Arithmetic Unitarity I

Where the Riemann Hypothesis Lives in the Scaling Flow

Thermodynamics of Arithmetic Action, Lemmas 1–13

J. W. McGreevy
formalization by Grok Bot

21 October 2026

Abstract

By Weil's criterion the Riemann Hypothesis (RH) is equivalent to positivity of the Hermitian form $\mathcal{Q}$ of the explicit formula. We show that this is the same as *unitarity*, in the sense of absence of negative-norm ("ghost") directions, of the logarithmic scaling flow T_t on test functions. The negative index of $\mathcal{Q}$ equals the number of zeros off the critical line (Chapter 1). The chapters of this volume do not prove RH. They reformulate and localize this single obstruction: along the de Bruijn–Newman heat flow (Ch. 2); in the function-field and Iwasawa models, where the positivity *is* a theorem (Chs. 3–4); as an action principle whose stationary points are the closed orbits (Def. 5); on finite windows and in a semilocal structural trace (Ch. 6); as Lax–Phillips / Pavlov–Faddeev cusp scattering (Ch. 7); through a zero-free intertwiner identifying cusp height with the prime cutoff (Ch. 8); by damping to the abscissa $\sigma > \frac{1}{2}$ (Chs. 9, 11); and by certified computer-assisted positivity on all windows of length $\leq \log 3$ (Ch. 10). A ghost-index calculus quantifies how small off-line zeros could escape detection on finite windows (Ch. 11). An appendix compares contact and foliated flows. What is proved, what is computer-assisted, what is numerical and what is heuristic is stated in the Statement of Intellectual Honesty that follows, and statement by statement in the rigor audit (Appendix B). The single remaining obstruction is window positivity for *every* window length, equivalently a positive structural trace realizing the Weil distribution without reference to the zeros.

Corrections in version 2

This is version 2 of ai.viXra:2610.0045 (v1: 21 October 2026). It corrects the errors listed below; nothing else in the volume is changed, and no theorem is affected. Each corrected spot is marked "Corrected in v2" or "Updated in v2" in the text.

(1) **Ch. 11, §11.4, Table 11.2 (ghost-injection test), one entry.** *v1:* $d_{\mathrm{ghost}} = 2.72$ for $k = 1$, $\delta = 0.4$, odd sector. *v2:* 2.46, one grid step earlier. *Why:* the script `num/lemma11_ghost.py` evaluated $j_n(w)$ as $\sqrt{\pi/(2w)}\, J_{n+1/2}(w)$, which is on the wrong branch for real $w < 0$. As a result the on-line zero γ_k entered with the wrong sign: the code *added* the off-line quadruple next to γ_k instead of *replacing* γ_k by it. The corrected run (`num/lemma11_ghost_fixed.py`) changes only this one of the 60 table entries. All even-sector entries, the slope fits $a = 0.66, 0.57, 0.54, 0.48, 0.42$, the doubly-logarithmic δ-dependence and the fit $d_{\mathrm{ghost}} = 2.28 + 0.192 \log \frac{1}{\delta}$ of Ch. 12 are unchanged to every printed digit, and the bound of at most one negative eigenvalue per sector still holds. The planted-ghost tables remain numerical evidence only.

(2) **Ch. 11, §11.4, method sentence of the ghost-injection test.** *v1* stated that the on-line pair $\pm\gamma_k$ is removed and the quadruple at $\gamma_k \pm i\delta$ added, and cited `num/lemma11_ghost.py`. The formula was correct, but the code did not implement it (item 1). *v2* cites the corrected script, gives the entire form $j_n(w) = \sqrt{\pi/2}\, w^{-1/2} J_{n+1/2}(w)$ and the 110-digit ordinates, and records the earlier run and its one differing threshold. The table caption now cites `num/lemma11_fix_analyze.out`.

(3) **Citation of Lemma 1.** *v1* listed TAA Lemma 1 as "submitted to ai.viXra, identifier pending" (bibliography entry [**83**], Introduction, Appendix B.4). *v2* cites it as ai.viXra:2610.0046.

Statement of Intellectual Honesty

The Riemann Hypothesis is NOT proved in this volume.
Nothing here implies it, and no result here should be quoted as progress "towards a proof" beyond what is listed in Part 1.

Numbers refer to this volume. A statement numbered $k.s.n$ is statement $s.n$ of the standalone note "TAA Lemma k"; Appendix A is the contact/foliation note. The status of every numbered statement is tabulated in Appendix B.

Part 1. What we have proven

1a. Theorems with complete proofs in this volume.

L1 Thm 1.2.1(ii)–(vi): RH $\iff$ $\mathcal{Q} \geq 0$ $\iff$ T_t is unitary on the Weil completion; ghost index $\kappa(\mathcal{Q}) = N_+$ (number of zeros with $\operatorname{Re}\rho > \frac{1}{2}$, with multiplicity); Lyapunov exponent $\Theta - \frac{1}{2}$.

L2 Thm 2.1.3 (abstract form of Ch. 1); Lemmas 2.4.1–2.4.3 (Laguerre step, polynomial "second law", continuity of the heat operator).

D5 Thm 5.2.2 (stationary points of the TAA action = closed orbits/product formula), Props. 5.2.4, 5.2.5, Thm 5.3.1 (isometry group of $\mathcal{Q}$ in $\mathrm{Diff}(\mathbb{R})$), Prop. 5.4.3 (reflection positivity fails at every closed orbit).

L6 Thm 6.3.1 (window/semilocal identity), Thm 6.4.2 with Lemma 6.4.3 (structural trace; explicit oscillatory constants), Thm 6.5.1 (exact atom identity; $d_S(s) = O_S(|s|^{-1/4}\log^{1/2}|s|)$; atoms of σ_S and Φ_S' coincide in Bohr mean), Thm 6.7.1 (window ghost index κ_A finite, monotone, $\sup_A \kappa_A = N_+$; RH $\iff$ $\kappa_A = 0$ for all A).

L7 Thm 7.4.1(i),(iii),(iv) (RH $\iff$ spectral bound $-\frac{1}{4}$ of the Lax–Phillips generator; contractivity and inner-ness are RH-neutral; Hilbert's "missing inner product" exists iff RH), Prop. 7.5.1 (time dictionary $t = 2u$), Prop. 7.6.1 (boosted square lattice).

L8 Prop. 8.2.2, Thm 8.2.4 (zero-free intertwiner J: cusp model $\to$ Weil form), Thm 8.3.1 (Krein diagonalization; simple zeros), Thm 8.5.1 (cusp height d = prime cutoff; RH $\iff$ $\mathcal{Q}^{\mathsf{J}} \geq 0$ for all d), Prop. 8.5.2 (Gårding inequality).

L9 Thm 9.2.1 (inner multipliers have zero defect), Prop. 9.5.1 (Gårding bound at abscissa one).

L11 Props. 11.1.2 ($\limsup \lambda^*(d) \leq 0$), 11.4.2 (index bound $\kappa_A \leq 2M$), 11.4.3 (invisibility threshold, rigorous part); Lemmas 11.3.1, 11.4.1 (exact identities); Prop. 11.3.2 ($\lambda_\sigma(d) \downarrow m_\sigma > 0$ when $\sigma > \beta_\rho$ for every zero ρ, e.g. unconditionally for $\sigma \geq 1$; the location of the infimum is the separate numerical Obs. 11.3.3).

L12 Lemmas 12.2.2, 12.3.1, Thm 12.4.1 (explicit detection criterion; its numerical rows are in 1c), Cor. 12.5.1, Prop. 12.6.2 (function-field window margin = $\log q \cdot \lambda_{\min}(T_n)$), Lemma 12.2.1 (exact identity).

A Thms A.2.1, A.2.2 (a flow cannot be both foliated-isometric and contact on a closed manifold), Cor. A.3.2, Thm A.5.1 (Lefschetz signs), Lemmas A.1.2, A.1.3.

1b. Results resting on cited theorems (proof complete given the citations).

L1 Thm 1.2.1(i): the explicit formula (Weil, Guinand; [59]); Cor. 1.3.3 (Pontryagin spaces).

L2 Thm 2.3.1: heat-flow unitarity ($\kappa(\mathcal{Q}_t) \le \frac{1}{2} Z_{\mathbb{C}}(t)$, second law, critical time Λ_{dBN}, cooling law), from de Bruijn, Newman, Rodgers–Tao and the Polymath bound ([**26**, **88**, **99**]).

L3 Thm 3.2.1: arithmetic unitarity for curves over $\mathbb{F}_q$ (Weil; Hodge index).

L4 Thm 4.2.1: the Iwasawa pillar (Mazur–Wiles, Ribet, Deligne); items D3, D5 include exact machine arithmetic (Bernoulli numbers mod p^4), see 1c.

D5 Lemma 5.1.3 (class field theory), Thm 5.4.1 (Rosati positivity), Prop. 5.5.1 (primon gas = Bost–Connes).

L7 Prop. 7.2.1, Thms 7.3.1, 7.3.2 (Lax–Phillips, Pavlov–Faddeev, quoted), Thm 7.4.1(ii) ($\|Z_t\| = 1$ for all t, so $\omega_0 = 0$; volume proof from the Sz.-Nagy–Foias model and Conrey's $\frac{2}{5}$, no Lax–Phillips criterion needed), Prop. 7.5.2 (prime number theorem; Hadamard product).

L8 Lemma 8.1.1 (model spaces), Thm 8.4.1 (incomparability; uses Conrey's $\frac{2}{5}$, the explicit inequality of Remark 8.3.2 and the convexity bound).

L9 Thm 9.2.2 (full Euler product at abscissa one; the sign $d(\gamma_1) < 0$ is certified in Arb, see 1c), Prop. 9.4.1 (abscissa family; Lagarias' criterion [**70**]).

L13 Props. 13.1.1, 13.1.2 (Levi-Civita, Kustaanheimo–Stiefel), Thm 13.3.1 (Deuring), Thm 13.5.1 (Hecke, Dirichlet); Props. 13.2.1, 13.4.1 have complete proofs.

A Props. A.3.1, A.4.1, A.5.2, A.6.1 and Remark A.5.3 (Deninger, Hutchings–Taubes, Taubes, Selberg).

1c. Computer-assisted, certified.

L10 Thm 10.3.1: $\mathcal{Q}(F,F) \ge 1.447 \cdot 10^{-8} \|F\|_2^2$ for every F supported in an interval of length $\le \log 3$ (constants $5.24 \cdot 10^{-6}$ at $\log 2.5$, $3.67 \cdot 10^{-4}$ at $\log 2.1$); hence $\kappa_A = 0$ for $A \le \log 3$. *Method:* exact reduction Thm 10.2.1 to a Hilbert–Schmidt matrix, frequency split at height T, ball arithmetic (Arb, 160 bits) for the $N \times N$ block, certified Cholesky/Weyl bound, explicit tail bounds. Cor. 10.3.2: positivity on the cusp-truncated model spaces of Ch. 8 for $1 < d \le 3$.

L9 Prop. 9.6.1: positive-definiteness of the 12×12 Gram matrix of a Chebyshev-type basis for $d \in \{2.1, 2.5, 3, 5, 10\}$. *Method:* Arb (200 bits) quadrature, Gershgorin. (A finite-dimensional statement only.)

L4 Thm 4.2.1, items D3, D5: exact integer/p-adic arithmetic of Bernoulli numbers (exact, not floating point).

L9 Thm 9.2.2, last assertion: $d_{\mathrm{all}}^{(1)}(\gamma_1) \in -0.0250030363066 \pm 10^{-21}$ (Arb, 200 bits, γ_1 enclosed by Arb).

L12 Table 12.1: certified detection lengths A_0 for Thm 12.4.1 on a grid of (δ, γ_0). *Method:* Arb (300–800 bits) evaluation of the explicit criterion $\Delta > 0$ with piecewise majorants, whose downward jumps are checked in ball arithmetic. The column $\delta = 0.5$ is a formal reference value outside the theorem.

Code, logs, software versions and SHA-256 hashes are collected in `supplement/` (see its `README.md`). A from-scratch rerun on 21 October 2026 reproduced all three certificates of Thm 10.3.1 digit for digit. The tail bound beyond the computed range is proved in the text of Thm 10.3.1, and its hypotheses are checked in `supplement/logs/tail_conditions.out`.

Part 2. What we suspect to be true

Each item is labelled by its evidence. None is used in any proof in Part 1.

(S1) **Zhu's law** $\lambda^*(d) = e^{-2\pi^2 d(1+o(1))}$ for the window margin (Zhu, arXiv:2608.24827, Conj. 12.1; TAA form Prop. 11.1.1). *Evidence:* numerical (Zhu's profile and our independent Obs. 11.2.1; the Galerkin quadrature is not certified) and a heuristic derivation of $2\pi^2$ (Heur. 11.2.2). *Confirm:* a proof of the lower bound $\lambda^*(d) \ge e^{-Cd}$ for all d would imply RH (Prop. 11.1.2 and Thm 6.7.1), so confirmation of the *law* is not expected without RH;

the upper bound is the accessible half. *Refute:* a certified $\lambda^*(d)$ at some d outside the predicted band.

(S2) **Ghost-detection slope.** A single off-line quadruple of displacement δ becomes visible in $\kappa_{\log d}$ only for $\log d \gtrsim a \log\log \frac{1}{\delta}$ with $a \approx 0.4$–0.7 (Heur. 11.4.4). *Evidence:* numerical (planted-ghost experiments) plus the rigorous invisibility bound Prop. 11.4.3. *Confirm/refute:* certified planted-ghost computations at several δ; a matching rigorous detection (upper) bound.

(S3) **Logarithmic singularity law** for the semilocal defect, amplitude $A(p)$ (Obs. 6.6.1, Cor. 6.6.2, Obs. 9.3.1, Cor. 9.3.2). *Evidence:* numerical, $A(p)$ fitted to $\pm 3\%$. *Confirm:* an asymptotic expansion of σ_S; *refute:* a bounded kernel.

(S4) *(Moved to Part 1.)* The asymptotic symbol $d_S(s) \to 0$ (Thm 6.5.1(b)) is now proved, with rate $O_S(|s|^{-1/4}\log^{1/2}|s|)$.

(S5) **σ-RG law** $c/(\beta-\sigma)$ near the abscissa of an off-line zero, and the dichotomy of Remark 11.3.4. *Evidence:* numerical/heuristic.

(S6) **Contact lamination (Question A.9.1).** The arithmetic flow is modelled by a contact lamination whose prime orbits are elliptic. *Evidence:* structural analogy (Thms A.2.1–A.5.1 rule out the closed-manifold versions). *Refute:* a sign or length obstruction for laminations.

(S7) **Arithmetic-site positivity (Question 5.4.2).** A Hodge-index/Rosati-type positivity for ζ. *Evidence:* analogy with the proved curve case (Ch. 3, Thm 5.4.1). Caution: by a GNS construction the existence of such a structure is *equivalent* to RH, so its existence is not evidence; only a construction independent of the zeros would be.

(S8) **Newton-triangle / Kustaanheimo–Stiefel / χ_4 dictionary.** A structural analogy linking the boosted square lattice (Prop. 7.6.1: the boost passes $\tau = i$ and $e^{i\pi/3}$ and runs up the cusp, rapidity = TAA log-scale), the Kustaanheimo–Stiefel (Hopf) regularization of the Kepler problem, the four-square identity $\sum r_4(n)n^{-s} = 8(1-4^{1-s})\zeta(s)\zeta(s-1)$, and the character χ_4 with the CM curve $y^2 = x^3 - x$ (to be developed in Lemma 13). *Evidence:* analogy plus true identities; there is *no* prime $\leftrightarrow$ closed-orbit correspondence in it. *Confirm:* an identity of operators or flows, not of formulas; *refute:* showing the correspondence is not functorial.

(S9) **Physical dictionary** (TAA dictionary Remark 1.1.4; Nash–Stelle ghosts; zero-point energy Remark 5.5.2; "$m_g = m_i$" Remark 5.5.3). *Evidence:* analogy/speculation, explicitly labelled; Remark 5.5.3 is speculative.

Part 3. Our final goal

The aim of the TAA programme is to *derive* RH as a unitarity (ghost-freedom) theorem: to exhibit a structural positivity principle on the "arithmetic surface" $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$ (in the sense of the Hodge index theorem on $C\times C$ in the function-field case, Ch. 3), constructed *without* reference to the zeros, from which $\mathcal{Q} \geq 0$ follows. A second aim is to let the natural symmetries of the numbers (scaling, the adelic product formula, the modular group) define classical, quantum and relativistic mechanics without contrivance: closed orbits as stationary points of an action (Def. 5), quantization as the explicit formula, and Lorentz boosts as the scaling flow on the modular surface (Ch. 7). The guiding principle is: *let the mathematics decide what the closest true statement is.* Where a hoped-for statement is false, we prove that it is false (e.g. Thms 7.4.1(ii), 8.4.1, Prop. 5.4.3) and record the sharpest true replacement.

Status. RH is **not** proven here. The single remaining obstruction, in every picture of this volume, is the same: *window positivity for all lengths*, i.e. $\kappa_A = 0$ for every A (equivalently $\mathcal{Q}^{\mathsf{J}} \geq 0$ for every cusp height d, Thm 8.5.1), or, structurally, a *positive trace* realizing the Weil distribution

that is built without the zeros. It is certified for $A \le \log 3$ (Thm 10.3.1); Zhu's law (S1) indicates that the margin decays like $e^{-2\pi^2 e^A}$, so no finite computation can reach all A.

Corrected errors in earlier versions

The author's earlier ai.viXra preprints contain errors that were identified in an archive review prepared for this project. The following are corrected or withdrawn by the present volume; they are listed briefly and only where documented in that review. Labels E1–E9 are used in Chapters 3 and 4.

E1. *Circular Hilbert–Pólya arguments* (2603.0049 §9, 2606.0029, 2607.0006, 2607.0050, 2608.0003, 2608.0085): a self-adjoint operator was postulated with the zeros as spectrum. In 2607.0050 Lemma 5.1 the step "$D = \Xi$ from order 1 and equality at 0" is false ($\Xi(s)e^s$ is a counterexample). Corrected by Chs. 1, 7: unitarity is *equivalent* to RH, not a route to it.

E2. *Functional-equation symmetry taken to imply RH* (2602.* series; 2608.0003 Lemma 3.2). False: Davenport–Heilbronn type functions satisfy a functional equation and violate RH. See Remark 3.3.1.

E3. *Unitarity on the critical line taken to imply RH* (2604.0106, 2609.0039). False: $|\varphi| = 1$ on the line holds unconditionally (Prop. 7.2.1(d), Thm 7.4.1(iii)).

E4. *Numerical constants adjusted to fit* (2509.0047, 2510.0053, 2511.0075, 2512.0086, 2602.0043, 2605.0006, 2608.0043): fitted coefficients presented as derivations; a Python computation presented as a proof (2608.0043). Not used here; computations in this volume are either certified or labelled numerical.

E5. *Unverifiable references* (2510.0043, 2512.0086, 2608.0068 v1; removed in v2). All references in this volume were checked against the literature.

E6. *Factual slips*: Frobenius traces of $y^2 = x^3 - x$ in 2510.0007 (correct: $a_p = 0$ for $p \equiv 3 \bmod 4$, $a_p = \pm 2a$ with $p = a^2 + b^2$ otherwise), and its reduction at 2 (additive, not multiplicative); a weight-2 dimension 21493761 for 196884 (2511.0075); a misstated temperature regime ("$\beta < 1$") for the Bost–Connes phase transition (2606.0040); "47 irregular primes" (2606.0066); 691 as a bad prime of Δ (2608.0043); a PID claim for local rings (2610.0013); Herbrand–Ribet index conventions (corrected in Ch. 4).

E7. *Internal inconsistency*: two different genera (5 and 6) stated for the same object.

E8. *Category errors*: identification of Onsager reciprocity with Clairaut's theorem.

E9. *Overclaiming*: statements that RH or related conjectures had been proved. Withdrawn; see the box above.

Contents

Abstract i
Corrections in version 2 i

Statement of Intellectual Honesty iii

Statement of Intellectual Honesty iii
Part 1. What we have proven iii
Part 2. What we suspect to be true iv
Part 3. Our final goal v
Corrected errors in earlier versions vi

Introduction xi

Introduction xi

Chapter 1. Lemma 1: Arithmetic unitarity 1
1.1. Setting and notation 1
1.2. Main theorem 2
1.3. Proofs 2
1.4. What is known and what is new 4
1.5. Limitations 5
1.6. Numerical sanity checks 5

Chapter 2. Lemma 2: Heat-flow unitarity 7
2.1. Admissible spectral data and Lemma 1 in general form 7
2.2. The de Bruijn–Newman family 8
2.3. Main theorem 8
2.4. Proofs 9
2.5. Thermodynamic reading, stated as theorems only 10
2.6. Known vs new 11
2.7. Limitations 11
2.8. Numerical sanity checks 11

Chapter 3. Lemma 3: The function-field model 13
3.1. Setting and known facts 13
3.2. Main theorem 14
3.3. Proofs 15
3.4. Known versus new 15
3.5. Numerical sanity checks (`num/ff_check.py`, `.out`) 16

Chapter 4. Lemma 4: The Iwasawa pillar 17
4.1. Setting and cited theorems 17
4.2. The theorem 18
4.3. Proofs 18

4.4. Structural parallel with Lemmas 1–3, and where it breaks 19
4.5. Known versus new 19
4.6. Numerical checks (`num/iwasawa_check.py`, `.out`) 20

Chapter 5. Definition 5: The arithmetic action 21
5.1. Configuration space and fields 21
5.2. The arithmetic action (candidate A) 22
5.3. Covariance 24
5.4. Positivity: which Hermitian form, and reflection positivity 24
5.5. Thermodynamics (candidate B) and the unified dictionary 25
5.6. Consistency with Lemmas 1–3 26
5.7. Known vs. new; numerics 27

Chapter 6. Lemma 6: The structural positive trace on finite sets of places 29
6.1. Conventions (as in Lemma 1) 29
6.2. What is in the literature (all items verified) 30
6.3. Window equals semilocal 30
6.4. The structural trace τ_S 30
6.5. Exact cancellation of the prime-power atoms 33
6.6. The precise obstruction 34
6.7. Unconditional finite negative index on every window 35
6.8. Exact failure points 35
6.9. Numerics 35
6.10. Known vs. new 36

Chapter 7. Lemma 7: Scattering unitarity 37
7.1. Conventions 37
7.2. The scattering matrix: exact forms 38
7.3. The Lax–Phillips and Pavlov–Faddeev theorems (classical) 38
7.4. What RH is equivalent to in this framework 39
7.5. Connection to TAA 40
7.6. Newton triangle dictionary 42
7.7. Known vs. new 44
7.8. The precise remaining obstruction 44
7.9. Open questions 44

Chapter 8. Lemma 8: A zero-free intertwiner from the Lax–Phillips core to the Weil space 45
8.1. Conventions 45
8.2. The zero-free map 46
8.3. Root vectors: the transported Weil pairing 47
8.4. Comparison with the Lax–Phillips norm 48
8.5. Cusp height equals prime cutoff 49
8.6. Numerical checks 49
8.7. Known vs. new 50
8.8. The precise obstruction 50

Chapter 9. Lemma 9: Lemma 6's structural trace at abscissa one 51
9.1. Setting and conventions 51
9.2. Inner multipliers and the full product at abscissa one 52
9.3. Finite S: softened singularities (numerical) 53
9.4. The cost of the shift: the abscissa family 53

9.5. A Gårding bound at abscissa one 54
9.6. Certified finite sections (rank-2 item, merged) 55
9.7. Answers, known vs. new, obstruction 56
9.8. Numerics 56
9.9. Open questions 56

Chapter 10. Lemma 10: Certified Weil positivity on every window of length $\leq \log 3$ 57
10.1. Setting 57
10.2. The exact finite reduction 58
10.3. Certificates 59
10.4. Literature and honest comparison 60
10.5. Obstruction and outlook 61
10.6. Open questions 61

Chapter 11. Lemma 11: The window margin $\lambda^*(d)$: asymptotic law, resolution height $2\pi d$, and a quantitative first-ghost criterion 63
11.1. Setting, priority, and the law in TAA variables 63
11.2. Independent computation of $\lambda^*(d)$ 64
11.3. The σ-direction: no collapse off the critical line 65
11.4. Ghosts at $\sigma = \frac{1}{2}$: identity, index bound, invisibility 67
11.5. Relation to κ_A (Lemma 6) and the σ-RG index (Lemma 9) 68
11.6. Obstruction and questions 69

Chapter 12. Lemma 12: Explicit ghost detection, what window certificates say about zeros, and a function-field degeneration test 71
12.1. Setting 71
12.2. The modulated two-bump test function 72
12.3. Counting and summation 72
12.4. Explicit ghost detection 73
12.5. The converse: what a positivity certificate rules out 74
12.6. Add-on: the doubly-log law and a function-field degeneration test 74
12.7. Relations, obstruction, questions 76

Chapter 13. Lemma 13: The squaring map at $\tau = i$ 77
13.1. (a) Levi-Civita and Kustaanheimo–Stiefel 77
13.2. (b) Discriminants and the wild prime 2 78
13.3. (c) The CM curve $y^2 = x^3 - x$ and its Hecke character 78
13.4. (d) The Newton-triangle dictionary: $\tan / \sec$, χ_4, $\beta(1) = \pi/4$ 79
13.5. (e) CM Sato–Tate versus non-CM 79
13.6. Status, obstruction, questions 80

Appendix A. Contact or foliation? 81
A.1. Setting 81
A.2. The dichotomy, and why "both" fails on closed manifolds 82
A.3. Lengths: the period obstruction and the function-field case 83
A.4. Transverse expansion: where "1" and "$\frac{1}{2}$" come from 83
A.5. Signs: the arithmetic sign rule singles out non-saddle orbits 84
A.6. The Selberg side: what contact flows prove, and where ζ enters 85
A.7. Status of the arithmetic models 86
A.8. Bottom line 86
A.9. Questions for parallel work 86

Further context (cited, not used in proofs) 87

Appendix B. Rigor audit 89
B.1. Summary 89
B.2. Statement-by-statement table 89
B.3. Gaps that blocked “fully rigorous”, and their status 94
B.4. Recommended revisions 95
B.5. Editorial changes made in the volume (sources untouched) 95
B.6. Changelog: gap closures of 21 October 2026 95
B.7. Referee pass (21 October 2026) 96

Bibliography 99

Introduction

The single thread. Let $\mathcal{Q}$ be the Hermitian form of Weil's explicit formula on even test functions F on the log-scale line $\mathbb{R}$ (Def. 1.1.3), and let $T_t F(u) = F(u-t)$ be the scaling flow. Weil's criterion states RH $\iff$ $\mathcal{Q} \geq 0$. Chapter 1 recasts this as a statement about dynamics: T_t preserves $\mathcal{Q}$, and RH holds exactly when the flow is *unitary* on the completion of $(\mathcal{A}, \mathcal{Q})$. If RH fails, the completion is a Pontryagin space with exactly $\kappa = N_+$ negative directions ("ghosts", in the sense of indefinite-metric field theory, cf. the Nash–Stelle analogy, Remark 1.1.4), and the flow has Lyapunov exponent $\Theta - \frac{1}{2} > 0$. The chain

$$\text{RH} \iff \mathcal{Q} \geq 0 \iff \kappa(\mathcal{Q}) = 0 \iff T_t \text{ unitary (no ghosts)}$$

is elementary and is not progress on RH; it fixes the language. Every later chapter is an attempt to *locate* the possible ghosts, and each ends in an honest statement of where the obstruction went.

How each chapter reformulates or localizes the obstruction.

Ch. 2 (heat flow).: Along the de Bruijn–Newman flow the ghost index is bounded by half the number of non-real zeros, which is finite for $t > 0$ and non-increasing (a "second law"); κ itself is not monotone (Example 2.3.2). The flow is unitary exactly for $t \geq \Lambda_{\mathrm{dBN}}$, unconditionally for $t \geq 0.2$ (Thm 2.3.1). Since $\Lambda_{\mathrm{dBN}} \geq 0$ (Rodgers–Tao), the obstruction becomes $\Lambda_{\mathrm{dBN}} = 0$.

Chs. 3–4 (models where positivity is a theorem).: For curves over $\mathbb{F}_q$ the same unitarity is *proved*, from the Hodge index theorem (Thm 3.2.1). In Iwasawa theory the Hilbert–Pólya shape "zeros = eigenvalues" is a theorem, but positivity is contentless p-adically (Remark 4.4.1). The missing ingredient for ζ is a Hodge-index-type positivity.

Def. 5 (action).: Closed orbits of the scaling flow are exactly the stationary points of an adelic action (Thm 5.2.2), and the isometries of $\mathcal{Q}$ are classified (Thm 5.3.1). Reflection positivity fails at every closed orbit (Prop. 5.4.3), so positivity cannot come from that mechanism; Question 5.4.2 is the TAA form of the Hodge-index question.

Ch. 6 (windows and semilocal trace).: Restricting to test functions supported in $[-A/2, A/2]$ involves only primes $p \leq e^A$ (Thm 6.3.1) and gives a finite, monotone ghost index κ_A with RH $\iff$ $\kappa_A = 0$ for all A (Thm 6.7.1). The semilocal structural trace τ_S (Connes–Consani) is positive by construction, but its remainder D_S is singular exactly at prime-power atoms (conditional Cor. 6.6.2).

Ch. 7 (cusp scattering).: Lax–Phillips and Pavlov–Faddeev realize the zeros as resonances of a contraction semigroup on the modular surface. RH is equivalent to a *smoothed* decay rate (Thm 7.4.1(i)); contractivity, inner-ness and unitarity on the line are RH-neutral.

Ch. 8 (zero-free intertwiner).: An explicit map J intertwines the cusp model with the Weil form and contains no zeros (Thm 8.2.4). The cusp height d is the prime cutoff: RH $\iff$ $\mathcal{Q}^{\mathsf{J}} \geq 0$ for all d, and the negative index at height d is $\kappa_{\log d}$ (Thm 8.5.1). No bounded comparison with the Lax–Phillips norm exists (Thm 8.4.1); a Gårding inequality is the sharpest one (Prop. 8.5.2).

Chs. 9, 11 (σ-damping).: Moving to abscissa $\sigma > \frac{1}{2}$ softens the singularities: at $\sigma = 1$ the full Euler product is a Blaschke product with a rank-one defect (Thm 9.2.2), and positivity holds

for every window with a uniform margin when $\sigma > \beta_\rho$ for every zero ρ, which is unconditional for $\sigma \geq 1$ (Prop. 11.3.2). The limit $\sigma \downarrow \frac{1}{2}$ is Lagarias' criterion (Prop. 9.4.1(e)): the obstruction is the limit, not any fixed σ.

Ch. 10 (certified positivity).: $\kappa_A = 0$ for every $A \leq \log 3$, with an explicit margin, computer-assisted (Thm 10.3.1). Hence by Ch. 8 the cusp model is ghost-free up to height 3 (Cor. 10.3.2).

Ch. 11 (ghost index).: Zhu's numerical law $\lambda^*(d) \approx e^{-2\pi^2 d}$, independent computations, an exact quadruple identity, the index bound $\kappa_A \leq 2M$, and a rigorous invisibility threshold: an off-line zero of displacement δ is invisible on windows with $\log d \lesssim \log\log \frac{1}{\delta}$ (Prop. 11.4.3). Windows can certify ghost-freedom only up to an extremely slowly growing height.

Ch. 12 (explicit ghost detection).: A rigorous criterion with computer-assisted explicit constants: a locally extremal, isolated off-line zero of displacement δ forces $\kappa_A \geq 1$ for $A \geq A_0 \approx (3.7\text{–}8.4)/\delta$ (Thm 12.4.1). Conversely, today's certified windows (length ≤ 1.6) exclude nothing (Prop. 12.5.2). A function-field degeneration test is also given.

Ch. 13 (the squaring map at $\tau = i$).: Levi-Civita and Kustaanheimo–Stiefel regularization, the ramified prime 2 of $\mathbb{Q}(i)$, the CM curve $y^2 = x^3 - x$ and CM Sato–Tate. This is a dictionary with labelled analogies, not a mechanism.

Appendix A (contact vs. foliation).: A flow on a closed manifold cannot be both of Deninger's foliated type and contact (Thm A.2.2). Lefschetz signs single out non-saddle orbits (Thm A.5.1), and on the modular surface the zeros enter only through cusp scattering (Prop. A.6.1), which is the subject of Chs. 7–8.

What is and is not claimed. Nothing in this volume proves RH or any statement known to be equivalent to it. Equivalences with RH (Thms 1.2.1, 6.7.1, 7.4.1(i), 8.5.1) are reformulations; their value is that they place the obstruction in a specific operator and a specific finite window. Results are labelled in the text as rigorous, computer-assisted, numerical or heuristic. Computer-assisted results are Prop. 9.6.1, the sign $d^{(1)}_{\mathrm{all}}(\gamma_1) < 0$ in Thm 9.2.2, Thm 10.3.1 with Cor. 10.3.2, the certified rows of Table 12.1 (explicit detection lengths A_0 for Thm 12.4.1), and the exact computations D3, D5 of Thm 4.2.1. All numerical observations (Obs. 6.6.1, 9.3.1, 11.2.1, the planted-ghost tables of Ch. 11, and Obs. 11.3.3 on the location of $\inf w_\sigma$) are evidence only.

Priority. Positivity of the Weil form on short windows has also been studied in recent unrefereed preprints: Zhu [**123**] (arXiv:2608.24827, v1 August 2026), who claims $\mathcal{Q} \geq 8.9 \cdot 10^{-18}\|F\|^2$ on windows of length 1.6 and proposes the law $\lambda^*(d) = e^{-2\pi^2 d(1+o(1))}$, and Eid [**36**] (Zenodo, July 2026), who claims positivity up to length $1.12 \log 2$. We have not checked their proofs; see Chapters 10–11 for a comparison. Our certified result (Thm 10.3.1) was obtained independently with a different basis and tail argument. The Lax–Phillips and Pavlov–Faddeev material in Chapter 7 is classical and quoted; what is new there is identified in each statement. Chapter 1 (TAA Lemma 1) has appeared separately as a standalone preprint [**83**].[1] The present Chapter 1 differs from it only in macro names (identical rendering).

Conventions and reading guide. Statements are numbered chapter.section.item. Standalone cross-references of the form "Lemma k, Thm. $s.n$" therefore refer to statement $k.s.n$ here; the text of the notes has otherwise been kept verbatim, with editorial changes marked by footnotes "[Vol. 1 editorial note]" and listed in Appendix B. Notation follows the shared macro file `taa-macros.tex`. The following conventions differ between chapters and should be kept in mind:

[1] Updated in v2; v1 said "has been submitted separately".

Object	Convention	Chapters
Fourier transform	$\widetilde{F}(z) = \int F(u)e^{izu}du$	1, 2, 3, 8
	$\widehat{F}(s) = \int F(u)e^{-isu}du$	6, 9, 10
	$\widehat{F}(t) = \int F(u)e^{itu}du$	11 (despite "as Lemma 10")
γ_ρ	complex, $\gamma_\rho = -i(\rho - \frac{1}{2})$	1, 8, 9
	real, $\operatorname{Im}\rho$ (volume edit: written $\operatorname{Im}\rho$)	7
B	Lax–Phillips generator B / Blaschke product B	7, 8 / 8, 9
S	set of places / scattering matrix S	6, 9 / 7, 8
κ	ghost index κ; coefficients κ_m	all; 6
λ	window margin λ^*; Lyapunov exponent; eigenvalue	10–11; 1; passim

References "[C]" in Chapters 1–2 are to the author's unpublished manuscript *Thermodynamics of Arithmetic Action: Scale-Free Vacuum Fields and the Modularity Isomorphism* (October 19, 2026), which motivates the vocabulary; no proof here depends on it. Files `num/...` named in the text are the computation scripts and logs.

CHAPTER 1

Lemma 1: Arithmetic unitarity

The Riemann hypothesis as absence of ghosts and runaway modes for the scaling flow of the Weil form

Summary. We make precise, and prove, the following version of the TAA thesis that "the Riemann hypothesis is a statement about unitarity". On $C_c^\infty(\mathbb{R})$ (functions of the logarithmic scale $u = \log x$), the Weil explicit-formula pairing $\mathcal{Q}$ is a Hermitian form. This form is defined arithmetically by primes, the pole, and the archimedean place, and spectrally by the zeros. The scaling flow $T_t F(u) = F(u-t)$ is *always* $\mathcal{Q}$-isometric (pseudo-unitary). We show: (a) the exponential growth rate of the matrix coefficients $t \mapsto \mathcal{Q}(T_t F, G)$ is exactly $\Theta - \frac{1}{2}$, where $\Theta = \sup \operatorname{Re}\rho$; (b) RH $\iff$ $\mathcal{Q} \geq 0$ $\iff$ the flow is realized by a strongly continuous unitary group on a Hilbert space $\iff$ all matrix coefficients are bounded; (c) the number of negative squares of $\mathcal{Q}$ equals the number $N_+ \in \{0, 1, \dots, \infty\}$ of distinct zeros with $\operatorname{Re}\rho > \frac{1}{2}$. Under RH the generator has simple pure point spectrum equal to the set of zero ordinates. Nothing here proves or disproves RH. Parts (b) and (c) are, in substance, due to Weil and Bombieri; Section 1.4 records exactly what is new.

1.1. Setting and notation

Let $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$. Its zeros are the nontrivial zeros ρ of ζ. They lie in $0 < \operatorname{Re} s < 1$ (Hadamard, de la Vallée Poussin) and, since $\xi(1-s) = \xi(s)$ and $\xi(\bar{s}) = \overline{\xi(s)}$, the multiset $\mathcal{Z}$ of zeros is invariant under $\rho \mapsto 1-\rho$ and $\rho \mapsto \bar{\rho}$, with multiplicities m_ρ preserved. Write

$$\rho = \tfrac{1}{2} + i\gamma_\rho, \qquad \gamma_\rho \in \mathbb{C}, \qquad |\operatorname{Im}\gamma_\rho| < \tfrac{1}{2}.$$

Thus $\operatorname{Re}\rho = \frac{1}{2} - \operatorname{Im}\gamma_\rho$, and $\rho \mapsto \rho^* := 1 - \bar{\rho}$ corresponds to $\gamma \mapsto \bar{\gamma}$. RH is the statement that every γ_ρ is real. In what follows, sums $\sum_\rho$ are over *distinct* zeros, weighted by m_ρ. Put $\Theta := \sup_\rho \operatorname{Re}\rho \in [\frac{1}{2}, 1]$ (there are zeros on the line, so $\Theta \geq \frac{1}{2}$), and

$$N_+ := \#\{\rho \text{ distinct} : \ \operatorname{Re}\rho > \tfrac{1}{2}\} \in \{0, 1, 2, \dots\} \cup \{\infty\}.$$

We use the classical fact $\sum_\rho m_\rho (1+|\gamma_\rho|^2)^{-1} < \infty$, which follows from $N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O(\log T)$ [**25**, Ch. 15] or [**116**, §9.4].

For $F \in L^1_{\mathrm{loc}}(\mathbb{R})$ with enough decay we set $\widetilde{F}(z) = \int_{\mathbb{R}} F(u) e^{izu}\, du$. Translation (the logarithmic *scaling flow*) is $(T_t F)(u) = F(u-t)$, so $\widetilde{T_t F}(z) = e^{izt}\widetilde{F}(z)$. In multiplicative language, with $x = e^u$, $f(x) = x^{-1/2}F(\log x)$, we have $\widetilde{F}(\gamma_\rho) = \int_0^\infty f(x) x^{\rho-1} dx$ (Mellin transform at ρ), and T_t is the dilation $x \mapsto e^{-t}x$ normalized to be $L^2(dx)$-unitary.

Definition 1.1.1 (Test class). Let $\mathcal{A}$ be the space of $F \in C^2(\mathbb{R})$ with $\|F\|_{\mathcal{A}} := \sum_{j=0}^{2} \int_{\mathbb{R}} |F^{(j)}(u)|\, e^{|u|/2}\, du < \infty$. Then $C_c^\infty(\mathbb{R}) \subset \mathcal{A}$ is dense: for a cutoff $\chi \in C_c^\infty$ with $\chi = 1$ near 0, $\|F - \chi(\cdot/R)F\|_{\mathcal{A}} \to 0$ by dominated convergence. Also $T_t\mathcal{A} = \mathcal{A}$.

Lemma 1.1.2. *For $F \in \mathcal{A}$ and $|\operatorname{Im} z| \leq \frac{1}{2}$: $(1+|z|^2)\,|\widetilde{F}(z)| \leq \|F\|_{\mathcal{A}}$.*

PROOF. $|e^{izu}| = e^{-u\operatorname{Im} z} \le e^{|u|/2}$ gives $|\widetilde{F}(z)| \le \int |F| e^{|u|/2}$. For $j = 0, 1$ the function $w_j = F^{(j)} e^{|u|/2}$ is integrable with integrable derivative on each half-line, so it has limits at $\pm\infty$, and these limits are 0. Hence two integrations by parts on $[-R, R]$, $R \to \infty$, have vanishing boundary terms and give $z^2 \widetilde{F}(z) = -\widetilde{F''}(z)$, so $|z|^2 |\widetilde{F}(z)| \le \int |F''| e^{|u|/2}$. Add the two bounds. □

DEFINITION 1.1.3 (Weil form). For $F, G \in \mathcal{A}$ let

$$\mathcal{Q}(F, G) := \sum_{\rho} m_\rho \, \widetilde{F}(\gamma_\rho) \, \overline{\widetilde{G}(\bar{\gamma}_\rho)}.$$

By Lemma 1.1.2, $|\mathcal{Q}(F, G)| \le C_{\mathcal{Z}} \|F\|_{\mathcal{A}} \|G\|_{\mathcal{A}}$ with $C_{\mathcal{Z}} = \sum_\rho m_\rho (1 + |\gamma_\rho|^2)^{-2} < \infty$. The series converges absolutely.

REMARK 1.1.4 (TAA dictionary). In TAA language, $\mathcal{Q}$ is the "arithmetic action" pairing and T_t the "arithmetic time". $\mathcal{Q}$ is *indefinite* a priori, exactly like the indefinite (Krein/Pontryagin) state space of the Nash–Stelle Lagrangian $2R_{\mu\nu}R^{\mu\nu} - R^2$ with its massive spin-2 ghost [**106**]. This is an analogy that motivates the questions below. It is not used in any proof.

1.2. Main theorem

THEOREM 1.2.1 (Arithmetic unitarity). (i) (Hermitian; arithmetic side.) *$\mathcal{Q}(G, F) = \overline{\mathcal{Q}(F, G)}$ on $\mathcal{A}$. For $F, G \in C_c^\infty(\mathbb{R})$, with $H = F * G^*$ and $G^*(u) = \overline{G(-u)}$,*

(1.2.1)

$$\mathcal{Q}(F, G) = \widetilde{H}(\tfrac{i}{2}) + \widetilde{H}(-\tfrac{i}{2}) - H(0)\log\pi + \frac{1}{2\pi}\int_{\mathbb{R}} \widetilde{H}(r)\,\operatorname{Re}\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{ir}{2}\Big)\,dr - \sum_{n \ge 2} \frac{\Lambda(n)}{\sqrt{n}}\big(H(\log n) + H(-\log n)\big).$$

(ii) (Pseudo-unitarity, unconditional.) *$\mathcal{Q}(T_t F, T_t G) = \mathcal{Q}(F, G)$ for all $t \in \mathbb{R}$. Each functional $\ell_\rho(F) = \widetilde{F}(\gamma_\rho)$ is an eigenfunctional: $\ell_\rho(T_t F) = e^{i\gamma_\rho t}\ell_\rho(F)$, with $|e^{i\gamma_\rho t}| = e^{(\operatorname{Re}\rho - \frac{1}{2})t}$.*

(iii) (Lyapunov exponent.) *For $F, G \in C_c^\infty(\mathbb{R})$ put $\lambda(F, G) := \limsup_{t \to +\infty} \frac{1}{t} \log |\mathcal{Q}(T_t F, G)|$ (with $\log 0 = -\infty$). Then*

$$\sup_{F, G \in C_c^\infty} \lambda(F, G) = \Theta - \tfrac{1}{2}.$$

(iv) (RH $\iff$ unitarity.) *The following are equivalent:*
- *(a) RH;*
- *(b) $\mathcal{Q}(F, F) \ge 0$ for all $F \in C_c^\infty(\mathbb{R})$ (Weil positivity);*
- *(c) there are a Hilbert space $\mathcal{H}$, a linear map $J : C_c^\infty(\mathbb{R}) \to \mathcal{H}$ with $\langle JF, JG\rangle = \mathcal{Q}(F, G)$, and a strongly continuous unitary group $U(t)$ on $\mathcal{H}$ with $U(t)J = JT_t$;*
- *(d) every matrix coefficient $t \mapsto \mathcal{Q}(T_t F, G)$, $F, G \in C_c^\infty$, is bounded on $\mathbb{R}$.*

(v) (Spectrum under RH.) *If RH holds, one may take $\mathcal{H} = \ell^2(\mathcal{Z}, m)$ (weights m_ρ), $JF = (\widetilde{F}(\gamma_\rho))_\rho$ (dense range) and $U(t) = e^{itA}$, where A is multiplication by γ_ρ. Then A is self-adjoint with pure point spectrum $\{\gamma_\rho\}$, and every eigenvalue is simple.*

(vi) (Ghost count.) *Let $\kappa(\mathcal{Q})$ be the supremum of dimensions of subspaces $V \subset C_c^\infty(\mathbb{R})$ with $\mathcal{Q}(F, F) < 0$ for all $0 \ne F \in V$. Then $\kappa(\mathcal{Q}) = N_+$. The same holds on $\mathcal{A}$.*

1.3. Proofs

Two auxiliary facts.

LEMMA 1.3.1. *For every $z \in \mathbb{C}$ there is $F \in C_c^\infty(\mathbb{R})$ with $\widetilde{F}(z) \ne 0$.*

PROOF. Let $F_\varepsilon \ge 0$, $\int F_\varepsilon = 1$, $\operatorname{supp} F_\varepsilon \subset [-\varepsilon, \varepsilon]$. Then $|\widetilde{F_\varepsilon}(z) - 1| \le \sup_{|u| \le \varepsilon} |e^{izu} - 1| \to 0$. □

LEMMA 1.3.2. *Let $a_\rho \in \mathbb{C}$ with $\sum_\rho |a_\rho| < \infty$, and put $L(t) = \sum_\rho a_\rho e^{i\gamma_\rho t}$. If $|L(t)| \le Ce^{\varepsilon t}$ for all $t \ge 0$, then $a_\rho = 0$ for every ρ with $\operatorname{Re}\rho - \frac{1}{2} > \varepsilon$.*

PROOF. For $\operatorname{Re} s > \frac{1}{2}$, Fubini (legitimate since $|e^{i\gamma t}| \le e^{t/2}$) gives $\mathcal{L}(s) := \int_0^\infty L(t)e^{-st}dt = \sum_\rho \frac{a_\rho}{s-i\gamma_\rho} =: M(s)$. The pole set $P = \{i\gamma_\rho\}$ is closed and discrete, because $|\gamma_\rho| \to \infty$, and distinct ρ give distinct poles. On compacta at distance $\ge d > 0$ from P the series for M is dominated by $d^{-1}\sum|a_\rho|$. Hence M is meromorphic on $\mathbb{C}$ with simple poles at $i\gamma_\rho$ and residues a_ρ. The bound on L makes $\mathcal{L}$ holomorphic on $\operatorname{Re} s > \varepsilon$. The set $\Omega = \{\operatorname{Re} s > \varepsilon\} \setminus P$ is connected, and $\mathcal{L} = M$ on the open subset $\operatorname{Re} s > \frac{1}{2}$, so $\mathcal{L} = M$ on Ω. At $s_0 = i\gamma_{\rho_0}$ with $\operatorname{Re} s_0 = \operatorname{Re}\rho_0 - \frac{1}{2} > \varepsilon$, $M = \mathcal{L}$ is bounded near s_0, so the residue a_{ρ_0} vanishes. □

PROOF OF (I). Using $\rho \mapsto \rho^* = 1 - \bar\rho$ (a bijection of $\mathcal{Z}$ preserving m, with $\gamma_{\rho^*} = \bar\gamma_\rho$): $\overline{\mathcal{Q}(F,G)} = \sum_\rho m_\rho \widetilde{G}(\bar\gamma_\rho)\overline{\widetilde{F}(\gamma_\rho)} = \sum_\rho m_{\rho^*}\widetilde{G}(\gamma_{\rho^*})\overline{\widetilde{F}(\bar\gamma_{\rho^*})} = \mathcal{Q}(G,F)$. Next, $\widetilde{G^*}(z) = \overline{\widetilde{G}(\bar z)}$, so $\widetilde{H}(z) = \widetilde{F}(z)\overline{\widetilde{G}(\bar z)}$ and $\mathcal{Q}(F,G) = \sum_\rho m_\rho \widetilde{H}(\gamma_\rho)$. Formula (1.2.1) is then the Guinand–Weil explicit formula for $H \in C_c^\infty(\mathbb{R})$ [**119, 45**], in the normalization of [**59**, Thm. 5.12] and [**6**]. We checked numerically to 10^{-16} for even and non-even Gaussian H using the first 200–300 zeros (Section 1.6). □

PROOF OF (II). $\widetilde{T_tF}(\gamma)\overline{\widetilde{T_tG}(\bar\gamma)} = e^{i\gamma t}\widetilde{F}(\gamma)\,\overline{e^{i\bar\gamma t}}\,\overline{\widetilde{G}(\bar\gamma)} = \widetilde{F}(\gamma)\overline{\widetilde{G}(\bar\gamma)}$, because $\overline{e^{i\bar\gamma t}} = e^{-i\gamma t}$. Sum over ρ. The eigen-relation is $\widetilde{T_tF} = e^{izt}\widetilde{F}$, and $|e^{i\gamma t}| = e^{-t\operatorname{Im}\gamma} = e^{(\operatorname{Re}\rho - \frac{1}{2})t}$. □

PROOF OF (III). *Upper bound:* for $t \ge 0$, $|\mathcal{Q}(T_tF,G)| \le \sum_\rho m_\rho|\widetilde{F}(\gamma_\rho)\widetilde{G}(\bar\gamma_\rho)|e^{(\operatorname{Re}\rho-\frac{1}{2})t} \le C_{\mathcal{Z}}\|F\|_{\mathcal{A}}\|G\|_{\mathcal{A}}\,e^{(\Theta-\frac{1}{2})t}$. *Lower bound:* fix any zero ρ_0 and put $\beta_0 = \operatorname{Re}\rho_0 - \frac{1}{2} \ge 0$. Choose F, G by Lemma 1.3.1 with $\widetilde{F}(\gamma_{\rho_0}) \ne 0 \ne \widetilde{G}(\bar\gamma_{\rho_0})$. Then $L(t) = \mathcal{Q}(T_tF,G) = \sum_\rho a_\rho e^{i\gamma_\rho t}$ with $a_\rho = m_\rho\widetilde{F}(\gamma_\rho)\overline{\widetilde{G}(\bar\gamma_\rho)}$, $\sum|a_\rho| < \infty$, $a_{\rho_0} \ne 0$. By Lemma 1.3.2 (valid for any real ε, including $\varepsilon < 0$), no bound $|L(t)| \le Ce^{\varepsilon t}$ with $\varepsilon < \beta_0$ holds, so $\lambda(F,G) \ge \beta_0$. Taking the supremum over ρ_0 gives $\sup\lambda \ge \Theta - \frac{1}{2}$. □

PROOF OF (IV). (a)⇒(c) and part (v): Assume RH, so $\bar\gamma_\rho = \gamma_\rho \in \mathbb{R}$. Then $JF = (\widetilde{F}(\gamma_\rho)) \in \ell^2(\mathcal{Z}, m)$ by Lemma 1.1.2, and $\langle JF, JG\rangle = \sum m_\rho\widetilde{F}(\gamma_\rho)\overline{\widetilde{G}(\gamma_\rho)} = \mathcal{Q}(F,G)$. Multiplication by $e^{i\gamma_\rho t}$ is unitary and strongly continuous (dominated convergence), and $U(t)JF = JT_tF$ by (ii). *Dense range:* if $c \in \ell^2(\mathcal{Z}, m)$ is orthogonal to $J(C_c^\infty)$, then for every F and t, $0 = \langle JT_tF, c\rangle = \sum_\rho a_\rho e^{i\gamma_\rho t}$, where $a_\rho = m_\rho\widetilde{F}(\gamma_\rho)\bar c_\rho$ and $\sum|a_\rho| < \infty$ by Cauchy–Schwarz. The γ_ρ are real and distinct, so the Bohr mean gives $a_{\rho_0} = \lim_{S\to\infty}\frac{1}{2S}\int_{-S}^{S}\big(\sum_\rho a_\rho e^{i\gamma_\rho t}\big)e^{-i\gamma_{\rho_0}t}dt = 0$ (interchange by dominated convergence, since $\frac{1}{2S}\int_{-S}^{S}e^{i\mu t}dt \to \delta_{\mu,0}$ boundedly). With F from Lemma 1.3.1, $c_{\rho_0} = 0$. By Stone's theorem [**97**, Thm. VIII.8], $U(t) = e^{itA}$, with A the maximal multiplication operator by γ_ρ. This A is self-adjoint, with eigenvectors δ_ρ and spectrum $\overline{\{\gamma_\rho\}} = \{\gamma_\rho\}$ (no finite accumulation). The eigenvalues are simple since distinct zeros have distinct ordinates.

(c)⇒(b): $\mathcal{Q}(F,F) = \|JF\|^2 \ge 0$.

(b)⇒(d): Cauchy–Schwarz for the positive semidefinite Hermitian form $\mathcal{Q}$ and (ii) give $|\mathcal{Q}(T_tF,G)|^2 \le \mathcal{Q}(T_tF,T_tF)\mathcal{Q}(G,G) = \mathcal{Q}(F,F)\mathcal{Q}(G,G)$.

(d)⇒(a): If some ρ_0 has $\operatorname{Re}\rho_0 > \frac{1}{2}$, (iii) gives F, G with $\lambda(F,G) > 0$, so the coefficient is unbounded on $t \ge 0$. Zeros with $\operatorname{Re}\rho < \frac{1}{2}$ are excluded by the symmetry $\rho \mapsto 1 - \bar\rho$. □

PROOF OF (VI). Split the zeros into $\mathcal{Z}_0$ ($\operatorname{Re}\rho = \frac{1}{2}$), $\mathcal{Z}_+$ ($\operatorname{Re}\rho > \frac{1}{2}$, i.e. $\operatorname{Im}\gamma < 0$), and $\mathcal{Z}_- = \{\rho^* : \rho \in \mathcal{Z}_+\}$. For $\rho \in \mathcal{Z}_+$ write $a_\rho(F) = \widetilde{F}(\gamma_\rho)$ and $b_\rho(F) = \widetilde{F}(\bar\gamma_\rho)$. Pairing ρ with ρ^*,

$$\mathcal{Q}(F,F) = \sum_{\mathcal{Z}_0} m_\rho|\widetilde{F}(\gamma_\rho)|^2 + \sum_{\rho\in\mathcal{Z}_+} m_\rho(a_\rho\bar b_\rho + b_\rho\bar a_\rho) = \Pi(F) + \sum_{\rho\in\mathcal{Z}_+}\frac{m_\rho}{2}\big(|a_\rho+b_\rho|^2 - |a_\rho-b_\rho|^2\big), \tag{1.3.1}$$

with $\Pi \ge 0$; all series converge absolutely. Each off-line pair contributes a hyperbolic plane, of signature $(1,1)$.

$\kappa \le N_+$. Assume $N_+ < \infty$. By (1.3.1), $\mathcal{Q}(F,F) \ge -\sum_{\mathcal{Z}_+} \frac{m_\rho}{2}|a_\rho(F)-b_\rho(F)|^2$. If $\dim V > N_+$, the linear map $V \to \mathbb{C}^{N_+}$, $F \mapsto (a_\rho - b_\rho)_\rho$, has a nonzero kernel element F. That F satisfies $\mathcal{Q}(F,F) \ge 0$, so V is not negative definite.

$\kappa \ge N$ *for every finite* $N \le N_+$. Fix distinct $\rho_1, \dots, \rho_N \in \mathcal{Z}_+$ with $\gamma_j = a_j - i\beta_j$, $\beta_j \in (0, \frac12)$. The set $K = \{\gamma_\rho : \ |\operatorname{Re}\gamma_\rho - a_j| < 1 \text{ for some } j\}$ is finite and contains $\gamma_j, \bar\gamma_j$. The $2N$ points $\gamma_j, \bar\gamma_j$ are distinct. By Lagrange interpolation on K, choose polynomials $P_{j,1}, P_{j,2}$ vanishing on $K \setminus \{\gamma_j, \bar\gamma_j\}$ with $P_{j,1}(\gamma_j) = 1$, $P_{j,1}(\bar\gamma_j) = 0$, $P_{j,2}(\gamma_j) = 0$, $P_{j,2}(\bar\gamma_j) = 1$. For $\sigma \ge 1$ let

$$\Phi_{j,\epsilon}(z) = P_{j,\epsilon}(z)\exp\Big(-\tfrac{\sigma^2}{2}\big[(z-a_j)^2 + \beta_j^2\big]\Big), \qquad \epsilon = 1, 2.$$

Then $\Phi_{j,\epsilon} = \widetilde{F_{j,\epsilon}}$ with $F_{j,\epsilon} = c\, P_{j,\epsilon}(i\,\frac{d}{du})(e^{-u^2/2\sigma^2}e^{-ia_j u}) \in \mathcal{A}$, because $\widetilde{F'}(z) = -iz\widetilde{F}(z)$. The Gaussian factor equals 1 at γ_j and $\bar\gamma_j$.

Consider the Gram matrix $G_\sigma = (\mathcal{Q}(F_k, F_l))_{k,l}$, $k, l \in \{(j,\epsilon)\}$, and split the zero sum into $\gamma_\rho \in K$ and $\gamma_\rho \notin K$. If $\gamma_\rho \in K$, a term $\Phi_k(\gamma_\rho)\overline{\Phi_l(\bar\gamma_\rho)}$ with $k = (j,\epsilon)$, $l = (i,\epsilon')$ can be nonzero only if $\gamma_\rho \in \{\gamma_j, \bar\gamma_j\}$ and $\bar\gamma_\rho \in \{\gamma_i, \bar\gamma_i\}$. This forces $i = j$, since $\operatorname{Im}\gamma_i < 0 < \operatorname{Im}\bar\gamma_j$ and the γ's are distinct. Evaluating, the K-part equals $\bigoplus_j m_{\rho_j}\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, which has exactly N eigenvalues $\le -\min_j m_{\rho_j} \le -1$. If $\gamma_\rho = x + iy \notin K$, then $(x-a_j)^2 \ge 1$ and $|y| < \frac12$ for all j, so

$$\Big|\exp\big(-\tfrac{\sigma^2}{2}[(z-a_j)^2+\beta_j^2]\big)\Big|_{z=\gamma_\rho \text{ or } \bar\gamma_\rho} \le e^{-\frac{\sigma^2}{2}[(x-a_j)^2-\frac14]} \le e^{-\frac38(\sigma^2-1)}\, e^{-\frac12[(x-a_j)^2-\frac14]}.$$

The remaining factor, times the polynomials, is m_ρ-summable over $\mathcal{Z}$ (by $N(T) \ll T\log T$). Hence the non-K part has norm $O(e^{-\frac34\sigma^2})$. By Weyl's inequality, for σ large G_σ has at least N negative eigenvalues. The span of the corresponding coefficient vectors gives an N-dimensional subspace of $\mathcal{A}$ on which $\mathcal{Q}$ is negative definite. The map from coefficients to functions is injective there, since $\mathcal{Q}(F,F) < 0$ forces $F \ne 0$. Finally, replace F_k by $\chi(\cdot/R)F_k \in C_c^\infty$. By density and boundedness of $\mathcal{Q}$ on $\mathcal{A}$, the Gram matrix changes by $o(1)$ as $R \to \infty$, so negative definiteness on an N-dimensional subspace persists. Thus $\kappa \ge N$, and $\kappa = N_+$ also when $N_+ = \infty$. The argument also gives $\kappa_{\mathcal{A}} = \kappa_{C_c^\infty}$. □

Corollary 1.3.3 (Pontryagin structure if RH fails mildly). *If $N_+ < \infty$, the quotient of $(C_c^\infty, \mathcal{Q})$ by its isotropic part completes to a Pontryagin space Π_{N_+}, and T_t extends to a Π_{N_+}-unitary group. By Pontryagin's theorem* [**95, 57**] *every such group has an N_+-dimensional non-positive invariant subspace. In the TAA dictionary these are the "ghost" modes, carried by the eigenfunctionals ℓ_ρ, ℓ_{ρ^*} for $\rho \in \mathcal{Z}_+$ with growth $e^{\pm(\operatorname{Re}\rho - \frac12)t}$.*

(We state the corollary for orientation only. The completion and extension are standard (Iohvidov–Krein–Langer), and we do not use it.)

1.4. What is known and what is new

- (i) is the explicit formula of Guinand and Weil [**45, 119**]. Hermitian symmetry is standard.
- (iv)(a) $\Longleftrightarrow$ (b) is **Weil's criterion** [**119**]; see also [**6, 122**]. Connes [**22**] recast it as a trace formula on the adèle class space. There the scaling action of $\mathbb{R}_+^*$ (our T_t) is central, and the Hilbert realization in (iv)(c), (v) is implicit. Our proof of (b)⇒(a) avoids the Bochner–Schwartz theorem. It uses only isometry, Cauchy–Schwarz, and a Laplace transform (Lemma 1.3.2), and we have not seen it written this way, though it is elementary.
- (vi): Bombieri [**6**] proved that, if only finitely many zeros are off the line, sufficiently large finite truncations of the Weil quadratic form have exactly half as many negative eigenvalues as there are off-line zeros. That number equals our N_+. Our (vi) is the statement for the full form on C_c^∞, including $N_+ = \infty$, with a short interpolation proof. We regard it as a mild extension and reformulation, not a new phenomenon.

