

Infinitely Many Primes $p = m^2 + n^2$ with $n < p^{0.1129}$

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Abstract

We prove that infinitely many primes $p = m^2 + n^2$ have $n < p^{0.1129}$, and that infinitely many primes satisfy $\{\sqrt{p}\} < p^{-0.2742}$. This sharpens the exponents 0.119 and -0.262 of Harman and Lewis (2001). Following their approach, we derive both statements from a lower bound of the correct order of magnitude for the number of Gaussian primes of norm between $X/2$ and X in every sector of angle $X^{-0.3871}$, and we prove this bound with Harman's sieve and their arithmetic information. The new ingredients are three groupings of terms that their sieve discards: a factorization of a two-dimensional sum for the linear estimate, and two applications of the bilinear estimate to four-dimensional sums, one of them after a further application of Buchstab's identity. These groupings give the grouped terms asymptotic formulas, so that fewer terms are discarded. In addition, we give every step of the sieve argument in full, filling in the details that Harman and Lewis present briefly and making several of their statements and discarded regions precise, and we evaluate the losses of the discarded sums by two numerical methods.

Keywords: Gaussian primes, narrow sectors, Harman's sieve, Hecke characters, fractional part of $\sqrt{p}$.

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1 Introduction

A prime $p \equiv 1 \pmod{4}$ is the norm $p = m^2 + n^2$ of a Gaussian prime $m + in$, a prime of the ring $\mathbb{Z}[i]$ of Gaussian integers. Landau asked whether $n = 1$ occurs for infinitely many primes, that is, whether infinitely many primes have the form $m^2 + 1$ [20]. The question is open. Iwaniec [13] showed that infinitely many integers $m^2 + 1$ have at most 2 prime factors. Infinitely many primes $p = m^2 + n^2$ are known to have n in sparse sets of arithmetic form: the squares [6], the squares of primes [11], and the integers $y^2 + 1$ [18]. These results restrict the form of n , not its size. A weaker form of Landau's question asks how small n can be taken for infinitely many primes $p = m^2 + n^2$. With $m > n > 0$, the condition $n < p^\theta$ places the Gaussian prime $m + in$ inside a sector of angle about $p^{\theta-1/2}$ next to the real axis. So the problem is one of Gaussian primes in narrow sectors.

Under the Riemann hypothesis for the Hecke L -functions of $\mathbb{Q}(i)$, Ankeny [1] and Kubilius [15] showed that infinitely many primes $p = m^2 + n^2$ have $n \ll \log p$. Without hypotheses, every sector of angle greater than $X^{-3/10+\varepsilon}$, for every fixed $\varepsilon > 0$, contains the expected number of Gaussian primes of norm in $(X, 2X]$, by a result of Ricci [21] that Coleman [5, Theorem 3] states for every imaginary quadratic field. This covers $n < p^\theta$ for every $\theta > 1/5$. For smaller θ , Coleman [5] obtained $n < p^{0.1631}$ with the Rosser–Iwaniec sieve in number fields. Harman and Lewis [10] used essentially the arithmetic information of Coleman with Harman's sieve [9], and obtained $n < p^{0.119}$. Among the results that hold for every sector, this bound has remained the best known [14, §1.2]. Harman et al. [8] apply the same sieve to sectors in which the norm also lies in a short interval. Results that hold for almost all sectors go much further. Huang et al. [12] prove an asymptotic formula for the number of Gaussian primes of norm in $(X, 2X]$ in almost all sectors of angle greater than $X^{-3/5+\varepsilon}$, for every fixed $\varepsilon > 0$. Järvinen and Teräväinen [14] find products of exactly 2 Gaussian primes in almost all sectors of area $(\log X)^{15.1}$ in the disc of radius $X^{1/2}$.

If $p = m^2 + n^2$ with positive integers m, n and $n < p^\theta$, then

$$0 < \sqrt{p} - m = \frac{n^2}{\sqrt{p} + m} < p^{2\theta-1/2}. \quad (1.1)$$

Let $\{x\}$ denote the fractional part of x . For $\theta < 1/4$, (1.1) gives $0 < \sqrt{p} - m < 1$, so

$$\{\sqrt{p}\} = \sqrt{p} - m < p^{-(1/2-2\theta)}. \quad (1.2)$$

Harman and Lewis deduced $\{\sqrt{p}\} < p^{-0.262}$ for infinitely many primes [10, Theorem 4] from their bound $n < p^{0.119}$ [10, Theorem 1]. Earlier, Balog [3] and Harman [7] had obtained the exponent $1/4 - \varepsilon$ for every fixed $\varepsilon > 0$. The exponent 0.262 has remained the best known [16]. No exponent above $1/2$ is possible in such a bound, since $|\sqrt{p} - k| \geq 1/(2\sqrt{p} + 1)$ for the integer k nearest to $\sqrt{p}$.

Our main result is the following.

Theorem 1 *There are infinitely many primes $p = m^2 + n^2$ with m, n positive integers and $n < p^{0.1129}$. Consequently, there are infinitely many primes p with $\{\sqrt{p}\} < p^{-0.2742}$.*

Theorem 1 follows from a lower bound for the number of Gaussian primes in narrow sectors. Here and below, the norm of a Gaussian integer $m + in$ is $|m + in|^2 = m^2 + n^2$. A non-zero Gaussian integer, and in particular a product of Gaussian integers, is identified with its associate of argument in $[0, \pi/2)$, and arguments are taken modulo $\pi/2$.

Theorem 2 *Let $\theta = 0.1129$. There is a constant $c > 0$ such that, uniformly in $\psi \in \mathbb{R}$ and for every sufficiently large X ,*

$$\#\{\varpi \in \mathbb{Z}[i] : \varpi \text{ Gaussian prime, } X/2 \leq |\varpi|^2 < X, \psi \leq \arg \varpi < \psi + X^{\theta-1/2}\} \geq \frac{cX^{1/2+\theta}}{\log X}. \quad (1.3)$$

The sector contains about $X^{1/2+\theta}/(\pi \log X)$ Gaussian primes of norm in $[X/2, X)$ on average over ψ , so the lower bound (1.3) has the correct order of magnitude. For every ψ , an upper bound of the same order follows from the linear upper-bound sieve over $\mathbb{Z}[i]$ [5, Lemma 3]. The linear estimate of Lemma 1(a), applied to the positive and the negative parts of the sieve weights, gives this sieve the level of distribution $X^{1/2+\theta-o(1)}$. The sieve then bounds the number of Gaussian primes of norm in $[X/2, X)$ in the sector by $(4/(1+2\theta) + o(1))X^{1/2+\theta}/(\pi \log X)$ uniformly in ψ .

Proof of Theorem 1 from Theorem 2. Let $\theta = 0.1129$, and apply (1.3) with $\psi = 0$ and $X = 2^k$ for large k . The Gaussian primes of argument 0 are the rational primes $\ell \equiv 3 \pmod{4}$, and fewer than $X^{1/2}$ of them have norm $\ell^2 < X$. Since $cX^{1/2+\theta}/\log X > X^{1/2}$ for large X , the sector contains a Gaussian prime $\varpi = m + in$ with $0 < \arg \varpi < X^{\theta-1/2}$, so that $m, n > 0$. Its norm $p = m^2 + n^2$ is a rational prime with $X/2 \leq p < X$. Since $\theta < 1/2$ and $p < X$,

$$n = \sqrt{p} \sin(\arg \varpi) < \sqrt{p} X^{\theta-1/2} < \sqrt{p} p^{\theta-1/2} = p^\theta.$$

The ranges $[X/2, X)$ for different k are disjoint. So the primes p are distinct for different k , and there are infinitely many of them. The bound $\{\sqrt{p}\} < p^{-0.2742}$ follows from (1.2), because $1/2 - 2\theta = 0.2742$. $\square$

1.1 Outline of the method

The proof of Theorem 2 refines the argument of Harman and Lewis [10]. Let $\mathcal{A}$ be the set of Gaussian integers in the sector of (1.3) with norm in $[X/2, X)$, and let $\mathcal{B}$ be the set of all Gaussian integers of argument in $[0, \pi/2)$ with norm in $[X/2, X)$, so $\mathcal{A} \subset \mathcal{B}$. The number of Gaussian primes in $\mathcal{A}$ is the number of elements of $\mathcal{A}$ without prime factors of norm below $X^{1/2}$. Buchstab's identity (2.7) decomposes this count into sums of two kinds. Sums of the first kind have asymptotic formulas (Definition 2): they equal $2X^{\theta-1/2}/\pi$ times the corresponding sums for $\mathcal{B}$, up to a negligible error. Harman and Lewis derive these formulas from bounds for Hecke L -functions, through linear (Type I) and bilinear (Type II) estimates. A sum has such a formula when its primes can be combined into variables whose norms lie in the ranges that these estimates admit.

Sums of the second kind consist of non-negative terms that enter with the sign $+$, so a lower bound may omit them. They are discarded. Each discarded sum, taken over $\mathcal{B}$ and divided by the number of Gaussian primes in $\mathcal{B}$, tends to an integral (3.13) of Buchstab's function ω of (3.14) over a region of exponents. This integral is the loss of that region (Definition 3), and the sum of these losses is the total loss $\Phi(\theta)$ (Definition 4).

Buchstab's identity decomposes the number of Gaussian primes in $\mathcal{B}$ into the same sums over $\mathcal{B}$. So, up to a negligible error, the number of Gaussian primes in $\mathcal{A}$ is at least $2X^{\theta-1/2}/\pi$ times the difference between the number of Gaussian primes in $\mathcal{B}$ and the discarded sums over $\mathcal{B}$. With the total loss, this gives the lower bound

$$\#\{\varpi \in \mathcal{A} : \varpi \text{ Gaussian prime}\} \geq \frac{2X^{\theta-1/2}}{\pi} \#\{\varpi \in \mathcal{B} : \varpi \text{ Gaussian prime}\} (1 - \Phi(\theta) - o(1)). \quad (1.4)$$

Since $\mathcal{B}$ contains $(1 + o(1)) X/(2 \log X)$ Gaussian primes by (9.1), (1.4) gives (1.3) for every $\theta \in (1/9, 1/8)$ with $\Phi(\theta) < 1$. For their decomposition, Harman and Lewis state that the total loss at $\theta = 0.119$ is below 0.943 [10, p. 134].

Besides the prime number theorem in progressions modulo 4 [23, Section 1.2], only the arithmetic information is quoted without proof. It consists of the sector inequality, the mean-value estimate for Hecke characters, the prime number theorem with Hecke characters, and Lemmas 6 and 7 of Harman and Lewis. Lemmas 20 to 23 of Appendix A state them.

1.2 Contributions

We add three groupings of terms that Harman and Lewis discard. A *grouping* combines the primes of a sum into the variables of the linear or bilinear estimate, after a further application of Buchstab's identity (2.7) where needed. The sum of the grouped terms then has an asymptotic formula.

- *A new factorization on a region R.* Harman and Lewis discard the sum of the terms pq , with p and q Gaussian primes, whose exponents lie in a two-dimensional region R . Two applications of Buchstab's identity introduce primes r and s . We apply the linear estimate to the factorization $p \cdot (qr)$ of the sum over (p, q, r) . The sum over (p, q, r, s) is discarded, except where the other two groupings apply. Harman and Lewis apply the linear estimate after Buchstab's identity only to the factorization $(pq) \cdot r$. On R the product pq is too large for the linear estimate, while p and qr satisfy its conditions, because q is small.
- *The bilinear estimate with any set of primes on a region G.* The other two groupings act on terms $pqrst$ in which two applications of Buchstab's identity have introduced primes r and s , and t is the remaining factor, the cofactor. Harman and Lewis discard these terms over four-dimensional regions, except on a few subregions. The bilinear estimate needs a prime variable and further variables u and v of norm at least about $X^{1/2-\theta}$, and it applies at every level of sifting of t . On a region G , we apply it with s as the prime variable and with the product of a non-empty subset of $\{p, q, r\}$ as u . Harman and Lewis print the subset $\{p, r\}$ in one of their regions, and omit the conditions from other groupings in another.
- *A further Buchstab step on a region L.* On a region L , we apply Buchstab's identity once more, at a level below every fixed positive power of X . This application introduces a prime s' with $|s'| \leq |s|$. The linear estimate for $pqrs$ then treats the sum sifted at this level, and the bilinear estimate with s' as its prime variable treats the sum over s' . Here u may contain s , which on G is the prime variable.

Section 3 defines R , G and L and places them in the decomposition.

