q/\bar{\alpha}\}$ gives two hyperbolic planes $(2,2)$.

So $\langle\cdot,\cdot\rangle$ has signature $(D_{\mathrm{on}}+k, k)$, the γ-block has $(k, D_{\mathrm{on}}+k)$, and adding $(1,1)$ gives the claim. □

REMARK 46.1.5. Thm. 46.1.4 sharpens Lemma 23 and Lemma 45 Thm. 2.4. The positive direction of $\mathrm{NS} \otimes \mathbb{R}$ lies in $\langle H, V_0\rangle$, and Hodge index is exactly the statement that the $H^1 \otimes H^1$ block is negative definite. Thm. 46.1.3 applied to the γ-block with $\varphi =$ pull-back by $F \times 1$ recovers Rosati: the block is negative definite $\iff$ $k = 0$. Lemma 40 Ex. 3.3 had the 2×2 elliptic case $[[2,a],[a,2p]]$.

46.1.4. The tower out to infinity: one-sided bounds.

PROPOSITION 46.1.6 (Dirichlet/Landau tower lemma, [THM]). *Let P be reciprocal with integer s_n.*

(a) If $N_n \geq q^n + 1 - Cq^{n/2}$ for all n (a lower bound only), then all $|\alpha_i| = \sqrt{q}$.

(b) If $N_n \leq q^n + 1 + Cq^{n/2}$ for all n (an upper bound only), then either all $|\alpha_i| = \sqrt{q}$, or the largest modulus $M > \sqrt{q}$ is attained by the positive real *root $\alpha = M$. Only the pure boost $\theta = 0$ survives, as in Lemma 42 Thm. 6.8.*

PROOF. (a) The series $\sum_n (Cq^{n/2} - s_n)t^n = \frac{C}{1-\sqrt{q}t} - \sum_i \frac{\alpha_i t}{1-\alpha_i t}$ has non-negative coefficients. By Pringsheim–Landau [170], its radius of convergence R is a singular point. Suppose $M = \max|\alpha_i| > \sqrt{q}$, so $R = 1/M$. As $t \uparrow 1/M$ along the reals, the terms with $|\alpha_i| = M$ and $\alpha_i \neq M$ stay bounded.

If M is itself a root of multiplicity m, its term $-mMt/(1-Mt) \to -\infty$, which contradicts the non-negativity of the sum. If M is not a root, $t = 1/M$ is a regular point, which contradicts Pringsheim. Hence $\max|\alpha_i| \le \sqrt{q}$, and FE gives $\min|\alpha_i| \ge \sqrt{q}$. (b) The same argument with $\sum(Cq^{n/2} + s_n)t^n$: now the term at $t = 1/M$ has a + sign, so no contradiction arises, but Pringsheim forces M to be a root. □

So the "collection out to infinity" $\{C/\mathbb{F}_{q^n}\}_n$ forces RH from a one-sided *lower* bound. This is the logical shape of the Stepanov–Bombieri proof as usually presented [BIB] [37]: Riemann–Roch on C gives an upper bound for all n, and a Galois-cover argument supplies the lower bound. MilpRH's epilogue confirms only that the method "requires only the Riemann–Roch theorem for curves".

46.2. Numerics for the Néron–Severi Gram

All signatures below are *exact*. The Gram matrices are integral, so we count sign changes (Descartes) in the integer characteristic polynomial, whose roots are all real (`num/lemma46_ns.py`, `num/l46/ns.json`).

data	n	inertia (n_+, n_-, n_0)	prediction
11a, $p \in \{2,3,5,7,13,17,19,23,29,31,37\}$	4	$(1,3,2)$ all	$k=0$, $D_{\text{on}}=2$
174 genus-1 fakes, $p = 5,13,37$, $2\sqrt{p} < \lvert a\rvert \le 2p$	4	$n_+ = 2$ all	$k=1$
parabolic $q=4$, $a=\pm4$ (double root)	4	$(1,2,3)$	$D=1$
genus-2 curves x^5-x+1, x^5+3x^3+x+1, $x^5-x^4+x^2+2$, 21 good $p \le 23$	6	$(1,5,2)$ all	$k=0$
$q=11$ box, 401 Weil	6	$n_+ = 1$	$k=0$
$q=11$ box, 2108 / 300 real ghosts	6	$n_+ = 2$ / 3	$k=1$ / 2
$q=11$ box, 782 complex quadruples	6	$n_+ = 3$	$k=2$
$[1,-3,13,-33,121]$	12	$(1,5,8)$	$D=4$
fake $[1,11,-13,121,121]$ ($N_2=-25$)	12	$(2,4,8)$	$k=1$, $D_{\text{on}}=2$

The box is $a_1 \in [-13,13]$, $a_2 \in [-66,66]$, 3591 polynomials. Every case matches $n_+ = 1+k$ and $n_0 = n-D$ [NUM]. The 782 quadruple ghosts, which are invisible to the real-temperature fermionic positivity of Lemma 45 Thm. 6.3, all have $n_+ = 3$. The Néron–Severi form sees every ghost.

46.3. Deligne's mechanism, and its Zeeman translation

THEOREM 46.3.1 (Deligne's Main Lemma, [CITED] MilpRH §5, Main Lemma 5.2 and Prop. 5.3; [83]). *Let U be a smooth affine curve over $\mathbb{F}_q$ and E a lisse $\mathbb{Q}_\ell$-sheaf on U. Assume:*

(i) (rationality) $\det(1 - F_u t \mid E) \in \mathbb{Q}[t]$ *for every closed point u;*
(ii) a non-degenerate π_1-invariant skew form $E \otimes E \to \mathbb{Q}_\ell(-d)$;
(iii) the geometric monodromy is open in $Sp(E)$ (big monodromy; MilpRH attributes the input to Kazhdan–Margulis).

Then every eigenvalue α_u of F_u has $|\alpha_u| = q^{\deg(u)d/2}$. The proof bounds the eigenvalues on H^1_c by $q^{d/2+1}$ and removes the slack by the tensor-power (Künneth) trick on $V^{\otimes m}$, which yields $q^{d/2 \pm 1/(2m)}$.

Inside the proof (MilpRH §5, (32)–(34)). Step 1: rationality makes traces real, so $\mathrm{Tr}(F_u|E^{\otimes 2m}) = \mathrm{Tr}(F_u|E)^{2m} \ge 0$, and $\log L(U, E^{\otimes 2m}, t)$ has non-negative coefficients. Step 2: by invariant theory of Sp, the coinvariants are $(E^{\otimes 2m})_{\pi_1} \cong \mathbb{Q}_\ell(-md)^M$, so the only pole on H^2_c sits at $|t| = q^{-(md+1)}$. Step 3: Landau's theorem then gives $|\alpha_u|^{2m} \le q^{\deg(u)(md+1)}$, and letting $m \to \infty$ gives $|\alpha_u| \le q^{\deg(u)d/2}$.

REMARK 46.3.2 (What kind of positivity, [THM] for the translation). (a) The positivity is a *real-square* positivity, $x^{2m} \ge 0$ for $x = \mathrm{Tr}\, F_u \in \mathbb{R}$. It is not a Hermitian norm. Reality of the trace is where c_∞ enters. Hodge index is not used; Hard Lefschetz enters only in Weil II [84]. (b) *Zeeman*

form. For rank 2 ($d = 1$), $E^{\otimes 2m}$ decomposes under SL_2 into multiplets Sym^{2j}, and $\mathrm{Tr}(F|\mathrm{Sym}^k) = p^{k/2}\sum_{|m'|\le k/2} e^{2im'\theta}$ (Lemma 45 Thm. 4.1). So Step 1 states that the *sum over all Zeeman sublevels of all multiplets* is non-negative. Steps 2–3 state that only the spin-0 (Sp-invariant) part reaches the pole. A boosted fibre $\alpha = \sqrt{p}\,e^{\eta}$ has $\mathrm{Tr}(F|\mathrm{Sym}^k) = p^{k/2}\sinh((k+1)\eta)/\sinh\eta$, which outgrows the pole bound. (c) For ζ the tensor-power step needs $(\mathrm{Spec}\,\mathbb{Z})^m$, and the monodromy step needs a family containing ζ as a fibre. Neither is available (§46.7).

46.3.1. Numerics: Landau slack for planted fakes. For a single fibre with $a = 2\sqrt{p}\cosh\eta$, the local Landau inequality $|\alpha|^{2m} \le p^{m+1}$ fails first at $m^* = \lceil \ln p/(2\eta)\rceil$. This value is computed exactly and matches the prediction in all six cases [NUM] (`num/lemma46_family.py`, `num/l46/family.json`):

p	a	η	$\lvert\alpha\rvert/\sqrt{p}$	m^*
1009	64	0.1216	1.129	29
1009	69	0.4121	1.510	9
1009	95	0.9583	2.607	4
10007	201	0.0964	1.101	48
10007	220	0.4427	1.557	11
10007	300	0.9620	2.617	5

Plant the fake fibre into the Legendre family and form $R_m = \sum_\lambda (a_\lambda/\sqrt{p})^{2m}/(C_m p)$, with C_m the Catalan numbers. At $m = 32$, $p = 10007$: the real family gives 0.807, $a = 201$ gives 0.852, and $a = 300$ gives 6.0×10^9. At $p = 1009$, $m = 32$: real 1.41, $a = 64$: 1.94, $a = 95$: 5.0×10^{10}. *Honest point:* a weak fake is hidden in the H^1_c fluctuations of the raw family sum for $m < m^*$. What excludes it in Deligne's argument is the *local* Landau step as $m \to \infty$, which uses the pole location supplied by monodromy, not the raw sum.

46.4. Katz–Sarnak: the collection out to infinity, statistically

Remark 46.4.1 ([CITED] KS99b). Katz–Sarnak place ζ in their Family I (quadratic Dirichlet $L(s, \chi_d)$) with symplectic symmetry: "the operator should preserve a symplectic form". They recall that Deligne's first proof used the monodromy of a family, and that the hyperelliptic family has full $Sp(2g)$ monodromy. Equidistribution as $q \to \infty$ uses Deligne's purity as an input. *So statistics over all curves presuppose unitarity and do not force it.*

Numerics [NUM], Legendre family $y^2 = x(x-1)(x-\lambda)$, all $\lambda \in \mathbb{F}_p \setminus \{0, 1\}$. Traces are computed by FFT cyclic correlation and checked against direct sums for $p \le 10007$:

p	KS vs Sato–Tate	KS$\cdot\sqrt{N}$	$\max\lvert a_\lambda\rvert$ / $2\sqrt{p}$	moments m_2, m_4, m_6, m_8
101	0.172	1.71	18 / 20.10	1.000, 1.988, 5.14, 14.79
1009	0.0629	2.00	62 / 63.53	1.000, 2.012, 5.01, 14.01
10007	0.0199	1.99	200 / 200.07	1.000, 2.013, 5.09, 14.41
100003	0.00682	2.16	632 / 632.47	1.000, 2.000, 5.007, 14.06
1000003	0.00267	2.67	2000 / 2000.003	1.000, 1.99967, 4.99575, 13.976

The Catalan values are $1, 2, 5, 14$. At $p = 10^6 + 3$, $R_m = 0.999998, \ldots, 0.99603$ for $m = 1, \ldots, 8$. For the genus-2 family $y^2 = x(x-1)(x-2)(x-3)(x-\lambda)$ the KS distance to the $USp(4)$ trace law is $0.137, 0.0471, 0.0155, 0.00450, 0.00147$ (KS$\cdot\sqrt{N} \approx 1.35$–$1.55$). The moments at $p = 10^6 + 3$ are $0.9999995, 2.991, 13.896, 83.03$, against $USp(4)$ trace moments $1, 3, 14, 84$ (quadrature). We do not prove full $USp(4)$ monodromy for this family; the numerics are consistent with it.

46.5. Real forms: Hilbert versus Krein for SL_2

Proposition 46.5.1 ([THM]). *Let $g \in SL_2(\mathbb{R})$ and $k \ge 1$.*

(a) $\mathrm{Sym}^k g$ *preserves a positive definite Hermitian form iff* g *is elliptic or* $\pm I$.
(b) The space of $SL_2(\mathbb{R})$*-invariant Hermitian forms on* $\mathrm{Sym}^k\mathbb{C}^2$ *is one-dimensional. It is spanned by* $\mathrm{Sym}^k h$ *with* $h(x,y) = i\,\omega(\bar{x},y)$*, and its signature is* $(\lceil\frac{k+1}{2}\rceil, \lfloor\frac{k+1}{2}\rfloor)$ *up to an overall sign.*

Proof. (a) A positive invariant form puts $\mathrm{Sym}^k g$ in a compact group, so its powers are bounded. The eigenvalues of $\mathrm{Sym}^k g$ include λ^k, so $|\lambda| = 1$. A parabolic Jordan block of g gives an unbounded Jordan block in $\mathrm{Sym}^k g$. Hence g is elliptic or $\pm I$. Conversely, an elliptic g is conjugate into $SO(2)$, which preserves the standard form on every Sym^k. (b) An invariant Hermitian form is an element of $\mathrm{Hom}_G(V, \bar{V}^*)$. Here $V = \mathrm{Sym}^k$ is irreducible for $\mathfrak{sl}_2(\mathbb{R})\otimes\mathbb{C} = \mathfrak{sl}_2(\mathbb{C})$, and $\bar{V}^* \cong V$, so Schur gives dimension 1. In the null basis $e_\pm = (1, \pm i)$ one computes $h(e_+, e_+) = -2$, $h(e_-, e_-) = 2$ and $h(e_+, e_-) = 0$. On the monomials $e_+^a e_-^{k-a}$ the form $\mathrm{Sym}^k h$ is diagonal with sign $(-1)^a$, which gives the stated counts. □

Numerically [NUM] (`num/lemma46_realforms.py`): for $k = 1, \dots, 8$ the solution space of the invariance equations has dimension 1 with the predicted signature. An elliptic g ($\mathrm{tr}\, g = 0.4$) admits a positive invariant form with residual $\le 5 \times 10^{-15}$. For hyperbolic g ($\mathrm{tr}\, g = 3$, $k = 1$), $\log_{10}\|g^n\| = 4.18, 8.36, 16.72$ at $n = 10, 20, 40$, and parabolic norms grow linearly.

