

**TAA LEMMA 1: ARITHMETIC UNITARITY.
THE RIEMANN HYPOTHESIS AS ABSENCE OF GHOSTS AND RUNAWAY
MODES
FOR THE SCALING FLOW OF THE WEIL FORM**

TAA WORKING NOTES (J. W. MCGREEVY; FORMALIZATION BY GROK BOT)

ABSTRACT. We make precise, and prove, the following version of the TAA thesis that “the Riemann hypothesis is a statement about unitarity”. On $C_c^\infty(\mathbb{R})$ (functions of the logarithmic scale $u = \log x$), the Weil explicit-formula pairing $\mathcal{Q}$ is a Hermitian form. This form is defined arithmetically by primes, the pole, and the archimedean place, and spectrally by the zeros. The scaling flow $T_t F(u) = F(u - t)$ is *always* $\mathcal{Q}$ -isometric (pseudo-unitary). We show: (a) the exponential growth rate of the matrix coefficients $t \mapsto \mathcal{Q}(T_t F, G)$ is exactly $\Theta - \frac{1}{2}$, where $\Theta = \sup \operatorname{Re} \rho$; (b) $\text{RH} \iff \mathcal{Q} \geq 0 \iff$ the flow is realized by a strongly continuous unitary group on a Hilbert space $\iff$ all matrix coefficients are bounded; (c) the number of negative squares of $\mathcal{Q}$ equals the number $N_+ \in \{0, 1, \dots, \infty\}$ of distinct zeros with $\operatorname{Re} \rho > \frac{1}{2}$. Under RH the generator has simple pure point spectrum equal to the set of zero ordinates. Nothing here proves or disproves RH. Parts (b) and (c) are, in substance, due to Weil and Bombieri; Section 4 records exactly what is new.

1. SETTING AND NOTATION

Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Its zeros are the nontrivial zeros ρ of ζ . They lie in $0 < \operatorname{Re} s < 1$ (Hadamard, de la Vallée Poussin) and, since $\xi(1-s) = \xi(s)$ and $\xi(\bar{s}) = \overline{\xi(s)}$, the multiset $\mathcal{Z}$ of zeros is invariant under $\rho \mapsto 1 - \rho$ and $\rho \mapsto \bar{\rho}$, with multiplicities m_ρ preserved. Write

$$\rho = \frac{1}{2} + i\gamma_\rho, \quad \gamma_\rho \in \mathbb{C}, \quad |\operatorname{Im} \gamma_\rho| < \frac{1}{2}.$$

Thus $\operatorname{Re} \rho = \frac{1}{2} - \operatorname{Im} \gamma_\rho$, and $\rho \mapsto \rho^* := 1 - \bar{\rho}$ corresponds to $\gamma \mapsto \bar{\gamma}$. RH is the statement that every γ_ρ is real. In what follows, sums $\sum_\rho$ are over *distinct* zeros, weighted by m_ρ . Put $\Theta := \sup_\rho \operatorname{Re} \rho \in [\frac{1}{2}, 1]$ (there are zeros on the line, so $\Theta \geq \frac{1}{2}$), and

$$N_+ := \#\{\rho \text{ distinct} : \operatorname{Re} \rho > \frac{1}{2}\} \in \{0, 1, 2, \dots\} \cup \{\infty\}.$$

We use the classical fact $\sum_\rho m_\rho(1+|\gamma_\rho|^2)^{-1} < \infty$, which follows from $N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$ [3, Ch. 15] or [10, §9.4].

For $F \in L_{\text{loc}}^1(\mathbb{R})$ with enough decay we set $\tilde{F}(z) = \int_{\mathbb{R}} F(u) e^{izu} du$. Translation (the logarithmic *scaling flow*) is $(T_t F)(u) = F(u - t)$, so $\widetilde{T_t F}(z) = e^{izt} \tilde{F}(z)$. In multiplicative language, with $x = e^u$, $f(x) = x^{-1/2} F(\log x)$, we have $\tilde{F}(\gamma_\rho) = \int_0^\infty f(x) x^{\rho-1} dx$ (Mellin transform at ρ), and T_t is the dilation $x \mapsto e^{-t} x$ normalized to be $L^2(dx)$ -unitary.