Table 4 shows the effect of each grouping. With the regions derived from the lemmas of Section 6 (Remark 27), the decomposition of Harman and Lewis has total loss 1 at $\theta = 0.1167825$. The value 0.1167825 is just below the range from 0.117 to 0.118 in which, by Harman and Lewis [10, p. 134], the constant in their lower bound ceases to be positive. They add that they did not spend much time on the numerical calculations. Adding R moves the value 0.1167825 to 0.1141073, and adding G and L moves it to 0.1128554, so that the total loss is below 1 at $\theta = 0.1129$. The losses are computed in floating point by two methods (Section 10). At $\theta = 0.1129$ the first method gives the total loss 0.9990033. The gap $1 - 0.9990033 = 9.967 \times 10^{-4}$ is more than 800 times each of the following: the difference between the totals of the methods, the error estimate of the second method, and the change of the total of the first method from the rule of Table 3 to its finest rule.

We also give in full several steps that Harman and Lewis [10] present briefly: the fundamental lemma invoked in their Lemmas 8, 9, 11 and 12, the proofs of their Lemmas 8 to 10, the removal of the conditions that couple the variables of their bilinear estimate, and the extension of their Lemma 14 that part 6 of

their §5 uses. Lemma 3, Lemmas 4 and 5 with Corollary 6, Lemma 7 and Lemma 12 fill in these steps. Appendix E compares the two arguments item by item, including several statements and discarded regions that are made precise here.

1.3 Structure of the paper

Section 2 fixes the notation. Section 3 states the decomposition: identity (3.5), the ranges of Table 2, the groupings R, G and L, and the total loss (3.15). Section 4 states the linear and bilinear estimates, a fundamental lemma over $\mathbb{Z}[i]$, and asymptotic formulas for sifting functions with a variable level, and Section 5 removes the conditions that couple the variables of a sum. Section 6 treats the sums over two primes. Section 7 proves identity (3.5), checks the hypotheses of the lemmas on each range of Table 2, and defines the discarded regions. Section 8 treats the four-dimensional sums on G and L. Section 9 turns the discarded sums into integrals and proves the lower bound, Lemma 16. Section 10 evaluates the integrals numerically and proves Theorem 2, and Section 11 concludes. Appendix A collects standard facts and derives the linear and bilinear estimates from the arithmetic information. Appendix B proves Lemmas 3, 4, 5 and 7, and Appendix C proves Lemma 15. Appendix D lists the inequalities between the parameters, and Appendix E compares the argument with that of Harman and Lewis.

Theorem 2 follows from Lemma 16 and Table 3. Lemma 16 rests on Lemmas 9 to 15, and Lemmas 9 to 14 rest on the results of Sections 4 and 5.

2 Notation

Throughout, $1/9 < \theta < 1/8$, a range in which all the inequalities of Table 5 hold, and X is large in terms of θ and of the other fixed parameters. Put

$$\begin{aligned} \sigma &:= \frac{1}{2} - \theta, & \tau &:= \frac{1}{2} + \theta, & \rho &:= \frac{1 + 22\theta}{12}, & \zeta &:= \frac{1 + 4\theta}{3}, & \nu &:= \frac{7 + 10\theta}{12}, \\ \mu &:= \exp((\log X)^{4/5}). \end{aligned} \quad (2.1)$$

Here $\sigma, \tau, \rho, \zeta$ and ν are the exponents of Harman and Lewis [10, p. 120]. They satisfy $\nu - \sigma = \rho$ and $\nu - \sigma/2 = \zeta$. The exponents τ, ζ, ρ and ν bound, as powers of X , the norms of the variables of the linear estimate, Lemma 1, and X^σ bounds from below, up to a constant factor, the norms of the variables u and v of the bilinear estimate, Lemma 2. The number $X^{-\sigma} = X^{\theta-1/2}$ is the angle of the sector in (1.3). The number μ exceeds every fixed power of $\log X$ and is smaller than every fixed positive power of X . The fundamental lemma, Lemma 3, sifts at levels up to μ^4 , and the prime variable of the bilinear estimate has norm at least $\mu^4/2$.

For a Gaussian integer n , $n \sim N$ means $N \leq |n|^2 < 2N$. From now on, p, q, r, s and ϖ denote Gaussian primes, $d(n)$ is the number of divisors of n , and $\mathbf{1}[\cdot]$ is the indicator of the condition in brackets. A sequence (a_n) is *divisor-bounded* if $|a_n| \leq d(n)^{C_0}$ for a fixed C_0 . Fix $\psi \in \mathbb{R}$. The lower bounds for X below do not depend on ψ . The sets $\mathcal{A}$ and $\mathcal{B}$ of Section 1.1 are

$$\mathcal{A} = \{n \in \mathbb{Z}[i] : X/2 \leq |n|^2 < X, \psi \leq \arg n < \psi + X^{-\sigma}\}, \quad \mathcal{B} = \{n \in \mathbb{Z}[i] : X/2 \leq |n|^2 < X\}. \quad (2.2)$$

For $\mathcal{E} \in \{\mathcal{A}, \mathcal{B}\}$ and a Gaussian integer $u \neq 0$, put

$$\mathcal{E}_u := \{n \in \mathcal{E} : u \mid n\}. \quad (2.3)$$

Buchstab's identity classifies a Gaussian integer by its least prime factor, so it needs a total order on the Gaussian primes. The norm does not give one, since a rational prime $\ell \equiv 1 \pmod{4}$ is the norm of 2 Gaussian primes. As in Harman and Lewis [10, p. 122], the Gaussian primes are ordered by norm, and primes of equal norm by argument:

$$\varpi \prec \varpi' \quad \text{means} \quad |\varpi| < |\varpi'|, \text{ or } |\varpi| = |\varpi'| \text{ and } \arg \varpi < \arg \varpi'. \quad (2.4)$$

Then $\varpi \preccurlyeq \varpi'$ means $\varpi \prec \varpi'$ or $\varpi = \varpi'$, and $\succ$ and $\succcurlyeq$ are the reversed relations.

Definition 1 (Sifting functions) For $\mathcal{E} \in \{\mathcal{A}, \mathcal{B}\}$, $z \geq 1$ and a prime q , the *sifting functions* of Harman and Lewis [10, p. 121] are

$$S(\mathcal{E}, z) := \#\{n \in \mathcal{E} : |\varpi|^2 \geq z \text{ for all } \varpi \mid n\}, \quad S(\mathcal{E}, q) := \#\{n \in \mathcal{E} : \varpi \nmid q \text{ for all } \varpi \mid n\}, \quad (2.5)$$

and z and q are their *levels*. With $\mathcal{E}_u$ of (2.3) in place of $\mathcal{E}$, (2.5) defines $S(\mathcal{E}_u, z)$ and $S(\mathcal{E}_u, q)$. For $n \in \mathcal{E}_u$, the quotient n/u is the *cofactor* of n , and the functions that sift the cofactor alone are

$$S'(\mathcal{E}_u, z) := \#\{n \in \mathcal{E}_u : |\varpi|^2 \geq z \text{ for all } \varpi \mid n/u\}, \quad S'(\mathcal{E}_u, q) := \#\{n \in \mathcal{E}_u : \varpi \nmid q \text{ for all } \varpi \mid n/u\}. \quad (2.6)$$

Using the letter S for both functions of (2.5) is a slight abuse of notation. It causes no ambiguity, since a prime level is always written as a symbol that denotes a Gaussian prime, such as q , s' or ϖ , and every other level is a real number. By (2.5) and (2.6),

$$\begin{aligned} S(\mathcal{E}_u, z) &= S'(\mathcal{E}_u, z) \text{ if } |\varpi|^2 \geq z \text{ for every } \varpi \mid u, & S(\mathcal{E}_u, z) &= 0 \text{ otherwise,} \\ S(\mathcal{E}_u, q) &= S'(\mathcal{E}_u, q) \text{ if } \varpi \nmid q \text{ for every } \varpi \mid u, & S(\mathcal{E}_u, q) &= 0 \text{ otherwise.} \end{aligned}$$

Classifying n/u by its least prime factor for the order (2.4) gives Buchstab's identities

$$S'(\mathcal{E}_u, z) = S'(\mathcal{E}_u, w) - \sum_{w \leq |\varpi|^2 < z} S'(\mathcal{E}_{u\varpi}, \varpi), \quad S'(\mathcal{E}_u, q) = S'(\mathcal{E}_u, w) - \sum_{\varpi < q, |\varpi|^2 \geq w} S'(\mathcal{E}_{u\varpi}, \varpi), \quad (2.7)$$

for $1 \leq w \leq z$ in the identity for $S'(\mathcal{E}_u, z)$ and for $1 \leq w \leq |q|^2$ in the identity for $S'(\mathcal{E}_u, q)$. These are Harman and Lewis [10, (13)], and the same identities hold for S in place of S' when $u = 1$.

By (2.2), the angle $X^{-\sigma}$ of the sector of $\mathcal{A}$ is the fraction $2/(\pi X^\sigma)$ of the angle $\pi/2$ of $\mathcal{B}$.

Definition 2 (Sums and asymptotic formulas) A *sum* is a weighted count $\Sigma(\mathcal{E})$ of the elements of $\mathcal{E} \in \{\mathcal{A}, \mathcal{B}\}$, in which each element is counted with a real weight that does not depend on $\mathcal{E}$. A sum *has an asymptotic formula* if, for every $A > 0$,

$$\Sigma(\mathcal{A}) = \frac{2}{\pi X^\sigma} \Sigma(\mathcal{B}) + \mathcal{O}(X^\tau (\log X)^{-A}), \quad (2.8)$$

where the implied constant may depend on A and on the parameters allowed by the convention on implied constants below. The sums of a family, indexed by parameters, have asymptotic formulas *uniformly* in these parameters when (2.8) holds with the same implied constant for every member.

For example, by (2.3) and (2.6), $|\mathcal{E}_u|$ gives the weight 1 to the multiples of u and 0 to the other elements, and $S'(\mathcal{E}_u, z)$ gives the weight 1 to the multiples of u whose cofactor has no prime factor of norm below z . A linear combination of sums is again a sum. If the members of a family have asymptotic formulas uniformly, then, for a fixed C , a sum of at most $(\log X)^C$ members, each with the sign $+$ or $-$, has an asymptotic formula, since A is arbitrary.

Throughout, $\varepsilon > 0$ denotes a fixed small margin. An inequality between exponents holds *with margin ε* if it still holds after ε is added to its smaller side. Implied constants depend only on A , on θ , on ε , on the exponent C_0 of the divisor bounds, on the numbers of variables and of constraints, and on the constants C_1 , C_l and ε_0 of Lemmas 2, 3, 4, 5 and 8. They never depend on dyadic ranges, intervals, levels, coefficients or the bounds Z_c and Z_M of Definition 5. This convention extends that of Harman and Lewis [10, p. 121]. The letter C denotes a positive constant with the same dependence, not necessarily the same at each occurrence.

The function $z(y)$ of (2.11) bounds the levels that Lemmas 4 and 5 admit: a sum $\sum_n a_n S'(\mathcal{E}_n, l(n))$ over $|n|^2 \leq X^{\tau-\varepsilon}$, with $a_n \geq 0$ divisor-bounded, has an asymptotic formula when the level $l(n)$ is at most a constant times $z(|n|^2)$. For real $y \geq 1$, define the integer $j(y)$ by

$$j(y) := \begin{cases} -1 & \text{if } y \geq X^\sigma, \\ \lceil \log(X^\sigma/y)/(\theta \log X) \rceil - 1 & \text{if } y < X^\sigma. \end{cases} \quad (2.9)$$

For $y < X^\sigma$, (2.9) gives $j(y) \geq 0$ and

$$X^{\sigma-(j(y)+1)\theta} \leq y < X^{\sigma-j(y)\theta} \quad (y < X^\sigma). \quad (2.10)$$

Following Harman and Lewis [10, p. 128], put

$$z(y) := \left(\frac{X^\tau}{y} \right)^{1/(j(y)+2)}. \quad (2.11)$$

By (2.9) and (2.11), $z(y) = X^\tau/y$ for $y \geq X^\sigma$. For $y < X^\sigma$, (2.11), the bound $y < X^{\sigma-j(y)\theta}$ of (2.10) and the relation $\tau = \sigma + 2\theta$ of (2.1) give

$$z(y) > \left(\frac{X^\tau}{X^{\sigma-j(y)\theta}} \right)^{1/(j(y)+2)} = X^\theta \quad (y < X^\sigma). \quad (2.12)$$

Table 1 lists the main notation of the paper.

