

The Aperiodic Einstein Monotile on the Cusp Torus of Alpöge’s Complex Six-Sphere

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October 15, 2026

Abstract

The cusp fibre W of Alpöge’s complex structure on the six-sphere is the [3.4.6.4] Laves tiling rolled up on a torus $\mathbb{T} = \mathbb{R}^2/\Lambda$, $\Lambda = B_0\mathbb{Z}^2$, and at the proportions of the Hat that tiling is exactly the substrate of the aperiodic monotile: hexagon centres are components, edge midpoints double curves, hexagon vertices triple points. No monotile tiling descends to $\mathbb{T}$, since an aperiodic tiling is invariant under no lattice. We fold a Hat tiling generated by substitution onto $\mathbb{T}$ anyway. Every tile is a union of whole kites, so the fold is well defined, and we compute its fibres. The “mystic” tile (the Turtle, ten kites) lands on exactly six points of the torus in the patches of three and four substitution steps: one per orientation, each anchored on the midpoint of a double curve of W , two orientations per curve. The six classes form one orbit of the rotation group of a hexagon, and six is the number of oriented edges of the dual complex of W , which we show is the torus with one vertex, three edges and two triangles. We prove that at most 72 classes are possible, state six as a conjecture, and show by the boundary statistics of the folded tiling that equal fibres do not mean equal surroundings. The paper also records why the lattice data behind the construction is forced, that its monodromy is achiral and its lattice not self-dual over $\mathbb{Z}$, and three exact contacts with standard objects (a polytope volume, a Kodaira configuration, and central charges).

1 Introduction

In 1948 Hopf asked whether the six-sphere admits a complex structure [1]. In August 2026 Alpöge posted a construction answering yes [2]: a compact complex threefold X , fibred over $\mathbb{P}^1$ by complex 2-tori, with a cusp fibre W and two multiple fibres over three points, proved diffeomorphic to S^6 . Engel gave a pedagogical account [3], and Alexeev formalised the headline statement in Lean 4 [4]. The construction contradicts a published corollary of Campana, Demailly and Peternell [5], and Alpöge locates where the two accounts part. At the time of writing the result is not refereed.

Nearly all of the proof is analysis and topology. It stands, however, on two 4×4 integer matrices T_1, T_2 of orders 3 and 4 whose product is unipotent, and much of what is proved is decided by linear algebra over $\mathbb{Z}$. We work only with that finite layer, exactly, and ask what it is connected to. The answer we pursue is the aperiodic monotile.

The substrate. The cusp fibre W is a degree-six del Pezzo surface glued to itself, and its strata have the incidence structure of the A_2 triangulation of the plane modulo Λ . The substrate of the Hat, the kite graph of Smith, Myers, Kaplan and Goodman-Strauss [7], is at the Hat’s own proportions the [3.4.6.4] Laves tiling, and Theorem 1 shows that tiling is the Hasse diagram of that triangulation (Fig. 1). The cusp fibre is the substrate *rolled up*.

Folding a tiling. A periodic tiling can be rolled up and an aperiodic one cannot, so the substrate descends and the tiling does not. That concerns whole tilings. A finite patch of a Hat tiling is a union of tiles, each a union of whole kites, and the quotient $\pi : \mathbb{R}^2 \rightarrow \mathbb{T}$ sends it to a

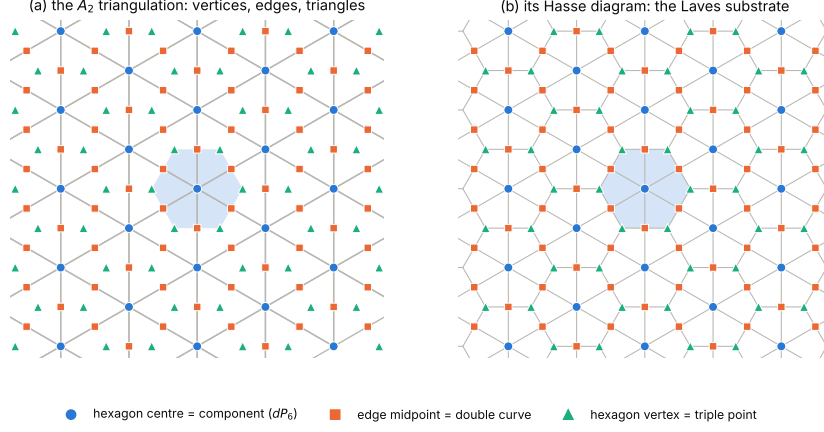


Figure 1: The dictionary at the Hat point (Theorem 1). (a) The cells of the A_2 triangulation: vertices (circles), edge midpoints (squares), triangle centroids (triangles). (b) The Laves substrate on the same sites; its bonds are the cover relations vertex $<$ edge $<$ triangle. The shaded hexagon has one centre, six edge midpoints and six vertices.

finite multiset on the torus. Section 4 does this for a substitution patch and records, for each of the twelve bond orbits of $\mathbb{T}$, how often the bond is a tile boundary (Fig. 2); none is always or never, which is the certificate of non-descent.

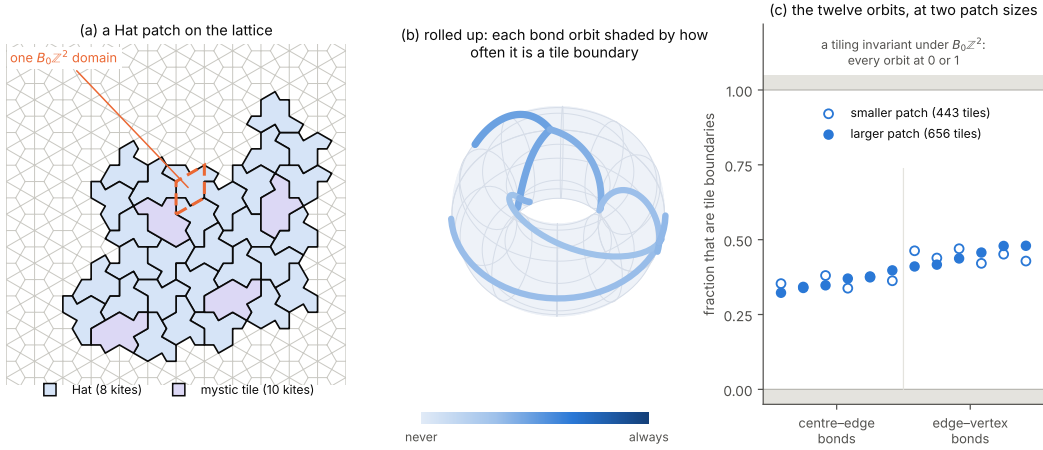


Figure 2: A Hat tiling rolled up on the cusp torus (Section 4). (a) A substitution patch on the Laves lattice, mystic tiles marked. (b) The torus, each bond orbit weighted by how often that bond is a tile boundary. (c) The twelve boundary fractions $f(o)$, all strictly between 0 and 1.

Mystic fibres. We follow the mystic, the ten-kite tile, through the fold. Two mystics are in the same *fibre class* if they differ by an element of Λ . In the three-step patch (443 tiles, 51 mystic) the mystics form exactly six fibre classes, the same six occur at four steps, and five of them at two steps. The classes are the six orientations of the mystic; in each, a distinguished vertex lies over the midpoint of one of the three double curves D_a, D_b, D_c , and orientations differing by 180° share a double curve (Fig. 3). The six classes are one orbit of the rotation group of a hexagon, and six is the number of oriented edges of the dual complex of W (Sections 3 and 6). We prove at most 72 classes and conjecture six (Proposition 3 and Conjecture 1); the dependence of the surroundings on the class is observed, not proved.

Mystic (Turtle) tiles of the Hat tiling, 3 substitution steps: 443 tiles, 51 mystic

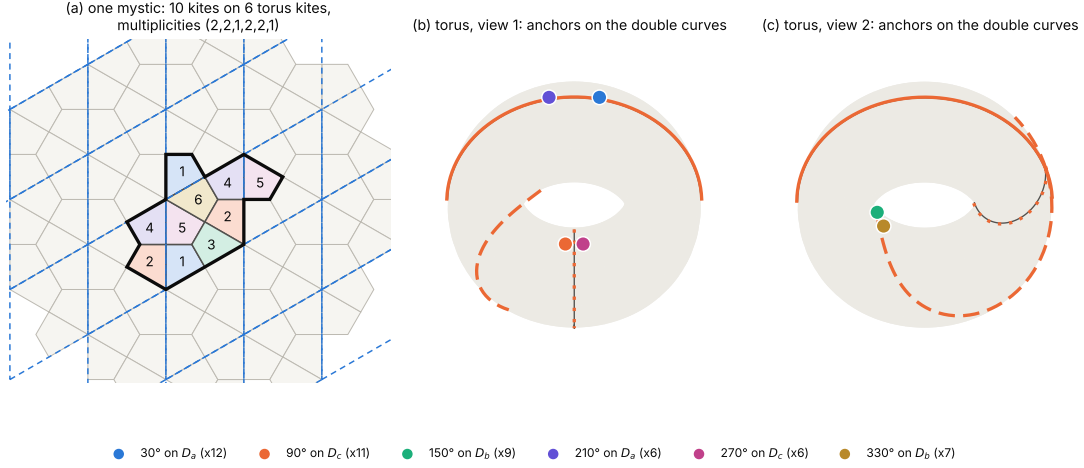


Figure 3: Mystic tiles after three substitution steps (443 tiles, 51 mystic). (a) One mystic, flat: its ten kites coloured by the torus kite they fold onto, multiplicities (2, 2, 1, 2, 2, 1); dashed blue lines are translates of the fundamental domain. (b), (c) The torus from two viewpoints that together show all six fibre points, each anchored over a double curve (solid D_a , dashed D_b , dotted D_c), two orientations per curve. The legend gives the first-edge angle, the double curve and the number of mystics in the class.