Definition 1.1 (Test class). Let $\mathcal{A}$ be the space of $F \in C^2(\mathbb{R})$ with $\|F\|_{\mathcal{A}} := \sum_{j=0}^2 \int_{\mathbb{R}} |F^{(j)}(u)| e^{|u|/2} du < \infty$. Then $C_c^\infty(\mathbb{R}) \subset \mathcal{A}$ is dense: for a cutoff $\chi \in C_c^\infty$ with $\chi = 1$ near 0, $\|F - \chi(\cdot/R)F\|_{\mathcal{A}} \rightarrow 0$ by dominated convergence. Also $T_t \mathcal{A} = \mathcal{A}$.

Lemma 1.2. For $F \in \mathcal{A}$ and $| \operatorname{Im} z | \leq \frac{1}{2}$: $(1 + |z|^2) |\tilde{F}(z)| \leq \|F\|_{\mathcal{A}}$.

Proof. $|e^{izu}| = e^{-u\operatorname{Im} z} \leq e^{|u|/2}$ gives $|\tilde{F}(z)| \leq \int |F| e^{|u|/2}$. For $j = 0, 1$ the function $w_j = F^{(j)} e^{|u|/2}$ is integrable with integrable derivative on each half-line, so it has limits at $\pm\infty$, and these limits are 0. Hence two integrations by parts on $[-R, R]$, $R \rightarrow \infty$, have vanishing boundary terms and give $z^2 \tilde{F}(z) = -\tilde{F}''(z)$, so $|z|^2 |\tilde{F}(z)| \leq \int |F''| e^{|u|/2}$. Add the two bounds. $\square$

Definition 1.3 (Weil form). For $F, G \in \mathcal{A}$ let

$$\mathcal{Q}(F, G) := \sum_{\rho} m_{\rho} \tilde{F}(\gamma_{\rho}) \overline{\tilde{G}(\gamma_{\rho})}.$$

By Lemma 1.2, $|\mathcal{Q}(F, G)| \leq C_{\mathcal{Z}} \|F\|_{\mathcal{A}} \|G\|_{\mathcal{A}}$ with $C_{\mathcal{Z}} = \sum_{\rho} m_{\rho} (1 + |\gamma_{\rho}|^2)^{-2} < \infty$. The series converges absolutely.

Remark 1.4 (TAA dictionary). In TAA language, $\mathcal{Q}$ is the “arithmetic action” pairing and T_t the “arithmetic time”. $\mathcal{Q}$ is *indefinite* a priori, exactly like the indefinite (Krein/Pontryagin) state space of the Nash–Stelle Lagrangian $2R_{\mu\nu}R^{\mu\nu} - R^2$ with its massive spin-2 ghost [9]. This is an analogy that motivates the questions below. It is not used in any proof.

2. MAIN THEOREM

Theorem 2.1 (Arithmetic unitarity). (i) (Hermitian; arithmetic side.) $\mathcal{Q}(G, F) = \overline{\mathcal{Q}(F, G)}$ on $\mathcal{A}$. For $F, G \in C_c^{\infty}(\mathbb{R})$, with $H = F * G^*$ and $G^*(u) = \overline{G(-u)}$,

(1)

$$\mathcal{Q}(F, G) = \tilde{H}(\tfrac{i}{2}) + \tilde{H}(-\tfrac{i}{2}) - H(0) \log \pi + \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{H}(r) \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} + \frac{ir}{2} \right) dr - \sum_{n \geq 2} \frac{\Lambda(n)}{\sqrt{n}} (H(\log n) + H(-\log n)).$$

