

Complex Four-Index Unified Field Theory: $SU(1, 1) \times SU(1, 2) \times SU(1, 3)$ Kähler Geometry, Four-Sector Normalization, Dark Matter, and Quantum Gravity

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Abstract

This paper formulates a purely complex unified field theory on a four-dimensional pseudo-Kähler manifold $(\mathcal{M}, J, g_{\alpha\bar{\beta}}, \omega)$ of signature $(1, 3)$ governed by the non-compact gauge and spacetime symmetry $SU(1, 1) \times SU(1, 2) \times SU(1, 3)$. Gravitation and matter are unified within the complex four-index field equation $G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa T_{\alpha\bar{\beta}\gamma\bar{\delta}}$ whose trace-free sector directly couples the Bochner curvature tensor to the complex vacuum energy-momentum tensor ($B_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa \hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$). Global regularity and normalizability are established through the four-sector orientation symmetry $N_g \in \mathbb{Z}_4 = \{0, 1, 2, 3\}$ across the invariant null cone $ds^2 = 0$ where the metric transforms as $g_{\alpha\bar{\beta}}^{(N_g)} = i^{2N_g} I_{N_g}^* g_{\alpha\bar{\beta}}[L]$ with horizon scale inversion $\tilde{L}_i = L_{0,i}^2/L_i$. This four-sector mapping bounds the complex Kretschmann scalar $K^{(N_g)}(z, \bar{z})$ in every sector and guarantees the convergence of the four-sector normalization integral $\alpha_4 < \infty$, yielding regular complex de Sitter-like central cores without a cosmological constant. Under the Kähler representation decomposition $100 = 1 \oplus 15 \oplus 84$, the compact spatial subgroup reproduces the Standard Model gauge group $U(1)_Y \times SU(2)_L \times SU(3)_C$ while the z^3 -axis coset $SU(1, 3)/SU(1, 2)$ and Bochner vacuum naturally predict a dark matter sector ($\chi = T_{3\bar{0}}$, dark gauge boson $A_{3\bar{3}}$, and stationary Bochner condensate $\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$). In galactic halos, the contravariant conservation law $\nabla^\alpha T_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0$ produces flat rotation curves ($v(r) \rightarrow v_\infty = \text{const}$) with a $2/3$ Bochner vacuum and $1/3$ sterile matter partition. Finally, we derive the four-sector normalized wave function, uncertainty principle, massless helicity- ± 2 gravitons, and four-sector ultraviolet (UV) loop cancellation from the rank-four quantum state tensor $\Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}$.

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1 Fundamental Postulates and Four-Sector $SU(1, 1) \times SU(1, 2) \times SU(1, 3)$ Geometry

1.1 Three Physical Postulates

The theory is founded on three physical postulates:

1. **Invariant speed of light:** Every local observer measures the vacuum speed of light to be constant (c , corresponding to the complex null cone $ds^2 = g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta = 0$), independently of the observer's state of motion or sector orientation.
2. **Observer-independent physical laws:** Fundamental physical equations preserve their covariant tensorial form across all reference frames and all four orientation sectors on the complex manifold.
3. **Coexistence of physical states:** All physically admissible configurations compatible with a system preparation coexist simultaneously within a single four-sector normalized complex rank-four quantum state tensor $\Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}(z, \bar{z})$, whereas a local observer resolves a single measurement channel at a given event.

1.2 Complex Pseudo-Kähler Spacetime and Nested Rotation Modules

The physical arena is a four-dimensional complex pseudo-Kähler manifold $(\mathcal{M}, J, g_{\alpha\bar{\beta}}, \omega)$ of complex dimension $N = 1 + 3 = 4$ ($\alpha, \bar{\beta} \in \{0, 1, 2, 3\}$) equipped with a pseudo-Hermitian metric $g_{\alpha\bar{\beta}} = \overline{g_{\beta\bar{\alpha}}}$, a metric-compatible torsion-free Kähler connection ($\nabla_\gamma g_{\alpha\bar{\beta}} = \nabla_{\bar{\gamma}} g_{\alpha\bar{\beta}} = 0$, $\Gamma_{\beta\gamma}^\alpha = g^{\alpha\bar{\delta}}\partial_{\beta\gamma}\bar{g}_{\bar{\delta}}$), and a closed Kähler two-form $\omega = ig_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$ ($d\omega = 0$):

$$ds^2 = g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta, \quad ds_{\text{flat}}^2 = \eta_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^\beta = -dz^0 d\bar{z}^0 + dz^1 d\bar{z}^1 + dz^2 d\bar{z}^2 + dz^3 d\bar{z}^3. \quad (1)$$

The flat pseudo-Hermitian metric $\eta_{\alpha\bar{\beta}} = \text{diag}(-1, +1, +1, +1)$ is preserved ($U^\dagger \eta U = \eta$, $\det U = 1$) by the nested chain of pseudo-unitary groups $SU(1, 1) \subset SU(1, 2) \subset SU(1, 3)$ and the full gauge-spacetime direct product group:

$$\mathcal{G} = SU(1, 1) \times SU(1, 2) \times SU(1, 3). \quad (2)$$

Every four-dimensional complex rotation $U \in SU(1, 3)$ acting on $dZ = (dz^0, dz^1, dz^2, dz^3)^T$ factorizes into three nested coset modules around the temporal axis z^0 , providing $3 + 5 + 7 = 15$ real degrees of freedom:

$$\mathcal{R}_{\text{total}} = \mathcal{M}_{SU(1,3)/SU(1,2)}(\eta_3, \alpha_3, \theta_{13}, \beta_{13}, \theta_{23}, \beta_{23}, \psi_3) \cdot \mathcal{M}_{SU(1,2)/SU(1,1)}(\eta_2, \alpha_2, \theta_{12}, \beta_{12}, \psi_2) \cdot \mathcal{M}_{SU(1,1)}(\eta_1, \alpha_1, \psi_1). \quad (3)$$

where $\mathcal{B}_{0k}(\eta_k, \alpha_k)$ are complex pseudo-unitary boosts in the (z^0, z^k) planes, $\mathcal{U}_{jk}(\theta_{jk}, \beta_{jk})$ are unitary spatial rotations in the (z^j, z^k) planes ($j < k \in \{1, 2, 3\}$), and $\Phi_k(\psi_k)$ are traceless diagonal Cartan phase generators.