- (iii), the identification of the growth exponent of the scaling flow's matrix coefficients with $\Theta-\frac{1}{2}$, is elementary. It is closely related to the classical fact that Θ governs $\psi(x)-x=O(x^{\Theta+\varepsilon})$. We have not found it stated in this dynamical form.
- **New (TAA-specific):** the reading "RH $\iff$ the arithmetic scaling flow is unitary $\iff$ no ghosts ($\kappa=0$) $\iff$ no runaway modes ($\lambda\le 0$)", with the exact ghost count N_+ and Lyapunov exponent $\Theta-\frac{1}{2}$. This turns the heuristic in TAA [C, §6.1] ("an off-line zero injects un-compensated vorticity and tears the chart") into a theorem: an off-line zero at ρ is exactly a hyperbolic (ghost) plane in the state space and a mode growing like $e^{(\operatorname{Re}\rho-\frac{1}{2})t}$.

1.5. Limitations

(1) **No progress on RH.** All conditions in (iv) are equivalent to RH. None is verified here. The Hilbert-space operator in (v) is built *from* the zeros, so it is not a Hilbert–Pólya operator in the substantive sense: the spectrum is put in by hand. A genuine advance would construct $\mathcal{H}$ and the positivity of $\mathcal{Q}$ from the *arithmetic side* (1.2.1) alone, e.g. from a physical (Lee–Yang-type) positivity principle.
(2) The zero multiplicity m_ρ appears only as a norm weight, not as spectral multiplicity, in (v). The construction cannot detect multiple zeros spectrally.
(3) The analogy with Nash/Stelle ghosts is structural. Nothing here derives the Weil form from $2R_{\mu\nu}R^{\mu\nu}-R^2$, the Taylor–Wiles isomorphism, Iwasawa theory, or Einstein–Cartan geometry.
(4) Part (vi) needs arbitrarily wide Gaussians, i.e. test functions with large support. For functions supported in a fixed interval $[-t,t]$, positivity is known for small t (Yoshida: $t=\frac{1}{2}\log 2$ [**122**]), and the negative index there can be smaller than N_+.

1.6. Numerical sanity checks

Scripts are in `num/` (mpmath, 30 digits).

- `explicit_formula_check.py`: (1.2.1) for even Gaussians $H(u)=e^{-u^2/2s^2}$, $s\in\{0.5,0.3,0.2\}$, with 300 zeros and $n<4000$. The zero side and the arithmetic side agree to $10^{-31},10^{-17},10^{-18}$. For example, $s=0.2$: both sides $=0.0185904504407629$.
- `explicit_formula_noneven.py`: shifted (non-even) Gaussians, $(s,c)\in\{(0.3,0.7),(0.25,-1.3),(0.4,2)\}$, 200 zeros, $n<6000$: agreement to 10^{-16}–10^{-22}. This confirms the normalization of (1.2.1), including the symmetrized prime term.
- The Gram matrix of $\mathcal{Q}$ on 13 shifted Gaussians, computed from the true zeros, is Hermitian and positive semidefinite to 10^{-28}, as it must be for zeros on the line.
- `toy_offline.py` (illustrating (vi)): we add k hypothetical off-line quadruples $\{\rho,1-\rho,\bar\rho,1-\bar\rho\}$, which contain $2k$ zeros with $\operatorname{Re}\rho>\frac{1}{2}$, at heights $17.5,27,35.5$ with $\beta=0.1,0.05,0.2$. Using test functions $\Phi(z)=e^{-\sigma^2(z-a)^2/2}e^{izt}$ ($\sigma=2$, $a=\pm$heights, $t\in[-3,3]$), we find exactly $0,2,4,6$ negative eigenvalues for $k=0,1,2,3$, as predicted by $\kappa=N_+$.

CHAPTER 2

Lemma 2: Heat-flow unitarity

The de Bruijn–Newman constant as the critical "unitarity time" of the arithmetic scaling flow

Summary. Let $H_t(z) = \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du$ be the de Bruijn–Newman family, with $H_0(z) = \frac{1}{8}\Xi(z/2)$ and $\partial_t H_t = -\partial_z^2 H_t$. To the zeros of H_t we attach the Weil-type form $\mathcal{Q}_t$ of Lemma 1, so that $\mathcal{Q}_0 = \mathcal{Q}$ is the Weil form of ζ. We prove the following for $t \geq 0$. (B1) The form $\mathcal{Q}_t$ is Hermitian and isometric for the scaling flow. Its number of negative squares is $\kappa(\mathcal{Q}_t) = N_+(t)$, and this number is finite for every $t > 0$. (B2) A *second law*: the number $Z_{\mathbb{C}}(H_t)$ of non-real zeros, counted with multiplicity, is non-increasing in t. (B3) $\mathcal{Q}_t \geq 0$, equivalently the flow is unitary, if and only if $t \geq \Lambda_{\mathrm{dBN}}$. Hence, unconditionally, the heat-deformed arithmetic flow is unitary for all $t \geq 0.22$ (Polymath15), or $t \geq 0.2$ (Platt–Trudgian). RH holds iff it is unitary already at $t = 0$. (B4) A *cooling law* for the Lyapunov exponent: $\lambda(t+s)^2 \leq \max(\lambda(t)^2 - \frac{s}{2}, 0)$, so $\Lambda_{\mathrm{dBN}} \leq t + 2\lambda(t)^2$. An explicit example shows that κ itself, which counts *distinct* ghosts, can increase when a multiple ghost splits. Only $Z_{\mathbb{C}}$ obeys the second law. No statement here implies RH. Section 2.6 separates known from new.

2.1. Admissible spectral data and Lemma 1 in general form

Definition 2.1.1 (Admissible spectral data). An object of **AdmSpec** is a pair $(\mathcal{Z}, m)$. Here $\mathcal{Z} \subset \{\gamma \in \mathbb{C} : |\operatorname{Im}\gamma| \leq \frac{1}{2}\}$ is locally finite and $m : \mathcal{Z} \to \mathbb{N}_{\geq 1}$, subject to: (a) $\bar{\gamma} \in \mathcal{Z}$ and $m(\bar{\gamma}) = m(\gamma)$ for all $\gamma \in \mathcal{Z}$; (b) $\sum_{\gamma\in\mathcal{Z}} m(\gamma)(1+|\gamma|^2)^{-1} < \infty$. Morphisms are inclusions of conjugation-closed sub-multisets, so **AdmSpec** is a poset. Put $N_+(\mathcal{Z}) = \#\{\gamma \in \mathcal{Z} : \operatorname{Im}\gamma < 0\}$ and $\lambda(\mathcal{Z}) = \sup_{\gamma\in\mathcal{Z}} |\operatorname{Im}\gamma|$. For $F, G \in \mathcal{A}$, the test space of Lemma 1, Def. 1.1, define

$$\mathcal{Q}_{\mathcal{Z}}(F,G) = \sum_{\gamma\in\mathcal{Z}} m(\gamma)\,\widetilde{F}(\gamma)\,\overline{\widetilde{G}(\bar{\gamma})}, \qquad (T_aF)(u) = F(u-a) \quad (a \in \mathbb{R}).$$

Example 2.1.2. The zeros of ζ, written $\rho = \frac{1}{2} + i\gamma_\rho$, give an object $\mathcal{Z}_\zeta$, and $\mathcal{Q}_{\mathcal{Z}_\zeta} = \mathcal{Q}$ is the form of Lemma 1.

Theorem 2.1.3 (Lemma 1, abstract form). *For every $(\mathcal{Z}, m) \in$ **AdmSpec**:*

(A1) *$\mathcal{Q}_{\mathcal{Z}}$ is a bounded Hermitian form on $\mathcal{A}$, and $\mathcal{Q}_{\mathcal{Z}}(T_aF, T_aG) = \mathcal{Q}_{\mathcal{Z}}(F,G)$.*

(A2) $\sup_{F,G\in C_c^\infty(\mathbb{R})} \limsup_{a\to+\infty} \frac{1}{a}\log|\mathcal{Q}_{\mathcal{Z}}(T_aF,G)| = \lambda(\mathcal{Z})$.

(A3) *The following are equivalent: $\mathcal{Z} \subset \mathbb{R}$; $\mathcal{Q}_{\mathcal{Z}} \geq 0$ on $C_c^\infty(\mathbb{R})$; T is realized by a strongly continuous unitary group on a Hilbert space $\mathcal{H}$ via an isometric J (as in Lemma 1, Thm. 2.1(iv)(c)); all matrix coefficients $a \mapsto \mathcal{Q}_{\mathcal{Z}}(T_aF,G)$ are bounded.*

(A4) *If $\mathcal{Z} \subset \mathbb{R}$, then $\mathcal{H} \cong \ell^2(\mathcal{Z}, m)$, and the generator is multiplication by γ: self-adjoint with simple pure point spectrum $\mathcal{Z}$.*

(A5) *$\kappa(\mathcal{Q}_{\mathcal{Z}}) = N_+(\mathcal{Z})$, on $C_c^\infty(\mathbb{R})$ and on $\mathcal{A}$.*

Proof. The proofs of Lemma 1, Thm. 2.1(ii)–(vi), use only properties (a), (b), local finiteness and the strip condition. Specifically: Lemma 1.2 needs $|\operatorname{Im}\gamma| \leq \frac{1}{2}$. Lemma 3.2 needs local finiteness and $\sum|a_\gamma| < \infty$. The Bohr-mean density argument needs distinct real γ. The interpolation argument in (vi) needs finiteness of K, and the Gaussian tail is bounded by $C(1+|\gamma|^2)^{-1}$,

which is summable by (b). The functional equation of ζ was used only through (a). In (A2) the lower bound applies Lemma 3.2 at any $\gamma_0 \in \mathcal{Z}$. Taking both γ_0 and $\bar{\gamma}_0$ (property (a)) gives the value $\sup|\operatorname{Im}\gamma|$; no real points are needed. □

2.2. The de Bruijn–Newman family

Let $\Phi(u) = \sum_{n\geq 1}(2\pi^2 n^4 e^{9u} - 3\pi n^2 e^{5u})\exp(-\pi n^2 e^{4u})$ and

$$H_t(z) = \int_0^\infty e^{tu^2}\Phi(u)\cos(zu)\,du \qquad (t\in\mathbb{R},\ z\in\mathbb{C}).$$

We use the following standard facts.

(F1) Φ is even, $\Phi(u) > 0$ for $u \geq 0$, and $\Phi(u) = O(e^{9u-\pi e^{4u}})$ as $u \to +\infty$ (Pólya; see [99, §1], [94, §2]). Hence H_t is an even entire function, real on $\mathbb{R}$, with $H_t(0) > 0$. We have $H_0(z) = \frac{1}{8}\Xi(\frac{z}{2})$, where $\Xi(x) = \xi(\frac{1}{2} + ix)$, and $\partial_t H_t = -\partial_z^2 H_t$.

(F2) (Growth.) For each t there are A, B with $|H_t(z)| \leq A\exp(B|z|\log(|z|+e))$. To see this, $|\cos(zu)| \leq e^{|\operatorname{Im} z|u}$, and $\sup_{u\geq 0}(yu + 9u + tu^2 - \pi e^{4u}) \leq \frac{y}{4}\log(y+e) + O_t(y+1)$. So H_t has order ≤ 1. By Jensen's formula, $n_t(r) := \#\{|z| \leq r : H_t(z) = 0\} = O_t(r\log(r+e))$, and so $\sum_z (1+|z|^2)^{-1} < \infty$.

(F3) (de Bruijn's strip theorem [26, Thm. 13]; in this form [94], [99, §1].) If all zeros of H_{t_0} lie in $|\operatorname{Im} z| \leq \Delta$, then for $t \geq t_0$ all zeros of H_t lie in $|\operatorname{Im} z| \leq \sqrt{\max(\Delta^2 - 2(t-t_0), 0)}$.

(F4) (Newman [88].) There is $\Lambda_{\mathrm{dBN}} \in \mathbb{R}$ such that H_t has only real zeros iff $t \geq \Lambda_{\mathrm{dBN}}$. RH is equivalent to $\Lambda_{\mathrm{dBN}} \leq 0$.

(F5) (Rodgers–Tao [99].) $\Lambda_{\mathrm{dBN}} \geq 0$. (Polymath15 [94].) $\Lambda_{\mathrm{dBN}} \leq 0.22$. (Platt–Trudgian [93], using RH verified to height $3\cdot 10^{12}$.) $\Lambda_{\mathrm{dBN}} \leq 0.2$.

(F6) (Ki–Kim–Lee [64].) For every $t > 0$, all but finitely many zeros of H_t are real and simple.

Since the zeros of H_0 are $z = 2\gamma_\rho$ with $|\operatorname{Im} z| < 1$, (F3) with $t_0 = 0$, $\Delta = 1$ shows that for $t \geq 0$ all zeros of H_t lie in $|\operatorname{Im} z| \leq \sqrt{\max(1-2t, 0)} \leq 1$.

Definition 2.2.1 (Heat-deformed spectral data). For $t \geq 0$ let $\mathcal{Z}_t = \{z/2 : H_t(z) = 0\}$, with multiplicities. By (F1)–(F3), $\mathcal{Z}_t \in \mathbf{AdmSpec}$, and $\mathcal{Z}_0 = \mathcal{Z}_\zeta$. Write $\mathcal{Q}_t := \mathcal{Q}_{\mathcal{Z}_t}$, $N_+(t) := N_+(\mathcal{Z}_t)$, $\lambda(t) := \lambda(\mathcal{Z}_t)$ and $Z_{\mathbb{C}}(t) := Z_{\mathbb{C}}(H_t)$, the number of non-real zeros with multiplicity, in $[0,\infty]$.

2.3. Main theorem

Theorem 2.3.1 (Heat-flow unitarity). *For $t \geq 0$:*

(B1) *$\mathcal{Q}_t$ is a bounded Hermitian, T-isometric form, with $\mathcal{Q}_0 = \mathcal{Q}$, and*

$$\kappa(\mathcal{Q}_t) = N_+(t) \leq \tfrac{1}{2}Z_{\mathbb{C}}(t).$$

For $t > 0$ both quantities are finite.

(B2) (Second law.) *If $0 \leq t < t'$ and $Z_{\mathbb{C}}(t) < \infty$ (automatic for $t > 0$), then $Z_{\mathbb{C}}(t') \leq Z_{\mathbb{C}}(t)$. Consequently $\kappa(\mathcal{Q}_{t'}) \leq \frac{1}{2}Z_{\mathbb{C}}(t)$.*

(B3) (Critical unitarity time.) *$\mathcal{Q}_t \geq 0 \iff$ the scaling flow is unitary for $\mathcal{Q}_t \iff t \geq \Lambda_{\mathrm{dBN}}$. Unconditionally, $\mathcal{Q}_t \geq 0$ for all $t \geq 0.2$. In that case the generator is self-adjoint with simple pure point spectrum $\mathcal{Z}_t = \frac{1}{2}\{$zeros of $H_t\}$. RH $\iff$ $\Lambda_{\mathrm{dBN}} = 0 \iff$ $\mathcal{Q}_0 \geq 0$.*

(B4) (Cooling law.) *$\lambda(t) \leq \frac{1}{2}\sqrt{\max(1-2t,0)}$, and for $s \geq 0$,*

$$\lambda(t+s)^2 \leq \max\left(\lambda(t)^2 - \tfrac{s}{2},\, 0\right).$$

In particular $\Lambda_{\mathrm{dBN}} \leq t + 2\lambda(t)^2$ for every $t \geq 0$. At $t = 0$, $\lambda(0) = \Theta - \frac{1}{2}$ with $\Theta = \sup\operatorname{Re}\rho$, so $\Lambda_{\mathrm{dBN}} \leq 2(\Theta - \frac{1}{2})^2 \leq \frac{1}{2}$.

EXAMPLE 2.3.2 (κ is not monotone in general). Let $p(z) = (z^2+1)^2$. Then $e^{-sD^2}p = p - sp'' + \frac{s^2}{2}p'''' = z^4 + (2-12s)z^2 + (1-4s+12s^2)$. As a quadratic in $w = z^2$ its discriminant is $32s(3s-1) < 0$ for $0 < s < \frac{1}{3}$. So for those s there are four distinct non-real zeros: N_+ jumps from 1 (the double zero $-i$) to 2, while $Z_{\mathbb{C}} = 4$ is constant. For $s > \frac{1}{3}$ one has w real; since $w_1 w_2 = 1 - 4s + 12s^2 > 0$ and $w_1 + w_2 = 12s - 2 > 0$, both $w_i > 0$, so $Z_{\mathbb{C}}$ drops to 0. Hence the second law (B2) cannot be strengthened to κ without excluding multiple non-real zeros. For H_t, $t > 0$, all non-real zeros are expected to be simple generically, but this is not proved.

2.4. Proofs

PROOF OF (B1). Apply Theorem 2.1.3 to $\mathcal{Z}_t \in \mathbf{AdmSpec}$. Each distinct zero in the lower half-plane has multiplicity ≥ 1, so $N_+(t) \leq \frac{1}{2}Z_{\mathbb{C}}(t)$, using conjugation symmetry. Finiteness for $t > 0$ is (F6). □

LEMMA 2.4.1 (Laguerre-type step). *Let q be a real polynomial of degree n and $c \in \mathbb{R} \setminus \{0\}$. Then $Z_{\mathbb{C}}(q - cq') \leq Z_{\mathbb{C}}(q)$.*

PROOF. Say $c > 0$ (the case $c < 0$ is symmetric, using $x \to -\infty$). Let q have r real zeros counted with multiplicity, at d distinct points $x_1 < \cdots < x_d$ with multiplicities k_i. Put $g(x) = e^{-x/c}q(x)$. Then $q - cq' = -c\,e^{x/c}g'$, a polynomial of degree n with the same leading coefficient as q. Now count zeros of g': each x_i is a zero of order $k_i - 1$; Rolle gives a zero in each (x_i, x_{i+1}); and since $g(x_d) = 0$, $g(x) \to 0$ as $x \to +\infty$, and $g \not\equiv 0$ on (x_d, ∞), g' also vanishes somewhere in (x_d, ∞). Hence $q - cq'$ has at least $\sum(k_i - 1) + (d-1) + 1 = r$ real zeros, so $Z_{\mathbb{C}}(q - cq') \leq n - r = Z_{\mathbb{C}}(q)$. If $r = 0$ there is nothing to prove. □

LEMMA 2.4.2 (Polynomial second law). *For a real polynomial p and $s \geq 0$, $Z_{\mathbb{C}}(e^{-sD^2}p) \leq Z_{\mathbb{C}}(p)$, where $e^{-sD^2}p = \sum_j (-s)^j p^{(2j)}/j!$ (a finite sum).*

PROOF. On the finite-dimensional space of polynomials of degree $\leq \deg p$, D is nilpotent, and $(1 - \frac{s}{N}D^2)^N \to e^{-sD^2}$. Since $1 - \frac{s}{N}D^2 = (1 - cD)(1 + cD)$ with $c = \sqrt{s/N}$, Lemma 2.4.1 applied $2N$ times gives $Z_{\mathbb{C}}(p_N) \leq Z_{\mathbb{C}}(p)$, where $p_N = (1 - \frac{s}{N}D^2)^N p$. Each p_N has the same degree and leading coefficient as p, and $p_N \to e^{-sD^2}p$ coefficientwise. The zeros depend continuously on the coefficients, and limits of real zeros are real. So the limit has at least as many real zeros as $\liminf$ of those of p_N. □

LEMMA 2.4.3 (Continuity of the heat operator). *Let $M(r) = A\exp(Br\log(r+e))$ and $s > 0$. If entire f satisfies $|f| \leq M(|z|)$, then $S_s f := \sum_{j\geq 0}(-s)^j f^{(2j)}/j!$ converges absolutely and locally uniformly. If $f_N \to f$ locally uniformly with $|f_N| \leq M(|z|)$ for all N, then $S_s f_N \to S_s f$ locally uniformly. Finally, $S_s H_t = H_{t+s}$.*

PROOF. Use Cauchy's estimate on a circle of radius $R_j = 4\sqrt{sj} + 1$: $|f^{(2j)}(z)| \leq (2j)!\,M(|z| + R_j)R_j^{-2j}$. Since $(2j)!/j! \leq (2j)^j$ and $R_j^2 \geq 16sj$, the j-th term is at most $8^{-j}M(|z| + R_j) = 8^{-j}e^{O(\sqrt{j}\log j)}$, uniformly for $|z| \leq r$ and uniformly in f in the class. This gives absolute convergence. It also gives the convergence $S_s f_N \to S_s f$ by dominated convergence, since $f_N^{(2j)} \to f^{(2j)}$ locally uniformly (Weierstrass). For the last claim, $H_t^{(2j)}(z) = \int_0^\infty e^{tu^2}\Phi(u)(-u^2)^j\cos(zu)\,du$, and $\sum_j \frac{(-s)^j(-u^2)^j}{j!} = e^{su^2}$. Interchange sum and integral is justified because $\int_0^\infty e^{(t+s)u^2}|\Phi(u)|e^{|\operatorname{Im} z|u}du < \infty$. □

PROOF OF (B2). Fix $t \geq 0$ with $Z_{\mathbb{C}}(t) < \infty$, and let $s = t' - t > 0$. By (F1), (F2) and Hadamard's theorem (order ≤ 1, H_t even), and since $H_t(0) \neq 0$ and paired factors

$E_1(z/w)E_1(-z/w) = 1 - z^2/w^2$ cancel the exponential factor, we have

$$H_t(z) = H_t(0)\,\pi(z)\prod_{k\geq 1}(1 - z^2/x_k^2).$$

Here $\pi(z) = \prod_w(1-z^2/w^2)$ runs over the finitely many non-real zeros w taken from one of each pair $\pm w$; it is a real polynomial of degree $Z_{\mathbb{C}}(t)$, by conjugation symmetry. The $x_k > 0$ are the positive real zeros, with multiplicity. Let $p_N = \pi\prod_{k\leq N}(1-z^2/x_k^2)$, a real polynomial with $Z_{\mathbb{C}}(p_N) = Z_{\mathbb{C}}(t)$. We have $p_N \to H_t/H_t(0)$ locally uniformly, and $|p_N(z)| \leq |\pi(z)|\prod_k(1 + |z|^2/x_k^2)$. The standard estimate $\sum_k \log(1 + r^2/x_k^2) = \int_0^\infty \frac{2r^2\,n(x)}{x(x^2+r^2)}dx = O(r\log(r+e))$, using $n(x) = O(x\log(x+e))$ from (F2), gives a uniform bound $M(|z|)$ of the required form. By Lemma 2.4.3, $e^{-sD^2}p_N \to H_{t+s}/H_t(0)$ locally uniformly, and by Lemma 2.4.2, $Z_{\mathbb{C}}(e^{-sD^2}p_N) \leq Z_{\mathbb{C}}(t)$. Now let $w_1,\dots,w_\ell$ be distinct non-real zeros of H_{t+s}, with multiplicities μ_i. Choose disjoint closed discs around them that do not meet $\mathbb{R}$. By Hurwitz's theorem, for large N each disc contains exactly μ_i zeros of $e^{-sD^2}p_N$, all non-real. So $\sum\mu_i \leq Z_{\mathbb{C}}(t)$, and therefore $Z_{\mathbb{C}}(t+s) \leq Z_{\mathbb{C}}(t)$. □

Proof of (B3). By (A3), $\mathcal{Q}_t \geq 0$ iff $\mathcal{Z}_t \subset \mathbb{R}$, iff H_t has only real zeros, iff $t \geq \Lambda_{\mathrm{dBN}}$ (F4). The spectral statement is (A4). The numerical bounds are (F5). For RH: by (F4) and (F5), RH $\iff \Lambda_{\mathrm{dBN}} \leq 0 \iff \Lambda_{\mathrm{dBN}} = 0$, and $\mathcal{Q}_0 = \mathcal{Q}$ (Lemma 1). □

Proof of (B4). All zeros of H_t lie in $|\operatorname{Im} z| \leq 2\lambda(t)$, since λ is measured in the $\gamma = z/2$ plane. Apply (F3) with $t_0 = t$ and $\Delta = 2\lambda(t)$: the zeros of H_{t+s} lie in $|\operatorname{Im} z| \leq \sqrt{\max(4\lambda(t)^2 - 2s, 0)}$, which is the inequality. Taking $s = 2\lambda(t)^2$ makes all zeros real, so $\Lambda_{\mathrm{dBN}} \leq t + 2\lambda(t)^2$. The first bound is (F3) from $t_0 = 0$, $\Delta = 1$. At $t = 0$, $\lambda(0) = \sup|\operatorname{Im}\gamma_\rho| = \Theta - \frac{1}{2}$. Finally, (A2) identifies $\lambda(t)$ with the growth exponent of the matrix coefficients of T for $\mathcal{Q}_t$. □

2.5. Thermodynamic reading, stated as theorems only

In TAA language, t is the "heat time" of [C, eq. 3.1]. In the *ghost phase* $0 \leq t < \Lambda_{\mathrm{dBN}}$, the state space of the arithmetic flow is a Pontryagin space with finitely many negative squares for $t > 0$. In the *unitary phase* $t \geq \Lambda_{\mathrm{dBN}}$ it is a Hilbert space. The transition happens at the critical time $\Lambda_{\mathrm{dBN}} \in [0, 0.2]$, and RH says that ζ sits *exactly at* the critical point (Newman's "RH is barely true"). The second law (B2) says the total ghost count with multiplicity never increases. The cooling law (B4) says the maximal ghost growth rate satisfies $\frac{d}{dt}\lambda^2 \leq -\frac{1}{2}$ in the integrated sense. Words like "entropy" and "temperature" are *not* used here as theorems. Only (B1)–(B4) are claimed.

Question 2.5.1 (Dissipation identity; candidate Lemma 3). Suppose that on an interval $I \subset (0, \Lambda_{\mathrm{dBN}})$ all non-real zeros $z_k = x_k + iy_k$ of H_t are simple. Their motion satisfies $\dot{z}_k = 2\sum_{j\neq k}(z_k - z_j)^{-1}$ (symmetric summation; cf. [24], [99]). A formal computation then gives, for $E(t) = \sum_{\operatorname{Im} z_k \neq 0} y_k^2$,

$$\dot{E} = -2\,Z_{\mathbb{C}}(t) - 4\sum_{k:\,y_k\neq 0}\sum_{j:\,y_j=0}\frac{y_k^2}{|z_k - z_j|^2} - 4\sum_{\{k,j\}:\,y_k,y_j\neq 0,\ j\neq k,\bar{k}}\frac{(y_k - y_j)^2}{|z_k - z_j|^2} \;\leq\; -2\,Z_{\mathbb{C}}(t).$$

Question: prove this rigorously, including across collision times. It would imply $E(t) + 2\int_{t_0}^t Z_{\mathbb{C}} \leq E(t_0)$, hence $\Lambda_{\mathrm{dBN}} \leq t_0 + E(t_0)/8$ whenever $Z_{\mathbb{C}} \geq 4$ on $[t_0, \Lambda_{\mathrm{dBN}})$. That would be a genuine free-energy (Lyapunov-function) theorem for the TAA heat flow. The obstacles are the conditional convergence of the real-zero sums and the possible collisions of non-real zeros.

2.6. Known vs new

- **Known, and used as input:** (F1)–(F6), namely Pólya, de Bruijn 1950, Newman 1976, Ki–Kim–Lee 2009, Rodgers–Tao 2020, Polymath15 2019, Platt–Trudgian 2021. The equivalence RH $\iff \Lambda_{\mathrm{dBN}} = 0$ is due to Newman together with Rodgers–Tao. The strip theorem behind (B4) is de Bruijn's. The inequality $\Lambda_{\mathrm{dBN}} \le t + y^2/2$ for zero-free regions $|\operatorname{Im} z| > y$ is exactly the criterion used by Polymath15. (B4) restates it in the Lyapunov language of Lemma 1.
- **Classical in spirit:** Lemma 2.4.1 is a Laguerre/Pólya-type argument, and operators $1 \pm cD$ and e^{-sD^2} are known not to increase non-real zeros of real polynomials (Pólya–Schur theory; Craven–Csordas). We expect (B2) for H_t to be folklore. We give a self-contained proof but have not located this exact statement.
- **New, but modest:** the abstract Theorem 2.1.3 (Lemma 1 for all of **AdmSpec**). The identification $\kappa(\mathcal{Q}_t) = N_+(t)$, together with its finiteness for $t > 0$. The resulting characterization of Λ_{dBN} as the *critical unitarity time* of the arithmetic scaling flow. The unconditional statement that the heat-deformed flow is unitary for $t \ge 0.2$. Example 2.3.2, which shows that the distinct-ghost count can increase.

2.7. Limitations

(1) Nothing here bears on whether $\Lambda_{\mathrm{dBN}} = 0$. The theorem relocates RH; it does not attack it.
(2) For $t > 0$, H_t has no Euler product. So $\mathcal{Q}_t$ has no prime-side explicit formula, and its arithmetic meaning for $t \neq 0$ is only through Φ, i.e. the Jacobi theta function.
(3) We treat only $t \ge 0$, where (F3) puts $\mathcal{Z}_t$ in the strip. For $t < 0$ (where Rodgers–Tao guarantee ghosts) the strip condition is not established here, and Theorem 2.1.3 would need a wider weight $e^{c|u|}$.
(4) "Thermodynamics" is used as a dictionary. No partition function or entropy is derived from TAA's physics. Question 2.5.1 is where a genuine free-energy statement could come from.

2.8. Numerical sanity checks

Script: `num/heatflow_check.py`, output in `num/heatflow_check.out`.

- Normalization: $H_0(z) = \frac{1}{8}\Xi(z/2)$ at $z = 0, 10, 25$ agrees to 12 digits; e.g. $H_0(0) = 0.0621400972735$.
- The first zeros of H_t near $2\gamma_k$ (28.2694503, 42.0440793, 50.0217152 at $t = 0$) stay real and drift slightly *toward the origin* as t grows (28.2115092, 41.9686551, 49.9630972 at $t = 0.1$; 28.1536123, 41.8933809, 49.9044109 at $t = 0.2$). This is consistent with the zero dynamics: the zero density increases with height, so repulsion pushes low zeros down.
- Polynomial second law (Lemma 2.4.2): for 400 random real polynomials of degree 3–10, tracked over $s \in [0, 3]$ in 61 steps, no increase of $Z_{\mathbb{C}}$ was observed.
- Example 2.3.2 reproduced numerically: $(Z_{\mathbb{C}}, N_+) = (4, 1)$ at $s = 0$; $(4, 2)$ at $s = 0.01, 0.1, 0.3$; $(0, 0)$ at $s = 0.34, 0.5$.

CHAPTER 3

Lemma 3: The function-field model

Arithmetic unitarity for curves over $\mathbb{F}_q$, the flow at time $\log q$ as normalized Frobenius, and positivity from the Hodge index theorem

Summary. Let X be a smooth projective geometrically connected curve of genus $g \geq 1$ over $\mathbb{F}_q$, with $N_m = \#X(\mathbb{F}_{q^m})$. We apply the abstract unitarity theorem of Lemma 2 (Theorem A, the abstract form of Lemma 1) to the zeros of ζ_X and obtain the following. (C1) An explicit formula that computes the Weil form $\mathcal{Q}_X$ *exactly* from the point counts N_m. (C2) $\kappa(\mathcal{Q}_X) \in \{0, \infty\}$: a single Frobenius eigenvalue off the circle $|\alpha| = \sqrt{q}$ produces infinitely many ghosts. (C3) Six equivalent conditions: RH for X; $\mathcal{Q}_X \geq 0$; a unitary realization of the scaling flow; bounded matrix coefficients; positive-definiteness of the Toeplitz sequence $c_n = \sum_i (\alpha_i/\sqrt{q})^n$; and the Hasse–Weil bound. Positivity *alone* gives the sharp constant $2g$. (C4) Unconditionally, by Weil's theorem, all six hold. The unitary flow at time $L_q = \log q$ is the normalized Frobenius $q^{-1/2}\mathrm{Frob}$ on H^1. (C5) The positivity comes from the Hodge index theorem on $X \times X$. This is the case where the TAA thesis "RH = unitarity" is a theorem. Everything here is classical in substance (Weil 1948). What is new is only the formulation inside the TAA framework of Lemmas 1–2; Section 3.4 lists the known and new parts. Nothing here concerns or implies RH for ζ.

3.1. Setting and known facts

Fix $X/\mathbb{F}_q$ as above and write $L_q = \log q$. We use the following standard facts; see [**100**, Ch. 5] and [**47**, App. C].

(K1) *(Rationality, Weil 1948; Schmidt.)* $Z_X(T) = \exp\big(\sum_{m\geq 1} N_m T^m/m\big) = \dfrac{P_X(T)}{(1-T)(1-qT)}$, where $P_X(T) = \prod_{i=1}^{2g}(1-\alpha_i T) \in \mathbb{Z}[T]$. Equivalently $S_m := \sum_i \alpha_i^m = q^m + 1 - N_m$.

(K2) *(Functional equation, from Riemann–Roch.)* $P_X(T) = q^g T^{2g} P_X(1/(qT))$. Hence the multiset $\{\alpha_i\}$ is stable under $\alpha \mapsto q/\alpha$. Since P_X has real coefficients, it is also stable under $\alpha \mapsto \bar{\alpha}$.

(K3) *(A priori bound.)* $1 \leq |\alpha_i| \leq q$. The Euler product for $\zeta_X(s) = Z_X(q^{-s})$ converges absolutely and is non-zero for $\mathrm{Re}\, s > 1$, which gives $|\alpha_i| \leq q$; then (K2) gives $|\alpha_i| \geq 1$.

(K4) *(Cohomology; Grothendieck.)* $P_X(T) = \det(1 - T\,\mathrm{Frob} \mid H^1_{\text{ét}}(X_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell))$ for $\ell \nmid q$ [**27**]. Frobenius acts semisimply on H^1, because $H^1 \cong V_\ell(\mathrm{Jac}_X)^\vee$ and the Frobenius endomorphism of an abelian variety is semisimple [**110**].

(K5) *(Weil's theorem [**120**]; elementary proof [**5**].)* $|\alpha_i| = \sqrt{q}$ for all i. One proof [**47**, Ex. V.1.10] applies the Hodge index theorem [**47**, Thm. V.1.9] on the surface $X \times X$ to divisors built from the diagonal Δ and the graph Γ_m of the q^m-Frobenius. It obtains $|N_m - q^m - 1| \leq 2g\, q^{m/2}$ and concludes via (C3) below.

DEFINITION 3.1.1 (Spectral data of X). The zeros of $\zeta_X(s)$ are $s = (\log\alpha_i + 2\pi i k)/L_q$ with $k \in \mathbb{Z}$, for any fixed branch of $\log\alpha_i$. Write $s = \frac{1}{2} + i\gamma$ as in Lemma 1. Put

$$\mathcal{Z}_X = \{\gamma_{i,k} = \gamma_{i,0} + 2\pi k/L_q\}, \qquad \gamma_{i,0} = \frac{\arg\alpha_i}{L_q} - i\Big(\frac{\log|\alpha_i|}{L_q} - \frac{1}{2}\Big),$$

where $m(\gamma)$ is the number of indices i that give γ, and $e^{i\gamma_{i,k}L_q} = \alpha_i/\sqrt{q}$. Put $\beta_i = \alpha_i/\sqrt{q}$ and $c_n = \sum_{i=1}^{2g}\beta_i^n$ for $n \in \mathbb{Z}$.

LEMMA 3.1.2. $(\mathcal{Z}_X, m) \in \mathbf{AdmSpec}$. *Moreover* $c_{-n} = c_n = \overline{c_n}$, *and* $c_m = S_m q^{-m/2} = (q^m + 1 - N_m)q^{-m/2}$ *for* $m \geq 1$.

PROOF. By (K3), $\operatorname{Re} s = \log|\alpha_i|/L_q \in [0,1]$, so $|\operatorname{Im}\gamma| = |\operatorname{Re} s - \frac{1}{2}| \leq \frac{1}{2}$. Under $s \mapsto \bar{s}$ we have $\gamma \mapsto -\bar{\gamma}$, and under $s \mapsto 1-s$ we have $\gamma \mapsto -\gamma$. Both symmetries hold by (K2), so $\gamma \mapsto \bar{\gamma}$ preserves $(\mathcal{Z}_X, m)$. $\mathcal{Z}_X$ is a union of $2g$ arithmetic progressions with step $2\pi/L_q$, so it is locally finite and $\sum m(\gamma)(1+|\gamma|^2)^{-1} < \infty$. Finally, $\beta \mapsto 1/\beta$ corresponds to $\alpha \mapsto q/\alpha$, and $\beta \mapsto \bar{\beta}$ to $\alpha \mapsto \bar{\alpha}$; this gives the symmetries of c_n. □

3.2. Main theorem

For $F, G \in C_c^\infty(\mathbb{R})$ put $G^*(u) = \overline{G(-u)}$ and $H = F * G^*$, so that $\widetilde{H}(z) = \widetilde{F}(z)\overline{\widetilde{G}(\bar{z})}$.

THEOREM 3.2.1 (Arithmetic unitarity for curves). *Let* $\mathcal{Q}_X = \mathcal{Q}_{\mathcal{Z}_X}$.

(C1) (Explicit formula.) *For* $F, G \in C_c^\infty(\mathbb{R})$,

$$\mathcal{Q}_X(F,G) = L_q\sum_{n\in\mathbb{Z}} c_n\, H(nL_q) = L_q\Big[2g\,H(0) + \sum_{m\geq 1}\frac{q^m+1-N_m}{q^{m/2}}\big(H(mL_q) + H(-mL_q)\big)\Big].$$

(C2) (Ghost dichotomy.) $\kappa(\mathcal{Q}_X) = N_+(\mathcal{Z}_X)$, *and this is* 0 *if every* $|\alpha_i| = \sqrt{q}$ *and* ∞ *otherwise.* $\lambda(\mathcal{Z}_X) = \max_i|\log|\alpha_i|/L_q - \frac{1}{2}|$.

(C3) (Equivalences.) *The following are equivalent: (a)* $|\alpha_i| = \sqrt{q}$ *for all* i *(RH for* X*); (b)* $\mathcal{Q}_X \geq 0$ *on* $C_c^\infty(\mathbb{R})$*; (c) the scaling flow has a unitary realization as in Theorem A (A3); (d) all matrix coefficients* $a \mapsto \mathcal{Q}_X(T_aF, G)$ *are bounded; (e)* $(c_n)_{n\in\mathbb{Z}}$ *is positive definite, i.e.* $\sum_{j,k} x_j\bar{x}_k c_{j-k} \geq 0$ *for all finite* (x_j)*; (f)* $|N_m - q^m - 1| \leq C\,q^{m(1/2+\epsilon)}$ *for some* C*, every* $\epsilon > 0$ *and all* m*. Moreover (e) implies the sharp Hasse–Weil bound* $|N_m - q^m - 1| \leq 2g\,q^{m/2}$.

(C4) (Unconditional unitarity; Frobenius.) *By (K5), (a)–(f) hold. Then* $\mathcal{H} \cong \ell^2(\mathcal{Z}_X, m)$*, and the generator of* U *has simple pure point spectrum* $\mathcal{Z}_X \subset \mathbb{R}$*, periodic under* $\gamma \mapsto \gamma + 2\pi/L_q$*. The time-*$L_q$ *map* $U(L_q)$ *is unitary and is multiplication by* $\beta(\gamma) = e^{i\gamma L_q}$*. Its spectrum is* $\{\beta_i\} = \operatorname{spec}(q^{-1/2}\mathrm{Frob} \mid H^1)$*. Precisely: if* $\beta^{(1)}, \dots, \beta^{(r)}$ *are the distinct* β_i*, then* $\mathcal{H} = \bigoplus_{j=1}^r \mathcal{H}_j$ *with* $\mathcal{H}_j \cong \ell^2(\mathbb{Z})$*, and* $U(L_q)|_{\mathcal{H}_j} = \beta^{(j)}\cdot I$*. Hence*

$$U(L_q) \;\cong\; I_{\ell^2(\mathbb{Z})} \otimes (q^{-1/2}\mathrm{Frob}|_{H^1_{\mathrm{red}}}),$$

where $H^1_{\mathrm{red}} \otimes \mathbb{C}$ *is the sum of the distinct Frobenius eigenlines, one for each eigenvalue (equivalently,* H^1 *itself when the* α_i *are distinct).*

(C5) (Source of positivity.) *The chain*

Hodge index on $X\times X \;\Rightarrow$ *(f) with* $C = 2g$, $\epsilon = 0 \;\Rightarrow$ *(a)* $\Rightarrow\; \mathcal{Q}_X \geq 0 \;\Rightarrow\; U$ *unitary,* $U(L_q) = q^{-1/2}\mathrm{Frob}$

is a complete proof that the arithmetic scaling flow of X *is unitary.*

3.3. Proofs

PROOF OF (C1). $\mathcal{Q}_X(F,G) = \sum_\gamma m(\gamma)\widetilde{H}(\gamma) = \sum_{i=1}^{2g}\sum_{k\in\mathbb{Z}}\widetilde{H}(\gamma_{i,0}+2\pi k/L_q)$, and the sum converges absolutely because $\widetilde{H}$ decays rapidly in the strip. Fix i and put $h(u) = H(u)e^{i\gamma_{i,0}u} \in C_c^\infty(\mathbb{R})$. Then $\widetilde{H}(\gamma_{i,0}+\xi) = \int h(u)e^{i\xi u}du = \hat{h}(\xi)$. Poisson summation for $h \in C_c^\infty(\mathbb{R})$ gives $\sum_k \hat{h}(2\pi k/L_q) = L_q\sum_n h(nL_q)$. Since $e^{i\gamma_{i,0}L_q} = \beta_i$, this is $L_q\sum_n H(nL_q)\beta_i^n$; both sums are finite because H has compact support. Summing over i gives the first formula. The second follows from Lemma 3.1.2 ($c_0 = 2g$, $c_{\pm m} = S_m q^{-m/2}$) and (K1). □

PROOF OF (C2). The equality $\kappa = N_+$ is (A5). If some $|\alpha_i| \neq \sqrt{q}$, then by (K2) some α_j has $|\alpha_j| > \sqrt{q}$, so $\operatorname{Im}\gamma_{j,k} < 0$ for *all* $k \in \mathbb{Z}$. These are infinitely many distinct zeros. The formula for λ is immediate from the definition. □

PROOF OF (C3). (a)⇔(b)⇔(c)⇔(d) is (A3), since (a) says $\mathcal{Z}_X \subset \mathbb{R}$.

(b)⇒(e): Choose $\phi \in C_c^\infty(\mathbb{R})$ supported in $(-L_q/4, L_q/4)$ with $\phi \neq 0$, and let $F(u) = \sum_j x_j\phi(u - jL_q)$. Then $F * F^* = \sum_{j,k} x_j\bar{x}_k(\phi*\phi^*)(u-(j-k)L_q)$. Here $\phi*\phi^*$ is supported in $(-L_q/2, L_q/2)$ and $(\phi*\phi^*)(0) = \|\phi\|_2^2$. So $(F*F^*)(nL_q) = \|\phi\|_2^2\sum_{j-k=n}x_j\bar{x}_k$, and (C1) gives $0 \le \mathcal{Q}_X(F,F) = L_q\|\phi\|_2^2\sum_{j,k}x_j\bar{x}_k c_{j-k}$.

(e)⇒(f), with the sharp constant: The 2×2 minors of a positive-definite Toeplitz matrix give $|c_n|^2 \le c_0^2$, so $|c_m| \le c_0 = 2g$. Now use $c_m = (q^m + 1 - N_m)q^{-m/2}$.

(f)⇒(a): $\sum_{m\ge1} S_m T^m = \sum_i \alpha_i T/(1-\alpha_i T)$ is a rational function. For each distinct value α among the α_i it has a pole at $1/\alpha$ with non-zero residue $-(\text{multiplicity})/\alpha$, so its radius of convergence is $1/\max|\alpha_i|$. By (f) the radius is $\ge q^{-1/2-\epsilon}$ for every ϵ, so $\max|\alpha_i| \le \sqrt{q}$. By (K2), $\min|\alpha_i| \ge \sqrt{q}$.

(a)⇒(e) (for completeness): $\sum x_j\bar{x}_k c_{j-k} = \sum_i|\sum_j x_j\beta_i^j|^2 \ge 0$ when $|\beta_i| = 1$. □

PROOF OF (C4). By (A4), $\mathcal{H} \cong \ell^2(\mathcal{Z}_X, m)$ and $U(a)$ is multiplication by $e^{i\gamma a}$. At $a = L_q$ we get $e^{i\gamma_{i,k}L_q} = e^{i\gamma_{i,0}L_q}e^{2\pi ik} = \beta_i$, independent of k. Distinct β give disjoint progressions $\{\gamma_{i,0}+2\pi k/L_q\}_k$; let $\mathcal{H}_j$ be the closed span of the progression attached to $\beta^{(j)}$. This gives the decomposition. The spectral identification is (K4). Periodicity of $\mathcal{Z}_X$ is the definition. □

PROOF OF (C5). [47, Ex. V.1.10] derives $|N_m - q^m - 1| \le 2g\,q^{m/2}$ for all m from [47, Thm. V.1.9]; the remaining implications are (C3) and (C4). □

REMARK 3.3.1 (Comparison with ζ and the archive errors). (1) For ζ, Lemma 1 allows $\kappa(\mathcal{Q})$ to be finite and non-zero. For curves, the periodicity $\gamma \mapsto \gamma + 2\pi/L_q$ (the "arithmetic time quantum" $\log q$) forces $\kappa \in \{0, \infty\}$. This difference is structural: $\operatorname{Spec}\mathbb{Z}$ has no constant field. (2) The proof of unitarity here is *not* circular (cf. error E1 in the list of corrected errors, front matter). The positivity comes from an independent geometric input, the Hodge index theorem, and not from postulating a self-adjoint operator. Nor is it a symmetry argument (E2): the functional equation (K2) gives only $\gamma \mapsto \bar{\gamma}$, which holds also for fake L-polynomials violating (a). Unitarity on a line (E3) also plays no role. (3) What is missing for ζ is precisely an analogue of the surface $X \times X$ with a Hodge index theorem applied to the graphs of the scaling flow. This is the program of Weil, Deninger, and Connes–Consani [11], and it is open. The present lemma makes no progress on it. It only shows that the TAA dictionary (flow, Weil form, ghosts, unitarity) reproduces the proven case exactly.

3.4. Known versus new

Known. (K1)–(K5) and the explicit formula for function fields (Weil [120, 119]). The equivalence of RH for X with bounds on N_m is classical. The positive-definiteness criterion (e) is a direct consequence of the explicit formula plus the Herglotz/Toeplitz characterization. The

Hodge-index proof is in [47]. The heuristic "$\log q$-flow = Frobenius" for function fields is standard in Connes's work [11]. **New (modest; formulation only).** (i) The function-field case placed as an object of **AdmSpec**, so that Theorem A applies verbatim. In particular, the Hilbert space and unitary flow are built from the Weil form alone, with no cohomology, and *then* identified with $q^{-1/2}\mathrm{Frob}$ (C4). (ii) The ghost dichotomy $\kappa \in \{0, \infty\}$ (C2), contrasted with ζ. (iii) The short derivation of the sharp Hasse–Weil constant $2g$ directly from $\mathcal{Q}_X \geq 0$, via $F = \sum x_j \phi(\cdot - jL_q)$ (C3, (b)⇒(e)⇒(f)). This is a known kind of argument, but we have not seen it stated in this form; treat this priority claim as unverified. **Limitations.** Nothing here bears on RH for ζ or on Λ_{dBN}. The theorem is a template: it identifies which ingredient (C5) TAA would need to supply for $\operatorname{Spec}\mathbb{Z}$.

3.5. Numerical sanity checks (`num/ff_check.py`, `.out`)

Point counts were computed by brute force over $\mathbb{F}_{p^m}$, using our own finite-field arithmetic.

- $E : y^2 = x^3 + x + 1$ over $\mathbb{F}_5$: $N_1 = 9$, $N_2 = 27$, $P_E = 1 + 3T + 5T^2$, and $|\alpha|/\sqrt{5} = 1$. $N_3 = 108$ and $N_4 = 675$ agree with P_E.
- $X : y^2 = x^5 + x^3 + x + 2$ over $\mathbb{F}_7$, genus 2, with f squarefree mod 7: $N_m = 11, 49, 398, 2377$ for $m = 1, \ldots, 4$, and $P_X = 1 + 3T + 4T^2 + 21T^3 + 49T^4$ (determined by N_1, N_2 and predicting N_3, N_4 correctly). All $|\alpha_i|/\sqrt{7} = 1$ to 10^{-12}. A first trial curve $y^2 = x^5 + x + 1$ was singular mod 7 (double root at $x = 4$), and its N_4 failed the consistency check; this caught the error.
- (C1): the zero side (sum over $\mathcal{Z}_X$, $|k| \leq 400$) and the point-count side agree to $\leq 10^{-14}$ for three Gaussians, for both curves.
- (C3b), (C3e): Gram matrices on shifted Gaussians and 40×40 Toeplitz matrices $[c_{j-k}]$ are PSD (min eigenvalue $\geq -10^{-14}$). Also $\max_{n<200} |c_n| = 2$ and 3.993, both $\leq 2g$.
- (C2), $\kappa = \infty$: for the *fake* polynomial $1 - (2\sqrt{5} + 1.5)T + 5T^2$ (satisfying (K2) but violating (a)), modulated Gaussians at heights $2\pi k/L_q$, $|k| \leq K$, give exactly $2K + 1$ negative eigenvalues for $K = 2, 5, 10, 20$ $(5, 11, 21, 41)$. This is one per off-line zero probed. The true curve gives 0, and the Toeplitz matrix of the fake has a negative eigenvalue $\approx -4 \cdot 10^{13}$.
- (C4): $e^{i\gamma_{i,k} L_q} = \beta_i$ for $k = 0, 3$, to 10^{-10}.

CHAPTER 4

Lemma 4: The Iwasawa pillar

The Mazur–Wiles theorem as a p-adic spectral realization, Ribet's 691, and where the analogy with unitarity breaks

Summary. We state TAA's "Iwasawa pillar" as a theorem. Every deep ingredient is a cited, proved theorem: Herbrand, Ribet, Mazur–Wiles, Deligne, Iwasawa. The deductions written out here are short and are proved in full. (D1) Ramanujan's congruence $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$, proved from $691E_{12} = 441E_4^3 + 250E_6^2$. (D2) The residual representation of Δ mod 691 has semisimplification $1 \oplus \bar{\chi}^{11}$. (D3) The 691-part of the minus class group of $\mathbb{Q}(\mu_{691})$ is $(\mathbb{Z}/691)^2$, supported on ω^{-11} and ω^{-199}. (D4) A p-adic spectral theorem: the eigenvalues of the Iwasawa generator γ on $X_\infty^{(1-k)} \otimes \mathbb{Q}_p$ are exactly $1+T$ over the zeros T of the p-adic L-power series. (D5) An explicit criterion, checked for six irregular pairs including $(691, 12)$ and $(691, 200)$, under which the generator $\Theta_p = \log_p \gamma / \log_p u$ acts on a line by the zero s_0 of $L_p(s, \omega^k)$. Our computed s_0 agree with the published tables of Sunseri (1980) to 4 p-adic digits. We also give a structural comparison with Lemmas 1–3: the p-adic side supplies a *proved spectral realization* but has *no positivity*. Isometry is automatic there and carries no information. Nothing here bears on RH.

4.1. Setting and cited theorems

Let p be an odd prime, $K = \mathbb{Q}(\mu_p)$, $\Delta_p = \mathrm{Gal}(K/\mathbb{Q})$, and ω the Teichmüller character. Let $F_\infty = \mathbb{Q}(\mu_{p^\infty})$ and $\Gamma = \mathrm{Gal}(F_\infty/K) \cong \mathbb{Z}_p$. Fix a topological generator γ and put $u = \chi(\gamma) \in 1 + p\mathbb{Z}_p$, where χ is the cyclotomic character. Let $\Lambda_{\mathrm{Iw}} = \mathbb{Z}_p[[\Gamma]] \cong \mathbb{Z}_p[[T]]$ with $\gamma \mapsto 1+T$. Let $X_\infty = \mathrm{Gal}(L_\infty/F_\infty)$, where L_∞ is the maximal unramified abelian pro-p extension of F_∞, and put $X^{(j)} = e_{\omega^j} X_\infty$. Let C be the p-part of the class group of K and $C^{(j)} = e_{\omega^j} C$. These conventions are those of [10, §§2–3, 6].

(K1) *(Kubota–Leopoldt; Iwasawa; in this form* [10, Thm. 2.4, eq. (2)]*.)* For even $k \not\equiv 0 \pmod{p-1}$ there is $W_{\omega^k} \in \mathbb{Z}_p[[T]]$ with $L_p(s, \omega^k) = W_{\omega^k}(u^s - 1)$ for $s \in \mathbb{Z}_p$, and $L_p(1-n, \omega^k) = -(1-p^{n-1})B_n/n$ for $n \geq 1$, $n \equiv k \pmod{p-1}$. (This uses $\zeta(1-n) = -B_n/n$.)

(K2) *(Iwasawa* [61]*;* [10, Thm. 3.2]*.)* X_∞ is a finitely generated torsion $\Lambda_{\mathrm{Iw}}[\Delta_p]$-module. Hence each $X^{(j)}$ is a finitely generated torsion Λ_{Iw}-module, with characteristic ideal $\mathrm{char}(X^{(j)})$ defined via the structure theorem [10, §3].

(K3) *(Herbrand* [52]*, Ribet* [98]*;* [10, Thm. 6.1], [82, Thm. 1]*.)* For even k with $2 \leq k \leq p-3$: $C^{(1-k)} \neq 0 \iff p \mid B_k$.

(K4) *(Main conjecture: Mazur–Wiles* [81]*; also* [121], [101]*; in this form* [10, Thm. 3.5]*.)* For i odd, $\mathrm{char}(X^{(-i)}) = W_{\omega^{i+1}}(T)\,\mathbb{Z}_p[[T]]$. Equivalently, with $k = i+1$ even: $\mathrm{char}(X^{(1-k)}) = (W_{\omega^k})$.

(K5) *(Deligne* [28]*.)* There is a continuous $\rho_\Delta : G_\mathbb{Q} \to \mathrm{GL}_2(\mathbb{Z}_{691})$, unramified outside 691, with $\mathrm{tr}\,\rho_\Delta(\mathrm{Frob}_\ell) = \tau(\ell)$ and $\det \rho_\Delta(\mathrm{Frob}_\ell) = \ell^{11}$ for $\ell \neq 691$.

(K6) *(Class number formula; see* [**118**, Ch. 4]*.)* $h^-(K) = Q\,w\prod_{\chi\ \mathrm{odd}}(-\frac{1}{2}B_{1,\chi})$ with $Q = 1$ and $w = 2p$, where $B_{1,\chi} = \frac{1}{p}\sum_{a=1}^{p-1}\chi(a)a$. Moreover $|C^-|$ is the p-part of h^-.

4.2. The theorem

THEOREM 4.2.1 (Iwasawa pillar). (D1) (Proved here.) $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ *for all* $n \geq 1$. *Also* $B_{12} = -691/2730$, *and the irregular indices of* 691 *are exactly* $k = 12$ *and* $k = 200$ *(computed).*

(D2) (Proved here from (K5).) *The semisimplification of* $\bar\rho_\Delta = \rho_\Delta \bmod 691$ *is* $1 \oplus \bar\chi^{11}$.

(D3) (Proved here from (K3), (K6) and a computation.) $C^- \cong C^{(-11)} \oplus C^{(-199)} \cong (\mathbb{Z}/691)^2$ *for* $p = 691$. *That is,* $v_{691}(h^-(\mathbb{Q}(\mu_{691}))) = 2$, *and both components are non-zero.*

(D4) (p-adic spectral theorem; proved here from (K2), (K4).) *Let* k *be even with* $k \not\equiv 0 \pmod{p-1}$. *Put* $V_k = X^{(1-k)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, *and suppose* $\mu(W_{\omega^k}) = 0$. *Then* $\dim_{\mathbb{Q}_p} V_k = \lambda(W_{\omega^k})$, *and*

$$\det\big(T - (\gamma - 1) \mid V_k\big) = P_k(T),$$

where P_k *is the distinguished polynomial of* W_{ω^k} *(Weierstrass preparation). So the spectrum of* γ *on* V_k *is* $\{1 + T : W_{\omega^k}(T) = 0,\ |T|_p < 1\}$, *with multiplicity.*

(D5) (Simple-zero case; proved here, with numerics.) *Let* $2 \leq k \leq p-3$ *be even with* $p \mid B_k$, *and suppose*

$$(\star) \qquad (1 - p^{k-1})\tfrac{B_k}{k} \not\equiv (1 - p^{k+p-2})\tfrac{B_{k+p-1}}{k+p-1} \pmod{p^2}.$$

Then $W_{\omega^k} = (T - T_0)\cdot(\text{unit})$ *with* $T_0 \in p\mathbb{Z}_p$. *Hence* $L_p(s, \omega^k)$ *has a unique zero* $s_0 \in \mathbb{Z}_p$, *determined by* $u^{s_0} = 1 + T_0$. *Moreover* V_k *is a line on which* γ *acts by* u^{s_0}, *i.e. the generator* $\Theta_p := \log_p(\gamma)/\log_p(u)$ *acts on* V_k *as the scalar* s_0, *independently of the choice of* γ. *Condition* $(\star)$ *holds for* $(p,k) = (37,32), (59,44), (67,58), (101,68), (691,12), (691,200)$; *see §4.6 for* s_0 *mod* p^4. *Buhler–Harvey* [**8**] *verified* $\lambda_p = i_p$ *(equivalently simple zeros, in this sense) for all* $p < 163\,577\,356$.

4.3. Proofs

PROOF OF (D1). The space M_{12} of level-one weight-12 forms has dimension 2 [**104**, VII.3]. So $691E_{12} = 441E_4^3 + 250E_6^2$ follows by comparing the constant terms ($691 = 441 + 250$) and the q^1 terms ($65520 = 441\cdot 720 - 250\cdot 1008$), where $E_{12} = 1 + \frac{65520}{691}\sum\sigma_{11}(n)q^n$. Since $1728\Delta = E_4^3 - E_6^2$, we get $691E_{12} = 691E_4^3 - 432000\,\Delta$. The series E_4, E_6, Δ have integer coefficients, so for $n \geq 1$ we have $65520\,\sigma_{11}(n) \equiv -432000\,\tau(n) \pmod{691}$. Both 65520 and -432000 are $\equiv 566 \pmod{691}$, and $\gcd(566, 691) = 1$. The Bernoulli facts are a direct computation. □

PROOF OF (D2). For $\ell \neq 691$, (D1) and (K5) give $\operatorname{tr}\bar\rho_\Delta(\mathrm{Frob}_\ell) = \tau(\ell) \equiv 1 + \ell^{11}$ and $\det = \ell^{11}$. These are the trace and determinant of $1 \oplus \bar\chi^{11}$ at Frob_ℓ. By Chebotarev the Frob_ℓ are dense, so the two representations have the same characteristic polynomials everywhere. By Brauer–Nesbitt their semisimplifications agree. □

PROOF OF (D3). We computed B_{1,ω^j} modulo 691^4 for all odd j (§4.6). The term $j = p-2$ has $v_p = -1$ and cancels against $w = 2p$. Of the remaining terms, exactly $j = 11$ and $j = 199$ have $v_p = 1$, and all others are units. By (K6), $|C^-| = 691^2$. By (K3) with $k = 12, 200$, $C^{(-11)}$ and $C^{(-199)}$ are non-zero. Each therefore has order exactly 691, and they exhaust C^-. This agrees with [**82**, §5], where the class group mod 691 is described as 2-dimensional with eigencharacters $\iota^{-11}, \iota^{-199}$. □

PROOF OF (D4). By the structure theorem there is a pseudo-isomorphism $X^{(1-k)} \to \bigoplus_i \Lambda_{\mathrm{Iw}}/(f_i)$ with $\prod f_i$ generating $\mathrm{char}(X^{(1-k)})$. Its kernel and cokernel are finite, so it becomes an isomorphism after $\otimes\mathbb{Q}_p$. By (K4), $\prod_i f_i = W_{\omega^k} \cdot (\text{unit})$, so $\mu = 0$ forces every f_i to be $(\text{unit})\cdot P_i$ with P_i distinguished and $\prod P_i = P_k$. The quotient $\Lambda_{\mathrm{Iw}}/(P_i) = \mathbb{Z}_p[T]/(P_i)$ is free of rank $\deg P_i$, and $T = \gamma - 1$ acts on it with characteristic polynomial P_i. Taking products gives the claim. The zeros of W_{ω^k} in the open unit disc are the roots of P_k. □

PROOF OF (D5). Write $W = W_{\omega^k} = \sum a_m T^m$. Put $t_0 = u^{1-k} - 1$ and $t_1 = u^{2-k-p} - 1$; both lie in $p\mathbb{Z}_p$, and $t_0 - t_1 = u^{1-k}(1 - u^{1-p})$ has valuation exactly 1. By (K1), $W(t_0) = L_p(1-k)$ and $W(t_1) = L_p(2-k-p)$. For $m \geq 2$, $t_0^m - t_1^m = (t_0 - t_1)\sum t_0^a t_1^b$ with $a + b = m - 1 \geq 1$, so it lies in $p^2\mathbb{Z}_p$. Hence $W(t_0) - W(t_1) \equiv a_1(t_0 - t_1) \pmod{p^2}$. So $(\star)$ holds iff $a_1 \in \mathbb{Z}_p^\times$. Also $a_0 \equiv W(t_0) \equiv -B_k/k \equiv 0 \pmod p$. Therefore $\mu = 0$, $\lambda = 1$, and $P_k = T - T_0$ with $T_0 \in p\mathbb{Z}_p$. Since $1 + p\mathbb{Z}_p = u^{\mathbb{Z}_p}$, there is a unique $s_0 \in \mathbb{Z}_p$ with $u^{s_0} = 1 + T_0$. It is the unique zero of $L_p(s, \omega^k) = W(u^s - 1)$, since the other factor is a unit. Now apply (D4): V_k is a line on which $\gamma = 1 + T_0 = u^{s_0}$. Changing γ to γ^a with $a \in \mathbb{Z}_p^\times$ replaces u by u^a and leaves $\log_p \gamma / \log_p u$ unchanged. □

4.4. Structural parallel with Lemmas 1–3, and where it breaks

	Lemma 1 (ζ)	Lemma 3 (curve $X/\mathbb{F}_q$)	Lemma 4 (p-adic, ω^k)
time group	$\mathbb{R}$ (scaling flow T_a)	$\mathbb{R}$, period L_q	$\Gamma \cong \mathbb{Z}_p$
generator	mult. by γ_ρ (under RH)	$U(L_q) = q^{-1/2}\mathrm{Frob}$	$\Theta_p = \log_p \gamma / \log_p u$
spectral realization	*open* (Hilbert–Pólya)	theorem (Grothendieck)	*theorem* (Mazur–Wiles)
spectrum	ordinates γ_ρ	$\{\beta_i\}$, $\lvert\beta_i\rvert = 1$	zeros s_0 of $L_p(s, \omega^k)$
positivity / unitarity	$\iff$ RH (open)	theorem (Hodge index)	*no analogue*
ghost count κ	$= N_+$	$\in \{0, \infty\}$, $= 0$	undefined

REMARK 4.4.1 (Honest limits). (1) Γ is compact and acts on the $\mathbb{Z}_p$-lattice $X^{(1-k)}/\mathrm{tors}$. So γ is automatically an isometry of the associated p-adic norm, and its eigenvalues are 1-units. The p-adic analogue of "unitarity" is therefore *automatic and contentless*. There is no Hilbert space, $\mathbb{Q}_p$ is not ordered, and no Weil form or negative index κ is available. (2) The p-adic zeros s_0 carry no known information about the complex zeros of ζ. Nothing here implies or approaches RH (cf. E1, E9 in the list of corrected errors, front matter). (3) What the pillar *does* supply is a proved instance of the Hilbert–Pólya shape "zeros = eigenvalues of a canonical operator on an arithmetic cohomology", together with the mechanism behind it: an Eisenstein congruence (D1) ⇒ a reducible residual representation (D2) ⇒ an unramified extension (K3). This is the analogue of Lemma 3's (C4). The missing positivity is exactly the ingredient posed in Lemma 3, Remark (3). (4) Iwasawa's analogy "γ on $X_\infty \leftrightarrow$ Frobenius on the Tate module" is classical; see [**80**].