- (ii) (Pseudo-unitarity, unconditional.) $\mathcal{Q}(T_t F, T_t G) = \mathcal{Q}(F, G)$ for all $t \in \mathbb{R}$. Each functional $\ell_{\rho}(F) = \tilde{F}(\gamma_{\rho})$ is an eigenfunctional: $\ell_{\rho}(T_t F) = e^{i\gamma_{\rho} t} \ell_{\rho}(F)$, with $|e^{i\gamma_{\rho} t}| = e^{(\operatorname{Re} \rho - \frac{1}{2})t}$.
- (iii) (Lyapunov exponent.) For $F, G \in C_c^{\infty}(\mathbb{R})$ put $\lambda(F, G) := \limsup_{t \rightarrow +\infty} \frac{1}{t} \log |\mathcal{Q}(T_t F, G)|$ (with $\log 0 = -\infty$). Then

$$\sup_{F, G \in C_c^{\infty}} \lambda(F, G) = \Theta - \frac{1}{2}.$$

(iv) (RH $\iff$ unitarity.) The following are equivalent:

- (a) RH;
- (b) $\mathcal{Q}(F, F) \geq 0$ for all $F \in C_c^{\infty}(\mathbb{R})$ (Weil positivity);
- (c) there are a Hilbert space $\mathcal{H}$, a linear map $J : C_c^{\infty}(\mathbb{R}) \rightarrow \mathcal{H}$ with $\langle JF, JG \rangle = \mathcal{Q}(F, G)$, and a strongly continuous unitary group $U(t)$ on $\mathcal{H}$ with $U(t)J = JT_t$;
- (d) every matrix coefficient $t \mapsto \mathcal{Q}(T_t F, G)$, $F, G \in C_c^{\infty}$, is bounded on $\mathbb{R}$.
- (v) (Spectrum under RH.) If RH holds, one may take $\mathcal{H} = \ell^2(\mathcal{Z}, m)$ (weights m_{ρ}), $JF = (\tilde{F}(\gamma_{\rho}))_{\rho}$ (dense range) and $U(t) = e^{itA}$, where A is multiplication by γ_{ρ} . Then A is self-adjoint with pure point spectrum $\{\gamma_{\rho}\}$, and every eigenvalue is simple.
- (vi) (Ghost count.) Let $\kappa(\mathcal{Q})$ be the supremum of dimensions of subspaces $V \subset C_c^{\infty}(\mathbb{R})$ with $\mathcal{Q}(F, F) < 0$ for all $0 \neq F \in V$. Then $\kappa(\mathcal{Q}) = N_+$. The same holds on $\mathcal{A}$.

3. PROOFS

Two auxiliary facts.

Lemma 3.1. For every $z \in \mathbb{C}$ there is $F \in C_c^{\infty}(\mathbb{R})$ with $\tilde{F}(z) \neq 0$.

Proof. Let $F_{\varepsilon} \geq 0$, $\int F_{\varepsilon} = 1$, $\operatorname{supp} F_{\varepsilon} \subset [-\varepsilon, \varepsilon]$. Then $|\tilde{F}_{\varepsilon}(z) - 1| \leq \sup_{|u| \leq \varepsilon} |e^{izu} - 1| \rightarrow 0$. $\square$

Lemma 3.2. Let $a_{\rho} \in \mathbb{C}$ with $\sum_{\rho} |a_{\rho}| < \infty$, and put $L(t) = \sum_{\rho} a_{\rho} e^{i\gamma_{\rho} t}$. If $|L(t)| \leq C e^{\varepsilon t}$ for all $t \geq 0$, then $a_{\rho} = 0$ for every ρ with $\operatorname{Re} \rho - \frac{1}{2} > \varepsilon$.