Remark 46.5.2 (Unitary trick and Wick rotation, [ANALOGY] beyond the stated facts). $SU(2)$ and $SL_2(\mathbb{R})$ are the compact and split real forms of $SL_2(\mathbb{C})$. Finite-dimensional irreducibles are unitary for the first and Krein (Prop. 46.5.1(b)) for the second. Their intersection, up to conjugacy, is the compact Cartan $SO(2)$. So "Frobenius is unitary" = "$g_p/\sqrt{p}$ lies in a compact Cartan" = elliptic (Lemma 42 Thm. 6.8), and Wick rotation $\theta \mapsto i\eta$ carries it to a boost.

Proposition 46.5.3 (Zeeman fan, [THM]). *If* $\alpha = \sqrt{p}\,e^{\eta}$ *and* $\beta = \sqrt{p}\,e^{-\eta}$*, then the eigenvalues of* $\mathrm{Sym}^k(F)$ *have archimedean slopes* $\log_{p^k}|\alpha^{k/2+m}\beta^{k/2-m}| = \frac{1}{2} + \frac{2m\eta}{k\ln p}$ *for* $|m| \le k/2$*. For* $p = 13$*,* $a = 8$ *the fan spans* $[0.31926, 0.68074]$ *(error* 5.6×10^{-17}*, [NUM]). These slopes are archimedean. The p-adic Newton slopes (*$0, 1$ *for* $p \nmid a$*; Lemma 42 Thm. 8.1–8.2) are the* same *for this fake as for a real ordinary curve, so the fan is invisible p-adically.*

46.6. Verschiebung

46.6.1. $FV = q$ **is algebra: it gives Krein unitarity for whatever** $\sharp$ **makes** $F^\sharp = V$**.** For a curve, $V := q\pi^{-1} \in \mathrm{End}(J)$ is the Verschiebung, the dual-isogeny partner of Frobenius with $FV = VF = [q]$ ([BIB], standard; [**85, 184**]). Thm. 46.1.1(c) gives $\pi^\dagger\pi = q$ with π invertible in $\mathrm{End}^0(J)$, so

$$\boxed{\pi^\dagger = q\pi^{-1} = V} \qquad \text{and therefore} \qquad (\pi/\sqrt{q})(\pi/\sqrt{q})^\dagger = 1.$$

Clarification of the chat claim. The composition $F \circ V = [q]$ is not itself an isometry. The isometry is $F/\sqrt{q}$, and $F \circ V = q$ is precisely the statement $FF^\dagger = q$ once one knows $V = F^\dagger$.

Theorem 46.6.1 (Abstract (F, V) pairs, [THM]). *Let* $P \in \mathbb{Z}[T]$ *be monic of degree* $2g$*, reciprocal* $(P(T) = q^{-g}T^{2g}P(q/T)$*,* $P(0) = q^g)$*, with distinct roots. Put* $A = \mathbb{Q}[T]/(P)$*,* $F = T$ *and* $V = qT^{-1} \in A$*.*

(i) $FV = VF = q$*. Moreover* V *is integral, its characteristic polynomial being* P *itself (by FE), so* $\mathbb{Z}[F, V]$ *is an order.*
(ii) $T \mapsto q/T$ *defines a* $\mathbb{Q}$*-algebra involution* $\sharp_0$ *of* A *with* $F^{\sharp_0} = V$*.*
(iii) $b(x, y) = \mathrm{Tr}_{A/\mathbb{Q}}(x\, y^{\sharp_0})$ *is symmetric and non-degenerate, and satisfies* $b(Fx, y) = b(x, Vy)$*. Hence* $b(Fx, Fy) = q\, b(x, y)$*, and* $F/\sqrt{q}$ *is a b-isometry.*
(iv) b *has signature* $(D_{\mathrm{on}} + k, k)$*, where* $D_{\mathrm{on}} = \#\{|\alpha| = \sqrt{q}\}$ *and* $k = \#\{|\alpha| > \sqrt{q}\}$*.*

So $FV = q$ *together with an adjointness* $V = F^{\sharp_0}$ *always holds, and the form it preserves is* Krein*, with boosts allowed by Thm. 46.1.3, whenever* P *is fake. It is* Hilbert *iff* $k = 0$*.*

PROOF. (i) $T \cdot qT^{-1} = q$. The eigenvalues of V are q/α_i, which are again the roots of P by FE. (ii) FE gives $P(q/T) = q^g T^{-2g} P(T)$, so the involutive substitution $f(T) \mapsto f(q/T)$ of $\mathbb{Q}[T^{\pm 1}]$ maps P into (P). Since T is invertible in A ($P(0) \neq 0$), it descends to A. (iii) $\sharp_0$ is a ring automorphism, so it preserves $\mathrm{Tr}_{A/\mathbb{Q}}$, and $b(y,x) = \mathrm{Tr}((yx^{\sharp_0})^{\sharp_0}) = \mathrm{Tr}(y^{\sharp_0}x) = b(x,y)$. Non-degeneracy holds because the trace form of the étale algebra A is non-degenerate and $\sharp_0$ is bijective. Adjointness: $b(Fx,y) = \mathrm{Tr}(Tx\, y^{\sharp_0}) = \mathrm{Tr}(x\,(Vy)^{\sharp_0})$, because $(Vy)^{\sharp_0} = V^{\sharp_0} y^{\sharp_0} = Ty^{\sharp_0}$. (iv) Under $A \otimes \mathbb{C} \cong \mathbb{C}^{2g}$ one has $b(x,y) = \sum_i x(\alpha_i) y(q/\alpha_i)$, which is the form of Thm. 46.1.4 with all $m_i = 1$. □

REMARK 46.6.2 (Positivity is the upgrade). For a Jacobian with squarefree P, V_ℓ is free of rank 1 over $A \otimes \mathbb{Q}_\ell$, so $\mathrm{Tr}_{V_\ell} = \mathrm{Tr}_{A/\mathbb{Q}}$ on $A = \mathbb{Q}[\pi] \subset \mathrm{End}^0(J)$. Also $\dagger|_A = \sharp_0$ by the boxed identity. Milne's positivity (Thm. 46.1.1(a)) is then the statement $b > 0$, i.e. $k = 0$. *The only non-algebraic input that separates real curves from abstract (F,V) pairs is* $\mathrm{Tr}(\alpha\alpha^\dagger) > 0$.

Numerics [NUM] (`num/lemma46_verschiebung.py`, `num/l46/verschiebung.json`; exact sympy matrices on the basis $1, T, \ldots, T^{2g-1}$). In every case $FV - q = 0$, $F^\top B - BV = 0$ and $F^\top BF - qB = 0$ hold exactly:

$P(T)$	$\lvert\alpha\rvert/\sqrt{q}$	signature of b
$T^2 - 3T + 13$	$1, 1$	$(2,0)$ Hilbert
$T^2 - 8T + 13$ (fake)	$0.629,\ 1.590$	$(1,1)$ Krein
$T^4 - 3T^3 + 13T^2 - 33T + 121$	$1, 1, 1, 1$	$(4,0)$
$T^4 + 11T^3 - 13T^2 + 121T + 121$ (real ghost)	$0.261, 1, 1, 3.833$	$(3,1)$
$T^4 - 13T^3 + 65T^2 - 143T + 121$ (complex quadruple)	$0.712^{\times 2},\ 1.404^{\times 2}$	$(2,2)$

For $g = 2$, $V = qT^{-1}$ is not in $\mathbb{Z}[T]/(P)$; it lies in the order $\mathbb{Z}[F,V]$.

46.6.2. On eigenvalues: V is the functional equation, positivity is c_∞.

PROPOSITION 46.6.3 ([THM], with RH-equivalence [RESTATEMENT]). [1] *Let v be an F-eigenvector with $Fv = \alpha v$. Then $Vv = (q/\alpha)v$. Hence:*

- *(a) As a $\mathbb{Q}$-linear operation on the spectrum, V (equivalently $\dagger|_{\mathbb{Q}[\pi]}$) realizes the functional equation* $\mathrm{FE} : \alpha \mapsto q/\alpha$. *In s-coordinates, $\alpha = q^s$, this is $s \mapsto 1 - s$.*
- *(b) The Hermitian form needs the antilinear involution $x \mapsto \bar{x}^\dagger$ on $\mathbb{Q}[\pi] \otimes \mathbb{C} \cong \mathbb{C}^{2g}$. It acts on coordinates by $\alpha \mapsto q/\bar{\alpha}$, i.e. by $\sharp = c_\infty \circ \mathrm{FE}$: $s \mapsto 1 - \bar{s}$, Lemma 40's mirror (Lemma 30 Cor. 2.3).*
- *(c) Rosati positivity $\iff$ $\sharp$ fixes every root $\iff$ $q/\alpha = \bar{\alpha}$ for all α $\iff$ V* acts as c_∞ on the spectrum. *By Milne Cor. 1.29 this is $\rho(\alpha^\dagger) = \overline{\rho(\alpha)}$, which is RH for C.*

PROOF. $Vv = qF^{-1}v = (q/\alpha)v$. For (b), with ρ_i the evaluation at α_i, we have $\rho_i(\bar{x}^\dagger) = \overline{x}(q/\alpha_i) = \overline{x(q/\bar{\alpha}_i)}$. So the involution permutes coordinates by $\alpha \mapsto q/\bar{\alpha}$, and the form $\mathrm{Tr}(x\bar{y}^\dagger) = \sum_i x(\alpha_i)\overline{y(q/\bar{\alpha}_i)}$ is diagonal iff that permutation is trivial. Part (c) is then Lemma 30 Thm. 2.2. □

Correction to the chat claim "V realizes Lemma 40's $\sharp = c_\infty \circ \mathrm{FE}$". V realizes FE. Lemma 40's $\sharp$ is $c_\infty \circ \dagger$, i.e. V composed with c_∞. RH is the statement that this composite acts trivially on the spectrum, or equivalently that V and c_∞ agree there. This matches Lemma 40 Prop. 7.1 ("positivity $\iff$ Rosati $= c_\infty$ on spectrum").

[1] **[Vol. 3 editorial note]** For a picture of this proposition in log-modulus coordinates (fixed point $(\frac{1}{2}, \frac{1}{2})$ of the transpose, phase fibres, why the product $\alpha\beta = q$ is blind to ghosts and the sum is not), see the Vol. 3 editorial remark "Transposition and precession" in the Introduction.

46.6.3. Global F and V: Witt vectors, the BC system, and the degree mismatch.

THEOREM 46.6.4 (The tower is one F-orbit, [THM]). *Identify the big Witt ring $W(\mathbb{Z})$ with $1+t\mathbb{Z}[[t]]$, with ghost components defined by $t\frac{d}{dt}\log f=\sum_{n\geq 1}\mathrm{gh}_n(f)t^n$. The Witt Frobenius and Verschiebung satisfy $\mathrm{gh}_n(F_mf)=\mathrm{gh}_{nm}(f)$ and $\mathrm{gh}_n(V_mf)=m\,\mathrm{gh}_{n/m}(f)$ (zero if $m\nmid n$), hence $F_mV_m=m$ (CC14 Prop. 5.1, [CITED]). For the zeta function:*

(a) $\mathrm{gh}_n(Z(C/\mathbb{F}_q,t))=N_n$.
(b) $F_mZ(C/\mathbb{F}_q)=Z(C/\mathbb{F}_{q^m})$: *Witt Frobenius* is *base change.*
(c) $V_mZ(X/\mathbb{F}_{q^m})(t)=Z(X/\mathbb{F}_{q^m})(t^m)$ *is the zeta function of X regarded as an $\mathbb{F}_q$-scheme (base-forgetting). Thus $F_mV_m=m$ is the identity $X\times_{\mathbb{F}_q}\mathbb{F}_{q^m}=\bigsqcup_{\sigma\in\mathrm{Gal}}X^\sigma$.*

PROOF. (a) is the definition of Z. Ghost maps are injective on the torsion-free ring $W(\mathbb{Z})$, so (b) follows from $\#C(\mathbb{F}_{q^{nm}})=\mathrm{gh}_{nm}$. For (c): an $\mathbb{F}_{q^n}$-point of $X\to\operatorname{Spec}\mathbb{F}_{q^m}\to\operatorname{Spec}\mathbb{F}_q$ is a choice of one of the m embeddings $\mathbb{F}_{q^m}\hookrightarrow\mathbb{F}_{q^n}$ (possible iff $m\mid n$) together with an $\mathbb{F}_{q^n}$-point over it, so the count is $m\,N_{n/m}$. □

So J.W.'s "entire collection out to infinity" over a single curve is the $F_\bullet$-orbit of one Witt vector. RH for C is a growth condition on that orbit, $|\mathrm{gh}_n-q^n-1|\leq 2g\,q^{n/2}$ for all n (Lemma 3), and Prop. 46.1.6 shows that a one-sided bound already suffices.

THEOREM 46.6.5 (Integral BC system, [CITED] CC14). *In the integral Bost–Connes algebra of Connes–Consani [67]:*

(a) $\sigma_n(\tilde\rho_n(x))=n\,x$ *(Prop. 3.1, eq. (19)), and $\mu_n^*\tilde\mu_n=n$ (relations (22));*
(b) on $W_0(A)$, $F_n\circ V_n=n$ (Prop. 2.2(1)), and under $W_0(\overline{\mathbb{F}}_p)\cong\mathbb{Z}[\overline{\mathbb{F}}_p^\times]$ one has $\sigma_n\leftrightarrow F_n$, $\tilde\rho_n\leftrightarrow V_n$ (Prop. 4.4, Thm. 4.5);
(c) over $\mathbb{Q}$, $\mu_n=\frac{1}{n}\tilde\mu_n$ satisfies $\mu_n^*\mu_n=1$, and in the representation on $\ell^2(\mathbb{N})$, $\mu_n\varepsilon_m=\varepsilon_{nm}$ (eq. (86)).