1.3 Four-Sector Symmetry and Horizon Scale Inversion

The invariant complex null cone $ds^2 = 0$ partitions each complex time-space plane into four disjoint orientation sectors $N_g \in \mathbb{Z}_4 = \{0, 1, 2, 3\}$ generated by the discrete center of $SU(1, 3)$, $Z(SU(1, 3)) = \{I_4, iI_4, -I_4, -iI_4\} \cong \mathbb{Z}_4$. A transition from the base sector $N_g = 0$ to sector N_g multiplies the complex coordinate differentials by i^{N_g} , the conjugate differentials by i^{-N_g} , and the pseudo-Kähler metric by $i^{2N_g} = (-1)^{N_g}$:

$$dz_{(N_g)}^\alpha = i^{N_g} dz^\alpha, \quad d\bar{z}_{(N_g)}^\beta = i^{-N_g} d\bar{z}^\beta, \quad g_{\alpha\bar{\beta}}^{(N_g)} = i^{2N_g} g_{\alpha\bar{\beta}}^{(0)} = (-1)^{N_g} g_{\alpha\bar{\beta}}^{(0)}, \quad g_{(N_g)}^{\alpha\bar{\beta}} = i^{-2N_g} g_{(0)}^{\alpha\bar{\beta}} = (-1)^{N_g} g_{(0)}^{\alpha\bar{\beta}}. \quad (4)$$

Combining the background geometry sector $N_g \in \mathbb{Z}_4$ with a particle's relative sector index $s_X \in \mathbb{Z}_4$ ($s_X = 0$: ordinary timelike state; $s_X = 1, 3$: lateral/tachyonic sectors; $s_X = 2$: PT-reflected anti-sector) gives the composite sector phase $N_X = N_g + s_X \pmod{4}$.

In curved complex spacetime, let $L_i(z, \bar{z})$ denote the proper geometric scale along each principal direction and $L_{0,i}$ the corresponding critical null horizon scale, defining the dimensionless compactness ratio $q_i = L_{0,i}/L_i \in [0, 1]$. When L_i reaches the null horizon $L_i = L_{0,i}$ ($q_i = 1$), the complex clock-ruler plane rotates by $\pi/2$ across $ds^2 = 0$, and the manifold transitions to the adjacent sector ($N_g \rightarrow N_g + 1$) via the inverse scale mapping I_{N_g} :

$$\tilde{L}_i = \frac{L_{0,i}^2}{L_i}, \quad d\tilde{L}_i = -\frac{L_{0,i}^2}{L_i^2} dL_i, \quad g_{\alpha\bar{\beta}}^{(N_g)}(L) = i^{2N_g} I_{N_g}^* g_{\alpha\bar{\beta}}^{(0)}[L] = (-1)^{N_g} g_{\alpha\bar{\beta}}^{(0)}(\tilde{L}). \quad (5)$$

Because the Kähler connection $\Gamma_{\beta\gamma}^\alpha[i^{2N_g}g] = \Gamma_{\beta\gamma}^\alpha[g]$ is invariant under constant phase-metric factors, the covariant geometry is preserved across all four sectors while the former singular point $L_i \rightarrow 0$ is mapped to the regular asymptotic boundary $\tilde{L}_i \rightarrow \infty$.

2 Complex Four-Index Field Equations and Four-Sector Normalization

2.1 Bochner Decomposition and Complex Four-Index Field Equations

On a complex pseudo-Kähler manifold of complex dimension N , the curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ obeys the Kähler symmetries $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}} = R_{\alpha\bar{\delta}\gamma\bar{\beta}}$ and decomposes into the completely trace-free Bochner tensor $B_{\alpha\bar{\beta}\gamma\bar{\delta}}$ ($g^{\alpha\bar{\beta}}B_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0$) and the two-index Ricci tensor $R_{\alpha\bar{\beta}} = g^{\gamma\bar{\delta}}R_{\alpha\bar{\delta}\gamma\bar{\beta}}$ ($R = g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}$):

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = B_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{1}{N+2}(R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + R_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + R_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}) - \frac{1}{(N+1)(N+2)}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}})R. \quad (6)$$

The divergence-free complex four-index Einstein tensor $G_{\alpha\bar{\beta}\gamma\bar{\delta}}$ and the total four-index energy-momentum source tensor $T_{\alpha\bar{\beta}\gamma\bar{\delta}}$ are defined as:

$$G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-N}[R_{\alpha\bar{\beta}\gamma\bar{\delta}} - R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} - R_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}}R], \quad (7)$$

$$T_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{1-N}\left[\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{N+1}{N+2}(T_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}}) + \frac{1}{N+2}(T_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}})\right] + \frac{-(N^2 + N + 1)g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + (2N + 1)g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}}{(1-N)(N+1)(N+2)}T, \quad (8)$$

where $T_{\alpha\bar{\beta}}$ is the two-index Hermitian matter tensor ($T = g^{\alpha\bar{\beta}}T_{\alpha\bar{\beta}}$) and $\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is the completely trace-free vacuum energy-momentum tensor ($g^{\alpha\bar{\beta}}\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0$).