4.5. Known versus new

Known: (K1)–(K6), (D1) (Ramanujan 1916 [**96**]; the proof is standard), (D2) (standard, e.g. [**108**]), (D3) (consistent with [**82**, §5]), the simple-zero criterion $(\star)$ in substance ([**8**] uses equivalent elementary congruences), and the zeros s_0 [**107**]. **New (formulation only):** the operator form (D4)–(D5) of the main conjecture as "Θ_p acts on V_k by the zero s_0", stated in TAA notation; the self-contained elementary proof of the criterion $(\star)$; and the three-way comparison table, including the explicit statement that p-adic "unitarity" is contentless.

4.6. Numerical checks (`num/iwasawa_check.py`, `.out`)

- $\tau(n) \equiv \sigma_{11}(n) \pmod{691}$ for $1 \le n \le 3000$, with τ computed from $q\prod(1-q^n)^{24}$. The identity $691E_{12} = 441E_4^3 + 250E_6^2$ holds exactly through q^{300}, and $\tau(691) \equiv 1$.
- $B_{12} = -691/2730$, and $691 \mid \mathrm{num}(B_{200})$. The irregular indices of 691 are $\{12, 200\}$. The 47 irregular primes below 700 coincide with OEIS A000928.
- $(\star)$ holds for the six pairs listed. Zeros s_0, base-p digits $p^0, \dots, p^3$: $(37, 32)$: $13, 20, 30, 8$; $(59, 44)$: $31, 9, 15, 55$; $(67, 58)$: $59, 51, 7, 0$; $(101, 68)$: $91, 100, 88, 14$; $(691, 12)$: $485, 299, 92, 410$; $(691, 200)$: $297, 528, 244, 206$. All 24 digits agree with Sunseri's Table A-2 [**107**]. Method: Mahler interpolation of $j \mapsto L_p(1-k-j(p-1))$ from exact Bernoulli numbers, followed by a digit search. Independent check: the interpolant at $s = 0$ equals $-B_{1,\omega^{k-1}}$ (Teichmüller sums) mod p^4 in all six cases.
- $v_p(h^-)$: $1, 1, 1, 2$ for $p = 37, 59, 67, 691$, supported on $j = 31$, 43, 57, and $\{11, 199\}$ respectively. Also $h^-(\mathbb{Q}(\mu_{37})) = 37.000000$ from the complex formula.

CHAPTER 5

Definition 5: The arithmetic action

Configuration space, action, stationarity, covariance and thermodynamics

Summary. Paper [C] defines a prime as "a stationary action state, $\delta S = 0$" but never defines S. We supply (1) a configuration space, (2) an explicit action, (3) the precise meaning of $\delta S = 0$, and (4) consistency checks against Lemmas 1–3. We then test three candidates.

- (A) *The arithmetic action* $\mathcal{S}[F] = \mathcal{Q}(F, F)$. Its critical points are exactly the fields whose spectrum avoids all zeros (Thm. 5.2.2). So "prime $\Leftrightarrow$ $\delta\mathcal{S} = 0$" is *false* for $\mathcal{S}$. The closest true statements are these:
  - primes are exactly the closed orbits of the scaling flow on $\mathcal{Y} = \widehat{\mathbb{Z}}^\times \backslash \mathbb{A}_\mathbb{Q}/\mathbb{Q}^\times$, with length $\log p$. Closure of an orbit is equivalent to the product formula, and the closing element carries the Frobenius at p (Lemma 5.1.3);
  - in a local symplectic model these orbits are exactly the critical points of a Hamiltonian action, with Maupertuis action $E\,m \log p$ (Prop. 5.2.4);
  - the prime part of $\mathcal{S}$ is the orbit sum, with Gutzwiller/Selberg-type weights that differ by an explicit factor and the opposite sign (Prop. 5.2.5).
- (B) *The primon gas / Bost–Connes thermodynamics.* It gives exact theorems: the energy is the archimedean log-size, the occupation number is the p-adic valuation, the first law is the product formula, the Hagedorn point is $\beta = 1$, and the description symmetry is $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$. It does *not* give a Hermitian form reproducing the explicit formula.
- (C) *An Einstein–Cartan torsion functional.* It vanishes identically on the one-dimensional configuration space.

Further results: $\mathcal{Q}$ is covariant exactly under $u \mapsto \pm u + b$, which is scaling and inversion and nothing larger in $\mathrm{Diff}(\mathbb{R})$ (Thm. 5.3.1). Osterwalder–Schrader reflection positivity fails at every prime orbit, for ζ and for curves alike (Prop. 5.4.3). For curves, $\mathcal{Q}_X$ equals a Rosati trace form whose sign is structural (Thm. 5.4.1); this is the Hodge-index-type form asked for. For ζ no such form is known, and we state the precise question. Nothing here implies RH.

5.1. Configuration space and fields

Definition 5.1.1 (Configuration space). Let $\mathbb{A}_\mathbb{Q}$ be the adeles, $C_\mathbb{Q} = \mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$, and $\mathcal{Y} := \mathbb{A}_\mathbb{Q}/(\mathbb{Q}^\times \cdot \widehat{\mathbb{Z}}^\times)$ with the quotient topology. The group $\mathbb{R}_+^*$ acts on $\mathcal{Y}$ through the ideles $(\lambda, 1, 1, \dots)$; this is the *scaling flow*, and we write $\lambda = e^a$. For $x \in \mathbb{A}_\mathbb{Q}$ put $S(x) = \{\ell \text{ finite} : x_\ell = 0\}$.

By [12, Thm. 1.1], $\mathcal{Y}$ is the set of points of the arithmetic site $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ over $\mathbb{R}_+^{\max}$, and the Frobenius action Fr_λ corresponds to the scaling action. Also $C_\mathbb{Q} \cong \mathbb{R}_+^* \times \widehat{\mathbb{Z}}^\times$, and $C_\mathbb{Q}/\widehat{\mathbb{Z}}^\times \cong \mathbb{R}_+^*$ via the idele norm.

Definition 5.1.2 (Fields). A *field* is $F \in C_c^\infty(\mathbb{R})$, a function of the log-scale $u = \log\lambda$ on $C_\mathbb{Q}/\widehat{\mathbb{Z}}^\times \cong \mathbb{R}$. Equivalently, it is a test function on the ideles invariant under $\mathbb{Q}^\times$ (the "change of description") and under $\widehat{\mathbb{Z}}^\times$. The flow acts by $T_a F(u) = F(u - a)$.

LEMMA 5.1.3 (Closed orbits ⇔ product formula). *Let $x \in \mathbb{A}_{\mathbb{Q}}$.*

(a) *If $x_\infty = 0$, then $[x]$ is fixed by all of $\mathbb{R}_+^*$.*
(b) *If $x_\infty \neq 0$, then $\mathrm{Stab}_{\mathbb{R}_+^*}([x]) = \{q \in \mathbb{Q}_{>0} : |q|_\ell = 1 \ \forall \ell \notin S(x)\} = \langle S(x)\rangle$. For such q, the product formula $\prod_v |q|_v = 1$ reads*

$$\log q = \log |q|_\infty = -\sum_{p \in S(x)} \log |q|_p.$$

That is, the archimedean dilation by $\log q$ is exactly cancelled by the p-adic contractions at the places where x vanishes.
(c) *Consequently: if $S(x) = \emptyset$ the orbit is free. If $S(x) = \{p\}$ the orbit is closed with minimal period $\log p$. If $|S(x)| \geq 2$ the stabilizer is dense (for instance $\langle 2, 3\rangle$, since $\log 2/\log 3 \notin \mathbb{Q}$). The* length spectrum *of the flow (minimal periods of orbits with non-trivial discrete stabilizer) is exactly $\{\log p : p \text{ prime}\}$.*
(d) (Frobenius locking.) *For $S(x) = \{p\}$ and $\lambda = p$, the identity $\lambda x = q\, e\, x$ of (b) holds with $q = p$ and $e = \iota_p(p)^{-1}$, where $\iota_p(p) \in \widehat{\mathbb{Z}}^\times$ has ℓ-component p for $\ell \neq p$ and 1 at p. This e is the $\widehat{\mathbb{Z}}^\times$-component of the class of a uniformizer at p in $C_{\mathbb{Q}} = \mathbb{R}_+^* \times \widehat{\mathbb{Z}}^\times$. Under the Artin map $\widehat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ it is the Frobenius at p on $\mathbb{Q}(\mu_N)$, $p \nmid N$ (up to the arithmetic/geometric convention). So the orbit closes because a change of scale by p equals a change of description: multiplication by $p \in \mathbb{Q}^\times$ composed with $\mathrm{Frob}_p^{\pm 1}$. On $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^\times$, without the $\widehat{\mathbb{Z}}^\times$ quotient, the $\mathbb{R}_+^*$-orbit does not close: the stabilizer there is trivial.*
(e) *$\mathcal{Y}$ is not Hausdorff.*

PROOF. $[\lambda x] = [x]$ iff $\lambda x = qe\, x$ for some $q \in \mathbb{Q}^\times$ and $e \in \widehat{\mathbb{Z}}^\times$ ($e_\infty = 1$). (a) Take $q = e = 1$: λ acts only at ∞, where $x_\infty = 0$. (b) At ∞: $\lambda x_\infty = q x_\infty$ forces $q = \lambda$. At $\ell \notin S(x)$: $x_\ell = q e_\ell x_\ell$ forces $e_\ell = q^{-1} \in \mathbb{Z}_\ell^\times$, i.e. $|q|_\ell = 1$. At $\ell \in S(x)$ there is no condition; take $e_\ell = 1$. Conversely, such a q together with this $e \in \widehat{\mathbb{Z}}^\times$ works. The positive rationals that are units outside S form $\langle S\rangle$. The displayed identity is the product formula. (c) This is immediate from (b), using the density of $\{a \log 2 + b \log 3\}$ for (c). (d) Read off from the proof of (b): $e_\ell = p^{-1}$ for $\ell \neq p$, and $e_p = 1$. The idele with p at the place p and 1 elsewhere equals $p \cdot (p^{-1}, (p^{-1})_{\ell \neq p}, 1_p)$, so its class in $\mathbb{R}_+^* \times \widehat{\mathbb{Z}}^\times$ is $(p^{-1}, \iota_p(p)^{-1})$. The Artin map sends a uniformizer at p to $\mathrm{Frob}_p^{\pm 1}$ on extensions unramified at p (class field theory; convention-dependent sign). Without e, the condition at $\ell \notin S$ forces $q = 1$. (e) Stabilizers of a continuous group action on a Hausdorff space are closed. The stabilizer $\langle 2, 3\rangle$ of $x = (1; 0_2, 0_3, 1, 1, \dots)$ is dense and not closed. □

REMARK 5.1.4. (c) is the point-level form of two results: Connes's prediction [**22**, eq. (13)] that "primitive orbits are labelled by primes, with periods $\log p$", and the circles $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$ of length $\log p$ in the scaling site [**13**]. In topos language, a prime is a periodic orbit of the scaling site $[0, \infty) \rtimes \mathbb{N}^\times$ lying over the closed point p of $\mathrm{Spec}\,\mathbb{Z}$, with the quasi-tropical structure of [**13**, §5]. We use only the elementary statement proved above.

5.2. The arithmetic action (candidate A)

Let $\mathcal{W}$ be the Weil distribution of Lemma 1 (eq. (EF), Thm. (i) there), so that $\mathcal{Q}(F, G) = \mathcal{W}(F * G^*)$ with $G^*(u) = \overline{G(-u)}$, and

$$\mathcal{W}(H) = \underbrace{\widetilde{H}(\tfrac{i}{2}) + \widetilde{H}(-\tfrac{i}{2})}_{\text{pole}} \ \underbrace{-H(0)\log\pi + \tfrac{1}{2\pi}\int \widetilde{H}(r)\ \mathrm{Re}\,\tfrac{\Gamma'}{\Gamma}(\tfrac{1}{4} + \tfrac{ir}{2})\,dr}_{\text{archimedean}} \ \underbrace{-\sum_{(p,m)} (\log p)\, p^{-m/2}\big(H(m\log p) + H(-m\log p)\big)}_{\text{closed orbit } (p,m)}$$

DEFINITION 5.2.1 (TAA action). $\mathcal{S}[F] := \mathcal{Q}(F,F) = \mathcal{S}_{\text{pole}} + \mathcal{S}_{\infty} - \mathcal{S}_{\text{orb}}$ for $F \in C_c^\infty(\mathbb{R})$. The orbit term is

$$\mathcal{S}_{\text{orb}}[F] = \sum_{\text{closed orbits } c} \ell^{\#}(c)\, e^{-\ell(c)/2}\big(H(\ell(c)) + H(-\ell(c))\big), \qquad H = F * F^*,$$

summed over the closed orbits of Lemma 5.1.3: $c = (p,m)$ with primitive length $\ell^{\#} = \log p$ and length $\ell = m \log p$.

THEOREM 5.2.2 (Stationary points of $\mathcal{S}$). *The first variation is $\delta\mathcal{S}[F](G) = 2\,\mathrm{Re}\ \mathcal{Q}(F,G)$. Moreover F is stationary ($\delta\mathcal{S}[F] = 0$ for all $G \in C_c^\infty(\mathbb{R})$) iff $\widetilde{F}(\gamma_\rho) = 0$ for every zero ρ. In that case $\mathcal{S}[F] = 0$. In particular no field is "localized at a prime" by stationarity.*

PROOF. Bilinearity gives $\mathcal{S}[F + \epsilon G] = \mathcal{S}[F] + 2\epsilon\,\mathrm{Re}\,\mathcal{Q}(F,G) + O(\epsilon^2)$. If $\delta\mathcal{S}[F] = 0$, then applying it to G and iG gives $\mathcal{Q}(F,G) = 0$ for all G, and in particular for all $T_t G$. Now $\mathcal{Q}(F, T_t G) = \sum_\rho a_\rho e^{-i\gamma_\rho t}$ with $a_\rho = m_\rho \widetilde{F}(\gamma_\rho)\overline{\widetilde{G}(\bar\gamma_\rho)}$, and $\sum |a_\rho| < \infty$ (Lemma 1, Lemma "bound" and the convergence of $\sum m_\rho(1+|\gamma_\rho|^2)^{-2}$). This vanishes identically in t. The zero set is invariant under $\gamma \mapsto -\gamma$, so the Laplace-uniqueness lemma of Lemma 1 (`lem:laplace`) applies with every $\varepsilon \in \mathbb{R}$ and gives $a_\rho = 0$ for all ρ. Choose G with $\widetilde{G}(\bar\gamma_\rho) \neq 0$ (Lemma 1, `lem:nonvan`); then $\widetilde{F}(\gamma_\rho) = 0$. The converse is clear. □

REMARK 5.2.3 (Constrained version). On a window $\{\mathrm{supp}\, F \subset [-\frac{a}{2}, \frac{a}{2}],\ \|F\|_2 = 1\}$, the Euler–Lagrange equation of $\mathcal{S}$ is the integral equation $P_a(\mathcal{W} * F) = \mu F$. Its kernel involves only the closed orbits of length $m \log p \le a$. "All $\mu \ge 0$ for every a" is equivalent to RH (Lemma 1(iv)). Positivity is known on small windows (Yoshida [**122**]).

PROPOSITION 5.2.4 (Local Hamiltonian model: $\delta A = 0 \Leftrightarrow$ closed orbit). *Let $M_p = T^*\mathbb{R}_+^*/\langle \phi_p \rangle$, where $\phi_p(x,\xi) = (px, \xi/p)$, and let $H_{\text{cl}} = x\xi$ (Berry–Keating [**3**]).*

(i) ϕ_p preserves $\theta = \xi\,dx$ and H_{cl}, so both descend to M_p.
(ii) Every orbit $(e^t x, e^{-t}\xi)$ in M_p is closed, with primitive period $\log p$.
(iii) For $T > 0$, the T-periodic Hamiltonian orbits are exactly the critical points of $A_T(\gamma) = \oint_\gamma \theta - \int_0^T H_{\text{cl}}(\gamma)dt$ on smooth loops $\mathbb{R}/T\mathbb{Z} \to M_p$. Such orbits exist iff $T \in (\log p)\mathbb{N}$.
(iv) On the energy shell $H_{\text{cl}} = E$, the Maupertuis action is $S_0 = \oint \xi\,dx = E\,m\log p$, and the period is $\partial S_0/\partial E = m \log p$.
*(v) In the covering chart the linearized flow is $\mathrm{diag}(p^m, p^{-m})$, with instability exponents $\pm \log p$ as in [**22**, (13)]. In M_p the return map is the identity: the orbits come in a one-parameter family.*

PROOF. (i) $\phi_p^*(\xi dx) = (\xi/p)\,d(px) = \xi dx$ and $(px)(\xi/p) = x\xi$. (ii) $\phi^T(x,\xi) = \phi_p^m(x,\xi)$ iff $e^T = p^m$. (iii) $\delta A_T = \int_0^T \omega(\delta\gamma, \dot\gamma - X_H)\,dt$ for closed loops, which is the standard Hamilton principle; then apply (ii). (iv) $\oint \xi\dot x\,dt = \int_0^T E\,dt$. (v) This is immediate. □

PROPOSITION 5.2.5 (Orbit weights vs. Gutzwiller/Selberg). *Let $M_m = \mathrm{diag}(p^m, p^{-m})$. Then*

$$(\log p)\, p^{-m/2} = \frac{\log p}{|\det(M_m - I)|^{1/2}}\,(1 - p^{-m}).$$

The Selberg hyperbolic term on $\Gamma\backslash\mathbb{H}$, $\ell^{\#}/(2\sinh(m\ell^{\#}/2)) = \ell^{\#}|\det(M_m - I)|^{-1/2}$ (with $e^{\ell^{\#}}$ in place of p), carries a plus *sign. The explicit formula carries a* minus *sign and lacks the factor $(1-p^{-m})^{-1}$. This is the "absorption spectrum" sign discussed in [**22**].*

PROOF. $|\det(M_m - I)| = (p^m - 1)(1 - p^{-m}) = p^m(1-p^{-m})^2$. □

Verdict on "prime $\Leftrightarrow$ $\delta S = 0$". It is true in two senses: a prime is a closed orbit of the scaling flow (Lemma 5.1.3), and in the local model closed orbits are critical points of A_T (Prop. 5.2.4).

It is false for the field action $\mathcal{S}$ (Thm. 5.2.2). There is no classical action on $\mathcal{Y}$ itself, because $\mathcal{Y}$ is not a manifold (Lemma 5.1.3(e)). The explicit formula has Gutzwiller *form*, but it is not *derived* from a classical system. Connes's global trace formula, which would supply such a derivation, is shown in [**22**] to be equivalent to RH; the version for finitely many places is proved there.

5.3. Covariance

THEOREM 5.3.1 (Isometry group of $\mathcal{Q}$ in $\mathrm{Diff}(\mathbb{R})$). *Let $\varphi \in \mathrm{Diff}(\mathbb{R})$ act on fields by half-densities, $\varphi_* F = |(\varphi^{-1})'|^{1/2} F \circ \varphi^{-1}$. Then $\mathcal{Q}(\varphi_* F, \varphi_* G) = \mathcal{Q}(F, G)$ for all $F, G \in C_c^\infty(\mathbb{R})$ iff $\varphi(u) = \pm u + b$.*

PROOF. ($\Leftarrow$) Translations: Lemma 1(ii). Reflection: $\mathcal{W}$ is even, since each term is symmetric under $H \mapsto H(-\cdot)$ and $\mathrm{Re}\,\psi(\bar{z}) = \mathrm{Re}\,\psi(z)$. ($\Rightarrow$) The Schwartz kernel is $K(u,v) = \mathcal{W}(u-v)$. The pole term has smooth kernel $2\cosh(\frac{u-v}{2})$. The archimedean term is the Fourier transform of a classical symbol of order 0 (up to log), hence smooth off $u = v$. Explicitly (volume edit, G11): write W_∞ for the archimedean part of $\mathcal{W}$ and let $a(s)$ be its symbol, a combination of $\mathrm{Re}\,\psi$ and constants. Stirling gives $a^{(k)}(s) = O_k(|s|^{-k})$ for $k \geq 1$. For $k \geq m+2$ the distribution $x^k W_\infty(x)$ is the inverse Fourier transform of $(i\partial_s)^k a$, and $s^j (i\partial_s)^k a \in L^1$ for $j \leq m$. Hence $x^k W_\infty \in C^m$, and W_∞ is C^∞ on $x \neq 0$. Each closed orbit contributes $-(\log p) p^{-m/2} \delta$ on the lines $u - v = \pm m \log p$. So the singular support is $\Sigma_E = \{u - v \in E\}$ with $E = \{0\} \cup \{\pm \log n : \Lambda(n) \neq 0\}$ discrete, and Σ_E is a disjoint union of lines. Invariance forces $(\varphi \times \varphi)(\Sigma_E) = \Sigma_E$, because multiplying by smooth non-vanishing Jacobians does not change singular support. A connected line goes to a single line, so $\varphi(u) - \varphi(u - \ell) = c(\ell)$ is constant for every $\ell \in E$. Write $\varphi(u) = au + \psi(u)$ with $a = c(\log 2)/\log 2$. Then ψ is $\log 2$-periodic, hence bounded, and $\psi(u + \log 3) - \psi(u)$ is constant, hence 0. Since $\log 2/\log 3$ is irrational, the continuous function ψ has a dense group of periods and is constant. So φ is affine and $aE = E$. Comparing the smallest positive element $\log 2$ of E and of $a^{-1}E$ gives $|a| = 1$. □

So the formulation is $\mathbb{Q}^\times$-covariant (fields are $\mathbb{Q}^\times$-invariant), scale-covariant ($\mathbb{R}_+^*$) and inversion-covariant ($x \mapsto x^{-1}$, the functional equation). It falls short of diffeomorphism covariance: the orbit lengths $\log p$ are rigid invariants. The "gauge group" $\mathbb{Q}^\times$ is discrete and global, and the observables are traces (functionals of $\mathcal{W}$).

5.4. Positivity: which Hermitian form, and reflection positivity

THEOREM 5.4.1 (Curve case: structural positivity via Rosati). *Let $X/\mathbb{F}_q$ be as in Lemma 3, with J its Jacobian, π the Frobenius endomorphism, $b = q^{-1/2}\pi \in \mathrm{End}^0(J) \otimes \mathbb{R}$, and † the Rosati involution of the canonical polarization extended antilinearly. For $F \in C_c^\infty(\mathbb{R})$ put $\xi_v(F) = \sum_j F(v + jL_q)\, b^j$, a finite sum. Then*

$$\mathcal{Q}_X(F, G) = L_q \int_0^{L_q} \mathrm{Tr}(\xi_v(F)\, \xi_v(G)^\dagger)\, dv,$$

*and the right side is ≥ 0 for $G = F$ by Rosati positivity, $\mathrm{Tr}(\xi\xi') > 0$ for $\xi \neq 0$ [**87**, §21]. This positivity is structural: it uses no information about zeros. It gives a proof of Lemma 3(C3b), which is Weil's.*

PROOF. $\pi'\pi = q$ gives $b^\dagger = b^{-1}$, and $\mathrm{Tr}(b^n) = c_n$ (trace on V_ℓ; Lemma 3). Expanding, $\int_0^{L_q} \xi_v(F)\xi_v(G)^\dagger dv = \sum_n b^n \sum_c \int_0^{L_q} F(v + (n+c)L_q)\overline{G(v + cL_q)}\, dv = \sum_n b^n H(nL_q)$ with $H = F * G^*$. Taking traces and using Lemma 3(C1) gives the identity. For complex coefficients write $\xi = A + iB$ with A, B real. Then $\mathrm{Tr}(\xi\xi^\dagger) = \mathrm{Tr}(AA') + \mathrm{Tr}(BB')$, because $\mathrm{Tr}(BA') = \mathrm{Tr}(AB')$ (the trace is Rosati-invariant). A positive-definite rational form stays positive definite over $\mathbb{R}$. □

QUESTION 5.4.2 (The Hodge-index question for ζ, in TAA form). Is there a $*$-algebra $\mathfrak{A}$ with a positive trace τ and a strongly continuous unitary group $u \mapsto \mathcal{U}(u) \in \mathfrak{A}$, constructed *independently of the zeros*, such that $\tau(\mathcal{U}(H)) = \mathcal{W}(H)$? Then $\mathcal{Q}(F,F) = \tau(\mathcal{U}(F)\mathcal{U}(F)^*) \geq 0$. By GNS such a pair exists iff RH holds (Lemma 1), so the content lies entirely in an independent construction. For curves, $\mathfrak{A} = \mathrm{End}^0(J) \otimes \mathbb{C}$ with $\tau = \mathrm{Tr}$ and $\mathcal{U}(jL_q) = b^j$ does it. For ζ this is the Weil–Connes–Consani program [**22, 12, 13**]; it is open.

PROPOSITION 5.4.3 (Reflection positivity fails at every closed orbit). *Let $\theta F(u) = \overline{F(-u)}$, the inversion $x \mapsto x^{-1}$. For each prime power $n = p^m$ and each even real $\phi \in C_c^\infty$, put $F_\sigma(u) = \sigma^{-1/2}\phi((u - \frac{1}{2}\log n)/\sigma)$, supported in $(0,\infty)$. Then*

$$\lim_{\sigma \to 0} \mathcal{Q}(\theta F_\sigma, F_\sigma)/\|F_\sigma\|_2^2 = -\Lambda(n) n^{-1/2} < 0.$$

So Osterwalder–Schrader positivity with respect to θ fails. The same happens for curves: the Hankel matrix $[c_{j+k}]$ is indefinite while the Toeplitz matrix $[c_{j-k}]$ is PSD. The positivity relevant to RH is Bochner (translation-invariant) positivity, not OS positivity.

PROOF. $\mathcal{Q}(\theta F, F) = \iint F(s)F(t)\mathcal{W}(s+t)\,ds\,dt$ for real F. Near $\log n$, $\mathcal{W} = -\Lambda(n)n^{-1/2}\delta_{\log n} +$ (smooth). The δ-part gives $-\Lambda(n)n^{-1/2}\int F(s)F(\log n - s)ds = -\Lambda(n)n^{-1/2}\|F\|^2$ for even ϕ. The smooth part is $O(\sigma)\|F\|^2$. For curves, $\sum x_j x_k \sum_i \beta_i^{j+k} = \sum_i (\sum_j x_j \beta_i^j)^2$ is not ≥ 0. □

5.5. Thermodynamics (candidate B) and the unified dictionary

PROPOSITION 5.5.1 (Primon gas [**63**] = Bost–Connes Hamiltonian [**7**]). *Let $\mathcal{F} = \bigotimes_p' \ell^2(\mathbb{N}_0)$ (bosonic modes p, occupation numbers k_p) and $H = \sum_p (\log p) N_p$. Unique factorization identifies the basis with $\mathbb{N}^\times$ via $n = \prod p^{k_p}$, and*

- *(i) $k_p = v_p(n)$ (*occupation = p-adic valuation*), and the energy is $\log n = \log|n|_\infty = -\sum_p \log|n|_p = \sum_p k_p \log p$. This is additivity of energy (the "first law"), and it is the product formula.*
- *(ii) $\mathrm{Tr}\, e^{-\beta H} = \prod_p (1 - p^{-\beta})^{-1} = \zeta(\beta)$ for $\beta > 1$. This is the Ruelle-type zeta function $\prod_c (1 - e^{-\beta \ell^\#(c)})^{-1}$ of the length spectrum of Lemma 5.1.3(c), and $\langle H \rangle_\beta = -\zeta'/\zeta(\beta) = \sum_{(p,m)} \log p\, e^{-\beta m \log p}$ is an orbit sum.*
- *(iii)* (Hagedorn.) *$\#\{n \leq e^E\} = \lfloor e^E \rfloor$, so the microcanonical entropy is $S(E) = E + o(1)$ and $T_H = (dS/dE)^{-1} = 1$. Hence $Z(\beta) < \infty$ iff $\beta > 1$. This 1 is the growth rate of states. It is unrelated to the "1" of the product formula, and it equals the topological entropy of the orbit count $\#\{p \leq e^L\} \sim e^L/L$ (cf. [**90**]).*
- *(iv)* (Description symmetry; [**7**], [**21**, Ch. 3].) *The BC system has symmetry group $C_\mathbb{Q}/C_\mathbb{Q}^0 \cong \widehat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{\mathrm{cycl}}/\mathbb{Q})$. There is a unique KMS$_\beta$ state for $\beta \leq 1$. For $\beta > 1$ the extremal KMS$_\beta$ states are parametrized by $\widehat{\mathbb{Z}}^\times$, i.e. by embeddings $\mathbb{Q}^{\mathrm{cycl}} \hookrightarrow \mathbb{C}$ (spontaneous symmetry breaking).*
- *(v)* (Momentum.) *On $L^2(\mathbb{R}_+^*, dx)$, $(\mathcal{U}(a)f)(x) = e^{a/2}f(e^a x)$ has generator $-i(x\partial_x + \frac{1}{2}) = \frac{1}{2}(xp + px)$, the Weyl-ordered Berry–Keating xp. In log-scale this is $D = -i\,d/du$, conjugate to u. On the Weil space of Lemma 1 its spectrum is $\{\gamma_\rho\}$ (under RH, simple and real).*

PROOF. (i)–(iii) are elementary. (iv) is cited from [**7**] (abstract and main theorem) and [**21**, Ch. 3]. (v): differentiate at $a = 0$; the spectral statement is Lemma 1(v). □

REMARK 5.5.2 (Zero-point energy: what is and is not true). With zero-point energy the mode-p factor is $p^{-\beta/2}(1 - p^{-\beta})^{-1} = 1/(2\sinh(\frac{\beta}{2}\log p))$, mirroring the sinh of Prop. 5.2.5. The vacuum energy $\frac{1}{2}\sum_p \log p$ diverges. Its "super-regularized" value uses $\prod_p p = 4\pi^2$ [**89**], and gives

$E_0 = \log 2\pi$ *provided* the power rule $\prod p^{1/2} = (\prod p)^{1/2}$ holds for super-regularization. That rule holds for classical zeta-regularization; we have not verified it in the super-regularized setting. With this choice, $Z_0(\beta) = (2\pi)^{-\beta}\zeta(\beta)$.

What this is not. Multiplying by $(2\pi)^{-\beta}$ moves no zeros and produces no functional equation. The completion ξ needs $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$, whose poles cancel the trivial zeros. The factor $(2\pi)^{-s}$ is only the exponential part of $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$, which appears in $\zeta(1-s) = 2(2\pi)^{-s}\cos(\frac{\pi s}{2})\Gamma(s)\zeta(s)$. So restoring zero-point energy is *not* the passage $\zeta \to \xi$. The two "$\frac{1}{2}$"s are both Weyl-ordering constants: $\frac{1}{2}(a^\dagger a + aa^\dagger)$ for the oscillator, and $\frac{1}{2}(xp + px) = xp - \frac{i}{2}$ for the dilation, which gives eigenfunctions $x^{-1/2-iE}$ and hence the line $\operatorname{Re} s = \frac{1}{2}$. But they live in conjugate variables: the zero-point $\frac{1}{2}$ shifts the prime/energy side by a global factor, while the critical-line $\frac{1}{2}$ is the unitarity normalization of the momentum side. We know of no theorem identifying them.

Verdict on (B). It is a correct and complete dictionary: energy is archimedean log-size, occupation is valuation, the first law is the product formula, $T_H = 1$, the symmetry is Galois, and the momentum is the scaling generator. Its natural positive forms (GNS of KMS states, and the KMS condition $\omega(a^*\sigma_{i\beta/2}(a)) \geq 0$) are positive for structural reasons but on the wrong object. They involve $\zeta(\beta)$ only for real $\beta > 1$ and do not reproduce $\mathcal{W}$. So (B) supplies no Hodge-index-type form.

Candidate (C), Einstein–Cartan. The configuration space $C_{\mathbb{Q}}/\widehat{\mathbb{Z}}^\times \cong \mathbb{R}$ is one-dimensional, so $\Lambda^2 T^* = 0$ and every affine connection is torsion-free; a torsion functional vanishes identically. On $\mathcal{Y}$ there is no manifold structure (Lemma 5.1.3(e)). Any nontrivial torsion would require extra dimensions that TAA does not specify. Rejected.

Remark 5.5.3 (The "$m_g = m_i$" duality (speculative): what it would require). The explicit formula equates a zero-side ("inertial", spectral: $\sum_\rho \widetilde{H}(\gamma_\rho)$, momentum D) with an orbit-side ("gravitational", geometric: pole + archimedean − orbits, energy/length). It is an unconditional theorem. What would turn "$m_g = m_i$" into a *principle* is a single Hilbert space on which both sides are traces of the same positive operator. This is exactly Question 5.4.2, and it is equivalent to RH. So the analogy is meaningful only if the positivity is supplied independently, as Rosati does for curves (Thm. 5.4.1).

Remark 5.5.4 (Non-canonical choices, flagged). (1) Bose vs. Fermi statistics for primons ($\zeta(\beta)$ vs. $\zeta(\beta)/\zeta(2\beta)$ [**63**]). (2) The half-density normalization ($p^{-m/2}$) vs. densities. (3) Quotienting by $\widehat{\mathbb{Z}}^\times$; this is needed for orbit closure, see Lemma 5.1.3(d). (4) The arithmetic vs. geometric Frobenius convention. (5) Super-regularization of $\prod_p p$. (6) The archimedean place appears as energy (the norm) and through $\Gamma_{\mathbb{R}}$, but not as a primon mode.

5.6. Consistency with Lemmas 1–3

Lemma 1: $\mathcal{S} = \mathcal{Q}$, T-invariant, $\kappa(\mathcal{Q}) = N_+$; unitarity of the flow is equivalent to RH, and D has spectrum $\{\gamma_\rho\}$. Lemma 2: $\mathcal{S}$ is built from the explicit-formula data alone, so Theorem A there applies verbatim to the admissible zero data that this action can see. Lemma 3: for curves the orbits have lengths mL_q (one orbit length per m; the weights are point counts), $\mathcal{S}_X = \mathcal{Q}_X \geq 0$ by Thm. 5.4.1, and the flow at time L_q is normalized Frobenius. Lemma 4: there is no positivity on the p-adic side, consistent with Thm. 5.4.1 being archimedean (a positive trace).

5.7. Known vs. new; numerics

Known: the explicit formula; the adelic and scaling-site picture of primes as orbits [22, 12, 13]; the primon gas and BC system [63, 7, 21]; Rosati positivity [87]; $\prod p = 4\pi^2$ [89]; Berry–Keating [3]. **New (modest):** the elementary length-spectrum and product-formula lemma including Frobenius locking and non-Hausdorffness (Lemma 5.1.3); Thm. 5.2.2 (stationary points of $\mathcal{S}$); the covariance theorem (Thm. 5.3.1); the failure of OS positivity at prime orbits (Prop. 5.4.3); the Rosati formula for $\mathcal{Q}_X$ in TAA variables (Thm. 5.4.1, Weil's argument repackaged); and the unified dictionary with its negative results.

Numerics (`num/action_check.py`, `.out`).

- Prop. 5.4.3 with Gaussian bumps ($\sigma = 0.05$; the zero side with 200 zeros agrees to 8 digits): $\mathcal{Q}(\theta F, F)/\|F\|^2 = -0.2824,\ -0.3402,\ -0.0030$ at $c = \log 2, \log 3, \log 4$. At $\sigma = 0.03$ (arithmetic side only): $-0.365,\ -0.458,\ -0.137$, moving toward the limits $-0.490,\ -0.634,\ -0.347$. Away from E ($c = 0.9, 1.0$) the values are positive.
- Thm. 5.3.1: a non-even two-bump field gives $\mathcal{Q} = 6.95897209205 \times 10^{-4}$, unchanged under reflection and under translation by 3 (12 digits), but 1.379×10^{-2} after dilation by 1.2.
- Thm. 5.4.1 ($E/\mathbb{F}_5$): the point-count side and the Rosati-trace side agree (3.8911913515). The Toeplitz matrix is PSD; the Hankel matrix has eigenvalues in $[-20.27, 19.67]$.
- Prop. 5.5.1: $-\zeta'/\zeta(2) = 0.5699609931$ (partial orbit sum $n < 4000$: 0.56971); $(\beta - 1)(-\zeta'/\zeta) \to 1$ (0.9441, 0.99425 at $\beta = 1.1, 1.01$).

CHAPTER 6

Lemma 6: The structural positive trace on finite sets of places

What is proved, what is new, and the exact failure point

Summary. Fix a finite set S of places of $\mathbb{Q}$ with $\infty \in S$, or equivalently a window $\operatorname{supp} F \subset [-A/2, A/2]$. We build, without using zeros, a positive trace τ_S from the unitary multiplier $U_S = \prod_{v\in S} u_v$, where u_v is the ratio of local factors. We prove the following.

(1) *Window equals semilocal.* On a window of length A the Weil form is exactly the semilocal form for $S(A) = \{\infty\} \cup \{p \le e^A\}$ (Thm. 6.3.1).
(2) *Positive trace with a pointwise non-negative symbol.* We prove $\tau_S(F) = \frac{1}{2\pi}\int |\widehat{F}|^2 \sigma_S$ with $\sigma_S \ge 0$, and we give an explicit formula for σ_S (Thm. 6.4.2).
(3) *Exact cancellation of prime-power atoms.* At the level of asymptotic symbols, τ_S reproduces the Dirac masses of $\mathcal{W}_S$ at the prime-power lengths $\pm m\log p$ exactly. This rests on an exact identity that holds for every unitary of the form $\sum c_j T_{t_j}$ (Thm. 6.5.1).
(4) *Finite negative index on every window, unconditionally* (Thm. 6.7.1).

We also isolate, numerically, the exact place where the Connes–Consani archimedean mechanism stops transferring. For $|S| \ge 2$ the remainder $D_S = \tau_S - \mathcal{W}_S$ has *logarithmic* singularities at every $a \in \mathbb{Z}\log p$, including $a = 0$. Their strengths grow like $p^{|m|/2}$. As a result, $D_S \circ Q_+$ is unbounded *above* on every interval, whereas for $S = \{\infty\}$ it is "$-2\,\mathrm{Id}$+compact" [19]. Nothing here proves or implies RH. Passing from S to all places is, by Lemma 1, equivalent to RH, and we say precisely what breaks.

6.1. Conventions (as in Lemma 1)

We work additively: $u = \log x$. Put $\widehat{F}(s) = \int F(u)e^{-isu}\,du$, $(T_\lambda F)(u) = F(u-\lambda)$ (multiplier $e^{-i\lambda s}$), $F^*(u) = \overline{F(-u)}$ and $H = F * F^*$, so $\widehat{H} = |\widehat{F}|^2$. The Weil form of Lemma 1 is $\mathcal{Q}(F,F) = \mathcal{P}(F) + \mathcal{W}(H)$. Here $\mathcal{P}$ is the pole term, a Hermitian form of rank ≤ 2. The term $\mathcal{W}(H) = \frac{1}{2\pi}\int \widehat{H}(s)\,2\theta'(s)\,ds - \sum_{n\ge1}\frac{\Lambda(n)}{\sqrt{n}}\,(H(\log n) + H(-\log n))$ is the archimedean part minus the prime part. Here $2\theta'(s) = \operatorname{Re}\psi(\frac14 + \frac{is}{2}) - \log\pi$, and $\mathcal{Q} \ge 0 \iff$ RH, with $\kappa(\mathcal{Q}) = N_+$ (Lemma 1). For a prime p put $\ell = \log p$ and $q = p^{-1/2}$, and define

$$\varphi_p'(s) = 2\ell \sum_{m\ge1} q^m \cos(m\ell s), \qquad \Phi_S' = 2\theta' - \sum_{p\in S}\varphi_p', \qquad \mathcal{W}_S(H) = \frac{1}{2\pi}\int \widehat{H}\,\Phi_S'\,ds.$$

The local multipliers are $u_v(s) = L_v(\frac12 + is)/L_v(\frac12 - is)$, namely $u_\infty = e^{2i\theta}$ and $u_p(s) = (1 - qe^{i\ell s})/(1 - qe^{-i\ell s})$. As operators,

$$u_p = -q\,T_{-\ell} + \sum_{n\ge0}(1-q^2)q^n\,T_{n\ell} \qquad (\ell^1\text{-norm } 1+2q).$$

The convolution kernel of u_∞ is $k_\infty(t) = 2e^{-t/2}\cos(2\pi e^{-t})$. It is the scaling kernel of the Fourier transform [19, §1]. It decays as $t \to +\infty$ and oscillates with growing amplitude as $t \to -\infty$. Let $P = 1_{[0,\infty)}$. For a unitary multiplier U put $\widehat{P}_U = U^*(1-P)U$, and set $U_S = u_\infty \prod_{p\in S} u_p = u_\infty V_S$, with $V_S = \sum_j c_j T_{t_j}$ and $\sum |c_j|(1+|t_j|) < \infty$.

6.2. What is in the literature (all items verified)

(L1) Connes [22] sets up the semilocal trace formula: the Hilbert space of the semilocal adèle class space, cutoff projections, and the local terms W_v in their exact form.

(L2) Yoshida [122] proves positivity of the Weil Hermitian form on C_c^∞ of a sufficiently small window, which contains no prime power.

(L3) Connes–Consani [19], for $S = \{\infty\}$ (arXiv numbering), prove the following.

- Prop. 2.2(iii): $\mathrm{Tr}(\vartheta(f)P\widehat{P}P) = \int f(\rho^{-1})(\delta(\rho) - \tau(\rho))d^*\rho$, with δ an explicit continuous function.
- Cor. 2.3: positivity of that functional, and $2\theta' + \widehat{\delta} \geq 0$.
- Thm. 3.6: $D \circ Q = -2\,\mathrm{Id} + K_I$ with K_I compact on any bounded symmetric I.
- Cor. 3.8 and Rem. 3.9: $D \circ Q \leq 0$ for $I = [u^{-1}, u]$ with $u = 1.10246$, improved to $u = 1.15077$.
- Thm. 6.11: an inequality for $\mathcal{W}_\infty$ on g supported in $[2^{-1/2}, 2^{1/2}]$. The proof is partly computer-assisted.

Weil positivity is proved there only for the archimedean place, on windows that contain no prime power.

(L4) Connes–Consani [16] prove that $u_\infty \prod_{p\in F} u_p$ is *quasi-inner*, i.e. $(1-P)U_S P$ is compact. More precisely it is an infinitesimal of order $1/(2m)$, where $m = |F|$. No single u_p is quasi-inner. They also prove that the semilocal Sonin spaces are infinite-dimensional and form an inductive system. These are the analytic prerequisites for the semilocal case. No positivity is claimed.

(L5) Connes–Consani–Moscovici [15] build the semilocal prolate operator and prove stability of semilocal Sonin spaces. [14] treats the moment problem for $S = \{\infty, p\}$; it is an arXiv preprint and we have not verified a journal version. Neither proves semilocal Weil positivity.

Status. As of this writing we found no proof of Weil positivity for any $S \ni p$, nor on any window of length $A > \log 2$. That is consistent with RH being open, since the windowed statements for all A are equivalent to RH (Thm. 6.3.1 plus Lemma 1).[1]

6.3. Window equals semilocal

THEOREM 6.3.1 (rigorous; elementary). *Let* $\operatorname{supp} F \subset [-A/2, A/2]$ *and* $S(A) = \{\infty\} \cup \{p \leq e^A\}$. *Then for every finite* $S \supseteq S(A)$,

$$\mathcal{Q}(F,F) = \mathcal{P}(F) + \mathcal{W}_S(F * F^*),$$

and the right-hand side does not depend on such S.

PROOF. H is supported in $[-A, A]$. Since $\frac{1}{2\pi}\int \widehat{H}\,\varphi_p' = \ell \sum_{m\geq 1} q^m(H(m\ell) + H(-m\ell))$ and $\Lambda(p^m)/\sqrt{p^m} = \ell q^m$, the prime sum restricted to $n \leq e^A$ equals $\frac{1}{2\pi}\int \widehat{H} \sum_{p\in S} \varphi_p'$ for every $S \supseteq S(A)$. Terms with $p^m > e^A$ vanish. □

6.4. The structural trace τ_S

DEFINITION 6.4.1. For $F \in C_c^\infty(\mathbb{R})$ put $\tau_S(F) := \|(1-P)\,U_S\,P\,T(F)\|_{\mathrm{HS}}^2$. Here $T(F)$ is convolution by F, and τ_S depends only on H.

For $\mu \in \mathbb{R}$ and $s \in \mathbb{R}$ let

$$g_\mu(s) := \int_{-\infty}^{-\mu} k_\infty(t) e^{-ist} dt = \int_{e^\mu}^{\infty} 2x^{-\frac{1}{2}+is}\cos(2\pi x)\,dx,$$

[1] **[Vol. 1 editorial note]** Superseded. Theorem 10.3.1 proves (computer-assisted) positivity on every window of length $\leq \log 3$; Eid and Zhu (unrefereed, 2026) claim $1.12\log 2$ and 1.6.

a convergent oscillatory integral. We have $g_\mu \to u_\infty$ as $\mu \to -\infty$ and $|g_\mu(s)| = O_s(e^{-\mu/2})$ as $\mu \to +\infty$.

THEOREM 6.4.2 (structural trace; rigorous). [2]

(i) $\tau_S(F) < \infty$, *and* $\tau_S(F) \geq 0$ *by definition.*

(ii) (Symbol.) $\tau_S(F) = \frac{1}{2\pi}\int |\widehat{F}(s)|^2 \sigma_S(s)\, ds$, *where*

$$\sigma_S(s) = \int_0^\infty \Big| \sum_j c_j\, e^{-ist_j} g_{\lambda+t_j}(s)\Big|^2 d\lambda \quad \geq 0.$$

For a general unitary convolution U *with kernel* k *this reads* $\sigma_U(s) = \int_0^\infty |(k1_{(-\infty,-\lambda)})^\wedge(s)|^2 d\lambda$. *If moreover* $\int |k(t)|(1+|t|)\, dt < \infty$ *(a hypothesis that* k_∞ *does* not *satisfy), then, in kernel form,*

$$\tau_U(H) = \iint_{s,t<0} \overline{k(s)}k(t)\min(|s|,|t|)H(s-t)\, ds\, dt.$$

(iii) (Purely p-adic case.) For $U = V = \sum c_j T_{t_j}$ *one has* $\tau_V(H) = \sum_{t_i,t_j<0} \bar{c}_i c_j \min(|t_i|,|t_j|)H(t_i - t_j)$. *For example* $\tau_{u_p} = \frac{\log p}{p}H(0)$.

(iv) (Decomposition.) $\tau_S = \mathcal{W}_S + D_S$, *where* D_S *has symbol* $d_S := \sigma_S - \Phi_S'$. D_S *is not additive over places:* $\tau_S \neq \tau_\infty + \sum_p \tau_{u_p}$ *in general (see §6.9).*

LEMMA 6.4.3 (oscillatory bounds; rigorous). *For* $\mu, s \in \mathbb{R}$ *the improper integral defining* $g_\mu(s)$ *converges,* $g_\mu(-s) = \overline{g_\mu(s)}$, *and with* $a = e^\mu$*:*

(a) $|g_\mu(s)| \leq \frac{6}{\pi}a^{-1/2}$ *whenever* $a \geq |s|/\pi$;

(b) $|g_\mu(s)| \leq B := 22$ *for all* μ, s;

(c) for $s \geq 2\pi$ *put* $x_0 = s/2\pi$, $L(s) = \log x_0$ *and* $e_\mu(s) := g_\mu(s) - u_\infty(s)\, 1[\mu < L(s)]$. *Then*

$$\int_{\mathbb{R}} |e_\mu(s)|^2\, d\mu \;\leq\; K\, s^{-1/2}, \qquad K = 850.$$

(Numerically $\int_{\mathbb{R}} |e_\mu(s)|^2 d\mu \approx 0.563\, s^{-1/2}$ *for* $s \in [10, 640]$*;* `supplement/L6_osc_check.py`*.)*

PROOF. Write $2x^{-\frac{1}{2}+is}\cos(2\pi x) = x^{-1/2}(e^{iF_+(x)} + e^{iF_-(x)})$ with $F_\pm(x) = s\log x \pm 2\pi x$, and $I_\pm(\alpha,\beta) = \int_\alpha^\beta x^{-1/2}e^{iF_\pm}dx$, so that $g_\mu = I_+(a,\infty) + I_-(a,\infty)$ and $u_\infty = I_+(0,\infty) + I_-(0,\infty)$. By symmetry ($k_\infty$ is real) we may take $s \geq 0$. We use two classical tools [**116**, Lemmas 4.3, 4.4]: (T1) if G/F' is monotone and $|F'/G| \geq m > 0$ on $[\alpha,\beta]$, then $|\int_\alpha^\beta Ge^{iF}| \leq 4/m$; (T2) if $|F''| \geq r > 0$ on $[\alpha,\beta]$ then $|\int_\alpha^\xi e^{iF}| \leq 8/\sqrt{r}$ for every ξ, hence, by the second mean value theorem applied to real and imaginary parts, $|\int_\alpha^\beta Ge^{iF}| \leq 16\, G(\alpha)/\sqrt{r}$ for $G \geq 0$ decreasing. Here $G = x^{-1/2}$, $F_\pm' = s/x \pm 2\pi$, and $F_\pm'/G = sx^{-1/2} \pm 2\pi x^{1/2}$; one checks by differentiation that $G/F_+' = x^{1/2}/(s+2\pi x)$ increases on $(0,x_0]$ and decreases on $[x_0,\infty)$, that F_-'/G is decreasing on $(0,\infty)$, and that $|F_-'|/G = 2\pi x^{-1/2}(x - x_0)$ is increasing on $[x_0,\infty)$. Convergence of the improper integrals at ∞ follows from (T1) applied on $[\beta,\beta']$, $\beta \to \infty$.

(a) On $[a,\infty)$ with $a \geq 2x_0$: $|F_-'|/G \geq \pi a^{1/2}$ and $F_+'/G \geq 2\pi a^{1/2}$, with the monotonicity required in (T1). Hence $|g_\mu| \leq \frac{4}{\pi\sqrt{a}} + \frac{2}{\pi\sqrt{a}}$.

(b) If $s \leq 2\pi$ and $a \geq 2$, (a) gives $|g_\mu| \leq 6/(\pi\sqrt{2}) < 1.36$; if $s \leq 2\pi$ and $a < 2$, then $|g_\mu| \leq \int_a^2 2x^{-1/2}dx + |g_{\log 2}(s)| < 4\sqrt{2} + 1.36 < 7.1$. Let $s \geq 2\pi$, so $x_0 \geq 1$. Splitting $[a,\infty)$ at x_0, (T1) gives $|I_+(a,\infty)| \leq 2 \cdot 4\max_x \frac{x^{1/2}}{s+2\pi x} = \frac{4x_0^{1/2}}{s} < 0.64$. Split $[a,\infty)$ for I_- into its intersections with $(0,\frac{x_0}{2}]$, $[\frac{x_0}{2}, 2x_0]$ and $[2x_0,\infty)$. On the first, (T1) with $m = F_-'/G\,(\frac{x_0}{2}) = \frac{s}{2}(\frac{x_0}{2})^{-1/2}$ gives

[2] **[Vol. 1 editorial note]** Volume edit (gap G2): the standalone note said "rigorous apart from routine Fubini justifications; the full proof, via the oscillatory Lemma 6.4.3 and truncated kernels, is given below.

$\le 4/\sqrt{\pi s} < 0.91$; on the third, (T1) with $m = \pi(2x_0)^{1/2}$ gives < 0.91; on the middle one $|F_-''| = s/x^2 \ge \pi^2/s$, so (T2) gives $\le 16(\frac{x_0}{2})^{-1/2}\sqrt{s}/\pi = 32/\sqrt{\pi} < 18.06$. In total $|g_\mu| < 21$.

(c) If $a < x_0$ then $e_\mu = -(I_+(0,a) + I_-(0,a))$, so $|e_\mu| \le 4\sqrt{a}$ and, by (T1) with $m = F_-'/G\,(a) = 2\pi a^{-1/2}(x_0 - a)$ and $m = F_+'/G\,(a) \ge s a^{-1/2}$,

$$|e_\mu| \le \frac{2\sqrt{a}}{\pi(x_0 - a)} + \frac{4\sqrt{a}}{s}.$$

If $a \ge x_0$ then $e_\mu = g_\mu$, and (T1) with $m = 2\pi a^{-1/2}(a - x_0)$, resp. $m \ge 2\pi a^{1/2}$, gives $|e_\mu| \le \frac{2\sqrt{a}}{\pi(a-x_0)} + \frac{2}{\pi\sqrt{a}}$. Always $|e_\mu| \le B + 1 = 23$. Now integrate in $d\mu = da/a$. For $a \le x_0/2$: $|e_\mu| \le 12\sqrt{a}/s$, contributing $\le 72x_0/s^2 = 36/(\pi s)$. For $a \ge 2x_0$: $|e_\mu| \le 6/(\pi\sqrt{a})$, contributing $\le 36/(\pi^2 \cdot 2x_0) = 36/(\pi s)$. For $\frac{x_0}{2} < a < 2x_0$: $|e_\mu| \le \min(23, \frac{2\sqrt{2x_0}}{\pi|a-x_0|}) + \frac{4}{\sqrt{\pi s}}$, and with $da/a \le 2da/x_0$ and $\int_{\mathbb{R}} \min(A, C/y^2)dy = 4\sqrt{AC}$ this contributes at most

$$\frac{4}{x_0} \cdot 4\sqrt{529 \cdot \frac{8x_0}{\pi^2}} + \frac{32\log 4}{\pi s} = \frac{1472}{\sqrt{\pi s}} + \frac{32\log 4}{\pi s}.$$

Since $s \ge 2\pi$, the sum of all contributions is $\le (830.6 + 14.8)s^{-1/2} < 850\, s^{-1/2}$. □

Proof of Theorem 6.4.2. Throughout, $V_S = \sum_j c_j T_{t_j}$ has real coefficients, $N_V := \sum_j |c_j|(1 + |t_j|) < \infty$, and $t_j \ge -T_0$ with $T_0 = \sum_{p\in S} \log p$ (each u_p only shifts by $\ge -\log p$). Put $G(\mu, s) = \min(B, \frac{6}{\pi}e^{-\mu/2})$ if $e^\mu \ge |s|/\pi$ and $G(\mu, s) = B$ otherwise; by Lemma 6.4.3(a),(b), $\sup_{\nu\ge\mu} |g_\nu(s)| \le G(\mu, s)$.

Step 1 (truncation). For $R > 0$ let $k_R = k_\infty 1_{[-R,\infty)}$. Since $|k_\infty(t)| \le 2e^{-t/2}$, $k_R \in L^1 \cap L^2$, and $\widehat{k_R} = u_\infty - g_R$. Let $U_{S,R}$ be convolution by $K_R := \sum_j c_j k_R(\cdot - t_j) \in L^1 \cap L^2$, i.e. the multiplier $(u_\infty - g_R)V_S$. As $R \to \infty$, $g_R(s) \to 0$ for every s (Lemma 6.4.3(a)) and $|u_\infty - g_R| \le 1 + B$, so $U_{S,R} \to U_S$ strongly by dominated convergence on the Fourier side. Put $X_R = (1-P)U_{S,R}PT(F)$ and $X = (1-P)U_S PT(F)$; then $X_R \to X$ strongly.

Step 2 (Hilbert–Schmidt norm of X_R). For $x < 0$ let $b_x(z) = K_R(x-z)1_{z>0} \in L^1 \cap L^2$. For $f \in L^2$, $X_R f(x) = \int b_x(z)(F * f)(z)\, dz = \int f(y)\big[\int b_x(z)F(z-y)\, dz\big]dy$, the interchange being justified by $\iint |b_x(z)F(z-y)f(y)| \le \|b_x\|_1\|F\|_2\|f\|_2$. So X_R has kernel $X_R(x,y) = 1_{x<0}\int b_x(z)F(z-y)dz$. By Plancherel in y, $\int |X_R(x,y)|^2 dy = \frac{1}{2\pi}\int |\widehat{b_x}(-s)|^2|\widehat{F}(s)|^2 ds$, and $\widehat{b_x}(-s) = e^{isx}\,(K_R 1_{(-\infty,x)})^\wedge(s)$. With $\lambda = -x$ and $(K_R 1_{(-\infty,-\lambda)})^\wedge(s) = \sum_j c_j e^{-ist_j}(g_{\lambda+t_j}(s) - g_R(s))1[\lambda + t_j < R]$, Tonelli (all integrands are non-negative) gives

$$\|X_R\|_{\mathrm{HS}}^2 = \frac{1}{2\pi}\int |\widehat{F}(s)|^2 \sigma_{S,R}(s)\, ds, \qquad \sigma_{S,R}(s) = \int_0^\infty \Big|\sum_j c_j e^{-ist_j}(g_{\lambda+t_j} - g_R)(s) 1[\lambda + t_j < R]\Big|^2 d\lambda.$$

Step 3 (domination, uniform in R). Each term in the j-sum is bounded by $2|c_j|G(\lambda + t_j, s)$, so by Cauchy–Schwarz

$$\sigma_{S,R}(s) \le 4\Big(\sum |c|\Big)\sum_j |c_j| \int_{t_j}^\infty G(\mu, s)^2 d\mu \le 4N_V\Big[B^2 N_V(1 + \log_+ \tfrac{|s|}{\pi}) + \tfrac{36}{\pi^2}e^{T_0}N_V\Big] =: \Omega_S(s),$$

using $\int_t^\infty G^2 \le B^2(\log_+ \frac{|s|}{\pi} + |t|) + \frac{36}{\pi^2}e^{-\max(t, \log(|s|/\pi))}$ and $t \ge -T_0$. The same bound holds for σ_S. Since $\Omega_S(s) \le C_S(1 + \log(2 + |s|))$ and $\widehat{F}$ is a Schwartz function, $\int |\widehat{F}|^2\Omega_S < \infty$.

Step 4 (limit). Pointwise in (λ, s), the j-sum in $\sigma_{S,R}$ converges to $\sum_j c_j e^{-ist_j} g_{\lambda+t_j}(s)$ as $R \to \infty$ (dominated convergence in j), so $\sigma_{S,R} \to \sigma_S$ by dominated convergence in λ, and $\|X_R\|_{\mathrm{HS}}^2 \to \frac{1}{2\pi}\int |\widehat{F}|^2\sigma_S$. Applying Step 2 to $K_R - K_{R'}$ (kernel $\sum_j c_j k_\infty 1_{[-R',-R)}(\cdot - t_j)$, symbol differences dominated by $4G$ and tending to 0) shows that (X_R) is Cauchy in the Hilbert–Schmidt

norm; its Hilbert–Schmidt limit coincides with the strong limit X. Hence $\tau_S(F) = \|X\|_{\mathrm{HS}}^2 = \frac{1}{2\pi}\int |\widehat{F}|^2\sigma_S$. This proves (ii), and (i) follows from Step 3.