Proof. For $\operatorname{Re} s > \frac{1}{2}$, Fubini (legitimate since $|e^{i\gamma t}| \leq e^{t/2}$) gives $\mathcal{L}(s) := \int_0^\infty L(t)e^{-st}dt = \sum_\rho \frac{a_\rho}{s - i\gamma_\rho} =: M(s)$. The pole set $P = \{i\gamma_\rho\}$ is closed and discrete, because $|\gamma_\rho| \rightarrow \infty$, and distinct ρ give distinct poles. On compacta at distance $\geq d > 0$ from P the series for M is dominated by $d^{-1} \sum |a_\rho|$. Hence M is meromorphic on $\mathbb{C}$ with simple poles at $i\gamma_\rho$ and residues a_ρ . The bound on L makes $\mathcal{L}$ holomorphic on $\operatorname{Re} s > \varepsilon$. The set $\Omega = \{\operatorname{Re} s > \varepsilon\} \setminus P$ is connected, and $\mathcal{L} = M$ on the open subset $\operatorname{Re} s > \frac{1}{2}$, so $\mathcal{L} = M$ on Ω . At $s_0 = i\gamma_{\rho_0}$ with $\operatorname{Re} s_0 = \operatorname{Re} \rho_0 - \frac{1}{2} > \varepsilon$, $M = \mathcal{L}$ is bounded near s_0 , so the residue a_{ρ_0} vanishes. $\square$

Proof of (i). Using $\rho \mapsto \rho^* = 1 - \bar{\rho}$ (a bijection of $\mathcal{Z}$ preserving m , with $\gamma_{\rho^*} = \bar{\gamma}_\rho$): $\overline{\mathcal{Q}(F, G)} = \sum_\rho m_\rho \tilde{G}(\bar{\gamma}_\rho) \overline{\tilde{F}(\gamma_\rho)} = \sum_\rho m_{\rho^*} \tilde{G}(\gamma_{\rho^*}) \overline{\tilde{F}(\bar{\gamma}_{\rho^*})} = \mathcal{Q}(G, F)$. Next, $\tilde{G}^*(z) = \overline{\tilde{G}(\bar{z})}$, so $\tilde{H}(z) = \overline{\tilde{F}(z)} \tilde{G}(\bar{z})$ and $\mathcal{Q}(F, G) = \sum_\rho m_\rho \tilde{H}(\gamma_\rho)$. Formula (1) is then the Guinand–Weil explicit formula for $H \in C_c^\infty(\mathbb{R})$ [11, 4], in the normalization of [5, Thm. 5.12] and [1]. We checked numerically to 10^{-16} for even and non-even Gaussian H using the first 200–300 zeros (Section 6). $\square$

Proof of (ii). $\overline{T_t F(\gamma)} \overline{T_t G(\bar{\gamma})} = e^{i\gamma t} \tilde{F}(\gamma) \overline{e^{i\bar{\gamma} t} \tilde{G}(\bar{\gamma})} = \tilde{F}(\gamma) \overline{\tilde{G}(\bar{\gamma})}$, because $\overline{e^{i\bar{\gamma} t}} = e^{-i\gamma t}$. Sum over ρ . The eigen-relation is $\overline{T_t F} = e^{izt} \tilde{F}$, and $|e^{i\gamma t}| = e^{-t \operatorname{Im} \gamma} = e^{(\operatorname{Re} \rho - \frac{1}{2})t}$. $\square$

Proof of (iii). *Upper bound:* for $t \geq 0$, $|\mathcal{Q}(T_t F, G)| \leq \sum_\rho m_\rho |\tilde{F}(\gamma_\rho) \tilde{G}(\bar{\gamma}_\rho)| e^{(\operatorname{Re} \rho - \frac{1}{2})t} \leq C_{\mathcal{Z}} \|F\|_{\mathcal{A}} \|G\|_{\mathcal{A}} e^{(\Theta - \frac{1}{2})t}$. *Lower bound:* fix any zero ρ_0 and put $\beta_0 = \operatorname{Re} \rho_0 - \frac{1}{2} \geq 0$. Choose F, G by Lemma 3.1 with $\tilde{F}(\gamma_{\rho_0}) \neq 0 \neq \tilde{G}(\bar{\gamma}_{\rho_0})$. Then $L(t) = \mathcal{Q}(T_t F, G) = \sum_\rho a_\rho e^{i\gamma_\rho t}$ with $a_\rho = m_\rho \tilde{F}(\gamma_\rho) \overline{\tilde{G}(\bar{\gamma}_\rho)}$, $\sum |a_\rho| < \infty$, $a_{\rho_0} \neq 0$. By Lemma 3.2 (valid for any real ε , including $\varepsilon < 0$), no bound $|L(t)| \leq C e^{\varepsilon t}$ with $\varepsilon < \beta_0$ holds, so $\lambda(F, G) \geq \beta_0$. Taking the supremum over ρ_0 gives $\sup \lambda \geq \Theta - \frac{1}{2}$. $\square$