PROPOSITION 46.6.6 (Weight 0 and no spectrum on a circle, [THM]). *On $\ell^2(\mathbb{N})$, μ_n ($n\geq 2$) is an isometry that is unitarily equivalent to a direct sum of countably many unilateral shifts, one along each orbit $\{mn^j\}_{j\geq 0}$ with $n\nmid m$. Hence $\sigma(\mu_n)$ is the closed unit disc, μ_n has no eigenvalues, and $\mu_n\mu_n^*$ is the projection onto $\overline{\mathrm{span}}\{\varepsilon_m : n\mid m\}$ (checked on a truncation, [NUM]). The positive form is the bosonic Fock form: $\ell^2(\mathbb{N})=$ Fock space over the primes, with basis $\mathrm{Div}^+(\operatorname{Spec}\mathbb{Z})$ (Lemma 45 Prop. 1.2).*

PROOF. The orbits $\{mn^j\}$ with $n\nmid m$ partition $\mathbb{N}$, and μ_n shifts each one. The spectrum of a unilateral shift is the closed disc, and a direct sum of copies has the same spectrum. □

THEOREM 46.6.7 (Borger, [CITED] [41], §2.9, §1.2 and Thm. 0.1; cf. [42]). *A Λ-structure (= descent to $\mathbb{F}_1$) on a scheme flat over $\mathbb{Z}$ is a commuting family of Frobenius lifts ψ_p. A connected smooth proper curve of genus $g\geq 1$ over $\mathbb{Z}[1/M]$ has* no *Λ-structure. If a smooth proper $X/\mathbb{Z}[1/M]$ admits one, its p-adic cohomology is potentially cyclotomic and its Hodge numbers $h^{i,j}$ vanish for $i\neq j$ (only abelian Artin–Tate motives). In this framework $\operatorname{Spec}\mathbb{Z}\times_{\mathbb{F}_1}\operatorname{Spec}\mathbb{Z}$ is defined to be the Witt space $W^*(\operatorname{Spec}\mathbb{Z})$. Borger notes that the theory "says nothing about the archimedean place".*

OPEN PROBLEM 46.6.8 (The H^0/H^1 degree mismatch, [OPEN]). For curves, Frobenius has weight 0 on H^0, weight 1 on H^1 and weight 2 on H^2, and the normalized $F/\sqrt{q}$ is unitary with point spectrum on $|z|=1$ *on* H^1. All global F_n/V_n structures with a positive form that are actually constructed are of H^0 type:

- the BC/Witt operators on $\ell^2(\mathbb{N})$ are weight 0 (isometries, not $\sqrt{n}$-similitudes) and have no point spectrum (Prop. 46.6.6);
- Λ-structures exclude every H^1 with $h^{1,0}\neq 0$ at finite type (Thm. 46.6.7).

Problem: construct a Hilbert space $\mathcal{H}^1$ with commuting operators F_n satisfying (1) $F_nF_n^* = n$, i.e. $V_n := F_n^*$ gives $F_nV_n = n$ for a *positive* form; (2) $F_n/\sqrt{n}$ normal; and (3) a (regularized) Lefschetz identity matching $\mathrm{Tr}(F_n/\sqrt{n}\,|\,\mathcal{H}^1)$ with the zero side $\sum_\rho m_\rho\, n^{\rho-1/2}$ of the explicit formula, as for curves, where $F^m/q^{m/2}$ has eigenvalues $q^{m(\rho-1/2)}$. Conditions (1)+(2) make $F_n/\sqrt{n}$ unitary, so (3) would place the zeros on the line. Hence the problem is at least as strong as RH. Its content is Lemma 30's axiom G5 rewritten in Witt language. It is not a conjecture of ours. Thm. 46.6.7 shows that $\mathcal{H}^1$ cannot come from a finite-type Λ-scheme, so any solution must be of infinite type (e.g. $W^*(\mathrm{Spec}\,\mathbb{Z})$) and must see the archimedean place.

46.6.4. Dieudonné: $FV = VF = p$, ordinary and supersingular.

Proposition 46.6.9 ([THM]; Dieudonné theory [BIB] [**85**, **184**]). *For $E/\mathbb{F}_p$, the Dieudonné module of $E[p^\infty]$ is free of rank 2 over $\mathbb{Z}_p$ ($\sigma = \mathrm{id}$ on $W(\mathbb{F}_p) = \mathbb{Z}_p$), with F and $V = pF^{-1}$, $FV = VF = p$. E is ordinary (slopes $0, 1$) iff $p \nmid a_p$, and supersingular (slopes $\frac{1}{2}, \frac{1}{2}$) iff $p \mid a_p$ (Lemma 42 Thm. 8.1).* For every integer a, *the module $M_a = \mathbb{Z}_p[T]/(T^2 - aT + p)$ with $F = T$, $V = a - T$ satisfies $FV = VF = p$ and $\dim_{\mathbb{F}_p} M_a/FM_a = 1 = g$. Its Newton polygon has vertices among $(0,1), (1, v_p(a)), (2,0)$, so its slopes are $0, 1$ if $v_p(a) = 0$ (and Hensel's lemma gives a unit root in $\mathbb{Z}_p$), and $\frac{1}{2}, \frac{1}{2}$ if $v_p(a) \geq 1$. Hence:*

(a) *the fake $a = 8$, $p = 13$ is "ordinary" with Dieudonné data indistinguishable from a real curve;*
(b) *the fake $a = p = 13$ is "supersingular" (p-adically pure, slopes $\frac{1}{2}, \frac{1}{2}$), yet its roots $(13 \pm \sqrt{117})/2$ have $|\alpha|/\sqrt{13} = 3.303, 0.303$, so it is archimedeanly boosted.*

Dieudonné $FV = p$ encodes integrality (Newton above Hodge, Lemma 42 Thm. 8.2), not positivity. p-adic purity does not imply archimedean purity. For real curves, supersingular with $p \geq 5$ forces $a_p = 0$ because Hasse already holds.

46.7. ζ: what each mechanism needs, and which ingredient fails

mechanism	ingredients	status for ζ
Weil / Hodge index	surface $X \times X$, ample class, Hodge index, $\mathrm{Tr}(\alpha\alpha^\dagger) > 0$	$\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ has no intersection theory. Connes–Consani [**71**] prove Weil positivity at the archimedean place for test functions supported in $[2^{-1/2}, 2^{1/2}]$ [CITED]. Global positivity is [OPEN] $\equiv$ RH (Lemma 30 G5; [**62**]).
Weil 1948 / zero–pole count	principal divisors have degree 0 on $C \times C$	product formula $\prod_v \lvert x\rvert_v = 1$ holds on $\mathbb{Q}$, but no function Φ on a "$\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$" is known [ANALOGY].
Stepanov–Bombieri	Riemann–Roch on C + constant-field tower $\mathbb{F}_{q^n}$	no constant field and no tower. The Witt F_m-orbit (Thm. 46.6.4) exists for $W(\mathbb{Z})$, but ζ is not a single Witt vector over a base field [OPEN].
Deligne	(i) rationality, (ii) symplectic pairing, (iii) big monodromy in a family, (iv) tensor powers $V^{\otimes m}$	(i) holds (real Dirichlet coefficients). (ii)–(iii) would need a family with ζ as a fibre and geometric monodromy, which is unknown. (iv) needs $(\operatorname{Spec}\mathbb{Z})^m$ [OPEN].
Katz–Sarnak	purity + monodromy $\Rightarrow$ equidistribution	symplectic Family I statistics [**155, 154**]. Low-lying zeros of families [**138**] [BIB]. Statistics presuppose purity, so they give no proof route [RESTATEMENT]/[ANALOGY].
Deninger	foliated space, flow, Hodge-$*$ positivity	conjectural [**91**] [CONJ].
$\mathbb{F}_1$ / Λ	Frobenius lifts ψ_p, $W^*(\operatorname{Spec}\mathbb{Z})$	no H^1 at finite type and no archimedean place (Thm. 46.6.7); [**68**] [BIB]. Degree mismatch, Problem 46.6.8 [OPEN].

The failing ingredient. Every proved mechanism has an algebraic half that ζ already possesses: an involution with $FF^\sharp = q$ (Lemma 30 Prop. 3.2: $\delta_\lambda * \delta_\lambda^\sharp = \lambda$), FE, and reality. That algebraic half yields only Krein unitarity (Thm. 46.6.1). The missing half is a *reason for positivity*: an ample class with Hodge index, a degree-zero count, or a family with big monodromy and tensor powers. For ζ this is precisely the positivity of the Weil functional (Lemma 30 G5) [RESTATEMENT]. In Hilbert/Krein terms: the Néron–Severi-type form for ζ, if it existed, would have $n_+ = 1 + k$ with k the number of off-line zero pairs counted as in Thm. 46.1.4, and RH $\iff$ $n_+ = 1$ ([RESTATEMENT]; Lemma 40 Thm. 5.2 and Lemma 45 Thm. 6.4 give the Pick/Fock versions).

46.8. Corrections to the chat claims

E1. "Ghost $\iff$ index 2" is corrected to $n_+ = 1 + k$: a real pair gives 2 and a complex quadruple gives 3 (Thm. 46.1.4; 3591/3591 at $q = 11$).

E2. "$F/\sqrt{q}$ is a Rosati isometry by algebra, positivity from Hodge index": *correct*. More precisely, the positivity comes from ampleness plus positivity of a non-empty intersection (MilpRH 1.27), or from Weil's degree-zero count. Considering the whole collection of curves is not needed (Rem. 46.1.2).

E3. "Deligne's positivity is Hermitian" is corrected: it is real-square positivity from rationality, combined with Landau, monodromy coinvariants and tensor powers. Hodge index is not used.

E4. "The collection out to infinity forces unitarity statistically" is corrected: Katz–Sarnak presupposes purity. What the tower over one curve does force is Prop. 46.1.6: a one-sided lower bound for all n implies RH.

E5. "Zeeman fan $\frac{1}{2} + 2m\eta/(k \ln p)$": correct as archimedean slopes. It is not a p-adic phenomenon, and the Newton slopes are blind to it.

E6. "$F \circ V = [q]$ makes things a Rosati isometry" is corrected: $F/\sqrt{q}$ is the isometry, and $FV = q$ means $FF^\dagger = q$ once $V = F^\dagger$. Without positivity this gives Krein unitarity (Thm. 46.6.1).

E7. "V realizes Lemma 40's $\sharp = c_\infty \circ \mathrm{FE}$" is half right: V realizes FE, $\sharp = c_\infty \circ \dagger$, and RH $\iff$ $V = c_\infty$ on the spectrum (Prop. 46.6.3).

E8. "Connes–Consani have $\sigma_n \tilde{\rho}_n = n$": *verified*. The paper is J. Noncommut. Geom. **8** (2014) 873–945, Prop. 3.1/eq. (19), and $\sigma_n \leftrightarrow F_n$, $\tilde{\rho}_n \leftrightarrow V_n$ (Prop. 4.4). The normalized maps are isometries on $\ell^2(\mathbb{N})$ of weight 0 and type H^0 (Prop. 46.6.6). The degree mismatch is stated as Problem 46.6.8 ([OPEN], not a conjecture).

E9. Dieudonné: $FV = VF = p$ is integral. Fakes possess it, and a supersingular-type fake can be archimedeanly boosted (Prop. 46.6.9).

46.9. Open questions

Q1. Accept "index $= 1 + k$" and the abstract Néron–Severi lattice $G_n(P)$ as the canonical Krein space Π_{1+k} of a fake (replacing "index 2")?

Q2. Which positivity source should the program target for ζ: (a) ample + Hodge index, (b) Weil 1948's degree-zero count of a principal divisor (product-formula flavour), or (c) Deligne's family + tensor powers, which would need $(\operatorname{Spec}\mathbb{Z})^m$ and a family through ζ?

Q3. Adopt the dictionary "V = functional equation, c_∞ = reality, RH = ($V = c_\infty$ on the spectrum)" for Volume 2?

Q4. Pursue Problem 46.6.8 (an H^1-type Hilbert space with $F_n F_n^* = n$), or treat Borger's Artin–Tate theorem as evidence that $\Lambda/\mathbb{F}_1$ structures cannot host H^1 at finite type?

Q5. Use the Witt-vector reading of the tower (Thm. 46.6.4: F_m = base change) as the official meaning of "all curves out to infinity"?

CHAPTER 47

Lemma 47: Reflecting $\mathbb{N}$ into $\mathbb{Z}$

Dilation of the Bost–Connes/Verschiebung isometries, the per-prime Dirac sea, the boson–fermion correspondence, Toeplitz versus Hankel, and the finite-field test

Standalone note `lemma47.tex`, October 22, 2026.

Summary. **Statement of intellectual honesty.** RH is not proved here and no step below is claimed to bring it closer except where explicitly said. Tags: [THM] proved here (elementary, proof given); [CITED] checked at the source (`lit/lemma47/verified_refs.md`); [BIB] bibliographic record verified (Crossref), content standard; [RESTATEMENT] equivalent to RH or to a known statement; [ANALOGY] analogy only; [NUM] numerical (`num/lemma47_*.py`, data `num/147/`); [CONJ] conjecture; [OPEN] open. Lemmas 42, 45, 46 are cited by number and not redone.

Question (J.W., [t170u]/[t171u]). Lemma 46 Prop. 6.6 found the normalized Bost–Connes/Verschiebung operators μ_n on $\ell^2(\mathbb{N})$ to be weight-0 isometries of "bosonic H^0" type, while the zeros live on a "fermionic H^1". Is this an artifact of starting on $\mathbb{N}$? Would a *reflective* transformation pairing $\mathbb{N}$ with $\mathbb{Z}$ (positive and negative integers) make the same operators act fermionically? *Follow-up:* is the $\mathbb{N} \to \mathbb{Z}$ half-turn the 180° rotation of the zeta zeros?