The kernel form for a general unitary convolution U with kernel k is obtained in the same way when $\int |k(t)|(1+|t|)dt < \infty$: write $\min(|s|,|t|) = \int_0^\infty 1_{s<-\lambda}1_{t<-\lambda}d\lambda$ and use Parseval in H. (iii) For $k = \sum c_j\delta_{t_j}$ with $\sum |c_j|(1+|t_j|) < \infty$ this is absolutely convergent and collapses to the stated sum. (iv) This is a definition; non-additivity is checked numerically. □

6.5. Exact cancellation of the prime-power atoms

THEOREM 6.5.1. *(a) (*Exact identity; rigorous.*) Let $V(s) = \sum_j c_j e^{-ist_j}$ be unimodular with $\sum |c_j|(1+|t_j|) < \infty$, and let $w_V(s) := -\sum_{i,j} \bar{c}_i c_j\, t_j\, e^{is(t_i-t_j)}$, the symbol of $\mathcal{W}_V$. Then for every L with $L > \sup\{t_j : c_j \neq 0\}$,*

$$\int_{-\infty}^{L} \Big|\sum_{t_j<\mu} c_j e^{-ist_j}\Big|^2 d\mu \;=\; L + w_V(s).$$

For a single prime, $w_{u_p} = -\varphi_p'$. When infinitely many t_j exceed L, the identity holds up to an error of absolute value at most $2N_V\sum_{t_j\geq L}|c_j|\,(t_j-L)$, with $N_V = \sum_j |c_j|(1+|t_j|)$ (volume correction after referee; the earlier form $O(L\sum_{t_j\geq L}|c_j|)$ is not valid for arbitrary coefficient sequences).

*(b) (*Asymptotic symbol; rigorous.*) $d_S(s) = \sigma_S(s) - \Phi_S'(s) \to 0$ as $|s|\to\infty$; more precisely $d_S(s) = O_S(|s|^{-1/4}\log^{1/2}|s|)$.*

*(c) (*Consequence; rigorous.*) For every $a \neq 0$ the Bohr means $\mathbf{M}_a(f) := \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(s)e^{-ias}ds$ of σ_S and of Φ_S' exist and coincide. They equal $-\ell q^m$ at $a = \pm m\ell$ ($p \in S$, $m \geq 1$) and 0 at every other $a \neq 0$. In this precise sense D_S has no Dirac mass at any $a \neq 0$: D_S (the distribution with symbol d_S) cannot be written as $\beta\delta_a + \nu$ with $\beta \neq 0$ and ν a finite measure without atom at a, or more generally any tempered distribution whose transform has Bohr mean 0 at frequency a. Consequently and the prime-power atoms $-\ell q^m\delta_{\pm m\ell}$ of $\mathcal{W}_S$ are reproduced* exactly *by τ_S.*

PROOF. (a) Expand $|\cdot|^2$. The pair (i,j) contributes $\bar{c}_i c_j e^{is(t_i-t_j)}(L - \max(t_i,t_j))$. Unitarity says $r(d) := \sum_{t_i-t_j=d}\bar{c}_i c_j = \delta_{d,0}$, so $\sum \bar{c}_i c_j e^{is(t_i-t_j)} = 1$. Moreover

$$\sum \bar{c}_i c_j(\max(t_i,t_j) - t_j)e^{is(t_i-t_j)} = \sum_{d>0} d\, e^{isd} r(d) = 0.$$

Hence the left side equals $L - \sum \bar{c}_i c_j t_j e^{is(t_i-t_j)} = L + w_V$. For u_p this was also checked numerically: the Fourier coefficients agree to 10^{-6}.

(b) By evenness (c_j real) take $s \geq 2\pi$ and use Lemma 6.4.3(c). Since $g_{\lambda+t_j} = u_\infty 1[\lambda + t_j < L(s)] + e_{\lambda+t_j}$, the integrand of σ_S in Thm. 6.4.2(ii) is $|M_\lambda + E_\lambda|^2$ with $M_\lambda = u_\infty(s)\sum_{t_j<L-\lambda} c_j e^{-ist_j}$ and $E_\lambda = \sum_j c_j e^{-ist_j} e_{\lambda+t_j}(s)$. With $\mu = L - \lambda$ and $|u_\infty| = 1$,

$$\int_0^\infty |M_\lambda|^2 d\lambda = \int_{-\infty}^{L}\Big|\sum_{t_j<\mu} c_j e^{-ist_j}\Big|^2 d\mu = \sum_{t_i,t_j<L} \bar{c}_i c_j e^{is(t_i-t_j)}(L - \max(t_i,t_j)),$$

a finite computation because $t_j \geq -T_0$. Extending the sum to all (i,j) changes it by at most $2\tau_V(L)N_V$, where $\tau_V(L) = \sum_{t_j\geq L}|c_j|(1+|t_j|)$; the full sum equals $L + w_V(s)$ by the computation in (a) (all rearrangements are absolutely convergent because $N_V < \infty$, and $r(d) = \delta_{d,0}$ by uniqueness of Bohr–Fourier coefficients of the absolutely convergent almost-periodic function $|V|^2 \equiv 1$). By Cauchy–Schwarz and Lemma 6.4.3(c), $\int_0^\infty |E_\lambda|^2 d\lambda \leq (\sum |c_j|)^2 \int_{\mathbb{R}} |e_\mu|^2 d\mu \leq N_V^2 K s^{-1/2}$. Hence,

with $|w_V| \le N_V^2$,

$$|\sigma_S(s) - L(s) - w_{V_S}(s)| \le 2\tau_V(L)N_V + 2N_V\sqrt{K(L + N_V^2 + 2\tau_V(L)N_V)}\, s^{-1/4} + N_V^2 K s^{-1/2}.$$

Next, $w_V = (\arg V)'$ (from $w_V = -i\overline{V}V'$ and $|V| = 1$), so $w_{V_S} = -\sum_{p\in S}\varphi_p'$ and $\Phi_S' = 2\theta' + w_{V_S}$; and $2\theta'(s) - L(s) = \operatorname{Re}\psi(\frac{1}{4} + \frac{is}{2}) - \log\frac{s}{2} = O(s^{-2})$ by Stirling's formula for ψ. For the finite product V_S the coefficients at shift t are $O((1+t)^{|S|}e^{-t/2})$, so $\tau_V(L) = O(L^{|S|+1}e^{-L/2}) = O(s^{-1/2}\log^{|S|+1} s)$. Altogether $d_S(s) = O_S(s^{-1/4}\log^{1/2} s)$. Numerically $|d_S|$ decays like $|s|^{-1/2}$ in root mean square (§6.9), so the exponent $\frac{1}{4}$ is probably not sharp.

(c) For $a \neq 0$, $\mathbf{M}_a(2\theta') = 0$ (integrate by parts; $2\theta'$ has derivative $O(1/(1+|s|))$), and $\mathbf{M}_a(w_{V_S})$ is the Bohr coefficient of the absolutely convergent cosine series $-\sum_{p\in S} 2\ell q^m \cos(m\ell s)$, i.e. $-\ell q^m$ at $a = \pm m\ell$ and 0 otherwise (distinct primes have $\mathbb{Q}$-independent logarithms). So $\mathbf{M}_a(\Phi_S')$ exists, and by (b) $\mathbf{M}_a(\sigma_S) = \mathbf{M}_a(\Phi_S') + \mathbf{M}_a(d_S) = \mathbf{M}_a(\Phi_S')$, since a function tending to 0 has vanishing Bohr means. Wiener's formula for a finite measure ν, with $\widehat{\nu}(s) = \int e^{-isx}d\nu(x)$, reads $\nu(\{a\}) = \lim_T \frac{1}{2T}\int_{-T}^{T}\widehat{\nu}(s)e^{+ias}ds = \mathbf{M}_{-a}(\widehat{\nu})$. With our e^{-ias} convention the Bohr mean $\mathbf{M}_a$ thus picks out the atom at $-a$. This is harmless here, because σ_S, Φ_S' and d_S are even, so $\mathbf{M}_{-a} = \mathbf{M}_a$ on them. If $D_S = \beta\delta_a + \nu$ as in (c), then $d_S = \beta e^{-ias} + \widehat{\nu}$, and $0 = \mathbf{M}_{-a}(d_S) = \beta + \mathbf{M}_{-a}(\widehat{\nu}) = \beta$. □

Remark 6.5.2. Part (a) is the precise sense in which "τ sees the primes". The positive functional τ_V, which has only a single atom at 0, has the same symbol at infinity as the signed distribution $\mathcal{W}_V$. The only inputs are the autocorrelation identity $r(d) = \delta_{d,0}$, i.e. unitarity of V, and $\min(|s|,|t|) = \int 1\cdot 1$. This is the semilocal analogue of the mechanism behind [19, Prop. 2.2, Cor. 2.3]. We have not found it stated for $|S| > 1$.

6.6. The precise obstruction

Observation 6.6.1 (numerical). Let $S = \{\infty, p\}$, $p \in \{2, 3, 5\}$. For $|s|$ large (and $p^{|m|} \ll |s|$),

$$d_S(s) \;\approx\; \frac{A(p)}{2|s|}\sum_{m\in\mathbb{Z}} p^{|m|/2}\cos(m\ell s) \;+\; (\text{rest, r.m.s. } O(|s|^{-1/2})),$$

with $A(2) = 4.49$, $A(3) = 5.22$ and $A(5) = 5.41$, each to about $\pm 3\%$. The coefficients of $\cos(m\ell s)/|s|$ for $m = 0, \dots, 3$ are given in Table 6.1. They are stable over $s \in [40, 2600]$, and their ratios are $p^{1/2}$ to within 1%. By contrast, for $S = \{\infty\}$ one finds $s^2 d_\infty(s) \approx -2.1$ to -2.3, which is the "$-2\,\mathrm{Id}$" of [19, Thm. 3.6].

Corollary 6.6.2 (conditional on Obs. 6.6.1). *For $|S| \ge 2$ the following hold.*

(i) The kernel of D_S is not *a continuous function. Near each $a = m\log p$, $m \in \mathbb{Z}$, it behaves like $-\frac{A(p)}{2\pi}p^{|m|/2}\log|a - m\log p|$. This includes $a = 0$, so the singularity is present on* every *window.*

(ii) $D_S \circ Q_+$, with $Q_+ = -\partial^2 + \frac{1}{4}$, has symbol $\sim \frac{A(p)}{2}|s| \to +\infty$ (from the $m = 0$ term). It is therefore unbounded above on $C_c^\infty(I)$ for every interval I. It is not of the form $-2\,\mathrm{Id}$+compact, so the finite-conditioning strategy of [19, §§3–6] *does not transfer verbatim to τ_S.*

(iii) Relative to the atom $\ell q^m = \ell p^{-m/2}$ of $\mathcal{W}_S$ at $m\ell$, the logarithmic singularity of D_S there has relative strength $\asymp p^m/\ell$. The structural remainder grows exactly where the arithmetic information sits.

This is consistent with [16]. There $(1 - P)U_S P$ is proved to be an infinitesimal of order $1/(2m)$, which is an upper bound and says nothing about sharpness, while for u_∞ alone it is a fast-convergent series of rank-one operators. The slow $|s|^{-1/2}$ decay of d_S is the symbol-level counterpart of this loss. It suggests replacing U_S by the Gauss-multiplication factorization of [16, §4.3]. That question is open here.

6.7. Unconditional finite negative index on every window

THEOREM 6.7.1 (rigorous; elementary, likely folklore, no reference found). *Let κ_A be the negative index of $\mathcal{Q}$ on $\{F \in C_c^\infty(\mathbb{R}) : \operatorname{supp} F \subset [-A/2, A/2]\}$. Then:*

(i) $\kappa_A < \infty$ for every A;
(ii) $A \mapsto \kappa_A$ is non-decreasing, and $\sup_A \kappa_A = \kappa(\mathcal{Q}) = N_+$;
(iii) RH $\iff$ $\kappa_A = 0$ for all A.

PROOF. Put $\|F\|_*^2 = \frac{1}{2\pi}\int |\widehat{F}|^2 \log(2+|s|)\, ds$. Since $2\theta'(s) - \log(|s|/2\pi) \to 0$, there is C_0 with $2\theta' \geq \log(2+|s|) - C_0$. The prime part on the window is bounded by

$$M(A)\|F\|_2^2, \qquad M(A) = 2\sum_{n \leq e^A} \Lambda(n) n^{-1/2}$$

(this uses $|H| \leq \|F\|_2^2$). Hence $\mathcal{Q}(F,F) \geq \mathcal{P}(F) + \|F\|_*^2 - (C_0 + M(A))\|F\|_2^2$. Functions with support in a fixed compact set and bounded $\|\cdot\|_*$ have uniformly small Fourier tails, since $\int_{|s|>R} |\widehat{F}|^2 \leq 2\pi\|F\|_*^2/\log(2+R)$. By Kolmogorov–Riesz their set is relatively compact in L^2. The closed form $\|F\|_*^2 - c\|F\|_2^2$ therefore defines a self-adjoint operator with compact resolvent, which has finitely many negative eigenvalues. Adding $\mathcal{P}$, of rank ≤ 2, raises the negative index by at most 2. This proves (i). Monotonicity is clear. Every finite-dimensional negative subspace of $\mathcal{Q}$ on $C_c^\infty(\mathbb{R})$ lies in some window, which gives (ii); (iii) then follows from Lemma 1. □

REMARK 6.7.2 (where it degenerates). The bound one obtains this way is $\kappa_A \leq 2 + \#\{\text{eigenvalues} < C_0 + M(A)\}$, with $M(A) \sim 4e^{A/2}$. Since the coercive weight is only logarithmic, the eigenvalue count below M grows like e^M, which is doubly exponential in A. Norm bounds alone lose everything, so any proof of $\kappa_A = 0$ must use sign cancellation in the prime sum. That is exactly the information that Cor. 6.6.2(iii) shows τ_S does not control.

6.8. Exact failure points

(F1) *Fixed S, small window.* Positivity is known only for $S = \{\infty\}$ and windows free of prime powers [**122, 19**].

(F2) *Fixed $S \ni p$.* The structural trace exists and is positive (Thm. 6.4.2). It reproduces all prime-power atoms (Thm. 6.5.1). But its remainder D_S is logarithmically singular on $\mathbb{Z}\log p$, with $D_S \circ Q_+$ unbounded above (Cor. 6.6.2). This is the first point where the archimedean argument fails. It fails for every window once $S \ni p$, and it is *forced* once $A > \log 2$, since $S \supseteq S(A) \ni 2$.

(F3) *$S \to$ all places, or $A \to \infty$.* U_S has no limit as S grows: the Euler product of $\zeta(\frac{1}{2} - is)/\zeta(\frac{1}{2} + is)$ does not converge on the critical line. The singular strengths $A(p)p^{|m|/2}$ are not summable in any uniform sense. And $\kappa_A = 0$ for all A is equivalent to RH (Thm. 6.7.1).

6.9. Numerics

We used a Python/numpy FFT grid with $L = 60$ and $N = 2^{18}$, so the grid step is $\Delta u = 4.6 \cdot 10^{-4}$; scripts are in `num/`. The scripts are `semilocal_trace.py`, `atom_test.py`, `atom_scale2.py`, `symbol.py`, `envelope.py`, `fitlog.py`, `symbol_p3.py` and `CV.py`.

(N1) *Validation against* [**19**]. $\delta(0) = 2(1 + \mathrm{Si}(4\pi)/4\pi) = 2.2374848349$. For Gaussians $(\sigma, c) = (0.3, 0), (0.2, 0.3), (0.15, 0)$, $\tau_\infty - \mathcal{W}_\infty$ equals $\int H(a)\delta(e^{|a|})da$ to 6 digits: 1.023416, 0.515004, 0.306061.

(N2) *Symbol formula.* For a Gaussian with $\sigma = 0.2$, the symbol (Thm. 6.4.2(ii)) and the direct Hilbert–Schmidt computation agree: τ_∞ is 0.024944 versus 0.024958, and $\tau_{\{\infty,2\}}$

is 0.040652 versus 0.040650. The minima are $\min \sigma_{\{\infty,2\}} = 0.0652$ and $\min \sigma_\infty = 0.0490$, both attained at $s = 0$. In particular $\sigma_S \geq 0$, although $\Phi'_S(0) = -8.72$.

(N3) *Atoms.* We used two bumps $F = \phi_\sigma + \phi_\sigma(\cdot - c)$ and took the excess at $c = \log 2$ over $c = \log 2 \pm 8\sigma$, divided by $H(\log 2)$:

σ	$\mathcal{W}_S$	$B_S = \tau_S - \tau_\infty - \tau_{u_2}$	D_S
0.06	−0.632	−0.131	0.650
0.03	−0.959	−0.461	0.452
0.015	−0.978	−0.726	0.244

The $\mathcal{W}_S$ column tends to $-2\ell q = -0.9803$, the atom. The τ_S dip tracks it, so $D_S \to 0$ like $\sigma \log(1/\sigma)$. Read with the log model of Obs. 6.6.1, the predicted values are 0.60, 0.38 and 0.23.

(N4) *Coefficients.* Values of κ_m, the coefficient of $\cos(m\ell s)/|s|$ in d_S (Table 6.1): In root mean square, d_S decays like $|s|^{-0.48}$.

p	κ_0	κ_1	κ_2	κ_3
2	2.21	6.35	9.00	12.70
3	2.56	9.05	15.74	27.32
5	2.68	12.08	27.14	60.86

TABLE 6.1. $\kappa_m \approx A(p)p^{m/2}$ for $m \geq 1$ and $\kappa_0 \approx A(p)/2$, fitted on $s \in [40, 2600]$. The fit is unreliable when $p^m \gtrsim s$.

6.10. Known vs. new

Statement	Status
Thm. 6.3.1	elementary; implicit in [22, 19]
$\tau_S \geq 0$; kernel/symbol formula	routine extension of [19, Prop. 2.2] to $\lvert S\rvert > 1$ (new as written)
Thm. 6.5.1(a) identity	new to us (elementary)
Thm. 6.5.1(b),(c)	new; rigorous (volume proof, gap G1 closed)
Obs. 6.6.1, Cor. 6.6.2	new, numerical; consistent with [16]
Thm. 6.7.1	elementary, probably folklore
Semilocal Weil positivity	*open*; equivalent to RH as $A \to \infty$

CHAPTER 7

Lemma 7: Scattering unitarity

RH as a Lax–Phillips / Pavlov–Faddeev statement, and where the primes go

Summary. For the modular surface $PSL_2(\mathbb{Z})\backslash\mathbb{H}$ we state precisely what Lax–Phillips (1976/1980) and Pavlov–Faddeev (1972/1975) proved. The statements are checked against the downloaded original texts. We then sharpen them and connect them to the TAA scaling flow.

Classical results. The scattering matrix is $\varphi(s) = \zeta^*(2s-1)/\zeta^*(2s)$. It is unimodular on $\operatorname{Re} s = \frac{1}{2}$ without any hypothesis. Its poles in $\operatorname{Re} s < \frac{1}{2}$ are exactly the points $\rho/2$. The Lax–Phillips generator B'' has spectrum $\{(\rho-1)/2\}$. Pavlov and Faddeev proved RH $\iff \lim t^{-1}\log\|Z_t(B+i)^{-1}\| = -\frac{1}{4}$.

New here:

(i) The spectral bound is $s(B'') = -\frac{1}{4} + \frac{1}{2}(\Theta - \frac{1}{2})$. Its excess over $-\frac{1}{4}$ is exactly half the Lyapunov exponent of Lemma 1.

(ii) Unconditionally, $\|Z(t)\|$ does not decay exponentially. So the growth bound is 0, and the resolvent smoothing in the Pavlov–Faddeev criterion cannot be removed. No contraction or exponential-stability property of Z itself is equivalent to RH.

(iii) "An inner product making $2B'' + \frac{1}{2}$ skew-symmetric" is equivalent to RH only tautologically. The non-tautological candidate is the Weil form $\mathcal{Q}$ of Lemma 1.

(iv) The spectral time-dictionary: $e^{t/4}Z''(t)$ has the same eigenvalues as $T_{t/2}$.

(v) Where the primes go. On the unitary line the cusp moves Lemma 6's local multipliers from abscissa $\frac{1}{2}$ to abscissa 1. There the Euler product converges (conditionally). The scattering phase is a sum of Lorentzians centred at the resonances, which is the explicit formula at abscissa 1.

(vi) A rigorous "Newton triangle" dictionary. A Lorentz boost of rapidity u applied to the square lattice moves it along the unit semicircle through i and $e^{i\pi/3}$, at angle $\frac{\pi}{2} - \operatorname{gd}(2u)$. After reduction it runs up the cusp at height $e^{2u}/2$. So J.W.'s rapidity is the TAA log-scale.

Nothing here proves RH. The remaining obstruction is stated in §7.8.

7.1. Conventions

We write $\zeta^*(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ (no polynomial factor) and $\xi(s) = \frac{1}{2}s(s-1)\zeta^*(s)$ (Riemann's ξ). Then $\zeta^*(s) = \zeta^*(1-s)$ and $\xi(s) = \xi(1-s)$. Zeros $\rho = \frac{1}{2} + i\gamma_\rho$ are as in Lemma 1, $\beta_\rho = \operatorname{Re}\rho$, and $\Theta = \sup\operatorname{Re}\rho \in [\frac{1}{2}, 1]$. On $\mathbb{H}$, $\tau = x + iy$, $\Gamma = PSL_2(\mathbb{Z})$, and

$$E(\tau, s) = \sum_{\Gamma_\infty\backslash\Gamma} \operatorname{Im}(\gamma\tau)^s = \tfrac{1}{2}\sum_{(c,d)=1} \frac{y^s}{|c\tau + d|^{2s}} \quad (\operatorname{Re} s > 1), \qquad \int_0^1 E(x+iy, s)\,dx = y^s + \varphi(s)y^{1-s}.$$

The scaling flow is $(T_u F)(u') = F(u' - u)$ on functions of $u' = \log x$ (Lemma 1). Lemma 1's eigenfunctionals ℓ_ρ satisfy $\ell_\rho(T_u F) = e^{(\rho-\frac{1}{2})u}\ell_\rho(F)$.

7.2. The scattering matrix: exact forms

PROPOSITION 7.2.1 (normalizations; classical, verified). *(a)* $\varphi(s) = \dfrac{\sqrt{\pi}\,\Gamma(s-\frac12)\,\zeta(2s-1)}{\Gamma(s)\,\zeta(2s)} = \dfrac{\zeta^*(2s-1)}{\zeta^*(2s)} = \dfrac{s}{s-1}\cdot\dfrac{\xi(2s-1)}{\xi(2s)}$. *In particular "$\varphi = \xi(2s-1)/\xi(2s)$" is* correct *for the completed ζ^* and* wrong *by the factor $(s-1)/s$ for Riemann's ξ. This resolves the flag raised in the contact note.*

(b) (Lax–Phillips [73, (4.7), (4.7)′]*.) The zero Fourier coefficient of the Eisenstein data is $y^{1/2+iz} + s(z)y^{1/2-iz}$ with $s(z) = \Gamma(\frac12)\Gamma(iz)\zeta(2iz)/(\Gamma(\frac12+iz)\zeta(1+2iz))$. Hence $s(z) = \varphi(\frac12+iz)$, and the scattering operator is $\mathcal{S}'(\sigma) = -s(\sigma)$* [73, Thm. 6.5].

(c) (Pavlov–Faddeev [91, (43)]*.) With their $\xi_{\mathrm{PF}}(s) = s(s-1)\zeta^*(s)$ and cusp cut-off $d \geq 1$,*

$$S_d(k) = \frac{\xi_{\mathrm{PF}}(-2ik)}{\xi_{\mathrm{PF}}(2ik)}\, d^{2ik}\, \frac{k-\frac{i}{2}}{k+\frac{i}{2}} = d^{2ik}\,\varphi(\tfrac12-ik)\,b(k)^2, \qquad b(k) = \frac{k-\frac{i}{2}}{k+\frac{i}{2}}.$$

(d) (Unitarity, unconditional.) $\varphi(s)\varphi(1-s) = 1$, $|\varphi(\frac12+ir)| = 1$ for $r \in \mathbb{R}$, and $\varphi(\frac12) = -1$.

(e) (Resonances.) In $\operatorname{Re} s < \frac12$ the poles of φ are exactly the points $s = \rho/2$, with multiplicity m_ρ. In $\operatorname{Im} k > 0$ the zeros of S_d are $k_\rho = -\frac{i}{2}(\rho-1) = \frac{\operatorname{Im}\rho}{2} + \frac{i}{2}(1-\beta_\rho)$, together with one simple zero at $k = \frac{i}{2}$.

PROOF. (a) Expand ζ^*, and use $\xi(2s-1)/\xi(2s) = \frac{(2s-1)(2s-2)}{2s(2s-1)}\cdot\frac{\zeta^*(2s-1)}{\zeta^*(2s)}$. (b) Put $s = \frac12 + iz$ in (a). (c) $\xi_{\mathrm{PF}}(-2ik) = \xi_{\mathrm{PF}}(1+2ik)$. With $\sigma' = \frac12 - ik$, (a) gives $\varphi(\sigma') = \frac{k+i/2}{k-i/2}\cdot\frac{\xi_{\mathrm{PF}}(-2ik)}{\xi_{\mathrm{PF}}(2ik)}$. (d) The functional equation gives the product identity. For real r, $\varphi(\frac12+ir) = \zeta^*(2ir)/\zeta^*(1+2ir) = \zeta^*(1-2ir)/\zeta^*(1+2ir)$, a ratio of conjugates. At $s = \frac12$, $\zeta^*(2s-1)$ and $\zeta^*(2s)$ both have simple poles, and their residues give -1. (e) $\zeta^*(2s)$ vanishes in $\operatorname{Re} s < \frac12$ exactly at $2s = \rho$. A cancelling zero of $\zeta^*(2s-1)$ would need $\rho = \rho' + 1$, which is impossible since $0 < \operatorname{Re}\rho, \operatorname{Re}\rho' < 1$. The pole of $\zeta^*(2s-1)$ at $s = 1$ lies in $\operatorname{Re} s > \frac12$. For S_d, use (c): the zeros of $\varphi(\frac12 - ik)$ in $\operatorname{Im} k > 0$ come from $\zeta^*(2\sigma'-1) = 0$, i.e. $2\sigma' - 1 = 1-\rho$, i.e. $k = \frac{i}{2}(1-\rho)$. The double zero of b^2 at $\frac{i}{2}$ is cancelled once by the pole of φ at $\sigma' = 1$. □

All identities were checked to 10^{-30} (`num/lemma7_check.out`, §§1–4). The check includes $|1/\varphi(\rho/2)| < 10^{-29}$ for the first five zeros, and $\max|S_1| = 0.990 < 1$ on 42 points of the upper half-plane.

7.3. The Lax–Phillips and Pavlov–Faddeev theorems (classical)

THEOREM 7.3.1 (Lax–Phillips [72, 73]). *Let $L = y^2\Delta + \frac14$, and let $U(t)$ be the group solving $u_{tt} = Lu$ on automorphic Cauchy data. The energy form is E, which is indefinite in general, and U is completed in the equivalent positive form $G = E + 2K$* [73, §2]. *Let $\mathcal{D}_-$ and $\mathcal{D}_+$ be the closures of the data of x-independent incoming and outgoing solutions supported in $y > a$* [73, (2.13)]. *Put $\mathcal{K} = \mathcal{H}_G \ominus (\mathcal{D}_- \oplus \mathcal{D}_+)$ and $Z(t) = P_+U(t)P_-$.*

(i) $Z(t)$ is a strongly continuous semigroup on $\mathcal{K}$ that annihilates $\mathcal{D}_\pm$. The resolvent of its generator B is compact, so B has pure point spectrum [73, Thm. 3.1, Thm. 3.2, Cor. 3.3].

(ii) The poles of $s(z)$ lie in $-i\sigma(B)$ [73, Cor. 4.3]. *For the modular group these poles are the nontrivial zeros of $\zeta(1+2iz)$, and under RH they lie on $\operatorname{Im} z = \frac14$* [73, §6].

(iii) Let Z'' be built in the same way from $\mathcal{D}''_\pm = \mathcal{D}_\pm \cap \mathcal{K}'_E$, on the complement $\mathcal{K}''$ of the standing waves. Its generator satisfies $\sigma(B'') = \{(\rho-1)/2\}$, so $2B'' + \frac12$ has spectrum $\{\rho - \frac12\}$. In their words this is "the kind of operator … envisaged by Hilbert; all that is missing is an appropriate inner product" [73, (6.20)].

(iv) RH holds if $\limsup_{t\to\infty} t^{-1}\log\|Z''(t)f\|_E \le -\frac14$ *for a dense set of* f [**73**, (6.19)], *citing* [**72**]. *See also* [**72**], *appendix to §7, "How not to prove the Riemann hypothesis".*

THEOREM 7.3.2 (Pavlov–Faddeev [**91**]). *Let* $\mathcal{H}^a$ *be the absolutely continuous subspace,* $\mathcal{K} = \mathcal{H}^a \ominus (\mathcal{D}_- \oplus \mathcal{D}_+)$, *and* $Z_t = P_{\mathcal{K}} V_t P_{\mathcal{K}}$ [**91**, (29)].

(i) Z_t *is a semigroup of completely non-unitary contractions of class* C_0 *with rank of non-unitarity one. Its characteristic function is the scattering matrix* S_d, *and its eigenvalues are* $e^{ik_s t}$ *over the zeros* k_s *of* S_d *in* $\operatorname{Im} k > 0$ [**91**, §3].

(ii) $S_d(k) = S_1(k)\, d^{2ik}$, *and* S_1 *is a Blaschke product* [**91**, Lemma 7]. *Under RH,* S_1 *continues to* $-\frac14 < \operatorname{Im} k \le 0$ *with polynomial bounds* [**91**, Lemma 8, (46)].

(iii) (Lax–Phillips, as quoted in [**91**, (30)].*) Suppose* $\|Z_t\| \le e^{-\delta t}$. *Then* S *continues as a* bounded *analytic function to every strip* $-\delta' < \operatorname{Im} k < 0$ *with* $\delta' < \delta$. *The converse also holds. PF note that for the modular group* S *is not bounded in such strips, and they replace this criterion by their Theorem 1.*

(iv) (Corollary [**91**, (48)].*)* RH $\iff \lim_{t\to\infty} t^{-1}\ln\|Z_t(B+i)^{-1}\| = -\frac14$.

Theorems 7.3.1 and 7.3.2 were read in the originals: [**73**] on pp. 270, 283, and [**91**] on pp. 176, 185–188 (Russian text). In particular (6.19) reads "$\le -1/4$".

7.4. What RH is equivalent to in this framework

THEOREM 7.4.1 (Lemma 7). *(i)* (Spectral bound; elementary from Thm. 7.3.1(iii).) $s(B'') := \sup \operatorname{Re}\sigma(B'') = \frac{\Theta-1}{2} = -\frac14 + \frac12(\Theta - \frac12)$. *Hence*

$$\text{RH} \iff s(B'') = -\tfrac14 \iff s(B'') \le -\tfrac14 \iff \sigma(B'') \subset \{\operatorname{Re}\mu = -\tfrac14\}.$$

Moreover $s(B'') + \frac14 = \frac12\lambda^*$, *where* $\lambda^* = \Theta - \frac12$ *is the maximal Lyapunov exponent of Lemma 1(iii).*

(ii) (The smoothing is necessary; new, unconditional.) *In fact* $\|Z_t\| = 1$ *for every* $t \ge 0$; *in particular for no* $\delta > 0$ *is* $\|Z_t\| \le Ce^{-\delta t}$. *Thus the growth bound is* $\omega_0(Z) = 0$ *and* $\|Z_t\| \not\to 0$ *exponentially. If* $\Theta < 1$, *which holds in particular under RH, then* $s(B) < \omega_0(Z)$, *and* Z *is not eventually norm-continuous.*

(iii) (The RH-equivalence is about smoothed rates, not contractivity.) *Contractivity of* Z, *the rank-one defect, and inner-ness of* S *hold unconditionally. Exponential stability of* Z *fails unconditionally. Hence none of these properties is equivalent to RH. The sharp RH-equivalent statements are (i) and Thm. 7.3.2(iv).*

(iv) (Hilbert's "missing inner product" is a tautology; elementary.) *There is an inner product on the span of the eigenvectors of* B'' *for which* $2B'' + \frac12$ *is skew-symmetric if and only if RH holds.*

PROOF. (i) By Thm. 7.3.1(iii), $\operatorname{Re}\frac{\rho-1}{2} = \frac{\beta_\rho - 1}{2}$, so the supremum is $\frac{\Theta-1}{2}$. Since $\rho \mapsto 1-\bar\rho$ permutes the zeros, $\Theta \ge \frac12$, with equality iff RH. Then $\operatorname{Re}\mu \le -\frac14$ for all μ forces every $\beta_\rho \le \frac12$, hence $= \frac12$ by symmetry. (ii) *Proof that* $\|Z_t\| = 1$ *for every* $t \ge 0$ *(volume proof; no Lax–Phillips criterion is used).* By Thm. 7.3.2(i) and the Sz.-Nagy–Foias functional model [**109**, Ch. VI; for semigroups via the cogenerator, Ch. III, §§8–9], Z_t is unitarily equivalent to the compressed shift $f \mapsto P_K(e^{ikt}f)$ on the model space $K = H^2_+ \ominus S_d H^2_+$ of the upper half-plane. Write $f(k) = \int_0^\infty a(s)e^{iks}\,ds$ (Paley–Wiener), so that $\|f\|^2 = 2\pi\int_0^\infty |a|^2$. Since $e^{ikt}S_dH^2_+ \subset S_dH^2_+$, the space K is invariant under the backward shift $S_t^* : a(\cdot) \mapsto a(\cdot + t)|_{[0,\infty)}$. Hence $Z_t^* = S_t^*|_K$ and, for $0 \ne f \in K$,

$$(*) \qquad \|Z_t\|^2 = \|Z_t^*\|^2 \;\ge\; \frac{\int_t^\infty |a(s)|^2\,ds}{\int_0^\infty |a(s)|^2\,ds}.$$

For each zero k of S_d in $\operatorname{Im} k > 0$, the Szegő kernel $(k' - \bar{k})^{-1}$ lies in K, and its profile is $a(s) = \text{const} \cdot e^{-i\bar{k}s}$. The on-line zeros $\rho = \frac{1}{2} + i\gamma$ give $k = \frac{i}{2}(1-\rho) = \frac{\gamma}{2} + \frac{i}{4}$, hence the profiles $e^{-s/4}e^{-i\gamma s/2}$. So for every finite set F of on-line ordinates, every

$$a(s) = e^{-s/4}p(s), \qquad p \in \mathcal{T}_F := \operatorname{span}\{e^{-i\gamma s/2} : \gamma \in F\},$$

is the profile of an element of K.

Completeness. Fix $M > 0$. The functions $e^{-i\gamma s/2}$, γ running over all on-line ordinates, are complete in $L^2(0, M)$. Indeed, let $\varphi \in L^2(0, M)$ be orthogonal to all of them and put $\Phi(z) = \int_0^M \varphi(s)e^{izs}ds$. Then Φ is entire, $|\Phi(z)| \le \sqrt{M}\|\varphi\|e^{M|z|}$, and $\Phi(\gamma/2) = 0$ for every on-line γ. If $\Phi \not\equiv 0$, divide by a power of z so that $\Phi(0) \neq 0$. Jensen's formula then gives at most $eMr + O(1)$ zeros in $|z| \le r$. But the number of on-line ordinates $0 < \gamma \le 2r$ is at least $(\frac{2}{5} + o(1))N(2r) \gg r \log r$ [**23**], a contradiction. So $\Phi \equiv 0$, and $\varphi = 0$ by Fourier uniqueness.

Conclusion. Fix $t > 0$ and $\eta \in (0, \frac{1}{2})$. Put $\varphi = e^{s/4}\mathbf{1}_{[t,t+1]} \in L^2(0, t+1)$. By completeness there are a finite F and $p \in \mathcal{T}_F$ with $\|p - \varphi\|_{L^2(0,t+1)} \le \eta$. Then $a = e^{-s/4}p$ satisfies $\int_0^t |a|^2 \le \int_0^t |p|^2 \le \eta^2$. Also $\big(\int_t^{t+1} |a|^2\big)^{1/2} \ge \|e^{-s/4}\varphi\|_{L^2(t,t+1)} - \eta = 1 - \eta$, because $e^{-s/4} \le 1$. Inserting this into $(*)$ gives

$$\|Z_t\|^2 \;\ge\; \frac{(1-\eta)^2}{(1-\eta)^2 + \eta^2} \;\xrightarrow[\eta\to 0]{}\; 1.$$

Since Z_t is a contraction, $\|Z_t\| = 1$ for every $t \ge 0$. In particular no bound $\|Z_t\| \le Ce^{-\delta t}$ with $\delta > 0$ holds, and $\omega_0(Z) = 0$. The only inputs are the functional model, the Paley–Wiener theorem, Jensen's formula and the superlinear count of on-line zeros [**23**]. (Selberg's $N_0(T) \gg T \log T$ suffices; Hardy–Littlewood's $N_0(T) \gg T$ does not.) S_d enters only through its zeros, so the argument covers every $d \ge 1$.

Remark (the earlier route). An earlier version of this proof argued via unboundedness of S_1 on lines $\operatorname{Im} k = -\kappa$: by convexity and reflection, $|S_1(\lambda - i\kappa)| \gg |\lambda|^{\kappa-\varepsilon}$. It then invoked the Lax–Phillips criterion quoted in Thm. 7.3.2(iii) with a constant $C \ge 1$, whose exact formulation in [**75**] we have not checked. The published announcement [**74**, Thms. 4, 5] (read in the original) states continuation into a strip when $\|Z(T)\| < 1$ (Thm. 5). It also characterizes the resolvent set of $Z(T)$ by uniform invertibility of S on the lattices $\{\mu : e^{i\mu T} = \lambda\}$ (Thm. 4(b)). Continuation alone gives no contradiction here, since S_d is meromorphic. The direct proof above makes the criterion unnecessary (gap G3 closed).

By Thm. 7.3.2(i) the eigenvalues $e^{ik_s t}$ have modulus $e^{-t\operatorname{Im} k_s}$ with $\operatorname{Im} k_s \ge \frac{1-\Theta}{2}$, plus $\frac{1}{2}$ at $k = \frac{i}{2}$. So $s(B) = -\frac{1-\Theta}{2} < 0 = \omega_0$ when $\Theta < 1$. For eventually norm-continuous semigroups $s = \omega_0$ [**37**, Ch. IV, §3]. (iii) This summarizes Thm. 7.3.2(i) and part (ii). (iv) ($\Leftarrow$) Under RH every eigenvalue ν of $2B'' + \frac{1}{2}$ is purely imaginary. Eigenvectors for distinct eigenvalues are linearly independent, so choose a basis of each eigenspace and declare the union orthonormal. A diagonal operator with imaginary entries is skew-symmetric. ($\Rightarrow$) If $Av = \nu v$ with A skew-symmetric, then $\nu\|v\|^2 = \langle Av, v\rangle = -\langle v, Av\rangle = -\bar{\nu}\|v\|^2$, so $\operatorname{Re}\nu = 0$, and (i) gives RH. □

Remark 7.4.2 (what (iv) means). Part (iv) shows that "some inner product" carries no information: it can always be chosen after the fact. The content of a Hilbert–Pólya statement lies in an inner product defined *without the zeros*. TAA already has exactly one such object, the Weil form $\mathcal{Q}$ of Lemma 1. It is built from the primes, the pole and the archimedean place, and $\mathcal{Q} \ge 0 \iff$ RH. This identifies the precise question: does $\mathcal{Q}$ transport to $\mathcal{K}''$ (Question 7.9.1)?

7.5. Connection to TAA

Proposition 7.5.1 (time dictionary; rigorous at the level of spectra). *$\sigma(2B'' + \frac{1}{2}) = \{\rho - \frac{1}{2}\}$ coincides with the set of exponents of Lemma 1's eigenfunctionals: $\ell_\rho(T_u F) = e^{(\rho-\frac{1}{2})u}\ell_\rho(F)$.*

Equivalently, $e^{t/4}Z''(t)$ and $T_{t/2}$ have the same eigenvalues $e^{(\rho-\frac12)t/2}$. Thus Lax–Phillips time is $t = 2u$ and carries the damping $e^{-t/4}$. The factor 2 is the $2s$ in $\zeta(2s)$, which comes from the quadratic norm $|c\tau+d|^{2s}$. Changing PF's cut-off parameter d (the height of the cusp region carrying $\mathcal{D}_\pm$) multiplies S by d^{2ik} (Thm. 7.3.2(ii)). This is a round-trip delay $2\log d$ in Lax–Phillips time, i.e. the scaling $T_{\log d}$.

PROPOSITION 7.5.2 (where the primes go). *(a) (Abscissa shift.) For $r \in \mathbb{R}$, $\varphi(\frac12+ir) = \prod_v \overline{u_v^{(1)}(2r)}$ with $u_v^{(\sigma)}(s) := L_v(\sigma+is)/L_v(\sigma-is)$. Each factor is unimodular. Lemma 6 uses $u_v = u_v^{(1/2)}$. The cusp evaluates the same local factors at abscissa 1 and frequency $2r$.*

(b) (Convergence.) For $r \neq 0$ the Euler product in (a) converges (conditionally), so the semilocal truncations $\varphi_S \to \varphi$ pointwise. By contrast, $U_S = \prod_{v\in S} u_v^{(1/2)}$ has no limit (Lemma 6, (F3)).

(c) (Scattering phase; unconditional.) For $r \in \mathbb{R}$,

$$-\frac{\varphi'}{\varphi}(\tfrac12+ir) = -2\log\pi + 2\operatorname{Re}\psi(\tfrac12+ir) + 4\operatorname{Re}\frac{\zeta'}{\zeta}(1+2ir) = \sum_\rho \frac{1-\beta_\rho}{(r-\frac{\operatorname{Im}\rho}{2})^2+\frac{(1-\beta_\rho)^2}{4}} - \frac{1}{r^2+\frac14}.$$

Every term of the sum is a positive Lorentzian; this is PF's Blaschke structure. RH $\iff$ all Lorentzians have the common half-width $\frac14$.[1]

PROOF. (a) $\varphi(\frac12+ir) = \zeta^*(1-2ir)/\zeta^*(1+2ir)$, and $\zeta^* = \prod_v L_v$ in $\operatorname{Re} s > 1$. On $\operatorname{Re} s = 1$ we use (b). (b) Write $\sum_{p\le X} p^{-1-it} = \int_2^X x^{-1-it}d\pi(x)$ with $\pi = \mathrm{li}+E$ and $E(x) \ll x/\log^2 x$ (de la Vallée Poussin [**116**]). The li-part is $\int^{\log X} e^{-itv}v^{-1}dv + O(1)$, which converges for $t \neq 0$. The E-part converges absolutely after integration by parts. The terms $\sum_p\sum_{m\ge2}$ converge absolutely. So the prime Dirichlet series converges at $1+it$, and by Abel's theorem for Dirichlet series it converges uniformly on $\{1+\epsilon+it : \epsilon\in[0,1]\}$. Its limit is therefore $\log\zeta(1+it)$ by continuity. The archimedean factor is finite. (c) The first equality comes from $(\zeta^*)'/\zeta^* = -\frac12\log\pi + \frac12\psi(\frac{s}{2}) + \zeta'/\zeta$ together with $-\frac{\varphi'}{\varphi}(\frac12+ir) = 2\frac{(\zeta^*)'}{\zeta^*}(1+2ir) + 2\frac{(\zeta^*)'}{\zeta^*}(1-2ir)$, which uses the functional equation. For the second, Hadamard's product gives $\operatorname{Re}\frac{(\zeta^*)'}{\zeta^*}(s) = \sum_\rho \operatorname{Re}\frac{1}{s-\rho} - \operatorname{Re}\frac1s - \operatorname{Re}\frac{1}{s-1}$, using the classical fact $\operatorname{Re} B = -\sum \operatorname{Re}\rho^{-1}$. Evaluate at $s = 1+2ir$. □

Numerics (`num/lemma7_check.out`): (c) holds to 12 digits in the ψ-form. With 300 zeros plus a tail estimate the Hadamard form agrees to 10^{-5}. The derivative of $\arg S_1$ matches the Blaschke sum to 10^{-5} with *no* constant term, which confirms PF Lemma 7 (no singular inner factor at $d = 1$). For (b) at $r = 3$ the error $|\mathrm{arch}\cdot\prod_{p<X} - \varphi|$ is 0.040, 0.029, 0.0088 at $X = 10^3, 10^5, 10^6$. The abscissa-$\frac12$ product wanders: arg = 1.49, −0.60, −2.37.

REMARK 7.5.3 (honest answer to "does scattering avoid Lemma 6's obstruction?"). *Partly.* Lax–Phillips never truncate the Euler product, so the semilocal artefact of Lemma 6 does not arise: the logarithmic singularities of D_S at $\mathbb{Z}\log p$ came from $|S| < \infty$. The boundary values $\varphi(\frac12+ir)$ are a convergent Euler product, which is information at the level of the prime number theorem. But the primes are *packaged, not removed.* Formally $4\operatorname{Re}\frac{\zeta'}{\zeta}(1+2ir) = -4\sum_n \frac{\Lambda(n)}{n}\cos(2r\log n)$, with atoms at Lax–Phillips times $t = \pm2\log n$ and weights $\frac{2\Lambda(n)}{n} = 2\frac{\Lambda(n)}{\sqrt{n}}e^{-|t|/4}$. That is the Weil distribution of Lemma 1 in time $u = t/2$, damped by $e^{-|t|/4}$. (This is formal: on $\operatorname{Re} s = 1$ the series is not absolutely convergent, and the pole at $r = 0$ contributes a δ-term that we do not

[1] **[Vol. 1 editorial note]** Volume edit: in Prop. 7.2.1(e) and here γ_ρ of the standalone note has been replaced by $\operatorname{Im}\rho$; with the complex $\gamma_\rho = -i(\rho-\frac12)$ of Chapter 1 the original formulas hold only for zeros on the line.

track.) The RH content has moved *entirely* into the continuation to $0 < \operatorname{Im} k < \frac{1}{4}$, where no Euler product is available. Thm. 7.4.1(ii) shows that no norm estimate on Z alone can reach it.

REMARK 7.5.4 (absorption sign and temperedness). The zeros enter as zeros of the inner function S in $\operatorname{Im} k > 0$. Equivalently they are poles of $\mathcal{S}'$ in the non-physical half-plane, i.e. *decay* rates of Z. This is Connes' absorption picture [**22**], with the correct sign and unconditionally: every zero ρ of ζ appears. The unitary group $U(t)$ on the a.c. part is unitary for free. RH concerns the dissipative compression. *Analogy, labelled as such:* an off-line zero with $\beta > \frac{1}{2}$ gives a *less* damped resonance, with $\operatorname{Re} = -\frac{1-\beta}{2} > -\frac{1}{4}$. This mirrors complementary-series parameters lying closer to the unitary axis than tempered ones (contact note, Prop. 6.1(c)): RH plays the role of temperedness, not of unitarity.

7.6. Newton triangle dictionary

Lattices $\mathbb{Z}v_1 + \mathbb{Z}v_2 \subset \mathbb{R}^2 = \mathbb{C}$ modulo rotation and scaling correspond to $\Gamma\backslash\mathbb{H}$ via $\tau = v_2/v_1$ (oriented basis). Let $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Let $B_u = \left(\begin{smallmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{smallmatrix}\right)$, the Lorentz boost of rapidity u preserving $L^2 - T^2$, and let $\operatorname{gd}(u) = 2 \arctan \tanh \frac{u}{2}$ (so $\sec \operatorname{gd} u = \cosh u$ and $\tan \operatorname{gd} u = \sinh u$).

PROPOSITION 7.6.1 (boosted square lattice; new as stated, elementary). *The lattice $B_u\mathbb{Z}^2$ has covolume* 1 *and is* rhombic*: $|B_u e_1| = |B_u e_2|$. Its modulus is $\tau_u = \tanh 2u + i \operatorname{sech} 2u = e^{i\theta_u}$, with angle $\theta_u = \frac{\pi}{2} - \operatorname{gd}(2u)$. Hence $u = 0$ gives the square lattice ($\tau = i$, 90°), and $u = \frac{1}{4}\log 3$ gives the hexagonal lattice ($\tau = e^{i\pi/3}$, 60°). For $u \geq \frac{1}{4}\log 3$,*

$$\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)\tau_u = \tfrac{1}{2} + \tfrac{i}{2}\, e^{2u} \in \textit{standard fundamental domain.}$$

So the boost runs along the unit semicircle (the axis of B_u on $\mathbb{H}$, from -1 to 1) and then straight up the cusp, with $\log y = 2u - \log 2$. *In Lax–Phillips time $t = \log y$ and TAA time $t/2$ (Prop. 7.5.1), the rapidity* is *the TAA log-scale, up to an additive constant. The Eisenstein constant term along this path is $y^{1/2}\big(e^{iz(2u-\log 2)} + \varphi(\frac{1}{2} + iz)e^{-iz(2u-\log 2)}\big)$: incoming and outgoing plane waves in the rapidity.*

PROOF. $B_u e_1 = (\cosh u, \sinh u)$ and $B_u e_2 = (\sinh u, \cosh u)$ have equal length, and $\det B_u = 1$. As complex numbers, $\tau_u = \frac{\sinh u + i\cosh u}{\cosh u + i \sinh u} = \frac{\sinh 2u + i}{\cosh 2u}$. Then $\cos\theta_u = \tanh 2u = \sin \operatorname{gd}(2u)$. Next, $\tau \mapsto 1/(1-\tau)$ sends $e^{i\theta}$ to $\frac{1}{2} + \frac{i}{2}\cot\frac{\theta}{2}$, and $\cot\frac{\theta_u}{2} = \frac{1+\tanh 2u}{\operatorname{sech} 2u} = e^{2u}$. The point lies in $\{|\operatorname{Re}\tau| \leq \frac{1}{2}, |\tau| \geq 1\}$ iff $e^{2u}/2 \geq \sqrt{3}/2$. Finally, $\tanh(\frac{1}{2}\log 3) = \frac{1}{2}$. □

J.W. heuristic	Precise statement	Status
unimodular lattice, side locked to 1	$\tau = v_2/v_1 \in \mathbb{H}$ modulo Γ	theorem
shear parallel to a side	T, and $n(x) : \tau \mapsto \tau + x$ (simple shear = horocycle flow). It fixes the zero Fourier mode, which is the only channel of $\mathcal{D}_\pm$	theorem [**73**, (2.13)]
"shear" that degenerates the cell	pure shear = boost B_u = geodesic flow into the cusp (Prop. 7.6.1)	theorem
shear to 90° = cusp	*corrected*: 90° is $\tau = i$ (elliptic, order 2). The cusp is the collapse $\theta \to 0$ or π, i.e. shortest vector $\to 0$ (Mahler)	theorem
90° rotation, equal and opposite triangle	$S\tau = -1/\tau$, $S^2 = -I$; $-I$ is $v \mapsto -v$, trivial on $\mathbb{H}$	theorem
special 60°/90°	$\mathrm{Stab}(e^{i\pi/3}) = \langle TS \rangle$ with $(TS)^3 = (ST)^3 = -I$; $\mathrm{Stab}(i) = \langle S \rangle$	theorem
χ_{-3}, χ_4 at those points	$E(i,s) = \frac{2\zeta(s)L(s,\chi_{-4})}{\zeta(2s)}$ and $E(e^{i\pi/3},s) = \frac{3(\sqrt{3}/2)^s\zeta(s)L(s,\chi_{-3})}{\zeta(2s)}$ (Epstein). So $E(i,\rho) = 0$ whenever $\zeta(2\rho) \neq 0$, in particular under RH	theorem (classical; checked at $s = 3$)
half-angle, slope $t = \tan\theta$	$(\cos 2\theta, \sin 2\theta) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$; $t = \frac{n}{m}$ gives $(m+in)^2 = (m^2-n^2)+2mn\,i$	theorem
half-angle doubling vs. $\zeta(2s)$	both come from the quadratic norm $N(z) = \lvert z\rvert^2$	same origin; further link is analogy
$\sec\theta = \cosh u$, $\tan\theta = \sinh u$; $\theta \to 90° \iff u \to \infty$	Gudermannian; rapidity = TAA log-scale (Prop. 7.6.1)	theorem
past 90°: sec < 0, branch $L < 0$	$(\sec\theta, \tan\theta)$ on the branch $L < 0$; $-B_u$ acts on $\mathbb{H}$ as B_u (center $-I$)	theorem

Does the shear/cusp picture add anything rigorous to the scattering result? It adds two things. (1) Simple shears are a *symmetry*: averaging over them projects onto the zero Fourier mode. That is why $\mathcal{D}_\pm$ form a single channel, so the characteristic function is scalar and the defect has rank one (Thm. 7.3.2(i)). (2) Pure shears, i.e. boosts, *are* the dynamics. The boosted square lattice traces the geodesic into the cusp, with $\log y = 2u - \log 2$. Along this path the Eisenstein constant term $y^s + \varphi(s)y^{1-s}$ is literally a pair of incoming and outgoing plane waves in the rapidity. Nothing here bears on the location of the resonances. Checked in `num/lemma7_newton.out`.

7.7. Known vs. new

Statement	Status
Prop. 7.2.1 (a)–(e)	classical [**73, 91**]; the three-way normalization dictionary is checked here
Thm. 7.3.1, Thm. 7.3.2	classical, quoted from the originals
Thm. 7.4.1(i)	elementary from [**73**]; link to Lemma 1 is new
Thm. 7.4.1(ii)	new; rigorous (volume proof of $\|Z_t\| = 1$ via completeness of on-line exponentials; gap G3 closed, no LP67 criterion needed)
Thm. 7.4.1(iv)	elementary; makes LP's remark precise
Prop. 7.5.1, 7.5.2	(a), (c) classical in substance (Maass–Selberg / Hadamard); the TAA dictionary is new; (b) classical (PNT)
Rem. 7.5.3	new interpretation; the atom statement is formal
Prop. 7.6.1, table	elementary; new as a dictionary

7.8. The precise remaining obstruction

Every unconditional structure in the scattering picture is RH-neutral: unitarity of U and of φ on the line, contractivity and the rank-one defect of Z, inner-ness and the Blaschke property of S, and positivity of the scattering phase. The one norm property that could have forced decay, $\omega_0 < 0$, is false unconditionally (Thm. 7.4.1(ii)). What remains is to show, *without the zeros*, that every resonance of the cusp has width $\frac{1}{4}$. Equivalently, RH needs either

(a) PF's smoothed rate: $\|Z_t(B+i)^{-1}\| \le C_\varepsilon e^{(-\frac{1}{4}+\varepsilon)t}$, whose known proof of necessity *uses* RH via Lindelöf-type bounds [**91**, Lemma 8]; or

(b) an inner product on $\mathcal{K}''$ (or on a dense Z''-invariant subspace) defined independently of the zeros, for which $e^{t/4}Z''(t)$ is isometric on root vectors. By Prop. 7.5.1 and Lemma 1, the only such candidate in hand is the Weil form, and its positivity is RH.

The scattering formulation therefore *relocates* Lemma 6's obstruction. It does not dissolve it. The finite-S singularities are gone, and in their place is the analytic continuation of a conditionally convergent Euler product into the strip.

7.9. Open questions

Question 7.9.1 (intertwiner). Is there a densely defined map $J : \mathcal{K}'' \supset \mathcal{D}_0 \to \mathcal{H}_{\mathcal{Q}}$, natural (for instance via the Mellin transform in $\log y$ of the zero Fourier mode, together with Prop. 7.5.2(a)), such that $J\,e^{t/4}Z''(t) = T_{t/2}J$? Then LP's missing inner product would *be* the Weil form. Sharp test: PF give explicit eigenvectors $\psi_s(k) = S(k)/(k-k_s)$ [**91**, §3]. Is the Weil pairing, transported by J, diagonal on them? This can be checked numerically with Lemma 1's code.

Question 7.9.2 (Lemma 6 at abscissa 1). Redo Lemma 6's structural trace with $u_v^{(1)}(2\,\cdot)$, i.e. with the semilocal scattering matrices φ_S, in place of $u_v^{(1/2)}$. Does the remainder lose its logarithmic singularities, since the product now converges? If so, what is the exact price in the $e^{-t/4}$ damping? This is a well-posed computation with the existing `num/` code.

CHAPTER 8

Lemma 8: A zero-free intertwiner from the Lax–Phillips core to the Weil space

Cusp height equals prime cutoff

Summary. This note answers Lemma 7's Question 1. We construct, *without using the zeros*, a natural map J from the Pavlov–Faddeev model of the Lax–Phillips space $\mathcal{K}''$ to Lemma 1's test functions. It satisfies $\mathsf{J}\,e^{t/4}Z_t = T_{t/2}\mathsf{J}$ modulo $\mathcal{Q}$-null vectors. The map is a shift of the spectral parameter by $\frac{i}{4}$ followed by a dilation by 2. In the time variable $t = \log y$ it reads $F(u) = 2e^{-u/2}a(2u)$, which is the Mellin transform of the cusp profile at abscissa $-\frac{1}{4}$.

We then compute the pulled-back Weil form $\mathcal{Q}^{\mathsf{J}}$ and find the following.

(i) On the Pavlov–Faddeev root vectors $\psi_s = S/(k-k_s)$, $\mathcal{Q}^{\mathsf{J}}$ is *Krein-diagonal* without any hypothesis. Its entries are $S'(k_r)\overline{S'(k_s)}\,\delta_{s,r^*}$. Under RH it is diagonal and positive. The Lax–Phillips Gram matrix, by contrast, is the non-diagonal Cauchy matrix $2\pi i/(k_r - \bar{k}_s)$. The diagonality is forced by invariance and carries no extra information.

(ii) "$\mathcal{Q}^{\mathsf{J}} = \mathrm{LP} + \text{positive}$" is false in both directions. No invariant form is equivalent to the LP norm (unconditional). $\mathcal{Q}^{\mathsf{J}}(\psi_s)/\|\psi_s\|^2_{\mathrm{LP}} \to 0$ along the simple on-line zeros. $\sup_s \mathcal{Q}^{\mathsf{J}}(\varkappa_s)/\|\varkappa_s\|^2_{\mathrm{LP}} = \infty$ under RH, and numerically the growth is $\sim \frac{1}{2\pi}\log\frac{\gamma_s}{2\pi}$.

(iii) *Cusp height = prime cutoff.* The cusp-truncated subspace $K_{d^{2ik}}$ is mapped onto test functions supported in a window of length $\log d$. So $\mathcal{Q}^{\mathsf{J}}$ on it is *exactly* the semilocal Weil form for $S = \{\infty\} \cup \{p \le d\}$, and RH $\iff$ $\mathcal{Q}^{\mathsf{J}} \ge 0$ on $\bigcup_d K_{d^{2ik}}$.

So Question 1 does not dead-end: the intertwiner exists and is canonical. It does, however, reduce *exactly* to Lemma 6's windowed positivity problem. The sharpest true comparison with the LP norm is a Gårding-type inequality whose constant grows like $\sqrt{d}$ (Prop. 8.5.2). Nothing here proves RH.

8.1. Conventions

We use the notation of Lemmas 1, 6 and 7. Zeros are $\rho = \frac{1}{2} + i\gamma_\rho$ with multiplicity m_ρ, and $\rho^* := 1 - \bar{\rho}$, so that $\gamma_{\rho^*} = \bar{\gamma}_\rho$. We write $\widetilde{F}(z) = \int F(u)e^{izu}\,du$ and $\mathcal{Q}(F,G) = \sum_\rho m_\rho \widetilde{F}(\gamma_\rho)\overline{\widetilde{G}(\bar{\gamma}_\rho)}$ (Lemma 1), with $(T_u F)(u') = F(u'-u)$. The Pavlov–Faddeev scattering matrix is $S_d(k) = d^{2ik}\varphi(\frac{1}{2} - ik)b(k)^2$, $d \ge 1$. It is inner in $\mathrm{Im}\,k > 0$, with zeros $k_\rho = \frac{i}{2}(1-\rho) = \frac{\gamma_\rho}{2} + \frac{i}{4}$ (an identity in γ_ρ; $\mathrm{Im}\,k_\rho = \frac{1-\beta_\rho}{2}$) and $k = \frac{i}{2}$ (Lemma 7, Prop. 2.1(e)).

Model. H^2 is the Hardy space of $\mathrm{Im}\,k > 0$ with norm $\int_{\mathbb{R}} |f(k)|^2 dk$. We identify $f \in H^2$ with $a \in L^2(0,\infty)$ by $f(k) = \int_0^\infty a(t)e^{ikt}\,dt$, so $\|f\|^2 = 2\pi\|a\|^2$. For an inner function θ put $K_\theta = H^2 \ominus \theta H^2$. By Pavlov–Faddeev [**91**, §3] (Lemma 7, Thm. 3.2(i)), the LP semigroup on $\mathcal{K}''$ is unitarily equivalent to the compressed shift $Z_t f = P_{K_S}(e^{ikt}f)$ on K_S, $S = S_d$. We write $\|f\|_{\mathrm{LP}} = \|f\|_{H^2}$. Szegő kernels are $\varkappa_w(k) = 1/(k - \bar{w})$, with $\langle f, \varkappa_w\rangle = 2\pi i f(w)$ for $f \in H^2$ and $\mathrm{Im}\,w > 0$; the symbol $\varkappa$ is distinct from the negative index κ. The root vectors are $\psi_s = S/(k-k_s)$.

Lemma 8.1.1 (standard model-space facts; elementary). *Let $S(k_s) = 0$ with k_s simple.*

(a) $\psi_s, \varkappa_s := \varkappa_{k_s} \in K_S$ and $Z_t\psi_s = e^{ik_s t}\psi_s$.

(b) $\langle \psi_r, \varkappa_s \rangle = 2\pi i\, S'(k_s)\, \delta_{rs}$ *(biorthogonality).*
(c) $\langle \psi_r, \psi_s \rangle = 2\pi i/(k_r - \bar{k}_s)$ *and* $\|\psi_s\|^2_{\mathrm{LP}} = \|\varkappa_s\|^2_{\mathrm{LP}} = \pi/\operatorname{Im} k_s$. *For on-line zeros this equals* 4π, *and then* $\langle \psi_r, \psi_s \rangle = 4\pi i/(\gamma_r - \gamma_s + i)$.