2 The cusp fibre is the monotile's substrate

2.1 The substrate at the Hat point

The Hat of [7, 8] is a union of eight kites (Fig. 4). A kite has legs a from a hexagon centre to an edge midpoint and b from that edge midpoint to a hexagon vertex, meeting at right angles at the edge midpoint, along directions that are multiples of 30° ; coordinates lie in $\mathbb{Q}(\sqrt{3})$ and are exact. Around a hexagon centre the six edge midpoints are $a e_k$, $e_k = u_{30^\circ + 60^\circ k}$, and each carries two outer kite corners $a e_k \pm b e_k^\perp$, twelve per hexagon. Adjacent corners coincide exactly when

$$\overbrace{a e_k + b e_k^\perp}^{\text{corner of kite } k} = \overbrace{a e_{k+1} - b e_{k+1}^\perp}^{\text{corner of kite } k+1}, \quad \text{i.e.} \quad a = \frac{a}{2} + \frac{\sqrt{3}}{2}b \quad \text{and} \quad b = \frac{\sqrt{3}}{2}a - \frac{b}{2} \iff a = \sqrt{3}b. \quad (1)$$

So at the Hat's proportions $a : b = \sqrt{3} : 1$, and at no others, the twelve corners merge in pairs into the six vertices of the [3.4.6.4] Laves tiling. With $a = \sqrt{3}$, $b = 1$ the kite has area $\underbrace{2}_{\text{two triangles}} \cdot \frac{1}{2}ab = \sqrt{3}$, and the Hat, eight kites, has area $8\sqrt{3}$.

On the six-sphere's side, the cusp fibre is built from a toric fan whose cross-section is the A_2 triangulation of the plane [2]: its vertices give the components of W , its edges the double curves, its triangles the triple points, and the quotient by $B_0\mathbb{Z}^2$ identifies them.

Theorem 1. *At the Hat point $a : b = \sqrt{3} : 1$ the monotile substrate is the Hasse diagram of the A_2 triangulation: its sites are the cells (hexagon centres the vertices, edge midpoints the edges, hexagon vertices the triangles) and its bonds are exactly the cover relations $\text{vertex} < \text{edge} < \text{triangle}$. At no other proportions is it.*

Computation. Embed A_2 by $e_1 \mapsto 2a u_{30^\circ}$, $e_2 \mapsto 2a u_{90^\circ}$ with vertices at hexagon centres; edge midpoints and triangle centroids follow (Fig. 1). For the two-ring patch at the Hat point all 61 sites coincide exactly, in $\mathbb{Q}(\sqrt{3})$, with cells of the matching kind, and the 102 cover relations

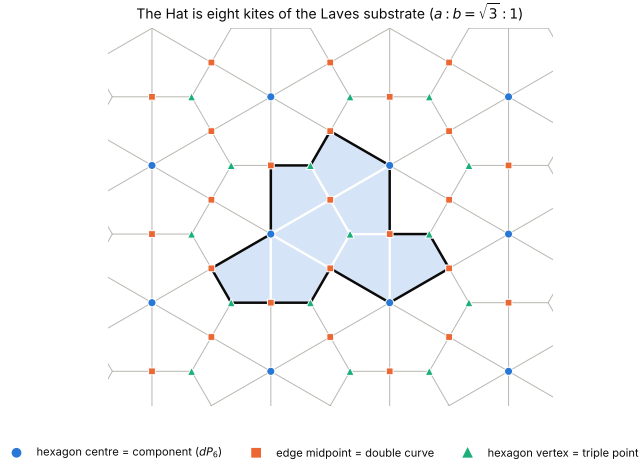


Figure 4: The Hat at $a : b = \sqrt{3} : 1$ as eight whole kites of the $[3.4.6.4]$ Laves lattice. Circles are hexagon centres (valence 6), squares edge midpoints (valence 4), triangles hexagon vertices (valence 3). The thirteen-sided outline has area $8\sqrt{3}$ in units where $b = 1$.

among them are exactly its 102 bonds. That the corners merge at no other ratio is (1). At the Spectre's proportions $a = b$, and at $17/10 : 1$ (within 2% of $\sqrt{3}$), the corners stay apart: the patch has 97 and 97 sites, its "vertex" sites have valence 1 rather than 3, and the graph is a different graph, not a metric distortion of the same one. The substrate is periodic and the patch contains a fundamental domain with every cell incident to it, so agreement on the patch is agreement everywhere. $\square$

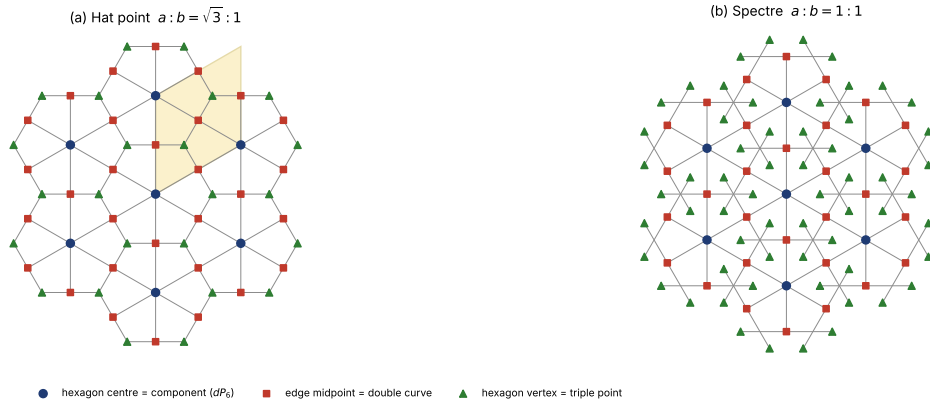


Figure 5: (a) The substrate at the Hat point, sites marked by the stratum of W they become. The shaded parallelogram is one fundamental domain of $B_0\mathbb{Z}^2$, holding one hexagon centre, three edge midpoints and two hexagon vertices (one component, three double curves, two triple points). (b) The same patch at the Spectre's proportions: the outer corners stay paired and apart, and the graph is the Hasse diagram of nothing.

2.2 The dictionary

Since $|\det B_0| = 1$, translation by $B_0\mathbb{Z}^2$ is exactly translation of the Laves tiling by its period lattice, so the cusp fibre is that tiling rolled up on $\mathbb{R}^2/B_0\mathbb{Z}^2$ (Figs. 6 and 7):

Laves tiling at the Hat point	cuspidal fibre W	per domain
hexagon centre (valence 6)	a dP_6 component	1
edge midpoint (valence 4)	a double curve	3
hexagon vertex (valence 3)	a triple point	2
kite	a corner of the dP_6 hexagon	6
centre–edge bond	a (-1) -curve of the hexagon	6
edge–vertex bond	a double curve through a triple point	6

Three statements about W proved in [2] become statements about the tiling. First, the rolled-up tiling is a torus:

$$\chi(\mathbb{T}) = \underbrace{\left(\underbrace{1}_{E_0} + \underbrace{3}_D + \underbrace{2}_T \right)}_{6 \text{ sites}} - \underbrace{\left(\underbrace{6}_{\text{centre-edge}} + \underbrace{6}_{\text{edge-vertex}} \right)}_{12 \text{ bonds}} + \underbrace{6}_{\text{kites}} = 0. \quad (2)$$

Second, the torus orbits of W have Euler characteristic 0 except the triple points, so $e(W) = e(E_0^\circ) + e(D^\circ) + \#T = 2$ counts hexagon vertices in a fundamental domain. Third, Friedman's triple-point formula [6], valid on every double curve C of W with components V, V' through it, reads

$$\underbrace{(C|_V)^2}_{-1} + \underbrace{(C|_{V'})^2}_{-1} + \underbrace{T_C}_2 = 0 : \quad (3)$$

each hexagon side is a (-1) -curve and each edge midpoint is bonded to two hexagon vertices.