Answer in brief. (1) *The dilation is exact and canonical* (§47.1). The μ_n dilate minimally to the regular representation of $\mathbb{Q}_{>0}^{\times} \cong \bigoplus_p \mathbb{Z} = \mathrm{Div}(\mathrm{Spec}\,\mathbb{Z})$ on $\ell^2(\mathbb{Q}_{>0}^{\times})$; $F_n := \sqrt{n}\,U_n$, $V_n := F_n^*$ satisfy $F_nV_n = V_nF_n = n$ with $F_n/\sqrt{n}$ unitary. Conditions (1)–(2) of Lemma 46 Open Problem 6.8 thus cost nothing; but the joint spectrum is the whole compact torus $\mathbb{T}^{\mathcal{P}} = \widehat{\mathbb{Q}_{>0}^{\times}}$ with Haar (Lebesgue) spectral type and no eigenvalues. The critical line sits inside it as a dense, Haar-null winding line: the dilation builds the *stage*, not the *actors*. (2) *J.W.'s reflection exists, but only prime by prime* (§47.2). On each $\ell^2(\mathbb{Z})$ the map $k \mapsto -1-k$ intertwines U_p and U_p^* and exchanges $\ell^2(\mathbb{N}_0)$ with its orthocomplement (the Dirac sea). For two or more primes no unitary or antiunitary reversing all U_r exchanges the "positive orthant" with its complement (Thm. 47.2.3); the global substitute is a restricted product of per-prime seas. (3) *The boson–fermion correspondence applied to the dilated system* gives exactly computable partition functions: charge-0 sector $\prod_{k\geq 1}\zeta(k\beta) = \sum a(n)n^{-\beta}$ ($a(n)$ = number of abelian groups of order n); full fermionic trace $\prod_m[\zeta((m-\frac{1}{2})\beta)/\zeta((2m-1)\beta)]^2$; supertrace $\prod_m \zeta((m-\frac{1}{2})\beta)^{-2}$. The original primon gas is the $m=1$ bosonic subsector. Zeros of ζ appear *only* as complex-β singularities (e.g. poles at $\beta = 2\rho/(2m-1)$), never as spectral data (Thm. 47.2.7). (4) *Toeplitz versus Hankel, decided* (§47.3). A reflection turns every translation-invariant (Toeplitz) form into a Hankel form. A kernel is both Toeplitz and Hankel iff it is *square-periodic*, i.e. factors through $\mathbb{Q}_{>0}^{\times}/(\mathbb{Q}_{>0}^{\times})^2 \cong \bigoplus_p \mathbb{Z}/2$ — Lemma 45's fermionic cube (Liouville's λ is the basic example). Weil's form and even its finite-prime part are not square-periodic, so *no form produced by the reflection equals Weil's form*; the Toeplitz forms the construction can produce are the ones put in before reflecting (Thm. 47.3.1). (5) *Finite-field test* (§47.4). For a curve, H^1 with its positive Rosati form *is* recovered from H^0/divisor data, but by a different mechanism: dilation $\mathrm{Div}^+ \to \mathrm{Div}$ (J.W.'s step, needed: degree-0 divisors need negative coefficients), then the *quotient by principal divisors* (Jacobian), then the theta polarization (ample class) for positivity. The reflection

that exists globally is Riemann–Roch's $D \mapsto K - D$; it yields the functional equation (Lemma 46's V), not positivity. For $\mathbb{Q}$ the quotient step degenerates ($\mathbb{Q}_{>0}^{\times}/\mathbb{Q}_{>0}^{\times} = 0$; class number one), which is the precise point where Connes' cokernel/absorption picture takes over and RH reappears as a positivity. Numerics: $h = P(1)$ agrees with an independent Mumford-representation count of Pic^0 on 9 smooth genus-2 curves. (6) 180° *rotation* (§47.5). $s \mapsto 1 - s$ is the half-turn about $\frac{1}{2}$, i.e. the functional equation; it is an involution, not the precession $x^{i\gamma}$. RH $\iff$ the half-turn agrees with complex conjugation on every zero ([RESTATEMENT], = Lemma 46 Prop. 6.3). The Mellin normalization in the chat is corrected.

47.1. The dilation of the multiplicative system

Throughout, $\mathcal{P}$ is the set of primes, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, \dots\}$, $\{\varepsilon_r\}$ the standard basis, and v_p the p-adic valuation. Recall (Lemma 46 Thm. 6.5, Prop. 6.6, [CITED] CC14) that the normalized BC isometries on $\ell^2(\mathbb{N})$ are $\mu_n \varepsilon_m = \varepsilon_{nm}$.

47.1.1. One prime: Wold and Sz.-Nagy.

Proposition 47.1.1 ([BIB] [259], Ch. I; proof elementary). *Let $S\varepsilon_k = \varepsilon_{k+1}$ on $\ell^2(\mathbb{N}_0)$. (a) S is a pure isometry (Wold: $\bigcap_j S^j \ell^2(\mathbb{N}_0) = 0$, wandering space $\mathbb{C}\varepsilon_0$); $\sigma(S) = \overline{\mathbb{D}}$, no eigenvalues; S^* has every $|z| < 1$ as eigenvalue. (b) Its minimal unitary dilation is the bilateral shift $U\varepsilon_k = \varepsilon_{k+1}$ on $\ell^2(\mathbb{Z}) \supset \ell^2(\mathbb{N}_0)$: $U|_{\ell^2(\mathbb{N}_0)} = S$, $PU^*P = S^*$, and $\bigcup_j U^{-j}\ell^2(\mathbb{N}_0)$ is dense. (c) Fourier series identifies $\ell^2(\mathbb{Z}) \cong L^2(\mathbb{T}, d\theta/2\pi)$, $U \leftrightarrow$ multiplication by z, $\ell^2(\mathbb{N}_0) \leftrightarrow H^2$. Hence $\sigma(U) = \mathbb{T}$, purely absolutely continuous (Lebesgue), no eigenvalues.*

47.1.2. All primes at once.

Theorem 47.1.2 (Minimal unitary dilation of the BC isometries, [THM]; C*-level [CITED] [164]). *Let $U_r\varepsilon_s = \varepsilon_{rs}$ on $\ell^2(\mathbb{Q}_{>0}^{\times})$, $r, s \in \mathbb{Q}_{>0}^{\times}$, and regard $\ell^2(\mathbb{N}) \subset \ell^2(\mathbb{Q}_{>0}^{\times})$, with projection P.*

- *(a) $U_n P = P U_n P$ and $U_n|_{\ell^2(\mathbb{N})} = \mu_n$; $PU_n^*P = \mu_n^*$ on $\ell^2(\mathbb{N})$. The μ_n are* doubly commuting *for coprime arguments: $\mu_m^*\mu_n = \mu_n\mu_m^*$ if $(m, n) = 1$.*
- *(b) (Minimality) $\overline{\mathrm{span}} \bigcup_n U_n^*\ell^2(\mathbb{N}) = \ell^2(\mathbb{Q}_{>0}^{\times})$, since $\varepsilon_{a/b} = U_b^*\varepsilon_a$. Any other unitary representation W of $\mathbb{Q}_{>0}^{\times}$ on $\mathcal{K} \supset \ell^2(\mathbb{N})$ extending μ minimally is unitarily equivalent to U by an equivalence fixing $\ell^2(\mathbb{N})$.*
- *(c) (Product structure) $\ell^2(\mathbb{Q}_{>0}^{\times}) = \bigotimes_p'(\ell^2(\mathbb{Z}), \varepsilon_0)$, the restricted tensor product with respect to ε_0, and $U_p = 1 \otimes \cdots \otimes U^{(p)} \otimes \cdots$, $\ell^2(\mathbb{N}) = \bigotimes_p' \ell^2(\mathbb{N}_0)$ (Lemma 45 Prop. 1.2: bosonic Fock space over the primes).*
- *(d) (Spectrum) Let $\mathbb{T}^{\mathcal{P}} = \prod_p \mathbb{T}$ be the compact dual of the discrete group $\mathbb{Q}_{>0}^{\times} \cong \bigoplus_p \mathbb{Z}$, with Haar measure m, and $\chi_r(z) = \prod_p z_p^{v_p(r)}$. Fourier transform $\ell^2(\mathbb{Q}_{>0}^{\times}) \cong L^2(\mathbb{T}^{\mathcal{P}}, m)$ carries U_r to multiplication by χ_r. The joint spectrum of $\{U_r\}$ is all of $\mathbb{T}^{\mathcal{P}}$, of Haar type; for $n \geq 2$, U_n is unitarily equivalent to a countable direct sum of bilateral shifts, $\sigma(U_n) = \mathbb{T}$, absolutely continuous, no eigenvalues.*
- *(e) (Verschiebung normalization) $F_n := \sqrt{n}\, U_n$, $V_n := F_n^* = \sqrt{n}\, U_n^{-1}$ satisfy $F_nV_n = V_nF_n = n$, $F_mF_n = F_{mn}$, and $F_n/\sqrt{n}$ is unitary.*

Proof. (a) $U_n\varepsilon_m = \varepsilon_{nm} \in \ell^2(\mathbb{N})$ for $m \in \mathbb{N}$; $\langle U_n^*\varepsilon_m, \varepsilon_k\rangle = \langle \varepsilon_m, \varepsilon_{nk}\rangle$, so $PU_n^*P\varepsilon_m = \varepsilon_{m/n}$ if $n \mid m$, else 0, which is μ_n^*. For coprime m, n: both $\mu_m^*\mu_n\varepsilon_k$ and $\mu_n\mu_m^*\varepsilon_k$ equal $\varepsilon_{nk/m}$ if $m \mid k$ and 0 otherwise (as $m \mid nk \iff m \mid k$). (b) Density is the displayed formula. Uniqueness: the inner products $\langle W_b^*\varepsilon_a, W_d^*\varepsilon_c\rangle = \langle W_d\varepsilon_a, W_b\varepsilon_c\rangle = \langle \varepsilon_{ad}, \varepsilon_{bc}\rangle$ are forced by μ, so $W_b^*\varepsilon_a \mapsto \varepsilon_{a/b}$ is a well-defined isometry from a dense set onto a dense set, intertwining W and U; this is the standard argument, and at the C*-level the general Ore-semigroup statement is [164]. (c) The basis ε_r, $r = \prod p^{k_p}$ with finitely many $k_p \neq 0$, is the product basis. (d) Pontryagin duality for

the discrete group $\bigoplus_p \mathbb{Z}$ gives the compact dual $\prod_p \mathbb{T}$. A multiplication operator by χ_r ($r \neq 1$) on $L^2(\mathbb{T}^{\mathcal{P}}, m)$ has spectrum the essential range of χ_r, which is $\mathbb{T}$, and the push-forward of m under χ_r is Lebesgue measure on $\mathbb{T}$ (a nontrivial character of a compact group pushes Haar to Haar on its image). Alternatively: the orbits of $s \mapsto ns$ on $\mathbb{Q}_{>0}^{\times}$ are all bi-infinite, so U_n is a direct sum of bilateral shifts. (e) Immediate. □

Remark 47.1.3 (precision on the C*-level dilation, [CITED]). Laca [**164**] (abstract and Thm. 2.2.1, read via arXiv math/9911135) proves: for an Ore semigroup $S \subset G = S^{-1}S$, isometric representations of S dilate uniquely (under minimality) to unitary representations of G; actions by injective endomorphisms dilate to automorphic actions; the semigroup crossed product is a full corner of the group crossed product. His explicit application: the BC Hecke C*-algebra $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^{\times}$ dilates to $C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_{>0}^{\times}$, the multiplicative action of $\mathbb{Q}_{>0}^{\times}$ on the *finite adeles*. So the chat claim "μ_n on $\ell^2(\mathbb{N}^{\times})$ dilate to unitaries on $\ell^2(\mathbb{Q}_{>0}^{\times})$" is correct for the isometries alone (Thm. 47.1.2); for the full BC system the dilated space is $\mathbb{A}_f$, which is one step from Connes' adele class space ($\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^{\times}$, adding the archimedean place) [**75**]. Laca–Raeburn [**165**] realize the BC algebra as a semigroup crossed product [BIB].

Corollary 47.1.4 (Problem 6.8 (1)–(2) are cheap, [THM]). *The family F_n of Thm. 47.1.2(e) satisfies conditions (1) $F_n F_n^* = n$ for a positive form and (2) $F_n/\sqrt{n}$ normal (indeed unitary) of Lemma 46 Open Problem 6.8, on a Hilbert space, unconditionally. Condition (3) (a regularized Lefschetz identity reproducing $\sum_\rho m_\rho n^{\rho-1/2}$) fails: U_n ($n \neq 1$) has no eigenvalues, its matrix in the basis ε_r has zero diagonal, so every diagonal regularization $\mathrm{Tr}(U_n e^{-\beta H})$, $H\varepsilon_r = f(r)\varepsilon_r$, is 0; and the spectral measure class is Haar on $\mathbb{T}^{\mathcal{P}}$, which assigns no special weight to any point. Hence the entire content of Problem 6.8 lies in (3).*

Proposition 47.1.5 (The critical line inside the stage, [THM]). *The map $\iota : \mathbb{R} \to \mathbb{T}^{\mathcal{P}}$, $t \mapsto (p^{-it})_p$, is the dual of the inclusion $\mathbb{Q}_{>0}^{\times} \hookrightarrow \mathbb{R}_{>0}$ ($\chi_r(\iota(t)) = r^{-it}$). It is injective with dense image (the $\log p$ are $\mathbb{Q}$-linearly independent by unique factorization), and $\iota(\mathbb{R})$ is Haar-null. Thus $\{\frac{1}{2} + it\}$ embeds densely in the joint spectrum $\mathbb{T}^{\mathcal{P}}$, with measure zero.*

Proof. Injectivity: $p^{-it} = 1$ for all p forces $t \log 2 \in 2\pi\mathbb{Z}$ and $t \log 3 \in 2\pi\mathbb{Z}$, so $t = 0$ by irrationality of $\log 3/\log 2$. Density: a character of $\mathbb{T}^{\mathcal{P}}$ trivial on $\iota(\mathbb{R})$ is χ_r with $r^{-it} = 1$ for all t, i.e. $r = 1$. Null: let $\pi : \mathbb{T}^{\mathcal{P}} \to \mathbb{T}^2$ be the projection to the coordinates $p = 2, 3$; $\pi_* m$ is Lebesgue measure, and $\pi(\iota(\mathbb{R})) = \{(2^{-it}, 3^{-it})\}$ is the image of $\mathbb{R}$ under a smooth map into a surface, hence Lebesgue-null. So $\iota(\mathbb{R}) \subset \pi^{-1}(\pi\iota(\mathbb{R}))$ is contained in a Haar-null Borel set. □

47.2. The reflection: per prime, and why not globally

Definition 47.2.1. On $\ell^2(\mathbb{Z})$ put $J\varepsilon_k = \varepsilon_{-1-k}$ (Dirac/particle–hole reflection, centre $-\frac{1}{2}$) and $J^0\varepsilon_k = \varepsilon_{-k}$ (centre 0). At the prime p, written multiplicatively on $p^{\mathbb{Z}}$: $J_p : r \mapsto 1/(pr)$ (fixed "point" $p^{-1/2}$) and $J_p^0 : r \mapsto 1/r$.