Proof of (iv). (a) $\Rightarrow$ (c) and part (v): Assume RH, so $\bar{\gamma}_\rho = \gamma_\rho \in \mathbb{R}$. Then $JF = (\tilde{F}(\gamma_\rho)) \in \ell^2(\mathcal{Z}, m)$ by Lemma 1.2, and $\langle JF, JG \rangle = \sum m_\rho \tilde{F}(\gamma_\rho) \overline{\tilde{G}(\gamma_\rho)} = \mathcal{Q}(F, G)$. Multiplication by $e^{i\gamma_\rho t}$ is unitary and strongly continuous (dominated convergence), and $U(t)JF = JT_t F$ by (ii). *Dense range:* if $c \in \ell^2(\mathcal{Z}, m)$ is orthogonal to $J(C_c^\infty)$, then for every F and t , $0 = \langle JT_t F, c \rangle = \sum_\rho a_\rho e^{i\gamma_\rho t}$, where $a_\rho = m_\rho \tilde{F}(\gamma_\rho) \bar{c}_\rho$ and $\sum |a_\rho| < \infty$ by Cauchy–Schwarz. The γ_ρ are real and distinct, so the Bohr mean gives $a_{\rho_0} = \lim_{S \rightarrow \infty} \frac{1}{2S} \int_{-S}^S (\sum_\rho a_\rho e^{i\gamma_\rho t}) e^{-i\gamma_{\rho_0} t} dt = 0$ (interchange by dominated convergence, since $\frac{1}{2S} \int_{-S}^S e^{i\mu t} dt \rightarrow \delta_{\mu,0}$ boundedly). With F from Lemma 3.1, $c_{\rho_0} = 0$. By Stone’s theorem [8, Thm. VIII.8], $U(t) = e^{itA}$, with A the maximal multiplication operator by γ_ρ . This A is self-adjoint, with eigenvectors δ_ρ and spectrum $\overline{\{\gamma_\rho\}} = \{\gamma_\rho\}$ (no finite accumulation). The eigenvalues are simple since distinct zeros have distinct ordinates.

(c) $\Rightarrow$ (b): $\mathcal{Q}(F, F) = \|JF\|^2 \geq 0$.

(b) $\Rightarrow$ (d): Cauchy–Schwarz for the positive semidefinite Hermitian form $\mathcal{Q}$ and (ii) give $|\mathcal{Q}(T_t F, G)|^2 \leq \mathcal{Q}(T_t F, T_t F) \mathcal{Q}(G, G) = \mathcal{Q}(F, F) \mathcal{Q}(G, G)$.

(d) $\Rightarrow$ (a): If some ρ_0 has $\operatorname{Re} \rho_0 > \frac{1}{2}$, (iii) gives F, G with $\lambda(F, G) > 0$, so the coefficient is unbounded on $t \geq 0$. Zeros with $\operatorname{Re} \rho < \frac{1}{2}$ are excluded by the symmetry $\rho \mapsto 1 - \bar{\rho}$. $\square$