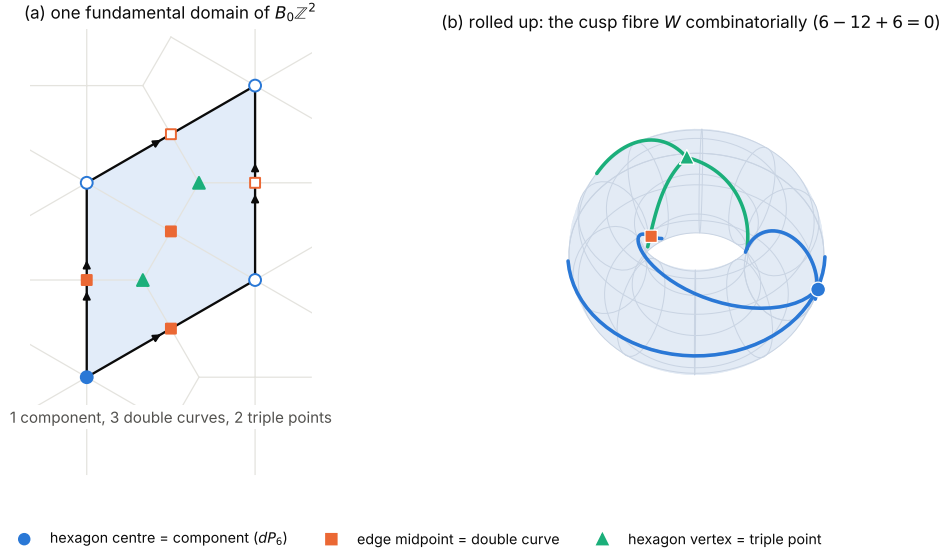


Figure 6: (a) One fundamental domain of $B_0\mathbb{Z}^2$. Filled marks are its representatives, hollow marks their images on the boundary; sides with equal arrows are identified. It holds one component, three double curves and two triple points. (b) The same sites and bonds on $\mathbb{R}^2/B_0\mathbb{Z}^2$, far side hidden.

2.3 What descends, and what does not

Theorem 1 is an identity of substrates, not of tilings. The Laves tiling is periodic and its quotient by $B_0\mathbb{Z}^2$ is a compact object, the cuspidal fibre. A Hat tiling is aperiodic: it is invariant under no translation lattice, $B_0\mathbb{Z}^2$ included, so it does not descend to W or any torus. Both inherit their lattice from A_2 , and the six-sphere knows nothing about monotiles; what they share is that

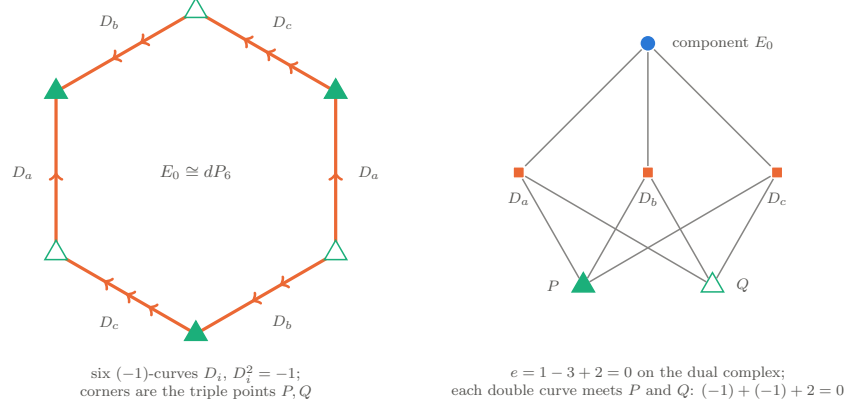


Figure 7: The component $E_0 \cong dP_6$ and the incidence structure of W . Left: the hexagon of six (-1) -curves; opposite sides are glued (equal arrow counts) into the three double curves D_a, D_b, D_c , and the six corners fall on two triple points, P (filled) and Q (hollow), alternating. Right: the Hasse diagram of the strata, with 1, 3, 2 elements per rank; every double curve passes through both triple points.

lattice and, by Theorem 1, the single point of the monotile family at which the substrate is its incidence structure. Section 4 tests non-descent on a computed tiling and Section 6 asks what survives when the tiling is folded onto the torus anyway.

3 The dual complex of the cusp fibre

The dual complex of a normal-crossing fibre has a vertex for each component, an edge for each double curve and a triangle for each triple point. For W the incidences are those of the A_2 triangulation modulo Λ .

Proposition 1. *The dual complex $\Delta(W)$ has 1 vertex E_0 , 3 edges D_a, D_b, D_c (along $e_1, e_2 - e_1, e_2$) and 2 triangles P (lower) and Q (upper); every edge borders both triangles. It is the torus, with*

$$\partial P = \underbrace{D_a + D_b - D_c}_{\partial_2 P}, \quad \partial Q = \underbrace{D_c - D_a - D_b}_{\partial_2 Q} = -\partial P, \quad \partial_1 = 0, \quad (4)$$

and therefore

$$\underbrace{H_0 = \mathbb{Z}}_{\text{rk } 1}, \quad \underbrace{H_1 = \mathbb{Z}^3 / \langle (1, 1, -1) \rangle \cong \mathbb{Z}^2}_{\text{rk } 2}, \quad \underbrace{H_2 = \ker \partial_2 = \mathbb{Z} \langle P + Q \rangle}_{\text{rk } 1}, \quad (5)$$

$$\chi = \underbrace{1 - 2 + 1}_{\text{Betti}} = \underbrace{1 - 3 + 2}_{V-E+F} = 0,$$

and $\pi_1 \Delta(W) = \langle a, b, c \mid ab = c, ba = c \rangle = \mathbb{Z}^2 \cong \Lambda$.

Proof. The vertices of the A_2 triangulation form Λ itself ($|\det B_0| = 1$), so there is one vertex per domain. Its edges fall into three Λ -orbits of directions, and its triangles into the two orbits “lower” $(0, 0), (1, 0), (0, 1)$ and “upper” $(1, 0), (1, 1), (0, 1)$ of the unit parallelogram cut along its short diagonal. Reading the boundary of P round its vertices gives e_1 , then $e_2 - e_1$, then $-e_2$, which is (4); the boundary of Q is the negative. Each edge is a loop, so $\partial_1 = 0$. The matrix of ∂_2 has the single primitive column $(1, 1, -1)^T$ up to sign, giving (5). The fundamental group is generated by the three loops with one relation per triangle, $ab = c$ from P and $ba = c$ from Q , hence $ab = ba$. The counts and incidences are asserted against `sixsphere.cusp_fibre()` by the figure script before anything is drawn. $\square$

$V - E + F = 1 - 3 + 2 = 0$: the dual complex of W is a torus

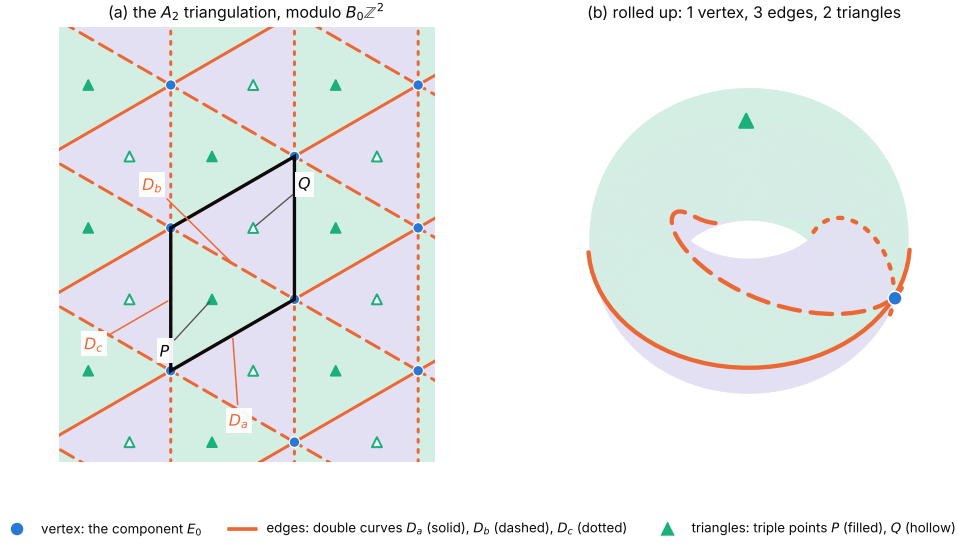


Figure 8: The dual complex of W . (a) The A_2 triangulation modulo $B_0\mathbb{Z}^2$: triangles tinted by class P or Q , double curves as orange lines in three styles (solid D_a , dashed D_b , dotted D_c), the domain as the black parallelogram. (b) The same complex rolled up: one vertex, three loops, two triangles.

It has $2E = \underbrace{6}_{\text{oriented edges}} = 3F = \underbrace{6}_{(\text{triangle, side})}$ oriented edges; this number returns in Section 6.

4 Rolling up a Hat tiling

The patch. Tilings are generated by the substitution system of `spectre.py`, which builds `Tile(a, b)` for any proportions; at $(1, \sqrt{3})$ it is the Hat, with the mystic Γ_2 at $(\sqrt{3}, 1)$. The script asserts that every tile vertex lies on a site of the $[3.4.6.4]$ lattice at $a : b = \sqrt{3} : 1$, every Hat is eight whole kites and every mystic ten. A patch after n steps is a disc: 443 tiles for $n = 3$ and 656 for $n = 4$.