Proposition 47.2.2 ([THM]). *(a) $JUJ = U^*$ and $J^0 U J^0 = U^*$. (b) $J\,\ell^2(\mathbb{N}_0) = \ell^2(\mathbb{Z}_{<0}) = \ell^2(\mathbb{N}_0)^{\perp}$; $J^0\ell^2(\mathbb{N}_0) \cap \ell^2(\mathbb{N}_0) = \mathbb{C}\varepsilon_0$. (c) In Fourier variables, $(Jf)(z) = \bar{z}\,\check{f}(z)$ with $\check{f}(z) := f(\bar{z})$ (linear), so $JH^2 = \overline{\mathrm{span}}\{z^{-1-j}\}_{j\geq 0} = (H^2)^{\perp}$: J is the classical flip $H^2 \leftrightarrow (H^2)^{\perp}$. (d) If D is the generator with $D\varepsilon_k = (k + \frac{1}{2})\varepsilon_k$ then $JDJ = -D$: the reflection is a* charge/energy conjugation *with Fermi level $-\frac{1}{2}$.*

Proof. $JUJ\varepsilon_k = JU\varepsilon_{-1-k} = J\varepsilon_{-k} = \varepsilon_{k-1} = U^*\varepsilon_k$; same for J^0. (b),(d) are direct. (c) $\sum_k f_{-1-k} z^k = \sum_j f_j z^{-1-j} = \bar{z}\,\check{f}(z)$ on $|z| = 1$. □

So the chat claim "$JSJ = S^*$, creation becomes annihilation" is correct ([NUM]: checked on a window, `num/l47/checks.json`).

THEOREM 47.2.3 (No global Dirac reflection, [THM]). *Let $S \subseteq \mathcal{P}$ with $|S| \geq 2$, $\mathbb{Q}_S = \{r \in \mathbb{Q}_{>0}^\times : v_p(r) = 0\ \forall p \notin S\}$, $\mathbb{N}_S = \mathbb{Q}_S \cap \mathbb{N}$. There is no unitary or antiunitary $\mathcal{J}$ on $\ell^2(\mathbb{Q}_S)$ with $\mathcal{J}U_r\mathcal{J}^{-1} = U_r^*$ for all $r \in \mathbb{Q}_S$ and $\mathcal{J}\ell^2(\mathbb{N}_S) = \ell^2(\mathbb{N}_S)^\perp$. For $|S| = 1$ there is one (J_p). For $S = \mathcal{P}$ the formal product $\bigotimes_p J_p$ is not defined on the restricted product, since it would send the reference vector $\varepsilon_1 = \otimes\varepsilon_0$ to $\otimes\varepsilon_{-1}$, i.e. to "$\varepsilon_{1/\prod_p p}$".*

PROOF. Let $M = \ell^2(\mathbb{N}_S)$; it is invariant under U_p ($p \in S$) and its *joint wandering space* $M \ominus \overline{\sum_p U_p M}$ is $\mathbb{C}\varepsilon_1$. If $\mathcal{J}$ existed, $M' = \mathcal{J}M$ would be invariant under the U_p^* with joint wandering space $M' \ominus \overline{\sum_p U_p^* M'} = \mathcal{J}(\mathbb{C}\varepsilon_1) \neq 0$. But $M' = \ell^2(\mathbb{N}_S)^\perp$ is spanned by ε_r with some $v_p(r) < 0$, and $\sum_p U_p^* M'$ already contains every such ε_r: if $v_p(r) \leq -2$ for some p then $\varepsilon_r = U_p^*\varepsilon_{pr}$ with $\varepsilon_{pr} \in M'$; if $v_p(r) = -1$ pick $q \in S \setminus \{p\}$, then $\varepsilon_r = U_q^*\varepsilon_{qr}$ and $v_p(qr) = -1 < 0$, so $\varepsilon_{qr} \in M'$. Thus the wandering space of M' is 0, a contradiction. The last statement: $\otimes\varepsilon_{-1}$ is orthogonal to every finitely-supported product vector, hence not in the restricted product. □

REMARK 47.2.4 (what does exist globally, [THM]). $J^0 : \varepsilon_r \mapsto \varepsilon_{1/r}$ exists globally (it is $\otimes_p J_p^0$, fixing ε_1), and more generally $\varepsilon_r \mapsto \varepsilon_{c/r}$ for any fixed $c \in \mathbb{Q}_{>0}^\times$ (a permutation unitary with $U_r \mapsto U_r^*$ must be of this form: $\phi(rs) = \phi(s)/r$ forces $\phi(r) = \phi(1)/r$). None of these maps $\ell^2(\mathbb{N})$ onto its complement ($c/\mathbb{N}$ has bounded numerators, $\mathbb{Q}_{>0}^\times \setminus \mathbb{N}$ does not). Compare §47.4: on a curve the global reflection is $D \mapsto K - D$ with $c = K$ the canonical divisor.

47.2.1. Boson–fermion correspondence at one prime. Let $\mathcal{H}_p = \ell^2(\mathbb{Z})$ with one-particle energy $h_p = \log p \cdot D$, $D\varepsilon_k = (k + \frac{1}{2})\varepsilon_k$. Let $\mathcal{F}_p = \Lambda^{\infty/2}\mathcal{H}_p$ be the semi-infinite wedge (Dirac sea) with vacuum $|0\rangle = \varepsilon_{-1} \wedge \varepsilon_{-2} \wedge \cdots$, normal-ordered energy $H_p \geq 0$ (particles at $k \geq 0$ cost $(k + \frac{1}{2})\log p$, holes at $k < 0$ cost $(-k - \frac{1}{2})\log p$), charge $c = \#\text{particles} - \#\text{holes}$.

THEOREM 47.2.5 ([BIB] [**106**, **145**, **146**]; characters [NUM]). *(a) $\mathcal{F}_p = \bigoplus_{c\in\mathbb{Z}} \mathcal{F}_p^{(c)}$, and each $\mathcal{F}_p^{(c)}$ is isomorphic, compatibly with H_p up to the shift $\frac{c^2}{2}\log p$, to the bosonic Fock space $\mathbb{C}[\alpha_{-1}, \alpha_{-2}, \dots]|0\rangle$ of oscillators α_{-m} of energy $m \log p$ ($m \geq 1$), $\alpha_{-m} = \sum_k \psi_{k+m}\psi_k^*$. (b) With $x = p^{-\beta}$:*

$$\operatorname{Tr}_{\mathcal{F}_p^{(c)}} e^{-\beta H_p} = \frac{x^{c^2/2}}{\prod_{m\geq 1}(1 - x^m)}, \qquad \operatorname{Tr}_{\mathcal{F}_p} y^c e^{-\beta H_p} = \prod_{m\geq 1}(1 + yx^{m-\frac{1}{2}})(1 + y^{-1}x^{m-\frac{1}{2}})$$

(Jacobi triple product). (c) J induces the particle–hole conjugation $\mathcal{C}_p$ on $\mathcal{F}_p$: $\mathcal{C}_p\mathcal{F}_p^{(c)} = \mathcal{F}_p^{(-c)}$, $\mathcal{C}_p H_p \mathcal{C}_p^{-1} = H_p$.

The character identity in (b) was verified by enumerating all Maya diagrams with energy $\leq 12\log p$ in charges $|c| \leq 3$ ([NUM], `checks.json`: `true`), and the supertrace form at $x = 0.3$ to 20 digits. Item (c) follows from Prop. 47.2.2(d).

REMARK 47.2.6 (a level shift, [THM]). In the BC/primon gas, $\ell^2(\mathbb{N}_0)$ at the prime p is the Fock space of *one* boson of energy $\log p$ (occupation k). In Thm. 47.2.5, $\ell^2(\mathbb{Z}) \supset \ell^2(\mathbb{N}_0)$ is used as the *one-particle* space of a new second quantization. The original space reappears as the $m = 1$ oscillator subsector: $\varepsilon_{p^k} \mapsto (k!)^{-1/2}\alpha_{-1}^k|0\rangle$ is an energy-preserving isometry $\ell^2(\mathbb{N}_0) \hookrightarrow \mathcal{F}_p^{(0)}$ (norm: $[\alpha_1, \alpha_{-1}] = 1$). So the correspondence does not turn the primon bosons into fermions; it embeds them as the lowest current mode of a chiral Dirac fermion living on $\ell^2(\mathbb{Z})$.

47.2.2. All primes: partition functions and the supertrace. Put $\mathcal{F} = \bigotimes_p'(\mathcal{F}_p, |0\rangle)$, $H = \sum_p H_p$, $(-1)^F$ the total fermion parity (particles + holes), which on $\mathcal{F}_p^{(c)}$ equals $(-1)^c$.

THEOREM 47.2.7 (Partition functions, [THM]; [NUM]). *(a) Charge-0 sector: $Z^{(0)}(\beta) = \prod_{k\geq 1}\zeta(k\beta) = \sum_{n\geq 1} a(n)n^{-\beta}$ ($\beta > 1$), $a(n) = \prod_p \mathrm{p}(v_p(n)) =$ number of abelian groups*

of order n (OEIS A000688, Kendall's Dirichlet series, [BIB]). (b) Full trace: $Z(\beta) = \prod_{m\geq 1}\left[\zeta((m-\frac{1}{2})\beta)/\zeta((2m-1)\beta)\right]^2$ *($\beta > 2$). (c) Supertrace:* $\mathrm{Str}\, e^{-\beta H} = \prod_{m\geq 1}\zeta((m-\frac{1}{2})\beta)^{-2} = \prod_p \sum_{c\in\mathbb{Z}}(-1)^c p^{-\beta c^2/2}\Big/\prod_m(1-p^{-m\beta})$ *($\beta > 2$). (d) (Spectrum) H has pure point spectrum contained in $\frac{1}{2}\log\mathbb{N}$, with finite multiplicities on each $\mathcal{F}^{(c)}$-sector. Non-trivial zeros of ζ enter only through the meromorphic continuation in complex β: in $\mathrm{Re}\,\beta > 0$, $Z^{(0)}$ has zeros exactly at $\beta = \rho/k$ and poles at $1/k$; the supertrace has poles of order $2m_\rho$ at $\beta = 2\rho/(2m-1)$. No zero is an eigenvalue, resonance or spectral point of H, of any U_r, or of $\mathcal{C}$.*

Proof. Multiply Thm. 47.2.5(b) over p: $\prod_p\prod_m(1-p^{-m\beta})^{-1} = \prod_m\zeta(m\beta)$; $\prod_p(1+p^{-s}) = \zeta(s)/\zeta(2s)$; $\prod_p(1-p^{-s}) = 1/\zeta(s)$; set $y = 1$ or $y = -1$ (the latter is $(-1)^F$). Convergence of the restricted product: the per-prime factor is $1 + O(p^{-\beta/2})$ in (b),(c), $1 + O(p^{-\beta})$ in (a). (d) Eigenvalues are finite sums of $\frac{2j-1}{2}\log p$ and $m\log p$. Zeros/poles: each $\zeta(k\beta)$ is meromorphic, the tail $\prod_{k>K}\zeta(k\beta)$ is holomorphic non-vanishing where $\mathrm{Re}(K\beta) > 1$; non-real zeros of distinct factors cannot cancel against the real poles. [NUM]: $\prod_k\zeta(3k) = 1.22570470513$ vs. $\sum_{n\leq 2\cdot 10^5} a(n)n^{-3} = 1.22570470510$; supertrace at $\beta = 4$: 0.356326 (60 factors) vs. Euler product over $p < 2\cdot 10^4$: 0.356330; $a(1..32) =$ A000688. □

Remark 47.2.8 (relation to Lemma 45, [THM]/[ANALOGY]). Lemma 45's $Z_B = \zeta(\beta)$ is the $m = 1$ factor of (a) (Rem. 47.2.6); Lemma 45's Spector grading $(-1)^F = \mu$ lives on the squarefree cube of $\ell^2(\mathbb{N})$, while the grading here lives on Maya diagrams. The half-integer energies $(m-\frac{1}{2})\log p$, i.e. multiplicative weights $p^{-1/2}, p^{-3/2}, \ldots$, are forced by the Fermi level $-\frac{1}{2}$ of J. That the Dirac sea "knows" a $\sqrt{p}$ is [THM]; that this is the $\sqrt{p}$ of Frobenius normalization is [ANALOGY] only: the supertrace poles sit at $2\rho/(2m-1)$, a rescaling of the zero set, not its placement on a line.

47.3. Toeplitz versus Hankel: the decision

Let G be $\mathbb{Q}_{>0}^\times$ (or $\mathbb{Q}_S$, or $\mathbb{R}_{>0}$ in Lemma 42). A sesquilinear form B on finitely supported functions on G (or on a sub-semigroup Σ with $\Sigma^{-1}\Sigma = G$) is *Toeplitz* if $B(\varepsilon_r, \varepsilon_s) = w(s/r)$ and *Hankel* if $B(\varepsilon_r, \varepsilon_s) = h(rs)$. On $\ell^2(\mathbb{N}_0) = H^2$ these are the classical notions ($S^*TS = T$ resp. $S^*\Gamma = \Gamma S$, [49] [BIB]).