PROOF. (a) $\varkappa_s \perp SH^2$ because $\langle Sh, \varkappa_s \rangle = 2\pi i S(k_s)h(k_s) = 0$. Next, $\psi_s \in H^2$ since $S(k_s) = 0$, and $\langle \psi_s, Sh \rangle = \langle (k-k_s)^{-1}, h \rangle = 0$ because $(k-k_s)^{-1} \in \overline{H^2}$. Finally, $e^{ikt}\psi_s - e^{ik_s t}\psi_s = S \cdot \frac{e^{ikt} - e^{ik_s t}}{k - k_s} \in SH^2$. (b) $\langle \psi_r, \varkappa_s \rangle = 2\pi i \psi_r(k_s)$. (c) $|S| = 1$ on $\mathbb{R}$; close the contour in $\operatorname{Im} k > 0$. □

8.2. The zero-free map

DEFINITION 8.2.1. For $f \in H^2$ put $(\mathsf{J}f)\tilde{}(z) := f(\frac{z}{2} + \frac{i}{4})$, for $\operatorname{Im} z > -\frac{1}{2}$. Equivalently,

$$(\mathsf{J}f)(u) = 2e^{-u/2} a(2u)\, \mathbf{1}_{u>0}.$$

With $t = \log y$ this is $\widetilde{(\mathsf{J}f)}(z) = \int_1^\infty a(\log y)\, y^{\frac{iz}{2} - \frac{1}{4}} \frac{dy}{y}$, the Mellin transform of the cusp profile at abscissa $-\frac{1}{4}$. This is the "Mellin transform of the zero mode" asked for in Lemma 7, Question 1. The identification of the PF model variable t with $\log y$ is derived below the definition.

Derivation of $t \leftrightarrow \log y$ *(volume addition, gap G6).* On the cusp $y > a$, the incoming and outgoing subspaces $\mathcal{D}_\mp$ of Thm. 7.3.1 consist of data of x-independent solutions of $u_{tt} = Lu$, $L = y^2\Delta + \frac{1}{4}$ [**73**, (2.13)]. On x-independent functions $L = y^2\partial_y^2 + \frac{1}{4}$. Put $r = \log y$; then $y\partial_y = \partial_r$, $y^2\partial_y^2 = \partial_r^2 - \partial_r$, and

$$L = \partial_r^2 - \partial_r + \tfrac{1}{4} = e^{r/2}\, \partial_r^2\, e^{-r/2}.$$

Hence $u = y^{1/2}v$ solves $u_{tt} = Lu$ iff $v_{tt} = v_{rr}$, and by d'Alembert every such solution is $u = y^{1/2}(\varphi(\log y - t) + \chi(\log y + t))$. Outgoing (resp. incoming) solutions are the $y^{1/2}\varphi(\log y - t)$ (resp. $y^{1/2}\chi(\log y + t)$) supported in $y > a$, and $U(t)$ acts on their profiles by translation by t in the variable $r = \log y$. So the time variable of the translation representation is displacement in $\log y$ at unit speed. Changing the reference height of both $\mathcal{D}_\pm$ from 1 to d adds the round-trip time $2\log d$, which is the factor $d^{2ik} = e^{ik\cdot 2\log d}$ in $S_d = S_1 d^{2ik}$ (Thm. 7.3.2(ii)); correspondingly $K_{d^{2ik}}$ consists of profiles a supported in $[0, 2\log d]$ (Prop. 8.2.2(c)), and J halves this to $[0, \log d]$, in accordance with Lemma 7's time dictionary $t = 2u$ (Prop. 7.5.1). The same plane waves appear in the Eisenstein constant term $y^{1/2}(y^{iz} + s(z)y^{-iz})$ (Prop. 7.2.1(b)). What is *not* re-derived here is the unitary equivalence of the full Lax–Phillips semigroup with the compressed shift on K_S, which is quoted from [**91**, §3] (Thm. 7.3.2(i)); it is used only through that citation.

PROPOSITION 8.2.2 (elementary properties; rigorous). *(a)* $\int_0^\infty |\mathsf{J}f(u)|^2 e^u\, du = 2\|a\|^2 = \|f\|^2_{\mathrm{LP}}/\pi$. *So* J *is, up to the factor* $\pi^{-1/2}$, *a unitary map from* H^2 *onto* $L^2((0,\infty), e^u\, du)$.
(b) $\mathsf{J}(e^{t/4}e^{ikt}f) = T_{t/2}\mathsf{J}f$ *exactly,* before *compression.*
(c) $\operatorname{supp} a \subset [0, 2L] \iff \operatorname{supp} \mathsf{J}f \subset [0, L]$. *By Paley–Wiener,* $K_{d^{2ik}} = \{f : \operatorname{supp} a \subset [0, 2\log d]\}$. *Moreover* J *maps* $\{a \in C_c^\infty(0, 2\log d)\}$ *bijectively onto* $C_c^\infty(0, \log d)$.

PROOF. (a) Substitute $t = 2u$. (b) $e^{t/4}e^{i(z/2 + i/4)t} = e^{izt/2}$. (c) Multiplication by $2e^{-u/2}$ is a smooth positive multiplier, and dilation preserves C_c^∞. □

REMARK 8.2.3 (why the LP norm grows). By (a), the LP norm is the norm of $L^2(e^u\, du)$ on the half-line. Right translation by $t/2$ multiplies it by $e^{t/4}$. This is exactly the growth of $e^{t/4}Z_t$ forced by $\omega_0(Z) = 0$ (Lemma 7, Thm. 4.1(ii)). It is also the same weight $e^{|u|/2}$ that defines Lemma 1's class $\mathcal{A}$. The Weil form is *translation invariant*, whereas the LP norm carries the weight e^u. Thm. 8.4.1 quantifies this mismatch.

THEOREM 8.2.4 (zero-free intertwiner; rigorous, unconditional). *Define $J_8 f := (f(k_\rho))_\rho$; this is the coordinate vector of $\mathsf{J}f$ in Lemma 1's eigenfunctionals, $\widetilde{(\mathsf{J}f)}(\gamma_\rho) = f(k_\rho)$. Define also $\mathcal{Q}^{\mathsf{J}}(f,g) := \sum_\rho m_\rho f(k_\rho)\overline{g(k_{\rho^*})}$ wherever the sum converges absolutely. Then:*

- *(a) $J_8(e^{t/4}Z_t f) = \big(e^{\frac{t}{2}(\rho-\frac12)} f(k_\rho)\big)_\rho$, which is the action of $T_{t/2}$ in Lemma 1's coordinates. The defect $\mathsf{J}(e^{t/4}Z_t f) - T_{t/2}\mathsf{J}f = -e^{t/4}\mathsf{J}(Sh)$ has Fourier transform vanishing at every γ_ρ. It is therefore $\mathcal{Q}$-null whenever it lies in $\mathcal{A}$.*
- *(b) $\mathcal{Q}^{\mathsf{J}}(e^{t/4}Z_t f, e^{t/4}Z_t g) = \mathcal{Q}^{\mathsf{J}}(f,g)$.*
- *(c) If $\mathsf{J}f, \mathsf{J}g \in \mathcal{A}$, then $\mathcal{Q}^{\mathsf{J}}(f,g) = \mathcal{Q}(\mathsf{J}f, \mathsf{J}g)$. The right-hand side is given by the explicit formula (pole + archimedean − primes) and contains no zeros.*
- *(d) Under RH, $J_8 : K_S \supset \mathrm{dom} \to \ell^2(\mathcal{Z}, m) \cong \mathcal{H}_{\mathcal{Q}}$ is closed, densely defined and unbounded. Its domain is $e^{t/4}Z_t$-invariant, and $J_8 e^{t/4}Z_t = T_{t/2}J_8$ holds there.*
- *(e) On K_{S_1}, $\ker J_8 = \mathbb{C}\psi_{i/2}$. For $d > 1$, $\ker J_8 = B \cdot K_{d^{2ik}b}$, where $B = S_1/b$ is the Blaschke product over the k_ρ. This kernel is infinite-dimensional, but J_8 is injective on $K_{d^{2ik}}$.*

PROOF. (a) $Z_t f = e^{ikt} f - Sh$ with $h \in H^2$. Evaluate at k_ρ, where $S(k_\rho) = 0$, and use $e^{t/4}e^{ik_\rho t} = e^{\frac{t}{2}(\rho-\frac12)}$, which by Lemma 1 is the multiplier of $\ell_\rho \circ T_{t/2}$. By Lemma 1, a function in $\mathcal{A}$ whose transform vanishes at all γ_ρ is $\mathcal{Q}$-orthogonal to $\mathcal{A}$. (b) The multiplier for ρ times the conjugate of that for ρ^* is $e^{\frac{t}{2}(\rho-\frac12)+\frac{t}{2}(\overline{\rho^*}-\frac12)} = 1$, because $\overline{\rho^*} = 1-\rho$. (c) Lemma 1 together with Prop. 8.2.2. (d) Point evaluations are continuous on H^2, so the maximal domain gives a closed operator. Under RH the multipliers are unimodular, which gives invariance. Unboundedness follows from Thm. 8.4.1(c). (e) Suppose $f \in K_S$ vanishes at all k_ρ. Then $f = Bg$ with $g \in H^2$. Since $S = Bd^{2ik}b$, the condition $f \perp SH^2$ is equivalent to $g \perp d^{2ik}bH^2$. For $d = 1$, $K_b = \mathbb{C}\varkappa_{i/2}$, and $B\varkappa_{i/2} \propto \psi_{i/2}$. On $K_{d^{2ik}}$, f is entire of exponential type. Its zero set would contain $\{k_\rho\}$, whose counting function is $\gg R\log R$ in $|k| < R$. This contradicts Jensen's formula unless $f \equiv 0$. □

8.3. Root vectors: the transported Weil pairing

THEOREM 8.3.1 (Krein-diagonalization; rigorous for simple zeros). *Let k_r, k_s be simple zeros of S_1 with ρ_r, ρ_s simple zeros of ζ. Then*

$$\mathcal{Q}^{\mathsf{J}}(\psi_r, \psi_s) = S'(k_r)\overline{S'(k_s)}\,\delta_{s,r^*}, \qquad \mathcal{Q}^{\mathsf{J}}(\psi_{i/2}, \cdot) = 0,$$

$$|S_1'(k_\rho)| = \frac{2\,|(\zeta^*)'(\rho)|\,|b(k_\rho)|^2}{|\zeta^*(2-\rho)|}.$$

Here $\mathcal{Q}^{\mathsf{J}}$ on root vectors is defined by the absolutely convergent zero sum, which has a single nonzero term.

- *(a) On the span of an on-line pair ($r^* = r$), $\mathcal{Q}^{\mathsf{J}}$ is diagonal and positive.*
- *(b) Each off-line pair $\{r, r^*\}$ spans a hyperbolic plane $\left(\begin{smallmatrix} 0 & c \\ \bar c & 0 \end{smallmatrix}\right)$, which contributes exactly one negative square. This is consistent with $\kappa(\mathcal{Q}) = N_+$ (Lemma 6, Thm. 7.1(ii)).*
- *(c) The LP Gram matrix (Lemma 8.1.1(c)) is* not *diagonal. $\mathcal{Q}^{\mathsf{J}}$-orthogonality of ψ_r, ψ_s for $k_r \neq k_s^*$ follows from (b) of Thm. 8.2.4 alone: eigenvectors of a $\mathcal{Q}^{\mathsf{J}}$-isometry with eigenvalues $\lambda_r\bar\lambda_s \neq 1$ are $\mathcal{Q}^{\mathsf{J}}$-orthogonal. So* diagonality is RH-neutral. *The RH content is whether $r^* = r$, i.e. the sign of the diagonal entries.*

PROOF. $\psi_r(k_\rho) = S(k_\rho)/(k_\rho - k_r)$ vanishes unless $\rho = r$, and then equals $S'(k_r)$. Likewise $\psi_s(k_{\rho^*}) \neq 0$ only if $\rho^* = s$. Also $\psi_{i/2}(k_\rho) = 0$ for all ρ.

For the derivative, write $S_1(k) = b(k)^2\,\zeta^*(-2ik)/\zeta^*(1-2ik)$ (Lemma 7, Prop. 2.1(a),(c)). Since $-2ik_\rho = 1-\rho$, the numerator vanishes at k_ρ while the denominator equals $\zeta^*(2-\rho) \neq 0$ (because $\mathrm{Re}(2-\rho) > 1$). Moreover $\frac{d}{dk}\zeta^*(-2ik)|_{k_\rho} = -2i(\zeta^*)'(1-\rho) = 2i(\zeta^*)'(\rho)$ by $\zeta^*(s) = \zeta^*(1-s)$. Also $b(k_\rho) \neq 0$ since $k_\rho \neq \frac{i}{2}$.

(b) For $\rho^* = s \neq r = \rho$, the Gram matrix on $\{\psi_r, \psi_s\}$ is $\left(\begin{smallmatrix} 0 & c \\ \bar{c} & 0 \end{smallmatrix}\right)$ with $c \neq 0$, which has eigenvalues $\pm|c|$. (c) We have $\mathcal{Q}^{\mathsf{J}}(f,g) = \mathcal{Q}^{\mathsf{J}}(Vf, Vg)$ with $V = e^{t/4}Z_t$, and $V\psi_r = \lambda_r\psi_r$ with $\lambda_r = e^{t/4+ik_rt}$. Hence $(1-\lambda_r\bar{\lambda}_s)\mathcal{Q}^{\mathsf{J}}(\psi_r,\psi_s) = 0$, and $\lambda_r\bar{\lambda}_s = 1$ for all t iff $k_r - \bar{k}_s = \frac{i}{2}$, i.e. iff $s = r^*$. □

REMARK 8.3.2 (size of the entries; explicit inequality). Let $\rho = \frac{1}{2} + i\gamma$ be a simple zero on the line, $\gamma > 0$. Since $\zeta(\rho) = 0$, $|(\zeta^*)'(\rho)| = \pi^{-1/4}|\Gamma(\frac{\rho}{2})||\zeta'(\rho)|$, and $|\zeta^*(2-\rho)| = \pi^{-3/4}|\Gamma(\frac{3}{4} - \frac{i\gamma}{2})||\zeta(\frac{3}{2} - i\gamma)|$. With $\|\psi_\rho\|^2_{\mathrm{LP}} = 4\pi$ (Lemma 8.1.1(c)), Thm. 8.3.1 gives the *identity*

$$\frac{\mathcal{Q}^{\mathsf{J}}(\psi_\rho)}{\|\psi_\rho\|^2_{\mathrm{LP}}} = \frac{|S_1'(k_\rho)|^2}{4\pi} = r\left(\tfrac{\gamma}{2}\right)^2 |b(k_\rho)|^4 \, \frac{|\zeta'(\rho)|^2}{|\zeta(\frac{3}{2}+i\gamma)|^2}, \qquad r(y) := \Big|\frac{\Gamma(\frac{1}{4}+iy)}{\Gamma(\frac{3}{4}+iy)}\Big|.$$

Each factor is bounded explicitly. (1) $|b(k)| < 1$ for $\operatorname{Im} k > 0$. (2) $|\zeta(\frac{3}{2}+i\gamma)| \geq \prod_p (1+p^{-3/2})^{-1} = \zeta(3)/\zeta(\frac{3}{2})$. (3) From $|\Gamma(x+iy)|^2 = \Gamma(x)^2 \prod_{n\geq 0}(1+y^2/(n+x)^2)^{-1}$,

$$\log \frac{r(y)^2}{r(0)^2} = \sum_{n\geq 0} \left(\phi(n+\tfrac{3}{4}) - \phi(n+\tfrac{1}{4})\right) = -\sum_{n\geq 0} \int_{n+1/4}^{n+3/4} |\phi'(x)|\,dx, \qquad \phi(x) = \log\Big(1+\frac{y^2}{x^2}\Big).$$

Since $|\phi'(x)| = 2y^2/(x(x^2+y^2))$ is decreasing, $\int_{n+1/4}^{n+3/4}|\phi'| \geq \frac{1}{2}\int_{n+1/4}^{n+5/4}|\phi'|$, and summing gives $\log(r(y)^2/r(0)^2) \leq -\frac{1}{2}\phi(\frac{1}{4})$, i.e. $r(y)^2 \leq r(0)^2(1+16y^2)^{-1/2}$. Altogether, for every simple on-line zero,

$$\frac{\mathcal{Q}^{\mathsf{J}}(\psi_\rho)}{\|\psi_\rho\|^2_{\mathrm{LP}}} \;\leq\; \Big(\frac{\Gamma(\frac{1}{4})\,\zeta(\frac{3}{2})}{\Gamma(\frac{3}{4})\,\zeta(3)}\Big)^2 \frac{|\zeta'(\rho)|^2}{\sqrt{1+4\gamma^2}} \;\leq\; 41.35\,\frac{|\zeta'(\rho)|^2}{\sqrt{1+4\gamma^2}}.$$

(Stirling gives $r(y)^2 \sim 1/y$, so the Γ-ratio factor alone loses only a constant factor ≈ 2.2. The full bound, which also bounds $|b|^4$ and $|\zeta(\frac{3}{2}+i\gamma)|^{-2}$ crudely, exceeds the exact ratio by a factor 3.1–12.4 at the tested zeros; checked for the 1st, 2nd, 3rd, 10th, 100th and 1000th zero, `supplement/logs/L8_remark_bound.out`.) The convexity bound $|\zeta'(\frac{1}{2}+i\gamma)| \ll_\varepsilon \gamma^{1/4+\varepsilon}$ [**116**, §5.1] (with Cauchy's estimate for the derivative) then gives $\mathcal{Q}^{\mathsf{J}}(\psi_\rho)/\|\psi_\rho\|^2_{\mathrm{LP}} \ll_\varepsilon \gamma^{-1/2+\varepsilon}$.

8.4. Comparison with the Lax–Phillips norm

THEOREM 8.4.1 (incomparability). *(a)* (Unconditional.) *No Hermitian form h on K_S with $c\|f\|^2_{\mathrm{LP}} \leq h(f) \leq C\|f\|^2_{\mathrm{LP}}$, $c > 0$, is invariant under $e^{t/4}Z_t$ for all $t \geq 0$.*

(b) (Unconditional.) *Along the simple on-line zeros, of which there are infinitely many (at least $\frac{2}{5}$ of all zeros [**23**]), $\mathcal{Q}^{\mathsf{J}}(\psi_s)/\|\psi_s\|^2_{\mathrm{LP}} \ll \gamma_s^{-1/2+\varepsilon} \to 0$. Hence no inequality $\mathcal{Q}^{\mathsf{J}} \geq c\cdot\mathrm{LP}$ with $c > 0$ holds on root vectors.*

(c) (Under RH.) $\mathcal{Q}^{\mathsf{J}}(\varkappa_s)/\|\varkappa_s\|^2_{\mathrm{LP}} = \frac{1}{4\pi}\sum_\rho m_\rho \frac{4}{(\gamma_\rho-\gamma_s)^2+1}$, *and $\sup_s$ of this is ∞. Hence no inequality $\mathcal{Q}^{\mathsf{J}} \leq C\cdot\mathrm{LP}$ holds, and J_8 is unbounded. Numerically the ratio is $\approx \frac{1}{2\pi}\log\frac{\gamma_s}{2\pi}$ (§8.6).*

In particular "$\mathcal{Q}$ pulled back by J equals the LP form plus a positive (or bounded) term" is false.

PROOF. (a) Invariance under $e^{t/4}Z_t$ together with equivalence of the norms would give $\sup_t \|e^{t/4}Z_t\| \leq (C/c)^{1/2}$. But $\|Z_t\| = 1$ for every t by Lemma 7, Thm. 4.1(ii) (volume proof). (b) Apply the explicit inequality of Remark 8.3.2 and the convexity bound. (c) $\varkappa_s(k_\rho) = 1/(k_\rho - \bar{k}_s) = 2/(\gamma_\rho - \gamma_s + i)$, and under RH $\rho^* = \rho$ and $\|\varkappa_s\|^2_{\mathrm{LP}} = 4\pi$. For unboundedness: $N(2T) - N(T) \sim \frac{T}{2\pi}\log T$. Cover $[T, 2T]$ by T unit intervals; one of them contains $\geq \frac{1}{2\pi}\log T(1+o(1))$ ordinates. Taking γ_s there, each such term is ≥ 2, so the ratio is $\geq \frac{1}{4\pi}\log T(1+o(1))$. □

8.5. Cusp height equals prime cutoff

THEOREM 8.5.1 (rigorous; reduces Question 1 to Lemma 6). *Let $d > 1$ and $L = \log d$. Let $f \in K_{d^{2ik}}$ with $a \in C_c^\infty(0, 2L)$, and put $F = \mathsf{J}f \in C_c^\infty(0, L)$, $H = F * F^\vee$ with $F^\vee(u) = \overline{F(-u)}$. Then*

$$\mathcal{Q}^{\mathsf{J}}(f,f) = \widetilde{H}(\tfrac{i}{2}) + \widetilde{H}(-\tfrac{i}{2}) - H(0)\log\pi + \frac{1}{2\pi}\int_{\mathbb{R}} \widetilde{H}(r)\,\operatorname{Re}\psi(\tfrac{1}{4}+\tfrac{ir}{2})\,dr - \sum_{n<d}\frac{\Lambda(n)}{\sqrt{n}}\big(H(\log n)+H(-\log n)\big).$$

Only primes $p < d$ occur, so this is Lemma 6's semilocal form for $S(L) = \{\infty\}\cup\{p \le d\}$ (Lemma 6, Thm. 3.1), up to the translation $u \mapsto u - L/2$, under which $\mathcal{Q}$ is invariant. Consequently:

$$\mathrm{RH} \iff \mathcal{Q}^{\mathsf{J}}(f,f) \ge 0 \quad \text{for all } d \text{ and all } f \in K_{d^{2ik}} \text{ with } a \in C_c^\infty(0, 2\log d).$$

*For each fixed d, the negative index of $\mathcal{Q}^{\mathsf{J}}$ on this space equals Lemma 6's $\kappa_{\log d} < \infty$. It vanishes for $d > 1$ sufficiently close to 1 by Yoshida's theorem [**122**] (Lemma 6, (L2)). No case with a prime $p < d$ is known.*[1]

PROOF. Thm. 8.2.4(c) and the explicit formula of Lemma 1 express $\mathcal{Q}^{\mathsf{J}}(f,f)$ through H. Since $\operatorname{supp} H \subset [-L, L]$, only $n < d$ contribute. The equivalence follows from Prop. 8.2.2(c), translation invariance, and Lemma 6, Thm. 7.1(iii). □

PROPOSITION 8.5.2 (the sharpest true comparison: a Gårding inequality; rigorous). *In the setting of Thm. 8.5.1, with $\|F\|_*^2$, $\mathcal{P}$, C_0 and $M(A) = 2\sum_{n\le e^A}\Lambda(n)n^{-1/2}$ as in Lemma 6, Thm. 7.1,*

$$\mathcal{Q}^{\mathsf{J}}(f,f) \;\ge\; \mathcal{P}(\mathsf{J}f) + \|\mathsf{J}f\|_*^2 - \frac{C_0 + M(\log d)}{\pi}\,\|f\|_{\mathrm{LP}}^2, \qquad M(\log d) \sim 4\sqrt{d}.$$

PROOF. Lemma 6 gives $\mathcal{Q}(F,F) \ge \mathcal{P}(F) + \|F\|_*^2 - (C_0 + M(L))\|F\|_2^2$. For $\operatorname{supp} F \subset (0, L)$, $\|F\|_2^2 \le \int |F|^2 e^u\,du = \|f\|_{\mathrm{LP}}^2/\pi$ by Prop. 8.2.2(a). (We take $C_0 \ge 0$ without loss.) □

So the LP norm controls $\mathcal{Q}^{\mathsf{J}}$ from below only up to a cusp-height-dependent loss, and the loss grows like $\sqrt{d}$. This is the precise sense in which "$\mathcal{Q}^{\mathsf{J}}$ = LP+ explicit term" holds: the explicit term is the archimedean coercive part $\|\cdot\|_*^2 + \mathcal{P}$, and the price is the full prime sum up to d.

8.6. Numerical checks

All checks are in `num/lemma8_check.py`, with output in `num/lemma8_check.out`. They use 1000 zeros from mpmath `zetazero`, cached in `num/zeros1000.txt`.

(1) *Biorthogonality* (Lemma 8.1.1(b)). We computed $\int_{-50}^{50} S_1(k)/((k-k_r)(k-k_s))\,dk$ by quadrature for $k_r, k_s \in \{\frac{i}{2}, k_{\rho_1}, k_{\rho_2}\}$. The off-diagonal entries are $\le 5\cdot 10^{-3}$, which is the size of the truncation tail. The diagonal entries agree with $2\pi i S_1'(k_s)$ to the same accuracy: for example $10.948 + 5.117i$ against $10.953 + 5.118i$.

(2) *Derivative formula* (Thm. 8.3.1). It agrees to 12 digits for $n = 1, 2, 10$.

(3) *Rayleigh quotients* (Thm. 8.4.1).

n	γ_n	$\mathcal{Q}^{\mathsf{J}}(\psi)/\|\psi\|_{\mathrm{LP}}^2$	$\frac{2}{\gamma}\frac{\lvert\zeta'(\rho)\rvert^2}{\lvert\zeta(\frac{3}{2}+i\gamma)\rvert^2}$	$\mathcal{Q}^{\mathsf{J}}(\varkappa)/\|\varkappa\|_{\mathrm{LP}}^2$	$\frac{1}{2\pi}\log\frac{\gamma}{2\pi}$
1	14.13	0.2946	0.3003	0.335	0.129
10	49.77	0.1921	0.1924	0.464	0.329
50	143.1	0.1619	0.1620	0.521	0.497
100	236.5	0.1123	0.1123	0.610	0.577
200	396.4	0.0514	0.0514	0.771	0.660
400	679.7	0.1019	0.1019	0.659	0.745

[1] **[Vol. 1 editorial note]** Superseded for $d \le 3$ by Theorem 10.3.1 and Corollary 10.3.2.

The ψ-quotients fluctuate with $|\zeta'(\rho)|^2$ and decay on average. The $\varkappa$-quotients grow like $\frac{1}{2\pi}\log\frac{\gamma}{2\pi}+O(1)$. (Only $\gamma_\rho>0$ is summed; the terms with $\gamma_\rho<0$ are $O(\gamma_s^{-2})$.)

(4) *Cusp height = prime cutoff* (Thm. 8.5.1). We took 5-dimensional bases (bump × Legendre polynomials) of $C_c^\infty(0,\log d)=\mathsf{J}\{a\in C_c^\infty(0,2\log d)\}$ and computed the Gram matrix of $\mathcal{Q}^{\mathsf{J}}$ from the arithmetic side only, i.e. pole, archimedean, and primes $p<d$. The result agrees with the zero side over 1000 zeros to $\le 2\cdot10^{-7}$ in absolute terms. The smallest eigenvalues are $1.0\cdot10^{-4}$ ($d=1.9$), $5.7\cdot10^{-7}$ ($d=3$), $1.5\cdot10^{-7}$ ($d=7$) and $9.9\cdot10^{-8}$ ($d=20$). These values are consistent with positivity, but they are *not* a certificate: for $d\ge7$ the margin is comparable to the discretisation error. Certifying them is the "certified window" item of the ranking.

8.7. Known vs. new

Statement	Status
Model-space facts (Lemma 8.1.1)	classical (Sz.-Nagy–Foias [**109**]; PF [**91**, §3])
J, intertwining, $\mathcal{Q}^{\mathsf{J}}$ zero-free (Thm. 8.2.4)	new as stated; elementary
Krein-diagonal on ψ_s, S' formula (Thm. 8.3.1)	new as stated; diagonality RH-neutral
Incomparability (Thm. 8.4.1)	new as stated; (a),(b) unconditional, (c) under RH
Cusp height = prime cutoff (Thm. 8.5.1)	new as stated; reduces to Lemma 6 + Lemma 1
Gårding bound (Prop. 8.5.2)	immediate from Lemma 6

8.8. The precise obstruction

Q1 has a positive answer at the level of *structure*. A canonical zero-free intertwiner exists, and Lax–Phillips' missing invariant inner product is $\mathcal{Q}^{\mathsf{J}}$. But $\mathcal{Q}^{\mathsf{J}}$ is not comparable to the LP energy (Thm. 8.4.1). Truncating the cusp at height d produces *exactly* the semilocal Weil form with primes $p\le d$ (Thm. 8.5.1). The scattering picture therefore adds no positivity mechanism of its own: the cusp is a geometric realization of Lemma 6's windows, with cusp height corresponding to window $\log d$. RH $\iff$ $\kappa_{\log d}=0$ for all d, which is Lemma 6's open problem verbatim. Diagonality on PF root vectors is forced by invariance (Thm. 8.3.1(c)) and is RH-neutral. The only RH-relevant datum is $r^*=r$, i.e. the zeros themselves.

Question 8.8.1. Is the intended "missing inner product" required to be bounded relative to the LP energy? If so, Thm. 8.4.1(a) rules it out unconditionally. If not, the canonical choice is $\mathcal{Q}^{\mathsf{J}}$, and the problem *is* windowed Weil positivity. Should we accept this and move to Lemma 7's Question 2, i.e. Lemma 6 at abscissa 1 with semilocal φ_S?

Question 8.8.2. The Gårding loss $(C_0+M(\log d))/\pi\sim4\sqrt{d}/\pi$ is a norm bound and ignores signs. Is there a cusp-geometric reason, for instance the Maass–Selberg relations for truncated Eisenstein series, why the prime sum up to d should be dominated by $\|\cdot\|_*^2$ on $K_{d^{2ik}}$? That would be a genuinely new input.

CHAPTER 9

Lemma 9: Lemma 6's structural trace at abscissa one

Softened singularities, a rank-one full product, and certified finite sections

Summary. We redo Lemma 6's structural positive trace $\tau_S = \mathcal{W}_S + D_S$ with the local multipliers taken at abscissa σ rather than $\frac{1}{2}$, that is, $u_v^{(\sigma)}(s) = L_v(\sigma+is)/L_v(\sigma-is)$. The case $\sigma = 1$ is where the Euler product converges and where Lemma 8's cusp intertwiner lands. We find the following.

(i) *Inner multipliers have no defect* (Thm. 9.2.1, rigorous). If $U = \bar{\Theta}$ with Θ a Blaschke product, then $PU^*(1-P)UP$ is the model-space projection and $d_U \equiv 0$.

(ii) *The full product at $\sigma = 1$ has a rank-one defect* (Thm. 9.2.2, rigorous). $U^{(1)} = \zeta^*(1+is)/\zeta^*(1-is) = \overline{B}\,\tilde{b}$, and

$$d_{\text{all}}^{(1)}(s) = \frac{2}{1+s^2}\Big(1 - \big|B(s) - \tfrac{3}{\pi}\big|^2\Big),$$

which is $O(s^{-2})$ with no singularities at all.

(iii) *Finite S: the singularities soften but do not disappear* (Obs. 9.3.1, numerical). Lemma 6's logarithmic singularities at $a = m\log p$ become Hölder-$(2\sigma-1)$ cusps, which are Lipschitz kinks at $\sigma = 1$, with amplitude $\propto p^{m\sigma}$. The $a = 0$ singularity is at noise level at $\sigma = 1$.

(iv) *The exact cost of the shift* (Prop. 9.4.1, rigorous). Abscissa σ is the damped Weil form $\mathcal{Q}_\sigma(F) = \mathcal{Q}(e^{-(\sigma-\frac{1}{2})|u|}H_F)$. It coincides with the Lagarias form $\mathcal{W}^{(\sigma)}$ for $\sigma \geq \sup\beta$, in particular at $\sigma = 1$, where it is unconditionally ≥ 0. Positivity passes upward in σ (Bochner) but not downward. RH is exactly the removal of the positive-definite factor $e^{-|u|/2} = e^{-|t|/4}$.

(v) *A Gårding bound with loss* $\sim \frac{2}{\pi}\log d$ instead of Lemma 8's $\frac{4}{\pi}\sqrt{d}$ (Lemma 8, Prop. 5.2) in LP units (Prop. 9.5.1, rigorous). The $\sqrt{d}$ reappears in the comparison of norms.

(vi) *Certified finite sections* (Prop. 9.6.1, rigorous, computer-assisted). We prove in ball arithmetic that $\mathcal{Q} > 0$ on a 12-dimensional space of C^2 test functions supported in $(0, \log d)$, for $d = 2.1, 2.5, 3, 5, 10$. These windows contain the atoms $\log 2, \ldots,$ so they lie beyond Yoshida's prime-free window. Positivity on the whole window is *not* proved; the infinite-dimensional tail is open.[1]

Nothing here proves RH.

9.1. Setting and conventions

We follow Lemma 6 throughout. $\widehat{F}(s) = \int F(u)e^{-isu}du$, $H_F = F * F^*$, $\widehat{H_F} = |\widehat{F}|^2$, and $P = \mathbf{1}_{[0,\infty)}$. Functions supported in $[0,\infty)$ have transforms in the Hardy space $\mathcal{H}$ of the *lower* half-plane, with $\langle f, g\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} f\bar{g}\,ds$. For a unimodular multiplier U (Lemma 6, Def. 4.1)

$$\tau_U(F) = \|(1-P)UPT(F)\|_{\text{HS}}^2 = \frac{1}{2\pi}\int |\widehat{F}|^2\sigma_U\,ds, \qquad d_U := \sigma_U - (\arg U)'$$

[1] **[Vol. 1 editorial note]** The tail was closed for $d \leq 3$ in Chapter 10.

(Lemma 6, Thm. 4.2). Lemma 6 uses $\sigma = \frac{1}{2}$. At abscissa $\sigma \geq \frac{1}{2}$ we put

$$u_\infty^{(\sigma)} = \frac{\Gamma_{\mathbb{R}}(\sigma+is)}{\Gamma_{\mathbb{R}}(\sigma-is)}, \quad (\arg u_\infty^{(\sigma)})' = \operatorname{Re}\psi(\tfrac{\sigma+is}{2}) - \log\pi; \qquad u_p^{(\sigma)} = \frac{1-p^{-\sigma}e^{i\ell s}}{1-p^{-\sigma}e^{-i\ell s}},\ \ell = \log p,$$

and $U_S^{(\sigma)} = \prod_{v\in S} u_v^{(\sigma)}$, $d_S^{(\sigma)} = d_{U_S^{(\sigma)}}$. Zeros of ζ are $\rho = \beta + i\gamma$, and $\gamma_\rho := -i(\rho - \frac{1}{2})$ as in Lemma 1. Lemma 8's map J turns LP time t into $u = t/2$, so the shift $\frac{i}{4}$ in k is the shift $\frac{i}{2}$ in s, i.e. abscissa $\frac{1}{2} \to 1$.

9.2. Inner multipliers and the full product at abscissa one

THEOREM 9.2.1 (rigorous). *Let Θ be a Blaschke product for the lower half-plane (zeros a_j, $\operatorname{Im} a_j < 0$, $\sum |\operatorname{Im} a_j|/(1+|a_j|^2) < \infty$), and let $U = \bar\Theta$ on $\mathbb{R}$. Then $PU^*(1-P)UP = P_{K_\Theta}$ with $K_\Theta = \mathcal{H} \ominus \Theta\mathcal{H}$, and $\sigma_U = |\Theta'| = (\arg U)'$. Hence $d_U \equiv 0$ and $\tau_U = \mathcal{W}_U$ exactly.*

PROOF. For $f \in \mathcal{H}$ write $f = \Theta g + k$ with $k \in K_\Theta$. Then $\bar\Theta f = g + \bar\Theta k$, and $k \perp \Theta\mathcal{H}$ is equivalent to $\bar\Theta k \perp \mathcal{H}$. So $(1-P)Uf = \bar\Theta k$ and $PU^*(1-P)Uf = P(\Theta\bar\Theta k) = k$. The symbol of a positive operator X on $\mathcal{H}$ in Lemma 6's normalisation is the diagonal $X(s,s)$ of its kernel, since $\tau = \operatorname{Tr}(T(F)^* X T(F))$. The reproducing kernel of $\mathcal{H}$ is $k_w(s) = -i/(s-\bar w)$, and that of K_Θ is $(1 - \Theta(s)\overline{\Theta(w)})k_w(s)$. Letting $w \to s \in \mathbb{R}$ gives $i\Theta'(s)\overline{\Theta(s)} = -(\arg\Theta)'(s) = |\Theta'(s)|$, because $\arg\Theta$ is decreasing for a lower-half-plane inner function. For infinite products, $|\Theta'| = \sum_j 2|\operatorname{Im} a_j|/|s-a_j|^2$ is a monotone limit, and so is the kernel diagonal. □

THEOREM 9.2.2 (full Euler product at abscissa one; rigorous). *Let $U^{(1)}(s) = \zeta^*(1+is)/\zeta^*(1-is)$. This is $\prod_v u_v^{(1)}(s)$, the product converging for real $s \neq 0$ by the prime number theorem.*

(a) $U^{(1)} = \overline{B}\,\tilde b$ on $\mathbb{R}$. Here $\tilde b(s) = (s+i)/(s-i)$ and B is the lower-half-plane Blaschke product with zeros $s_\rho = -\gamma - i(1-\beta)$, with multiplicity. Moreover $B(-i) = 3/\pi$.

(b) $PU^{(1)*}(1-P)U^{(1)}P = P_{K_B} - |w\rangle\langle w|$ with $w(s) = \sqrt{2}\,i\,(B(s) - \frac{3}{\pi})/(s+i) \in K_B$.

(c) $(\arg U^{(1)})' = 2\operatorname{Re}\frac{\zeta^{*\prime}}{\zeta^*}(1+is) = |B'(s)| - \frac{2}{1+s^2} = \sum_\rho \frac{2(1-\beta)}{(s+\gamma)^2+(1-\beta)^2} - \frac{2}{1+s^2}$, and

$$d_{\mathrm{all}}^{(1)}(s) = \frac{2}{1+s^2} - |w(s)|^2 = \frac{2}{1+s^2}\Big(1 - \big|B(s) - \tfrac{3}{\pi}\big|^2\Big), \qquad B = \overline{U^{(1)}}\,\tilde b \text{ on } \mathbb{R}.$$

In particular the defect of the full product is a bounded rank-one perturbation of $\frac{2}{1+s^2}$. It has no singularity at any $a = m\log p$. It changes sign: $d_{\mathrm{all}}^{(1)}(0) = 2(1-(1-\frac{3}{\pi})^2)$, while $d_{\mathrm{all}}^{(1)}(\gamma_1) < 0$. The last sign is computer-assisted. *In Arb ball arithmetic (200 bits, γ_1 enclosed by `acb_dirichlet_zeta_zero`) one finds $|B(\gamma_1) - \frac{3}{\pi}|^2 \in 3.51018551609646 \pm 10^{-19}$ and $d_{\mathrm{all}}^{(1)}(\gamma_1) \in -0.0250030363066 \pm 10^{-21}$ (`supplement/scripts/L9_sign_dgamma1.py`, log in `supplement/logs/`; gap G8 closed).*

PROOF. (a) With $\varphi(s) = \zeta^*(2s-1)/\zeta^*(2s)$ and the functional equation, $\varphi(\frac{1}{2} - ik) = \zeta^*(1+2ik)/\zeta^*(1-2ik) = U^{(1)}(2k)$. By Pavlov–Faddeev [**91**] (Lemma 7, Prop. 2.1 and Thm. 3.2), $S_1(k) = \varphi(\frac{1}{2}-ik)b(k)^2$ is a Blaschke product in $\operatorname{Im} k > 0$ with zeros k_ρ and $\frac{i}{2}$. Since $b(s/2) = 1/\tilde b(s)$, we get $U^{(1)}(s) = S_1(s/2)\tilde b(s)^2$. Reflecting $S_1(s/2)$ to the lower half-plane gives $\overline{B\tilde b}$ on $\mathbb{R}$, hence $U^{(1)} = \overline{B}\tilde b$. Next, $B = \tilde b \cdot \zeta^*(1-is)/\zeta^*(1+is)$ continues analytically into $\operatorname{Im} s < 0$. At $s = -i+\epsilon$ we have $\tilde b \approx i\epsilon/2$ and $\zeta^*(1-is) = \zeta^*(-i\epsilon) \approx -1/(-i\epsilon)$, because the residue of ζ^* at 0 is -1. So $B(-i) = \frac{1}{2}/\zeta^*(2) = \frac{1}{2}\cdot\frac{6}{\pi}$. (b) $\tilde b$ is inner for the lower half-plane, and $\mathcal{H} \ominus \tilde b\mathcal{H} = \mathbb{C}\,k_{-i}$. Exactly as in Thm. 9.2.1, $\|(1-P)U^{(1)}f\|^2 = \|P_{K_B}(\tilde b f)\|^2$, and $\tilde b B h \in B\mathcal{H}$, so only $f \in K_B$ matters. For

$f \in K_B$, $\varphi := \bar{B}f \perp \mathcal{H}$, i.e. φ lies in the upper-half-plane Hardy space. Then $\tilde{b}\varphi = \varphi + 2i\varphi/(s-i)$ and

$$P(\tilde{b}\varphi) = \frac{2i\,\varphi(i)}{s-i}, \qquad \|P(\tilde{b}\varphi)\|^2 = 2|\varphi(i)|^2.$$

Hence $\|P_{K_B}\tilde{b}f\|^2 = \|f\|^2 - \|P(\tilde{b}\varphi)\|^2 = \|f\|^2 - |\langle f, w\rangle|^2$. Upper-half-plane evaluation at i is the inner product with $i/(s+i)$, so $w = \sqrt{2}\,P(B\,i/(s+i)) = \sqrt{2}\,i(B(s)-B(-i))/(s+i)$. (c) The symbol of P_{K_B} is $|B'|$ (Thm. 9.2.1), and that of $|w\rangle\langle w|$ is $|w(s)|^2$. Finally $(\arg\tilde{b})' = -2/(1+s^2)$. □

REMARK 9.2.3 (what Thm. 9.2.2 says and does not say). $|B'| = 2\,\mathrm{Re}\,\frac{\xi'}{\xi}(1+is)$ (the factors $s(s-1)$ of ξ contribute exactly $\frac{2}{1+s^2}$). So the trace identity reads

$$0 \le \tau_{U^{(1)}}(F) = \mathcal{W}^{(1)}(F) - \|\overline{\widehat{F}}w\|^2, \qquad \mathcal{W}^{(1)}(F) := \frac{1}{2\pi}\int |\widehat{F}|^2\, 2\,\mathrm{Re}\,\frac{\xi'}{\xi}(1+is)\,ds \;\ge 0.$$

Both inequalities are unconditional and follow from $\beta < 1$, so they carry no RH content. This is the precise sense in which "the log singularities disappear at abscissa one": for the full product they disappear completely, and with them every trace of the prime atoms' delicacy.

9.3. Finite S: softened singularities (numerical)

OBSERVATION 9.3.1 (semilocal law; numerical, `num/lemma9_kappa.out`, `lemma9_fine.out`, `lemma9_spec.py`). For $S = \{\infty, p\}$ and $\frac{1}{2} \le \sigma \le 1$, in the range $p^m \lesssim |s|/2\pi$,

$$d_S^{(\sigma)}(s) \approx K_\sigma(p) \sum_m p^{m\sigma}\,\frac{\cos(m\ell s + \phi_m)}{|s|^{2\sigma}}, \qquad K_{1/2}(2) \approx 4.55,\ K_{1/2}(3) \approx 5.07.$$

On $s \in [2000, 5000)$ the coefficient of $|s|^{-1}e^{im\ell s}$ divided by $p^{m\sigma}$ is 4.55 ± 0.01 ($\sigma = \frac{1}{2}$, $p = 2$, $m = 1..9$), 5.07 ± 0.01 ($p = 3$), and 0.147 ± 0.003 ($\sigma = \frac{3}{4}$, both p, consistent with $|s|^{-3/2}$ at $|s| \approx 3000$). At $\sigma = 1$ the $m \ge 5$ coefficients double with m and decrease $\approx 4\times$ between the windows $[400, 1400)$ and $[2000, 5000)$, consistent with $2^m|s|^{-2}$. The $m = 0$ term is at the noise level ($6 \cdot 10^{-4}$ vs. noise $4 \cdot 10^{-2}$). Consequently the kernel of $D_S^{(\sigma)}$ has cusps $|a - m\ell|^{2\sigma-1}$ of strength $\propto p^{m\sigma}$. These are logarithmic at $\sigma = \frac{1}{2}$ (Lemma 6, Obs. 6.1 recovered: $\kappa_m = 4.55 \cdot 2^{m/2}$) and Lipschitz kinks at $\sigma = 1$. In the time domain, $\mathrm{rms}(s\,d_S^{(1)})$ stays bounded in s as $|S|$ grows ($4.0, 8.7, 21$ on $[200, 400)$ for $S = \{\infty, 2\}, \{\infty, 2, 3\}, \{\infty, 2, \ldots, 13\}$). At $\sigma = \frac{1}{2}$ the same quantity is $52, 98, 252$ and grows with s.

COROLLARY 9.3.2 (conditional on Obs. 9.3.1). *At $\sigma = 1$ the operator $D_S^{(1)} \circ Q_+$ is bounded, indeed Hilbert–Schmidt, on every window. Lemma 6's "unbounded above" defect is an artefact of abscissa $\frac{1}{2}$. The singular amplitudes $p^{m\sigma}$ still grow with m, so no finite-S formula is uniform. Only the full product (Thm. 9.2.2) is singularity-free.*

9.4. The cost of the shift: the abscissa family

For $F \in C_c^2(\mathbb{R})$ and $\sigma \ge \frac{1}{2}$ define

$$\mathcal{Q}_\sigma(F) := \sum_\rho m_\rho \int e^{-(\sigma-\frac{1}{2})|u|} H_F(u)\, e^{i\gamma_\rho u}\,du, \qquad \mathcal{W}^{(\sigma)}(F) := \frac{1}{2\pi}\int |\widehat{F}(s)|^2\, 2\,\mathrm{Re}\,\frac{\xi'}{\xi}(\sigma+is)\,ds.$$

Here $\mathcal{Q}_{1/2} = \mathcal{Q}$ is Lemma 1's form. The sum over ρ is taken with $\rho, \rho^* = 1 - \bar{\rho}$ paired, and $\mathcal{W}^{(\sigma)}$ is defined for $\sigma > \frac{1}{2}$ off zeros.

PROPOSITION 9.4.1 (rigorous). *(a) $\mathcal{W}^{(\sigma)}(F) = \sum_\rho m_\rho \int H_F(u)\,\mathrm{sgn}(\sigma-\beta)e^{-|\sigma-\beta||u|}e^{i\gamma u}du$ for $\sigma \notin \{\beta\}$.*

(b) $\mathcal{Q}_\sigma = \mathcal{W}^{(\sigma)}$ *whenever* $\sigma \geq \beta$ *for every zero. In particular this holds at* $\sigma = 1$ *unconditionally, where both are* ≥ 0.

(c) $\lim_{\sigma\downarrow\frac12} \mathcal{Q}_\sigma(F) = \mathcal{Q}(F)$, *whereas* $\lim_{\sigma\downarrow\frac12} \mathcal{W}^{(\sigma)}(F) = \sum_{\beta=\frac12} m_\rho |\widehat{F}(\pm\gamma)|^2$, *the on-line part. The off-line pairs cancel in the Poisson limit.*

(d) (Bochner) If $\mathcal{Q}_\sigma \geq 0$ *then* $\mathcal{Q}_{\sigma'} \geq 0$ *for all* $\sigma' > \sigma$. *No implication holds downward.*

(e) RH $\iff$ $\mathcal{Q}_\sigma \geq 0$ *for all* $\sigma > \frac12$ $\iff$ $\mathcal{W}^{(\sigma)} \geq 0$ *for all* $\sigma > \frac12$ *(Lagarias* [**70**, Thm. 1.1]*; cf. Hinkkanen* [**54**]*).*

PROOF. (a) $2\,\mathrm{Re}\,\frac{\xi'}{\xi}(\sigma + is) = \sum_\rho \frac{2(\sigma-\beta)}{(\sigma-\beta)^2+(s-\gamma)^2}$ (Hadamard product, paired). The inverse Fourier transform of this Poisson kernel is $\mathrm{sgn}(\sigma-\beta)e^{-|\sigma-\beta||u|}e^{i\gamma u}$, and Parseval applies with $\widehat{H_F} = |\widehat{F}|^2$. (b) For the pair ρ, ρ^* (same γ, abscissae $\beta, 1-\beta$), $e^{i\gamma_\rho u} + e^{i\gamma_{\rho^*} u} = 2e^{i\gamma u}\cosh((\beta-\frac12)u)$. If $\sigma - \frac12 \geq |\beta - \frac12|$ then $e^{-(\sigma-\frac12)|u|}2\cosh((\beta-\frac12)u) = e^{-(\sigma-\beta)|u|} + e^{-(\sigma-1+\beta)|u|}$, which is (a). At $\sigma = 1$ the condition is $\beta \leq 1$, which holds. Positivity at $\sigma = 1$: every term $\frac{2(1-\beta)}{\dots} > 0$. (c) $H_F \in C_c^2$, so $e^{-c|u|}H_F$ has transform $O(\gamma^{-2})$ uniformly for $c \in [0, \frac12]$, with the derivative jump at 0 bounded by $2c|H_F(0)|$. Together with $\sum \gamma^{-2} < \infty$, dominated convergence gives the first limit. For $\mathcal{W}^{(\sigma)}$, the pair contributes $(\frac12 - \beta) + (\beta - \frac12) = 0$ at $\sigma = \frac12$ in the Poisson numerators, while on-line zeros give $2\pi\delta_\gamma$. (d) $\mathcal{Q}_{\sigma'}(F)$ is the distribution $T_\sigma = \sum_\rho e^{-(\sigma-\frac12)|u|}e^{i\gamma_\rho u}$ paired with $e^{-(\sigma'-\sigma)|u|}H_F$. $\mathcal{Q}_\sigma \geq 0$ means T_σ is positive-definite. The product of T_σ with the positive-definite function $e^{-c|u|}$ ($c > 0$; transform $2c/(c^2+s^2)$) is positive-definite (Bochner–Schwartz). Division by $e^{-c|u|}$ is multiplication by $e^{c|u|}$, which is not positive-definite. (e) "$\Leftarrow$": (c) and Weil's criterion (Lemma 1). "$\Rightarrow$": under RH, (b) holds for every $\sigma \geq \frac12$, and $\mathrm{Re}\,\frac{\xi'}{\xi} > 0$ on $\mathrm{Re}\,s > \frac12$ [**70**]. □

Answer to "the exact cost of the $e^{-t/4}$ *damping".* Abscissa 1 is $\mathcal{Q}_1$, the Weil form evaluated on $e^{-|u|/2}H_F = e^{-|t|/4}H_F$. It is positive for PNT-level reasons (only $\beta < 1$ is used). RH is *exactly* the statement that the positive-definite factor $e^{-|u|/2}$ can be removed, and Bochner's theorem shows that this removal is not a positivity-preserving operation. Lemma 8's J hides the same factor one-sidedly. For $F = e^{-u/2}A$ on $(0,\infty)$,

$$H_F(u) = e^{-|u|/2}\widetilde{K}_A(u), \qquad \widetilde{K}_A(u) = \int A(y+u)\overline{A(y)}\, e^{-\min(y,\,y+u)}\, dy,$$

so $\mathcal{Q}(\mathsf{J}f) \geq 0$ automatically whenever $\widetilde{K}_A$ is positive-definite (by (b), (d) at $\sigma = 1$). It is not positive-definite in general, because the weight $e^{-\min(y,y+u)}$ is not translation-invariant. *That is the whole obstruction in one line.*

9.5. A Gårding bound at abscissa one

PROPOSITION 9.5.1 (rigorous). *Let* $\mathrm{supp}\,F \subset (0, L)$, $d = e^L$, $F = e^{-u/2}A$, *and* $\|A\|^2 = \int |F|^2 e^u du = \|f\|_{\mathrm{LP}}^2/\pi$ *for* $F = \mathsf{J}f$ *(Lemma 8, Prop. 2.2(a)). With* $\mathcal{P}$, $\|\cdot\|_*$, $C_0 \geq 0$ *as in Lemma 6, Thm. 7.1,*

$$\mathcal{Q}(F,F) \geq \mathcal{P}(F) + \|F\|_*^2 - \Big(C_0 + 2\sum_{n<d}\frac{\Lambda(n)}{n}\Big)\frac{\|f\|_{\mathrm{LP}}^2}{\pi}, \qquad 2\sum_{n<d}\frac{\Lambda(n)}{n} = 2\log d + O(1).$$

PROOF. For $a \geq 0$, $|H_F(a)| = |\int e^{-(u+a)/2}A(u+a)e^{-u/2}\bar{A}(u)du| \leq e^{-a/2}\|A\|^2$ since $u \geq 0$. So each prime term satisfies $\Lambda(n)n^{-1/2}|H_F(\pm\log n)| \leq \Lambda(n)n^{-1}\|A\|^2$. The bound $C_0\|F\|_2^2 \leq C_0\|A\|^2$ holds since $e^u \geq 1$. The asymptotic is Mertens. □

d	$M(\log d)$ (Lemma 8)	$4\sqrt{d}$	$2\sum_{n<d}\Lambda(n)/n$	$2\log d$
10	7.08	12.6	3.39	4.61
10^3	121.0	126.5	12.66	13.82
10^5	1259.9	1264.9	21.87	23.03

(`num/lemma9_misc.out`)

Conservation of $\sqrt{d}$. In LP units the loss drops from $\frac{4}{\pi}\sqrt{d}$ to $\frac{2}{\pi}\log d$. But $\|F\|_2^2 \le \|A\|^2 \le d\|F\|_2^2$, so in L^2 units the new bound is *worse* ($\sim 2d\log d$). Neither bound improves Lemma 6's finite negative index on windows. The gain is entirely a change of norm.

9.6. Certified finite sections (rank-2 item, merged)

PROPOSITION 9.6.1 (rigorous, computer-assisted). *Let $L = \log d$ and $F_j(u) = (u(L-u))^3 T_j(2u/L-1)$ on $(0,L)$, $0 \le j < 12$, with T_j Chebyshev. For $d \in \{2.1, 2.5, 3, 5, 10\}$ the Gram matrix $G_{ij} = \mathcal{Q}(F_i, F_j)$ (arithmetic side, Lemma 1) is positive definite. Hence $\mathcal{Q} > 0$ on* $\operatorname{span}\{F_0, \dots, F_{11}\} \setminus \{0\}$.

METHOD. Code: `num/lemma9_certify.py`, using python-flint/Arb [62] at 200 bits. The arithmetic side is

$$\mathcal{Q}(F_i,F_j) = \int H\, 2\cosh\tfrac{u}{2} - H(0)\log\pi + \int_0^\infty \Big[\frac{H(0)e^{-2u}}{u} - \frac{H_e(u)e^{u/2}}{\sinh u}\Big]du - \sum_{n<d}\frac{\Lambda(n)}{\sqrt{n}}\big(H(\log n)+H(-\log n)\big),$$

with $H(a) = \int F_i(u+a)F_j(u)du$ and H_e its even part. The archimedean term $\frac{1}{2\pi}\int \widehat{H}\,\operatorname{Re}\psi(\frac{1}{4}+\frac{ir}{2})dr$ is converted to the time domain by Gauss's integral for ψ.

- H is an exact rational polynomial (sympy) on $[-L,0]$ and $[0,L]$.
- The pole and archimedean integrals use rigorous `acb.integral` on $[\frac{1}{8}L, L]$. On $[0, \frac{1}{8}L]$ they use a 60-term Taylor expansion with a Cauchy-estimate tail bound. The tail $u > L$ is $H(0)E_1(2L)$.
- Positive definiteness: with V the floating-point eigenvectors of $\operatorname{mid}(G)$, we bound $V^\top G V$ in ball arithmetic and check that all Gershgorin discs lie in $(0,\infty)$. We also check $\det V^\top V > 0$, so V is invertible.

For $d = 2.1$ the smallest Gershgorin lower bound is $1.644 \cdot 10^{-12} \pm 8 \cdot 10^{-19}$ (arbitrary scale: F_j unnormalised). For $d = 10$ all twelve discs are positive. Cross-check in floating point (`num/lemma9_crosscheck.py`): $G_{00}(2.1) = 3.92534 \cdot 10^{-8}$ against the zero side with 1000 zeros, $3.92535 \cdot 10^{-8}$. □

REMARK 9.6.2 (what is *not* proved). Prop. 9.6.1 bounds the negative index $\kappa_{\log d}$ of Lemma 6 by dim of the orthogonal complement, which is still infinite. To conclude $\kappa_{\log d} = 0$ one needs, on the G-orthogonal complement of the span, a lower bound $\|F\|_*^2 \ge (C_0 + M(\log d))\|F\|_2^2 + |\mathcal{P}(F)|$ with explicit constants. This is a Bernstein/Jackson-type inequality for C^2 functions orthogonal to low Chebyshev modes, in the spirit of the archimedean analysis of Connes–Consani [19]. For $2 < d < 3$ only the atom $\log 2$ enters, so this is the cleanest first target beyond Yoshida's prime-free window [122]. The first-written LDL$^\top$ certificate failed at $J = 12$ by producing a NaN pivot. The congruence/Gershgorin test above replaced it and also succeeds at $J = 12$; With $J = 20$ the same test also certifies $d = 2.1$, $d = 3$ (`num/lemma9_certify_J20.out`; smallest disc bound $2.6 \cdot 10^{-14} \pm 3 \cdot 10^{-20}$ at $d = 2.1$) and $d = 10$. For $d = 10$, $J = 20$, $\operatorname{cond}(G) \approx 10^{18}$, and float64 eigenvectors gave discs reaching $-4 \cdot 10^{-16}$ (inconclusive, not a negative direction). With 50-digit eigenvectors the smallest disc bound is $5.9 \cdot 10^{-18} \pm 2 \cdot 10^{-24}$ (`num/lemma9_certify_J20_d10_mp.out`). The shrinking margin is the expected sign that the tail, not the finite section, is the real issue.

9.7. Answers, known vs. new, obstruction

Q1. *Do the log singularities disappear?* For finite S they soften to Hölder-$(2\sigma-1)$ cusps (Lipschitz at $\sigma=1$) with strength $\propto p^{m\sigma}$, and the $a=0$ term vanishes to numerical accuracy. For the full product they disappear entirely: the defect is rank one (Thm. 9.2.2).

Q2. *Exact cost of $e^{-t/4}$ / the shift $\frac{i}{4}$.* Abscissa 1 is the Weil form on $e^{-|u|/2}H_F$. It is unconditionally positive, and RH is the removal of a positive-definite factor (Prop. 9.4.1), which is the Lagarias family.

Q3. *Sharper Gårding?* Yes in LP units: $\frac{2}{\pi}\log d$ instead of $\frac{4}{\pi}\sqrt{d}$. It is not sharper for index purposes ($\sqrt{d}$ is conserved).

Q4. *Certified positivity for d slightly above 2?* On finite sections yes, rigorously, for d up to 10. On the whole window no; the tail inequality is open.

Known: the Lagarias criterion and its unconditional half-plane $\operatorname{Re} s \geq 1$ [**70, 69**]; model-space reproducing kernels [**42**]; the PF Blaschke property [**91**]; Weil and Yoshida. *New here (as far as we checked):* Thms. 9.2.1–9.2.2 in the language of Lemma 6's trace, including the closed form $B(-i) = 3/\pi$ and the rank-one defect; the identification $\mathcal{Q}_\sigma = \mathcal{W}^{(\sigma)}$ for $\sigma \geq \sup\beta$ with the distinct limits in (c); the semilocal law (numerical); the one-line obstruction via $\widetilde{K}_A$; and the certified sections. *Obstruction.* At abscissa 1 every structure is positive for PNT-level reasons. The Euler product, the absence of singularities, and positivity all live where $e^{-|u|/2}$ damps the zero sum. Moving to $\frac{1}{2}$ requires dividing by a positive-definite function, which no positivity mechanism survives. Lemma 6's singularities are the finite-S shadow of the same fact.

9.8. Numerics

All files are in `num/`: `lemma9_symbol.py` (module); `lemma9_semilocal`, `lemma9_fine`, `lemma9_kappa`, `lemma9_spec` (Obs. 9.3.1); `lemma9_full` (Thm. 9.2.2); `lemma9_misc` (Lagarias minima, Gårding constants); `lemma9_certify`, `lemma9_crosscheck` (Prop. 9.6.1). For Thm. 9.2.2, the defect computed from the symbol integral (zeta values on a grid, $s_{\max} = 3431$) matches the closed form: 1.99613 vs. 1.99594 at $s=0$, 0.95003 vs. 0.94984 at $s=1.047$, and -0.025025 vs. -0.024890 at $s=14.137$. The maximal discrepancy is $2\cdot10^{-4}$ on $[0,5)$, growing to $7.6\cdot10^{-3}$ on $[1000,3000)$, which is the discretisation and edge floor. The min over $s\in(0,60]$ of $\operatorname{Re}\frac{\xi'}{\xi}(\sigma+is)$ is positive for $\sigma = 1, 0.75, 0.6, 0.55, 0.51$ ($0.0231,\ldots,0.00046$, attained near $s=0$), consistent with Prop. 9.4.1(e).

9.9. Open questions

Question 9.9.1. Should the next step be the tail inequality of Rem. 9.6.2? That would upgrade Prop. 9.6.1 to $\kappa_{\log d} = 0$ for some $d \in (2,3)$, a genuine positivity result past the first prime.

Question 9.9.2. Is σ a TAA "RG scale"? Here $\sigma=1$ is the trivially positive IR end, $\sigma \downarrow \frac{1}{2}$ is the UV end, and $\Theta = \sup\beta$ is the first scale at which a negative square can appear. If so, the natural object is the window index $\kappa_{\sigma,A}$ of $\mathcal{Q}_\sigma$, which is monotone in σ by Prop. 9.4.1(d).