Proof of (vi). Split the zeros into $\mathcal{Z}_0$ ($\operatorname{Re} \rho = \frac{1}{2}$), $\mathcal{Z}_+$ ($\operatorname{Re} \rho > \frac{1}{2}$, i.e. $\operatorname{Im} \gamma < 0$), and $\mathcal{Z}_- = \{\rho^* : \rho \in \mathcal{Z}_+\}$. For $\rho \in \mathcal{Z}_+$ write $a_\rho(F) = \tilde{F}(\gamma_\rho)$ and $b_\rho(F) = \tilde{F}(\bar{\gamma}_\rho)$. Pairing ρ with ρ^* ,

$$(2) \quad \mathcal{Q}(F, F) = \sum_{\mathcal{Z}_0} m_\rho |\tilde{F}(\gamma_\rho)|^2 + \sum_{\rho \in \mathcal{Z}_+} m_\rho (a_\rho \bar{b}_\rho + b_\rho \bar{a}_\rho) = \Pi(F) + \sum_{\rho \in \mathcal{Z}_+} \frac{m_\rho}{2} (|a_\rho + b_\rho|^2 - |a_\rho - b_\rho|^2),$$

with $\Pi \geq 0$; all series converge absolutely. Each off-line pair contributes a hyperbolic plane, of signature $(1, 1)$.

$\kappa \leq N_+$. Assume $N_+ < \infty$. By (2), $\mathcal{Q}(F, F) \geq -\sum_{\mathcal{Z}_+} \frac{m_\rho}{2} |a_\rho(F) - b_\rho(F)|^2$. If $\dim V > N_+$, the linear map $V \rightarrow \mathbb{C}^{N_+}$, $F \mapsto (a_\rho - b_\rho)_\rho$, has a nonzero kernel element F . That F satisfies $\mathcal{Q}(F, F) \geq 0$, so V is not negative definite.

$\kappa \geq N$ for every finite $N \leq N_+$. Fix distinct $\rho_1, \dots, \rho_N \in \mathcal{Z}_+$ with $\gamma_j = a_j - i\beta_j$, $\beta_j \in (0, \frac{1}{2})$. The set $K = \{\gamma_\rho : |\operatorname{Re} \gamma_\rho - a_j| < 1 \text{ for some } j\}$ is finite and contains $\gamma_j, \bar{\gamma}_j$. The $2N$ points $\gamma_j, \bar{\gamma}_j$ are distinct. By Lagrange interpolation on K , choose polynomials $P_{j,1}, P_{j,2}$ vanishing on $K \setminus \{\gamma_j, \bar{\gamma}_j\}$ with $P_{j,1}(\gamma_j) = 1$, $P_{j,1}(\bar{\gamma}_j) = 0$, $P_{j,2}(\gamma_j) = 0$, $P_{j,2}(\bar{\gamma}_j) = 1$. For $\sigma \geq 1$ let

$$\Phi_{j,\epsilon}(z) = P_{j,\epsilon}(z) \exp\left(-\frac{\sigma^2}{2}[(z - a_j)^2 + \beta_j^2]\right), \quad \epsilon = 1, 2.$$

Then $\Phi_{j,\epsilon} = \widetilde{F_{j,\epsilon}}$ with $F_{j,\epsilon} = c P_{j,\epsilon}(i \frac{d}{du})(e^{-u^2/2\sigma^2} e^{-ia_j u}) \in \mathcal{A}$, because $\widetilde{F'}(z) = -iz\widetilde{F}(z)$. The Gaussian factor equals 1 at γ_j and $\bar{\gamma}_j$.

Consider the Gram matrix $G_\sigma = (\mathcal{Q}(F_k, F_l))_{k,l}$, $k, l \in \{(j, \epsilon)\}$, and split the zero sum into $\gamma_\rho \in K$ and $\gamma_\rho \notin K$. If $\gamma_\rho \in K$, a term $\Phi_k(\gamma_\rho) \overline{\Phi_l(\bar{\gamma}_\rho)}$ with $k = (j, \epsilon)$, $l = (i, \epsilon')$ can be nonzero only if $\gamma_\rho \in \{\gamma_j, \bar{\gamma}_j\}$ and $\bar{\gamma}_\rho \in \{\gamma_i, \bar{\gamma}_i\}$. This forces $i = j$, since $\operatorname{Im} \gamma_i < 0 < \operatorname{Im} \bar{\gamma}_j$ and the γ 's are distinct. Evaluating, the K -part equals $\bigoplus_j m_{\rho_j} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which has exactly N eigenvalues $\leq -\min_j m_{\rho_j} \leq -1$. If $\gamma_\rho = x + iy \notin K$, then $(x - a_j)^2 \geq 1$ and $|y| < \frac{1}{2}$ for all j , so