The fundamental complex four-index field equation, its trace-free Bochner projection, and the contravariant conservation law are:

$$G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa T_{\alpha\bar{\beta}\gamma\bar{\delta}}, \quad B_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa \hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}, \quad \nabla^\alpha G_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa \nabla^\alpha T_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0. \quad (9)$$

Contracting with $g^{\alpha\bar{\beta}}$ recovers the two-index Kähler-Einstein equation $G_{\gamma\bar{\delta}} = R_{\gamma\bar{\delta}} - g_{\gamma\bar{\delta}}R = \kappa T_{\gamma\bar{\delta}}$ ($R = \frac{\kappa}{1-N}T$). Inverting the trace yields the trace-reversed complex field equation for $N = 4$:

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa \mathcal{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa \left[\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{1}{6}(T_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + T_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}) - \frac{1}{10}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}})T \right]. \quad (10)$$

In terms of the traceless Hermitian matter tensor $\hat{T}_{\alpha\bar{\beta}} = T_{\alpha\bar{\beta}} - \frac{1}{4}g_{\alpha\bar{\beta}}T$ ($g^{\alpha\bar{\beta}}\hat{T}_{\alpha\bar{\beta}} = 0$), the trace-reversed source tensor $\mathcal{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ splits into three orthogonal $SU(1,3)$ -invariant irreducible representations:

$$\mathcal{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \underbrace{\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}}_{\text{Bochner sector (84)}} + \underbrace{\frac{1}{6}(\hat{T}_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + \hat{T}_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + \hat{T}_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + \hat{T}_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}})}_{\text{Traceless Ricci sector (15)}} - \underbrace{\frac{1}{60}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}})T}_{\text{Scalar trace sector (1)}}. \quad (11)$$

2.2 Contravariant Conservation Law, Four-Sector Regularity, and Global Normalization

The contravariant conservation law $\nabla^\alpha T_{\alpha\bar{\beta}\gamma\bar{\delta}} = g^{\alpha\bar{\sigma}}\nabla_{\bar{\sigma}}T_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0$ couples the divergence of the trace-free Bochner vacuum tensor $\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$ directly to the covariant gradients of the matter tensor $T_{\alpha\bar{\beta}}$. Defining the matter part of the trace-reversed tensor for $N = 4$ as:

$$\tilde{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} \equiv \frac{1}{6}(T_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + T_{\gamma\bar{\delta}}g_{\alpha\bar{\beta}} + T_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}} + T_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}) - \frac{1}{10}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}})T, \quad (12)$$

and using the two-index conservation law $\nabla^\alpha T_{\alpha\bar{\beta}} = 0$ together with the differential Bianchi identity $\nabla^\alpha R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nabla_{\bar{\beta}}R_{\gamma\bar{\delta}} - \nabla_{\bar{\delta}}R_{\gamma\bar{\beta}}$ on Kähler manifolds, the four-index conservation equation $\nabla^\alpha T_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0$ reduces to the local exchange balance between the Bochner vacuum field and matter gradients:

$$\nabla^\alpha \hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{5}{6}\nabla_{\bar{\beta}}T_{\gamma\bar{\delta}} - \frac{1}{6}\nabla_{\bar{\delta}}T_{\gamma\bar{\beta}} \iff \nabla^\alpha \hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nabla^\alpha \tilde{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}. \quad (13)$$

Theorem 2.1 (Four-Sector Regularity and Global Normalization Theorem). *Let $(\mathcal{M}, J, g_{\alpha\bar{\beta}}^{(N_g)}, \omega)$ be a four-sector complex pseudo-Kähler manifold equipped with the sector transformation law (4) and the horizon scale inversion rule (5) across the critical null boundary $L_i = L_{0,i}$ ($q_i = L_{0,i}/L_i = 1$). Then:*

1. **Sector Covariance of the Four-Index Equations:** *In every sector $N_g \in \{0, 1, 2, 3\}$, the four-index Einstein tensor and source tensor scale as $G_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(N_g)} = i^{2N_g}I_{N_g}^*G_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(0)}$ and $T_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(N_g)} = i^{2N_g}I_{N_g}^*T_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(0)}$, leaving $G_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(N_g)} = \kappa T_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(N_g)}$ and $\nabla_{(N_g)}^\alpha T_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(N_g)} = 0$ invariant in form.*
2. **Boundedness of the Complex Kretschmann Scalar Across All Four Sectors:** *The complex Kretschmann scalar in sector N_g ,*

$$K^{(N_g)}(z, \bar{z}) = R_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(N_g)}R^{\alpha\bar{\beta}\gamma\bar{\delta}}_{(N_g)} = (i^{2N_g})^2(i^{-2N_g})^2I_{N_g}^*K^{(0)}(z, \bar{z}) = \kappa^2\mathcal{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(N_g)}\mathcal{T}^{\alpha\bar{\beta}\gamma\bar{\delta}}_{(N_g)}, \quad (14)$$

is invariant under the phase factor $(i^{2N_g})^2(i^{-2N_g})^2 = 1$ and bounded everywhere on $\mathcal{M}$. In the exterior sectors $N_g \in \{0, 2\}$ ($L_i \geq L_{0,i}$), $K^{(0,2)}(L_i) \propto (L_{0,i}/L_i)^6 \rightarrow 0$ as $L_i \rightarrow \infty$. In the interior inverted sectors $N_g \in \{1, 3\}$ ($0 < L_i \leq L_{0,i}$, where $\tilde{L}_i = L_{0,i}^2/L_i \geq L_{0,i}$), the scale inversion maps $L_i \rightarrow 0$ to $\tilde{L}_i \rightarrow \infty$, yielding:

$$K^{(1,3)}(L_i) = K^{(0,2)}(\tilde{L}_i) \propto \left(\frac{L_{0,i}}{\tilde{L}_i}\right)^6 = \left(\frac{L_i}{L_{0,i}}\right)^6 \xrightarrow{L_i \rightarrow 0} 0. \quad (15)$$

Consequently, $K^{(N_g)}(z, \bar{z})$ attains its finite maximum $K_{\max} = K(L_{0,i}) < \infty$ at the horizon boundary $L_i = L_{0,i}$ and is free of curvature singularities throughout all four sectors.