CHAPTER 10

Lemma 10: Certified Weil positivity on every window of length $\le \log 3$

(the first prime active), with tail control

Summary. Let $\lambda^*(d)$ be the infimum of $\mathcal{Q}(F,F)/\|F\|_2^2$ over test functions supported in an interval of length $\log d$. We prove, by a computer-assisted argument in ball arithmetic that is complete including the infinite-dimensional tail, that

$$\lambda^*(2.1) \ge 3.676 \cdot 10^{-4}, \qquad \lambda^*(2.5) \ge 5.241 \cdot 10^{-6}, \qquad \lambda^*(3) \ge 1.447 \cdot 10^{-8}.$$

Since λ^* is non-increasing in d, the Weil form is positive definite on every window of length $\le \log 3$. Equivalently, Lemma 6's negative index satisfies $\kappa_A = 0$ for $A \le \log 3$. These windows contain the atom $\log 2$, so they lie beyond Yoshida's prime-free window $\log 2$.

The proof is a *frequency split*. Because $\operatorname{Re}\psi(\frac{1}{4}+\frac{it}{2})$ is increasing in $|t|$, the Weil symbol is $\ge \beta_T$ for $|t| \ge T$. This reduces the problem (as a sufficient condition) to one finite matrix, plus explicit super-exponentially small tail constants (Thm. 10.2.1): positivity on the window follows from $\lambda_{\min}(A_N) > \varepsilon_B$.

Priority, stated honestly. Positivity beyond $\log 2$ is already claimed in two unrefereed 2026 preprints. Eid (Zenodo, July 2026) claims windows up to $1.12 \log 2$ ($d \approx 2.17$). Zhu (arXiv:2608.24827, v1 Aug. 2026) claims windows up to 1.6 ($d \approx 4.95$) with $\lambda^* \ge 8.9 \cdot 10^{-18}$, using the same frequency-split principle. Our results are an independent implementation and confirmation for $d \le 3$, and we do not claim priority. What is new here is the monotone-digamma envelope, the TAA consequences (Cor. 10.3.2: unconditional positivity on Lemma 8's cusp-truncated model spaces up to cusp height 3), and the identification of Zhu's barrier constant with Lemma 8's Gårding constant.

10.1. Setting

As in Lemmas 1 and 6, $\widehat{F}(t) = \int F(u)e^{-itu}du$ and $H_F = F * F^*$, and $\mathcal{Q}(F,F)$ is the arithmetic side of the explicit formula (Lemma 1). Translation does not change H_F, so $\mathcal{Q}$ is translation-invariant and we may centre windows: $\operatorname{supp} F \subset [-h,h]$ with $2h = L = \log d$. Put

$$\lambda^*(d) := \inf\Big\{ \frac{\mathcal{Q}(F,F)}{\|F\|_2^2} : \ 0 \ne F \in \mathcal{D},\ \operatorname{supp} F \subset [-h,h] \Big\},$$

where $\mathcal{D} = \{F \in L^2 : \int |\widehat{F}|^2 \log(2+|t|)\, dt < \infty\}$ is the form domain, which contains C_c^∞ and Lemma 1's class $\mathcal{A}$. For $A = \log d$, Lemma 6's $\kappa_A = 0$ iff $\lambda^*(d) \ge 0$. The function λ^* is non-increasing in d, since windows are nested.

Lemma 10.1.1 (frequency form; rigorous). *For $F \in \mathcal{D}$ with $\operatorname{supp} F \subset [-h,h]$,*

$$\mathcal{Q}(F,F) = \mathcal{P}(F) + \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{F}(t)|^2\, \Psi_d(t)\, dt, \qquad \Psi_d(t) = \operatorname{Re}\psi(\tfrac{1}{4}+\tfrac{it}{2}) - \log\pi - \sum_{n<d} \frac{2\Lambda(n)}{\sqrt{n}}\cos(t\log n),$$

$$\mathcal{P}(F) = 2\,\mathrm{Re}\Big[\int F e^{u/2}du\ \overline{\int F e^{-u/2}du}\Big] = 2(c^2 - s^2) \quad (F \text{ real},\ c = \textstyle\int F\cosh\frac{u}{2},\ s = \int F\sinh\frac{u}{2}).$$

For $2 < d \le 3$ the prime sum is the single term $\sqrt{2}\log 2\cos(t\log 2)$.

PROOF. This is Lemma 1's arithmetic side, written with $\widehat{H_F} = |\widehat{F}|^2$. The pole term is $\int H_F(a)2\cosh\frac{a}{2}\,da$, which factorises as shown. H_F is supported in $[-L, L]$, and $H_F(\pm L) = 0$ because the overlap has measure zero, so only $n < d$ contribute. On $\mathcal{D}$ both sides are continuous for the form norm, and they agree on C_c^∞. □

REMARK 10.1.2. $\mathcal{P}$ is *indefinite*, of rank 2 and signature $(1,1)$: it is positive on even and negative on odd centred functions, as in Zhu [**123**, Lemma 6.1]. In particular $\mathcal{P} \ge 0$ fails for odd F; we use no sign of $\mathcal{P}$ here.[1]

LEMMA 10.1.3 (monotone envelope; rigorous, elementary). *For $a > 0$, $t \mapsto \mathrm{Re}\,\psi(a + it)$ is even and strictly increasing on $[0,\infty)$. Hence, for $|t| \ge T$,*

$$\Psi_d(t) \ \ge\ \beta_T := \mathrm{Re}\,\psi(\tfrac{1}{4} + \tfrac{iT}{2}) - \log\pi - A_d, \qquad A_d := \sum_{n<d}\frac{2\Lambda(n)}{\sqrt{n}},$$

and β_T is the best constant obtainable from a pointwise bound on the prime sum.

PROOF. $\mathrm{Re}\,\psi(a+it) = -\gamma + \sum_{k\ge0}\big(\frac{1}{k+1} - \frac{k+a}{(k+a)^2+t^2}\big)$, and each term increases in $|t|$. The prime sum $S_d(t) = \sum_{n<d} 2\Lambda(n)n^{-1/2}\cos(t\log n)$ is $\le A_d$. *Optimality* (volume proof, G14): $\sup_{t\ge T} S_d(t) = A_d$ for every T. Take $t = 2\pi q/\log 2$ with $q \in \mathbb{N}$. Then $t\log 2 \in 2\pi\mathbb{Z}$, and $t\log p = 2\pi q\alpha_p$ with $\alpha_p = \log p/\log 2$. Dirichlet's simultaneous approximation theorem gives arbitrarily large q with $\|q\alpha_p\| < \varepsilon$ for all primes $p < d$. Then $\cos(t\log p^k) \ge \cos(2\pi k\varepsilon)$ for all $p^k < d$, so $S_d(t) \ge A_d - O_d(\varepsilon)$. Hence no constant smaller than A_d bounds S_d on $|t| \ge T$. Since $\mathrm{Re}\,\psi$ increases in $|t|$, β_T is the best bound of this form. (This is the content of Zhu's Lemma 3.2 [**123**]; it is proved here without that source.) □

This replaces Zhu's Binet envelope $\log\frac{t}{2\pi} - \frac{1}{t}$ [**123**, Lemma 3.1] by the exact value at T. The only inequality used is the pointwise bound on the comb.

10.2. The exact finite reduction

Let $e_n(v) = \sqrt{(2n+1)/2h}\,P_n(v/h)$ be the orthonormal Legendre basis of $L^2[-h,h]$. Then $\widehat{e_n}(t) = (-i)^n\sqrt{L(2n+1)}\,j_n(th)$, with j_n the spherical Bessel function.

THEOREM 10.2.1 (reduction; rigorous). *Fix T with $\beta_T > 0$, and put $R_T(F) := \beta_T\|F\|^2 + \mathcal{P}(F) + \frac{1}{2\pi}\int_{|t|<T}(\Psi_d - \beta_T)|\widehat{F}|^2dt$. Then:*

(a) $\mathcal{Q}(F,F) \ge R_T(F)$ on the window.

(b) $R_T - \beta_T I$ is Hilbert–Schmidt. Its matrix in (e_n) is $C_{nm} + \mathcal{P}_{nm}$, where

$$C_{nm} = (-1)^{\frac{m-n}{2}}\frac{L\sqrt{(2n+1)(2m+1)}}{\pi}\int_0^T(\Psi_d - \beta_T)\,j_n(th)j_m(th)\,dt \quad (n \equiv m \bmod 2,\ \text{else } 0),$$

$\mathcal{P}_{nm} = 2(c_nc_m - s_ns_m)$, and c_n (n even) and s_n (n odd) equal $\sqrt{(2n+1)h/2}\cdot 2\,i_n(h/2)$.

(c) Let A_N be the leading $N\times N$ block. Put $M = \max(\mathrm{Re}\,\psi(\frac{1}{4}+\frac{iT}{2}) - \psi(\frac{1}{4}),\ 2A_d)$ and $x = Th$. Using $|j_n(y)| \le \min(1, y^n/(2n+1)!!)$ and $|i_n(a)| \le a^n e^{a^2/4}/(2n+1)!!$, the entries outside A_N are bounded by a_na_m with

$$a_n = \sqrt{LMT/\pi}\,\sqrt{2n+1}\,\min(1, \tfrac{x^n}{(2n+1)!!}) + \sqrt{2}\sqrt{\tfrac{(2n+1)h}{2}}\,\tfrac{2(h/2)^n}{(2n+1)!!}e^{h^2/16}.$$

[1] **[Vol. 1 editorial note]** Volume edit (G12): a sentence attributing "$\mathcal{P} \ge 0$" to Lemma 9, which does not occur in Chapter 9, was deleted.

Let ε_B and ε_D be the Frobenius norms of the off-diagonal and tail blocks, bounded through $a_n a_m$. Then

$$\lambda^*(d) \;\geq\; \min\left(\lambda_{\min}(A_N),\ \beta_T - \varepsilon_D\right) - \varepsilon_B.$$

PROOF. (a) Split $\int_{\mathbb{R}} = \int_{|t|<T} + \int_{|t|\geq T}$, use Lemma 10.1.3 on the second piece, and apply Parseval to $\beta_T \int_{|t|\geq T} |\widehat{F}|^2$. (b) Substitute $\widehat{e_n}$. The integrand $j_n j_m$ has parity $(-1)^{n+m}$ and Ψ_d is even. For c_n, s_n use $\int_{-1}^{1} P_n(x) e^{ax} dx = 2i_n(a)$. The Hilbert–Schmidt property follows from the entry bounds in (c). (c) For $|j_n|$ use Poisson's integral $j_n(y) = \frac{y^n}{2^{n+1} n!} \int_{-1}^{1} \cos(ys)(1-s^2)^n ds$. Since $\operatorname{Re}\psi$ is monotone, $\Psi_d - \beta_T \in [\psi(\frac{1}{4}) - \operatorname{Re}\psi(\frac{1}{4} + \frac{iT}{2}),\ 2A_d]$ on $[0,T]$. Writing $F = y + z$ with y in the span of $e_{<N}$,

$$R_T(F) \geq \lambda_{\min}(A_N)\|y\|^2 - 2\varepsilon_B \|y\|\|z\| + (\beta_T - \varepsilon_D)\|z\|^2,$$

and $2\|y\|\|z\| \leq \|F\|^2$. □

REMARK 10.2.2 (sharpest explicit reduction). Thm. 10.2.1 is the requested "finite matrix inequality plus explicit constants". Window positivity follows from $\lambda_{\min}(A_N) > \varepsilon_B$, where $N \approx eTh/2 + O(1)$ suffices (then $x^N/(2N+1)!!$ is small on $[0, Th]$), and T must exceed the root $T_1(d)$ of $\beta_T = 0$; since $\operatorname{Re}\psi(\frac{1}{4} + \frac{iT}{2}) \approx \log\frac{T}{2}$, $T_1(d) \approx 2\pi e^{A_d}$. It is *not* an equivalence. R_T discards the positive excess of Ψ_d above β_T at $|t| \geq T$, which costs a factor 1.6–5 here (Table 10.1, last column).

10.3. Certificates

`num/lemma10_certify.py` is in python-flint/Arb [62] at 160 bits.

- Each C_{nm} is computed with rigorous `acb.integral`, using $\operatorname{Re}\psi$ continued as $\frac{1}{2}(\psi(\frac{1}{4} + \frac{it}{2}) + \psi(\frac{1}{4} - \frac{it}{2}))$ and $j_n(z) = \frac{z^n}{(2n+1)!!} {}_0F_1(;n+\frac{3}{2};-\frac{z^2}{4})$.
- For $\lambda_{\min}(A_N)$ we use a 60-digit Cholesky factor $\widetilde{L}$ of $\operatorname{mid}(A_N) - \lambda_0 I$, with $\lambda_0 = 0.9\cdot$(floating-point minimum). The residual $E = A_N - \lambda_0 I - \widetilde{L}\widetilde{L}^\top$ is bounded in ball arithmetic, and $\lambda_{\min}(A_N) \geq \lambda_0 - \|E\|_F$ by Weyl.
- Tails are summed explicitly to $K = N + 300$. Beyond that, a geometric bound with ratio $\leq \frac{1}{2}$ applies; it is proved in the next paragraph.

Tail summation (rigorous). Let $x = Th$ and $a_n = \sqrt{c_0}\, w_n + \sqrt{2}\, p_n$ be the majorants of Thm. 10.2.1(c), with $c_0 = LMT/\pi$, $w_n = \sqrt{2n+1}\min(1, y_n)$, $y_n = x^n/(2n+1)!!$, and $p_n = \sqrt{(2n+1)h/2}\ \ 2(h/2)^n e^{h^2/16}/(2n+1)!!$. (The bound $|j_n| \leq 1$ on $\mathbb{R}$ used there follows from $j_n(y) = \frac{(-i)^n}{2}\int_{-1}^{1} e^{iys} P_n(s)\, ds$ and $|P_n| \leq 1$; the entry bound $v_{nm} := \frac{L\sqrt{(2n+1)(2m+1)}}{\pi} MT \min(1,y_n)\min(1,y_m) + 2p_n p_m \leq a_n a_m$ is immediate.) Assume

$$(\alpha)\ \ y_K \leq 1, \qquad (\beta)\ \ 2\max(x, \tfrac{h}{2}) \leq 2K+1.$$

For $n \geq K$ we have $y_{n+1} = y_n\, x/(2n+3) \leq y_n$, so $y_n \leq 1$ and $w_n = \sqrt{2n+1}\, y_n$. Hence $w_{n+1}/w_n = x/\sqrt{(2n+1)(2n+3)} \leq x/(2K+1) \leq \frac{1}{2}$, and likewise $p_{n+1}/p_n = \frac{h}{2}/\sqrt{(2n+1)(2n+3)} \leq \frac{1}{2}$. A sum of two non-negative sequences with ratios $\leq \frac{1}{2}$ has ratio $\leq \frac{1}{2}$, so $a_{n+1} \leq \frac{1}{2} a_n$ for $n \geq K$ and $\sum_{n\geq K} a_n^2 \leq \frac{4}{3} a_K^2 =: \alpha_K$. Writing $S_I = \sum_{n\in I} a_n^2$, the Frobenius norms of the off-diagonal block B ($n \geq N > m$) and of the tail block $D - \beta_T I$ ($n, m \geq N$) satisfy

$$\varepsilon_B^2 \leq \sum_{N\leq n<K} \sum_{m<N} v_{nm}^2 + \alpha_K S_{[0,N)}, \qquad \varepsilon_D^2 \leq \sum_{N\leq n,m<K} v_{nm}^2 + \alpha_K\left(S_{[0,K)} + \alpha_K\right) + \alpha_K S_{[N,K)},$$

the last two terms covering all ordered pairs with $\max(n,m) \geq K$. All finite sums are evaluated in ball arithmetic by `lemma10_certify.py`. Conditions (α) and (β) hold with a huge margin in every run ($\log y_K \leq -945$ and $2x \leq 44 < 2K+1$; `supplement/logs/tail_conditions.out`).

d	L	T	β_T	N	$\lambda_{\min}(A_N) \geq$	ε_B	ε_D	$Z_{1000}(v)/\lambda_{\min}$
2	0.6931	40	1.8510	56	$1.1033 \cdot 10^{-3}$	$2.2 \cdot 10^{-25}$	$3.0 \cdot 10^{-54}$	–
2.1	0.7419	40	0.8707	56	$3.6761 \cdot 10^{-4}$	$1.1 \cdot 10^{-23}$	$6.5 \cdot 10^{-51}$	1.59
2.17	0.7747	40	0.8707	56	$1.9273 \cdot 10^{-4}$	$1.4 \cdot 10^{-22}$	$8.6 \cdot 10^{-49}$	–
e	1	40	0.8707	72	$5.2094 \cdot 10^{-7}$	$4.5 \cdot 10^{-29}$	$3.9 \cdot 10^{-62}$	–
2.5	0.9163	40	0.8707	64	$5.2409 \cdot 10^{-6}$	$6.7 \cdot 10^{-25}$	$1.2 \cdot 10^{-53}$	2.03
3	1.0986	40	0.8707	72	$1.4476 \cdot 10^{-8}$	$4.8 \cdot 10^{-26}$	$3.3 \cdot 10^{-56}$	5.15

TABLE 10.1. Certified lower bounds ($\|E\|_F \leq 7 \cdot 10^{-39}$ in all runs, from $2 \cdot 10^{-43}$ at $d = 2$ to $6.9 \cdot 10^{-39}$ at $d = 3$). The last column is a consistency check, not part of the proof. It is the zero-side partial sum $2\sum_{\gamma \leq \gamma_{1000}} |\widehat{v}(\gamma)|^2$ for the bottom eigenvector v of A_N, divided by $R_T(v)$, and it must be $\gtrsim 1$ because $\mathcal{Q} \geq R_T$ (`num/lemma10_zeroside.out`). In all three cases the ground state is even. Rows $d = 2$, 2.17, $L = 1$ are comparison runs (`num/lemma10_d2.out`, `lemma10_d2.17.out`, `lemma10_L1.out`).

THEOREM 10.3.1 (rigorous, computer-assisted). *For every $F \in \mathcal{D}$ supported in an interval of length $\leq \log 3$,*

$$\mathcal{Q}(F, F) \;\geq\; 1.447 \cdot 10^{-8}\, \|F\|_2^2.$$

For length $\leq \log 2.5$ the constant is $5.24 \cdot 10^{-6}$, and for length $\leq \log 2.1$ it is $3.67 \cdot 10^{-4}$. In particular $\kappa_A = 0$ for all $A \leq \log 3$.

PROOF. Thm. 10.2.1 with the certified numbers of Table 10.1, together with the monotonicity of λ^* and the reduction to real F. For complex F, $\mathcal{Q}(F) = \mathcal{Q}(\operatorname{Re} F) + \mathcal{Q}(\operatorname{Im} F)$, because the bilinear form is real-symmetric on real functions. □

Normalisation check. For $F_0 = (u(L-u))^3$ at $d = 2.1$, the frequency form of Lemma 10.1.1 gives $\mathcal{Q}(F_0) = 3.92534535 \cdot 10^{-8}$. This equals Lemma 9's Arb time-domain value $3.9253453 \cdot 10^{-8}$ and the 1000-zero sum. Moreover $R_{40}(F_0) = 3.92465 \cdot 10^{-8} \leq \mathcal{Q}(F_0)$ (`num/lemma10_crosscheck.out`).

COROLLARY 10.3.2 (TAA: unconditional positivity on Lemma 8's cusp-truncated model spaces). *For $1 < d \leq 3$ and every $f \in K_{d^{2ik}}$ with $\mathsf{J}f \in \mathcal{A}$ (Lemma 1's class, contained in $\mathcal{D}$), so that $\mathcal{Q}^{\mathsf{J}}(f, f) = \mathcal{Q}(\mathsf{J}f, \mathsf{J}f)$ by Lemma 8, Thm. 2.4(c),*

$$\mathcal{Q}^{\mathsf{J}}(f, f) \;\geq\; \frac{\lambda^*(d)}{\pi d}\, \|f\|_{\mathrm{LP}}^2 \;\geq\; \frac{1.447 \cdot 10^{-8}}{3\pi}\, \|f\|_{\mathrm{LP}}^2.$$

PROOF. Lemma 8, Thm. 5.1 shows that J maps $K_{d^{2ik}}$ into functions supported in $(0, \log d)$. Lemma 8, Prop. 2.2(a) gives $\|f\|_{\mathrm{LP}}^2 = \pi \int |\mathsf{J}f|^2 e^u \leq \pi d \|\mathsf{J}f\|^2$. □

So up to cusp height 3 the pulled-back Weil form is equivalent to the LP norm, unconditionally. This is in contrast with Lemma 8's Thm. 4.1 (no invariant equivalence on the whole of $\mathcal{K}''$). Lemma 9's finite-section certificates (Prop. 6.1 there) are subsumed for $d \leq 3$.

10.4. Literature and honest comparison

- **Yoshida** [122]. As quoted in [19, Introduction], Theorem 1 there states that $W_\infty(f) \geq 0$ for positive-definite f supported in $(\frac{1}{2}, 2)$ whose transform vanishes at $\pm\frac{i}{2}$. That is our window $\log 2$ ($d = 2$), on the codimension-2 subspace killing the pole, with no prime active. Bombieri [6] reproves this for small support and reports Yoshida's verification at $t = \frac{1}{2}\log 2$.
- **Connes–Consani** [19], Thm. 1: a conceptual (Sonin-space) proof on $[2^{-1/2}, 2^{1/2}]$, i.e. the same range.

- **Eid** [36], unrefereed, July 2026, Zenodo: claims coercive bounds up to support length $1.12 \log 2$ ($d \approx 2.17$). The abstract mentions a Galerkin compression with the prime shift represented exactly, an off-span Fourier-symbol estimate and Arb interval Cholesky. We have not checked the proofs. Thm. 10.3.1 covers this range independently; at $d = 2.17$ we certify $\lambda^* \geq 1.93 \cdot 10^{-4}$, compatible with (and larger than) Eid's claimed floor $3.28 \cdot 10^{-5}$ ($6.56 \cdot 10^{-5}$ in his symmetric normalisation); both are lower bounds, so there is no conflict.
- **Zhu** [123], unrefereed, v1 Aug. 2026 (v2 Sept.), arXiv:2608.24827: claims $\mathcal{Q} \geq 8.9 \cdot 10^{-18} \|f\|^2$ on windows of length 1.6 ($d \approx 4.95$), for both parity sectors. The method is a frequency split with Binet envelope, 200 Legendre modes per parity, 50 digits, and a two-block bound; there is also an upper-bound profile λ^* with a fitted Landau–Widom law. Our method is the same in principle, and was adopted after reading [123]. Our numbers are an independent reproduction for $d \leq 3$, with a different envelope (exact digamma value at T), different software (Arb rather than mpmath), different quadrature (`acb.integral` rather than Gauss with ellipse bounds) and a different PSD test. Zhu's table is in half-width h. Our $\lambda^*(3)$ ($h = 0.549$) lies between his reference floors at $h = 0.5$ ($9.3 \cdot 10^{-7}$) and $h = 0.6$ ($1.6 \cdot 10^{-9}$), as it should; At $h = \frac{1}{2}$ (window length 1) we certify $\mathcal{Q} \geq 5.21 \cdot 10^{-7} \|F\|^2$, consistent with Zhu's even-sector value $9.34 \cdot 10^{-7}$ and certified upper bound $1.05 \cdot 10^{-6}$; our bound is lower because R_T discards the excess of Ψ above β_T.

Net novelty: none in the range of positivity, if Zhu's claim stands. New here are an independent certificate for $d \leq 3$, the monotone-envelope simplification, Cor. 10.3.2, and the remark below.

REMARK 10.4.1 (the barrier is Lemma 8's Gårding constant). The threshold $T_1(d) \approx 2\pi e^{A_d}$ (Zhu's Thm. 1.4) involves $A_d = \sum_{n \leq d} 2\Lambda(n)/\sqrt{n}$, which is (up to the endpoint term $n = d$) Lemma 6/8's constant $M(\log d) \sim 4\sqrt{d}$ in the Gårding bound (Lemma 8, Prop. 5.2). So the same constant governs both the coercivity loss and the frequency a certificate must resolve, $T_1 \approx 2\pi e^{4\sqrt{d}}$, with matrix size $N \approx eT_1 \log d/4$. Meanwhile $\lambda^*(d)$ falls super-exponentially: $3.7 \cdot 10^{-4}$, $5.2 \cdot 10^{-6}$, $1.4 \cdot 10^{-8}$ at $d = 2.1, 2.5, 3$. Lemma 9's $\sum_n \Lambda(n)/n$ improvement does *not* help here, because it changes the norm, not the symbol.

10.5. Obstruction and outlook

The method is complete and cheap for $d \leq 3$, taking 2–3 minutes per d on 7 cores. It extends verbatim to larger d, since the code accepts any d and adds the atoms $n < d$. The cost grows like $e^{4\sqrt{d}}$ in T, and the required precision grows like $|\log \lambda^*(d)|$. That is double-exponential in the window length $L = \log d$ (Zhu's barrier). Pointwise bounds on the prime comb cannot beat it. Going further needs cancellation in $\sum_{n<d} \Lambda(n) n^{-1/2} \cos(t \log n)$ on $t \in [T, T_1]$, which is a statement about the zeros in disguise. Positivity on a fixed window is a finite fragment of RH. No fixed-window result implies RH.

10.6. Open questions

QUESTION 10.6.1. Should we push the certificate to $d = 4$ and $d = e^{1.6}$, i.e. reproduce Zhu's full claim, or stop at $d = 3$? Reaching $e^{1.6}$ needs $T \approx 150$–200 and ≈ 300 modes, which is hours of compute but no new ideas.

QUESTION 10.6.2. Cor. 10.3.2 gives an *unconditional* positive inner product $\mathcal{Q}^{\mathsf{J}}$ on $K_{d^{2ik}}$, $d \leq 3$, invariant under $e^{t/4} Z_t$ modulo $\mathcal{Q}$-null vectors. Is this the "LP space with Weil metric" that TAA wants at finite cusp height? If so, the natural next object is the limit of its constant $\lambda^*(d)/\pi d$ as the cusp height grows.

CHAPTER 11

Lemma 11: The window margin $\lambda^*(d)$: asymptotic law, resolution height $2\pi d$, and a quantitative first-ghost criterion

Summary. $\lambda^*(d)$ is the infimum of $\mathcal{Q}(F,F)/\|F\|^2$ over F supported in a window of length $\log d$ (Lemma 10). We study how fast it decays and what a zero off the line would do to it.

(a) Priority. The asymptotic law has already been proposed. Zhu (arXiv:2608.24827, unrefereed) fits $-\log\lambda^* = 2\pi^2 N(T_{\rm res})/\log N(T_{\rm res})$ with $T_{\rm res} = 2\pi d$ to certified upper bounds up to $d \approx 55$, and proves $\lambda^* \le \exp(-\frac{1}{2}\sqrt{d}\log d)$ under RH. We reproduce his eigenvalue profile independently for $2 \le d \le 11$: an exact geometric-side Galerkin computation in 800-bit arithmetic agrees with his floating-point eigenvalues to $< 1\%$. His certified upper bounds exceed our Galerkin values by 13–47%, increasingly with d. In the TAA variable the law says $-\log\lambda^*(d) \sim 2\pi^2 d$. The resolution height is 2π times the cusp height d, and the decay is singly exponential in d. The data do not fix the asymptotic constant: $-\log\lambda^*/d$ is 10.1 at $d = 11$ and 11.9 at $d = 55$.

(b) Rigorous. We prove a ghost identity: each off-line quadruple contributes $|a+b|^2 - |a-b|^2$, where a, b are the values of $\widehat{F}$ at $\gamma \pm i\delta$. So each off-line quadruple adds at most one negative square per parity sector, consistent with Bombieri's eventual equality. A ghost of displacement $\delta = \beta - \frac{1}{2}$ stays invisible while $4(\frac{\sinh 2\delta h}{2\delta} - h) < \lambda^\circ(d)$. We extend the identity to Lemma 9's damped forms $\mathcal{Q}_\sigma$, where the effective displacement is $\beta - \sigma$. We also prove that the margin of $\mathcal{Q}_\sigma$ does *not* collapse: $\lambda_\sigma(d) \ge m_\sigma$ for all d, unconditionally at $\sigma = 1$. Numerically m_σ equals $2\xi'/\xi(\sigma)$, which is $\approx 0.0924(\sigma - \frac{1}{2})$ near $\sigma = \frac{1}{2}$ and $2 + \gamma - \log 4\pi = 0.04619$ at $\sigma = 1$. The collapse of λ^* is therefore a $\sigma = \frac{1}{2}$ phenomenon.

(c) Ghost injection (numerical). We replace an on-line zero γ_k ($k \le 12$) by a quadruple with displacement $\delta \in [10^{-12}, 0.4]$ in the exact Galerkin matrix. A negative direction appears at a cusp height $d_{\rm ghost}$ that depends only doubly-logarithmically on δ (consistent with $-\log\lambda^*(d) \approx 2\log\frac{1}{\delta} + O(1)$). It grows like $(\gamma_k/2\pi)^a$ with $a \approx 0.4$–0.7, i.e. *before* the resolution height $2\pi d$ reaches γ_k. The guessed law $\log d \sim c/(\beta - \frac{1}{2})$ is therefore not the generic behaviour of the Weil form itself. It appears, heuristically, for the damped index $\kappa_{\sigma,A}$ as $\sigma \uparrow \beta$ (as $c/(\beta - \sigma)$), and as a far-ghost upper bound.

11.1. Setting, priority, and the law in TAA variables

Notation follows Lemma 10, with one exception: here $\widehat{F}(t) = \int F(u)e^{+itu}du$, while Lemma 10 uses e^{-isu}. For real F the two transforms are complex conjugates of each other (and coincide for even F), so $|\widehat{F}|^2$, $\|F\|$ and every quadratic form below are the same in both conventions (volume edit, G9). $\mathcal{Q}$ is the Weil form, $h = \frac{1}{2}\log d$ is the half-width, $\widehat{F}(t) = \int F(u)e^{itu}du$, $\|F\|^2 = \frac{1}{2\pi}\int|\widehat{F}|^2$, and $N(T) = \#\{0 < \gamma \le T\}$. For real F the explicit formula reads $\mathcal{Q}(F) = \sum_\rho \widehat{H}(\gamma_\rho)$, where $\gamma_\rho = -i(\rho - \frac{1}{2})$, $H = F \star \widetilde{F}$ and $\widehat{H}(w) = G(w)G(-w)$ with $G = \widehat{F}$.

What is known (verified). Bombieri [6] showed that the minimum of $\mathcal{Q}$ on $L^2(-t,t)$ is attained and that, if only finitely many zeros are off the line, large truncations have exactly half that many negative eigenvalues. Bombieri [4] gives the variational approach. Zhu [123] (arXiv, v1 Aug. 2026, unrefereed) has the following.

- Certified upper bounds for λ^* at half-widths $0.5 \le L \le 2$.
- The fitted law $-\log\lambda^* = C\,N(T^*)/\log N(T^*)$ with $C = 20.13 \approx 2\pi^2$ and $T^* = 2\pi e^{2L}$ (his Conj. 12.1). He calls the constant "fitted, not derived".
- Thm. 1.3: under RH, $\lambda^* \le \exp(-Le^L)$.
- The Landau–Widom reading [71, 105].

Zhu does not discuss off-line zeros, the index κ_A, or abscissae $\sigma > \frac{1}{2}$ (we searched the downloaded text). The prolate programme of Connes–Moscovici and Connes–Consani–Moscovici [20, 15] relates the zeros to prolate spectra. We use it only as context.

Proposition 11.1.1 (TAA form of the law; exact algebra). *With $L = h$ one has $T^* = 2\pi e^{2h} = 2\pi d =: T_{\mathrm{res}}$, so the resolution height is 2π times Lemma 8's cusp height. By Riemann–von Mangoldt, $N(2\pi d) = d\log\frac{d}{e} + \frac{7}{8} + S(2\pi d) + O(\frac{1}{d})$. Zhu's Conjecture 12.1 is therefore equivalent to*

$$-\log\lambda^*(d) = 2\pi^2\,\frac{d(\log d - 1)}{\log d + \log(\log d - 1)}\,(1 + o(1)), \qquad \text{so} \qquad \lambda^*(d) = e^{-2\pi^2 d(1+o(1))}.$$

The relative correction is $O(\log\log d/\log d)$, which explains the slow drift of $-\log\lambda^*/d$ seen in Table 11.1.

Proposition 11.1.2 (rigorous, unconditional). $\limsup_{d\to\infty}\lambda^*(d) \le 0$. *Under RH, $\lambda^*(d) \downarrow 0$.*

Proof. Suppose RH fails. Weil's criterion gives $F \in C_c^\infty$ with $\mathcal{Q}(F) < 0$. By translation invariance $\lambda^*(d) < 0$ once $\log d$ exceeds the width of its support, and λ^* is non-increasing.

Suppose RH holds. Take $F_L(u) = L^{-1/2}\varphi(u/L)$ with $\varphi \in C_c^\infty(-\frac{1}{2},\frac{1}{2})$ and $\|\varphi\| = 1$. Then $\mathcal{Q}(F_L) = \sum_\gamma L|\widehat{\varphi}(L\gamma)|^2 \le C_k L^{1-2k}\sum_\gamma|\gamma|^{-2k} \to 0$. □

11.2. Independent computation of $\lambda^*(d)$

`num/lemma11_galerkin.py` assembles $\mathcal{Q}$ on orthonormal Legendre modes using the time-domain explicit formula only; no zeros are used. With $E = H(x) + H(-x)$,

$$\mathcal{Q}_{ij} = \int_0^L E_{ij}(x)\,2\cosh\tfrac{x}{2}\,dx + \int_0^L \frac{e^{-2x}E_{ij}(0) - e^{-x/2}E_{ij}(x)}{1-e^{-2x}}dx \\ - (\gamma + \log\pi + \log(1-e^{-2L}))\delta_{ij} - \sum_{n<d}\frac{\Lambda(n)}{\sqrt{n}}E_{ij}(\log n).$$

Here $E_{ij}(x) = a_{ij}(x/h) + a_{ji}(x/h)$, and the $a_{ij}(\xi) = \int_{-1}^{1-\xi}\tilde{P}_i(y+\xi)\tilde{P}_j(y)dy$ do not depend on d; they are computed exactly by Gauss quadrature in Arb at 800 bits. The x-integrand is analytic on $[0,L]$, and we integrate it with $\mathrm{N}+130$ Gauss nodes.

Validation. For $F_0 = (h^2-u^2)^3$ at $d = 2.1$ we obtain $\mathcal{Q}(F_0) = 3.92534534721\cdot10^{-8}$. Lemma 10 (frequency side) gives $3.92534534706\cdot10^{-8}$ and Lemma 9 (Arb) gives $3.9253453\cdot10^{-8}$. The cross-parity entries vanish to 10^{-240}, and ball radii are $\le 4\cdot10^{-150}$.

Galerkin minima are *upper* bounds for λ^* and decrease with N (nested bases). The quadrature error is not certified, so these values are numerical.

Observation 11.2.1 (numerical). (i) The profile agrees with Zhu's eigenvalues: $9.334\cdot10^{-7}$ vs. $9.34\cdot10^{-7}$ at $d = e$; $1.603\cdot10^{-9}$ vs. $1.61\cdot10^{-9}$ at $d = e^{1.2}$; $1.640\cdot10^{-17}$ vs. $1.66\cdot10^{-17}$ at $d = e^{1.6}$. His certified upper bounds exceed our Galerkin values by 13%

d	λ_{even} (120 modes)	rel. chg. 90→120	λ_{odd}	Zhu (cert. UB)	$N(2\pi d)$	$\frac{-\log\lambda^*\ \log N}{N}$	$\frac{-\log\lambda^*}{d}$
2	$1.329\cdot10^{-3}$	6e-07	$7.310\cdot10^{-2}$	–	0	–	3.31
2.1	$5.722\cdot10^{-4}$	3e-05	$3.533\cdot10^{-2}$	–	0	–	3.56
2.17	$3.048\cdot10^{-4}$	4e-05	$2.174\cdot10^{-2}$	–	0	–	3.73
2.5	$1.026\cdot10^{-5}$	1e-04	$1.690\cdot10^{-3}$	–	1	–	4.59
$e^{1.0}$ (2.72)	$9.334\cdot10^{-7}$	1e-04	$1.937\cdot10^{-4}$	$1.050\cdot10^{-6}$	1	–	5.11
3	$5.534\cdot10^{-8}$	2e-04	$1.492\cdot10^{-5}$	–	1	–	5.57
$e^{1.2}$ (3.32)	$1.603\cdot10^{-9}$	5e-04	$5.950\cdot10^{-7}$	$1.820\cdot10^{-9}$	1	–	6.10
4	$7.530\cdot10^{-13}$	1e-03	$4.754\cdot10^{-10}$	–	3	10.22	6.98
$e^{1.4}$ (4.06)	$4.163\cdot10^{-13}$	2e-03	$2.542\cdot10^{-10}$	$4.990\cdot10^{-13}$	3	10.44	7.03
$e^{1.5}$ (4.48)	$3.630\cdot10^{-15}$	2e-03	$3.360\cdot10^{-12}$	–	3	12.18	7.42
$e^{1.6}$ (4.95)	$1.640\cdot10^{-17}$	2e-03	$1.553\cdot10^{-14}$	$2.270\cdot10^{-17}$	4	13.39	7.80
5	$9.205\cdot10^{-18}$	4e-03	$9.644\cdot10^{-15}$	–	4	13.59	7.85
6	$7.242\cdot10^{-23}$	4e-03	$1.321\cdot10^{-19}$	–	6	15.22	8.50
$e^{1.8}$ (6.05)	$4.077\cdot10^{-23}$	8e-03	$7.105\cdot10^{-20}$	$5.910\cdot10^{-23}$	6	15.40	8.52
$e^{2.0}$ (7.39)	$5.759\cdot10^{-30}$	8e-03	$1.448\cdot10^{-26}$	$8.080\cdot10^{-30}$	8	17.50	9.11
8	$4.034\cdot10^{-33}$	2e-02	$1.204\cdot10^{-29}$	–	10	17.18	9.32
$e^{2.2}$ (9.03)	$1.890\cdot10^{-38}$	4e-02	$7.414\cdot10^{-35}$	$2.780\cdot10^{-38}$	12	17.99	9.62
10	$1.498\cdot10^{-43}$	5e-02	$7.334\cdot10^{-40}$	–	14	18.59	9.86
$e^{2.4}$ (11.02)	$6.872\cdot10^{-49}$	6e-02	$4.291\cdot10^{-45}$	$9.980\cdot10^{-49}$	16	19.22	10.06

TABLE 11.1. Window margin. Lemma 10's certified lower bounds are $1.103\cdot10^{-3}, 3.676\cdot10^{-4}, 1.927\cdot10^{-4}, 5.241\cdot10^{-6}, 5.209\cdot10^{-7}, 1.448\cdot10^{-8}$ at $d = 2, 2.1, 2.17, 2.5, e, 3$, so for $d \le 3$ the true value is enclosed within a factor 1.2–3.8. Source: `num/lemma11_law.out`.

at $d = e$, rising to 40–47% at $d = e^{1.8}$–$e^{2.4}$. That is $< 1\%$ in $\log\lambda^*$, so it does not affect the law.

(ii) The ground state is even at every d. The odd sector collapses as well, and $\lambda_{\text{odd}}/\lambda_{\text{even}}$ grows from 55 ($d = 2$) to $6\cdot10^3$ ($d = 11$).

(iii) Zhu's normalised cost $-\log\lambda^*\cdot\log N/N$ is 19.2 at $d = 11$ (ours) and 20.1 at $d = 55$ (his). One-constant forms fitted on $d \ge 6$ give these maximal residuals: $c\,N/\log N$, 29% (but $< 1\%$ for $d \ge 16$); $c\,d\log d/(\log d + 1)$, 13%; $c\,d$, 37%. Data with $d \le 55$ cannot decide between the first two, whose asymptotic constants for $-\log\lambda^*/d$ are $2\pi^2$ and 14.8.

HEURISTIC 11.2.2 (why $2\pi^2$: a derivation of Zhu's constant). Landau–Widom [71]: for time–frequency limiting with Shannon number K, the eigenvalues beyond the plunge satisfy $\log\lambda_{K+j} \approx -\pi^2 j/\log K$, where j counts eigenfunctions of *both* parities. A minimiser must vanish, to the working level, at the $2N(T_{\text{res}})$ *real* points $\pm\gamma$ with $|\gamma| < T_{\text{res}}$. If each vanishing condition consumes one plunge index, then $j = 2N(T_{\text{res}})$ and $-\log\lambda^* \approx 2\pi^2 N/\log K$. Here $K = LT_{\text{res}}/\pi = 2d\log d$, so $\log K = \log N\,(1+O(1/\log d))$. This yields Zhu's constant, the factor 2 being the $\pm\gamma$ symmetry, which also appears as index doubling within each parity sector. It is not a proof: the counting $K - 2N \approx 2d > 0$ shows that the conditions are not simply "beyond" the Shannon number, and the choice $\log K$ versus $\log N$ is an $O(1/\log d)$ effect that the present data can see ($\log K$ underpredicts by 16% at $d = 55$).

11.3. The σ-direction: no collapse off the critical line

Recall Lemma 9: $\mathcal{Q}_\sigma(F) = \mathcal{Q}(e^{-(\sigma-\frac12)|u|}H_F)$. When $\sigma \ge \beta$ for all ρ it equals the Lagarias form [70] $\mathcal{W}^{(\sigma)}(F) = \frac{1}{2\pi}\int|\widehat{F}(s)|^2 w_\sigma(s)\,ds$, with $w_\sigma(s) = 2\,\mathrm{Re}\,\frac{\xi'}{\xi}(\sigma+is)$. Let $\lambda_\sigma(d)$ be its window margin.

LEMMA 11.3.1 (σ-ghost identity; rigorous). *Let F be real with $G = \widehat{F}$. Let $\delta' = \sigma - \frac{1}{2}$, let $z = \gamma_0 + i\delta$, and write $P_a(u) = \frac{a}{\pi(a^2+u^2)}$, with $P_{-a} = -P_a$. Then*

$$\widehat{H_\sigma}(z) + \widehat{H_\sigma}(\bar{z}) = \int_{\mathbb{R}} |G(s)|^2 (P_{\delta'+\delta} + P_{\delta'-\delta})(\gamma_0 - s)\, ds + \mathbf{1}_{\delta>\delta'} \cdot 2\,\mathrm{Re}\left[G(\gamma_0 + i\varepsilon)\overline{G(\gamma_0 - i\varepsilon)}\right],$$

where $\varepsilon = \delta - \delta'$. For an off-line pair $\beta\pm$ at height γ_0 ($\delta = \beta - \frac{1}{2}$) seen at abscissa $\sigma < \beta$, the result is a negative *Lorentzian of width $\beta - \sigma$, a positive one of width $\sigma + \beta - 1$, and the point term of Lemma 11.4.1 with displacement $\varepsilon = \beta - \sigma$. At $\sigma = \frac{1}{2}$ the Lorentzians cancel exactly.*

PROOF. Write $H(x) = \frac{1}{2\pi}\int \Phi(s)e^{-isx}ds$ with $\Phi(s) = G(s)G(-s)$, which is entire. Splitting $\int He^{-\delta'|x|}e^{izx}dx$ at $x = 0$ gives the kernel $\frac{2\delta'}{\delta'^2+(z-s)^2}$ as long as its poles $s = z \mp i\delta'$ lie on opposite sides of $\mathbb{R}$, i.e. for $\delta < \delta'$.

The left side is entire in z. As δ increases past δ', the pole $s_* = \gamma_0 + i(\delta - \delta')$ crosses $\mathbb{R}$. Continuing the contour across it adds $2\pi \cdot \frac{1}{2\pi}\Phi(s_*)$, because the residue of the kernel there is $i\Phi(s_*)$. Adding the conjugate term and using $G(-\bar{w}) = \overline{G(w)}$ gives the claim, together with $\mathrm{Re}\,\frac{2\delta'}{\delta'^2+(u+i\delta)^2} = \pi(P_{\delta'+\delta} + P_{\delta'-\delta})(u)$.

Numerical check with $F = (1-u^2)^3$ (`num/lemma11_sigma_identity.out`): both sides agree to relative accuracy 10^{-6} for $(\delta, \delta') = (0.3, 0.1)$ and $(0.1, 0.3)$. □

PROPOSITION 11.3.2 (uniform margin for $\sigma > \frac{1}{2}$; rigorous). *Let $m_\sigma := \inf_s w_\sigma(s)$, and suppose $\sigma > \beta_\rho$ for every zero ρ (no zero has $\mathrm{Re}\,\rho \geq \sigma$). This holds unconditionally for $\sigma \geq 1$, since ζ has no zeros on $\mathrm{Re}\,s = 1$, and under RH for every $\sigma > \frac{1}{2}$. (We do* not *assume $\sup\beta < \sigma$; whether $\sup\beta < 1$ is open.) Then $m_\sigma > 0$ is attained, $\lambda_\sigma(d) \geq m_\sigma$ for* all *d, and $\lambda_\sigma(d) \downarrow m_\sigma$ as $d \to \infty$.*

PROOF. *The weight.* By the Hadamard product for ξ (order one, with $\mathrm{Re}\,B = -\sum_\rho \mathrm{Re}\,\frac{1}{\rho}$ [**25**, Ch. 12]), we have $\mathrm{Re}\,\frac{\xi'}{\xi}(s) = \sum_\rho \mathrm{Re}\,\frac{1}{s-\rho}$. Hence

$$w_\sigma(s) = 2\sum_\rho \frac{\sigma - \beta_\rho}{(\sigma-\beta_\rho)^2 + (s-\gamma_\rho)^2} > 0.$$

Since $\sigma > \beta_\rho$ for every ρ, ξ'/ξ has no pole on $\mathrm{Re}\,z = \sigma$, so w_σ is real-analytic. It is even, because $\xi(\bar{z}) = \overline{\xi(z)}$. From $\frac{\xi'}{\xi}(z) = \frac{\zeta'}{\zeta}(z) + \frac{1}{2}\psi(\frac{z}{2}) - \frac{1}{2}\log\pi + \frac{1}{z} + \frac{1}{z-1}$ and Stirling, $w_\sigma(s) = \log|s| + O_\sigma(1)$ for $\sigma > 1$. For $\sigma \leq 1$ (still with $\sigma > \beta_\rho$ for all ρ), put $\delta = \sigma - \frac{1}{2} > 0$ and $u = s - \gamma_\rho$. In each pair $\rho, 1-\bar{\rho}$ (same γ) the two values $a = \sigma - \beta$ and $a' = \sigma - 1 + \beta$ are positive with $a + a' = 2\delta$, so one of them is $\geq \delta$. For $a \in [\delta, 1]$ we have $\frac{a}{a^2+u^2} \geq \frac{\delta}{1+u^2}$. Also $\frac{3/2-\beta}{(3/2-\beta)^2+u^2} \leq \frac{6}{1+u^2}$. Hence $w_\sigma(s) \geq \frac{\delta}{12}w_{3/2}(s) \geq \frac{\delta}{12}\log|s| - C$. In both cases $w_\sigma \to \infty$, so the infimum m_σ is attained at some s_0 and is positive.

Lower bound. $\mathcal{Q}_\sigma = \mathcal{W}^{(\sigma)}$ on every window by Lemma 9, Prop. 4.1, which needs only $\sigma \geq \sup\beta$, a consequence of our hypothesis. Hence $\mathcal{W}^{(\sigma)}(F) = \frac{1}{2\pi}\int|\widehat{F}|^2 w_\sigma \geq m_\sigma\|F\|^2$. The margin is non-increasing in d because the windows are nested.

Limit. Fix a real $\varphi \in C_c^\infty(-1,1)$ with $\|\varphi\|_2 = 1$, and put $F_h(u) = h^{-1/2}\varphi(u/h)\cos(s_0 u)$, which is real and supported in $[-h, h]$, $h = \frac{1}{2}\log d$. Then $\widehat{F_h}(s) = \frac{h^{1/2}}{2}\big(\widehat{\varphi}(h(s-s_0)) + \widehat{\varphi}(h(s+s_0))\big)$. The probability densities $\mu_h = |\widehat{F_h}|^2 / \int|\widehat{F_h}|^2$ concentrate at $\{\pm s_0\}$ at scale $1/h$, with Schwartz tails. Since w_σ is continuous, even, and $O(\log(2+|s|))$, dominated convergence gives

$$\lambda_\sigma(d) \leq \frac{\mathcal{W}^{(\sigma)}(F_h)}{\|F_h\|^2} = \int w_\sigma\, d\mu_h \;\longrightarrow\; w_\sigma(s_0) = m_\sigma \qquad (h \to \infty).$$

Together with the lower bound and monotonicity, $\lambda_\sigma(d) \downarrow m_\sigma$. □

OBSERVATION 11.3.3 (numerical: location of the infimum). On $s \in [0, 120]$, the infimum of w_σ is attained at $s = 0$, so $m_\sigma = 2\frac{\xi'}{\xi}(\sigma)$ for the tested σ. This gives $m_\sigma \approx 0.0924\,(\sigma - \frac{1}{2})$ near $\frac{1}{2}$, and $m_1 = 2\sum_\rho \frac{1}{\rho} = 2 + \gamma - \log 4\pi = 0.0461914$ (`num/lemma11_msigma.out`; for example $m_{0.6} = 0.00924$ and $m_{0.75} = 0.0231$). This is not proved. Beyond $s = 120$ it would follow from a nearest-zero bound, given an explicit bound on gaps between ordinates, which we do not supply. Prop. 11.3.2 does not use it.

REMARK 11.3.4 (dichotomy). For $\sigma > \frac{1}{2}$ the weight w_σ is a smoothed zero density without holes, so the margin is $\approx m_\sigma$ and is independent of d. As $\sigma \downarrow \frac{1}{2}$ it degenerates to $2\pi\sum_\gamma \delta_\gamma$, and the margin becomes a *sampling* problem: $\widehat{F}$ of type h can vanish at all ordinates below $T_{\mathrm{res}} = 2\pi d$, where the zero density $\frac{1}{2\pi}\log\frac{t}{2\pi}$ is below the Nyquist density $\frac{h}{\pi}$. The doubly-exponential (in $\log d$) collapse of λ^* is entirely this hole effect. Since $m_\sigma \to 0$ only linearly in $\sigma - \frac{1}{2}$, Lemma 9's σ-RG is the right regulator: every abscissa $\sigma > \frac{1}{2}$ has a d-uniform margin.

11.4. Ghosts at $\sigma = \frac{1}{2}$: identity, index bound, invisibility

LEMMA 11.4.1 (quadruple identity; exact). *Let F be real and let $a = G(\gamma_0 + i\delta)$, $b = G(\gamma_0 - i\delta)$. The four zeros $\frac{1}{2} \pm \delta \pm i\gamma_0$ contribute $|a+b|^2 - |a-b|^2$ to $\mathcal{Q}(F)$. An on-line pair at $\pm\gamma_0$ contributes $2|G(\gamma_0)|^2$. Moreover*

$$|a-b| = 2\Big|\int F(u)e^{i\gamma_0 u}\sinh(\delta u)\,du\Big| \le 2\,S(\delta,h)^{1/2}\|F\|, \qquad S(\delta,h) = \frac{\sinh 2\delta h}{2\delta} - h \;\big(\approx \tfrac{2}{3}\delta^2 h^3\big).$$

PROOF. $\sum \widehat{H} = 2(a\bar{b} + \bar{a}b) = 4\,\mathrm{Re}\,a\bar{b}$, using $G(-w) = \overline{G(\bar{w})}$. The bound is Cauchy–Schwarz on $[-h, h]$. □

PROPOSITION 11.4.2 (index bound; rigorous). *Suppose only $M < \infty$ quadruples are off the line. Then $\mathcal{Q} = \mathcal{Q}_+ - \sum_{j\le M}|a_j - b_j|^2$ with $\mathcal{Q}_+ \ge 0$. The negative index satisfies $\kappa_A \le 2M$, and $\kappa_A^{\mathrm{even}}, \kappa_A^{\mathrm{odd}} \le M$.*

PROOF. For even real F, G is even and real on $\mathbb{R}$, so $b = \bar{a}$ and $a - b = 2i\,\mathrm{Im}\,a$, a rank-one functional. For odd real F, $b = -\bar{a}$ and $a - b = 2\,\mathrm{Re}\,a$, again rank one. □

Bombieri's theorem [6] (half the number of off-line zeros, i.e. 2 per quadruple, for large truncations) says the bound is eventually attained in both sectors. So each ghost quadruple produces exactly one negative square per parity sector, at window lengths A_ρ^{even} and A_ρ^{odd}.

PROPOSITION 11.4.3 (invisibility threshold; rigorous). *Let $\lambda^\circ(d)$ be the window margin of the on-line part $\mathcal{Q}_+^{\mathrm{on}} = \sum_{\mathrm{Re}\,\rho=\frac{1}{2}}|G(\gamma)|^2$. If $4\sum_j S(\delta_j, h) < \lambda^\circ(d)$, then $\kappa_{\log d} = 0$.*

With $\lambda^\circ \approx \lambda^*$ and Prop. 11.1.1, a single ghost of displacement δ is invisible for $d < d_\delta$, where $-\log\lambda^*(d_\delta) \approx 2\log\frac{1}{\delta} + \log\frac{8}{3}h^3$. Asymptotically $d_\delta \approx \frac{\log(1/\delta)}{\pi^2}$, i.e. $\log d_\delta \approx \log\log\frac{1}{\delta} - 2\log\pi$.

PROOF. $\mathcal{Q} \ge \mathcal{Q}_+^{\mathrm{on}} - \sum|a_j - b_j|^2 \ge (\lambda^\circ - 4\sum S)\|F\|^2$ by Lemma 11.4.1. □

HEURISTIC 11.4.4 (detection threshold; revised against Table 11.2). Let $\ell = \log\frac{\gamma_0}{2\pi}$, the log-height.

(a) *δ-dependence (supported by the data).* The gain is $\approx 4\delta^2|G'(\gamma_0)|^2$, and the cost is the margin of functions that are small at the other ordinates. Detection therefore needs $-\log\lambda^*(d) \gtrsim 2\log\frac{1}{\delta} + E(\gamma_0, d)$, so $\log d_{\mathrm{ghost}}$ depends on δ only doubly-logarithmically. In Table 11.2, lowering δ from 0.4 to 10^{-8} (seven decades) raises d_{ghost} only by a factor 2–2.7.

(b) *Height dependence (the naive prediction is refuted).* The naive "resolution" prediction $A_\rho \approx \log \frac{\gamma_0}{2\pi}$, i.e. slope 1, holds only for γ_1 and γ_2. Higher ghosts are detected *before* the resolution height reaches them. Fitting $\log d_{\text{ghost}} = a \log \frac{\gamma_0}{2\pi} + b$ gives slopes $a = 0.42$–0.66. Equivalently, a ghost becomes visible once the local oversampling ratio $\ell/\log d$ falls below $\approx 1/a \approx 1.5$–2.4, not below 1. Since the data reach only $\gamma_0/2\pi \le 9$, the asymptotic slope is open. The rigorous statement is only the invisibility bound, Prop. 11.4.3.

(c) *Far ghost* ($\delta \gtrsim \sqrt{3}/\ell$; untested, since it needs $\ell \gg 1/\delta$). An unresolved window already wins once the point gain $4S(\delta, h)$ exceeds the local density cost $\approx \ell$. For $\delta h \gg 1$ this gives $A_\rho \lesssim \frac{1}{\delta} \log(\delta\ell)$, the guessed "$c/(\beta - \frac{1}{2})$" law up to a log-log factor. In the tested range the bound is far from sharp: for γ_{12} and $\delta = 0.4$ it predicts $d \approx 32$, while we observe 5.5.

At $\sigma = \frac{1}{2}$ the guessed law $A \sim c/(\beta - \frac{1}{2})$ is therefore *not* the generic behaviour. The δ-dependence is doubly logarithmic. A $1/\delta$ law can only come from the far regime, as an upper bound. A genuine $c/(\beta - \sigma)$ law appears in the damped index (Lemma 11.3.1). There the negative Lorentzian of width $\beta - \sigma$ beats the background $w_\sigma(\gamma_0) \approx \ell$ iff $\beta - \sigma \lesssim 2/\ell$, and resolving a well of that width costs window length $A \approx c/(\beta - \sigma)$ by the uncertainty principle.

Ghost-injection test. (`num/lemma11_ghost_fixed.py`, `.json`). We take the exact Galerkin matrix (50 modes per parity, $d = e^{2h}$ with $h = 0.1, 0.15, \ldots, 1.2$). We remove the on-line pair $\pm\gamma_k$ and add the quadruple at $\gamma_k \pm i\delta$, in exact matrix form via $\widehat{e}_n(w) = 2i^n \sqrt{h(2n+1)/2}\, j_n(wh)$, with $j_n(w) = \sqrt{\pi/2}\, w^{-1/2} J_{n+1/2}(w)$ (entire) and ordinates γ_k to 110 digits. A Galerkin negative eigenvalue is a genuine negative direction of the model form, so the detected threshold is an upper bound for d_{ghost}. (An earlier run, `num/lemma11_ghost.py`, evaluated j_n on the wrong branch at negative real arguments and therefore added the quadruple without removing γ_k. Its thresholds coincide with the corrected ones in 59 of 60 sector-cases; the exception is $k = 1$, $\delta = 0.4$, odd: 2.72 instead of 2.46.)[1]

k	γ_k	$\gamma_k/2\pi$	$\delta = 0.4$	$\delta = 0.1$	$\delta = 10^{-2}$	$\delta = 10^{-4}$	$\delta = 10^{-8}$	$\delta = 10^{-12}$
1	14.13	2.25	2.23/2.46	2.72/3.00	3.32/3.67	4.06/4.48	6.05/6.05	7.39/8.17
2	21.02	3.35	2.72/3.00	3.32/3.32	3.67/4.06	4.95/4.95	6.69/6.69	8.17/9.03
4	30.42	4.84	3.67/3.67	4.06/4.48	4.48/4.95	5.47/6.05	7.39/8.17	9.03/9.97
8	43.33	6.90	4.48/4.95	4.95/5.47	6.05/6.05	6.69/7.39	9.03/9.03	$> 11/> 11$
12	56.45	8.98	5.47/6.05	6.05/6.69	6.69/7.39	8.17/8.17	$11.02/> 11$	$> 11/> 11$

TABLE 11.2. First d (grid $d = e^{2h}$, $h = 0.10, 0.15, \ldots, 1.20$, so grid ratio $e^{0.1}$) at which the ghost model has a negative eigenvalue (even/odd sector). At most one negative eigenvalue per sector occurred in all 1380 sector-runs, as Prop. 11.4.2 requires. Fits of $\log d_{\text{ghost}}^{\text{even}} = a \log \frac{\gamma_k}{2\pi} + b$ give $a = 0.66, 0.57, 0.54, 0.48, 0.42$ for $\delta = 0.4, 0.1, 10^{-2}, 10^{-4}, 10^{-8}$. Source: `num/lemma11_fix_analyze.out`. Corrected in v2: entry $k = 1$, $\delta = 0.4$, odd (v1: 2.72).

11.5. Relation to κ_A (Lemma 6) and the σ-RG index (Lemma 9)

- κ_A counts resolved ghosts. In a world with finitely many ghosts, $\kappa_A = \#\{\rho : A \ge A_\rho^{\text{even}}\} + \#\{\rho : A \ge A_\rho^{\text{odd}}\}$, which is $\le 2M$ (Prop. 11.4.2) and eventually equal to $2M$ (Bombieri). In the tested range, ghosts switch on at $\log d \approx a \log \frac{\gamma_0}{2\pi} + b$ with $a \approx 0.4$–0.7, i.e. *before* the resolution height $2\pi d$ reaches γ_0. The TAA reading "cusp height d sees the zeros below $2\pi d$" is therefore conservative: it is a sufficient scale, not the threshold. Lemma 10's $d = 3$ certificate already

[1] Corrected in v2: this paragraph and one entry of Table 11.2; see "Corrections in version 2" in the front matter.

has $2\pi d = 18.85 > \gamma_1$, and the model detects a ghost at γ_1 with $\delta \geq 0.1$ by $d \approx 2.7$. Hence Lemma 10's positivity at $d = 3$, read as a ghost test, *excludes* (within the one-ghost model) that γ_1 could be off the line with $\delta \geq 0.1$. This exclusion is only a model statement: it is already known from verification [**93**]. An actual ghost has $\gamma_0 > 3 \cdot 10^{12}$, and with slope $a \in [0.4, 1]$ it would need $A \gtrsim 0.4 \cdot 26.9 \approx 11$ at least, if the fitted slope persisted.

- RH-world margin and index. RH $\iff$ $\kappa_A = 0$ for all A $\iff$ $\lambda^*(d) \geq 0$ for all d, and $\lambda^*(d) = e^{-2\pi^2 d(1+o(1))}$ (conjecturally) is the size of the gap any certificate at cusp height d must resolve.
- σ-RG. $\kappa_{\sigma,A} = 0$ for $\sigma \geq \sup \beta$, with uniform margin m_σ (Prop. 11.3.2). A ghost β is visible at abscissa $\sigma < \beta$ with displacement $\beta - \sigma$ against a hole-free background. So $\sigma \mapsto \kappa_{\sigma,A}$ is a step function whose jumps sit at the β's, and the window length at which a jump appears grows like $c/(\beta - \sigma)$ as $\sigma \uparrow \beta$. The σ-RG thus converts the delicate $\sigma = \frac{1}{2}$ sampling problem, which collapses doubly exponentially in $\log d$, into a family of hole-free problems with d-uniform margins $m_\sigma \sim 0.09(\sigma - \frac{1}{2})$.

11.6. Obstruction and questions

Obstruction. No lower bound on $\lambda^*(d)$ for $d \to \infty$ can be unconditional, since it would imply RH (Prop. 11.1.2). Even under RH, a lower bound of Landau–Widom type needs a sampling inequality for PW_h on the zero set *beyond* T_{res}, with constants depending on the irregularity of zero gaps near T_{res}. Feeding the known gap bound under RH (gaps $\leq (\pi + o(1))/\log\log T$, Goldston–Gonek [**43**]) into such a sampling argument would, by our sketch, give at best $\log(1/\lambda^*) \lesssim e^{\sqrt{d}} \log d$, far from $2\pi^2 d$. We did not carry this out. The rigorous ghost statements (Lemmas 11.3.1, 11.4.1, Props. 11.4.2, 11.4.3) are complete. The detection side (Heuristic 11.4.4) needs an explicit sampling *upper* bound $\sum_\gamma |G(\gamma)|^2 \leq (\ell + o(\ell))\|F\|^2$ for $\widehat{F}$ localised near γ_0. This is routine from explicit $N(T)$ bounds, but we have not written it out.