Symbol	Definition	Defined in	Depends on	Role
θ	$1/9 < \theta < 1/8$; $\theta = 0.1129$ in Theorem 2	Section 2	—	the sector of (1.3) has angle $X^{\theta-1/2}$
σ, τ	$1/2 - \theta, 1/2 + \theta$	(2.1)	θ	$X^{-\sigma}$ is the angle of the sector; u, v of Lemma 2 have norm at least X^σ/C_1 ; X^τ bounds the variable of Lemma 1(a)
ζ, ρ, ν	$(1+4\theta)/3, (1+22\theta)/12, (7+10\theta)/12$	(2.1)	$\theta; \rho = \nu - \sigma, \zeta = \nu - \sigma/2$	X^ζ, X^ρ, X^ν bound u, u', uu' in Lemma 1(b)
μ	$\exp((\log X)^{4/5})$	(2.1)	X	Lemma 3 sifts at levels up to μ^4 ; the prime of Lemma 2 has norm at least $\mu^4/2$
$\mathcal{A}, \mathcal{B}$	norms in $[X/2, X)$, in the sector and of any argument	(2.2)	X, σ, ψ	$S(\mathcal{A}, X^{1/2})$ counts the Gaussian primes in the sector
$\mathcal{E}_u$	multiples of u in $\mathcal{E}$	(2.3)	$\mathcal{E} \in \{\mathcal{A}, \mathcal{B}\}$	sets of the sifting functions
$\prec$	by norm, then by argument	(2.4)	—	least prime factor in Buchstab's identity (2.7)
S, S'	elements of $\mathcal{E}_u$ without prime factors below the level; for S' , in the cofactor n/u	Definition 1	$\prec, \mathcal{E}_u$	terms of the decomposition
$d(n), C_0$	number of divisors; divisor-bounded means $ a_n \leq d(n)^{C_0}$	Section 2	—	coefficients of the estimates
$n \sim N$	$N \leq n ^2 < 2N$	Section 2	—	dyadic ranges and boxes
A, ε	saving $(\log X)^{-A}$; fixed small margin	Definition 2, Section 2	—	error of an asymptotic formula; margin of an inequality between exponents
$j(y)$	-1 for $y \geq X^\sigma$; for $y < X^\sigma$, the integer with $X^{\sigma-j(y)+1}\theta \leq y < X^{\sigma-j(y)\theta}$	(2.9), (2.10)	σ, θ	exponent $1/(j(y)+2)$ in $z(y)$
$z(y)$	$(X^\tau/y)^{1/(j(y)+2)}$	(2.11)	$j(y), \tau$	largest level, up to a constant factor, of Lemmas 4 and 5
$\sum^*$	sum of $S(\mathcal{E}_{pq}, q)$ over $p \in \Pi_1, q \prec p, q ^2 \geq z(p ^2)$	(3.5)	Π_1, z	the sum that the decomposition splits
$\alpha, \beta, \gamma, \delta$	$ p ^2 = X^\alpha, q ^2 = X^\beta, r ^2 = X^\gamma, s ^2 = X^\delta$	(3.6)	p, q, r, s	coordinates of the regions
$\beta_0(\alpha)$	$\log z(X^\alpha)/\log X$	(3.7)	z	lower limit of β in $\sum^*$
D, D_k, E_k	domain of $\sum^*$; regions of discarded sums	(3.8), Section 7	β_0 and the parameters	regions of Table 2
R, G, I, L, P	regions of the groupings	(3.10), (3.11), (3.12)	the parameters	the sums on them have asymptotic formulas by Lemmas 11 and 14
ω	Buchstab's function	(3.14)	—	integrand of the losses
$\Phi(F), \Phi_P(\theta)$	loss of a region; total losses	Definitions 3 and 4	$\omega, D_k, E_k, G, L, P$	$\Phi(0.1129) < 1$ gives Theorem 2
$H, \xi_\lambda, J_\lambda$	$(\log X)^B, (1 + 1/H)^\lambda, [\xi_\lambda, \xi_{\lambda+1})$	(5.1)	X, B	short intervals that remove the conditions coupling the variables

Table 1: The main notation: each symbol, its definition, the display, definition or section where it is introduced, the symbols it depends on, and its role.

3 The decomposition and the total loss

By (2.2) and Definition 1, the number of Gaussian primes in $\mathcal{A}$ is the sifting function $S(\mathcal{A}, X^{1/2})$. Lemma 16 proves the lower bound (1.4) in the form (9.3), in which $\Phi(\theta)$ is replaced by a number $\Phi_\varepsilon(\theta)$ that tends to the total loss $\Phi(\theta)$ of (3.15) as $\varepsilon \rightarrow 0$.

3.1 The linear and bilinear estimates in exponent form

The asymptotic formulas rest on the linear and bilinear estimates, Lemmas 1 and 2. The arithmetic information quoted from Harman and Lewis [10] enters the argument only through these two lemmas. Consider a sum over primes $\varpi_1, \dots, \varpi_K$ and a cofactor t with $\varpi_1 \cdots \varpi_K t \in \mathcal{E}$, where $|\varpi_i|^2$ is about X^{x_i} . For a set $\mathcal{S}$ of indices, put

$$x_{\mathcal{S}} := \sum_{i \in \mathcal{S}} x_i. \quad (3.1)$$

The linear estimate applies when t is *free*, that is, subject to no sieve condition. Through Lemma 3, it also applies when t has no prime factor of norm below μ^4 . In exponent form, it needs $\sum_{i=1}^K x_i \leq \tau$, or, for a partition of the indices into sets $\mathcal{U}$ and $\mathcal{V}$,

$$x_{\mathcal{U}} \leq \zeta, \quad x_{\mathcal{V}} \leq \rho, \quad \sum_{i=1}^K x_i \leq \nu. \quad (3.2)$$

The bilinear estimate applies at every level of sifting of t , since t becomes part of one of its variables, whose coefficients may be any non-negative divisor-bounded sequence. It needs an index i_0 and a non-empty set $\mathcal{S}$ of other indices with

$$\sigma \leq x_{\mathcal{S}} \leq \tau - x_{i_0}. \quad (3.3)$$

The product of the primes of $\mathcal{S}$ then has norm at least X^σ , up to a constant factor. So does the quotient of $\varpi_1 \cdots \varpi_K t$ by this product and by ϖ_{i_0} , since (2.2), (3.3) and (2.1) give, up to a constant factor,

$$\frac{|\varpi_1 \cdots \varpi_K t|^2}{|\varpi_{i_0} \prod_{i \in \mathcal{S}} \varpi_i|^2} \geq \frac{X/2}{X^{x_{\mathcal{S}} + x_{i_0}}} \geq \frac{X^{1-\tau}}{2} = \frac{X^\sigma}{2}.$$

The product and the quotient are the variables u and v of Lemma 2, and ϖ_{i_0} is its prime variable ϖ . Lemma 8 states the condition $\sum_{i=1}^K x_i \leq \tau$ and (3.2), with margin ε , as (P1) for a free cofactor and as (P2) for a cofactor without prime factors of norm below μ^4 . It states (3.3), up to constant factors in the norms, as (P3). Through Lemmas 4 and 5, its condition (P4) covers a cofactor sifted at a constant level that is at most a constant times $z(|\varpi_1 \cdots \varpi_K|^2)$, for products of norm at most $X^{\tau-\varepsilon}$.

3.2 Decomposition of the sifting function

The decomposition begins with two applications of Buchstab's identity (2.7). Put

$$\Pi_1 := \{p : X^\theta \leq |p|^2 < X^{1/2}, z(|p|^2) \leq |p|^2\}, \quad \Pi_2 := \{p : X^\theta \leq |p|^2 < X^{1/2}, z(|p|^2) > |p|^2\}. \quad (3.4)$$

For $\mathcal{E} \in \{\mathcal{A}, \mathcal{B}\}$,

$$S(\mathcal{E}, X^{1/2}) = S(\mathcal{E}, X^\theta) - \sum_{p \in \Pi_1} S(\mathcal{E}_p, z(|p|^2)) - \sum_{p \in \Pi_2} S(\mathcal{E}_p, p) + \sum_{p, q}^* S(\mathcal{E}_{pq}, q), \quad (3.5)$$

where $\sum^*$ runs over the pairs with $p \in \Pi_1$, $q \prec p$ and $|q|^2 \geq z(|p|^2)$. The first application of Buchstab's identity gives the terms $S(\mathcal{E}_p, p)$ with $X^\theta \leq |p|^2 < X^{1/2}$. Since $|p|^2 < z(|p|^2)$ for $p \in \Pi_2$ by (3.4), the sum over $p \in \Pi_2$ in (3.5) has an asymptotic formula without a second application. For $p \in \Pi_1$, the second application lowers the level of $S(\mathcal{E}_p, p)$ to $z(|p|^2)$ and subtracts the terms of $\sum^*$. Lemma 13 proves (3.5) and gives the terms other than $\sum^*$ asymptotic formulas by Corollary 6 and Lemmas 1, 4 and 5. The rest of the argument concerns $\sum^*$.

The lemmas of Section 6 treat a pair (p, q) of $\sum^*$ according to the sizes of $|p|^2$ and $|q|^2$ as powers of X . For primes p, q, r, s , define the exponents $\alpha, \beta, \gamma, \delta$ by

$$|p|^2 = X^\alpha, \quad |q|^2 = X^\beta, \quad |r|^2 = X^\gamma, \quad |s|^2 = X^\delta. \quad (3.6)$$

For $0 < \alpha < 1/2$, the lower limit of β in $\sum^*$ is, by (2.9) and (2.11),

$$\beta_0(\alpha) := \frac{\log z(X^\alpha)}{\log X} = \begin{cases} \tau - \alpha & \text{for } \alpha \geq \sigma, \\ (\tau - \alpha)/(j(X^\alpha) + 2) & \text{for } \alpha < \sigma. \end{cases} \quad (3.7)$$

Section 7 discards the pairs of $\sum^*$ with $|p|^2|q|^4 \geq X$, whose terms have cofactor 1 and whose sum over $\mathcal{B}$ is $\mathcal{O}(X/(\log X)^2)$. The other pairs with $|q| < |p|$ have exponents (3.6) in

$$\mathbf{D} := \{(\alpha, \beta) : \theta \leq \alpha < 1/2, \beta_0(\alpha) \leq \beta < \min(\alpha, (1 - \alpha)/2)\}. \quad (3.8)$$

The other pairs with $|q| = |p|$ have exponents on the line $\beta = \alpha$, on the boundary of $\mathbf{D}$. Following Harman and Lewis [10, §5], whose numbering we keep, we split $\mathbf{D}$ into Ranges 1 to 6, listed in Table 2, and each of them into parts, which are polytopes. Figure 1 shows the ranges at $\theta = 0.1129$, with the regions $\mathbf{D}_k$ of Section 7 where $\sum^*$ is discarded and the region $\mathbf{R}$ of (3.10). On each part, $\sum^*$ receives one of three *treatments*: an asymptotic formula for the whole sum; identity (6.7) of Lemma 11 or identity (6.12) of Lemma 12; or a discard of the whole sum. Both identities apply Buchstab's identity (2.7) twice. For $|pq|^2 \geq X^\sigma$, the level $X^\tau/|pq|^2$ is the largest, up to a constant factor, that Lemma 4 admits for a sum of $S(\mathcal{A}_{pq}, \cdot)$ over (p, q) . The conditions $|pq|^2 \leq X^\tau \leq |p|^2|q|^4$ make this level at least 1 and at most $|q|^2$. For a pair (p, q) with $q \prec p$ that satisfies them, the two applications of Buchstab's identity (2.7) at the level $X^\tau/|pq|^2$ give

$$S(\mathcal{A}_{pq}, q) = S\left(\mathcal{A}_{pq}, \frac{X^\tau}{|pq|^2}\right) - \sum_{\substack{r \prec q \\ |pqr|^2 \geq X^\tau}} S\left(\mathcal{A}_{pqr}, \frac{X^\tau}{|pq|^2}\right) + \sum_{\substack{s \prec r \prec q \\ |pqrs|^2 \geq X^\tau}} S(\mathcal{A}_{pqrs}, s). \quad (3.9)$$

In each term $S(\mathcal{A}_u, \cdot)$ of (3.9), every prime factor of u satisfies the condition that the level imposes, by $s \prec r \prec q \prec p$, the bound $|p|^2|q|^4 \geq X^\tau$ and the conditions of summation. So these terms equal $S'(\mathcal{A}_u, \cdot)$ by (2.5) and (2.6), as (2.7) requires. In the third sum, $s \prec r$ and $|pqrs|^2 \geq X^\tau$ give $|pqr|^2 \geq X^\tau$. A *box* is the set of pairs (p, q) with $p \sim P$ and $q \sim Q$, for powers P and Q of 2. Its *exponent rectangle* is the set of pairs (α, β) with $X^\alpha \in [P, 2P)$ and $X^\beta \in [Q, 2Q)$, and the distance of a box from a set of exponents is that of its exponent rectangle. Summing (3.9), with PQ in place of $|pq|^2$, over the pairs (p, q) of a box with $X^\sigma \leq PQ \leq X^\tau$ that satisfy $q \prec p$ and $|q|^2 \geq X^\tau/(PQ)$ gives identity (6.7). For the boxes with $PQ < X^\sigma$, (2.9) gives $j(PQ) \geq 0$, so the level $X^\tau/(PQ)$ exceeds $z(PQ)$ by (2.11). Identity (6.12) is built in the same way, at the level $z(PQ)$ and at a level that depends on $|r|^2$. On the parts whose treatment is an identity, the sums over (p, q) and over (p, q, r) of that identity have asymptotic formulas. The *third sum* of identity (6.7) or (6.12), of $S(\mathcal{A}_{pqrs}, s)$ over primes $s \prec r \prec q \prec p$, has non-negative terms and enters the identity with the sign $+$. It is discarded over a four-dimensional region of exponents $(\alpha, \beta, \gamma, \delta)$.