3. **Four-Sector Global Normalization:** *Because $0 \leq K^{(N_g)}(z, \bar{z}) \leq K_{\max} < \infty$ and $K^{(N_g)}$ decays as L_i^{-6} for $L_i \rightarrow \infty$ and as L_i^6 for $L_i \rightarrow 0$, the total four-sector normalization integral α_4 over the complete four-sector manifold $\mathcal{M}_4 = \bigsqcup_{N_g=0}^3 \mathcal{M}^{(N_g)}$ is strictly finite and positive:*

$$\alpha_4 = \frac{1}{4} \sum_{N_g=0}^3 \int_{ct_A}^{ct_B} \int_{\Sigma_t^{(N_g)}} \sqrt{|g^{(N_g)}|} K^{(N_g)}(z, \bar{z}) d^6z dz^0 d\bar{z}^0 < \infty. \quad (16)$$

Proof. Under $g_{\alpha\bar{\beta}}^{(N_g)} = i^{2N_g} I_{N_g}^* g_{\alpha\bar{\beta}}^{(0)}$, the inverse metric transforms with the opposite factor $g_{(N_g)}^{\alpha\bar{\beta}} = i^{-2N_g} I_{N_g}^* g_{(0)}^{\alpha\bar{\beta}}$. The Kähler connection $\Gamma_{\beta\gamma}^\alpha = g^{\alpha\bar{\delta}} \partial_{\bar{\beta}} g_{\gamma\bar{\delta}}$ and the mixed curvature tensor $R^\kappa_{\alpha\gamma\bar{\delta}}$ have one upper and one lower metric factor, so the phase factors cancel: $i^{-2N_g} \cdot i^{2N_g} = 1$. Lowering the first index with $g_{\kappa\beta}^{(N_g)}$ yields $R_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(N_g)} = i^{2N_g} I_{N_g}^* R_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(0)}$ and $G_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(N_g)} = i^{2N_g} I_{N_g}^* G_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(0)}$, proving Statement 1. Contracting two lowered and two raised curvature tensors gives $(i^{2N_g})^2 (i^{-2N_g})^2 = i^0 = 1$, while the pullback $I_{N_g}^*$ replaces L_i by $\tilde{L}_i = L_{0,i}^2/L_i$ inside the horizon $L_i < L_{0,i}$, converting the L_i^{-6} divergence into the regular L_i^6 decay as $L_i \rightarrow 0$, which proves Statement 2. Finally, integrating the bounded density $\sqrt{|g^{(N_g)}|} K^{(N_g)}$ over both exterior ($L_i \in [L_{0,i}, \infty)$) and interior inverted ($L_i \in (0, L_{0,i}]$) sectors yields two convergent integrals over $[L_{0,i}, \infty)$, establishing $\alpha_4 < \infty$ in Statement 3. $\square$

3 Particle Spectrum vs. Standard Model and the Geometric Dark Matter Sector

3.1 Comparison Between $U(1) \times SU(2) \times SU(3)$ and $SU(1,1) \times SU(1,2) \times SU(1,3)$

In the Standard Model, $U(1)_Y \times SU(2)_L \times SU(3)_C$ is a compact internal group detached from spacetime geometry. In our complex four-index theory, all four indices of $\Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}$ and $T_{\alpha\bar{\beta}\gamma\bar{\delta}}$ belong to the complex tangent/cotangent bundle of $\mathbb{C}^{1,3}$ and transform simultaneously under $U \in SU(1,3)$ containing the nested chain $SU(1,1) \subset SU(1,2) \subset SU(1,3)$:

$$T'_{\alpha\bar{\beta}\gamma\bar{\delta}} = U_\alpha{}^\kappa \bar{U}_{\bar{\beta}}{}^{\bar{\lambda}} U_\gamma{}^\eta \bar{U}_{\bar{\delta}}{}^{\bar{\theta}} T_{\kappa\bar{\lambda}\eta\bar{\theta}}. \quad (17)$$

The maximal compact subgroup acting on the spatial coordinates (z^1, z^2, z^3) contains the exact Standard Model gauge group:

$$K_{\text{compact}} = U(1) \times (SU(2) \times U(1)) \times (SU(3) \times U(1)) \supset U(1)_Y \times SU(2)_L \times SU(3)_C. \quad (18)$$

Physical Feature	Standard Model: $U(1)_Y \times SU(2)_L \times SU(3)_C$	Complex Theory: $SU(1,1) \times SU(1,2) \times SU(1,3)$
Symmetry nature	Compact internal fiber symmetry detached from gravity	Non-compact pseudo-unitary spacetime Kähler symmetry
Generators	$1 + 3 + 8 = 12$ compact gauge bosons	$3 + 8 + 15 = 26$ (15 in nested chain: 9 compact + 6 boost)
Field carrier	Separate spin-0, 1/2, 1 fields added by hand	Single 4-index Kähler tensor $\Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}$ ($100 = 1 \oplus 15 \oplus 84$)
Matter tensor	Symmetric real $T_{\mu\nu}$ (10 components)	Hermitian $T_{\alpha\bar{\beta}} = \Theta_{(\alpha\beta)} + i\Omega_{[\alpha\beta]}$ (16 components)
Dark matter sector	None (requires ad hoc beyond-SM fields)	Built-in sterile z^3 sector $(\chi, A_{3\bar{3}}) +$ Bochner vacuum $\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$

Table 1: Comparison between the Standard Model and the complex four-index $SU(1,1) \times SU(1,2) \times SU(1,3)$ theory.