Theorem 47.3.1 (Reflection converts, and the intersection is the fermionic cube, [THM]). *(a) If B is Toeplitz with kernel w and $\mathcal{J}_c\varepsilon_r = \varepsilon_{c/r}$, then $B(\cdot, \mathcal{J}_c\cdot)$ is Hankel with $h(u) = w(c/u)$, and conversely. Compression to $\ell^2(\Sigma)$ preserves both types. Hence every form obtainable from a U-invariant form by finitely many reflections $\mathcal{J}_c$ and compressions is Toeplitz (even number of reflections) or Hankel (odd number). (b) Let $\Sigma \subseteq G$ be a sub-semigroup with $1 \in \Sigma$ and $\Sigma^{-1}\Sigma = G$ (e.g. $\Sigma = \mathbb{N}$, $G = \mathbb{Q}_{>0}^\times$). If $w(s/r) = h(rs)$ for all $r, s \in \Sigma$, then $h = w$ on Σ and $w(u\,t^2) = w(u)$ for all $u \in G$, $t \in G$: w factors through G/G^2. For $G = \mathbb{Q}_{>0}^\times$, $G/G^2 \cong \bigoplus_p \mathbb{Z}/2$, represented by the squarefree integers — Lemma 45's fermionic (Pauli) cube. Conversely every such w is both Toeplitz and Hankel, and the form is $\sum_\chi c_\chi\,\chi(r)\chi(s)$ over the quadratic characters χ of $\mathbb{Q}_{>0}^\times$ ($\chi \in \widehat{G/G^2} = \prod_p \mathbb{Z}/2$): a mixture of rank-one forms, positive iff $c \geq 0$. (c) For $G = \mathbb{R}_{>0}$ every element is a square, so Toeplitz ∩ Hankel = constants (this is the mechanism of Lemma 42 Thm. 2.2, [CITED] internal). (d) (Weil) Weil's explicit-formula form is a Toeplitz form on $\mathbb{R}_{>0}$ with non-constant kernel; by (c) it is not Hankel, and by (a) it is not produced by an odd number of reflections from any U-invariant form. Its finite-prime part on $\mathbb{Q}_{>0}^\times$, $w_f(n^{\pm 1}) = -\Lambda(n)n^{-1/2}$, is not square-periodic ($w_f(p) = -\log p/\sqrt{p} \neq w_f(p^3) = -\log p/p^{3/2}$), so by (b) it is not Hankel on $\mathbb{Q}_{>0}^\times$ either.*

Proof. (a) $B(\varepsilon_r, \mathcal{J}_c\varepsilon_s) = w(c/(rs))$. (b) $r = 1$: $h = w$ on Σ. Replacing (r, s) by (rt, st), $t \in \Sigma$: $h(rst^2) = w(s/r) = h(rs)$. For $u = s/r \in G$ and $t \in \Sigma$, $w(ut^2) = w(st^2/r) = h(rst^2) = h(rs) =$

$w(u)$. For $t = a/b \in G$: $w(ut^2) = w((u/b^2)a^2) = w(u/b^2) = w(u)$. So w is G^2-periodic. The converse and the character expansion are Pontryagin duality for the elementary abelian 2-group G/G^2, with $\chi(s/r) = \chi(s)\chi(r)$ for $\chi^2 = 1$. (c),(d) as stated. □

REMARK 47.3.2 ([NUM]). A linear-algebra check on the box $\{-3..3\}^2$ of exponents for $p = 2, 3$ confirms that every solution of $w(s - r) = h(s + r)$ is constant on the four parity classes (`checks.json`).

REMARK 47.3.3 (Liouville is the canonical Toeplitz–Hankel kernel, [THM] / [RESTATEMENT]). $\lambda(r) = (-1)^{\sum_p v_p(r)}$ is a quadratic character of $\mathbb{Q}^\times_{>0}$, so $(r, s) \mapsto \lambda(s/r) = \lambda(r)\lambda(s)$ is simultaneously Toeplitz, Hankel and J-even, of rank one. Its Dirichlet series is $\zeta(2s)/\zeta(s)$. If $L(x) = \sum_{n\le x} \lambda(n) = O(x^{1/2+\varepsilon})$ then $\zeta(2s)/\zeta(s)$ is holomorphic in $\operatorname{Re} s > \frac{1}{2}$, so ζ has no zeros there ([THM] by partial summation); the converse is the standard Perron-formula argument used for $M(x)$ in Lemma 45 Thm. 3.2, not reproved here ([RESTATEMENT]). So the reflection-invariant kernels carry RH only as a *growth* statement on a rank-one object, exactly as Möbius did in Lemma 45 — not as positivity of a form.

COROLLARY 47.3.4 (verdict on task (c), [THM]). *No form produced by the dilation–reflection–boson/fermion construction equals Weil's form unless Weil's Toeplitz kernel is inserted by hand before reflecting (then the reflection is idle). The reflection-produced forms are Hankel (reflection positivity / Osterwalder–Schrader type; Lemma 42 Thm. 2.1), and the overlap with Toeplitz is the square-periodic (fermionic-cube) class, which excludes Weil's kernel. Lemma 42's pincer is therefore reproduced, not escaped, and refined to the discrete group $\mathbb{Q}^\times_{>0}$.*

47.4. The finite-field test

Let $C/\mathbb{F}_q$ be smooth projective geometrically connected of genus g, K a canonical divisor, $Z(C, T) = P(T)/((1 - T)(1 - qT))$, $P(T) = \prod_{i=1}^{2g}(1 - \alpha_i T)$.

PROPOSITION 47.4.1 (the $\mathbb{N}$-tower and its dilation, [BIB] [**237**] Ch. 5). *(a) $\ell^2(\mathrm{Div}^+ C) = \bigotimes'_x \ell^2(\mathbb{N}_0)$ over closed points x is the bosonic Fock space; with $H = \deg \cdot \log q$, $\operatorname{Tr} T^{\deg} = \sum_{D\ge 0} T^{\deg D} = Z(C, T)$. (b) Its minimal dilation is $\ell^2(\mathrm{Div}\, C) = \bigotimes'_x \ell^2(\mathbb{Z})$, joint spectrum $\prod_x \mathbb{T}$ (Haar), exactly as in Thm. 47.1.2. (c) The per-point Dirac construction of §47.2 applies verbatim with $N(x) = q^{\deg x}$; its charge-0 partition function is $\prod_{k\ge 1} Z(C, T^k)$, and no α_i is an eigenvalue of anything (same proof as Thm. 47.2.7(d)).*

So the analogue of the chat's construction, performed literally, does *not* produce H^1 for a curve either. Yet H^1 is recoverable from divisor (H^0-type) data; here is how.

THEOREM 47.4.2 (Recovering H^1 with its positive form from divisors, [BIB] standard; extra inputs isolated [THM]). *(i)* Dilation is needed. $\mathrm{Div}^0(C) = \ker(\deg : \mathrm{Div}\, C \to \mathbb{Z})$ *consists of divisors with negative coefficients; it is invisible on* Div^+. *(ii)* Quotient by the function field. $\mathrm{Pic}^0(C) = \mathrm{Div}^0(C)/\operatorname{Prin}(C)$, $\operatorname{Prin} = \{\operatorname{div} f : f \in \mathbb{F}_q(C)^\times\}$, *is $J(\mathbb{F}_q)$, of order $h = P(1) = \prod(1 - \alpha_i)$; over $\mathbb{F}_{q^n}$, $\#J(\mathbb{F}_{q^n}) = \prod_i(1 - \alpha_i^n)$, and $T_\ell J$ is the ($\mathbb{Z}_\ell$-dual of) H^1, with F acting by α_i. (iii)* Positivity *is not in (i)–(ii): it comes from the principal polarization (theta divisor, ample) and Rosati positivity (Lemma 46 §1, MilpRH 1.27, [CITED]). (iv)* The global reflection *on* $\mathrm{Div}\, C$ *is $D \mapsto K - D$; Riemann–Roch $\ell(D) - \ell(K - D) = \deg D + 1 - g$ turns it into the functional equation $Z(C, 1/(qT)) = q^{1-g}T^{2-2g}Z(C, T)$, i.e. $\alpha \mapsto q/\alpha$, Lemma 46's V (Prop. 6.3): Krein, not Hilbert. Its centre is degree $g - 1$, i.e. $|T| = q^{-1/2}$.*

COMMENTS. (i) is definitional. (ii)–(iv) are standard ([**237**] Ch. 5: $Z = \sum_{D\ge 0} T^{\deg D}$, the class-number formula and the FE via Riemann–Roch). What is proved here is the bookkeeping: among the four steps (dilate, quotient, polarize, reflect), only "polarize" carries positivity, and "reflect" is

J.W.'s reflection in its globally existing form $c = K$ (compare Thm. 47.2.3, where $c = \prod_p p$ does not exist). □

PROPOSITION 47.4.3 (numerical test of (ii), [NUM] `num/lemma47_ff.py`, `l47/ff.json`). *For $y^2 = f(x)$, $\deg f = 5$, $f \in \{x^5 + x + 1,\ x^5 + x^2 + 2,\ x^5 + x^3 + 3x + 1\}$ over $\mathbb{F}_q$, $q \in \{5, 7, 11, 13\}$: in all 9 smooth cases $h = P(1)$ computed from point counts N_1, N_2 ("H^0" data) equals $\#\operatorname{Pic}^0$ counted independently by enumerating reduced Mumford pairs (u, v); the effective-divisor counts satisfy $A_n = h(q^{n-1} - 1)/(q - 1)$ for $3 \le n \le 11$; and all reciprocal roots have modulus $\sqrt{q}$. E.g. $q = 13$, $f = x^5 + x^3 + 3x + 1$: $P = 1 + T + 18T^2 + 13T^3 + 169T^4$, $h = 202 = \#\operatorname{Pic}^0$. Three singular choices ($\gcd(f, f') \neq 1$) were detected and excluded (two give $P(1) \neq$ Mumford count, as expected for a generalized Jacobian).*

THEOREM 47.4.4 (Transfer to $\mathbb{Q}$: where the mechanism degenerates, [THM] for (a), [BIB]/[CITED] for (b)). *(a) $\operatorname{Div}(\operatorname{Spec}\mathbb{Z}) = \mathbb{Q}^\times_{>0}$ and $\operatorname{Prin} = \{\operatorname{div}(r) : r \in \mathbb{Q}^\times\} = \mathbb{Q}^\times_{>0}$, so the analogue of step (ii) is $\mathbb{Q}^\times_{>0}/\mathbb{Q}^\times_{>0} = 0$: class number one, no finite Jacobian, no finite H^1. (b) With the archimedean place adjoined, the idele class group is $C_\mathbb{Q} = \mathbb{A}^\times/\mathbb{Q}^\times \cong \hat{\mathbb{Z}}^\times \times \mathbb{R}_{>0}$; its norm-one part $\hat{\mathbb{Z}}^\times$ is compact and its characters are Dirichlet characters (giving L-functions, not zeros). Connes [75] therefore performs the quotient at the level of* function spaces *on the adele class space, and the zeros appear as an absorption spectrum (cokernel), with RH equivalent to the validity of a positivity/trace formula; CCM09 [66] make the cokernel the analogue of H^1 [CITED] (carried from Lemmas 30, 42).*

REMARK 47.4.5 (answer to task (d)). Yes for curves, with mechanism: *dilate* ($\operatorname{Div}^+ \to \operatorname{Div}$, J.W.'s step), *quotient* by principal divisors (the step that creates H^1), *polarize* (the step that creates positivity); the global *reflection* $D \mapsto K - D$ supplies only the FE. For $\mathbb{Q}$ the dilation and the reflection survive, the quotient collapses at the group level, and positivity has no ample class to come from. This is a sharper form of Lemma 46 Problem 6.8: the missing $\mathcal{H}^1$ must be a *cokernel at the function-space level* carrying a positive form, not a reflected copy of H^0.

47.5. The 180° rotation and the zeros

PROPOSITION 47.5.1 (half-turns, [THM]). *(a) $k \mapsto -k$ on $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$ is the rotation of $\mathbb{C}$ by π about 0 (multiplication by $e^{i\pi}$), equivalently the point reflection; $k \mapsto -1 - k$ is the half-turn about $-\frac{1}{2}$. Multiplicatively these are $r \mapsto 1/r$ and $r \mapsto 1/(pr)$ (Prop. 47.2.2). (b) (Mellin conventions; [NUM] `l47/checks.json`, `l47/mellin_riemann.txt`.) With $\mathcal{M}f(s) = \int_0^\infty f(x)x^{s-1}dx$, the unitary inversion on $L^2(\mathbb{R}_{>0}, dx)$ is $(Rf)(x) = x^{-1}f(1/x)$ and $\mathcal{M}(Rf)(s) = \mathcal{M}f(1 - s)$ (unitary line $\operatorname{Re} s = \frac{1}{2}$). In the same convention $x^{-1/2}f(1/x)$ gives $s \mapsto \frac{1}{2} - s$. In Riemann's normalization $\int_0^\infty f(x)x^{s/2}\,dx/x$, the map $f \mapsto x^{-1/2}f(1/x)$ gives $s \mapsto 1 - s$, and $\theta(1/x) = \sqrt{x}\,\theta(x)$ is its fixed-point property. (Checked with $f = e^{-x}$: $\Gamma(1 - s)$, $\Gamma(\frac{1}{2} - s)$, $\Gamma(\frac{1-s}{2})$ resp.) (c) $s \mapsto 1 - s$ is the rotation by π about $\frac{1}{2}$: $(1 - s) - \frac{1}{2} = -(s - \frac{1}{2})$. It maps the critical line to itself, $\frac{1}{2} + i\gamma \mapsto \frac{1}{2} - i\gamma$; there it coincides with complex conjugation.*

PROPOSITION 47.5.2 (rotation versus mirror, [RESTATEMENT]). *The zero multiset $\mathcal{Z}$ is invariant under the half-turn $\rho \mapsto 1 - \rho$ (FE) and under $\rho \mapsto \bar{\rho}$ (reality), hence under their composite $\rho \mapsto 1 - \bar{\rho}$, the mirror in the critical line. Pointwise, $1 - \rho = \bar{\rho} \iff \operatorname{Re}\rho = \frac{1}{2}$. Hence RH $\iff$ the half-turn agrees with conjugation on every zero $\iff$ the mirror fixes every zero. For curves ($\alpha = q^\rho$) this is Lemma 46 Prop. 6.3: $V : \alpha \mapsto q/\alpha$ is the half-turn, c_∞ is conjugation, and RH $\iff V = c_\infty$ on the spectrum.*

REMARK 47.5.3 (not the precession; stage, not actors). (i) The half-turn is an *involution*, not the one-parameter group $x^{i\gamma} = e^{i\gamma \log x}$ (the "precession" of a zero); on the line it acts as $\gamma \mapsto -\gamma$, a time reversal of the precession (Lemma 42's flow-reversing ♯). It pairs zeros; it does not move them.