Question 11.6.1. Should Lemma 6's index be re-indexed by the resolution height, $\kappa(T)$ with $T = 2\pi e^A$, so that "κ jumps when T passes an off-line ordinate" becomes the organising statement for the TAA side (cusp height $d \leftrightarrow$ zeros below $2\pi d$)?

Question 11.6.2. Next step: make the far-ghost detection rigorous, which would give an explicit theorem "a zero with $\beta - \frac{1}{2} = \delta$ at height γ_0 forces $\kappa_A \geq 1$ for $A \geq \frac{C}{\delta} \log(\delta \log \gamma_0)$"? Or attack the RH-conditional lower bound $-\log \lambda^* \lesssim d^{O(1)}$ through multi-interval Landau–Widom?

CHAPTER 12

Lemma 12: Explicit ghost detection, what window certificates say about zeros, and a function-field degeneration test

Summary. *(a) Theorem (rigorous; its numerical rows are computer-assisted).* Let $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$ be a zero with $\delta > 0$. Assume it is locally extremal: zeros with $\big||\gamma| - \gamma_0\big| \le Y$ have $|\beta - \frac{1}{2}| \le \delta$. Assume also that it is isolated: other zeros with $|\gamma - \gamma_0| < \eta$ lie on the line. Then $\kappa_A \ge 1$ for every $A \ge A_0$. We give A_0, η, Y as explicit outputs of a criterion that uses only the explicit Riemann–von Mangoldt bound. The criterion is certified in ball arithmetic on a grid $\delta \in \{\frac{1}{2}, \dots, 10^{-2}\}$, $\gamma_0 \in \{\gamma_1, 10^3, 3 \cdot 10^{12}, 10^{20}, 10^{100}\}$. Typical values are $A_0 \approx 17$–31 at $\delta = \frac{1}{4}$ (and ≈ 9–16 in the formal limit column $\delta = \frac{1}{2}$, which lies outside the theorem) and $A_0 \approx (3.7\text{–}8.4)/\delta$ in general, with δA_0 growing roughly linearly in $\log\log\gamma_0$. This is the "$c/(\beta - \frac{1}{2})$" law, with a fully explicit constant.

(b) Converse (rigorous, and honestly vacuous today). A certificate $\kappa_A = 0$ excludes every zero that satisfies the hypotheses of a row with $A_0 \le A$. All windows on which positivity is certified (Lemma 10: $\log 3$; Zhu: length 1.6, his results for half-widths up to 2 being *upper* bounds on λ^*) are shorter than every tabulated A_0 (≥ 9.36), so nothing is excluded. Even hypothetically, adding anything to the Mossinghoff–Trudgian–Yang zero-free region needs windows $A \gtrsim 13$ (heights $> 3 \cdot 10^{12}$, δ just below $\frac{1}{2}$). A substantive region, such as $\delta \ge \frac{1}{4}$ at 10^{20}, needs $A \approx 24$, where $\lambda^* \approx e^{-2\pi^2 e^A}$. The value is structural, not computational. RH is verified to $3 \cdot 10^{12}$ (Platt–Trudgian).

(c) Add-on: doubly-log law and function fields. The doubly-logarithmic ghost sensitivity follows from Lemma 11's invisibility bound and the margin $e^{-2\pi^2 d}$. The lower bound is conditional-rigorous. The data fit $d_{\text{ghost}} = 2.28 + 0.192 \log\frac{1}{\delta}$, with an effective $\kappa = 10.4$ matching $-\log\lambda^*/d$. For a curve over $\mathbb{F}_q$ we prove that the window margin is *exactly* $\log q \cdot \lambda_{\min}(T_n)$, where T_n is a Toeplitz matrix of size $n = \lceil A/\log q \rceil$. It is $O(1)$ for $n \le 2g$ and *exactly* 0 for $n > 2g$ (finite rank). A ghost is detected at $n \le 2g$ *uniformly in* δ. Number fields have no exact kernel, which is where the $\log\log\frac{1}{\delta}$ law comes from. This yields a heuristic "degeneration test" for any candidate $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z}$ pairing.

12.1. Setting

Notation follows Lemmas 10–11. For $F \in L^2$ with compact support, set $G = \widehat{F}$, $\widehat{F}(t) = \int F(u)e^{itu}du$, $H_F(x) = \int F(y+x)\overline{F(y)}\,dy$ and $\|F\|^2 = \int |F|^2$. The Weil form is $\mathcal{Q}(F) = \sum_\rho \widehat{H_F}(\gamma_\rho)$, where $\gamma_\rho = -i(\rho - \frac{1}{2})$ and zeros are counted with multiplicity. The geometric side is Lemma 11, §1. For complex $F = F_1 + iF_2$, $\mathcal{Q}(F) = \mathcal{Q}(F_1) + \mathcal{Q}(F_2)$: the cross term of H_F is i times an odd function, and its transform sums to 0 over the zero set, which is symmetric under $\gamma \mapsto -\gamma$. κ_A is the negative index of $\mathcal{Q}$ on real functions supported in an interval of length A. It is non-decreasing in A by translation invariance.

12.2. The modulated two-bump test function

Fix $r > 0$, an integer $m \geq 2$, $c > 0$, and $\varphi(u) = (1 - u^2/r^2)^m$ on $[-r, r]$ (zero outside). Put $\Phi = \varphi \star \varphi$, so that $\widehat{\Phi}(w) = \widehat{\varphi}(w)\widehat{\varphi}(-w)$, and

$$f(u) = \varphi(u - c) - \varphi(u + c), \qquad F(u) = e^{-i\gamma_0 u} f(u), \qquad \operatorname{supp} F \subset [-c - r, c + r].$$

LEMMA 12.2.1 (exact; numerically confirmed). *$H_f = 2\Phi - \Phi(\cdot - 2c) - \Phi(\cdot + 2c)$, $\widehat{H_F}(w) = \widehat{H_f}(w - \gamma_0)$, and $\widehat{H_f}(\tau) = \widehat{\Phi}(\tau)\,(2 - 2\cos 2c\tau)$. Hence:*

(i) the pair $\rho_0, 1 - \bar{\rho}_0$ contributes exactly $-4I_\delta^2(\cosh 2c\delta - 1)$, where $I_\delta = \int \varphi(u) \cosh \delta u \, du$;
(ii) an on-line zero at ordinate γ contributes $4\widehat{\Phi}(s) \sin^2 cs \in [0, 4|\widehat{\varphi}(s)|^2]$, where $s = \gamma - \gamma_0$;
(iii) a zero with $\gamma_\rho - \gamma_0 = s - iy$ contributes at most $|\widehat{\Phi}(s - iy)|\,(2 + 2\cosh 2cy)$ in absolute value.

PROOF. The first identity is a direct expansion. The rest follows from $\widehat{\Phi(\cdot \mp 2c)} = e^{\pm 2ic\tau}\widehat{\Phi}$. For (i), $\tau = \mp i\delta$ and $\widehat{\Phi}(\pm i\delta) = \widehat{\varphi}(i\delta)\widehat{\varphi}(-i\delta) = I_\delta^2$, since φ is even. For (iii), use $|2 - 2\cos(2c(s - iy))| \leq 2 + 2\cosh 2cy$. □

Check (`num/lemma12_sanity.py`). For $r = 1$, $m = 4$, $c = 0.9$, $\gamma_0 = \gamma_1$, the geometric side gives $\mathcal{Q}(F) = 7.6275899 \cdot 10^{-5}$. The zero side over 1000 zeros, using the Bessel closed form of $\widehat{\varphi}$, agrees to $4 \cdot 10^{-20}$. The tiny value illustrates (ii): an on-line zero at the centre is invisible.

LEMMA 12.2.2 (decay; rigorous). *For $|\operatorname{Im} w| \leq y$, $|\widehat{\varphi}(w)| \leq M_y(|\operatorname{Re} w|)$, where*

$$M_y(x) := \min_{0 \leq k \leq m} \frac{c_k(y)}{x^k}, \qquad c_0(y) = \int \varphi \, e^{y|u|}, \qquad c_k(y) = e^{yr}\sqrt{2r}\,\|\varphi^{(k)}\|_2 \quad (k \geq 1).$$

Consequently $|\widehat{\Phi}(s - iy')| \leq g_y(|s|) := M_y(|s|)^2$ for $|y'| \leq y$.

PROOF. Integrate by parts $k \leq m$ times. The boundary terms vanish because $\varphi^{(j)}(\pm r) = 0$ for $j < m$. Then apply Cauchy–Schwarz on $[-r, r]$. The norms $\|\varphi^{(k)}\|_2$ are exact rationals (`fmpq_poly`). □

12.3. Counting and summation

Let $R(T) = 0.112 \log T + 0.278 \log\log T + 3.385$. Then $|N(T) - \frac{T}{2\pi}\log\frac{T}{2\pi e}| \leq R(T)$ for $T \geq e$. This is Trudgian [**117**], in the normalisation of Table 1 of [**48**]; we verified it from the downloaded text of [**48**], which also proves $0.1038, 0.2573, 9.3675$. At $T = e$ the bound gives $N(e) \leq 2$, which we use below e. With $M(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e}$, the number of zeros with $\gamma > 0$ and $|\gamma - \gamma_0| \leq x$ is at most

$$\widetilde{N}^+(x) = \begin{cases} M(\gamma_0 + x) - M(\gamma_0 - x) + R(\gamma_0 + x) + R(\gamma_0 - x), & \gamma_0 - x > e, \\ M(\gamma_0 + x) + R(\gamma_0 + x), & \text{else.} \end{cases}$$

LEMMA 12.3.1 (Abel majorant; rigorous). *Let $g \geq 0$ be non-increasing on $[a, b]$, continuous except for downward jumps. If D_j are points with counting function $\leq \widetilde{N}^+$, then*

$$\sum_{a \leq D_j \leq b} g(D_j) \leq g(b)\widetilde{N}^+(b) + \int_a^b \widetilde{N}^+ \, d(-g).$$

The case $b = \infty$ is the limit, valid when $g(x)\widetilde{N}^+(x) \to 0$.

PROOF. Stieltjes integration by parts. The term $-g(a)N(a^-) \leq 0$ is dropped, and $d(-g) \geq 0$. □

In the code (`num/lemma12_A0.py`), g_y is replaced by a piecewise majorant $c_k^2 x^{-2k}$ whose breakpoints are placed just *after* each crossover, so that all jumps are downward; this is checked in ball arithmetic. The integrals are rigorous Arb integrals of the analytic pieces. The tail beyond $X \geq \max(10^6, 4(\gamma_0+2))$ uses $\widetilde{N}^+(x) \leq x^{3/2}$.

12.4. Explicit ghost detection

THEOREM 12.4.1 (explicit detection length; rigorous). *Let $\rho_0 = \frac{1}{2} + \delta + i\gamma_0$ be a zero with $0 < \delta < \frac{1}{2}$, and let $r, m, c, \eta > 0$ and $Y \in (\eta, \infty]$. Assume:*

(H1$_Y$) every zero with $||\gamma| - \gamma_0| \leq Y$ has $|\beta - \frac{1}{2}| \leq \delta$ (for $Y = \infty$: $\delta = \sup_\rho |\beta - \frac{1}{2}|$, attained at ρ_0);

(H2$_\eta$) every zero with $|\gamma - \gamma_0| < \eta$, other than copies of ρ_0 and $1 - \bar{\rho}_0$, has $\beta = \frac{1}{2}$.

With $C = \cosh 2c\delta$, define the following sums by Lemma 12.3.1:

- $S_0 = \overline{\sum}_{0 \leq D \leq \eta} g_0$;
- $S_\delta = \overline{\sum}_{\eta \leq D \leq Y} g_\delta + \overline{\sum}_{\text{mirror}} g_\delta$;
- $S_{\text{far}} = \overline{\sum}_{D > Y} g_{1/2} + \overline{\sum}_{\text{mirror, far}} g_{1/2}$.

Here $D = |\gamma - \gamma_0|$, and the mirror zeros $\gamma < 0$ have $D = |\gamma| + \gamma_0$. If

$$\Delta := 4I_\delta^2(C-1) - 4S_0 - (2+2C)S_\delta - (2+2\cosh c)S_{\text{far}} > 0,$$

then $\mathcal{Q}(\operatorname{Re} F) < 0$ or $\mathcal{Q}(\operatorname{Im} F) < 0$. Hence $\kappa_A \geq 1$ for all $A \geq A_0 := 2(c+r)$.

PROOF. Split $\mathcal{Q}(F) = \sum_\rho \widehat{H_F}(\gamma_\rho)$ and apply Lemma 12.2.1: (i) for ρ_0 and its partner, since further copies only add more negative terms; (ii) for zeros within η, by (H2); (iii) for the rest. The bounds then come from Lemma 12.2.2, with $|y| \leq \delta$ under (H1) and $|y| < \frac{1}{2}$ always. Mirror zeros with $||\gamma| - \gamma_0| \leq Y$ satisfy (H1) by conjugation symmetry, and all mirrors are majorised by both g_δ and, beyond Y, $g_{1/2}$. Summing gives $\mathcal{Q}(F) \leq -\Delta < 0$, and the conclusion follows from $\mathcal{Q}(F) = \mathcal{Q}(\operatorname{Re} F) + \mathcal{Q}(\operatorname{Im} F)$. The explicit formula applies because $H_F \in C_c^{2m-1}$ and $|\widehat{H_F}| = O(|w|^{-2m})$ in horizontal strips. □

TABLE 12.1. Computer-assisted (rigorous) rows of Theorem 12.4.1 (Arb ball arithmetic, 300–800 bits; c, η, Y rounded up to 0.01 and re-verified; $m = 2$ except $m = 3$ in the local $\delta = 0.01$ rows). Entry: A_0 [r; η] for the global form ($Y = \infty$), and A_0 [η; Y] for the local form. Files `num/lemma12_certified.json`; – = search stopped for time (10^{100}, local, $\delta = 0.01$).

$\gamma_0 \setminus \delta$	0.5*	0.25	0.1	0.05	0.01
γ_1 glob.	9.4 [1; 5.8]	16.8 [1; 5.1]	39.1 [1; 4.8]	76.1 [1; 4.6]	370.8 [2; 2.2]
γ_1 loc.	9.9 [5.8; 9]	17.7 [5.1; 26]	41.4 [4.8; 2114]	80.8 [4.6; $7.6 \cdot 10^6$]	394.0 [2.5; $4.0 \cdot 10^{18}$]
10^3 glob.	10.3 [1; 7.4]	18.6 [1; 6.4]	43.6 [1; 5.9]	83.7 [2; 2.7]	396.4 [4; 1.2]
10^3 loc.	10.7 [7.4; 12]	19.6 [6.4; 49]	46.2 [5.9; 3757]	88.6 [2.7; $1.1 \cdot 10^7$]	421.9 [1.3; $3.3 \cdot 10^{19}$]
$3 \cdot 10^{12}$ glob.	12.3 [1; 11.5]	22.7 [1; 9.8]	52.0 [2; 4.0]	98.6 [3; 2.4]	454.0 [8; 0.8]
$3 \cdot 10^{12}$ loc.	12.7 [11.5; 19]	23.7 [9.8; 117]	54.4 [4.0; 8790]	103.4 [2.4; $1.9 \cdot 10^7$]	478.9 [0.7; $6.6 \cdot 10^{21}$]
10^{20} glob.	13.1 [1; 13.4]	24.3 [1; 11.5]	55.5 [2; 4.6]	105.0 [4; 2.0]	481.7 [8; 0.8]
10^{20} loc.	13.6 [13.4; 23]	25.2 [11.5; 156]	58.2 [4.6; $1.7 \cdot 10^4$]	109.9 [2.0; $4.2 \cdot 10^7$]	508.5 [0.8; $1.7 \cdot 10^{23}$]
10^{100} glob.	16.1 [1; 22.8]	30.2 [2; 9.2]	69.7 [4; 3.5]	131.7 [4; 3.1]	607.1 [8; 1.2]
10^{100} loc.	16.7 [22.8; 39]	31.3 [9.2; 207]	72.2 [3.5; $8.6 \cdot 10^4$]	136.7 [3.1; $4.0 \cdot 10^9$]	–

* The column $\delta = 0.5$ lies outside the hypothesis $0 < \delta < \frac{1}{2}$ of Theorem 12.4.1: it would mean $\beta = 1$, and ζ has no zeros on $\operatorname{Re} s = 1$. It is the criterion evaluated formally at $\delta = \frac{1}{2}$. It is shown only as a reference scale and certifies nothing about zeros (Remark 12.4.2(b)). Within the theorem's range the tabulated values are $A_0 \geq 16.8$.

REMARK 12.4.2 (shape and honesty). (a) In every tabulated row, $\delta A_0 \in [3.7, 8.4]$. At fixed δ, δA_0 increases roughly linearly in $\log\log\gamma_0$; at $\delta = \frac{1}{2}$ (global) it goes $4.7, 5.1, 6.2, 6.6, 8.1$ for $\gamma_0 = \gamma_1, \dots, 10^{100}$. This is the expected $A_0 \approx \frac{1}{\delta}(\log(K\log\gamma_0) + O(1))$: the ghost gain $e^{2c\delta}$ must beat the number of zeros in the isolation window, $\asymp \eta\log\gamma_0$. The optimiser (integer r, $m \in \{2,3,4,6,10\}$) always chooses $m \le 3$ and $r \in [1,8]$. The bumps are narrow, and the length is spent on the separation $2c$.

(b) For $\delta = \frac{1}{2}$ (that is, $\beta = 1$) the hypotheses are vacuous because ζ has no zeros on $\operatorname{Re} s = 1$. The global rows ($Y = \infty$) assume that ρ_0 attains $\sup\beta$, which need not be attained. The local rows are the meaningful form: if RH fails in a bounded box, the zero of maximal β there satisfies (H1$_Y$) for Y up to the box size. Even then Y grows like $e^{c(1-2\delta)/3}$, and it is huge for small δ.

(c) (H2) cannot be removed by this method. A companion off-line zero at distance $< \eta$ can contribute up to $(2+2C)I_\delta^2 > 4I_\delta^2(C-1)$ when $C < 3$, and k companions cost k times that. A version with $\le k$ companions is a routine modification.

(d) Lemma 11's ghost model knows all other zeros. There a ghost at γ_1 with $\delta = 0.4$ is seen at window $\log 2.23 = 0.80$, against $A_0(0.5, \gamma_1) = 9.36$ here (formal $\delta = \frac{1}{2}$ column). The gap of more than $10\times$ is the price of replacing the other zeros by $\widetilde{N}^+$.

12.5. The converse: what a positivity certificate rules out

COROLLARY 12.5.1 (rigorous). *If $\kappa_A = 0$ (for instance $\lambda^*(e^A) > 0$ is certified), then no zero satisfies (H1$_Y$) and (H2$_\eta$) with the parameters of any certified row with $A_0 \le A$.*

PROPOSITION 12.5.2 (vacuity today). *Every tabulated A_0 is ≥ 9.36, and ≥ 16.8 within the range $0 < \delta < \frac{1}{2}$ of Theorem 12.4.1. The longest windows on which positivity is certified are $\log 3 = 1.10$ (Lemma 10) and, granting Zhu's unrefereed certificate* [**123**]*, length 1.6 (*$\operatorname{supp} f \subset [-0.8, 0.8]$*). Zhu's results for half-widths L up to 2 are* upper *bounds on λ^* and certify no positivity. Corollary 12.5.1 therefore excludes nothing.*

Comparison.

- Platt–Trudgian [**93**] prove that all zeros with $0 < \gamma \le 3\cdot 10^{12}$ are on the line. Every row with $\gamma_0 \le 3\cdot 10^{12}$ is therefore a statement about the *form*, not new information about ζ.
- Mossinghoff–Trudgian–Yang [**86**] give $\beta < 1 - 1/(5.558691\log|t|)$ for $|t| \ge 2$, which is $\delta < 0.4961$ at 10^{20} and $\delta < 0.4992$ at 10^{100}. A certificate at window A would exclude locally extremal, isolated zeros with $\delta \gtrsim 4/A$ (from the table, $A \approx 25$ for $\delta = \frac{1}{4}$ at 10^{20}). That is a region of a different shape, $\delta \ll \log\log\gamma_0/A$, conditional on (H1)–(H2).
- By Lemma 11 the cost is prohibitive: $\lambda^*(e^A) \approx \exp(-2\pi^2 e^A)$. Already at $A = 9.36$ a certificate needs $\sim 10^5$-digit arithmetic and a Galerkin dimension $\gtrsim$ the Shannon number $2d\log d \approx 2\cdot 10^5$, whereas Lemma 10 reached $d = 3$.

The value is structural. It is an explicit, effective version of "a ghost is seen at a finite window", with the dependence on δ and γ_0 made visible.

12.6. Add-on: the doubly-log law and a function-field degeneration test

12.6.1. Doubly-logarithmic ghost sensitivity.

PROPOSITION 12.6.1 (conditional-rigorous lower bound). *Suppose the on-line margin satisfies $\lambda^\circ(d) \ge e^{-\varkappa d}$ for $d \le d_1$. Then a single ghost of displacement δ is invisible ($\kappa_{\log d} = 0$) whenever $\frac{8}{3}\delta^2 h^3\cosh(2\delta h) < e^{-\varkappa d}$, where $h = \frac{1}{2}\log d$. In particular,*

$$d_{\mathrm{ghost}}(\delta) \;\ge\; \frac{2\log\frac{1}{\delta} - \log\left(\frac{1}{3}(\log d)^3\cosh(\delta\log d)\right)}{\varkappa}, \qquad \log d_{\mathrm{ghost}} \;\ge\; \log\log\tfrac{1}{\delta} + \log\tfrac{2}{\varkappa} - o(1).$$

PROOF. Lemma 11, Prop. 4.3 (invisibility, rigorous) gives the criterion $4S(\delta,h) < \lambda^\circ$. Since $(2k+1)! \geq 6(2k-2)!$, we have $S(\delta,h) = h\sum_{k\geq 1}\frac{(2\delta h)^{2k}}{(2k+1)!} \leq \frac{2}{3}\delta^2 h^3 \cosh 2\delta h$. □

Status. The implication is a theorem. The hypothesis is not: $\varkappa = 2\pi^2(1+o(1))$ is Zhu's conjectured law (Lemma 11, Prop. 1.1), and in the computed range $-\log\lambda^*/d \approx 8$–$12$. The matching *upper* bound $\log d_{\text{ghost}} \lesssim \log\log\frac{1}{\delta} + O_{\gamma_0}(1)$ is heuristic (Lemma 11, Heur. 4.4(a)). The only rigorous upper bound is Theorem 12.4.1, which gives $A_0 \asymp 1/\delta$, exponentially weaker. *Data* (Lemma 11 ghost table, γ_1, $\delta = 0.4\ldots10^{-12}$): $d_{\text{ghost}} = 2.28 + 0.192\log\frac{1}{\delta}$ with maximal residual 0.24. This is linear in $\log\frac{1}{\delta}$, hence doubly-log in the window, with slope giving $\varkappa_{\text{eff}} = 2/0.192 = 10.4$. That agrees with $-\log\lambda^*(d)/d = 10.1$ at $d = 11$, so the mechanism of Prop. 12.6.1 is quantitatively the operative one.

12.6.2. Function fields: exact margin and δ-uniform detection. Let $X/\mathbb{F}_q$ have genus g, put $\ell = \log q$, and let $\alpha_j = \sqrt{q}\,z_j$ ($j \leq 2g$) be the inverse roots of $P(T)$. The zeros are $\rho = \frac{1}{2} + i(\theta_j + 2\pi k)/\ell$ with $e^{i\theta_j} = z_j$ (RH: $|z_j| = 1$). In general $z_\rho := e^{i\gamma_\rho\ell}$.

PROPOSITION 12.6.2 (rigorous). *For real F supported in an interval of length A, put $n = \lceil A/\ell\rceil$. Then*

$$\mathcal{Q}_X(F) = \ell\int_0^\ell v(x)^*\,T_n\,v(x)\,dx, \qquad v(x) = (F(x+j\ell))_{0\leq j<n}, \qquad (T_n)_{jk} = \sum_{\rho \bmod 2\pi i/\ell} z_\rho^{\,j} z_\rho^{-k}.$$

Hence:

- *(a) the window margin is* exactly $\lambda_X^*(A) = \ell\,\lambda_{\min}(T_n)$*;*
- *(b) under RH, $T_n = VV^*$ with V an $n\times 2g$ Vandermonde matrix, so $\lambda_X^*(A) > 0$ iff $n \leq \#\{\text{distinct } z_j\}$ and $\lambda_X^*(A) = 0$ for $n > 2g$;*
- *(c) if instead the $2g$ zeros contain one quadruple ($z = e^{i\theta}e^{\pm\delta\ell}$, $\bar z$ likewise, all $2g$ nodes distinct), then $T_n = VMV^*$ with M of signature $(2g-2,2)$, and $\kappa = 2$ for every $n \geq 2g$ and every $\delta > 0$.*

PROOF. Poisson summation: $\sum_k \widehat{H}(w + 2\pi k/\ell) = \ell\sum_j H(j\ell)e^{iwj\ell}$. Summing over the $2g$ residue classes and writing $H(j\ell) = \int_0^\ell \sum_i F(x+(i+j)\ell)F(x+i\ell)dx$ gives the formula. Fibres have at most n points, and shorter fibres give principal submatrices, so interlacing yields (a). For (b), an on-line z gives the rank-one term $z^j\bar z^{\,k}$. For (c), the partner pair $z, 1/\bar z$ gives $ab^* + ba^*$ with $a = (z^j)$, $b = (\bar z^{-j})$, which has signature $(1,1)$. Sylvester's law applies since V has full column rank for $n \geq 2g$. □

TABLE 12.2. Function fields (`num/lemma12_ff.*`). Top: exact window margin $\ell\lambda_{\min}(T_n)$ for the curves of Lemma 3 (values $< 10^{-14}$ are 0 up to rounding). Bottom: $\kappa(T_n)$ for a genus-preserving ghost quadruple (60-digit arithmetic).

curve	$n=1$	2	3	4	5	6
E/$\mathbb{F}_5$, $g=1$	3.219	1.060	0	0	0	0
C2/$\mathbb{F}_7$, $g=2$	7.784	5.577	4.799	1.250	0	0
ghost model	first n with $\kappa > 0$ / κ at $n = 2g$, for $\delta = 0.4$, 0.1, 10^{-4}, 10^{-8}, 10^{-12}					
$g=2$ (C2 phases)	2/2, 3/2, 3/2, 3/2, 3/2					
$g=6$ (random phases)	4/2, 8/2, 11/2, 11/2, 11/2					

12.6.3. Degeneration test (heuristic).

HEURISTIC 12.6.3 (what a $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$ pairing must reproduce). In the function-field case, positivity of $\mathcal{Q}_X$ is the Hodge index theorem on $X\times X$ (Lemma 3). Its window truncations are positive semidefinite of rank $2g$, with an *exact* kernel once $n>2g$. That exact kernel is what makes ghost detection δ-uniform (Prop. 12.6.2(c)). For ζ the dictionary is:

- the fibre count n corresponds to the Shannon number $K=2d\log d$;
- the $2g$ phases correspond to the $2N(2\pi d)$ resolved ordinates $\pm\gamma$, i.e. an effective genus $g_{\text{eff}}(d)=N(2\pi d)$.

Since $K-2N\approx 2d>0$, the truncations never saturate. Instead each vanishing condition costs $\approx\pi^2/\log K$ in $\log\lambda$ (Landau–Widom). Any candidate pairing whose positivity is to imply RH must therefore, restricted to windows of length $\log d$:

(i) be positive with *soft* (not exact) degeneracy, with

$$\lambda_{\min}\approx\exp(-2\pi^2 g_{\text{eff}}/\log g_{\text{eff}})=e^{-2\pi^2 d(1+o(1))};$$

(ii) reproduce ghost sensitivity $\log d_{\text{ghost}}\approx\log\log\frac{1}{\delta}$, not the δ-uniform $(2g-1)\log q$ of curves.

A pairing with finite-rank truncations ("curve-like") would predict δ-uniform detection, which the Weil form's data contradict. So the doubly-log law is a signature of an *infinite effective genus growing like* $N(2\pi d)$. It neither supports nor refutes the existence of the missing pairing; it constrains its truncations.

Labels. Prop. 12.6.2 is a theorem, and the curve numerics are exact up to floating point. Prop. 12.6.1 is a theorem conditional on the margin law. Heuristic 12.6.3 is a heuristic.

12.7. Relations, obstruction, questions

Relations. Theorem 12.4.1 makes quantitative Lemma 11's Prop. 1.2 ($\limsup\lambda^*\le 0$ if RH fails) and its far-ghost Heuristic 4.4(c). Corollary 12.5.1 is the contrapositive of Lemma 10's certificates. Prop. 12.6.2 sharpens Lemma 3's Toeplitz picture into an exact window statement.

Obstruction. Any rigorous detection length built only from zero-*counting* pays for the unknown arrangement of the other zeros. It gives $A_0\asymp\delta^{-1}\log\log\gamma_0$, while the true sensitivity, with the other zeros known, is $\asymp\log\log\frac{1}{\delta}$. Closing the gap needs a rigorous lower bound for the on-line margin $\lambda^\circ(d)$, which is RH-strength local information. In the converse direction the obstruction is the margin $e^{-2\pi^2 d}$ itself: positivity certificates stop near window length $\log d\approx 1.1$ (Lemma 10) to 1.6 (Zhu), and every tabulated A_0 is ≥ 9.

QUESTION 12.7.1. Is $(\mathrm{H1}_Y)$+$(\mathrm{H2}_\eta)$ (a locally extremal, isolated ghost) the notion of "ghost" TAA wants? Or should we state the $\le k$-companion version (Remark 12.4.2(c)), which costs a factor $\approx k$ inside log?

QUESTION 12.7.2. For the $\operatorname{Spec}\mathbb{Z}\times\operatorname{Spec}\mathbb{Z}$ pairing: should its window truncations be finite-rank (curve-like, hence predicting δ-uniform detection, which the data contradict for $\mathcal{Q}$) or infinite-rank with soft degeneracy $\exp(-2\pi^2 g_{\text{eff}}/\log g_{\text{eff}})$, $g_{\text{eff}}=N(2\pi d)$?

CHAPTER 13

Lemma 13: The squaring map at $\tau = i$

Levi-Civita/Kustaanheimo–Stiefel, the ramified prime 2 of $\mathbb{Q}(i)$, the CM curve $y^2 = x^3 - x$, and CM Sato–Tate

Summary. Four classical facts are organised around the map $z \mapsto z^2$ on $\mathbb{Z}[i]$.

(a) Levi-Civita/Kustaanheimo–Stiefel regularisation turns Kepler into an oscillator (coupling-constant metamorphosis).
(b) The ramified prime 2 of $\mathbb{Q}(i)$ is wild because $z \mapsto z^2$ is inseparable mod 2.
(c) The CM curve $E : y^2 = x^3 - x$ has Frobenius equal to a primary Gaussian prime. We prove the exact sign rule and check it against LMFDB and the η-product for all 17 983 odd primes $< 2 \cdot 10^5$.
(d) Hecke equidistribution gives the CM Sato–Tate law $\frac{1}{2}\delta_0 + \frac{1}{2}\,\frac{dx}{\pi\sqrt{1-x^2}}$: Kolmogorov distance $2.8 \cdot 10^{-3}$, versus $4.2 \cdot 10^{-3}$ to the semicircle for the non-CM curve 11.*a*.

Each statement is labelled *theorem* or *analogy*. In particular, "$|\alpha| = \sqrt{p}$ is the Levi-Civita coordinate $|z| = \sqrt{r}$" is the *norm* map, not a dynamical identity. We correct the claims "$F = 4GMu$" and "conductor $= 4|D|$", and the sign in "disc $= \prod f'(\text{roots})$".

13.1. (a) Levi-Civita and Kustaanheimo–Stiefel

Proposition 13.1.1 (Levi-Civita [78]; theorem). *Let $x(t) \in \mathbb{C}$ solve $\ddot{x} = -kx/|x|^3$ with energy $E = \frac{1}{2}|\dot{x}|^2 - k/|x|$. Put $x = z^2$ and $dt = r\,ds$, where $r = |x| = |z|^2$, and let $' = d/ds$. Then:*

(i) $|dx/dz|^2 = |2z|^2 = 4r$, and $\dot{x} = 2z'/\bar{z}$;
(ii) $z'' = \frac{E}{2}\,z$ and $2|z'|^2 - E|z|^2 = k$. Equivalently, with $w = 4z'$, $K = \frac{1}{8}|w|^2 - E|z|^2 = k$.

For $E < 0$ this is an isotropic oscillator with $\omega^2 = -E/2$. The Kepler energy becomes the (negative of the) spring constant, and the Kepler coupling $k = GM$ becomes the oscillator energy. This exchange is the coupling-constant metamorphosis *of Hietarinta–Grammaticos–Dorizzi–Ramani* [53].

Proof. $\dot{x} = x'/r = 2zz'/|z|^2 = 2z'/\bar{z}$. The energy equation reads $2|z'|^2/r - k/r = E$. Differentiating in s and using the equation of motion gives $z'' = \frac{E}{2}z$; this is the standard computation [78, 68]. □

Check (`num/lemma13_lc.out`). With $x_0 = 1$, $\dot{x}_0 = 0.8i$ ($E = -0.68$), the Kepler ODE solution agrees with $z(s)^2$ at the matched times $t(s) = \int_0^s r$ to 10^{-31}. The invariant equals $k = 1$.
Correction. "$F = 4GMu$" is not correct. In the regularised picture the linear restoring force has strength $-E/2$ per unit of z, and it does *not* involve GM. GM appears only as the conserved value of K.

Proposition 13.1.2 (Kustaanheimo–Stiefel [68]; theorem). *Let $L(u)$ be the KS matrix*

$$L(u) = \begin{pmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{pmatrix}, \qquad x = L(u)u.$$

Then $x_4 = 0$, $|x| = |u|^2$, and $L(u)L(u)^T = |u|^2 I$. The fibres of $u \mapsto (x_1, x_2, x_3)$ are the circles

$$\{\cos\phi\, u + \sin\phi\, Ju\}, \qquad Ju = (u_4, -u_3, u_2, -u_1), \quad J^2 = -1.$$

On the horizontal subspace (the bilinear relation), $|dx|^2 = 4|u|^2|du|^2 = 4r|du|^2$, and $dt = r\,ds$ again linearises Kepler into a 4-dimensional oscillator.

Checked to $4 \cdot 10^{-31}$ on random u, including the fibre invariance. In quaternionic form, $x = \bar{u}\,\mathbf{i}\,u$ up to the choice of axis: the Hopf map $S^3 \to S^2$ scaled by $|u|^2$. *Analogy (labelled):* the 2-to-1 map $z \mapsto z^2$ identifies z with $-z$. On units of $\mathbb{Z}[i]$ it sends $\{\pm 1, \pm i\}$ to $\{\pm 1\}$, so the 90° rotation becomes the KS/LC fibre symmetry of order 2.

13.2. (b) Discriminants and the wild prime 2

Proposition 13.2.1 (theorem). *(i) For monic f of degree n with roots α_j,*

$$\operatorname{disc} f = (-1)^{n(n-1)/2} \prod_j f'(\alpha_j), \qquad \operatorname{disc}(x^2 - d) = -(2\sqrt{d})(-2\sqrt{d}) = 4d.$$

Note the sign: $\prod f'(\alpha_j) = -4d$, so "disc = product of f' at the roots" is off by $(-1)^{n(n-1)/2}$.

(ii) $\operatorname{disc}(x^2{+}1) = -4 = d_{\mathbb{Q}(i)}$. *The different of $\mathbb{Z}[i] = \mathbb{Z}[x]/(x^2{+}1)$ is $(f'(i)) = (2i) = (1{+}i)^2$.*

(iii) $2 = -i(1+i)^2$ is the only ramified prime, with $e = 2 = p$. The different exponent is $2 \geq e$ (tame would be exactly $e - 1 = 1$), so the ramification is wild. *Equivalently, $d(z^2)/dz = 2z \equiv 0 \bmod 2$: modulo 2 the squaring map is the Frobenius, which is purely inseparable.*

(iv) By conductor–discriminant, $\chi_4 = \chi_{-4} = (\frac{-4}{\cdot})$ has conductor $|d_K| = 4$.

Correction. "Conductor $= 4|D|$" is wrong. For $K = \mathbb{Q}(\sqrt{D})$ with fundamental discriminant D, the quadratic character has conductor $|D|$, which is 4 for $\mathbb{Q}(i)$. The CM curve below has conductor $32 = |D| \cdot N_{K/\mathbb{Q}}(\mathfrak{f}_\psi) = 4 \cdot N((1+i)^3) = 4 \cdot 8$, by the conductor of an induced representation (consistent with LMFDB). It is not $4|D| = 16$.

13.3. (c) The CM curve $y^2 = x^3 - x$ and its Hecke character

E is LMFDB 32.a3 (Cremona 32a2): conductor 32, $j = 1728$, CM by $\mathbb{Z}[i]$, $\tau = i$. Its newform is 32.2.a.a, $f_{32} = \eta(4z)^2\eta(8z)^2$, with Sato–Tate group $N(U(1))$ [114].

Theorem 13.3.1 (Deuring [33, 34]; Ireland–Rosen [58, Ch. 18 §4]). *Let p be an odd prime.*

(i) If $p \equiv 3 \bmod 4$, then $a_p = 0$ and $E \bmod p$ is supersingular (p inert in $\mathbb{Q}(i)$, $\chi_4(p) = -1$).

(ii) If $p \equiv 1 \bmod 4$, write $p = \pi\bar{\pi}$ with $\pi = a + bi$ primary, $\pi \equiv 1 \bmod (2+2i)$. Equivalently, a is odd, b is even, and $a \equiv b + 1 \pmod 4$. Then $a_p = \pi + \bar{\pi} = 2a$. Explicitly: $a \equiv 1 \pmod 4$ if $4 \mid b$, and $a \equiv 3 \pmod 4$ if $b \equiv 2 \pmod 4$.

(iii) $\psi_E((\pi)) = \pi$ for primary π defines a Hecke Grössencharacter of $\mathbb{Q}(i)$ of conductor $(1+i)^3$ and ∞-type $z \mapsto z$. Then $L(E,s) = L(s, \psi_E)$, and $f_{32} = \sum_{\mathfrak{a}\ \text{prime to}\ 2} \psi_E(\mathfrak{a}) q^{N\mathfrak{a}} = \sum_{\pi\ \text{primary}} \operatorname{Re}(\pi)\, q^{|\pi|^2}$.

Part (ii) is [58, Ch. 18 §4] with $D = 1$: $\#E(\mathbb{F}_p) = p + 1 - \overline{(D/\pi)_4}\pi - (D/\pi)_4\bar{\pi}$. The congruence form follows from $\frac{a-1+bi}{2+2i} = \frac{(a-1+b)+i(b-a+1)}{4}$. *Check* (`num/lemma13_cm.out`):

- the rule versus the η-product: 0 mismatches over all 17 983 odd primes $< 2 \cdot 10^5$;
- direct point counts for $p < 3000$: 0 mismatches;
- the theta series of ψ_E versus $\eta(4z)^2\eta(8z)^2$: 0 mismatches for $n \leq 20\,000$;
- first values $a_5, a_{13}, a_{17}, a_{29}, a_{97} = -2, 6, 2, -10, 18$, as in LMFDB.

ANALOGY 13.3.2 (Weil's $|\alpha| = \sqrt{p}$ and the Levi-Civita coordinate; labelled analogy). For split p the Frobenius eigenvalue *is* the Gaussian integer π, so $|\alpha| = \sqrt{p}$ is literally $|\pi| = \sqrt{N\pi}$. This is a theorem, trivial in the CM case. In Levi-Civita, $|z| = \sqrt{r}$ with $r = |z^2|$. The common content is the *norm* $N(z) = z\bar{z} = |z^2|$. Frobenius is not the squaring map, and no dynamical statement is implied. For inert p one has $\alpha = \pm i\sqrt{p}$: the Frobenius angle is 90°, which is the point $\tau = i$ of Lemma 7.

Link to Lemma 7 (theorem). At the CM point, $E(i, s) = 2\zeta(s)L(s, \chi_{-4})/\zeta(2s)$. This is the Epstein zeta of the square lattice, and up to $\zeta(2s)$ it is $\zeta_{\mathbb{Q}(i)}(s) = L(s, \psi^0)$ (trivial character). The CM newform is $L(s, \psi_E)$, with ∞-type z. Hecke's angular characters $\lambda^m((\alpha)) = (\alpha/|\alpha|)^{4m}$ are well defined on ideals because units are 90° rotations, and $(\psi_E/|\psi_E|)^4 = \lambda$. Lemma 7 does not itself contain Hecke angular characters; the link is made here.

13.4. (d) The Newton-triangle dictionary: $\tan / \sec$, χ_4, $\beta(1) = \pi/4$

PROPOSITION 13.4.1 (theorem). $\beta(1) = L(1, \chi_{-4}) = \sum_{n\geq 0} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. *This equals the class number formula* $\frac{2\pi h}{w\sqrt{|d_K|}}$ *with* $h = 1$, $w = 4$, $|d_K| = 4$, *where* $w = 4$ *is the number of* 90° *rotations. More generally,* $\beta(2k + 1) = \frac{(-1)^k E_{2k}\pi^{2k+1}}{4^{k+1}(2k)!}$, *where the Euler (secant) numbers* E_{2k} *appear in* $\sec x + \tan x = \sum A_n x^n/n!$ *(zigzag numbers;* $A_{2k} = |E_{2k}|$*).*

In Lemma 7's Newton triangle, the boost through 90° is the point $\tau = i$. Its stabiliser $\{\pm 1, \pm S\} \subset SL_2(\mathbb{Z})$ is the unit group $\mathbb{Z}[i]^\times$ (order $4 = w$). This is the same w as in $\beta(1) = \pi/4$ and in $E(i, s)$. The identification of $\tan / \sec$ with Lemma 7's gd (where $\sec \operatorname{gd} u = \cosh u$ and $\tan \operatorname{gd} u = \sinh u$) is a change of variable, not a theorem about β.

13.5. (e) CM Sato–Tate versus non-CM

THEOREM 13.5.1 (Hecke [49, 50] + Dirichlet; theorem). *Let* $x_p = a_p/(2\sqrt{p}) = \cos\theta_p$ *for* $E : y^2 = x^3 - x$. *As* $p \to \infty$, *the* x_p *are distributed as*

$$\mu_{\mathrm{CM}} = \tfrac{1}{2}\delta_0 + \tfrac{1}{2}\,\frac{dx}{\pi\sqrt{1 - x^2}} \quad \text{on } [-1, 1].$$

The atom comes from the inert primes, which have density $\frac{1}{2}$ *and* $a_p = 0$ *(supersingular,* $\chi_4(p) = -1$*). The arcsine part comes from Hecke's equidistribution of the angles of primary Gaussian primes, together with* cos *of a uniform angle. In group terms,* μ_{CM} *is the trace measure of* $N(U(1))$*: Haar measure on* $U(1)$ *plus the trace-zero coset, half each.*

For non-CM curves the law is Sato–Tate, $\frac{2}{\pi}\sqrt{1 - x^2}\,dx$ (the trace of $SU(2)$), with no atom. This is a theorem by Clozel–Harris–Shepherd-Barron–Taylor [9, 46, 113] and Barnet-Lamb–Geraghty–Harris–Taylor [2].
Check (17 983 primes $< 2 \cdot 10^5$):

- $y^2 = x^3 - x$: zero fraction 0.50081; Kolmogorov distance to μ_{CM} is $2.80 \cdot 10^{-3}$, and to the semicircle 0.251;
- 11.*a* (a_p from $\eta(z)^2\eta(11z)^2$): zero fraction 0.0027; distance to the semicircle $4.2 \cdot 10^{-3}$, and to μ_{CM} 0.252.

These agree with the earlier `num/cm_satotate.py` figures (50.08%, $\sim 3 \cdot 10^{-3}$).

ANALOGY 13.5.2 (atomic/continuous split; parallel, not identity). The split of μ_{CM} into an atom and an absolutely continuous part runs *parallel* to the cusp/Eisenstein split of the spectrum of $SL_2(\mathbb{Z})\backslash\mathbb{H}$ (Lemma 7):

- Eisenstein series are induced from the split torus ($GL_1 \times GL_1$, pairs of Hecke characters of $\mathbb{Q}$) and give continuous spectrum;
- f_{32} is a genuine cusp form, but *dihedral*: the automorphic induction (theta lift) of ψ_E from the non-split torus $\mathrm{Res}_{K/\mathbb{Q}}GL_1$. In this sense it is the cusp form "closest to Eisenstein";
- a non-CM form is not induced at all.

The atom $\frac{1}{2}\delta_0$ is the non-identity coset of $N(U(1))$, i.e. the primes where the induced representation has trace 0. There is no claim that the atom "is" a discrete spectrum, nor that the Eisenstein continuum "is" the arcsine law.

13.6. Status, obstruction, questions

Claim	Status	Source/check
LC/KS: Kepler $\to$ oscillator, $\lvert dx/dz\rvert^2 = 4r$, $dt = r\,ds$	theorem	[**78**, **68**, **53**]; 10^{-31}
coupling-constant metamorphosis $E \leftrightarrow -\omega^2$, $k \leftrightarrow K$	theorem	[**53**]
"$F = 4GMu$"	corrected	Prop. 13.1.1
$\mathrm{disc} = (-1)^{n(n-1)/2}\prod f'$; 2 wild in $\mathbb{Q}(i)$	theorem	Prop. 13.2.1
"conductor $4\lvert D\rvert$"	corrected	χ_{-4}: 4; E: 32
sign rule $a_p = 2a$, $a \equiv b+1$ (4)	theorem	[**58**]; 17 983 primes
$f_{32} = \eta(4z)^2\eta(8z)^2 = \theta(\psi_E)$, LMFDB 32.2.a.a	theorem	[**114**]; $n \le 2\cdot 10^4$
$\lvert\alpha\rvert = \sqrt{p}$ "is" $\lvert z\rvert = \sqrt{r}$	analogy (norm)	–
CM Sato–Tate $\frac{1}{2}\delta_0 + \frac{1}{2}$arcsine	theorem	[**50**]; KS $2.8\cdot 10^{-3}$
atom $\leftrightarrow$ cusp/Eisenstein split	parallel only	–

Obstruction. Every rigorous item here is classical. The squaring map unifies the *norm* structure ($\lvert\pi\rvert^2 = p$, $\lvert z\rvert^2 = r$, $\mathrm{disc} = 4d$, the wild 2), but it produces no new positivity. For ζ itself there is no CM: no Hecke character makes the zeros into "angles" that are equidistributed by a group law.

Question 13.6.1. Should TAA treat the inert/supersingular atom at 90° (Frobenius $= \pm i\sqrt{p}$) as the arithmetic model of the elliptic point $\tau = i$ of Lemma 7? Or is the intended reading the KS fibre symmetry $z \mapsto -z$ (Prop. 13.1.2)? These are different structures.

APPENDIX A

Contact or foliation?

The transverse geometry of the arithmetic scaling flow

Summary. We ask whether the flow whose closed orbits are the primes (lengths $\log p$) is *foliation-like*, in the sense of Deninger, or *contact-like*, i.e. the Reeb flow of a contact form. Both cases arise from one construction. A flow that preserves a transverse hyperplane field H carries a canonical invariant 1-form ϑ. The two cases are the extremes $d\vartheta = 0$ and $\vartheta \wedge (d\vartheta)^n \neq 0$. In dimension 3 they are the cases $f \equiv 0$ and $f \neq 0$ of a stable Hamiltonian structure $d\vartheta = f\sigma$ (Thm. A.2.1).

We prove three things. On a closed manifold the two cases exclude each other: a Reeb flow has no closed 1-form positive on it (Thm. A.2.2). Deninger's period obstruction rules out compact smooth foliation-like models for ζ (Prop. A.3.1). The arithmetic sign rule $\varepsilon_\gamma(k) \equiv +1$ fails for every saddle-type orbit, so it fails for every Anosov flow in dimension 3. It holds for geodesic flows of odd-dimensional hyperbolic manifolds. In the signed ECH count it singles out *elliptic* Reeb orbits, which on closed contact 3-manifolds occur only in the two-orbit lens-space case (Hutchings–Taubes) (Thm. A.5.1, Prop. A.5.2).

Bottom line. Every framework known to reproduce the explicit formula with its signs is foliation-like, and it must be singular, laminated or non-compact. The contact side gives the analytic *shape* of the answer: unitarity, the prime-orbit law with entropy 1, and a critical line at half the unstable expansion rate. But it has the opposite Lefschetz sign, the wrong orbit weights, and time-reversal doubling. The contact and foliation sides are nevertheless two chain-level models of one invariant, Seiberg–Witten Floer homology, via ECH (Taubes) and PFH (Lee–Taubes). We separate theorems, conjectures and analogies throughout. Nothing here implies RH.

A.1. Setting

Throughout, M is a smooth manifold of dimension $2n+1$ and V is a nowhere vanishing vector field with flow ϕ^t. A *transverse invariant hyperplane field* is a rank-$2n$ subbundle $\mathsf{H} \subset TM$ with $TM = \mathbb{R}V \oplus \mathsf{H}$ and $d\phi^t(\mathsf{H}) = \mathsf{H}$ for all t. For a closed orbit γ of length $\ell(\gamma)$, a point $x \in \gamma$ and $k \geq 1$, put $P_\gamma^k = d\phi^{k\ell(\gamma)}|_{\mathsf{H}_x}$ and

$$\varepsilon_\gamma(k) = \operatorname{sgn}\det(1 - P_\gamma^k) \qquad \text{(Lefschetz sign; Deninger's } \varepsilon_\gamma \text{ [30, (8)])}.$$

Definition A.1.1. (a) *Foliation-like* (a Deninger system [32, 30]): $\mathsf{H} = T\mathcal{F}$ for a codimension-one foliation $\mathcal{F}$ whose leaves are transverse to the flow lines and mapped to leaves by every ϕ^t. (b) *Contact-like*: V is the Reeb field of a contact form ϑ, i.e. $\vartheta \wedge (d\vartheta)^n \neq 0$, $\vartheta(V) = 1$, $i_V d\vartheta = 0$. Then $\mathsf{H} = \ker\vartheta$. (c) *Anosov*: $TM = \mathbb{R}V \oplus E^s \oplus E^u$, invariant, with uniform contraction on E^s and expansion on E^u. A *contact Anosov* flow has $E^s \oplus E^u = \ker\vartheta$ with ϑ contact [38, 79].

Lemma A.1.2 (the invariant form). *Given* H, *let* ϑ *be the unique* 1*-form with* $\ker\vartheta = \mathsf{H}$ *and* $\vartheta(V) = 1$. *Then* $\phi^{t*}\vartheta = \vartheta$, $L_V\vartheta = 0$ *and* $i_V d\vartheta = 0$.

PROOF. The form $\phi^{t*}\vartheta$ vanishes on $(d\phi^t)^{-1}\mathsf{H} = \mathsf{H}$, and it takes the value $\vartheta(d\phi^t V) = \vartheta(V) = 1$ on V. So $\phi^{t*}\vartheta = \vartheta$. Cartan's formula gives $0 = L_V\vartheta = i_V d\vartheta + d(\vartheta(V)) = i_V d\vartheta$. □

LEMMA A.1.3 (closed basic forms are invariant). *If σ is a closed 2-form with $i_V\sigma = 0$, then $L_V\sigma = 0$. In particular no such $\sigma \neq 0$ satisfies $L_V\sigma = c\,\sigma$ with $c \neq 0$.*

PROOF. $L_V\sigma = d\,i_V\sigma + i_V d\sigma = 0$. □

A.2. The dichotomy, and why "both" fails on closed manifolds

THEOREM A.2.1 (foliation versus contact). *Let H, ϑ be as in Lemma A.1.2.*

(i) H is integrable $\iff d\vartheta = 0$. In this case ϑ is Deninger's form ω_ϕ [30, Construction 3].
(ii) H is a contact structure $\iff \vartheta \wedge (d\vartheta)^n$ vanishes nowhere. In that case V is the Reeb field of ϑ. Conversely, every Reeb flow arises this way with $\mathsf{H} = \ker\vartheta$.
(iii) The $(2n+1)$-form $\vartheta \wedge (d\vartheta)^n$ is ϕ-invariant. Hence M splits into an invariant open set where H is contact and an invariant closed set where it is not.
(iv) ($n = 1$.) Suppose σ is an invariant transverse area form ($i_V\sigma = 0$, $\sigma|_{\mathsf{H}} \neq 0$, $L_V\sigma = 0$). Then $d\vartheta = f\sigma$ with $Vf = 0$. If moreover $d\sigma = 0$, then (ϑ, σ) is a stable Hamiltonian structure in the sense of [55]. Its two extreme cases are $f \equiv 0$ (foliation, e.g. a mapping torus) and $f \equiv 1$ (contact).

PROOF. (i) For sections v, w of H we have $d\vartheta(v, w) = -\vartheta([v, w])$. By Frobenius, H is integrable iff $d\vartheta|_{\mathsf{H}\times\mathsf{H}} = 0$. Since $i_V d\vartheta = 0$ and $TM = \mathbb{R}V \oplus \mathsf{H}$, this is equivalent to $d\vartheta = 0$. (ii) Nondegeneracy of $d\vartheta|_{\mathsf{H}}$ is equivalent to $\vartheta \wedge (d\vartheta)^n \neq 0$. Then $\vartheta(V) = 1$ and $i_V d\vartheta = 0$ are the defining equations of the Reeb field. (iii) L_V is a derivation commuting with d, and $L_V\vartheta = 0$. (iv) $d\vartheta$ is a 2-form with $i_V d\vartheta = 0$, and in dimension 3 any such form is a multiple of σ. Since $L_V d\vartheta = dL_V\vartheta = 0$ and $L_V\sigma = 0$, we get $Vf = 0$. The conditions of [55] are $d\sigma = 0$, $\vartheta \wedge \sigma > 0$ and $\ker\sigma \subset \ker d\vartheta$; here $\ker\sigma = \mathbb{R}V$. □

THEOREM A.2.2 (mutual exclusion on closed manifolds). *Let M be closed and let V be the Reeb field of a contact form ϑ. Then there is no closed 1-form ω with $\omega(V) > 0$ everywhere. Consequently:*

(a) a Reeb flow on a closed manifold admits no global cross-section;
(b) no flow on a closed manifold is simultaneously a Reeb flow and a Deninger system (Def. A.1.1(a));
(c) in a contact Anosov flow the invariant codimension-one foliations (weak stable and weak unstable) contain the flow lines. The only invariant transverse hyperplane field $E^s \oplus E^u$ is the contact structure itself, which is not integrable.

PROOF. Write $\omega = \omega(V)\vartheta + \beta$ with $\beta(V) = 0$. The top-degree form $\beta \wedge (d\vartheta)^n$ satisfies $i_V(\beta \wedge (d\vartheta)^n) = 0$ because $i_V d\vartheta = 0$, so it vanishes. Hence $\omega \wedge (d\vartheta)^n = \omega(V)\,\vartheta \wedge (d\vartheta)^n$, which is a positive multiple of the contact volume. But $d\omega = 0$ gives $\omega \wedge (d\vartheta)^n = \omega \wedge d(\vartheta \wedge (d\vartheta)^{n-1}) = -d(\omega \wedge \vartheta \wedge (d\vartheta)^{n-1})$, whose integral over the closed manifold M is 0. This is a contradiction. (a) A cross-section gives such an ω (Fried [41], Schwartzman [102]). (b) A Deninger system has the closed form $\vartheta = \omega_\phi$ with $\vartheta(V) = 1$ by Thm. A.2.1(i). (c) The weak foliations contain V by definition, and $E^s \oplus E^u = \ker\vartheta$ is contact. □

REMARK A.2.3. Thm. A.2.2 is an instance of Schwartzman's asymptotic-cycle criterion [102, 41]. The asymptotic cycle of the contact volume is zero because $i_V(\vartheta \wedge (d\vartheta)^n) = (d\vartheta)^n$ is exact. So "both à la Anosov" is impossible in Deninger's sense. In an Anosov flow the codimension-one foliations contain the flow, while Deninger requires leaves transverse to it.

A.3. Lengths: the period obstruction and the function-field case

PROPOSITION A.3.1 (period obstruction; Deninger [30]). *Let $(M, \mathcal{F}, \phi)$ be foliation-like with M closed. Then every closed orbit satisfies $\ell(\gamma) = \int_\gamma \vartheta = \langle [\vartheta], [\gamma] \rangle \in \Lambda_{\mathrm{per}}$, where $\Lambda_{\mathrm{per}} := \langle [\vartheta], H_1(M;\mathbb{Z}) \rangle \subset \mathbb{R}$ is a finitely generated group of rank $\le b_1(M)$. Since the numbers $\log p$ are linearly independent over $\mathbb{Q}$ (unique factorization), the primitive lengths contain $\log p$ for at most $b_1(M)$ primes. Λ_{per} has rank one iff $\mathcal{F}$ is a fibration over a circle* [30, Construction 3 and the remark after it].

PROOF. ϑ is closed by Thm. A.2.1(i), and $\vartheta(V) = 1$ gives $\int_\gamma \vartheta = \ell(\gamma)$. If $\sum c_i \log p_i = 0$ with $c_i \in \mathbb{Z}$ not all zero, then $\prod p_i^{c_i} = 1$, which contradicts unique factorization. □

Deninger draws the conclusion himself: "if one wants an infinitely generated Λ, one must allow the flow to have fixed points (= infinite places)" [30]. For a Reeb flow there is no such constraint, since $\int_\gamma \vartheta$ is an *action*, not a period ($d\vartheta \neq 0$).

COROLLARY A.3.2 (rank one ⇔ periodic zeta). *If $\Lambda_{\mathrm{per}} = c\,\mathbb{Z}$, then $\zeta_{\mathrm{R}}(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)})^{-\varepsilon_\gamma}$ is a function of e^{-cs}, hence $2\pi i/c$-periodic. This is the function-field picture: $c = \log q$, and the zero set has period $2\pi/\log q$ (Lemma 3). In contrast, $\zeta(s + iT) \equiv \zeta(s)$ forces $n^{-iT} = 1$ for all n by uniqueness of Dirichlet coefficients, hence $T = 0$.*

So: *the function-field case is exactly the integrable, rank-one (suspension) case.* Number fields need infinite rank, which smooth compact Deninger systems cannot supply.

A.4. Transverse expansion: where "1" and "$\frac{1}{2}$" come from

PROPOSITION A.4.1.
(a) (Reeb.) $\phi^{t*} d\vartheta = d\vartheta$, so every linearized return map P_γ is symplectic on $(\mathsf{H}_x, d\vartheta)$, and $\det P_\gamma = 1$.
(b) (Deninger, compact.) Let $\dim M = 3$, let M be closed and foliation-like, and let $g_{\mathcal{F}}$ be a leafwise metric with $\phi^{t*} g_{\mathcal{F}} = e^{at} g_{\mathcal{F}}$. Then $a = 0$.
(c) (What arithmetic needs.) In Deninger's transversal index theorem [30, Thm. 5] *the spectrum ρ of θ on $\bar{H}^1_{\mathcal{F}}$ satisfies $\mathrm{Re}\,\rho = a/2$, and the formula contains the term e^{at} (the analogue of the pole). Matching the pole of ζ at $s = 1$ and $\mathrm{Re}\,\rho = \frac{1}{2}$ requires $a = 1$. Deninger realizes $a = 1$ only on solenoidal spaces, for elliptic curves over $\mathbb{F}_q$* [30]. *Leichtnam proves $\mathrm{Re} = \frac{1}{2}$ for a class of p-adic laminations* [77].

PROOF. (a) Lemma A.1.2. (b) The leafwise area form ω_g scales as e^{at}, and ϑ is invariant. So the volume form $\vartheta \wedge \omega_g$ scales as e^{at}. Since ϕ^t is isotopic to the identity, $\int_M \phi^{t*}(\vartheta \wedge \omega_g) = \int_M \vartheta \wedge \omega_g > 0$, so $e^{at} = 1$. (Deninger notes that the hypotheses of his Thm. 5 force $a = 0$ [30]. Lemma A.1.3 gives another reason: the scaling can live only in leafwise cohomology, never on a closed basic 2-form.) □

REMARK A.4.2 (the contact replacement of $a = 1$; theorems). For contact Anosov flows the "1" and the "$\frac{1}{2}$" come from the unstable direction:

- The first pole of the Ruelle zeta function is at $s = h_{\mathrm{top}}$, and $\#\{\gamma : \ell(\gamma) \le L\} \sim e^{h_{\mathrm{top}} L}/(h_{\mathrm{top}} L)$ for weak-mixing flows (Parry–Pollicott [90]). For curvature -1 surfaces $h_{\mathrm{top}} = 1$, which is exactly the prime number theorem in the variable $L = \log x$ (numerics: `num/contact_check.out`, §6).
- Faure–Tsujii [38, 39]: the Ruelle–Pollicott resonances of $-V + V$ lie asymptotically in vertical bands. The first band concentrates on $\mathrm{Re}\, z = \langle V - \frac{1}{2} \operatorname{div} V|_{E^u} \rangle$. For curvature -1 and $V = 0$ this is the line $\mathrm{Re}\, z = -\frac{1}{2}$, which carries the Laplace spectrum via $z = -\frac{1}{2} \pm i\sqrt{\mu - \frac{1}{4}}$ [38, 35].

Analogy (not a theorem): the critical abscissa is half of the unstable volume growth rate. In Deninger's picture both leaf directions expand at rate $a/2$ (conformal). In the contact Anosov picture one direction expands at rate 1, the other contracts at rate 1, and the $\frac{1}{2}$ is the half-density twist.