3.2 Multiplet Decomposition $100 = 1 \oplus 15 \oplus 84$ and New Dark Matter Particles

In complex dimension $N = 4$, a Kähler curvature/source tensor belongs to $(4 \odot 4) \otimes (\bar{4} \odot \bar{4}) = 10 \otimes 10 = 100$. By Eq. (11), these 100 components decompose into three irreducible sectors under $SU(1,3) \supset SU(1,2) \supset SU(1,1)$ (with temporal index 0 and spatial indices $i, j \in \{1, 2, 3\}$):

1. **Scalar Trace Sector (1): Higgs-like Mass Scalar T .** The scalar trace $T = g^{\alpha\bar{\beta}} T_{\alpha\bar{\beta}} = -T_{0\bar{0}} + T_{1\bar{1}} + T_{2\bar{2}} + T_{3\bar{3}}$ is an invariant singlet under all three groups and generates rest-mass energy density via $-\frac{1}{60}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}})T$.
2. **Traceless Ricci Sector (15): Gauge Bosons, Visible Matter, and Sterile Dark Matter.** The 15-component traceless Hermitian tensor $\hat{T}_{\alpha\bar{\beta}}$ transforms in the adjoint representation of $SU(1,3)$ and writes in $(1+3)$ block matrix form as:

$$\hat{T}_{\alpha\bar{\beta}} = \begin{pmatrix} \hat{T}_{0\bar{0}} & \Phi_1 & \Phi_2 & \chi \\ \bar{\Phi}_1 & A_{1\bar{1}} + \frac{1}{3}\hat{T}_{0\bar{0}} & A_{1\bar{2}} & A_{1\bar{3}} \\ \bar{\Phi}_2 & A_{2\bar{1}} & A_{2\bar{2}} + \frac{1}{3}\hat{T}_{0\bar{0}} & A_{2\bar{3}} \\ \bar{\chi} & A_{3\bar{1}} & A_{3\bar{2}} & A_{3\bar{3}} + \frac{1}{3}\hat{T}_{0\bar{0}} \end{pmatrix}, \quad \text{Tr}(A_{i\bar{j}}) = 0. \quad (19)$$

- *Spatial block $A_{i\bar{j}}$ and $T_{0\bar{0}}$ ($8+1=9$ states):* The 3×3 traceless Hermitian block $A_{i\bar{j}}$ yields the 8 gluon-like gauge fields of $SU(3)_C$. Restricting to the $SU(1,2)$ spatial subspace $\{1,2\}$ yields the 3 weak gauge bosons ($W^{1,2,3}$) of $SU(2)_L$, while restricting to $SU(1,1)$ on $\{0,1\}$ gives the electromagnetic/hypercharge $U(1)$ photon mode.
- *Time-space block $\Phi_i = T_{i\bar{0}}$ ($3+3=6$ real states):* Under $SU(3)$ spatial rotations, $(\Phi_1, \Phi_2, \chi)^T$ transforms as the fundamental triplet $\mathbf{3}$ (and $\bar{\mathbf{3}}$ for $T_{0\bar{j}}$). Under the electroweak subgroup $SU(1,2)$ acting on (z^0, z^1, z^2) , the triplet branches as $\mathbf{3} \rightarrow \mathbf{2} \oplus \mathbf{1}$: the first two components $(\Phi_1, \Phi_2) = (T_{1\bar{0}}, T_{2\bar{0}})$ form the visible electroweak doublet $\mathbf{2}$, whereas the third component:

$$\chi \equiv \chi_3 = T_{3\bar{0}}, \quad \bar{\chi} = T_{0\bar{3}} \quad (20)$$

is a strict singlet ($\mathbf{1}$) under both $SU(1,2)_{\text{weak}}$ and $SU(1,1)_{\text{EM}}$. Because $\chi = T_{3\bar{0}}$ carries zero electric charge ($Q_{\text{EM}} = 0$) and zero weak isospin ($I_W = 0$), it decouples from photons and $W^\pm$ bosons, providing a **geometric sterile dark matter particle**.

- *Dark Cartan Boson $A_{3\bar{3}}$ (Z'_D):* The diagonal generator $T_{15} = \frac{1}{\sqrt{12}}\text{diag}(1, 1, 1, -3) \in \mathfrak{su}(1,3)$ commutes with the entire $SU(1,2)$ subgroup. Its corresponding component $A_{3\bar{3}} = T_{3\bar{3}} - \frac{1}{3}(T_{0\bar{0}} + T_{1\bar{1}} + T_{2\bar{2}})$ is an electrically neutral **dark gauge boson** (Z'_D) mediating interactions along the z^3 axis.

3. **Trace-Free Bochner Sector (84): Gravitons and Dark Vacuum Condensate.** The 84-component completely trace-free tensor $\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{1}{\kappa}B_{\alpha\bar{\beta}\gamma\bar{\delta}}$ has vanishing contraction with $g^{\alpha\bar{\beta}}$ and cannot couple linearly to electromagnetic currents. Aside from the 2 transverse radiative helicity- ± 2 graviton modes, its 82 stationary and longitudinal components form an electrically dark **Bochner vacuum energy condensate** coupled to matter gradients via Eq. (13).