(ii) The dilated unitaries have spectrum the whole circle (each U_n) and the torus $\mathbb{T}^{\mathcal{P}}$ (jointly), *not* "the critical line": the line is the dense Haar-null winding $\iota(\mathbb{R}) \subset \mathbb{T}^{\mathcal{P}}$ (Prop. 47.1.5). Nothing in the spectral data of $\{U_r\}$ singles out the ordinates γ; they must be selected by an extra structure (Connes' absorption spectrum), and that is where positivity is again required (Thm. 47.4.4). So the chat sentence "spectrum the whole circle/critical line" is corrected to "circle per operator, torus jointly, critical line as a null dense winding".

47.6. Corrections to the chat claims

(C1) *Wold / Sz.-Nagy dilation $S \to$ bilateral shift*: correct (Prop. 47.1.1). *BC isometries dilate to unitaries on $\ell^2(\mathbb{Q}^\times_{>0})$, $\mathbb{Q}^\times_{>0} = \bigoplus_p \mathbb{Z}$*: correct and minimal (Thm. 47.1.2). Precision: Laca's dilation of the *full* BC system is $C_0(\mathbb{A}_f) \rtimes \mathbb{Q}^\times_{>0}$ (finite adeles), Rem. 47.1.3; "Laca dilation theory, to be verified" is now verified (JLMS 61 (2000) 893–904).

(C2) *$JSJ = S^*$, particle–hole, Dirac sea*: correct per prime. Correction: there is no global complementary reflection for ≥ 2 primes (Thm. 47.2.3); the global object is a restricted product of per-prime seas.

(C3) *"Bosonic Fock $\cong$ fermionic Fock" via the semi-infinite wedge*: correct sector by sector ($\mathcal{F}^{(c)} \cong$ bosonic Fock, Thm. 47.2.5); but it second-quantizes *again*: the primon gas is only the $m = 1$ subsector (Rem. 47.2.6). It does not make the BC operators fermionic on H^1; zeros never become spectral (Thm. 47.2.7).

(C4) *Toeplitz within halves, Hankel across the reflection; Lemma 42 obstruction*: correct and sharpened: Toeplitz $\cap$ Hankel = square-periodic kernels (fermionic cube), excluding Weil (Thm. 47.3.1, Cor. 47.3.4).

(C5) *Connes: zeros as absorption spectrum/cokernel, RH still a positivity*: correct (carried [CITED]); refined by the finite-field comparison: the cokernel replaces the collapsed Jacobian quotient (Thm. 47.4.4).

(C6) *Mellin: $x^{-1/2}f(1/x)$ sends $s \mapsto 1 - s$*: correct only in Riemann's theta normalization $\int f x^{s/2} dx/x$; in the convention $\int f x^{s-1} dx$ the correct map is $x^{-1}f(1/x)$ (Prop. 5.1(b)).

(C7) *"The dilated spectrum is the whole circle/critical line"*: circle per operator, torus jointly; the critical line is a dense Haar-null winding (Prop. 47.1.5).

(C8) *Half-turn = FE = Lemma 46's V, not the precession; RH $\iff$ half-turn = conjugation on every zero*: correct (Prop. 47.5.2, [RESTATEMENT]).

47.7. Open questions

Q1. *Standing principle.* The reflection has now appeared three times and each time gives the functional equation (Krein), never positivity: Lemma 42 (Hankel/OS), Lemma 46 (V = FE), here (Riemann–Roch $D \mapsto K - D$; square-periodic forms). Adopt "reflection $\Rightarrow$ FE, positivity needs a polarization" as a Volume 2 standing theorem?

Q2. *Quotient, not reflection.* For curves H^1 is born from $\mathrm{Div}^0 / \mathrm{Prin}$ (the Jacobian); for $\mathbb{Q}$ that quotient is 0. Shall we pursue $\mathcal{H}^1$ as a function-space cokernel (Connes/CCM) and ask what positive form it would need, i.e. restate Problem 6.8 as "a cokernel with a polarization"?

Q3. *The Fermi level $-\frac{1}{2}$.* The Dirac sea forces weights $p^{-1/2}$ per prime. Do you want a test of whether this $\frac{1}{2}$ can be tied rigorously to the $\sqrt{p}$ Frobenius normalization (e.g. for $L(E, s)$ with local factors $1 - a_p T + pT^2$), or keep it as an analogy?

Q4. *Liouville.* Include "reflection-invariant kernels = quadratic characters; RH $\iff$ $L(x) = O(x^{1/2+\varepsilon})$" as the fermionic-cube restatement in Volume 2?

Q5. *Next step.* Finite-field "Connes cokernel" test: realize $H^1(C)$ as a cokernel of a map between function spaces on the adele class space of $\mathbb{F}_q(C)$ and identify where the theta polarization enters. Proceed?

APPENDIX A

Corrections

This appendix collects every errata or correction list of Chapters 30–47, as each note records it, with a pointer to the place. Chapter texts were not edited. Where a later chapter corrects an earlier one, the earlier chapter carries a **[Vol. 3 editorial note]** footnote at the corrected statement whenever the anchor is unambiguous. Item letters and numbers are those of the notes. Nothing is added beyond what the notes state, except Section D, which records the decisions taken in assembling the volume.

A.1. Errata within a chapter (task premises, earlier versions, citations)

Ch. 31, Lemma 31.3.2.: $\langle \xi, J\xi \rangle \geq 0$ is false in general (it is not even real); the correct positivities are $\langle \xi, \Delta^{1/2}\xi \rangle \geq 0$ and self-duality of the natural cone.

Ch. 32, remark "correction to the task's expectation".: $\mathcal{Q}$ is exactly translation invariant; the archimedean and prime terms break the *sign* of the OS form, not invariance.

Ch. 33, Thm. 33.4.1.: La05c repairs Lemma 3.1 and a sign in (3.8) of La99; Lagarias' printed g equals $-m$ (convention only).

Ch. 34, (C1)–(C4): (corrections to the task premise, §34.1): $E_{1/2}$ with $\nu = 1$ violates $\nu\omega > 1$, so $(E_{1/2})^3$ is used; the string mass is positive wherever defined, and the RH content is existence for all t; the Kaltenbäck–Woracek series is in two journals; the midpoint-Nyström code is superseded.

Ch. 34, (V1)–(V8): (v2 revision note, §34.2): official G5 is $\|\mathcal{K}[t]\| < 1$ for every t, not $\sup < 1$ (V1); onset exponent from the trace term, Conj. 34.8.2 replaces Conj. 34.5.2 (V2); precision estimate (V3); "certified" means Ritz-sound [NUM] (V4); onsets $n \leq 20$ resolved (V5); hypothesis (M) removed by Ch. 36 (V6); Ch. 37's Table 1 too late for $t \gtrsim 0.8$ (V7, see B and D); the $A = 2t$ remark (V8).

Ch. 37, (K8)–(K15) and (v3.1)–(v3.4): (§37.14 and the v3 front section). v1 Table 1 margins too large for $t \gtrsim 0.8$, with corrected values (K8); "every break is a short" only for first exits (K9); $(0.3, 4)$ vs $(0.2, 6)$ differ by $1.6\text{–}4.2 \cdot 10^{-4}$ (K10); Iwasawa's Γ is $\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}(\mu_p))$ (K11); the margin plateau was an artefact, and the "large gaps" mechanism is void (K12); $r30p12$ not converged at $P = 97$, so the decay-slowing question is open (K14, v3.1, v3.2); $7.3 \cdot 10^{-7}$, not $8 \cdot 10^{-7}$ (K15, v3.4); data finished after v2, no verdict changed (v3.3).

Ch. 38, citation flags (i)–(ix): (§38.10): uniqueness at $\beta = 1$ is CM08 Thm. 3.32, not BC95 Thm. 5(a) (i); σ_t is dual to scaling (ii); the chat claim "Iwasawa generator = p-adic rapidity" (iii; Correction 38.2.6); "reality = Galois complex conjugation" (iv; Prop. 38.8.3(c)); Washington Thm. 7.10 and Greenberg (6)–(7) used only through secondary renderings, [CITED, secondary] (v, vi); Lemma 4's byline (vii; see C); Ch. 37 Rem. 7.6 imprecise (viii; fixed by K11); CC21 title verified (ix).

Ch. 45, "Corrections to the setting" (a)–(c).: Pauli exclusion is about fermions; BEC is low-temperature and the primon gas has no condensation; the UV end is the Hagedorn pole $\beta = 1$.

A.2. Corrections between chapters, and corrections of chat claims

Ch. 36, (a)–(h): (§36.11). (a)–(b) and (e)–(f) concern Volumes I/II (Section C). (c) Lemma 31's byline: superseded, since the source of Ch. 31 now carries the standard byline. (d) Ch. 33's Bode/Fano question is answered by Thm. 36.4.3 (footnote in Ch. 33). (g) Ch. 34 Thm. 5.4(ii) holds with equality and (iv) without (M), Thm. 36.7.9. (h) Ch. 37's "every break is a short" holds for first exits only (footnote in Ch. 36).

Ch. 37, (K1)–(K7).: $\sup_t < 1$ is false in the passive case (K1, against Ch. 34); local Euler sections stable but not passive (K2); $\frac{1}{2}\log 13$ (K3); Ch. 34's question on t_*^{arch} answered (K4); $A = 2t$ only as an information horizon (K5); a Ch. 34 heuristic unproved (K6); Ch. 33 Thm. 6.2(b) strengthened (K7). (K13): Ch. 39 §5's margin pattern fails on the corrected data (footnote in Ch. 39).

Ch. 38, Corrections 38.2.6 and 38.8.2.: "The Iwasawa group is the p-adic rapidity group" is false inside $C_{\mathbb{Q}}$. The KMS_1 of Ch. 31 Thm. 6.5 is related to the BC critical temperature only shallowly.

Ch. 39, (K1)–(K4).: Relative entropy is invariant, not monotone, along the state's own modular flow; "line time = modular time" is an analogy; BC is a tensor product only on its Galois-invariant part; the margin is not shown to be a relative-entropy difference.

Ch. 40, (E1)–(E10): (§40.12): chat claims on complex conjugation vs. Galois -1 (E1), pairs vs. quadruples (E2), the missing Rosati involution (E3: the involution exists; its positivity is missing), Π_κ (E4, sharpened), Frobenius as a Galois element (E5); naming J vs. C (E6); the Hodge analogy refined (E7); the Lagrange Gram determinant is constant (E8); the returning branch identification is [ANALOGY] (E9); natural-cone positivity is not Rosati (E10; scope refined by Ch. 42 E3).

Ch. 41, (C1)–(C5): (§41.10): the limit is $h \to \infty$ (C1); Ch. 40's K1 sharpened, $\delta_c \simeq (\kappa/2\beta)^{1/3}$ (C2; a refinement, not an error); Lemma 27's remark refined (C3); the chat claim "coincidence would constrain ζ" fails, since coincidence is a theorem for all reflection-symmetric configurations (C4); Lemma 27 Thm. 9.5 is a special case (C5).

Ch. 42, (E1)–(E12): (§42.14). (E1)–(E2): HSMI positivity is Hankel, and the construction is impossible unconditionally. (E3): Ch. 40 Prop. 6.10 needs the scope "when the scaling flow is the modular flow" (footnote at Prop. 40.6.10). (E4)–(E6): Ch. 39 R3, Ch. 31 Prop. 6.3, Ch. 39 Thm. 7.5(i) and Ch. 40 Def. 6.12 refined or subsumed by Thm. 42.3.1. (E7)–(E8): steering on Ramanujan and Euclidean periodicity. (E9)–(E12), v2: the symplectic argument is empty in rank 2; the Jordan-block boundary never occurs arithmetically; "Katz–Mazur" is Katz's conjecture (Mazur, Ogus), an integrality with slope labels swapped; the max-plus purity error is $\leq \log_p 2$.

Ch. 43, (a)–(f): ("Corrections to chat claims"): only simple on-line zeros are centers; off-line pairs can both be centers, so "RH $\iff$ all centers" is false; the reversible field over $\mathbb{F}_q$ is $u \cdot h$; Prop. 2.8 is anti-holomorphic; no evidence for a link to Ch. 36's integrality; game-theory steering corrected.

Ch. 44, C1–C11: (§44.7). Correct: Hasse = cavity stability (C1); θ_p = Gouy phase (C2); beam form = Ch. 42's form (C3); Euler factor = Planck mode (C8); BC Galois action (C9); Berry–Keating hyperbolic as their reading (C11). Half right or wrong: Boyd–Gordon modes (C4); Fox–Li loss (C5); depth of focus and Lemma 21 (C6, false); Hecke T_{p^k} is not k round trips (C7); ghost mode energies are log-moduli, not energies (C10).

Ch. 45, (0a)–(5): (§45.7): attributions (0a); no condensation (0b); UV end and null part (0c); Bost–Connes analogy (0d); two second quantizations (1); Spector's question and Titchmarsh 14.25 (2); spin characters (3); Zeeman is an analogy, partly inverted (4); the Pauli test answered, the Krein violation a reformulation (5).