The certificate. Call a lattice bond b a *tile boundary*, $\beta(b) = 1$, if its two adjacent kites lie in different tiles, and $\beta(b) = 0$ otherwise. For a bond orbit o of $\mathbb{T}$ (six centre-edge and six edge-vertex orbits) put

$$f(o) = \frac{\overbrace{\#\{b \in o : \beta(b) = 1\}}^{\text{boundary bonds in the patch}}}{\underbrace{\#\{b \in o\}}_{N_o}}. \quad (6)$$

Proposition 2. *If a tiling is invariant under Λ , then $\beta(b + \lambda) = \beta(b)$ for all $\lambda \in \Lambda$, so β is constant on each orbit and $f(o) \in \{0, 1\}$ for all twelve orbits.*

Proof. A tile boundary is determined by the tiling, and λ maps the tiling to itself, hence maps boundary bonds to boundary bonds and the others to the others. Orbits of $\mathbb{T}$ are the Λ -orbits of bonds. $\square$

For the patches, every $f(o)$ lies strictly inside $(0, 1)$: from 0.338 to 0.470 for $n = 3$ and from

0.323 to 0.480 for $n = 4$ (Fig. 2(b,c)); the script asserts that none is 0 or 1. Their mean is

$$\underbrace{\frac{1}{12} \sum_o f(o)}_{0.403} \approx \underbrace{\frac{\#\{\text{boundary bonds}\}}{\#\{\text{bonds}\}}}_{0.402} \quad (n = 4), \quad (7)$$

the two differing only because the orbit sizes N_o differ by a few bonds.

Overlap with translates. For $t = iE_1 + jE_2 \neq 0$ let

$$\alpha(t) = \frac{\#\{b : b, b+t \in \text{patch}, \beta(b) = \beta(b+t)\}}{\underbrace{\#\{b : b, b+t \in \text{patch}\}}_{>0}}, \quad (8)$$

which would be 1 for every $t \in \Lambda$ if the tiling were invariant. For $n = 4$ the best non-identity agreement is $\alpha = 0.796$, at $(i, j) = (-3, -5)$ (Fig. 9), and no translate gives 1.

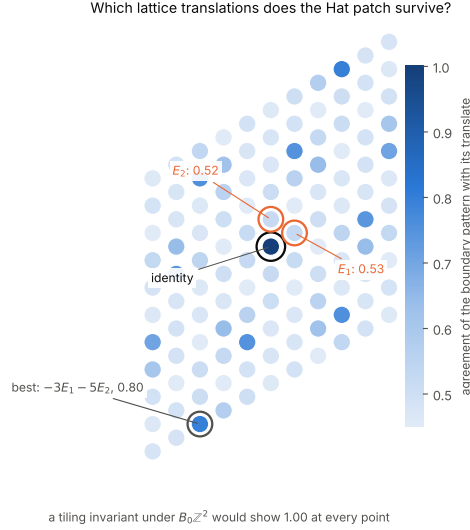


Figure 9: The agreement $\alpha(t)$ of the four-step patch's boundary pattern with its own lattice translates $iE_1 + jE_2$. The identity shift is trivially 1.00, while the optimal non-identity translation occurs at $t = -3E_1 - 5E_2$ with an overlap score of 0.796. A tiling invariant under $B_0\mathbb{Z}^2$ would exhibit 1.00 at every lattice point; the absence of any other unit score serves as a finite-patch certificate of non-descent, with partial peaks ($\sim 80\%$) capturing local self-similar cluster alignments.

These are finite-patch estimates: the fractions depend on the patch radius, and the statement they illustrate, that a Hat tiling is invariant under no translation, is the theorem of [7] and not something they prove.

5 Einstein Monotile: Hat, Spectre, Turtle

Theorem 1 singles out the Hat point for the *substrate*. The family has three famous members: the Hat Tile($1, \sqrt{3}$), the Spectre Tile($1, 1$), and the Turtle Tile($\sqrt{3}, 1$), the mystic of the Hat tiling. Figure 10 sets each beside its own substrate and its rolled-up domain.

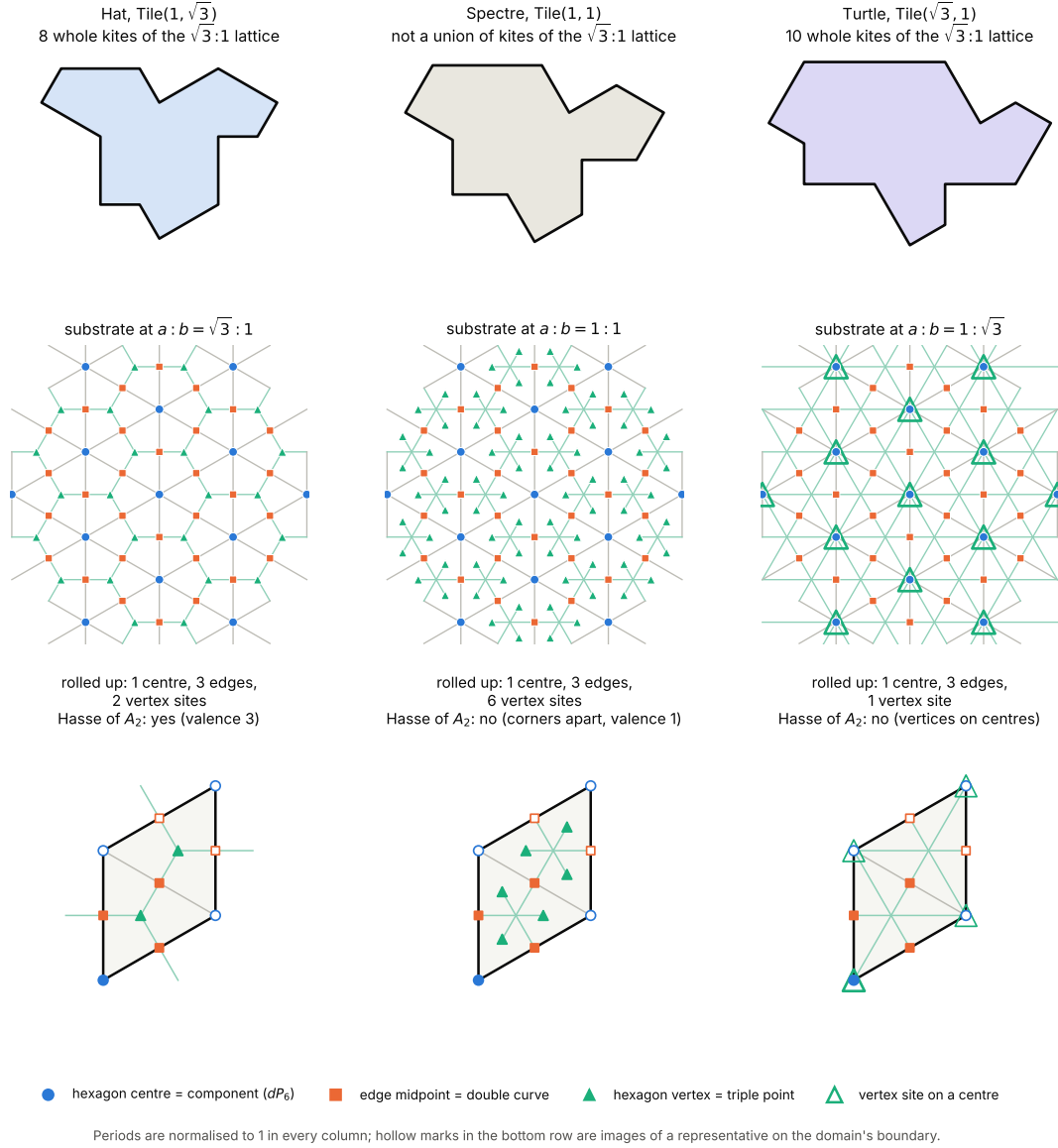


Figure 10: Hat, Spectre and Turtle. Top: the tile. Middle: the substrate at the proportions matching the tile. Bottom: that substrate rolled up on its own period lattice; hollow marks are boundary images. Only in the left column is the substrate the Hasse diagram of A_2 ; in the middle the corners stay apart, at right the vertex sites fall on the centres.

	substrate $a : b$	sites/domain (centre, edge, vertex)	Hasse of A_2	kites on $\sqrt{3}:1$ lattice
Hat	$\sqrt{3} : 1$	1, 3, 2	yes	8
Spectre	1 : 1	1, 3, 6	no	not a union
Turtle	1 : $\sqrt{3}$	1, 3, 1	no	10

At the Hat point the $\underbrace{12}_{\text{corners}}$ outer corners of a hexagon merge pairwise by (1) into $\underbrace{6}_{\text{vertices}}$, each shared by three hexagons, so $6/3 = 2$ vertex sites per domain; the computed counts are 6 at the Spectre's proportions, where the corners stay apart, and 1 at the Turtle's, where the vertex sites coincide with the neighbouring centres. Three computed remarks:

- The Hat's substrate gives the dictionary of Section 2; the Spectre's has six vertex sites of valence 1 per domain; the Turtle's own substrate is degenerate (one merged position per domain).
- The Turtle *as a tile* is ten whole kites of the Hat's lattice, of area $10\sqrt{3}$ against the Hat's $8\sqrt{3}$. That is what lets the mystic fold onto the same torus although its own substrate is not that lattice.
- The Spectre is not a union of kites of the Hat-point lattice, so it cannot be folded onto $\mathbb{T}$ in this way.