4 Complex De Sitter-like Solutions and Galactic Rotation Curves

4.1 Complex De Sitter-like Solution from Four-Sector Normalization

In $\mathbb{C}^{1,3}$ with coordinates (z^0, z^1, z^2, z^3) and spatial Hermitian radius $r = \sqrt{\delta_{i\bar{j}}z^i\bar{z}^j}$, consider a static, radially symmetric pseudo-Hermitian metric in sector N_g :

$$ds^2_{(N_g)} = (-1)^{N_g} \left[-f(r)dz^0d\bar{z}^0 + \frac{dr^2}{f(r)} + r^2d\Omega_{\mathbb{C}^3}^2 \right]. \quad (21)$$

In an orthonormal complex coframe $\{e^{\hat{0}}, e^{\hat{1}}, e^{\hat{2}}, e^{\hat{3}}\}$, let the Hermitian matter tensor have energy density $\rho(r) = T_{0\bar{0}}(r)$, radial pressure $p_r(r) = T_{1\bar{1}}(r) = -\rho(r)$, and transverse pressure $p_t(r) =$

Dark Candidate	Tensor Component	Subgroup Origin	Charges	Physical Role
Bochner Condensate	$\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \kappa^{-1} B_{\alpha\bar{\beta}\gamma\bar{\delta}}$	Irrep 84 of $SU(1, 3)$	$Q_{\text{EM}} = 0, I_W = 0$	Gravitational halo (2/3 of v_∞^2)
Sterile z^3 State χ	$T_{3\bar{0}}, T_{0\bar{3}}$ in $T_{\alpha\bar{\beta}}$	Coset $SU(1, 3)/SU(1, 2)$	$Q_{\text{EM}} = 0, I_W = 0$	Sterile DM particle (1/3 of v_∞^2)
Dark Boson Z'_D	$A_{3\bar{3}}$ in $T_{\alpha\bar{\beta}}$	$U(1)_D \subset SU(1, 3)$	$Q_{\text{EM}} = 0$, spin 1	Dark mediator along z^3
Coset Boostons	Non-compact generators	$SU(1, n)/[SU(n, 1) \times U(1)]$	Time-space mix	Vacuum-matter conversion

Table 2: New dark sector states predicted by the complex $SU(1, 1) \times SU(1, 2) \times SU(1, 3)$ four-index decomposition.

$T_{\hat{2}\hat{2}}(r) = T_{\hat{3}\hat{3}}(r)$, while $\epsilon_B(r)$ denotes the scalar amplitude of the trace-free Bochner vacuum tensor $\hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$.

The two-index conservation law $\nabla^\alpha T_{\alpha\bar{\beta}} = 0$ and the four-index Bochner conservation law (13) take the radial forms:

$$p_t(r) = -\rho(r) - \frac{r}{2} \frac{d\rho(r)}{dr}, \quad \frac{d}{dr}(r^3 \epsilon_B(r)) = -\frac{r^3}{6} \frac{d\rho(r)}{dr}. \quad (22)$$

Integrating Eq. (22) by parts with four-sector boundary regularity at $r = 0$ yields the exact relation between the angular trace-reversed source $\mathcal{T}_{\hat{2}\hat{2}\hat{3}\hat{3}}(r)$, the Bochner amplitude $\epsilon_B(r)$, and the enclosed mass-energy integral:

$$\mathcal{T}_{\hat{2}\hat{2}\hat{3}\hat{3}}(r) = 2\epsilon_B(r) + \frac{1}{3}\rho(r) = \frac{1}{r^3} \int_0^r \rho(r') r'^2 dr', \quad \epsilon_B(r) = -\frac{1}{6}\rho(r) + \frac{1}{2r^3} \int_0^r \rho(r') r'^2 dr'. \quad (23)$$

By Theorem 2.1, four-sector normalization across the critical horizon scale r_g prevents point-mass divergence and normalizes the central density to $\rho_{\text{core}} = \frac{3r_s}{\kappa r_g^3} < \infty$ (where $r_s = 2G_N M_b/c^2$). This yields the four-sector normalized baryonic profile:

$$\rho_b(r) = \frac{3r_s r_g^3}{\kappa(r^3 + r_g^3)^2}, \quad \epsilon_B(r) = \frac{r_s r^3}{2\kappa(r^3 + r_g^3)^2}, \quad f(r) = 1 - \kappa r^2 \left(2\epsilon_B(r) + \frac{1}{3}\rho_b(r) \right) = 1 - \frac{r_s r^2}{r^3 + r_g^3}. \quad (24)$$

In the central region ($r \ll r_g$), the Bochner tidal vacuum amplitude vanishes ($\epsilon_B(r) \propto r^3 \rightarrow 0$), the matter density is constant ($\rho_b(0) = \rho_{\text{core}}$ with $p_r = p_t = -\rho_{\text{core}}$), and the complex metric (21) becomes an exact **complex de Sitter-like pseudo-Kähler solution** of constant holomorphic sectional curvature:

$$ds_{\text{core}}^2 \approx (-1)^{N_g} \left[- \left(1 - \frac{r^2}{R_{dS, \text{core}}^2} \right) dz^0 d\bar{z}^0 + \left(1 - \frac{r^2}{R_{dS, \text{core}}^2} \right)^{-1} dr^2 + r^2 d\Omega_{\mathbb{C}^3}^2 \right], \quad (25)$$

$$R_{dS, \text{core}} = \sqrt{\frac{r_g^3}{r_s}} = \sqrt{\frac{3}{\kappa \rho_{\text{core}}}}, \quad K_{\text{core}}(0) = \frac{8}{3} \kappa^2 \rho_{\text{core}}^2 < \infty. \quad (26)$$

Thus, a complex de Sitter-like geometry emerges directly from four-sector normalization and the four-index conservation law without introducing an external cosmological constant Λ .

4.2 Galactic Rotation Curves from the Complex Dark Sector

In a real spacetime outside a baryonic galactic disk ($\rho_b(r) \approx 0$ for $r > r_{\text{disk}}$), Eq. (22) gives $\epsilon_W(r) = \frac{r_s}{2\kappa r^3}$ and a Keplerian decay $v_{\text{real}}^2(r) = \frac{G_N M_b}{r} \propto r^{-1}$.