$$\left| \exp\left(-\frac{\sigma^2}{2}[(z - a_j)^2 + \beta_j^2]\right) \right|_{z=\gamma_\rho \text{ or } \bar{\gamma}_\rho} \leq e^{-\frac{\sigma^2}{2}[(x-a_j)^2 - \frac{1}{4}]} \leq e^{-\frac{3}{8}(\sigma^2-1)} e^{-\frac{1}{2}[(x-a_j)^2 - \frac{1}{4}]}.$$

The remaining factor, times the polynomials, is m_ρ -summable over $\mathcal{Z}$ (by $N(T) \ll T \log T$). Hence the non- K part has norm $O(e^{-\frac{3}{4}\sigma^2})$. By Weyl's inequality, for σ large G_σ has at least N negative eigenvalues. The span of the corresponding coefficient vectors gives an N -dimensional subspace of $\mathcal{A}$ on which $\mathcal{Q}$ is negative definite. The map from coefficients to functions is injective there, since $\mathcal{Q}(F, F) < 0$ forces $F \neq 0$. Finally, replace F_k by $\chi(\cdot/R)F_k \in C_c^\infty$. By density and boundedness of $\mathcal{Q}$ on $\mathcal{A}$, the Gram matrix changes by $o(1)$ as $R \rightarrow \infty$, so negative definiteness on an N -dimensional subspace persists. Thus $\kappa \geq N$, and $\kappa = N_+$ also when $N_+ = \infty$. The argument also gives $\kappa_{\mathcal{A}} = \kappa_{C_c^\infty}$. $\square$

Corollary 3.3 (Pontryagin structure if RH fails mildly). *If $N_+ < \infty$, the quotient of $(C_c^\infty, \mathcal{Q})$ by its isotropic part completes to a Pontryagin space Π_{N_+} , and T_t extends to a Π_{N_+} -unitary group. By Pontryagin's theorem [7, 6] every such group has an N_+ -dimensional non-positive invariant subspace. In the TAA dictionary these are the “ghost” modes, carried by the eigenfunctionals ℓ_ρ, ℓ_{ρ^*} for $\rho \in \mathcal{Z}_+$ with growth $e^{\pm(\operatorname{Re} \rho - \frac{1}{2})t}$.*

(We state the corollary for orientation only. The completion and extension are standard (Iohvidov–Krein–Langer), and we do not use it.)

4. WHAT IS KNOWN AND WHAT IS NEW

- (i) is the explicit formula of Guinand and Weil [4, 11]. Hermitian symmetry is standard.
- (iv)(a) $\iff$ (b) is **Weil's criterion** [11]; see also [1, 12]. Connes [2] recast it as a trace formula on the adèle class space. There the scaling action of $\mathbb{R}_+^*$ (our T_t) is central, and the Hilbert realization in (iv)(c), (v) is implicit. Our proof of (b) $\implies$ (a) avoids the Bochner–Schwartz theorem. It uses only isometry, Cauchy–Schwarz, and a Laplace transform (Lemma 3.2), and we have not seen it written this way, though it is elementary.
- (vi): Bombieri [1] proved that, if only finitely many zeros are off the line, sufficiently large finite truncations of the Weil quadratic form have exactly half as many negative eigenvalues as there are off-line zeros. That number equals our N_+ . Our (vi) is the statement for the full form on C_c^∞ , including $N_+ = \infty$, with a short interpolation proof. We regard it as a mild extension and reformulation, not a new phenomenon.