"Eight kites" is therefore a statement about the Hat; the Turtle is a ten-kite polykite on the same lattice.

6 Monotile Mystic fibres

Set-up. Let $E_1 = (3, \sqrt{3})$, $E_2 = (0, 2\sqrt{3})$, $\Lambda = \mathbb{Z}E_1 + \mathbb{Z}E_2$ and $\pi : \mathbb{R}^2 \rightarrow \mathbb{T} = \mathbb{R}^2/\Lambda$. Around each hexagon centre $c \in \Lambda$ there are six kites with vertices c , $c + \sqrt{3}u_{30+60k}$, $c + 2u_{60k+60}$, $c + \sqrt{3}u_{90+60k}$. Areas fit:

$$\underbrace{|\det(E_1, E_2)|}_{6\sqrt{3}} = \underbrace{6}_{\text{kites per domain}} \times \underbrace{\sqrt{3}}_{\text{kite}}, \quad (9)$$

so π identifies a kite of the plane with one of the six kites of $\mathbb{T}$ according to its index mod 6. Corner angles are 60° (centre), 90° (edge midpoint, twice) and 120° (hexagon vertex).

A *mystic* is a copy $gM_0 + x$ of a fixed ten-kite polygon M_0 with a distinguished vertex v_0 (the first in the vertex list), g a rotation and x a site.

Definition 1. Two mystics M, M' of a patch are in the same *fibre class* if $M' = M + \lambda$ for some $\lambda \in \Lambda$. The fibre point of M is π of its centroid.

Lemma 1. *Six kites meet at a site only at a hexagon centre; four meet at an edge midpoint and three at a hexagon vertex: $\underbrace{360^\circ/60^\circ}_6, \underbrace{360^\circ/90^\circ}_4, \underbrace{360^\circ/120^\circ}_3$. Hence a translation preserving the kite tiling lies in Λ , and the rotation ρ by 60° about a hexagon centre preserves both the kite tiling and Λ .*

Proof. A translation preserving the tiling preserves the number of kites at each site, so it maps centres to centres, which form the single coset Λ (one per domain). For ρ : $\rho(E_1) = E_2$ and $\rho(E_2) = E_2 - E_1$, both in Λ , and the kite tiling is invariant under the 6-fold rotation about any centre. $\square$

Proposition 3. *In a tiling by whole kites whose mystics are rotations and translations of one polygon M_0 with anchor v_0 , the mystics fall into at most $\underbrace{12}_{\text{first-edge angles}} \times \underbrace{6}_{\text{sites mod } \Lambda} = 72$ fibre classes.*

Proof. Every tile edge is a union of kite edges, whose directions lie in $30^\circ\mathbb{Z}$, so the first edge of gM_0 has at most 12 angles. The anchor is a site, and mod Λ there are $1 + 3 + 2 = 6$ sites. Two mystics with equal angle and equal anchor mod Λ differ by an element of Λ . $\square$

n	tiles	mystics	fibre classes	class sizes
1	9	1	1	1
2	71	8	5	1, 1, 2, 2, 2
3	443	51	6	6, 6, 7, 9, 11, 12
4	656	76	6	6, 7, 12, 14, 17, 20

Table 1: Fibre classes of the mystics in the Hat-point patches (computed; every tile vertex asserted against the lattice). The $n = 4$ patch is a radius-cut disc.

The data.

Observation 1 (orientation and anchor). For $n = 3, 4$ the mystics occur in exactly six first-edge angles $\theta_k = 30^\circ + 60^\circ k$, $k \in \mathbb{Z}/6$ (the $n = 2$ patch shows five). In every patch the anchor v_0 is an edge-midpoint site and its class mod Λ is determined by the angle:

double curve	D_a	D_c	D_b
anchor mod Λ	$(\frac{1}{2}, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
θ	$30^\circ, 210^\circ$	$90^\circ, 270^\circ$	$150^\circ, 330^\circ$
k	0, 3	1, 4	2, 5

So a fibre class is exactly a pair (orientation, anchor), and there are six.

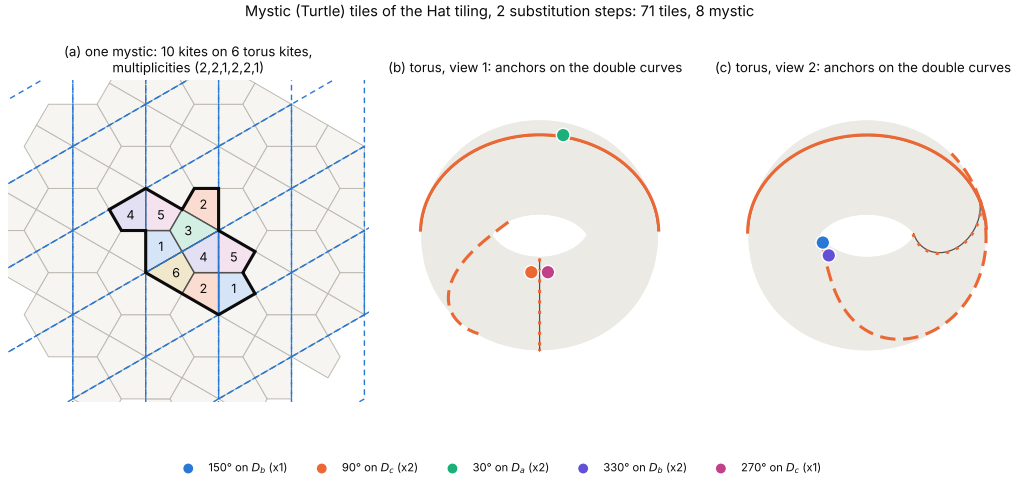


Figure 11: The same as Fig. 3 after two substitution steps (71 tiles, 8 mystic, five classes), where classes of size two were first noticed.

Why six, structurally. The six classes are not an arbitrary list.

Proposition 4. *The observed classes are one orbit of the rotation ρ of Lemma 1 acting on (θ, anchor) , and ρ acts freely:*

$$\#orbit = \frac{\overbrace{|\langle \rho \rangle|}^6}{\underbrace{|\text{Stab}(M_0)|}_1} = 6. \quad (10)$$

Under $k \mapsto \rho^k$ the anchor passes through the edge classes in the cyclic order $D_a \rightarrow D_c \rightarrow D_b \rightarrow D_a \rightarrow \dots$, so by the Chinese remainder theorem

$$\underbrace{\mathbb{Z}/6}_{\text{classes}} \cong \underbrace{\mathbb{Z}/3}_{k \bmod 3: \text{ double curve}} \times \underbrace{\mathbb{Z}/2}_{k \bmod 2: \text{ orientation of the edge}}, \quad (11)$$

that is, $(\text{double curve}) \times (\text{sign}) = 3 \times 2 = 2E$.

Proof. ρ preserves the kite tiling and Λ (Lemma 1), so it maps mystic placements to mystic placements, raising the first-edge angle by 60° . The stabiliser is trivial because M_0 has no rotational symmetry (any nontrivial rotation changes its edge-angle sequence). The edge midpoints of the hexagon at the origin are $\sqrt{3}u_{30^\circ+60^\circ j}$, which mod Λ are $E_1/2$, $E_2/2$ and $(E_2 - E_1)/2 \equiv (E_1 + E_2)/2$ for $j = 0, 1, 2$ and repeat with period 3 because $-E_1/2 \equiv E_1/2$. These are D_a, D_c, D_b . That the observed classes are exactly this orbit is the content of Observation 1. $\square$

Conjecture 1. *For every $n \geq 3$ the mystics of the full level- n patch fall into exactly the six classes of Observation 1, one for each oriented double curve of $\Delta(W)$.*

We have no proof. The data support ≤ 6 for $n = 3, 4$ on a disc, far below the bound 72 of Proposition 3: only $\underbrace{6}_{\text{observed}}/72 = 1/12$ of the possible classes occur. A proof would have to come from the metatile structure of the substitution, since the registration of a tile with the kite lattice is inherited from the level below.

Same fibre, different surroundings. Equal fibre points do not contradict non-descent. For each mystic in the inner 60% of the patch record the multiset of (label, angle) of the tiles it touches. For $n = 3$ three of the six classes contain two environments among their interior members (the rest one), and for $n = 4$ again three classes show two, with one to nine interior mystics per class. The counts are small and are an indication only. The logic is the safe part: a tiling that descends has identical surroundings throughout a class (Proposition 2), so a class with two environments is an obstruction, the tile-level form of the boundary-fraction certificate.

Folding one mystic. A mystic has ten kites and $\mathbb{T}$ six, so it covers $\mathbb{T} \underbrace{10}_{\text{kites}} / \underbrace{6}_{\text{per domain}} = \frac{5}{3}$ times. The mystic of 30° in the three-step patch folds onto the six torus kites with multiplicities $m = (2, 2, 1, 2, 2, 1)$, $\sum_k m_k = 10$ (Fig. 3(a)); other orientations give ρ -rotations of this pattern.

Open questions.

1. Prove or refute Conjecture 1; equivalently determine the anchor and orientation of the mystic in a general metatile.
2. Is the number of neighbourhood types per class bounded, or does it grow with n ? This is the quantitative form of non-descent.
3. The same count for the Hat itself (eight kites): the analogue of Proposition 3 is 72 (or 144 with reflections); it has not been computed.
4. Is the identification of the six classes with the oriented double curves of $\Delta(W)$ natural, for instance through the monodromy of the degeneration, or a coincidence of the Hat point? We have no argument either way.