In our complex pseudo-Kähler spacetime, the sterile z^3 -axis dark state $\chi = T_{3\bar{0}}$ (an $SU(1, 2)$ singlet) experiences boost tension along (z^0, z^3) ($p_r^{(\text{DM})} = -\rho_{\text{DM}}(r)$) but exerts **zero transverse pressure along the $SU(1, 2)$ subspace** (z^1, z^2):

$$p_t^{(\text{DM})}(r) = 0. \quad (27)$$

Substituting $p_t^{(\text{DM})} = 0$ into Eq. (22) and applying four-sector central normalization $\rho_{\text{DM}}(0) = \rho_0 < \infty$ at halo core scale r_c yields:

$$0 = -\rho_{\text{DM}}(r) - \frac{r}{2} \frac{d\rho_{\text{DM}}(r)}{dr} \implies \rho_{\text{DM}}(r) = \frac{\rho_0 r_c^2}{r^2 + r_c^2} \xrightarrow{r \gg r_c} \frac{\rho_0 r_c^2}{r^2}. \quad (28)$$

Inserting $\rho_{\text{DM}}(r)$ into the Bochner conservation equation (22) in the halo region ($r \gg r_c$) gives:

$$\frac{d}{dr}(r^3 \epsilon_B(r)) = -\frac{r^3}{6} \left(-\frac{2\rho_0 r_c^2}{r^3} \right) = +\frac{1}{3} \rho_0 r_c^2 = \text{const} \implies \epsilon_B(r) = \underbrace{\frac{r_s}{2\kappa r^3}}_{\text{Baryonic Keplerian part}} + \underbrace{\frac{1}{3} \frac{\rho_0 r_c^2}{r^2}}_{\text{Bochner vacuum halo}}. \quad (29)$$

Combining Eqs. (23), (27), and (28) determines the total gravitating halo density and the circular rotation velocity $v(r)$ of test stars:

$$\mathcal{T}_{\hat{2}\hat{2}\hat{3}\hat{3}}^{(\text{halo})}(r) = 2\epsilon_B^{(\text{DM})}(r) + \frac{1}{3}\rho_{\text{DM}}(r) = \frac{2}{3} \frac{\rho_0 r_c^2}{r^2} + \frac{1}{3} \frac{\rho_0 r_c^2}{r^2} = \frac{\rho_0 r_c^2}{r^2}, \quad (30)$$

$$v^2(r) = \frac{G_N M_b r^2}{r^3 + r_g^3} + v_\infty^2 \left(1 - \frac{r_c}{r} \arctan \frac{r}{r_c} \right), \quad v_\infty = \sqrt{\frac{3}{2} \kappa c^4 \rho_0 r_c^2} = \text{const}. \quad (31)$$

4.3 Two Testable Predictions for Galactic Halos

1. **Exact 2/3 Bochner Vacuum +1/3 Sterile Matter Partition:** In the flat rotation regime ($r \gg r_c$), 1/3 of the squared circular velocity v_∞^2 is sourced by the sterile $SU(1, 3)/SU(1, 2)$ matter density $\frac{1}{3}\rho_{\text{DM}}(r)$ ($\chi = T_{3\bar{0}}$), while 2/3 is sourced by the trace-free Bochner vacuum energy condensate $2\epsilon_B^{(\text{DM})}(r)$ induced via $\nabla^\alpha \hat{T}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nabla^\alpha \tilde{T}_{\alpha\bar{\beta}\gamma\bar{\delta}}$.
2. **Complex De Sitter-like Cored Galactic Centers (Core vs. Cusp):** By four-sector normalization (Theorem 2.1), as $r \rightarrow 0$ the dark halo profile (28) approaches a constant-density complex de Sitter-like core $\rho_{\text{DM}}(0) = \rho_0 < \infty$ with $f_{\text{DM}}(r) \approx 1 - \kappa \rho_0 r^2/3$ and solid-body rotation $v(r) \propto r$, resolving the central $\rho_{\text{NFW}} \propto r^{-1}$ cusp problem of standard cold dark matter simulations.

5 Geometric Quantum Mechanics and Rank-Four Quantum Gravity

5.1 Four-Sector Normalized Curvature Scalar as Wave Function and Uncertainty Principle

By Theorem 2.1, the four-sector normalization integral α_4 in Eq. (16) is finite across the entire manifold. The four-sector normalized curvature scalar $N_4(z, \bar{z})$ within a spatial domain V and time interval $[t_A, t_B]$ defines the geometric wave function:

$$N_4(z, \bar{z}) = \frac{1}{\alpha_4} \left(\frac{1}{4} \sum_{N_g=0}^3 \int_{ct_A}^{ct_B} \int_V \sqrt{|g^{(N_g)}|} K^{(N_g)}(z, \bar{z}) d^6 z dz^0 d\bar{z}^0 \right), \quad 0 \leq N_4(z, \bar{z}) \leq 1. \quad (32)$$

Detecting a particle on a localized detector requires accumulating at least a threshold curvature $N_0 \leq \alpha_4 N_4(z, \bar{z})$. Consequently, localizing a measurement within a small spatial volume $\epsilon_V \rightarrow 0$ requires a long integration time Δt , whereas a rapid measurement $\Delta t \rightarrow 0$ requires a large spatial volume V , reproducing the quantum uncertainty principle from four-sector normalized geometry.