3.3 Groupings on the regions $\mathbf{R}$, $\mathbf{G}$ and $\mathbf{L}$

The grouping in Range 2 of Table 2 acts on the region

$$\mathbf{R} := \{1/3 \leq \alpha < \sigma, \max(\zeta - \alpha, (\tau - \alpha)/2) \leq \beta \leq \rho/2\}. \quad (3.10)$$

Case (a) of (6.4) in Lemma 10, whose conditions read $\alpha + \beta \leq \zeta$, $\beta \leq \rho$ and $\alpha + 2\beta \leq \nu$ in exponents, applies the linear estimate to the factorization $(pq) \cdot r$. In Range 2, $\alpha + \beta \leq \zeta$ implies $\beta \leq \rho$ and $\alpha + 2\beta \leq \nu$ (Section 7). On $\mathbf{R}$, where $\alpha + \beta \geq \zeta$, case (b), whose conditions read $\alpha \leq \zeta$, $2\beta \leq \rho$ and $\alpha + 2\beta \leq \nu$ in exponents, applies the linear estimate to the factorization $p \cdot (qr)$. In Range 2, $\alpha < \sigma < \zeta$ by Table 5 and $\sigma + \rho = \nu$ by (2.1), so these conditions reduce to $2\beta \leq \rho$, the upper limit of β in (3.10). The lower limits of β there are $\zeta - \alpha$, below which case (a) applies, and $\beta_0(\alpha)$. In Range 2, $\sigma - \theta < 1/3 \leq \alpha < \sigma$ by Table 5, so $X^{\sigma-\theta} < X^\alpha < X^\sigma$ and (2.10) gives $j(X^\alpha) = 0$. Hence $\beta_0(\alpha) = (\tau - \alpha)/2$ by (3.7). Section 7 checks that

Range	Exponents	Treatment of $\sum^*$	Discarded	Third sums
1	$\sigma \leq \alpha < 1/2$	asymptotic formula (Lemma 9)	D_1	—
2	$1/3 \leq \alpha < \sigma$	identity (6.7), case (a), and case (b) on R (Lemma 11)	D_2	E_3, E_R
3	$\rho \leq \alpha < 1/3$	identity (6.7), case (a) (Lemma 11)	D_4	E_5
4	$\sigma - \theta \leq \alpha < \rho$	identity (6.7), case (a) (Lemma 11)	D_6	E_7
5	$\alpha < \sigma - \theta, \sigma \leq \alpha + \beta$	$S_<$ of (6.6), and identity (6.7), case (a) (Lemma 11)	D_8	E_9
5	$\alpha < \sigma - \theta \leq \alpha + \beta < \sigma$	$S_<$ of (6.11), and identity (6.12) with $j(PQ) = 0$ (Lemma 12)	—	E_{10}
6	$\alpha + \beta < \sigma - \theta$	identity (6.12) with $j(PQ) = 1$ (Lemma 12)	—	E_{11}

Table 2: The ranges of the domain D of (3.8), with Range 5 in 2 rows. For each range: the exponents of the pairs (p, q) of $\sum^*$; the treatment, where $S_<$ is the sum of (6.6) or (6.11) that has an asymptotic formula and the cases (a) and (b) are those of (6.4); the two-dimensional region where $\sum^*$ is discarded; and the four-dimensional regions where the third sum of (6.7) or (6.12) is discarded, except on $G \cup L$ of (3.11). The regions D_k and E_k are defined in Section 7, by (7.4), (7.5), (7.6), (7.7), (7.8) and (7.9). The index k of a region D_k or E_k is that of the loss φ_k of Harman and Lewis (Table 7), and E_R has no counterpart there.

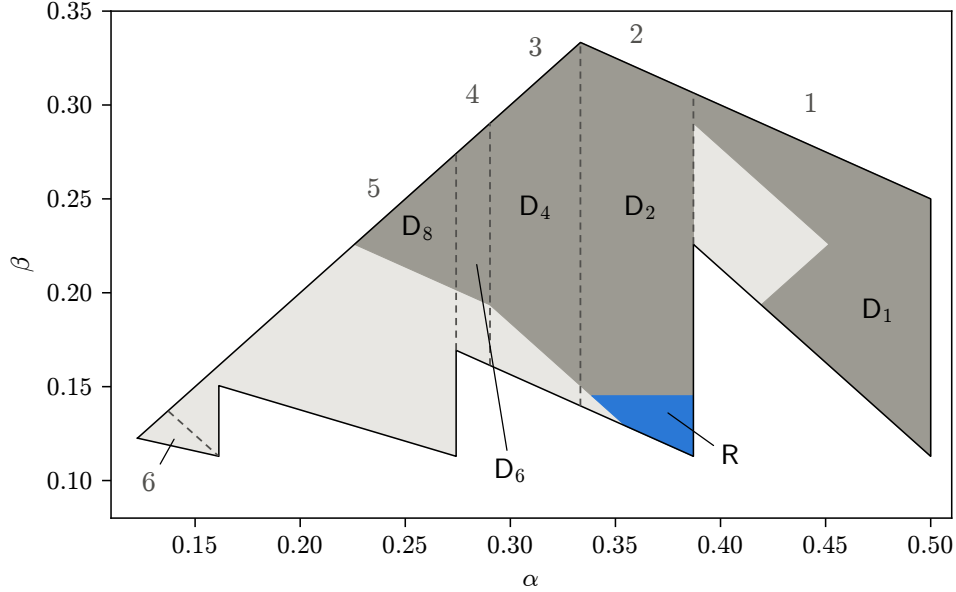


Figure 1: The domain D of (3.8) in the (α, β) -plane at $\theta = 0.1129$. Dashed lines separate the ranges of Table 2, numbered 1 to 6: the vertical ones are at $\alpha = \sigma = 0.3871$, $1/3$, $\rho = 0.29032$ and $\sigma - \theta = 0.2742$, and the oblique one is the line $\alpha + \beta = \sigma - \theta$ between Ranges 5 and 6. Dark gray: the regions D_1, D_2, D_4, D_6 and D_8 of Section 7. Blue: the region R of (3.10). Light gray: the rest of D.

$X^\sigma \leq PQ \leq X^{\tau-\varepsilon}$ on R, as Lemma 11 requires. So on R, the treatment of $\sum^*$ is identity (6.7), whose third sum alone is discarded, over the region E_R of (7.9).

The other two groupings give parts of the third sums of (6.7) and (6.12) asymptotic formulas. The exponents (3.6) of p, q, r, s are the x_i of (3.1). An index set is denoted by the set of its primes, as in $x_{\{p,q\}} = \alpha + \beta$, and i_0 is given by its prime, as in $\varpi_{i_0} = s$. On a region G, the bilinear estimate applies to the products $pqrst$ with s as its prime variable: (3.3) holds with $x_{i_0} = \delta$ for a non-empty set $\mathcal{S} \subset \{p, q, r\}$. On a region L, Buchstab's identity (2.7) at the level μ^4 writes $S(\mathcal{A}_{pqrs}, s)$ as the difference (8.1) of a sum sifted at the level μ^4 and a sum over a further prime $s' \prec s$. The sum over s' enters with the sign $-$, so

both sums need asymptotic formulas. The linear estimate treats the sum sifted at the level μ^4 on the region $\mathbf{I}$ where the exponents of p, q, r, s satisfy $\alpha + \beta + \gamma + \delta \leq \tau$ or (3.2). Since $|s'|^2 \leq |s|^2$, the bilinear estimate with s' as its prime variable treats the sum over s' where (3.3) holds with δ in place of x_{i_0} for a non-empty set $\mathcal{S} \subset \{p, q, r, s\}$. The regions $\mathbf{G}$, $\mathbf{I}$ and $\mathbf{L}$ are

$$\begin{aligned}\mathbf{G} &:= \bigcup_{\emptyset \neq \mathcal{S} \subset \{p, q, r\}} \{\sigma \leq x_{\mathcal{S}} \leq \tau - \delta\}, \\ \mathbf{I} &:= \{\alpha + \beta + \gamma + \delta \leq \tau\} \cup \bigcup_{\mathcal{U} \sqcup \mathcal{V} = \{p, q, r, s\}} \{x_{\mathcal{U}} \leq \zeta, x_{\mathcal{V}} \leq \rho, \alpha + \beta + \gamma + \delta \leq \nu\}, \\ \mathbf{L} &:= \mathbf{I} \cap \bigcup_{\emptyset \neq \mathcal{S} \subset \{p, q, r, s\}} \{\sigma \leq x_{\mathcal{S}} \leq \tau - \delta\}.\end{aligned}\tag{3.11}$$

Lemma 14 gives the third sums asymptotic formulas on $\mathbf{G}$ and $\mathbf{L}$. These regions use the smallest prime of the product pqr or $pqrss'$, namely s or s' , as the prime variable of the bilinear estimate. For a product with fixed exponents, such as pqr on $\mathbf{G}$, another prime in this role would not enlarge the region. Indeed, let ϖ_{i_1} be the smallest prime of the product and ϖ_{i_0} another prime of it, and suppose that (3.3) holds for a set $\mathcal{S}$ of primes of the product with $\varpi_{i_0} \notin \mathcal{S}$. If $\varpi_{i_1} \notin \mathcal{S}$, then (3.3) holds with i_1 in place of i_0 , since $x_{i_1} \leq x_{i_0}$. If $\varpi_{i_1} \in \mathcal{S}$, replacing it in $\mathcal{S}$ by ϖ_{i_0} raises $x_{\mathcal{S}}$ by $x_{i_0} - x_{i_1}$. By (3.3) for $\mathcal{S}$, the new value $x_{\mathcal{S}} + x_{i_0} - x_{i_1}$ satisfies

$$\sigma \leq x_{\mathcal{S}} \leq x_{\mathcal{S}} + x_{i_0} - x_{i_1} \leq \tau - x_{i_1}.$$

So the new set satisfies (3.3) with i_1 in place of i_0 .

The region $\mathbf{L}$ contains the polytope

$$\mathbf{P} := \{\alpha + \beta \leq \sigma, \gamma + \delta \leq \rho, \alpha + \beta + \delta \geq \sigma, \alpha + \beta + 2\delta \leq \tau\}.\tag{3.12}$$

On $\mathbf{P}$, the partition $\mathcal{U} = \{p, q\}$, $\mathcal{V} = \{r, s\}$ satisfies (3.2), since Table 5 and (2.1) give

$$x_{\mathcal{U}} = \alpha + \beta \leq \sigma < \zeta, \quad x_{\mathcal{V}} = \gamma + \delta \leq \rho, \quad \alpha + \beta + \gamma + \delta \leq \sigma + \rho = \nu.$$

The set $\mathcal{S} = \{p, q, s\}$ has $x_{\mathcal{S}} = \alpha + \beta + \delta$, so the bounds $\alpha + \beta + \delta \geq \sigma$ and $\alpha + \beta + 2\delta \leq \tau$ of (3.12) are (3.3) with δ in place of x_{i_0} . Section 10 also evaluates the losses with $\mathbf{P}$ in place of $\mathbf{L}$. On $\mathbf{P}$, the treatment of each four-dimensional box of Section 8 can be checked by hand.

3.4 The total loss

The lower bound of Lemma 16 drops the discarded sums over $\mathcal{A}$, and its main term is $2/(\pi X^\sigma)$ times the difference between $S(\mathcal{B}, X^{1/2})$ and the same sums over $\mathcal{B}$, as in (9.4). So each discarded sum is measured relative to $S(\mathcal{B}, X^{1/2})$. By Lemma 15, a discarded sum over $\mathcal{B}$ with exponents in a region, divided by $S(\mathcal{B}, X^{1/2})$, tends to the loss of that region (Definition 3).