7 The Flag Class: Block Structure of the Monodromy Matrices

Write $V = \mathbb{Z}^4$ with basis (γ, u, w, δ) . Alpöge's matrices fix γ , preserve $\langle \gamma, u, w \rangle$, and act trivially on the top quotient ([2, Lemma 2.7]); so each has the block form

$$T = \begin{pmatrix} 1 & r & s \\ 0 & M & c \\ 0 & 0 & 1 \end{pmatrix}, \quad M \in \text{SL}_2(\mathbb{Z}), \quad r \in \mathbb{Z}^{1 \times 2}, \quad c \in \mathbb{Z}^{2 \times 1}, \quad (12)$$

where M is the monodromy of the elliptic curve with period τ — modulo γ , the first period vector is $\tau u + w$ — and hence a Kodaira monodromy [13]. We call pairs (T_1, T_2) of this shape the *flag class*. Everything in Sections 8 and 9 is a statement about the flag class, and about nothing else.

The quantities we need are the paper's. The cusp monodromy is $T_0 = (T_1 T_2)^{-1} = I + N$; the fibre over the cusp is a single copy of the degree-six del Pezzo surface exactly when $|\det B_0| = 1$, where B_0 is induced by $M_0 - I$ on the dual lattice [2, Lemma 2.6]; and for twists ℓ_0, ℓ_1, ℓ_2 the presentation of [2, Theorem 7.17], abelianised, gives

$$|H_1(X)| = |p|, \quad p = \underbrace{m_1 m_2 \ell_0}_{m_1 m_2 \mid} - \underbrace{m_2 \ell_1}_{m_2 \mid} - \underbrace{m_1 \ell_2}_{m_1 \mid}, \quad (13)$$

with $(m_1, m_2) = (3, 4)$ for the paper. A twist is admissible when the logarithmic transform it defines is free, which for the paper's data is $3 \nmid \ell_1$ and ℓ_2 odd, a rule that follows from a fixed-point computation on the real torus.

8 Results: Why $(3, 4, \infty)$ Is the Only Allowed Triangle Group

Proposition 5. *In the flag class, a pair (T_1, T_2) of orders (m_1, m_2) , with (m_1, m_2, ∞) hyperbolic, a unipotent cusp, and $|p| = 1$ for some twist, has $\{m_1, m_2\} = \{3, 4\}$.*

Proof. The middle block M_j has the same finite order as T_j (a block-unipotent matrix of finite order is the identity), and the finite orders in $\mathrm{SL}_2(\mathbb{Z})$ are $1, 2, 3, 4, 6$. The only element of order two is $-I$, which is central; if $M_1 = -I$ then $M_0 = (M_1 M_2)^{-1} = -M_2^{-1}$ has trace $-\mathrm{tr} M_2 \in \{1, 0, -1\}$ for $m_2 \in \{3, 4, 6\}$, so the cusp is not unipotent. This removes every $(2, m)$. Finally, by (13) each term of p is divisible by $\gcd(m_1, m_2)$, so

$$\underbrace{\gcd(m_1, m_2)}_d \mid p \implies |p| = 1 \text{ requires } \underbrace{d=1}_{\text{coprime orders}}; \quad ;$$

among the hyperbolic pairs from $\{3, 4, 6\}$ only $(3, 4)$ is coprime. $\square$

The gcd alone therefore disposes of $(3, 3)$, $(4, 4)$ and $(6, 6)$. The survey finds that they fail earlier too, for a reason of independent interest:

The congruence is short enough to machine-check for every twist: the Lean facts `gcd_obstruction_3_6` and `gcd_obstruction_4_6` state $3 \mid 18\ell_0 - 6\ell_1 - 3\ell_2$ and $2 \mid 24\ell_0 - 6\ell_1 - 4\ell_2$ for all integers, by `omega`.

Observation 2. Enumerating $\mathrm{SL}_2(\mathbb{Z})$ elements in a box, $(4, 4)$ has no pair whose product is $\pm(\text{unipotent})$, and $(3, 3)$ and $(6, 6)$ have parabolic products only as $-(\text{unipotent})$: an I_2^* cusp, never I_n . The outcome is the same in the boxes ± 4 and ± 7 . For $(3, 3)$ the mechanism is visible in $\mathrm{PSL}_2(\mathbb{Z}) = \langle s \rangle * \langle t \rangle$, where $t \cdot sts^{-1} = (ts)^2$ is parabolic, but its lifts to $\mathrm{SL}_2(\mathbb{Z})$ carry the wrong sign. We have not turned this into a proof.

Table 2 shows the survey. Two of the excluded groups are worth a remark. $(3, 6, \infty)$ and $(4, 6, \infty)$ carry perfectly consistent elliptic surfaces — with fibres $\mathrm{IV}^* + \mathrm{II} + \mathrm{I}_2$ and $\mathrm{III}^* + \mathrm{II} + \mathrm{I}_1$, Euler numbers summing to 12 as Noether's formula requires — and fail only arithmetically: the best $|p|$ they reach is 3 and 2. So the paper's choice of triangle group is not a choice; it is the only coprime pair that survives the sign.

triangle group	unipotent pairs	cuspid	gcd	verdict
$(2, 3, \infty)$	0	—	1	no unipotent cuspid
$(2, 4, \infty)$	0	—	2	no unipotent cuspid
$(2, 6, \infty)$	0	—	2	no unipotent cuspid
$(3, 3, \infty)$	0	—	3	cuspid only I_2^*
$(3, 4, \infty)$	20	I_1	1	candidate
$(3, 6, \infty)$	20	I_2	3	$ p \geq 3$ (gcd)
$(4, 4, \infty)$	0	—	4	no unipotent cuspid
$(4, 6, \infty)$	20	I_1	2	$ p \geq 2$ (gcd)
$(6, 6, \infty)$	0	—	6	cuspid only I_2^*

Table 2: The nine hyperbolic (m_1, m_2, ∞) with $m_j \in \{2, 3, 4, 6\}$. “Unipotent pairs” counts middle blocks (M_1, M_2) in the box ± 4 whose product is unipotent; the cuspid is its Kodaira type [13]. 4 groups have no unipotent cuspid and 2 have one only as I_2^* ; the remaining two besides $(3, 4)$ are excluded by the gcd.

9 Uniqueness of the Lattice Constraints: the Alpöge Matrices

9.1 The families

Conjugating (12) by $U = \begin{pmatrix} 1 & x & z \\ 0 & I & y \\ 0 & 0 & 1 \end{pmatrix}$ fixes M and acts by

$$r \mapsto r + x(M - I), \quad c \mapsto c + (I - M)y, \quad (14)$$

with the *same* x and y on both generators (derived symbolically). Since $M_2 - I$ is invertible over $\mathbb{Q}$, with $\det(M_2 - I) = \underbrace{\det M_2}_1 - \underbrace{\text{tr } M_2}_0 + 1 = 2$ for the order-four block, the data of T_2 reduces to 2×2 residue classes — and that spends all of x and y . Nothing is left to normalise T_1 : its r and c are genuinely free, and the extensions come in infinite families. Given r and c , the corner s is fixed by $T^m = I$, whose corner reads $ms + f(r, c) = 0$ and need not have an integer solution.

9.2 The funnel

We fix M_2 to Alpöge’s middle block (every element of order four is conjugate to it in $\text{GL}_2(\mathbb{Z})$), take the 2 order-three M_1 whose product with it is unipotent of type I_1 , run T_2 over its residue classes and T_1 ’s (r, c) over the box $[-B, B]^4$, and apply the conditions of Section 7 in stages. At $B = 5$:

stage	survivors
candidates tried	87846
$T_1^3 = T_2^4 = I$ (corner integral)	49446
unipotent cuspid, $N^2 = 0$, rank $N = 2$	272
coinvariants exactly $\mathbb{Z}$	84
invariant closure of Λ_{tor} is $\ker \gamma$	84
$ \det B_0 = 1$	32

Observation 3. At the boxes $\pm 3, \pm 4, \pm 5, \pm 8, \pm 11$ the final count is 32, 32, 32, 32, 32, while the cuspid stage grows 136, 200, 272, 760, 1488 (Fig. 12). The condition $|\det B_0| = 1$ appears to cut the infinite families to a finite set. We have no proof. Separately, the closure stage has never removed a candidate, which suggests it follows from the coinvariant condition in the flag class; that too is unproved.

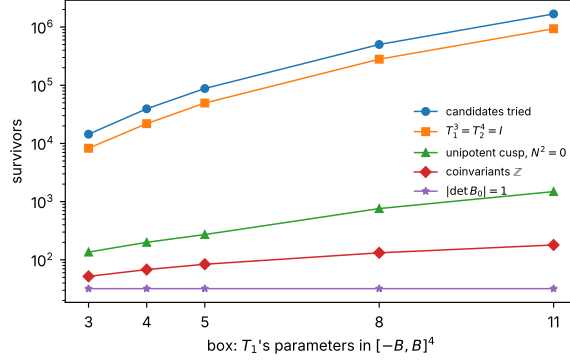


Figure 12: The funnel’s stage counts against the box. Every stage but the last keeps growing; the last is 32 throughout.