5.2 Complex Rank-Four Quantum State, Gravitons, and Four-Sector UV Cancellation

The complete quantum state of a system in complex dimension $N = 4$ is the four-sector normalized rank-four tensor superposition of admissible source configurations $T_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(q, S_X)}$ ($q \in \mathcal{Q}_{\text{prep}}, S_X \in \{0, 1, 2, 3\}$):

$$\Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}(z, \bar{z}) = \sqrt{l_P \kappa} \sum_{q \in \mathcal{Q}_{\text{prep}}} c_q T_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(q)}(z, \bar{z}), \quad c_q \in \mathbb{C}, \quad (33)$$

$$\langle \Phi, \Psi \rangle_{\Sigma, 4} = \frac{1}{4} \sum_{S_X=0}^3 \int_{\Sigma(S_X)} \sqrt{h(S_X)} \bar{\Phi}_{(S_X)}^{\alpha\bar{\beta}\gamma\bar{\delta}} \Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(S_X)} d^6 z = 1, \quad P(\Phi|\Psi) = \frac{|\langle \Phi, \Psi \rangle_{\Sigma, 4}|^2}{\sum_A |\langle \Phi_A, \Psi \rangle_{\Sigma, 4}|^2}. \quad (34)$$

Quantum time evolution is governed by the lower-index temporal covariant derivative $\nabla_0 = \frac{\partial}{\partial z^0} + \Gamma_0$ and the Hermitian rank-eight Hamiltonian operator $H_{\alpha\bar{\beta}\gamma\bar{\delta}}^{\kappa\bar{\lambda}\eta\bar{\theta}}$:

$$i l_P \nabla_0 \Psi_{\alpha\bar{\beta}\gamma\bar{\delta}} = H_{\alpha\bar{\beta}\gamma\bar{\delta}}^{\kappa\bar{\lambda}\eta\bar{\theta}} \Psi_{\kappa\bar{\lambda}\eta\bar{\theta}}, \quad H_{\alpha\bar{\beta}\gamma\bar{\delta}}^{\kappa\bar{\lambda}\eta\bar{\theta}} \Psi_{\kappa\bar{\lambda}\eta\bar{\theta}}^{(q)} = \frac{E_q}{E_P} \Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(q)}, \quad c_q(\tau) = c_q(0) e^{-i E_q \tau / \hbar} \quad (E_P l_P = \hbar c). \quad (35)$$

1. **Null Photon States:** Along a complex null geodesic ($g_{\alpha\bar{\beta}} k^\alpha \bar{k}^\beta = 0$), proper time vanishes identically ($\tau_\gamma = 0$) and the projected null state $\mathcal{S}_\gamma = \{(\gamma(s), [k(s)], \omega_U(s), \pi)\}$ carries frequency $\omega_U = -c g_{\alpha\bar{\beta}} U^\alpha \bar{k}^\beta$ relative to a timelike observer U^α .
2. **Massless Helicity- ± 2 Gravitons:** In a Ricci-flat vacuum ($T_{\alpha\bar{\beta}} = 0$), the quantum carrier reduces to the trace-free Bochner tensor $\Psi_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(q)} = -\frac{\sqrt{l_P}}{3} c_q B_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(q)}$ ($\nabla^\alpha B_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0$), whose transverse type-N modes $B_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(+2)} \propto k_{[\alpha} m_{\gamma]} \bar{k}_{[\bar{\beta}} \bar{m}_{\bar{\delta}]}$ and $B_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(-2)} \propto k_{[\alpha} \bar{m}_{\gamma]} \bar{k}_{[\bar{\beta}} m_{\bar{\delta}]}$ describe massless gravitons with $m_G = 0$, $\lambda_G = \pm 2$, and $E_G = \hbar \omega$.
3. **Four-Sector Normalization and Cancellation of UV Loop Divergences:** Under sector transformations $g_{\alpha\bar{\beta}}^{(S_X)} = i^{2S_X} g_{\alpha\bar{\beta}} = (-1)^{S_X} g_{\alpha\bar{\beta}}$ ($S_X \in \{0, 1, 2, 3\}$), the Kähler connection $\Gamma_{\beta\gamma}^\alpha(S_X) = \Gamma_{\beta\gamma}^\alpha$ and mixed curvature $R_{\alpha\gamma\bar{\delta}}^\kappa(S_X) = R_{\alpha\gamma\bar{\delta}}^\kappa$ are sector-invariant. In any L -loop 1PI diagram, internal vertex (i^{2S_X}) and propagator (i^{-2S_X}) factors cancel ($i^{2S_X} \cdot i^{-2S_X} = 1$), leaving the local UV pole tensor $H_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(L), \text{UV}}(S_X)$ (including the two-loop Goroff-Sagnotti cubic divergence) with the uncontracted external factor $i^{2S_X} = (-1)^{S_X}$. Normalizing over the four-sector orbit eliminates all local UV loop divergences identically:

$$\mathcal{P}_{\text{UV}} H_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(L), \text{UV}} = \frac{1}{4} \sum_{S_X=0}^3 i^{2S_X} H_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(L), \text{UV}}(0) = \frac{1}{4} (1 - 1 + 1 - 1) H_{\alpha\bar{\beta}\gamma\bar{\delta}}^{(L), \text{UV}}(0) = 0. \quad (36)$$

6 Conclusion

The purely complex four-index field theory on a $(1, 3)$ pseudo-Kähler manifold unifies gravitation, Standard Model gauge symmetries, and dark matter within a single tensorial framework governed by $SU(1, 1) \times SU(1, 2) \times SU(1, 3)$. Through four-sector horizon scale inversion ($\tilde{L}_i = L_{0,i}^2 / L_i$) and four-sector normalization ($\alpha_4 < \infty$), the theory guarantees global curvature regularity, yields complex de Sitter-like central cores without a cosmological constant, explains flat galactic rotation curves via the $2/3$ Bochner vacuum and $1/3$ sterile z^3 -axis partition, and achieves perturbative ultraviolet finiteness across the four-sector null-cone orbit.

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