Definition 3 (Losses) The *losses* of a two-dimensional region $\mathbf{F}$ and of a four-dimensional region $\mathbf{F}'$ are

$$\Phi(\mathbf{F}) := \iint_{\mathbf{F}} \omega\left(\frac{1 - \alpha - \beta}{\beta}\right) \frac{d\beta d\alpha}{\alpha\beta^2}, \quad \Phi(\mathbf{F}') := \iiint\limits_{\mathbf{F}'} \omega\left(\frac{1 - \alpha - \beta - \gamma - \delta}{\delta}\right) \frac{d\delta d\gamma d\beta d\alpha}{\alpha\beta\gamma\delta^2},\tag{3.13}$$

where ω is *Buchstab's function*, defined by $\omega(u) = 0$ for $u < 1$ and

$$u\omega(u) = 1 + \int_1^{u-1} \omega(t) dt \quad (u \geq 1).\tag{3.14}$$

Equation (3.14) defines ω implicitly, starting from $\omega(u) = 1/u$ for $1 \leq u \leq 2$, where the integral vanishes.

Definition 4 (Total loss) The *total loss* is

$$\Phi(\theta) := \sum_{k \in \{1, 2, 4, 6, 8\}} \Phi(\mathbf{D}_k) + \sum_{k \in \{3, \mathbf{R}, 5, 7, 9, 10, 11\}} \Phi(\mathbf{E}_k \setminus (\mathbf{G} \cup \mathbf{L})),\tag{3.15}$$

with $\mathbf{D}_k$ of (7.4), (7.5), (7.6), (7.7) and (7.8), $\mathbf{E}_k$ of (7.9), and $\mathbf{G}$, $\mathbf{L}$ of (3.11). Let $\Phi_{\mathbf{P}}(\theta)$ be defined in the same way with $\mathbf{P}$ of (3.12) in place of $\mathbf{L}$.

The letter Φ denotes the loss (3.13) when its argument is a region, and the total loss when its argument is θ .

The total loss collects the losses of the regions of all sums that remain discarded: the sums over (p, q) on the regions D_k , and the third sums on the regions E_k outside $G \cup L$, since Lemma 14 gives the third sums asymptotic formulas on $G \cup L$. So $\Phi(\theta)$ measures the proportion of $S(\mathcal{B}, X^{1/2})$ that the discarded sums take up, and the lower bound of Lemma 16 is positive when $\Phi(\theta) < 1$.

The proof of Lemma 16 splits $\sum^*$ and the third sums into dyadic boxes. It applies Lemmas 9, 11 and 12 to the boxes that lie inside a part where these lemmas apply, at distance at least ε from its boundary, and Lemma 14 to the four-dimensional boxes that lie in G or L with margin ε . It bounds the sums over the other boxes by Lemma 15.

4 Asymptotic formulas from the linear and bilinear estimates

Each result of this section gives a sum an asymptotic formula in the sense of Definition 2. Lemmas 1 and 2 below are the linear and bilinear estimates of Section 3, in the form used in this paper. The other results of this section treat sums in which the cofactor is sifted. Lemmas 4 and 5 sift the cofactors of the elements of $\mathcal{E}_n$ at a level $l(n)$ that depends on n .

Lemma 1 *Let $c_v \geq 0$ be divisor-bounded, and suppose that one of the following holds:*

- (a) c_v is supported on $|v|^2 \leq X^\tau \mu^{-1}$;
- (b) $c_v = \sum_{v=uu'} a_u b_{u'}$, where $a_u, b_{u'} \geq 0$ are divisor-bounded and supported on $u \sim U$, $u' \sim U'$, with $U \leq X^\zeta \mu^{-1}$, $U' \leq X^\rho \mu^{-1}$ and $UU' \leq X^\nu \mu^{-1}$.

Then $\sum_{v \neq 0} c_v |\mathcal{E}_v|$ has an asymptotic formula. (Proof in Appendix A.)

The bilinear estimate of Harman and Lewis [10, Lemma 5 and (30)] needs a prime variable. Its proof applies when the prime is restricted to any interval of norms contained in $[\mu^4/2, \infty)$, as the proof of Lemma 2 in Appendix A shows. Remark 25 explains the lower limit $\mu^4/2$, and Section 5 restricts the prime to short intervals.

Lemma 2 *Let $C_1 \geq 1$, let $a_u, b_v \geq 0$ be divisor-bounded and supported on $u \sim U$ and $v \sim V$, and let $\mu^4/2 \leq Z_1 < Z_2$. Suppose that $U, V \geq X^\sigma / C_1$. Then*

$$\sum_{u \sim U, v \sim V} a_u b_v \# \{ \varpi : Z_1 \leq |\varpi|^2 < Z_2, uv\varpi \in \mathcal{E} \} \quad (4.1)$$

has an asymptotic formula, uniformly in U, V, Z_1 and Z_2 . (Proof in Appendix A.)

Lemma 3 below is a fundamental lemma over $\mathbb{Z}[i]$. Every level at which it is applied is at most $\mu^4 = X^{o(1)}$, so Brun's pure sieve, described in Appendix B, suffices.

Lemma 3 *Let $0 < \varepsilon_0 < 1$, let $c_u \geq 0$ be divisor-bounded, and let z_0 be a function on the support of (c_u) with $1 \leq z_0(u) \leq \mu^4$. Then $\sum_{u \neq 0} c_u S'(\mathcal{E}_u, z_0(u))$ has an asymptotic formula if one of the following holds:*

- (a) c_u is supported on $|u|^2 \leq X^{\tau-\varepsilon_0} \mu^{-1}$;
- (b) z_0 is constant, and $c_u = \sum_{u=u_1 u_2} a_{u_1} b_{u_2}$, where $a_{u_1}, b_{u_2} \geq 0$ are divisor-bounded and supported on $u_1 \sim U_1$, $u_2 \sim U_2$, with $U_1 \leq X^\zeta \mu^{-1}$, $U_2 \leq X^{\rho-\varepsilon_0} \mu^{-1}$ and $U_1 U_2 \leq X^{\nu-\varepsilon_0} \mu^{-1}$.

(Proof in Appendix B.)

Lemma 3 sifts at levels up to μ^4 , and the decomposition needs levels $l(n)$ up to a constant times $X^\tau / |n|^2$ for $|n|^2 \geq X^\sigma$ and up to a constant times $z(|n|^2)$ for $|n|^2 < X^\sigma$. Lemmas 4 and 5 reach these levels through the bilinear estimate. Buchstab's identity (2.7) lowers the level $l(n)$ to μ^4 , where Lemma 3 applies, and subtracts a sum over a prime q with $\mu^4 \leq |q|^2 < l(n)$. Once q is confined to short intervals, n is the variable

u of Lemma 2 in this sum, and q is its prime variable. If $|n|^2 \geq X^\sigma/C_l$ and $l(n) \leq C_l X^\tau/|n|^2$ for a constant $C_l \geq 1$, then (2.2) and (2.1) give, for the cofactor v of an element of $\mathcal{E}_{nq}$,

$$|v|^2 > \frac{X/2}{|n|^2 l(n)} \geq \frac{X^{1-\tau}}{2C_l} = \frac{X^\sigma}{2C_l},$$

and Lemma 2 applies. The bounds on $|n|^2$ and $l(n)$ are among the hypotheses of Lemma 4.

Lemma 4 *Let $C_l \geq 1$, let $a_n \geq 0$ be divisor-bounded and supported on $X^\sigma/C_l \leq |n|^2 \leq X^{\tau-\varepsilon}$, and let l be any function on the support of (a_n) with $1 \leq l(n) \leq C_l X^\tau/|n|^2$. Then $\sum_{X^\sigma/C_l \leq |n|^2 \leq X^{\tau-\varepsilon}} a_n S'(\mathcal{E}_n, l(n))$ has an asymptotic formula, uniformly in (a_n) and l . (Proof in Appendix B.)*

For $|n|^2 < X^\sigma$, n can be too small to be the variable u of Lemma 2, which needs $U \geq X^\sigma/C_1$. The proof of Lemma 5 applies Buchstab's identity repeatedly, until the product of n and the primes it introduces has norm at least X^σ , and then applies Lemma 4 to that product. The bound $l(n) \leq C_l z(|n|^2)$ ensures that each such product and its level satisfy the hypotheses of Lemma 4.

Lemma 5 *Let $C_l \geq 1$, let $a_n \geq 0$ be divisor-bounded and supported on $1 \leq |n|^2 < X^\sigma$, and let l be any function on the support of (a_n) with $1 \leq l(n) \leq C_l z(|n|^2)$, where $z(|n|^2)$ is given by (2.11). Then $\sum_{1 \leq |n|^2 < X^\sigma} a_n S'(\mathcal{E}_n, l(n))$ has an asymptotic formula, uniformly in (a_n) and l . (Proof in Appendix B.)*

Corollary 6 *The sum $S(\mathcal{E}, X^\theta)$ of (3.5) has an asymptotic formula.*

Proof. By (2.3), (2.5) and (2.6), the sum $S(\mathcal{E}, X^\theta) = S'(\mathcal{E}_1, X^\theta)$ is the case $C_l = 1$, $a_n = \mathbf{1}[n = 1]$, $l \equiv X^\theta$ of Lemma 5. The hypotheses of Lemma 5 hold, since $1 < X^\sigma$ and $1 \leq X^\theta < z(1)$ by (2.12). $\square$

5 Removing the conditions that couple the variables

The sums of Harman's sieve run over products of primes with order constraints between them, conditions on products of their norms, and sieve conditions on the cofactor that depend on one of the primes. The bilinear estimate of Lemma 2 admits none of these conditions between its variables. This section removes them with short intervals. The order constraints and the conditions on products of norms are the cross-conditions of Merikoski [18, §7], and the short intervals give what he calls a finer-than-dyadic decomposition, which he builds from smooth weights [18, §3.1]. Heath-Brown and Li [11, §11.1] separate variables by short intervals of the same shape as ours, and Harman and Lewis [10, p. 130] remove such conditions by Perron's formula (Appendix E). Every order constraint and every condition on a product of norms is constant on most Cartesian products of short intervals, and the sums over the other products are small. On a single Cartesian product, the sieve condition differs from sifting at a constant level only at the prime factors of the cofactor whose norms lie in a few short intervals. Lemma 7 reduces a sum with such conditions to sums over single Cartesian products of short intervals, and Lemma 8 gives these sums asymptotic formulas under the conditions of Section 3.1.

For a fixed $B > 0$, put, for integers $\lambda \geq 0$,

$$H := (\log X)^B, \quad \xi_\lambda := (1 + 1/H)^\lambda, \quad J_\lambda := [\xi_\lambda, \xi_{\lambda+1}). \quad (5.1)$$

The intervals J_λ are the *short intervals*. For a prime q , let λ_q be the index with $|q|^2 \in J_{\lambda_q}$. The proofs of Lemmas 4, 5 and 7 in Appendix B choose B in terms of A .

Definition 5 (Configurations) A *configuration* $\mathcal{C}$ consists of the following data:

- (i) prime variables $\varpi_1, \dots, \varpi_k$, with k fixed and $|\varpi_i|^2 \in \mathcal{I}_i$ for intervals $\mathcal{I}_i \subset [\mu^4, X]$, the *ranges* of the variables. A *tuple* is a choice $\varpi = (\varpi_1, \dots, \varpi_k)$ of these primes;
- (ii) a cofactor t , which is a Gaussian integer;

- (iii) a fixed number of *monomial constraints* c , each of the form $m_c(\varpi) < Z_c$ or $m_c(\varpi) \leq Z_c$, where $m_c(\varpi) := \prod_{i=1}^k |\varpi_i|^{2e_{c,i}}$ with integers $|e_{c,i}| \leq 2$, not all 0, and $Z_c > 0$. We write $\mathbf{1}_c(\varpi)$ for the indicator that ϖ satisfies c . A lower bound $m_c(\varpi) \geq Z_c$ is the constraint $m_c(\varpi)^{-1} \leq Z_c^{-1}$. An *order constraint* $\varpi_i \prec \varpi_{i'}$ is the monomial constraint with $m_c(\varpi) = |\varpi_i|^2/|\varpi_{i'}|^2$ and $Z_c = 1$, in which the case $m_c(\varpi) = 1$ is decided by the order (2.4);
- (iv) a *sieve condition* on t , with indicator $\Xi(\varpi, t)$, of type (F), (C), (S) or (M): (F) none; (C) every prime factor of t has norm at least μ^4 ; (S) every prime factor ϖ' of t satisfies $\varpi' \succcurlyeq \varpi_{i_S}$, for a fixed index i_S ; (M) every prime factor of t has norm at least

$$m_M(\varpi) := Z_M \prod_{i=1}^k |\varpi_i|^{2f_i}, \quad (5.2)$$

with $Z_M > 0$ and reals $|f_i| \leq 2$ that do not depend on the tuple.