9.3 The classes

Proposition 6. *The 32 survivors form four $\mathrm{GL}_4(\mathbb{Z})$ -conjugacy classes of 8. Exactly one class can reach $|p| = 1$.*

Proof. Membership: for each survivor, the conjugating matrix is found by solving the linear system $PT_j = T_j'P$ exactly and searching its integer points; each is verified. Distinctness: the set of values of $|p|$, taken modulo $m_1m_2 = 12$ as $\min(x, 12 - x)$, that admissible twists can reach is a conjugacy invariant — the functional γ that reads off ℓ_j is canonical up to sign — and it takes a different value on each class: $\{1, 5\}$ on one, and $\{0, 6\}$, $\{2, 4\}$, $\{3\}$ on the others. Only the first contains 1. $\square$

The proof of distinctness does not depend on the conjugator search, which could in principle miss a large conjugator: two classes with different invariants cannot be conjugate, however hard one looks (Fig. 13).

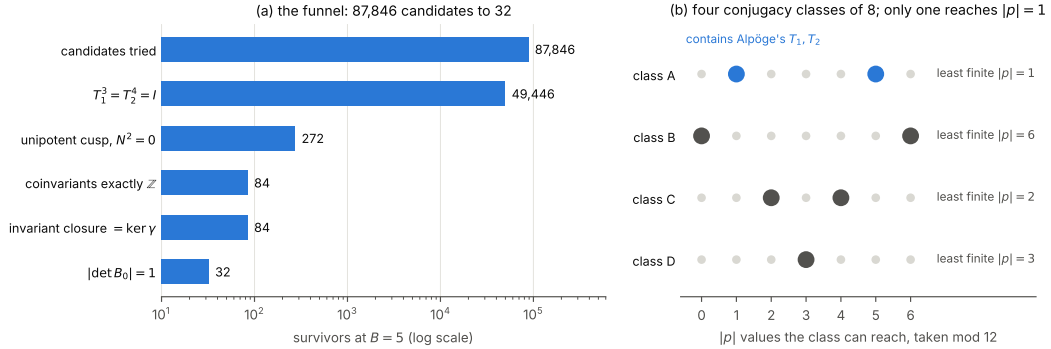


Figure 13: (a) The funnel at $B = 5$, on a log scale: the stage counts of the table above, from 87846 candidates to 32. (b) The 32 survivors form four $\mathrm{GL}_4(\mathbb{Z})$ -conjugacy classes of 8. Each is marked by the values of $|p|$, taken as $\min(x, 12 - x)$, that its admissible twists can reach; the four sets partition $\{0, \dots, 6\}$, and only the class containing Alpöge’s matrices reaches 1.

The class that reaches $|p| = 1$ is the paper’s:

Proposition 7. *The integer matrix*

$$P = \begin{pmatrix} 1 & 3 & -3 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \det P = -1,$$

conjugates Alpöge's (T_1, T_2) onto the normal-form representative of the class that reaches $|p| = 1$.

This is checked by multiplication, and in Lean (`paper_conjugator`, by `decide`). The other three classes bottom out at $|p| = 6, 2, 3$ respectively. Whether any of them supports the period functions of the construction — and so a compact complex threefold at all, a rational homology sphere in the cases $|p| = 2, 3$ — is a question about the analytic side that we do not address.

10 Orientation Reversal and Lattice Duality

Proposition 8. *Reversing the orientation of the base replaces (T_1, T_2) by (T_1^{-1}, T_2^{-1}) [2, Remark 3.18]. The two pairs are conjugate:*

$$P = \begin{pmatrix} 1 & 6 & -6 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \det P = 1,$$

satisfies $PT_jP^{-1} = T_j^{-1}$ for $j = 1, 2$.

The monodromy therefore cannot distinguish X from its orientation-reversed counterpart. This is a statement about the lattice, not a proof that X is biholomorphic to its conjugate. It is machine-checked (`lattice_achiral`). The involution is orientation reversal of the base; it preserves the reachable $|p|$ residues $\{1, 5\}$ (as it must, since $|p|$ is insensitive to sign), and the lattice data is a fixed point, in contrast with the Spectre, which is strictly chiral [8].

Proposition 9. *The monodromy lattice V and its dual are isomorphic over $\mathbb{Q}$ but not over $\mathbb{Z}$: every integral intertwiner has determinant $36s^4$ in the coordinates (s, t) of the rank-two lattice of intertwiners.*

Proof. The invariant alternating form Q_0 of [2, Lemma 2.8] satisfies $Q_0T = T^{-t}Q_0$, so it intertwines V and its dual over $\mathbb{Q}$; but $\det Q_0 = 36$. The intertwiners form a rank-two lattice, and the determinant of the general one factors as $36s^4$, which is never ± 1 . $\square$

Alpöge shows the construction carries no polarization, because the form $b(x, y) = Q_0(x, Ny)$ is indefinite. Proposition 9 is a sharper, integral statement: there is not even a non-positive *principal* one.

11 Arithmetic Invariants and Resemblances to the Spectre

The special values of the construction's period functions lie at the elliptic points, $\tau(z_1) = e^{i\pi/3}$ and $\tau(z_2) = i$, generating $\mathbb{Q}(\sqrt{-3}, i) = \mathbb{Q}(\zeta_{12})$. Its quadratic square classes are $\{-3, -1, 1, 3\}$ — signed: dropping the sign would report the classes of the real field $\mathbb{Q}(\sqrt{3})$ and manufacture an identity with the monotile's geometry. With signs kept:

field	square classes	shared with $\mathbb{Q}(\zeta_{12})$
six-sphere periods, $\mathbb{Q}(\zeta_{12})$	$\{-3, -1, 1, 3\}$	—
monotile geometry, $\mathbb{Q}(\sqrt{3})$	$\{1, 3\}$	$\{1, 3\}$
Hat inflation, $\mathbb{Q}(\sqrt{5})$	$\{1, 5\}$	$\{1\}$
Spectre inflation, $\mathbb{Q}(\sqrt{6}, \sqrt{10})$	$\{1, 6, 10, 15\}$	$\{1\}$

The six-sphere shares the monotiles' *geometry* field and none of their *inflation* fields — consistent with Section 2, where it shares the substrate and not the tiling.

Two further facts sharpen the picture. The Euclidean triangle groups are $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$; none contains both a 3 and a 4, so the base orbifold of the construction is necessarily hyperbolic, while the substrate's symmetry is the Euclidean $(2, 3, 6)$. And the cusp component's polygon is the hexagon of the toric surface B_3 , whose hopping vectors are those of the triangular lattice: the mirror curve of the component is the triangular-lattice Hofstadter model of [9, 10].

The $\sqrt{6}$. The form $b = 6w^2 - \delta^2$ has isotropic slopes $\pm\sqrt{6}$, and the Spectre's inflation factor is $\lambda = (\sqrt{6} + \sqrt{10})/2$, with $\lambda^2 = 4 + \sqrt{15}$: the two share the square class $\{1, 6\}$. The origins are unrelated. The six-sphere's 6 is $Q_0(u, w)$, forced by the entry -6 of T_1 , which the period laws pin down; the Spectre's is a consequence of its substitution matrix. Nothing downstream of either uses the other.

Valence four. The Laves valences $(6, 4, 3)$ echo the orbifold orders $(3, 4, \infty)$. But the valence-four sites are the double curves of W , which lie over the cusp p_0 — not over p_2 , the point of order four, whose fibre is a multiple fibre with no double curves at all.

12 Central Charges on Calabi–Yau Threefolds

We ask what the six-sphere's finite layer shares with three standard objects: a reflexive polygon, the lattice $H^2(K3, \mathbb{Z})$ with Kodaira's classification, and Bridgeland central charges on Calabi–Yau threefolds. Each number below is reached by at least two routes.

12.1 The cusp component and the volume product

The component of the cusp fibre is the degree-six del Pezzo surface dP_6 , the toric surface whose polygon is B_3 , the hexagon with 6 boundary points. Its anticanonical degree K^2 is the normalised volume $V(B_3) = 2 \text{Area}(B_3)$ (the identity $V(P) = (-K)^n$ for the Gorenstein toric Fano variety whose anticanonical polytope is the reflexive polytope P ; B_3 is self-dual, so there is no ambiguity about which of the pair is meant).

Proposition 10. $K_{dP_6}^2 = 6$. *This is computed four ways, with no shared code: by triangulating the hexagon ($V(B_3) = 6$); by Ehrhart counts of the same polygon; by Noether's formula $K^2 = 12 - n = 12 - 6$ for a smooth complete toric surface with n rays [11]; and as $9 - 3$ for $\mathbb{P}^2$ blown up at three points. The toric divisors have self-intersections $(-1)^{\times 6}$, and*

$$\underbrace{\sum_{i=1}^n D_i^2}_{(-1)^{\times 6} = -6} = \underbrace{12 - 3n}_{n=6} \quad \text{while} \quad K^2 = \underbrace{12 - n}_{n=6} = 6,$$

as they must.