- (iii), the identification of the growth exponent of the scaling flow’s matrix coefficients with $\Theta - \frac{1}{2}$, is elementary. It is closely related to the classical fact that Θ governs $\psi(x) - x = O(x^{\Theta+\varepsilon})$. We have not found it stated in this dynamical form.
- **New (TAA-specific):** the reading “RH $\iff$ the arithmetic scaling flow is unitary $\iff$ no ghosts ($\kappa = 0$) $\iff$ no runaway modes ($\lambda \leq 0$)”, with the exact ghost count N_+ and Lyapunov exponent $\Theta - \frac{1}{2}$. This turns the heuristic in TAA [C, §6.1] (“an off-line zero injects un-compensated vorticity and tears the chart”) into a theorem: an off-line zero at ρ is exactly a hyperbolic (ghost) plane in the state space and a mode growing like $e^{(\operatorname{Re} \rho - \frac{1}{2})t}$.

5. LIMITATIONS

- (1) **No progress on RH.** All conditions in (iv) are equivalent to RH. None is verified here. The Hilbert-space operator in (v) is built *from* the zeros, so it is not a Hilbert–Pólya operator in the substantive sense: the spectrum is put in by hand. A genuine advance would construct $\mathcal{H}$ and the positivity of $\mathcal{Q}$ from the *arithmetic side* (1) alone, e.g. from a physical (Lee–Yang-type) positivity principle.
- (2) The zero multiplicity m_ρ appears only as a norm weight, not as spectral multiplicity, in (v). The construction cannot detect multiple zeros spectrally.
- (3) The analogy with Nash/Stelle ghosts is structural. Nothing here derives the Weil form from $2R_{\mu\nu}R^{\mu\nu} - R^2$, the Taylor–Wiles isomorphism, Iwasawa theory, or Einstein–Cartan geometry.
- (4) Part (vi) needs arbitrarily wide Gaussians, i.e. test functions with large support. For functions supported in a fixed interval $[-t, t]$, positivity is known for small t (Yoshida: $t = \frac{1}{2} \log 2$ [12]), and the negative index there can be smaller than N_+ .

6. NUMERICAL SANITY CHECKS

Scripts are in `num/` (mpmath, 30 digits).

- `explicit_formula_check.py`: (1) for even Gaussians $H(u) = e^{-u^2/2s^2}$, $s \in \{0.5, 0.3, 0.2\}$, with 300 zeros and $n < 4000$. The zero side and the arithmetic side agree to $10^{-31}, 10^{-17}, 10^{-18}$. For example, $s = 0.2$: both sides = 0.0185904504407629.
- `explicit_formula_noneven.py`: shifted (non-even) Gaussians, $(s, c) \in \{(0.3, 0.7), (0.25, -1.3), (0.4, 2)\}$, 200 zeros, $n < 6000$: agreement to 10^{-16} – 10^{-22} . This confirms the normalization of (1), including the symmetrized prime term.
- The Gram matrix of $\mathcal{Q}$ on 13 shifted Gaussians, computed from the true zeros, is Hermitian and positive semidefinite to 10^{-28} , as it must be for zeros on the line.
- `toy_offline.py` (illustrating (vi)): we add k hypothetical off-line quadruples $\{\rho, 1-\rho, \bar{\rho}, 1-\bar{\rho}\}$, which contain $2k$ zeros with $\operatorname{Re} \rho > \frac{1}{2}$, at heights 17.5, 27, 35.5 with $\beta = 0.1, 0.05, 0.2$. Using test functions $\Phi(z) = e^{-\sigma^2(z-a)^2/2} e^{izt}$ ($\sigma = 2$, $a = \pm \text{heights}$, $t \in [-3, 3]$), we find exactly 0, 2, 4, 6 negative eigenvalues for $k = 0, 1, 2, 3$, as predicted by $\kappa = N_+$.

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