On every tuple that satisfies all the constraints, we require $m_M(\varpi) \geq \mu^4$ in type (M). The *sum induced by* $\mathcal{C}$ is

$$\Sigma_{\mathcal{C}}(\mathcal{E}) := \sum_{\varpi_1, \dots, \varpi_k} \sum_{t \in \mathbb{Z}[i]} \mathbf{1}[\varpi_1 \cdots \varpi_k t \in \mathcal{E}] \prod_{i=1}^k \mathbf{1}[|\varpi_i|^2 \in \mathcal{I}_i] \prod_c \mathbf{1}_c(\varpi) \Xi(\varpi, t). \quad (5.3)$$

A configuration has the type of its sieve condition, and we identify it with the sum $\Sigma_{\mathcal{C}}$ that it induces.

For example, for fixed primes p, q, r, s , the sum $S'(\mathcal{E}_{pqrs}, s)$ counts, by (2.6), the cofactors t with $pqrst \in \mathcal{E}$ whose prime factors are all $\succcurlyeq s$. A sum of $S'(\mathcal{E}_{pqrs}, s)$ over primes in given ranges with $s \prec r \prec q \prec p$, such as the third sum of (6.7) on a box, is therefore a configuration with $k = 4$, the order constraints $s \prec r, r \prec q$ and $q \prec p$, and the sieve condition of type (S) with i_S the index of s .

Definition 6 (Pieces, levels and windows) Let $\mathcal{C}$ be a configuration. For an *index vector* $\lambda = (\lambda_1, \dots, \lambda_k)$ of non-negative integers, put

$$\mathcal{T}(\lambda) := \{\varpi : |\varpi_i|^2 \in \mathcal{I}_i \cap J_{\lambda_i} \text{ for } 1 \leq i \leq k\}. \quad (5.4)$$

The *pieces* of $\mathcal{C}$ are the sums over a single set $\mathcal{T}(\lambda)$, without constraints, in which the cofactor is sifted at a constant level $y \geq 1$, or which have a further prime ϖ' with norm in a short interval $J_{\lambda'}$ and a free cofactor:

$$\mathcal{P}_{\lambda}(\mathcal{E}; y) := \sum_{\varpi \in \mathcal{T}(\lambda)} S'(\mathcal{E}_{\varpi_1 \cdots \varpi_k}, y), \quad \mathcal{P}'_{\lambda, \lambda'}(\mathcal{E}) := \sum_{\varpi \in \mathcal{T}(\lambda)} \sum_{|\varpi'|^2 \in J_{\lambda'}} |\mathcal{E}_{\varpi_1 \cdots \varpi_k \varpi'}|, \quad (5.5)$$

where ϖ' runs over primes and the integer $\lambda' \geq 0$ is the index of a single short interval. The index vector λ *satisfies* a constraint c if every tuple of $\mathcal{T}(\lambda)$ satisfies c , and *straddles* c if $\mathcal{T}(\lambda)$ contains tuples that satisfy c and tuples that do not. The *level* of λ is

$$y_{\lambda} := \begin{cases} 1 & \text{in type (F),} \\ \mu^4 & \text{in type (C),} \\ \xi_{\lambda_{i_S}} & \text{in type (S),} \\ \inf_{\varpi \in \mathcal{T}(\lambda)} m_M(\varpi) & \text{in type (M),} \end{cases} \quad (5.6)$$

and its *window* is

$$\mathcal{W}(\lambda) := \begin{cases} \emptyset & \text{in types (F) and (C),} \\ \{\lambda_{i_S}\} & \text{in type (S),} \\ \{\lambda' : J_{\lambda'} \cap [\inf_{\varpi \in \mathcal{T}(\lambda)} m_M(\varpi), \sup_{\varpi \in \mathcal{T}(\lambda)} m_M(\varpi)] \neq \emptyset\} & \text{in type (M).} \end{cases} \quad (5.7)$$

For $y = 1$, the cofactor of $\mathcal{P}_{\lambda}(\mathcal{E}; y)$ is free, since $S'(\mathcal{E}_u, 1) = |\mathcal{E}_u|$ by (2.6). By (5.4) and (5.1), each $|\varpi_i|^2$ varies by at most the factor $1 + 1/H$ on $\mathcal{T}(\lambda)$. Since $|f_i| \leq 2$ in Definition 5, the level $m_M(\varpi)$ of (5.2) varies there by at most the factor $(1 + 1/H)^{2k}$, so the window (5.7) has at most $2k + 2$ elements. Suppose

that $\mathcal{T}(\boldsymbol{\lambda}) \neq \emptyset$ and that $\boldsymbol{\lambda}$ satisfies all constraints. By (5.7), the ranges $\mathcal{I}_i \subset [\mu^4, X]$ in type (S) and the requirement $m_M(\boldsymbol{\varpi}) \geq \mu^4$ of Definition 5 in type (M), every $\lambda' \in \mathcal{W}(\boldsymbol{\lambda})$ then has $\xi_{\lambda'+1} > \mu^4$. Hence, by (5.1),

$$\xi_{\lambda'} \geq \mu^4/2 \quad (\lambda' \in \mathcal{W}(\boldsymbol{\lambda})). \quad (5.8)$$

On the tuples of $\mathcal{T}(\boldsymbol{\lambda})$, the sieve condition $\Xi(\boldsymbol{\varpi}, t)$ and sifting at the constant level $y_{\boldsymbol{\lambda}}$ of (5.6) agree in types (F) and (C). In type (S) they differ only at prime factors ϖ' of t with $\xi_{\lambda_{i_S}} \leq |\varpi'|^2 \leq |\varpi_{i_S}|^2$, and in type (M) only at prime factors ϖ' of t with $y_{\boldsymbol{\lambda}} \leq |\varpi'|^2 < m_M(\boldsymbol{\varpi})$. By (5.1) and (5.7), the norm of each such prime factor lies in $J_{\lambda'}$ for some $\lambda' \in \mathcal{W}(\boldsymbol{\lambda})$. On $\mathcal{T}(\boldsymbol{\lambda})$, the indicators $\mathbf{1}[|\varpi_i|^2 \in \mathcal{I}_i]$ of (5.3) equal 1 by (5.4), and the indicators $\mathbf{1}_c(\boldsymbol{\varpi})$ equal 1 since $\boldsymbol{\lambda}$ satisfies all constraints. So, by (5.3) and (5.5), the part of Σ_C on the tuples of $\mathcal{T}(\boldsymbol{\lambda})$ satisfies

$$\left| \sum_{\boldsymbol{\varpi} \in \mathcal{T}(\boldsymbol{\lambda})} \sum_{t \in \mathbb{Z}[i]} \mathbf{1}[\varpi_1 \cdots \varpi_k t \in \mathcal{E}] \Xi(\boldsymbol{\varpi}, t) - \mathcal{P}_{\boldsymbol{\lambda}}(\mathcal{E}; y_{\boldsymbol{\lambda}}) \right| \leq \sum_{\lambda' \in \mathcal{W}(\boldsymbol{\lambda})} \mathcal{P}'_{\boldsymbol{\lambda}, \lambda'}(\mathcal{E}).$$

Lemma 7 *Let $\mathcal{C}$ be a configuration, and suppose that, for every fixed B , the following pieces of $\mathcal{C}$ (Definition 6) have asymptotic formulas, uniformly in $\boldsymbol{\lambda}$ and λ' :*

- (a) $\mathcal{P}_{\boldsymbol{\lambda}}(\mathcal{E}; y_{\boldsymbol{\lambda}})$, with the level $y_{\boldsymbol{\lambda}}$ of (5.6), for every $\boldsymbol{\lambda}$ that satisfies all constraints;
- (b) $\mathcal{P}_{\boldsymbol{\lambda}}(\mathcal{E}; 1)$ for every $\boldsymbol{\lambda}$ that straddles some constraint;
- (c) $\mathcal{P}'_{\boldsymbol{\lambda}, \lambda'}(\mathcal{E})$ for every $\boldsymbol{\lambda}$ that satisfies all constraints and every λ' in its window $\mathcal{W}(\boldsymbol{\lambda})$ of (5.7).

Then the sum Σ_C of (5.3) has an asymptotic formula. (Proof in Appendix B.)

The pieces in (b) of Lemma 7 are the *straddling pieces* of $\mathcal{C}$, and the pieces in (c) are its *window pieces*. Summed over $\boldsymbol{\lambda}$ and λ' , the straddling and window pieces over $\mathcal{B}$ satisfy the bound (B.13) of Appendix B, and their asymptotic formulas carry this bound to $\mathcal{A}$.

Lemma 8 below concerns the pieces of hypotheses (a) to (c) of Lemma 7. A piece with $\mathcal{T}(\boldsymbol{\lambda}) = \emptyset$ vanishes, so only pieces with $\mathcal{T}(\boldsymbol{\lambda}) \neq \emptyset$ are considered. A piece has the primes $\varpi_1, \dots, \varpi_K$, with $K = k$ for $\mathcal{P}_{\boldsymbol{\lambda}}(\mathcal{E}; y)$, and with $K = k + 1$ and $\varpi_{k+1} = \varpi'$ for $\mathcal{P}'_{\boldsymbol{\lambda}, \lambda'}(\mathcal{E})$. Its cofactor t is sifted at the constant level y , with $y = 1$ for $\mathcal{P}'_{\boldsymbol{\lambda}, \lambda'}(\mathcal{E})$. The norm of ϖ_i lies in Λ_i , and the *exponents* of the piece are the x_i , where

$$\Lambda_i := \mathcal{I}_i \cap J_{\lambda_i} \quad (i \leq k), \quad \Lambda_{k+1} := J_{\lambda'}, \quad x_i := \frac{\log \inf \Lambda_i}{\log X}. \quad (5.9)$$

For $1 \leq i \leq K$,

$$\sup \Lambda_i \leq (1 + 1/H) \inf \Lambda_i, \quad \inf \Lambda_i \geq \mu^4/2. \quad (5.10)$$

The bound for $\sup \Lambda_i$ holds by (5.1). The bound for $\inf \Lambda_i$ holds for $i \leq k$ since $\Lambda_i \subset \mathcal{I}_i \subset [\mu^4, X]$, and for $i = k + 1$ since $\boldsymbol{\lambda}$ satisfies all constraints and $\lambda' \in \mathcal{W}(\boldsymbol{\lambda})$, which give $\xi_{\lambda'} \geq \mu^4/2$ by (5.8). So each $\log |\varpi_i|^2 / \log X$ lies within $\mathcal{O}(1/(H \log X))$ of x_i , and a condition that holds at (x_i) with margin ε holds with margin $\varepsilon/2$ for every tuple of the piece, once X is large.

Lemma 8 *Let $C_1 \geq 1$. A piece (5.5) has an asymptotic formula, uniformly in $\boldsymbol{\lambda}$ and λ' , if one of the following holds.*

- (P1) $y = 1$, and either the exponents (5.9) satisfy $\sum_{i=1}^K x_i \leq \tau - \varepsilon$, or the indices split as $\mathcal{U} \sqcup \mathcal{V}$ and the sums (3.1) satisfy

$$x_{\mathcal{U}} \leq \zeta - \varepsilon, \quad x_{\mathcal{V}} \leq \rho - \varepsilon, \quad \sum_{i=1}^K x_i \leq \nu - \varepsilon.$$

- (P2) $y = \mu^4$, and the exponents satisfy one of the alternatives of (P1).

- (P3) There are an index i_0 and a non-empty set $\mathcal{S}$ of other indices such that

$$\inf \left| \prod_{i \in \mathcal{S}} \varpi_i \right|^2 \geq \frac{X^\sigma}{C_1}, \quad \frac{X/2}{\sup |\varpi_{i_0} \prod_{i \in \mathcal{S}} \varpi_i|^2} \geq \frac{X^\sigma}{C_1},$$

with the infimum and the supremum over the tuples of the piece.

(P4) Every product $n = \varpi_1 \cdots \varpi_K$ of the piece satisfies

$$X^\sigma/C_1 \leq |n|^2 \leq X^{\tau-\varepsilon}, \quad y \leq C_1 X^\tau/|n|^2, \quad (5.11)$$

or

$$|n|^2 < X^\sigma, \quad y \leq C_1 z(|n|^2). \quad (5.12)$$

Proof. In (P1), $y = 1$, and the piece is $\sum_{v \neq 0} c_v |\mathcal{E}_v|$ by (5.5) and (2.6), where c_v is the number of tuples of the piece with product v . This c_v is non-negative and at most $d(v)^K$ by Lemma 18(c), and it is supported on $|v|^2 \leq X \sum_{i=1}^K x_i + o(1)$ by (5.9) and (5.10). By (5.4), (5.5) and (5.9), the tuples of the piece are all the tuples with $|\varpi_i|^2 \in \Lambda_i$ for every i . So, when the indices split as $\mathcal{U} \sqcup \mathcal{V}$, $c_v = \sum_{v=uu'} a_u b_{u'}$, where a_u is the number of subtuples $(\varpi_i)_{i \in \mathcal{U}}$ with product u , and $b_{u'}$ is the number of subtuples $(\varpi_i)_{i \in \mathcal{V}}$ with product u' . By Lemma 18(c) and (5.10), a_u and $b_{u'}$ are non-negative, divisor-bounded and supported on at most 2 dyadic ranges. Lemma 1 applies, since the margin $\varepsilon/2$ absorbs the factor μ^{-1} : in case (a) when $\sum_{i=1}^K x_i \leq \tau - \varepsilon$, and in case (b), on each of the at most 4 pairs of dyadic ranges of u and u' , when the indices split as $\mathcal{U} \sqcup \mathcal{V}$.