The hexagon is self-dual: $V(B_3^*) = 6$, so the volume product is $V(B_3)V(B_3^*) = 36$. It is centrally symmetric, and the Mahler bound for symmetric bodies is $4^2 \cdot 2! = 32$, so B_3 lies strictly above it (only two reflexive polygons attain it). The number that the dictionary of Section 2 reads off as six kites per domain, six (-1) -curves of the hexagon, six corners, is therefore also the volume of the polytope; the agreement is the toric fact that a smooth toric surface has as many torus-fixed points as rays, together with $K^2 = 12 - n$, and it is specific to $n = 6$. We claim nothing beyond that.

12.2 The elliptic fibration and the lattice module

The middle block of the monodromy is that of an elliptic family over $\mathbb{P}^1$ with singular fibres over the three points of the orbifold, so each local monodromy fixes a Kodaira type up to the usual inversion: order 3 means IV or IV*, order 4 means III or III*, and the cusp, being unipotent, is

of type I_n ($n = 1$ here). The Euler numbers of the singular fibres of an elliptic surface over $\mathbb{P}^1$ sum to $12\chi(\mathcal{O})$.

Proposition 11. *For orders $(3, 4)$ and an I_1 cusp there are 4 candidate triples, with Euler totals 8, 12, 14, 18. Exactly one is a multiple of 12:*

$$IV^* + III + I_1, \quad \underbrace{8}_{e(IV^*)} + \underbrace{3}_{e(III)} + \underbrace{1}_{e(I_1)} = \underbrace{12}_{12\chi(\mathcal{O})}.$$

Its Jacobian is a rational elliptic surface; a K3 (total 24) is impossible.

The same enumeration reproduces the two examples of Section 8, $IV^* + II + I_2$ for $(3, 6)$ and $III^* + II + I_1$ for $(4, 6)$, and so is checked against the paper’s own Euler arithmetic before being used for $(3, 4)$. The trivial lattice of such a surface is $U \oplus R$ with R the root lattice of the reducible fibres, and the Mordell–Weil rank is $10 - 2 - \text{rk } R$.

Proposition 12. *The root lattices $E_6 \oplus A_1$ and E_7 embed in E_8 both integrally and over $\mathbb{Q}$. Integrally: the 240 roots of E_8 are generated from its Cartan matrix; the roots orthogonal to one root α form E_7 (126 roots, rank 7, discriminant 2); those orthogonal to an $A_2 \ni \alpha$ form E_6 (72 roots, rank 6, discriminant 3), and $A_1 = \langle \alpha \rangle \subset A_2$ is orthogonal to it, giving $E_6 \oplus A_1$ of discriminant 6. Over $\mathbb{Q}$ the Hasse–Minkowski test of `pyCICY.k3lattice` accepts both. A control fails as it should: $E_7 \oplus \langle 3 \rangle$ has the right rank but the wrong discriminant and does not embed. In all three configurations $(IV^* + III + I_1, IV^* + II + I_2, III^* + II + I_1)$ the Mordell–Weil rank is 1.*

So the paper’s $(3, 4, \infty)$ data and its $(3, 6, \infty)$ example have the same root part, $E_6 \oplus A_1$, although the fibres differ. The scope is narrow. Euler counting shows that the local data is consistent with a rational elliptic surface of that configuration; it does not show that the six-sphere’s family *is* such a surface, and whether the configuration exists is a question about the Miranda–Persson classification [12] that we do not decide. The $\mathbb{Q}$ -embedding is a necessary condition only; the integral statement is the explicit construction inside E_8 .

12.3 Where the central-charge module stops

The central-charge module computes the charges $Z(E) = -\int e^{-B-i\omega} \text{ch}(E)$ (ordinary and square-root-Todd) of the large-volume stability conditions announced for smooth projective threefolds with trivial canonical bundle. They need a real class ω with $\omega^3 > 0$. The six-sphere has $H^2(X, \mathbb{R}) = 0$, so it carries no such class: it is not Kähler, and the module’s own guard refuses the empty class. We record this as a result rather than a gap. The finite layer behind the six-sphere is rigid, but the charges that give the CICY side its stability-theoretic content have no counterpart here.

13 Conclusion

The cusp fibre of Alpöge’s six-sphere is the aperiodic monotile’s substrate, at the monotile’s own distinguished point, rolled up on a torus (Theorem 1). Folding a computed Hat tiling onto that torus, the tiling does not descend, as it cannot: every one of the twelve bond orbits is sometimes a tile boundary and sometimes not (Proposition 2, Section 4). The mystic tile, however, lands on exactly six points of the torus in the patches computed, one per orientation, each anchored over a double curve of W and forming a single orbit of the hexagon’s rotation group (Proposition 4); six is the number of oriented edges of the dual complex $\Delta(W)$ (Proposition 1). Proved is the bound of 72 classes; the count of six is a conjecture backed by patches of three and four substitution steps, and the dependence of surroundings on class rests on small samples.

The lattice data was forced: within the flag class the triangle group by proof, the matrices up to conjugacy in every range searched, with finiteness of the search space the one step left

unproved. The lattice is achiral and not self-dual over $\mathbb{Z}$, its arithmetic meets the monotiles’ in their geometry and not in their inflation, and two resemblances with the Spectre are coincidences.

None of this bears on whether the construction is correct; that rests on analysis and topology and on the external formalization [4]. The six-sphere knows nothing of monotiles, and we draw no physical conclusion: the substrate identity, the six fibres and the contacts of Section 12 are identities of integers and of finite patches.

Declarations

Funding and Competing Interests

The author received no specific funding for this work or competing interests.

Source code

The exact computations are in the repository at <https://github.com/brentharts/CICY> (pyCICY.sixsphere_search, verification suite tests/test_sixsphere_search.py, Lean facts in pyCICY.sixsphere_lean, the contacts of Section 12 in pyCICY.sixsphere_links). The figures of Sections 2 to 6 are produced by the make_sixsphere_*.py scripts in paper/, using the substitution tilings of <https://github.com/brentharts/spectre>.

Limitations

Everything in Sections 8 and 9 is restricted to the flag class; lattice data outside it is not searched. Observations 2 and 3 are finite searches. Theorem 1 is a statement about substrates. The fibre counts of Sections 4 and 6 are computed on finite substitution patches (the four-step patch is a radius-cut disc); Conjecture 1 is a conjecture, and the neighbourhood counts of Section 6 rest on one to nine interior mystics per class. Section 12: the Kodaira configuration and the lattice embeddings are consistency statements for the local monodromy, not statements that the six-sphere’s fibration is that elliptic surface, and the rational embedding is a necessary condition only. Every analytic question — whether the near-miss classes support a complex structure, whether X is biholomorphic to its conjugate — is declined, as is the diffeomorphism $X \cong S^6$ itself.

References

- [1] H. Hopf, *Zur Topologie der komplexen Mannigfaltigkeiten*, in: Studies and Essays Presented to R. Courant, Interscience, New York (1948), 167–185.
- [2] L. Alpöge, *A compact complex threefold fibred by tori over the projective line, and the six-sphere*, [preprint] (2026), <https://alpo.ge/s6.pdf>.
- [3] P. Engel, pedagogical account of Alpöge’s construction, expository notes (2026), <https://philip-engel.github.io/S6.pdf>.
- [4] B. Alexeev, *Lean formalisation of the solution to the Hopf problem* (2026), <https://github.com/plby/HopfProblem>.
- [5] F. Campana, J.-P. Demailly and T. Peternell, *The algebraic dimension of compact complex threefolds with vanishing second Betti number*, Compositio Math. **112** (1998), 77–91; *Corrigendum*, Compositio Math. **156** (2020), 679–696.
- [6] R. Friedman, *Global smoothings of varieties with normal crossings*, Ann. of Math. **118** (1983), 75–114.
- [7] D. Smith, J. S. Myers, C. S. Kaplan and C. Goodman-Strauss, *An aperiodic monotile*, Combinatorial Theory **4** (2024) 6, arXiv:2303.10798.
- [8] D. Smith, J. S. Myers, C. S. Kaplan and C. Goodman-Strauss, *A chiral aperiodic monotile*, Combinatorial Theory **4** (2024) 13, arXiv:2305.17743.
- [9] F. Chen, W. O. Wang, J. -X. Zhang, L. Balents, and D. N. Sheng. (2025). *Topological Chiral Superconductivity in the Triangular-Lattice Hofstadter-Hubbard Model*. arXiv. <https://doi.org/10.48550/arxiv.2509.02757>
- [10] D. Shaffer, J. Wang, L. H. Santos (2022). *Unconventional self-similar Hofstadter superconductivity from repulsive interactions*. Nature Communications, 13. <https://doi.org/10.1038/s41467-022-35316-z>
- [11] W. Fulton, *Introduction to Toric Varieties*, Annals of Mathematics Studies **131**, Princeton University Press (1993).
- [12] R. Miranda and U. Persson, *On extremal rational elliptic surfaces*, Math. Z. **193** (1986), 537–558.
- [13] P. Graf and M. Schwald (2020). *On the Kodaira problem for uniruled Kähler spaces*. Arkiv för Matematik, 58(2), 267–284. <https://doi.org/10.4310/arkiv.2020.v58.n2.a3>