Ch. 46, E1–E9.: $n_+ = 1 + k$, not "index 2" (E1); Rosati isometry by algebra (E2); Deligne's positivity is real-square positivity (E3); Katz–Sarnak presupposes purity (E4); the Zeeman fan is archimedean (E5); $F/\sqrt{q}$ is the isometry, Krein without positivity (E6); V realizes the FE, not $\sharp$ (E7); Connes–Consani $\sigma_n \tilde{\rho}_n = n$ verified (E8); fakes have Dieudonné structure (E9).

Ch. 47, C1–C8.: The dilation is correct and minimal, and Laca is verified (C1); no global Dirac reflection (C2); boson–fermion second-quantizes again (C3); Toeplitz ∩ Hankel is the fermionic cube (C4); Connes' cokernel (C5); the Mellin normalization $x^{-1} f(1/x)$ (C6); circle, torus and winding (C7); the half-turn is V, not the precession (C8).

A.3. Items that touch Volumes I and II

Reported here only; the earlier volumes and their sources were not changed.

C1. Ch. 36(a): "$A \leq 2.8$ certified" (Lemma 24 thread) is the middle-band sampling inequality, not Weil positivity on a window of length 2.8; certified window positivity stands at $A = 1.6$. Ch. 36(b): the hybrid Galerkin minorant fails at $A = 1.6$ and 2.4 as a lower bound only.

C2. Ch. 36(e)–(f): Lemma 14's "+1 from the pole at $s = 1$" is shared between s and $s - 1$; its "Maslov $-\frac{1}{8}$" is a Stirling phase.

C3. Ch. 41 C3, C5: Lemma 27's canonical remark refined; Lemma 27 Thm. 9.5 is a special case of Thm. 41.4.1.

C4. Ch. 44 C6: Lemma 21's limit $A_{\text{crit}} = 2.113$ is not a depth-of-focus effect.

C5. Ch. 38 (vii): Lemma 4's byline reads "TAA working notes (...)" (checked in the Lemma 4 source); not edited.

A.4. Collation inconsistencies and their handling

D1. *"Volume 2" in the questions.* Chs. 31, 40, 42 and 44 speak of material for "Volume 2". They were written before the split of Lemmas 14–47 into two volumes, and meant the next volume. Under the split that is this volume (Vol. III). Handled by footnotes at each mention.

D2. *Superseded cascade values.* Chs. 34 (V7) and 37 (K8) correct the v1 Table 1 of Lemma 37. Chapter 37 prints the corrected table; Ch. 39 §5 still uses the old values, and its prediction fails (Ch. 37 K13, M6.5). Footnotes in Chs. 34 and 39.

D3. *Logged certificates.* Ch. 43 quotes "certified values" from the Lemma 24 progress log. Volume II lists these as logged, write-up pending; this volume does not count them as theorems (Statement, Part 1c). Footnote in Ch. 43.

D4. *Lemma 37 Rem. 7.6 flag.* Ch. 38 (viii) flags it; Ch. 37 v2 (K11) fixed it. Footnote in Ch. 38.

D5. *Pending run.* Ch. 37 says an $r42p16$ run is queued; it has since started and is still running. Footnote in Ch. 37 pointing to placeholder P1 (Appendix B).

D6. *Macro clashes.* The notes define a few macro names with different meanings: \tagO is OPEN, OPEN≡RH in Ch. 30, and OBSERVATION in Ch. 38; \tagP is PARTIAL in Ch. 36 and PRIOR ART in Ch. 43; \Qp is $\mathbb{Q}_{>0}^{\times}$ in Ch. 47 and $\mathbb{Q}_p$ in Ch. 38; \Cp is C_+ in Ch. 37 and $\mathbb{C}_p$ in Ch. 38. Other names differ only typographically (\Qform, \Ft, \Frob, \OSa, \Zh, \Zp, \Zhs, \kr) or in convention (\Np, with multiplicity in Ch. 34 and distinct zeros in Ch. 40). Handled by renaming inside the outlier chapter only, so every printed symbol is unchanged (Appendix C).

D7. *Typography.* The volume macro file prints Re, Im as upright "Re", "Im". Chapters 34, 37, 39, 40 and 42 originally printed $\mathfrak{R}$-style symbols; the meaning is unchanged. Conjecture environments are set in one style throughout.

D8. *Typesetting repairs that change no text.* (a) Ch. 39 writes $p^{+-1/2}$ with p^+ already superscripted, which gives a double superscript (the error is also in the standalone log). The

volume braces p^+. (b) Ch. 35's long tables carry their label in a repeated head, which gives a duplicate label (also in the standalone note). The volume labels only the first head.

D9. *Citation keys.* Different notes use one key for different works: CC14 is the arithmetic site in most chapters and the p-adic BC system in Ch. 38; CCM09 is the Weil-proof paper in most chapters and "Fun with $\mathbb{F}_1$" in Ch. 38. Ch. 38's keys become CC14bc and CCM09fun. Aliases merge one work cited under several keys (Bo/Bom00→Bo00, Yo→Yos92, Weil48a→Weil48, BK99S→BKsiam, DH→DH36, Tit86→Tit). Ch. 42 (and a few places in Chs. 39, 45 and 47) names works by their key in running text without a citation command (e.g. "Maz72, Maz73, BO78"). These works are kept in the bibliography so that each such key can be looked up there.

D10. *Bibliographic data.* Ch. 30's entry for Grieve (NYJM 23, 2017) misnames the author "Grigg"; corrected from the journal page. Davenport's *Multiplicative Number Theory* is cited as "3rd ed." with the DOI of the 2nd edition (1980); the entry gives the 2nd edition, matching the DOI. Titchmarsh is merged as the 2nd ed. (1986). Volume II has the same Davenport entry ("3rd ed., 2000" with the 1980 DOI); reported, not changed. Details in `bib_verification.md`.

D11. *Bylines.* Ch. 30's standalone byline reads "TAA working notes (J. W. McGreevy; formalization by Grok Bot)". Ch. 36(c) flags the same for Ch. 31, which now carries the standard byline. Chapter bylines are not printed in this volume.

D12. *Excluded material.* The working note on the author's skin-effect papers is not a numbered lemma and is not included. Lemmas 48 and later are not included, apart from the editorial addition from Lemma 50 in the Introduction.

D13. *Ch. 38 abstract.* Ch. 38 has no abstract environment; its section "Abstract and verdict" serves as one.

D14. *Ch. 40 K1.* Sharpened by Ch. 41 C2. The item sits in a table, so it gets no footnote.

APPENDIX B

Pending computations

One computation was still running when this volume was built. Its results appear in the volume only as placeholder macros, defined in the single file `pending.tex`. A placeholder prints as [**pending: label**]. **Rule:** no value is invented, extrapolated or "expected". A placeholder is replaced only by the number read from the finished run's output, and the matching status line is then updated. Until then the statements that depend on it stay open, as Chapter 37 says (v3.2, (M6.2)–(M6.3)).

Id	What	Placeholder	Where used
P1	Ch. 37, M6: cascade margin $t_*(P) - \frac{1}{2}\log p^+$ at $(\omega,\nu) = (\frac{1}{2},3)$, resolution $r42p16$ (step 0.0015, 480 bits)	$P = 97$: [**pending: P1,** $P = 97$] $P = 67$ (control): [**pending: P1,** $P = 67$]	Ch. 37, v3.2 and (M6.2) (footnote); Statement, Part 2 (S1)

Context (from Ch. 37 v3, [NUM]). At $P = 97$ the $r24$, $r30$ and $r36$ margins are 0.0054979, 0.0046952 and 0.0043502, so $r30$ is not converged there. At $P = 67$, $r30$ and $r36$ agree to $7.3 \cdot 10^{-7}$. The $r42$ run at $P = 97$ decides whether $r36$ is converged, and with it the decay-slowing question; $P = 67$ is the control.

Status. running: the $P = 97$ run started on 22 Oct 2026 after the Lemma 24 queue finished; $P = 67$ follows (status file `num/l37/m6h42_wait.status`). Launcher `num/l37/run_m6h42_wait.sh`. Output `num/l37/m6_w0.5_nu3_r42p16_s0.0015.jsonl` (key `margin`), which does not yet exist. The runs are sequential, $P = 97$ first. Each start requires ≥ 4.5 GB of available memory. Volume II listed this run as P4; it has moved here because Lemma 37 is in this volume.

APPENDIX C

Consolidated notation

Each chapter keeps its own notation, as defined in the chapter. This appendix lists (1) the global conventions, (2) every symbol that two chapters use differently, and (3) the macro renamings made in assembling the volume (none changes a printed symbol). The full symbol tables, chapter by chapter, are in the project file `NOTATION.md`.

C.1. Global conventions

- $s = \frac{1}{2} + iz$ or $s = \frac{1}{2} - iz$ (as stated per chapter); $\rho = \beta + i\gamma = \frac{1}{2} + \delta_\rho + it_\rho$; $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, $\Xi(t) = \xi(\frac{1}{2} + it)$.
- $\mathcal{Q}$: the Weil form (Vol. I). Chs. 40, 42, 44 print it as Q, the others as $\mathcal{Q}$ (one object).
- $f^\sharp(x) = x^{-1}\overline{f(1/x)}$ on $C_c^\infty(\mathbb{R}_+^\times)$ (Ch. 30); spectrally $\sharp$ is the mirror $\rho \mapsto 1 - \bar\rho$ (ρ^*). J_{FE} is the half-turn $\rho \mapsto 1 - \rho$, and c_∞ the reality $\rho \mapsto \bar\rho$; $\sharp = c_\infty \circ J_{\mathrm{FE}}$ (Chs. 40, 46, 47).
- $N_+(\sigma)$: off-line zeros with $\operatorname{Re}\rho > \sigma$. Ch. 34 counts with multiplicity; Ch. 40's N_+^{dist} counts distinct zeros. Ch. 36's $N(\sigma_0, T)$ has multiplicity and $0 < t \le T$.
- $\beta^* = \sup \operatorname{Re}\rho$ (Ch. 37; this is Ch. 36's Θ, renamed there to avoid Suzuki's Θ).
- The Fourier transform is printed $\hat F$ in Chs. 40, 41, 44 and $\tilde F$ elsewhere, with the convention stated in each chapter.
- Frobenius is F; Verschiebung $V = qF^{-1} = F^\dagger$ (Rosati), Ch. 46. In Ch. 46 the macro for Frobenius prints F.

C.2. Symbols used differently in different chapters

Symbol	Meanings (chapter)
Θ	Suzuki's unimodular symbol $E^{\#}/E$ (34, 37); $\sup \mathrm{Re}\,\rho$ (36; $=\beta^*$ in 37); round-trip Gouy phase (44); Deninger's generator Θ_χ (38)
κ	negative index $\nu N_+(\frac{1}{2}+\omega)$ (34); model constant of n_{det} (35); cyclotomic character (38); κ_{Pick} (40); detuning×leakage, $\kappa = n_C + n_-$ (41)
J	modular conjugation J_{mod} (31, 40, 42); complex structure C (39 §7(c), $= i\mathfrak{J}$ in 40); J_g quarter turn (42–44); J_F, J^ε (42); Dirac reflection, centre $-\frac{1}{2}$ (47); Lagrange/Suzuki $J \to 0$ (34, 37)
V	$V(z) = \sum m\, e^{i\gamma z}$ (32, 33, 35, 36); Verschiebung (46, 47); $V_0 = \mathrm{pt} \times C$ (45, 46)
H	Hamiltonian (34, 45); $H = C \times \mathrm{pt}$ (45, 46); Krein form $i\omega(\bar{x}, y)$ (42); first integral (43); verified RH height (35)
Z	ζ-network impedance $\xi'/\xi(\frac{1}{2}+p)$ (33, 36); line impedance Z_0 (34); Z_B, Z_F partition functions (45); $Z(C,T)$ (46, 47)
η	rapidity $\delta \log x$ (31); Frobenius rapidity / ghost boost (42, 44, 46, 47); Atkin–Lehner sign η_f (42); residues η_j (43)
ω	Suzuki/Lagarias shift h (34, 37, 41); symplectic form (42); Kepler frequency (31)
ε	derivative-Christoffel threshold (30, 41); Doppler/OS sign (31); root number (42)
$\mathbb{Q}_p$ vs $\mathbb{Q}^\times_{>0}$	the p-adic numbers (38); the positive rationals (45, 47). The two macros are separated by renaming (D6)
C_+ vs $\mathbb{C}_p$	upper half-plane class (37); p-adic complex numbers (38)
O, P tags	see the claim-label legend: OPEN≡RH (30), OBSERVATION (38), PARTIAL (36), PRIOR ART (43)

C.3. Macro renamings

Inside the chapter named, the macro on the left was renamed to the one on the right, because another chapter, or the shared macro file, uses the name for something else. The printed symbol is that chapter's own.

Ch.	Renaming	Reason
30	`\tagO`→`\tagOXXX`	tag prints OPEN≡RH
36	`\OSa`→`\OSaXXXVI`	Ch. 36 defines it without argument (OS_a); Ch. 32 takes the shift as argument
38	`\Cp`, `\Zhs`, `\tagO`	$\mathbb{C}_p$ (not C_+); $\hat{\mathbb{Z}}^\times$ with a different accent; tag prints OBSERVATION
39	`\Zh`, `\Zp`	same symbols $\widehat{\mathbb{Z}}$, $\mathbb{Z}_p$, defined through different base macros
40	`\kr`, `\Np`, `\Qform`, `\Ft`	Krein bracket $[x,y]$ without the subscript $\mathcal{Z}$; N_+^{dist}; Q; $\hat{F}$
41	`\Ft`	$\hat{F}$
42, 44	`\Qform` (and `\Ft` in 44)	Q; $\hat{F}$
43	`\tagP`→`\tagPXLIII`	tag prints PRIOR ART (Ch. 36: PARTIAL)
46	`\Frob`→`\FrobXLVI`	prints F
47	`\Qp`→`\QpXLVII`	$\mathbb{Q}^\times_{>0}$ (Ch. 38: $\mathbb{Q}_p$)

The machine-readable list is `volume3/macro_renames.json`.

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