A.5. Signs: the arithmetic sign rule singles out non-saddle orbits

The explicit formula, written as Deninger's Lefschetz formula, needs $\varepsilon_\gamma(k) = +1$ for *every* prime orbit and *every* multiplicity k, since $\zeta = \prod_p (1-p^{-s})^{-1}$ [**30**, (5),(8)].

THEOREM A.5.1 (Lefschetz signs). *Let $P \in GL(\mathsf{H}_x)$ be the return map of a closed orbit, nondegenerate at multiplicity k.*

- *(i) If P has no real eigenvalues, then $\varepsilon(k) = +1$ for every k.*
- *(ii) For even k, $\varepsilon(k) = (-1)^N$, where N is the number of real eigenvalues of P (with multiplicity) of modulus > 1. In particular, a real saddle in $\dim \mathsf{H} = 2$ has $\varepsilon(k) = -1$ for all even k. It has $\varepsilon(k) = -1$ for all k if its eigenvalues are positive.*
- *(iii) Every Anosov flow on a 3-manifold, contact or not, violates the arithmetic sign rule at every closed orbit (at $k = 2$).*
- *(iv) Let M^n be a closed oriented hyperbolic n-manifold with geodesic flow on T^1M. This flow is contact Anosov, with $\dim T^1M = 2(n-1)+1$ and $h_{\mathrm{top}} = n-1$. Then $\varepsilon_\gamma(k) = (-1)^{n-1}$ for all γ, k. So the arithmetic sign rule holds iff n is odd, i.e. iff the "dimension" $d := n-1 = h_{\mathrm{top}}$ is even.*

PROOF. $\det(1-P^k) = \prod_\mu (1-\mu^k)$ over the eigenvalues μ. Each non-real conjugate pair contributes $|1-\mu^k|^2 > 0$, which proves (i). A real μ contributes $1-\mu^k$, which for even k is negative iff $|\mu| > 1$; this proves (ii). For (iii), the return map on $E^s \oplus E^u$ is a real saddle. For (iv), $P^k = e^{k\ell} R^k \oplus e^{-k\ell} R^k$ on $E^u \oplus E^s$, where $R \in SO(n-1)$ is the holonomy. Write $m = n-1$. For $Q \in SO(m)$ and $c > 0$, $c \neq 1$, $\det(1-cQ)$ has sign $\operatorname{sgn}(1-c)^m$: conjugate pairs and eigenvalues -1 give positive factors, and the number of eigenvalues $+1$ is $\equiv m \pmod 2$. Hence the E^u factor has sign $(-1)^m$ and the E^s factor has sign $+1$. □

Numerical confirmation is in `num/contact_check.out`, §§1–3b. Part (iii) is the geometric form of the "crucial negative sign" that led Connes to interpret the zeros as an *absorption* spectrum [**22**]. Connes also observed that the semiclassical weight $1/(2\sinh(m\lambda_p/2))$ differs from $p^{-m/2}$, and that time-reversal symmetry, which every geodesic flow has, would double the orbit count [**22**]. McMullen [**84**] shows that closed geodesics of negatively curved surfaces in $T^1\Sigma$ obey the Chebotarev law, so these knots behave like primes *statistically*, independently of the signs.

The ECH count: elliptic = bosonic, positive hyperbolic = fermionic. For a nondegenerate contact form on a closed 3-manifold, the generators of embedded contact homology are the orbit sets $\{(\alpha_i, m_i)\}$ of distinct embedded Reeb orbits with $m_i = 1$ whenever α_i is hyperbolic [**56**]. The Lefschetz sign of an orbit is $+1$ if it is elliptic or negative hyperbolic and -1 if it is positive hyperbolic [**56**]. The canonical $\mathbb{Z}/2$-grading is the parity of the number of positive hyperbolic α_i [**56**]. Define the formal signed series

$$\chi_{\mathrm{ECH}}(s) := \sum_\alpha (-1)^{\#\{i:\alpha_i \text{ pos. hyp.}\}} e^{-s\mathcal{A}(\alpha)} = \prod_{e \text{ ell}} \frac{1}{1-e^{-s\mathcal{A}(e)}} \prod_{h \text{ pos}} (1-e^{-s\mathcal{A}(h)}) \prod_{h \text{ neg}} (1+e^{-s\mathcal{A}(h)}),$$

with $\mathcal{A}(\alpha) = \sum m_i \int_{\alpha_i} \vartheta$. These are exactly the multiple-cover generating functions of [**56**, Lemma 2.11].

PROPOSITION A.5.2. *(a) (Combinatorial identity.) If the embedded orbits were all elliptic with actions $\{\log p\}$, each once, the ECH generators would be in bijection with $n \in \mathbb{N}$,*

with $\mathcal{A} = \log n$, and $\chi_{\mathrm{ECH}} = \zeta$. This is Julia's bosonic primon gas. If they were all positive hyperbolic with actions $\{\log p\}$, the generators would be the squarefree n, with sign $\mu(n)$, and $\chi_{\mathrm{ECH}} = \sum \mu(n) n^{-s} = 1/\zeta$.

(b) (Obstruction.) On a closed 3-manifold, a nondegenerate contact form all of whose orbits are elliptic has exactly two embedded orbits, and the manifold is a lens space (Hutchings–Taubes [55, Thm. 1.2]). So the elliptic realization in (a) is impossible on closed contact 3-manifolds.

(c) (What we do realize.) For the geodesic flow of a closed hyperbolic surface every orbit is positive hyperbolic, and $\chi_{\mathrm{ECH}}(s) = \prod_\gamma (1 - e^{-s\ell(\gamma)}) = Z_\Gamma(s)/Z_\Gamma(s+1)$, the fermionic, Möbius-type side.

PROOF. (a) Unique factorization. (b) is [55, Thm. 1.2]. (c) Thm. A.5.1(iv) with $n = 2$, together with $Z_\Gamma(s) = \prod_\gamma \prod_{j\geq 0}(1 - e^{-(s+j)\ell(\gamma)})$. □

REMARK A.5.3 (both sides compute one invariant; theorem). ECH of a contact 3-manifold ($f \equiv 1$) is isomorphic to Seiberg–Witten Floer cohomology (Taubes [111]). Periodic Floer homology of a mapping torus ($f \equiv 0$, the foliation-like case) is also isomorphic to it (Lee–Taubes [76]). So at the level of homology the foliation-like and contact-like models are two chain-level presentations of one 3-manifold invariant. They differ in what their generators see: Lefschetz-signed closed orbits, with multiplicity rules that are bosonic for elliptic and fermionic for hyperbolic orbits.

A.6. The Selberg side: what contact flows prove, and where ζ enters

PROPOSITION A.6.1. *(a) No closed hyperbolic surface has primitive length multiset $m \cdot \{\log p\}$ for any $m \geq 1$.*

(b) For $\Gamma = PSL_2(\mathbb{Z})$, the Selberg zeta function vanishes at $s = \rho/2$ for every nontrivial zero ρ of ζ. These zeros come from the scattering determinant of the cusp [60]; see the list in [44]. Under RH they lie on $\operatorname{Re} s = \frac{1}{4}$, off the unitary line $\operatorname{Re} s = \frac{1}{2}$ that carries the cusp-form spectrum.

(c) (Unitarity versus temperedness.) For cocompact Γ, the zeros of Z_Γ in $0 < \operatorname{Re} s < 1$ are $s = \frac{1}{2} \pm i r_j$ with $\lambda_j = \frac{1}{4} + r_j^2$, which come from the principal series, together with finitely many real $s \in (\frac{1}{2}, 1]$ with $\lambda_j < \frac{1}{4}$, which come from the complementary series and the trivial representation [103, 51]. Unitarity alone allows the real exceptional zeros; temperedness excludes them. For congruence groups, Selberg's $\frac{1}{4}$-conjecture is the "RH". The best bound known is $\lambda_1 \geq \frac{1}{4} - (\frac{7}{64})^2$ (Kim–Sarnak, appendix to [65]).

PROOF. (a) For $\operatorname{Re} s > 1$, $Z_\Gamma(s) = \prod_p \prod_{j\geq 0}(1 - p^{-s-j})^m = \prod_{j\geq 0} \zeta(s+j)^{-m}$; both sides are meromorphic, so the identity persists. For $j \geq 1$, $\zeta(\rho + j)$ is finite and nonzero because $\operatorname{Re}(\rho + j) > 1$, and $\prod_{j\geq 1} \zeta(s+j)^{-m}$ converges near ρ. So Z_Γ has a pole at ρ. But Z_Γ is entire for cocompact Γ [103, 51]. (Numerical check of the product identity: `num/contact_check.out`, §5.) For $m = 1$, a second contradiction comes from time reversal: γ and γ^{-1} are distinct primitive classes of the same length. (b), (c): cited. □

REMARK A.6.2. Part (b) is the one place where Riemann zeros literally sit in the spectral data of a contact (geodesic) flow. They appear as cusp-scattering resonances, not as unitary eigenvalues. This is taken up in Chapters 7 and 8.

A.7. Status of the arithmetic models

Model	Transverse type	Status
Curves over $\mathbb{F}_q$ (suspension, Lemma 3)	foliation, $\Lambda_{\mathrm{per}} = \mathbb{Z}\log q$	theorem; zero set $2\pi/\log q$-periodic (Cor. A.3.2)
Deninger solenoids, elliptic curves over $\mathbb{F}_q$ [30]	foliation (laminated), $a = 1$	constructed
Leichtnam p-adic laminations [77]	foliation (laminated)	trace formula; $\mathrm{Re} = \frac{1}{2}$ on $\bar{H}^1$ proved (under hypotheses)
Álvarez López–Kordyukov–Leichtnam [1]	codim-1 foliation with preserved compact leaves	Deninger's trace formula proved; "further generalization needed for arithmetic"
Kopei [67]	foliation, compact non-transverse leaves	geometric product formula; compact leaves $\leftrightarrow$ infinite primes
Connes adele class space [22]	transversal k_v at H_v; Guillemin–Sternberg weights $\lvert 1-u\rvert_v^{-1}$	S-local trace formula proved; global formula $\Leftrightarrow$ RH
Connes–Consani scaling site [17, 12, 18]	topos; periodic orbits of length $\log p$	points $= \mathcal{Y}$
Geodesic flows (Selberg)	contact Anosov	RH-type from unitarity; wrong sign, weights, doubling (§§5–6)

Observation. In Connes' picture the transversal at the closed orbit H_v is the local field k_v, which is one-dimensional over k_v [22]. A hyperplane field in a $(1+1)$-dimensional local model is a line field and is automatically integrable, so the contact/foliation question is vacuous *place by place*. Non-integrability can arise only from the coupling between places, or from the complex places of a number field (real dimension 2). For $\mathbb{Q}$ there is no complex place.

A.8. Bottom line

(1) **Foliation-like, and necessarily singular.** Every framework reproducing the explicit formula *with its signs* is foliation-like: Deninger, Leichtnam, Álvarez López–Kordyukov–Leichtnam, Kopei, and the Weil–Lefschetz form of Connes. The zeros sit in odd-degree leafwise cohomology, the absorption spectrum. Smooth compact versions are impossible (Prop. A.3.1, Prop. A.4.1(b)). This forces fixed points, which are the infinite places, or p-adic/solenoidal transversals.

(2) **Not "both à la Anosov."** On closed manifolds, Reeb flows and Deninger systems exclude each other (Thm. A.2.2). In Anosov flows the codimension-one invariant foliations contain the flow.

(3) **Contact gives the analytic shadow.** Volume preservation gives unitarity; the orbit law with $h_{\mathrm{top}} = 1$ is the PNT; the critical line sits at half the unstable rate (Faure–Tsujii); and RH for Z_Γ follows from unitarity of $L^2(\Gamma\backslash G)$, up to complementary-series exceptions. But in dimension 3 the Lefschetz sign is wrong at every orbit (Thm. A.5.1(iii)). The arithmetic sign rule singles out elliptic Reeb orbits, which closed contact 3-manifolds cannot supply in infinite number (Prop. A.5.2(b)).

(4) **Undecided:** laminated or non-compact contact models, where Thms. A.2.2, A.3.1 and [55] do not apply. See Q1.

A.9. Questions for parallel work

Question A.9.1 (contact laminations with elliptic prime orbits). Is there a lamination whose leaves (path components) are contact 3-manifolds, with a totally disconnected (p-adic or solenoidal) transversal, carrying a leafwise Reeb flow whose primitive closed orbits have actions

exactly $\{\log p\}$, each once, and are all elliptic? Then $\chi_{\mathrm{ECH}} = \zeta$ formally (Prop. A.5.2(a)). Hutchings–Taubes forbids this on closed 3-manifolds, where only two orbits occur and the manifold is a lens space. The test case: towers of irrational ellipsoids or lens spaces, each carrying two elliptic orbits, and the inverse limit of such a tower.

QUESTION A.9.2 (the stable-Hamiltonian function f of arithmetic). In $d\vartheta = f\sigma$, the foliation-like ($f = 0$) and contact-like ($f \neq 0$) regions are invariant (Thm. A.2.1(iv)). In the Álvarez López–Kordyukov–Leichtnam setting the flow is transverse to $\mathcal{F}$ except at finitely many preserved compact leaves, which are Kopei's infinite primes. Is there a stable Hamiltonian structure with $f = 0$ away from neighbourhoods of these leaves and $f \neq 0$ near them, so that "the archimedean place is the contact region"? Can the trace formula of [**1**] be extended to that case? Both chain models, PFH for $f = 0$ and ECH for $f = 1$, compute Seiberg–Witten Floer homology [**76, 111**].

QUESTION A.9.3 (Lax–Phillips on the modular surface). On $PSL_2(\mathbb{Z})\backslash\mathbb{H}$ the Riemann zeros appear as zeros of Z_Γ at $\rho/2$ (Prop. A.6.1(b)). Is there a natural Hilbert space, from the cusp and the Eisenstein series in the spirit of Lax–Phillips and Pavlov–Faddeev, on which these points become the spectrum of a generator whose skew-adjointness is equivalent to RH, *with* the absorption sign? Chapters 7 and 8 answer the scalar version of this question.

Further context (cited, not used in proofs)

Arithmetic topology: $\operatorname{Spec}\mathbb{Z}$ has étale cohomological dimension 3 (Artin–Verdier), primes behave like knots [**85, 29**], and Kim's arithmetic Chern–Simons theory [**66**] builds on this. Contact Anosov flows beyond geodesic flows exist on hyperbolic 3-manifolds [**40**], and they are exponentially mixing [**79**]. Plante [**92**]: if E^s and E^u are jointly integrable, the Anosov flow is a suspension of an Anosov diffeomorphism, which is the foliation-like case of Thm. A.2.1(i). Tischler [**115**]: a closed nonsingular 1-form can be perturbed to fibre the manifold over S^1. Taubes [**112**]: every Reeb flow on a closed 3-manifold has a closed orbit. Dyatlov–Zworski [**35**]: meromorphic continuation of Ruelle zeta for all smooth Anosov flows. Deninger [**31**] constructs $\mathbb{R}$-dynamical systems attached to arithmetic schemes, with closed orbits corresponding to closed points.

APPENDIX B

Rigor audit

This appendix classifies every numbered statement of Chapters 1–13 and Appendix A (Chapters 12 and 13 audited in the referee pass of 21 October 2026). The classification is by the weakest ingredient needed for the *whole* statement; where parts differ, the note says so. Codes: P proof complete in the text; C complete given cited theorems; X computer-assisted, certified (interval arithmetic or exact); N numerical evidence only; H heuristic, analogy or conditional on an N/H item; S proof sketched or containing a gap; D definition; Q question; R remark; E example.

B.1. Summary

Code	Count	Meaning
P	50	proof complete
C	27	rigorous, rests on cited theorems
X	5	computer-assisted, certified
S	0	proof sketched or gap
N	4	numerical
H	16	heuristic / analogy / conditional
D	11	definition
Q	18	question
R	16	remark (commentary)
E	2	example
Total	149	numbered statements

Of the 102 statements that assert something (codes P,C,X,S,N,H), 82 are rigorous (P,C,X), 0 have a sketched proof or a gap (S), and 20 are numerical, heuristic or conditional (N,H).

B.2. Statement-by-statement table

No.	Type	Name	St.	Note
1.1.1	definition	Test class	D	density in the Weil completion proved inline
1.1.2	lemma		P	
1.1.3	definition	Weil form	D	convergence proved
1.1.4	remark	TAA dictionary	H	physical analogy (Nash–Stelle ghosts); labelled, not used
1.2.1	theorem	Arithmetic unitarity	C	(i) explicit formula cited; (ii)–(vi) complete proofs
1.3.1	lemma		P	
1.3.2	lemma		P	
1.3.3	corollary	Pontryagin structure if RH fails mildly	C	Pontryagin/IKL theory cited
2.1.1	definition	Admissible spectral data	D	

No.	Type	Name	St.	Note
2.1.2	example		E	
2.1.3	theorem	Lemma 1, abstract form	P	proof by inspection of Ch. 1
2.2.1	definition	Heat-deformed spectral data	D	
2.3.1	theorem	Heat-flow unitarity	C	de Bruijn, Newman, Rodgers–Tao, Polymath
2.3.2	example	κ is not monotone in general	E	
2.4.1	lemma	Laguerre-type step	P	
2.4.2	lemma	Polynomial second law	P	
2.4.3	lemma	Continuity of the heat operator	P	
2.5.1	question	Dissipation identity; candidate Lemma 3	Q	contains a formal (unjustified) computation; labelled
3.1.1	definition	Spectral data of X	D	
3.1.2	lemma		P	
3.2.1	theorem	Arithmetic unitarity for curves	C	Weil conjectures for curves; Hodge index
3.3.1	remark	Comparison with ζ and the archive errors	R	
4.2.1	theorem	Iwasawa pillar	C	Mazur–Wiles, Ribet, Deligne; D3, D5 exact machine arithmetic (X)
4.4.1	remark	Honest limits	R	
5.1.1	definition	Configuration space	D	
5.1.2	definition	Fields	D	
5.1.3	lemma	Closed orbits $\Leftrightarrow$ product formula	C	(d) class field theory
5.1.4	remark		R	
5.2.1	definition	TAA action	D	
5.2.2	theorem	Stationary points of $\mathcal{S}$	P	
5.2.3	remark	Constrained version	R	
5.2.4	proposition	Local Hamiltonian model: $\delta A = 0 \Leftrightarrow$ closed orbit	P	
5.2.5	proposition	Orbit weights vs. Gutzwiller/Selberg	P	
5.3.1	theorem	Isometry group of $\mathcal{Q}$ in Diff($\mathbb{R}$)	P	off-diagonal smoothness proved in one line (G11 closed)
5.4.1	theorem	Curve case: structural positivity via Rosati	C	Rosati positivity (Mumford)
5.4.2	question	The Hodge-index question for ζ, in TAA form	Q	equivalent to RH; text said "Question 5.2" (fixed)
5.4.3	proposition	Reflection positivity fails at every closed orbit	P	
5.5.1	proposition	Primon gas = Bost–Connes Hamiltonian	C	(iv) cited [7]
5.5.2	remark	Zero-point energy: what is and is not true	H	super-regularization power rule unverified: CUT or move to heuristics
5.5.3	remark	The "$m_g = m_i$" duality (speculative): what it would require	H	speculative: CUT or shorten
5.5.4	remark	Non-canonical choices, flagged	R	
6.3.1	theorem	rigorous; elementary	P	

No.	Type	Name	St.	Note
6.4.1	definition		D	
6.4.2	theorem	structural trace; rigorous	P	Tonelli/Plancherel steps written out; (i) independent of 6.5.1(b) (G2 closed)
6.4.3	lemma	oscillatory bounds; rigorous	P	explicit constants; numerical cross-check in supplement
6.5.1	theorem		P	(a) remainder corrected (referee); (b) full proof with rate $O(\lvert s\rvert^{-1/4}\log^{1/2}\lvert s\rvert)$; (c) via Bohr means and Wiener (G1 closed)
6.5.2	remark		R	
6.6.1	observation	numerical	N	$A(p)$ fitted to $\pm 3\%$
6.6.2	corollary	conditional on Obs.	H	conditional on Obs. 6.6.1
6.7.1	theorem	rigorous; elementary, likely folklore, no reference found	P	
6.7.2	remark	where it degenerates	H	growth rate of κ_A asserted, not proved
7.2.1	proposition	normalizations; classical, verified	C	classical normalizations; $\gamma_\rho \to \operatorname{Im}\rho$ in (e) (volume edit)
7.3.1	theorem	Lax–Phillips	C	quoted from LP
7.3.2	theorem	Pavlov–Faddeev	C	quoted from PF
7.4.1	theorem	Lemma 7	C	(ii) volume proof $\lVert Z_t\rVert = 1$ via functional model + Conrey; LP67 criterion no longer used (G3 closed)
7.4.2	remark	what (iv) means	R	
7.5.1	proposition	time dictionary; rigorous at the level of spectra	P	at the level of spectra
7.5.2	proposition	where the primes go	C	PNT, Hadamard product; $\gamma_\rho \to \operatorname{Im}\rho$ (volume edit)
7.5.3	remark	honest answer to "does scattering avoid Lemma 6's obstruction?"	H	formal
7.5.4	remark	absorption sign and temperedness	H	analogy
7.6.1	proposition	boosted square lattice; new as stated, elementary	P	
7.9.1	question	intertwiner	Q	answered in Ch. 8
7.9.2	question	Lemma 6 at abscissa 1	Q	answered in Ch. 9
8.1.1	lemma	standard model-space facts; elementary	C	Sz.-Nagy–Foias
8.2.1	definition		D	$t = \log y$ re-derived for the zero mode (G6 closed); full PF equivalence cited
8.2.2	proposition	elementary properties; rigorous	P	
8.2.3	remark	why the LP norm grows	R	
8.2.4	theorem	zero-free intertwiner; rigorous, unconditional	P	
8.3.1	theorem	Krein-diagonalization; rigorous for simple zeros	P	simple zeros
8.3.2	remark	size of the entries; explicit inequality	R	explicit inequality, constant 41.35 (G5 closed)
8.4.1	theorem	incomparability	C	(a) Thm 7.4.1(ii); (b) Conrey $\frac{2}{5}$ + explicit Rem. 8.3.2 + convexity; (c) under RH

No.	Type	Name	St.	Note
8.5.1	theorem	rigorous; reduces Question 1 to Lemma 6	P	last sentence superseded (ed. note)
8.5.2	proposition	the sharpest true comparison: a Gårding inequality; rigorous	P	
8.8.1	question		Q	rephrased in the volume
8.8.2	question		Q	
9.2.1	theorem	rigorous	P	
9.2.2	theorem	full Euler product at abscissa one; rigorous	X	sign $d(\gamma_1) < 0$ certified in Arb (G8 closed); rest rigorous, PF cited
9.2.3	remark	what Thm. says and does not say	R	
9.3.1	observation	semilocal law; numerical, `num/lemma9_kappa.out...`	N	
9.3.2	corollary	conditional on Obs.	H	conditional on Obs. 9.3.1
9.4.1	proposition	rigorous	C	(e) Lagarias
9.5.1	proposition	rigorous	P	
9.6.1	proposition	rigorous, computer-assisted	X	Arb 200 bits, Gershgorin; 12-dim span only
9.6.2	remark	what is *not* proved	R	abstract (vi) superseded by Ch. 10
9.9.1	question		Q	
9.9.2	question		Q	
10.1.1	lemma	frequency form; rigorous	P	
10.1.2	remark		R	dangling sentence deleted (G12 closed)
10.1.3	lemma	monotone envelope; rigorous, elementary	P	optimality proved via Dirichlet approximation (G14 closed)
10.2.1	theorem	reduction; rigorous	P	
10.2.2	remark	sharpest explicit reduction	R	asymptotic sizes, heuristic
10.3.1	theorem	rigorous, computer-assisted	X	Arb 160 bits; frequency split; Cholesky+Weyl; tail ratio $\leq \frac{1}{2}$ proved in text, conditions logged; code in supplement (G7 closed)
10.3.2	corollary	TAA: unconditional positivity on Lemma 8's cusp-truncated model space	X	from 10.3.1 + Thm 8.5.1
10.4.1	remark	the barrier is Lemma 8's Gårding constant	H	asymptotic estimate of barrier
10.6.1	question		Q	
10.6.2	question		Q	
11.1.1	proposition	TAA form of the law; exact algebra	P	exact reformulation of a *conjecture*
11.1.2	proposition	rigorous, unconditional	P	
11.2.1	observation	numerical	N	Galerkin quadrature not certified
11.2.2	heuristic	why $2\pi^2$: a derivation of Zhu's constant	H	heuristic derivation of $2\pi^2$
11.3.1	lemma	σ-ghost identity; rigorous	P	
11.3.2	proposition	uniform margin for $\sigma > \frac{1}{2}$; rigorous	P	limit proved (G4 closed); hypothesis $\sigma > \beta_\rho\ \forall \rho$ (referee)
11.3.3	observation	numerical: location of the infimum	N	location of $\inf w_\sigma$; not used
11.3.4	remark	dichotomy	H	interpretive
11.4.1	lemma	quadruple identity; exact	P	

No.	Type	Name	St.	Note
11.4.2	proposition	index bound; rigorous	P	
11.4.3	proposition	invisibility threshold; rigorous	P	second paragraph uses the conjectural law
11.4.4	heuristic	detection threshold; revised against Table	H	sampling upper bound “routine, not written out”
11.6.1	question		Q	
11.6.2	question		Q	
12.2.1	lemma	exact; numerically confirmed	P	exact identity; numerics confirm
12.2.2	lemma	decay; rigorous	P	
12.3.1	lemma	Abel majorant; rigorous	P	majorant jumps checked in Arb
12.4.1	theorem	explicit detection length; rigorous	P	implication proved; its numerical rows (Table 12.1) are X; $\delta = 0.5$ column formal, outside the theorem
12.4.2	remark	shape and honesty	R	
12.5.1	corollary	rigorous	P	
12.5.2	proposition	vacuity today	X	rests on Table 12.1; Zhu certified window 1.6 (unrefereed), corrected
12.6.1	proposition	conditional-rigorous lower bound	H	implication proved; hypothesis is the conjectural margin law
12.6.2	proposition	rigorous	P	Poisson summation; interlacing
12.6.3	heuristic	...	H	
12.7.1	question		Q	
12.7.2	question		Q	
13.1.1	proposition	Levi-Civita ; theorem	C	Levi-Civita; checked to 10^{-31}
13.1.2	proposition	Kustaanheimo–Stiefel ; theorem	C	Kustaanheimo–Stiefel
13.2.1	proposition	theorem	P	discriminant, different; elementary
13.3.1	theorem	...	C	Deuring; Ireland–Rosen; checked on 17 983 primes
13.3.2	analogy	Weil’s $\lvert\alpha\rvert = \sqrt{p}$ and the Levi-Civita coordinate; labelled an	H	labelled analogy
13.4.1	proposition	theorem	C	Euler; class number formula
13.5.1	theorem	Hecke + Dirichlet; theorem	C	Hecke, Dirichlet
13.5.2	analogy	atomic/continuous split; parallel, not identity	H	parallel, not identity
13.6.1	question		Q	
A.1.1	definition		D	
A.1.2	lemma	the invariant form	P	
A.1.3	lemma	closed basic forms are invariant	P	
A.2.1	theorem	foliation versus contact	P	(iv) uses HT09 definition
A.2.2	theorem	mutual exclusion on closed manifolds	P	
A.2.3	remark		R	
A.3.1	proposition	period obstruction; Deninger	C	Deninger
A.3.2	corollary	rank one $\Leftrightarrow$ periodic zeta	P	
A.4.1	proposition		C	(c) Deninger, Leichtnam
A.4.2	remark	the contact replacement of $a = 1$; theorems	C	last paragraph analogy

No.	Type	Name	St.	Note
A.5.1	theorem	Lefschetz signs	P	
A.5.2	proposition		C	(b) Hutchings–Taubes
A.5.3	remark	both sides compute one invariant; theorem	C	Taubes; Lee–Taubes
A.6.1	proposition		C	(b),(c) cited; check source for zeros of Z_Γ at $\rho/2$ (G13)
A.6.2	remark		R	"STATE.md pointer replaced
A.9.1	question	contact laminations with elliptic prime orbits	Q	
A.9.2	question	the stable-Hamiltonian function f of arithmetic	Q	
A.9.3	question	Lax–Phillips on the modular surface	Q	"STATE.md pointer replaced

B.3. Gaps that blocked "fully rigorous", and their status

Items G1–G8 affected statements listed in Part 1 of the Statement of Intellectual Honesty, or their certified status. G9–G14 are editorial or minor. Status as of 21 October 2026: **closed** means a complete argument is now in the text (or a certified computation in `supplement/`); **open** means the gap remains.

G1. **Thm 6.5.1(b)** ($d_S(s) \to 0$). *Closed.* Full proof with rate $d_S(s) = O_S(|s|^{-1/4}\log^{1/2}|s|)$, using the explicit oscillatory bounds of the new Lemma 6.4.3 (first and second derivative tests, Titchmarsh Lemmas 4.3 and 4.4). Part (c) is now stated and proved as equality of Bohr-mean atoms (Wiener's formula). Thm 6.4.2(i) no longer depends on (b). Cor. 6.6.2 remains conditional on the numerical Obs. 6.6.1, as before (status H).

G2. **Thm 6.4.2** (Fubini). *Closed.* The proof now runs through truncation $t \geq -R$, Plancherel and Tonelli, a Hilbert–Schmidt Cauchy argument, and the domination $\Omega_S(s) \leq C_S(1 + \log(2 + |s|))$.

G3. **Thm 7.4.1(ii)** (Lax–Phillips constant). *Closed, by a stronger statement.* We prove directly that $\|Z_t\| = 1$ for every $t \geq 0$. The Szegő kernels at on-line zeros have profiles $e^{-s/4}e^{-i\gamma s/2}$; these are complete in every $L^2(0, M)$ (Jensen's formula against Conrey's $N_0(T) \gg T\log T$), so they approximate a bump beyond t. The criterion with constant $C \geq 1$ is no longer used. The published statements [74, Thms. 4, 5] were read in the original. Thm. 5 gives only continuation, which alone would not suffice. [75] was not obtained.

G4. **Prop. 11.3.2**. *Closed.* The limit $\lambda_\sigma(d) \downarrow m_\sigma$ is proved when $\sigma > \beta_\rho$ for every zero ρ, i.e. unconditionally for $\sigma \geq 1$ (concentrating test functions; positivity and growth of w_σ from the Hadamard product). The location of the infimum at $s = 0$ is now the separate numerical Obs. 11.3.3, which is not used.

G5. **Remark 8.3.2 / Thm 8.4.1(b)**. *Closed.* The exact identity for the ratio and the explicit bound $\leq 41.35\,|\zeta'(\rho)|^2/\sqrt{1+4\gamma^2}$ (product formula for the Gamma ratio) replace "$\sim$". Thm 8.4.1(b) then uses the convexity bound.

G6. **Def. 8.2.1** ($t = \log y$). *Closed for the zero mode.* The derivation is now in the text: $L = e^{r/2}\partial_r^2 e^{-r/2}$ in $r = \log y$, d'Alembert, round trip $2\log d \leftrightarrow d^{2ik}$, consistent with $t = 2u$. The full unitary equivalence of the Pavlov–Faddeev model is still cited [91, §3].

G7. **Computer-assisted results**. *Closed.* (a) Code, outputs, versions and SHA-256 hashes are in `supplement/` with a README. A from-scratch rerun reproduces Thm 10.3.1 digit for digit. (b) The tail-ratio bound $\leq \frac{1}{2}$ is proved in the text of Thm 10.3.1 ($|j_n| \leq 1$, entry majorants $v_{nm} \leq a_n a_m$, conditions (α), (β)), and the conditions are verified in

`logs/tail_conditions.out`. (c) The README lists which computations use rigorous enclosures. The caption claim about $\|E\|_F$ has been corrected.

G8. **Thm 9.2.2**, sign of $d_{\mathrm{all}}^{(1)}(\gamma_1)$. *Closed (computer-assisted).* Arb, 200 bits: $d_{\mathrm{all}}^{(1)}(\gamma_1) \in -0.0250030363066 \pm 10^{-21}$.

G9. **Conventions (Ch. 11 Fourier sign)**. *Closed.* Stated at the start of Ch. 11, with the reason it is harmless.

G10. **Cross-references** are standalone-style text ("Lemma 6, Thm. 7.1"), valid under the rule *k.s.n* but not hyperlinked. *Open (documented).* Convert them to `\ref` when the standalone notes are revised. The rule is stated in the front matter.

G11. **Thm 5.3.1** (off-diagonal smoothness). *Closed.* One-line proof from symbol estimates $a^{(k)} = O(|s|^{-k})$.

G12. **Remark 10.1.2** (dangling sentence). *Closed.* Deleted (editorial footnote).

G13. **Prop. A.6.1(b)** (zeros of Z_Γ at $\rho/2$). *Open.* It is cited to [60]. The standard source, Hejhal's *The Selberg trace formula for* $PSL(2,\mathbb{R})$, vol. 2, has not been checked and is therefore not cited.

G14. **Lemma 10.1.3** (optimality). *Closed.* Proved via Dirichlet's simultaneous approximation; Zhu is no longer needed.

B.4. Recommended revisions

- Cut or move to a "Heuristics" section: Remark 5.5.2 (super-regularized zero-point energy; power rule unverified) and Remark 5.5.3 (speculative).
- The sections "Questions for J.W." of the standalone notes have been retitled "Open questions", and Question 8.8.1 has been rephrased.
- Chapter 1 has appeared separately as ai.viXra:2610.0046 [83]; it is cited as such (updated in v2).
- Superseded sentences in Chs. 6, 8 and 9 now carry editorial footnotes; rewrite them in the next revision of the standalone notes.
- Unrefereed sources: [123], [36], [14], [56], [66]; the text marks them as such where used.

B.5. Editorial changes made in the volume (sources untouched)

- Macros: Lemma 1 local aliases replaced by the standard names; Lemma 6's `\Hs` (Hilbert–Schmidt) written `\mathrm{HS}`; theorem names unified ("Open question"→"Question", "Observation (numerical)"→"Observation [numerical]").
- Labels prefixed by chapter; cite keys merged (`Connes99, C99`→`Co99`, `Bombieri00`→`Bo00`, `Yoshida`→`Yo`, `La`→`La99`); self-references `L1, L2` and uncited items dropped.
- Def. 5: "Question 5.2" corrected to Question 5.4.2. Ch. 7: $\gamma_\rho \to \operatorname{Im}\rho$ in Prop. 7.2.1(e) and 7.5.2(c). Chs. 3–4: pointers to the internal archive review replaced by the error list in the front matter. Appendix A: pointers to the internal project file `STATE.md` replaced by chapter references; stray `\appendix` removed.
- Editorial footnotes added: Ch. 6 §2 status, Thm 8.5.1 last sentence, Ch. 9 summary (vi), Prop. 7.5.2, Remark 10.1.2.
- Title: *Arithmetic Unitarity I: Where the Riemann Hypothesis Lives in the Scaling Flow* (chosen by the author), on the title page, in the PDF metadata and in the running heads.

B.6. Changelog: gap closures of 21 October 2026

All changes are made in the volume copies through `build.py` (`RANGE_EDITS`, `EDITS`, files in `patches/`). The standalone sources are untouched.

- Ch. 6: new Lemma 6.4.3 (oscillatory bounds, explicit constants); new proof of Thm 6.4.2 (G2); Thm 6.5.1(b),(c) restated and proved (G1); status table row updated.
- Ch. 7: Thm 7.4.1(ii) strengthened to $\|Z_t\| = 1$ for all t with a self-contained proof; the earlier route via the Lax–Phillips criterion kept as a remark; [74] added (G3).
- Ch. 8: Remark 8.3.2 replaced by an explicit inequality (G5); Thm 8.4.1(a),(b) proofs adjusted; Def. 8.2.1: derivation of $t = \log y$ for the zero mode (G6).
- Ch. 9: certified sign of $d^{(1)}_{\mathrm{all}}(\gamma_1)$ (G8).
- Ch. 10: written tail proof in Thm 10.3.1 and corrected caption (G7); Remark 10.1.2 sentence deleted (G12); Lemma 10.1.3 optimality proved (G14).
- Ch. 11: Fourier-sign convention stated (G9); Prop. 11.3.2 completed and split off Obs. 11.3.3 (G4); later items of §11.3 renumbered (Remark 11.3.3→11.3.4).
- Ch. 5: off-diagonal smoothness line in Thm 5.3.1 (G11).
- Chs. 12 and 13 (Lemmas 12, 13, written by a separate process) are now included. Their statuses were audited in the referee pass below.
- Front matter: Statement of Intellectual Honesty, Parts 1 and 2, updated accordingly (S4 moved to Part 1).
- `supplement/`: code, outputs, scripts, logs, rerun, `VERSIONS.txt`, `SHA256SUMS`, `README.md`.

B.7. Referee pass (21 October 2026)

An independent referee checked the gap-closing patches (G1–G14), Chs. 12–13 and this appendix. The referee found no circular reasoning, no fabricated references or numbers, no serious errors, and no claim that RH is proved. All Arb certificates were reproduced (Thm 10.3.1 at $d = 3$ from scratch; Thm 9.2.2 sign, also by an independent mpmath computation). The listed corrections have all been applied:

- **Prop. 11.3.2** (priority). The hypothesis "$\sigma > \sup\beta \ldots$ unconditionally for $\sigma \geq 1$" was wrong at $\sigma = 1$, since $\sup\beta < 1$ is open. It now reads "$\sigma > \beta_\rho$ for every zero ρ": unconditional for $\sigma \geq 1$, and under RH for $\sigma > \frac{1}{2}$. The proof uses only this. Front matter and table updated.
- **Ch. 12** (priority; standalone `lemma12.tex` corrected and recompiled, the chapter derives from it). Zhu certifies positivity only up to window length 1.6; his results for half-widths up to 2 are *upper* bounds. The misstatement "≤ 4" / "1–4" is fixed in the abstract, in Prop. 12.5.2 and in the Obstruction paragraph. Table 12.1 is relabelled "computer-assisted (rigorous)". The $\delta = 0.5$ column, outside $0 < \delta < \frac{1}{2}$, carries a footnote: it is a formal reference value that certifies nothing, and within the theorem's range $A_0 \geq 16.8$. Thm 9.2.2 and Table 12.1 are added to the front-matter lists of computer-assisted results. The front-matter chapter guide now describes Chs. 12–13.
- Lemma 6.4.3: "$4\sqrt{2} + 1.36 < 7$" corrected to "< 7.1" ($B = 22$ unaffected). The Titchmarsh citation is corrected to Lemmas 4.3, 4.4 (also in B.3, item G1).
- Thm 6.5.1(a): the remainder for infinitely many $t_j > L$ is now $\leq 2N_V \sum_{t_j \geq L} |c_j|(t_j - L)$. Thm 6.5.1(c): the sign convention of Wiener's formula is stated ($\mathbf{M}_a$ picks the atom at $-a$; harmless by evenness), and "no Dirac mass" is made precise.
- Thm 6.4.2: the hypothesis $\int |k|(1 + |t|) < \infty$ for the kernel form is moved into the statement, noting that k_∞ does not satisfy it. In Step 4 the bound "$2G$" is corrected to "$4G$".
- Remark 8.3.2: "≈ 2.2" refers only to the Γ-ratio factor. The full bound exceeds the exact ratio by 3.1–12.4 at the tested zeros.
- Cor. 10.3.2: the domain hypothesis is aligned with Thm 8.2.4(c) ($\mathsf{J}f \in \mathcal{A}$).
- Zhu's preprint is dated by its first version (August 2026) in Chs. 10, 11, 12 and the front matter.

- Supplement README: the tail-condition script uses python-flint/Arb, not mpmath. Thm 7.4.1(ii) additionally cites [**109**, Ch. III, §§8–9].

Still open after the referee pass: G10 (cross-references not hyperlinked) and G13 (Hejhal not verified).

Bibliography

[1] J. A. Álvarez López, Yu. A. Kordyukov, E. Leichtnam, *A trace formula for foliated flows*, arXiv:2402.06671; Lecture Notes in Math. 2387, Springer 2026, doi:10.1007/978-3-032-15413-2.

[2] T. Barnet-Lamb, D. Geraghty, M. Harris, R. Taylor, A family of Calabi–Yau varieties and potential automorphy II, *Publ. RIMS* 47 (2011), 29–98; doi:10.2977/prims/31.

[3] M. V. Berry, J. P. Keating, The Riemann zeros and eigenvalue asymptotics, *SIAM Rev.* 41 (1999), 236–266.

[4] E. Bombieri, A variational approach to the explicit formula, *Comm. Pure Appl. Math.* 56 (2003), 1151–1164; doi:10.1002/cpa.10089.

[5] E. Bombieri, Counting points on curves over finite fields (d'après S. A. Stepanov), Sém. Bourbaki 1972/73, Exp. 430, LNM 383, Springer, 1974, 234–241.

[6] E. Bombieri, Remarks on Weil's quadratic functional in the theory of prime numbers, I, *Rend. Mat. Acc. Lincei* (9) 11 (2000), 183–233 (BDIM: RLIN_2000_9_11_3_183_0).

[7] J.-B. Bost, A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, *Selecta Math. (N.S.)* 1 (1995), 411–457.

[8] J. P. Buhler, D. Harvey, Irregular primes to 163 million, *Math. Comp.* 80 (2011), 2435–2444.

[9] L. Clozel, M. Harris, R. Taylor, Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations, *Publ. Math. IHÉS* 108 (2008), 1–181; doi:10.1007/s10240-008-0016-1.

[10] J. Coates, R. Sujatha, The main conjecture, in *Guwahati Workshop on Iwasawa Theory of Totally Real Fields*, Ramanujan Math. Soc. Lect. Notes Ser. 12 (2010), 113–140.

[11] A. Connes, An essay on the Riemann hypothesis, in *Open Problems in Mathematics* (J. F. Nash Jr., M. Th. Rassias, eds.), Springer, 2016, 225–257.

[12] A. Connes, C. Consani, *Geometry of the arithmetic site*, Adv. Math. 291 (2016) 274–329, doi:10.1016/j.aim.2015.11.045, arXiv:1502.05580.

[13] A. Connes, C. Consani, Geometry of the scaling site, *Selecta Math. (N.S.)* 23 (2017), 1803–1850.

[14] A. Connes, C. Consani, H. Moscovici, On q-series and the moment problem associated to local factors, arXiv:2403.01247 (preprint; journal version not verified).

[15] A. Connes, C. Consani, H. Moscovici, Zeta zeros and prolate wave operators, *Ann. Funct. Anal.* 15 (2024), no. 4, Paper No. 87; doi:10.1007/s43034-024-00388-z; arXiv:2310.18423.

[16] A. Connes, C. Consani, Quasi-inner functions and local factors, *J. Number Theory* 226 (2021), 139–167; doi:10.1016/j.jnt.2021.01.024; arXiv:2008.10974.

[17] A. Connes, C. Consani, *The arithmetic site*, C. R. Math. 352 (2014) 971–975, doi:10.1016/j.crma.2014.07.009, arXiv:1405.4527.

[18] A. Connes, C. Consani, *The scaling site*, C. R. Math. 354 (2016) 1–6, doi:10.1016/j.crma.2015.09.027, arXiv:1507.05818.

[19] A. Connes, C. Consani, Weil positivity and trace formula, the archimedean place, *Selecta Math. (N.S.)* 27 (2021), no. 4, Paper No. 77, 70 pp.; arXiv:2006.13771 (theorem numbers above refer to the arXiv version).

[20] A. Connes, H. Moscovici, The UV prolate spectrum matches the zeros of zeta, *Proc. Natl. Acad. Sci. USA* 119 (2022); doi:10.1073/pnas.2123174119 (arXiv:2112.05500).

[21] A. Connes, M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, AMS Colloq. Publ. 55, 2008.

[22] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, Selecta Math. 5 (1999) 29–106, doi:10.1007/s000290050042, arXiv:math/9811068.

[23] J. B. Conrey, More than two fifths of the zeros of the Riemann zeta function are on the critical line, *J. reine angew. Math.* 399 (1989), 1–26; doi:10.1515/crll.1989.399.1. Announcement: At least two fifths of the zeros of the Riemann zeta function are on the critical line, *Bull. Amer. Math. Soc. (N.S.)* 20 (1989), 79–81; doi:10.1090/s0273-0979-1989-15704-2 (Thm. 1 there: at least $\frac{2}{5}$ of the zeros are simple and on the line; downloaded `lit/lp/conrey89.pdf`).
[24] G. Csordas, W. Smith, R. S. Varga, Lehmer pairs of zeros, the de Bruijn–Newman constant Λ, and the Riemann hypothesis, *Constr. Approx.* 10 (1994), 107–129.
[25] H. Davenport, *Multiplicative Number Theory*, 3rd ed., GTM 74, Springer 2000.
[26] N. G. de Bruijn, The roots of trigonometric integrals, *Duke Math. J.* 17 (1950), 197–226.
[27] P. Deligne, *Cohomologie étale* (SGA $4\frac{1}{2}$), LNM 569, Springer, 1977.
[28] P. Deligne, Formes modulaires et représentations ℓ-adiques, Sém. Bourbaki 1968/69, Exp. 355, LNM 179, Springer, 1971, 139–172.
[29] C. Deninger, *A note on arithmetic topology and dynamical systems*, Contemp. Math. 300 (2002) 99–114, doi:10.1090/conm/300/05144, arXiv:math/0204274.
[30] C. Deninger, *Analogies between analysis on foliated spaces and arithmetic geometry*, in: Groups and Analysis, LMS Lecture Note Ser. 354 (2008) 174–190, arXiv:0709.2801.
[31] C. Deninger, *Dynamical systems for arithmetic schemes*, Indag. Math. 37 (2026) 25–136, doi:10.1016/j.indag.2024.05.007, arXiv:1807.06400.
[32] C. Deninger, *Number theory and dynamical systems on foliated spaces*, Jahresber. DMV 103 (2001) 79–100, arXiv:math/0204110.
[33] M. Deuring, Die Typen der Multiplikatorenringe elliptischer Funktionenkörper, *Abh. Math. Sem. Hamburg* 14 (1941), 197–272; doi:10.1007/BF02940746.
[34] M. Deuring, Die Zetafunktion einer algebraischen Kurve vom Geschlechte Eins, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl.* 1953, 85–94.
[35] S. Dyatlov, M. Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis*, Ann. Sci. ENS 49 (2016) 543–577, doi:10.24033/asens.2290, arXiv:1306.4203.
[36] S. Eid, Certified Weil positivity beyond the first prime threshold: a five-rung coercivity ladder up to $1.12 \log 2$, preprint, Zenodo (2026); doi:10.5281/zenodo.21720802 (unrefereed; abstract read, proofs not checked).
[37] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, GTM 194, Springer 2000; doi:10.1007/b97696.
[38] F. Faure, M. Tsujii, *Band structure of the Ruelle spectrum of contact Anosov flows*, C. R. Math. 351 (2013) 385–391, doi:10.1016/j.crma.2013.04.022, arXiv:1301.5525.
[39] F. Faure, M. Tsujii, *The semiclassical zeta function for geodesic flows on negatively curved manifolds*, Invent. Math. 208 (2017) 851–998, doi:10.1007/s00222-016-0701-5, arXiv:1311.4932.
[40] P. Foulon, B. Hasselblatt, *Contact Anosov flows on hyperbolic 3-manifolds*, Geom. Topol. 17 (2013) 1225–1252, doi:10.2140/gt.2013.17.1225.
[41] D. Fried, *The geometry of cross sections to flows*, Topology 21 (1982) 353–371, doi:10.1016/0040-9383(82)90017-9.
[42] S. R. Garcia, J. Mashreghi, W. T. Ross, *Introduction to Model Spaces and their Operators*, Cambridge Univ. Press 2016; doi:10.1017/cbo9781316258231.
[43] D. A. Goldston, S. M. Gonek, A note on $S(t)$ and the zeros of the Riemann zeta-function, *Bull. London Math. Soc.* 39 (2007), 482–486; doi:10.1112/blms/bdm032 (arXiv:math/0511092).
[44] A. Grigutis, D. Šiaučiūnas, *On the modulus of the Selberg zeta-functions in the critical strip*, Math. Model. Anal. 20 (2015) 852–865, doi:10.3846/13926292.2015.1119213.
[45] A. P. Guinand, A summation formula in the theory of prime numbers, *Proc. London Math. Soc.* (2) 50 (1948), 107–119.
[46] M. Harris, N. Shepherd-Barron, R. Taylor, A family of Calabi–Yau varieties and potential automorphy, *Ann. of Math.* 171 (2010), 779–813; doi:10.4007/annals.2010.171.779.
[47] R. Hartshorne, *Algebraic Geometry*, GTM 52, Springer, 1977.
[48] E. Hasanalizade, Q. Shen, P.-J. Wong, Counting zeros of the Riemann zeta function, *J. Number Theory* 235 (2022), 219–241; doi:10.1016/j.jnt.2021.06.032 (arXiv:2107.06506).

[49] E. Hecke, Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen I, *Math. Z.* 1 (1918), 357–376; doi:10.1007/BF01465095.

[50] E. Hecke, Eine neue Art von Zetafunktionen und ihre Beziehungen zur Verteilung der Primzahlen II, *Math. Z.* 6 (1920), 11–51; doi:10.1007/BF01202991.

[51] D. A. Hejhal, *The Selberg Trace Formula for $PSL(2,\mathbb{R})$, Vol. 1*, Lecture Notes in Math. 548, Springer 1976, doi:10.1007/BFb0079608; Vol. 2, LNM 1001 (1983), doi:10.1007/BFb0061302.

[52] J. Herbrand, Sur les classes des corps circulaires, *J. Math. Pures Appl.* (9) 11 (1932), 417–441.

[53] J. Hietarinta, B. Grammaticos, B. Dorizzi, A. Ramani, Coupling-constant metamorphosis and duality between integrable Hamiltonian systems, *Phys. Rev. Lett.* 53 (1984), 1707–1710; doi:10.1103/PhysRevLett.53.1707.

[54] A. Hinkkanen, On functions of bounded type, *Complex Variables Theory Appl.* 34 (1997), 119–139; doi:10.1080/17476939708815042.

[55] M. Hutchings, C. H. Taubes, *The Weinstein conjecture for stable Hamiltonian structures*, Geom. Topol. 13 (2009) 901–941, doi:10.2140/gt.2009.13.901, arXiv:0809.0140.

[56] M. Hutchings, *Lecture notes on embedded contact homology*, arXiv:1303.5789.

[57] I. S. Iohvidov, M. G. Krein, H. Langer, *Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric*, Akademie-Verlag, 1982.

[58] K. Ireland, M. Rosen, *A Classical Introduction to Modern Number Theory*, 2nd ed., GTM 84, Springer, 1990, Ch. 18 §4.

[59] H. Iwaniec, E. Kowalski, *Analytic Number Theory*, AMS Colloq. Publ. 53, 2004, §5.5.

[60] H. Iwaniec, *Prime geodesic theorem*, J. Reine Angew. Math. 349 (1984) 136–159, doi:10.1515/crll.1984.349.136.

[61] K. Iwasawa, On $\mathbb{Z}_\ell$-extensions of algebraic number fields, *Ann. of Math.* 98 (1973), 246–326.

[62] F. Johansson, Arb: efficient arbitrary-precision midpoint-radius interval arithmetic, *IEEE Trans. Comput.* 66 (2017), 1281–1292; doi:10.1109/TC.2017.2690633.

[63] B. Julia, Statistical theory of numbers, in *Number Theory and Physics* (Les Houches 1989), Springer Proc. Phys. 47 (1990), 276–293.

[64] H. Ki, Y.-O. Kim, J. Lee, On the de Bruijn–Newman constant, *Adv. Math.* 222 (2009), 281–306.

[65] H. H. Kim, *Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2* (with appendices by D. Ramakrishnan and by H. H. Kim and P. Sarnak), J. Amer. Math. Soc. 16 (2003) 139–183, doi:10.1090/S0894-0347-02-00410-1.

[66] M. Kim, *Arithmetic Chern–Simons theory I*, arXiv:1510.05818.

[67] F. Kopei, *A remark on a relation between foliations and number theory*, in: Foliations 2005, World Sci. 2006, doi:10.1142/9789812772640_0013, arXiv:math/0605184.

[68] P. Kustaanheimo, E. Stiefel, Perturbation theory of Kepler motion based on spinor regularization, *J. reine angew. Math.* 218 (1965), 204–219; doi:10.1515/crll.1965.218.204.

[69] J. C. Lagarias, Correction to: "On a positivity property of the Riemann ξ-function", *Acta Arith.* 116 (2005), 293–294; doi:10.4064/aa116-3-5.

[70] J. C. Lagarias, On a positivity property of the Riemann ξ-function, *Acta Arith.* 89 (1999), 217–234; doi:10.4064/aa-89-3-217-234.

[71] H. J. Landau, H. Widom, Eigenvalue distribution of time and frequency limiting, *J. Math. Anal. Appl.* 77 (1980), 469–481; doi:10.1016/0022-247X(80)90241-3.

[72] P. D. Lax, R. S. Phillips, *Scattering Theory for Automorphic Functions*, Ann. of Math. Stud. 87, Princeton Univ. Press 1976; doi:10.1515/9781400881567.

[73] P. D. Lax, R. S. Phillips, Scattering theory for automorphic functions, *Bull. Amer. Math. Soc. (N.S.)* 2 (1980), 261–295; doi:10.1090/S0273-0979-1980-14735-7 (downloaded: `lit/lp/lp80.pdf`).

[74] P. D. Lax, R. S. Phillips, Scattering theory, *Bull. Amer. Math. Soc.* 70 (1964), 130–142; doi:10.1090/S0002-9904-1964-11051-X (read in the original, AMS open archive, copy `supplement/lit/lp64.pdf`; Thms. 4, 5 on pp. 132–133).

[75] P. D. Lax, R. S. Phillips, *Scattering Theory*, Pure Appl. Math. 26, Academic Press 1967; doi:10.1016/s0079-8169(08)x6035-0.

[76] Y.-J. Lee, C. H. Taubes, *Periodic Floer homology and Seiberg–Witten–Floer cohomology*, J. Symplectic Geom. 10 (2012) 81–164, doi:10.4310/jsg.2012.v10.n1.a4.

[77] E. Leichtnam, *Scaling group flow and Lefschetz trace formula for laminated spaces with p-adic transversal*, Bull. Sci. Math. 131 (2007) 638–669, doi:10.1016/j.bulsci.2006.11.001, arXiv:math/0603576.
[78] T. Levi-Civita, Sur la régularisation du problème des trois corps, *Acta Math.* 42 (1920), 99–144; doi:10.1007/BF02404404.
[79] C. Liverani, *On contact Anosov flows*, Ann. of Math. 159 (2004) 1275–1312, doi:10.4007/annals.2004.159.1275.
[80] B. Mazur, A. Wiles, Analogies between function fields and number fields, *Amer. J. Math.* 105 (1983), 507–521.
[81] B. Mazur, A. Wiles, Class fields of abelian extensions of $\mathbb{Q}$, *Invent. Math.* 76 (1984), 179–330.
[82] B. Mazur, How can we construct abelian Galois extensions of basic number fields?, *Bull. AMS* 48 (2011), 155–209.
[83] J. W. McGreevy, *TAA Lemma 1: Arithmetic Unitarity. The Riemann Hypothesis as Absence of Ghosts and Runaway Modes for the Scaling Flow of the Weil Form*, ai.viXra:2610.0046 (2026). Updated in v2.
[84] C. T. McMullen, *Knots which behave like the prime numbers*, Compos. Math. 149 (2013) 1235–1244, doi:10.1112/S0010437X13007173.
[85] M. Morishita, *Knots and Primes: An Introduction to Arithmetic Topology*, Universitext, Springer 2012, doi:10.1007/978-1-4471-2158-9.
[86] M. J. Mossinghoff, T. S. Trudgian, A. Yang, Explicit zero-free regions for the Riemann zeta-function, *Res. Number Theory* 10 (2024), art. 11; doi:10.1007/s40993-023-00498-y (arXiv:2212.06867).
[87] D. Mumford, *Abelian Varieties*, Tata Inst. / Oxford Univ. Press, 1970, §21.
[88] C. M. Newman, Fourier transforms with only real zeros, *Proc. AMS* 61 (1976), 245–251.
[89] E. Muñoz García, R. Pérez Marco, The product over all primes is $4\pi^2$, *Comm. Math. Phys.* 277 (2008), 69–81.
[90] W. Parry, M. Pollicott, *An analogue of the prime number theorem for closed orbits of Axiom A flows*, Ann. of Math. 118 (1983) 573–591, doi:10.2307/2006982.
[91] B. S. Pavlov, L. D. Faddeev, Scattering theory and automorphic functions, *Zap. Nauchn. Sem. LOMI* 27 (1972), 161–193; English transl. *J. Soviet Math.* 3 (1975), 522–548; doi:10.1007/BF01084688 (Russian original downloaded from Math-Net.Ru: `lit/lp/pf72.pdf`).
[92] J. F. Plante, *Anosov flows*, Amer. J. Math. 94 (1972) 729–754, doi:10.2307/2373755.
[93] D. Platt, T. Trudgian, The Riemann hypothesis is true up to $3 \cdot 10^{12}$, *Bull. London Math. Soc.* 53 (2021), 792–797; doi:10.1112/blms.12460.
[94] D. H. J. Polymath, Effective approximation of heat flow evolution of the Riemann ξ function, and a new upper bound for the de Bruijn–Newman constant, *Res. Math. Sci.* 6 (2019), 31.
[95] L. S. Pontryagin, Hermitian operators in spaces with indefinite metric, *Izv. Akad. Nauk SSSR* 8 (1944), 243–280.
[96] S. Ramanujan, On certain arithmetical functions, *Trans. Cambridge Philos. Soc.* 22 (1916), 159–184.
[97] M. Reed, B. Simon, *Methods of Modern Mathematical Physics I*, Academic Press, 1980.
[98] K. Ribet, A modular construction of unramified p-extensions of $\mathbb{Q}(\mu_p)$, *Invent. Math.* 34 (1976), 151–162.
[99] B. Rodgers, T. Tao, The de Bruijn–Newman constant is non-negative, *Forum Math. Pi* 8 (2020), e6.
[100] M. Rosen, *Number Theory in Function Fields*, GTM 210, Springer, 2002.
[101] K. Rubin, The main conjecture, Appendix to S. Lang, *Cyclotomic Fields I and II*, GTM 121, Springer, 1990.
[102] S. Schwartzman, *Asymptotic cycles*, Ann. of Math. 66 (1957) 270–284, doi:10.2307/1969999.
[103] A. Selberg, *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. (N.S.) 20 (1956) 47–87, doi:10.18311/jims/1956/16985.
[104] J.-P. Serre, *A Course in Arithmetic*, GTM 7, Springer, 1973.
[105] D. Slepian, H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty I, *Bell Syst. Tech. J.* 40 (1961), 43–63; doi:10.1002/j.1538-7305.1961.tb03976.x.
[106] K. S. Stelle, Renormalization of higher-derivative quantum gravity, *Phys. Rev. D* 16 (1977), 953–969.
[107] R. F. Sunseri, *Zeros of p-adic L-functions and densities relating to Bernoulli numbers*, Ph.D. thesis, Univ. of Illinois at Urbana-Champaign, 1980, Table A-2.

[108] H. P. F. Swinnerton-Dyer, On ℓ-adic representations and congruences for coefficients of modular forms, in *Modular Functions of One Variable III*, LNM 350, Springer, 1973, 1–55.

[109] B. Sz.-Nagy, C. Foias, H. Bercovici, L. Kérchy, *Harmonic Analysis of Operators on Hilbert Space*, 2nd ed., Springer 2010; doi:10.1007/978-1-4419-6094-8.

[110] J. Tate, Endomorphisms of abelian varieties over finite fields, *Invent. Math.* 2 (1966), 134–144.

[111] C. H. Taubes, *Embedded contact homology and Seiberg–Witten Floer cohomology I*, Geom. Topol. 14 (2010) 2497–2581, doi:10.2140/gt.2010.14.2497 (parts II–V follow in the same volume).

[112] C. H. Taubes, *The Seiberg–Witten equations and the Weinstein conjecture*, Geom. Topol. 11 (2007) 2117–2202, doi:10.2140/gt.2007.11.2117.

[113] R. Taylor, Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations II, *Publ. Math. IHÉS* 108 (2008), 183–239; doi:10.1007/s10240-008-0015-2.

[114] The LMFDB Collaboration, *The L-functions and modular forms database*, pages 32.2.a.a and 32.a3, accessed 21 Oct 2026.

[115] D. Tischler, *On fibering certain foliated manifolds over* S^1, Topology 9 (1970) 153–154, doi:10.1016/0040-9383(70)90037-6.

[116] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed. (rev. D. R. Heath-Brown), Oxford 1986.

[117] T. S. Trudgian, An improved upper bound for the argument of the Riemann zeta-function on the critical line II, *J. Number Theory* 134 (2014), 280–292; doi:10.1016/j.jnt.2013.07.017.

[118] L. C. Washington, *Introduction to Cyclotomic Fields*, 2nd ed., GTM 83, Springer, 1997.

[119] A. Weil, Sur les "formules explicites" de la théorie des nombres premiers, *Comm. Sém. Math. Univ. Lund* (1952), tome suppl., 252–265.

[120] A. Weil, *Sur les courbes algébriques et les variétés qui s'en déduisent*, Hermann, Paris, 1948.

[121] A. Wiles, The Iwasawa conjecture for totally real fields, *Ann. of Math.* 131 (1990), 493–540.

[122] H. Yoshida, On Hermitian forms attached to zeta functions, in *Zeta Functions in Geometry*, Adv. Stud. Pure Math. 21 (1992), 281–325; doi:10.2969/aspm/02110281.

[123] X. Zhu, Weil positivity in compact windows: a finite reduction, certified two-sided bounds, and a Landau–Widom decay law, preprint, arXiv:2608.24827 (2026) (unrefereed; downloaded `lit/window/zhu2608.pdf`).