In (P2), by (5.5), the piece is $\sum_{u \neq 0} c_u S'(\mathcal{E}_u, \mu^4)$, where c_u is the number of tuples of the piece with product u . This c_u is non-negative and divisor-bounded, as in (P1). Lemma 3 applies with the constant function $z_0 \equiv \mu^4$ and $\varepsilon_0 = \varepsilon/4$, since the margin $\varepsilon/2$ absorbs the factors X^{ε_0} and μ^{-1} : in case (a) when $\sum_{i=1}^K x_i \leq \tau - \varepsilon$, and in case (b) when the indices split as $\mathcal{U} \sqcup \mathcal{V}$, with u_1 and u_2 the products of the primes indexed by $\mathcal{U}$ and by $\mathcal{V}$, each in at most 2 dyadic ranges.

In (P3), by (5.5) and (2.6), the piece is

$$\sum_{u \neq 0, v \neq 0} a_u b_v \# \{ \varpi_{i_0} : |\varpi_{i_0}|^2 \in \Lambda_{i_0}, uv\varpi_{i_0} \in \mathcal{E} \},$$

where a_u is the number of subtuples $(\varpi_i)_{i \in \mathcal{S}}$ with product u , and b_v is the number of pairs of a subtuple $(\varpi_i)_{i \notin \mathcal{S} \cup \{i_0\}}$ and a cofactor t sifted at the level y with product v . By Lemma 18(c), as in (P1), the coefficients a_u and b_v are non-negative and divisor-bounded. Each $|\varpi_i|^2$ varies by at most the factor $1 + 1/H$ on the piece by (5.10), and $X/2 \leq |uv\varpi_{i_0}|^2 < X$ by (2.2), so u lies in at most 2 dyadic ranges and v in at most 4. By (P3), $|u|^2 \geq X^\sigma/C_1$ and $|v|^2 \geq (X/2)/\sup |u\varpi_{i_0}|^2 \geq X^\sigma/C_1$, so the dyadic ranges $u \sim U$ and $v \sim V$ that occur have $U, V \geq X^\sigma/(2C_1)$. The norm of ϖ_{i_0} ranges over Λ_{i_0} , with $\inf \Lambda_{i_0} \geq \mu^4/2$ by (5.10), so Lemma 2 applies with $2C_1$ in place of C_1 .

In (P4), by (5.5) and (2.6), the piece is $\sum_{n \neq 0} a_n S'(\mathcal{E}_n, y)$, where a_n is the number of tuples of the piece with product n . This a_n is non-negative and divisor-bounded, as in (P1), and the level y of the piece is at least 1. Lemma 4 applies to the n that satisfy (5.11), and Lemma 5 to the other n , which satisfy (5.12), both with $C_l = C_1$ and the constant level y . In each of (P1) to (P4), the asymptotic formula of the piece holds uniformly in λ and λ' by the convention of Section 2 on implied constants. $\square$

6 Sums of sifting functions over two primes

Throughout this section, $p \sim P$ and $q \sim Q$, and the pairs (p, q) form a box. The lemmas of this section treat the sum $\sum^*$ of (3.5) on boxes and supply the treatments of Table 2. Lemma 9 gives the sum of $S(\mathcal{A}_{pq}, q)$ over a whole box an asymptotic formula. Lemma 10 gives the sum (6.3) over (p, q, r) an asymptotic formula under (a) or (b) of (6.4). Lemmas 11 and 12 split the sum of $S(\mathcal{A}_{pq}, q)$ over the pairs of a box with $q \prec p$ as $S_{<} + S_{\geq}$ in (6.6) and (6.11) according to the size of $|q|^2$, give $S_{<}$ an asymptotic formula, and express $S_{\geq}$ by identity (6.7) for $PQ \geq X^\sigma$ and by identity (6.12) for $PQ < X^\sigma$. Lemma 11 also gives the sum over (p, q) in (6.7) an asymptotic formula. Under (a) or (b) of (6.4), Lemma 11 gives the sum over (p, q, r) in (6.7) an asymptotic formula too. Lemma 12 also gives the sums over (p, q) and over (p, q, r) in (6.12) asymptotic formulas.

In every sifting function $S(\mathcal{A}_u, \cdot)$ below, every prime factor of u satisfies the condition that the level imposes, so $S(\mathcal{A}_u, \cdot) = S'(\mathcal{A}_u, \cdot)$ by (2.5) and (2.6). The sums and identities of this section are written for $\mathcal{A}$. Each sum is defined for $\mathcal{B}$ in the same way, and every identity holds with $\mathcal{B}$ in place of $\mathcal{A}$, since (2.7) holds for $\mathcal{E} = \mathcal{A}$ and for $\mathcal{E} = \mathcal{B}$.

In the proofs below, Lemma 8 is applied with $C_1 = 10$, and with $\varepsilon/2$ in place of ε , so that a constant factor in a bound for a norm is absorbed by $X^{\varepsilon/2}$. Since $\sigma + \tau = 1$ by (2.1), a piece then satisfies the conditions of (P3) if, for an index i_0 and a non-empty set $\mathcal{S}$ of other indices, each of its tuples satisfies

$$\left| \prod_{i \in \mathcal{S}} \varpi_i \right|^2 \geq X^\sigma, \quad \left| \varpi_{i_0} \prod_{i \in \mathcal{S}} \varpi_i \right|^2 \leq \frac{C_1 X^{1-\sigma}}{2} = 5X^\tau. \quad (6.1)$$

The ranges of the configurations below lie in $[\mu^4, X]$, as Definition 5 requires. Each lower limit is μ^4 , is at least X^ε by the hypotheses of the lemma being proved and the inequality $2\theta < \sigma$ of Table 5, or is $z(PQ)$, which exceeds X^θ by (2.12), since $PQ < X^\sigma$ in Lemma 12. In Lemmas 10 to 12, the lower limit P of $|p|^2$ is at least X^ε , since $Q \leq 2P$, $PQ \geq X^{\sigma-2\theta}$ and $2\theta < \sigma$ give, for small ε ,

$$P^2 \geq \frac{PQ}{2} \geq \frac{X^{\sigma-2\theta}}{2} \geq X^{2\varepsilon}.$$

Each upper limit is at most $2P$, $2Q$, $(X/(PQ))^{1/2}$ or $X^\tau/(PQ)$, and each of these is below X , since $P, Q \geq 1$, $PQ \leq X^\nu$ in Lemma 9, $PQ \leq X^\tau$ in Lemmas 10 to 12, and $\tau, \nu < 1$. By the convention of Section 2 on implied constants, the asymptotic formulas of Lemmas 9 to 12 hold uniformly in P and Q , and the asymptotic formula of Lemma 10 also holds uniformly in the intervals $\mathcal{I}_q$ and $\mathcal{I}_r$ of its statement.

Lemma 9 supplies the treatment of Range 1 in Table 2. The proof of Lemma 9 applies Buchstab's identity (2.7) at the level μ^4 in (6.2). The linear estimate, Lemma 1, treats the sum sifted at the level μ^4 through Lemma 3. The bilinear estimate, Lemma 2, treats the subtracted sum of $S'(\mathcal{A}_{pqr}, r)$, with p as the variable u of Lemma 2.

Lemma 9 *Suppose that $X^\sigma \leq P \leq X^{\zeta-\varepsilon}$, $Q \geq PX^{-2\theta}$ and $PQ \leq X^{\nu-\varepsilon}$. Then the sum of $S(\mathcal{A}_{pq}, q)$ over $p \sim P$ and $q \sim Q$ has an asymptotic formula.*

Proof. Since $\nu - \sigma = \rho$ by (2.1), the bounds $PQ \leq X^{\nu-\varepsilon}$ and $P \geq X^\sigma$ give $Q \leq X^{\rho-\varepsilon}$. With the hypothesis $Q \geq PX^{-2\theta}$, and with $2\theta < \sigma$ and $\rho < \sigma$ from Table 5, every pair of the box satisfies

$$\mu^4 < X^{\sigma-2\theta} \leq PX^{-2\theta} \leq Q \leq |q|^2 < 2Q \leq 2X^{\rho-\varepsilon} < X^\sigma \leq P \leq |p|^2.$$

Hence $q \prec p$, and $S(\mathcal{A}_{pq}, q) = S'(\mathcal{A}_{pq}, q)$. Since $|q|^2 > \mu^4$, Buchstab's identity (2.7) at the level μ^4 gives

$$\sum_{p \sim P, q \sim Q} S'(\mathcal{A}_{pq}, q) = \sum_{p \sim P, q \sim Q} S'(\mathcal{A}_{pq}, \mu^4) - \sum_{p \sim P, q \sim Q} \sum_{r \prec q, |r|^2 \geq \mu^4} S'(\mathcal{A}_{pqr}, r). \quad (6.2)$$

The sum of $S'(\mathcal{A}_{pq}, \mu^4)$ in (6.2) is, in the sense of Definition 5, the configuration with variables p, q , ranges $[P, 2P)$ and $[Q, 2Q)$, no constraints, and the sieve condition of type (C). This configuration has no straddling pieces, since it has no constraints, and no window pieces, since the window (5.7) is empty in type (C). Each piece of hypothesis (a) of Lemma 7 has the level $y = \mu^4$ by (5.6) and falls under (P2) of Lemma 8 with $\mathcal{U} = \{p\}$ and $\mathcal{V} = \{q\}$, since

$$|p|^2 < 2P \leq 2X^{\zeta-\varepsilon}, \quad |q|^2 < 2Q \leq 2X^{\rho-\varepsilon}, \quad |pq|^2 < 4PQ \leq 4X^{\nu-\varepsilon}.$$

The bound $|r|^2 < 2Q$, from $r \prec q$, gives only $|pr|^2 < 4PQ$, which may exceed the bound $5X^\tau$ of (6.1), since PQ may be as large as $X^{\nu-\varepsilon}$ and $\nu > \tau$. So (P3) needs a smaller upper bound for $|r|^2$. In the sum of $S'(\mathcal{A}_{pqr}, r)$ in (6.2), only the primes r with $|r|^4 < X/(PQ)$ contribute. Indeed, suppose that $|r|^4 \geq X/(PQ)$. Let $pqrt$ be an element counted by $S'(\mathcal{A}_{pqr}, r)$ or by $S'(\mathcal{B}_{pqr}, r)$, with cofactor t . If $t \neq 1$, then t has only prime factors $\geq r$ by (2.6), so $|t|^2 \geq |r|^2$. Then $|pqrt|^2 \geq PQ|r|^4 \geq X$, so $pqrt$ lies neither in $\mathcal{A}$ nor in $\mathcal{B}$ by (2.2). If $t = 1$, then $X/2 \leq |pqr|^2 < 4PQ|r|^2$ by (2.2), hence $|r|^2 > X/(8PQ)$. Since $\nu + \rho = (2 + 8\theta)/3 < 1$ for $\theta < 1/8$ by (2.1), the bounds $PQ \leq X^{\nu-\varepsilon}$ and $Q \leq X^{\rho-\varepsilon}$ give

$$16PQ^2 \leq 16X^{\nu+\rho-2\varepsilon} < X, \quad |r|^2 > \frac{X}{8PQ} = \frac{X}{16PQ^2} \cdot 2Q > 2Q > |q|^2.$$

The bound $|r|^2 > |q|^2$ contradicts $r \prec q$.

Hence the sum of $S'(\mathcal{A}_{pqr}, r)$ is the configuration with variables p, q, r , ranges $[P, 2P)$, $[Q, 2Q)$ and $[\mu^4, (X/(PQ))^{1/2})$, the constraint $r \prec q$, and the sieve condition of type (S) with i_S the index of r . All pieces of the configuration of $S'(\mathcal{A}_{pqr}, r)$, the window pieces included, fall under (P3) by (6.1), with $\mathcal{S} = \{p\}$ and $\varpi_{i_0} = r$. Indeed, on every piece, $X^\sigma \leq P \leq |p|^2 < 2P$ and $|r|^2 < (X/(PQ))^{1/2}$. The hypothesis $Q \geq PX^{-2\theta}$ gives $P/Q \leq X^{2\theta}$, and $1 + 2\theta = 2\tau$ by (2.1), so

$$|pr|^2 < 2P \left(\frac{X}{PQ} \right)^{1/2} = 2 \left(\frac{XP}{Q} \right)^{1/2} \leq 2(X^{1+2\theta})^{1/2} = 2X^\tau.$$

Lemma 7 gives the sums of $S'(\mathcal{A}_{pq}, \mu^4)$ and of $S'(\mathcal{A}_{pqr}, r)$ in (6.2) asymptotic formulas. By (6.2), the sum of $S(\mathcal{A}_{pq}, q)$ over $p \sim P$ and $q \sim Q$ then has an asymptotic formula. $\square$

The proof of Lemma 10 follows that of Lemma 9, with pq in place of p as the variable u of the bilinear estimate.

Lemma 10 *Suppose that $Q \leq 2P$ and $X^\sigma \leq PQ \leq X^{\tau-\varepsilon}$. Let $\mathcal{I}_q \subset [Q, 2Q)$ and $\mathcal{I}_r \subset [X^\tau/(PQ), 2Q)$ be intervals, and suppose that $\inf \mathcal{I}_q \geq X^\tau/(PQ)$. Then*

$$\sum_{p \sim P, |q|^2 \in \mathcal{I}_q, q \prec p} \sum_{|r|^2 \in \mathcal{I}_r, r \prec q} S(\mathcal{A}_{pqr}, X^\tau/(PQ)) \quad (6.3)$$

has an asymptotic formula if one of the following holds:

$$(a) \quad PQ \leq X^{\zeta-\varepsilon}, \quad Q \leq X^{\rho-\varepsilon}, \quad PQ^2 \leq X^{\nu-\varepsilon}; \quad (b) \quad P \leq X^{\zeta-\varepsilon}, \quad Q^2 \leq X^{\rho-\varepsilon}, \quad PQ^2 \leq X^{\nu-\varepsilon}. \quad (6.4)$$

Proof. Every prime factor of pqr has norm at least $X^\tau/(PQ)$, since $|p|^2 \geq |q|^2 \geq \inf \mathcal{I}_q \geq X^\tau/(PQ)$ and $|r|^2 \geq \inf \mathcal{I}_r \geq X^\tau/(PQ)$. So $S(\mathcal{A}_{pqr}, X^\tau/(PQ)) = S'(\mathcal{A}_{pqr}, X^\tau/(PQ))$. Since $X^\tau/(PQ) \geq X^\varepsilon \geq \mu^4$ by the hypothesis $PQ \leq X^{\tau-\varepsilon}$, Buchstab's identity (2.7) at the level μ^4 gives

$$S'(\mathcal{A}_{pqr}, X^\tau/(PQ)) = S'(\mathcal{A}_{pqr}, \mu^4) - \sum_{\mu^4 \leq |s|^2 < X^\tau/(PQ)} S'(\mathcal{A}_{pqrs}, s). \quad (6.5)$$

Summed as in (6.3), the term $S'(\mathcal{A}_{pqr}, \mu^4)$ of (6.5) is the configuration with variables p, q, r , ranges $[P, 2P)$, $\mathcal{I}_q$ and $\mathcal{I}_r$, the constraints $q \prec p$ and $r \prec q$, and the sieve condition of type (C). Summed in the same way, the sum of $S'(\mathcal{A}_{pqrs}, s)$ in (6.5) is the configuration with the further variable s , of range $[\mu^4, X^\tau/(PQ))$, the same constraints, and the sieve condition of type (S) with i_S the index of s .

For the configuration of type (C), the pieces of hypothesis (a) of Lemma 7 have the level $y = \mu^4$ by (5.6) and fall under (P2), and the straddling pieces fall under (P1). There are no window pieces, since the window (5.7) is empty in type (C). For (P1) and (P2), split the indices according to the case of (6.4):

$$\begin{aligned} (a) \quad & \mathcal{U} = \{p, q\}, \quad \mathcal{V} = \{r\} : \quad |pq|^2 < 4PQ, \quad |r|^2 < 2Q, \quad |pqr|^2 < 8PQ^2; \\ (b) \quad & \mathcal{U} = \{p\}, \quad \mathcal{V} = \{q, r\} : \quad |p|^2 < 2P, \quad |qr|^2 < 4Q^2, \quad |pqr|^2 < 8PQ^2. \end{aligned}$$

In both cases, (6.4) bounds the norms of the products over $\mathcal{U}$, over $\mathcal{V}$ and over all indices by $X^{\zeta-\varepsilon}$, $X^{\rho-\varepsilon}$ and $X^{\nu-\varepsilon}$, up to constant factors, as the conditions on the exponents in (P1) and (P2) require.

For the configuration of type (S), all pieces, the window pieces included, fall under (P3) by (6.1), with $\mathcal{S} = \{p, q\}$ and $\varpi_{i_0} = s$. Indeed, on every piece,

$$|pq|^2 \geq PQ \geq X^\sigma, \quad |pqs|^2 < 4PQ \cdot \frac{X^\tau}{PQ} = 4X^\tau.$$

Lemma 7 gives the configurations of type (C) and of type (S) asymptotic formulas. By (6.5), the sum (6.3) then has an asymptotic formula. $\square$

For the boxes with $PQ \geq X^\sigma$, Lemma 11 applies Buchstab's identity (2.7) twice at the level $X^\tau/(PQ)$, as in (3.9). For a prime level q , (2.7) lowers the level to w only for $w \leq |q|^2$. So Lemma 11 can apply (2.7)

to $S(\mathcal{A}_{pq}, q)$ at the level $X^\tau/(PQ)$ only for the pairs with $|q|^2 \geq X^\tau/(PQ)$. The pairs with $|q|^2 < X^\tau/(PQ)$ form the sum $S_<$ of (6.6). In the sum over (p, q, r) in (6.7), $|p|^2 \geq P$, $|q|^2 \geq Q$ and $|r|^2 \geq X^\tau/(PQ)$, so

$$|pqr|^2 \geq PQ \cdot \frac{X^\tau}{PQ} = X^\tau.$$

Hence the norms $|pqr|^2$ exceed the bound $X^{\tau-\varepsilon}$ for $|n|^2$ in Lemma 4 and the bound X^σ in Lemma 5. After Buchstab's identity (2.7) lowers the level $X^\tau/(PQ)$ of $S(\mathcal{A}_{pqr}, \cdot)$ to μ^4 , the exponents of p , q and r also violate the alternative $\sum_{i=1}^K x_i \leq \tau - \varepsilon$ of (P1) and (P2). The sum over (p, q, r) in (6.7) therefore needs Lemma 10.

Lemma 11 *Suppose that $X^\varepsilon \leq Q \leq 2P$ and $X^\sigma \leq PQ \leq X^{\tau-\varepsilon}$. Split*

$$\sum_{p \sim P, q \sim Q, q \prec p} S(\mathcal{A}_{pq}, q) = S_< + S_\geq \quad (6.6)$$

according as $|q|^2 < X^\tau/(PQ)$ or $|q|^2 \geq X^\tau/(PQ)$. Then $S_<$ has an asymptotic formula, and

$$\begin{aligned} S_\geq = & \sum_{p,q} S(\mathcal{A}_{pq}, X^\tau/(PQ)) - \sum_{p,q} \sum_{\substack{r \prec q \\ |r|^2 \geq X^\tau/(PQ)}} S(\mathcal{A}_{pqr}, X^\tau/(PQ)) \\ & + \sum_{p,q} \sum_{\substack{r \prec q \\ |r|^2 \geq X^\tau/(PQ)}} \sum_{\substack{s \prec r \\ |s|^2 \geq X^\tau/(PQ)}} S(\mathcal{A}_{pqrs}, s), \end{aligned} \quad (6.7)$$

where each $\sum_{p,q}$ runs over $p \sim P$, $q \sim Q$, $q \prec p$, $|q|^2 \geq X^\tau/(PQ)$. The sum over (p, q) in (6.7) has an asymptotic formula, and so has the sum over (p, q, r) if (a) or (b) of (6.4) holds.

Proof. Every pair (p, q) of the box satisfies

$$X^\sigma \leq |pq|^2 < 4PQ \leq X^{\tau-\varepsilon/2}, \quad \frac{X^\tau}{PQ} < \frac{4X^\tau}{|pq|^2}. \quad (6.8)$$

The sum $S_<$ of (6.6) is the configuration with variables p, q , ranges $[P, 2P]$ and $[Q, \min(2Q, X^\tau/(PQ))]$, the constraint $q \prec p$, and the sieve condition of type (S) with i_S the index of q . By (5.6), a piece of hypothesis (a) of Lemma 7 has the level $y = \xi_{\lambda_q}$. The range $|q|^2 < X^\tau/(PQ)$ of q and (6.8) give, for every pair (p, q) of the piece,

$$X^\sigma \leq |pq|^2 \leq X^{\tau-\varepsilon/2}, \quad y \leq |q|^2 < \frac{X^\tau}{PQ} < \frac{4X^\tau}{|pq|^2},$$

which implies (5.11) with $n = pq$. The piece therefore falls under (P4). A straddling piece falls under (P1), since $|pq|^2 < 4PQ \leq X^{\tau-\varepsilon/2}$ by (6.8). By (5.7), the prime ϖ' of a window piece has $|\varpi'|^2 \in J_{\lambda_q}$, so $|\varpi'|^2 < (1 + 1/H)|q|^2$ by (5.1). With $|pq|^2 < 4PQ$ and $|q|^2 < X^\tau/(PQ)$, the bound for $|\varpi'|^2$ gives

$$|pq\varpi'|^2 < 4PQ(1 + 1/H)|q|^2 < 4(1 + 1/H)X^\tau \leq 5X^\tau.$$

The window piece falls under (P3) by (6.1), with $\mathcal{S} = \{p, q\}$ and $\varpi_{i_0} = \varpi'$, since $|pq|^2 \geq X^\sigma$. Lemma 7 gives $S_<$ an asymptotic formula.

For $|q|^2 \geq X^\tau/(PQ)$, Buchstab's identity (2.7) at the level $X^\tau/(PQ)$ gives

$$S(\mathcal{A}_{pq}, q) = S(\mathcal{A}_{pq}, X^\tau/(PQ)) - \sum_{\substack{r \prec q \\ |r|^2 \geq X^\tau/(PQ)}} S(\mathcal{A}_{pqr}, r), \quad (6.9)$$

and Buchstab's identity at the same level, applied to each term $S(\mathcal{A}_{pqr}, r)$ of (6.9), gives

$$S(\mathcal{A}_{pqr}, r) = S(\mathcal{A}_{pqr}, X^\tau/(PQ)) - \sum_{\substack{s \prec r \\ |s|^2 \geq X^\tau/(PQ)}} S(\mathcal{A}_{pqrs}, s). \quad (6.10)$$

Inserting (6.10) into (6.9) and summing over the pairs of $S_{\geq}$ in (6.6) gives (6.7).

The sum over (p, q) in (6.7) is $\sum_{n \neq 0} a_n S(\mathcal{A}_n, X^\tau/(PQ))$, where a_n is the number of pairs (p, q) in the range of $\sum_{p, q}$ with $pq = n$. Each a_n is 0 or 1, since $q \prec p$. By (6.8), every n with $a_n \neq 0$ has $X^\sigma \leq |n|^2 \leq X^{\tau-\varepsilon/2}$, and the level $X^\tau/(PQ)$ is less than $4X^\tau/|n|^2$. So Lemma 4 applies with $C_l = 4$ and $\varepsilon/2$ in place of ε .

The sum over (p, q, r) in (6.7) is the sum (6.3) with $\mathcal{I}_q = [\max(Q, X^\tau/(PQ)), 2Q]$ and $\mathcal{I}_r = [X^\tau/(PQ), 2Q]$, since $r \prec q$ gives $|r|^2 < 2Q$. Then $\inf \mathcal{I}_q \geq X^\tau/(PQ)$, and the hypotheses $Q \leq 2P$ and $X^\sigma \leq PQ \leq X^{\tau-\varepsilon}$ of Lemma 10 are among those of Lemma 11. Lemma 10 therefore gives the sum over (p, q, r) in (6.7) an asymptotic formula if (a) or (b) of (6.4) holds. $\square$

For a box with $PQ < X^\sigma$, the level $X^\tau/(PQ)$ of (6.7) exceeds $z(PQ)$, which is, up to a constant factor, the largest constant level that Lemmas 4 and 5 admit for a sum of $S(\mathcal{A}_{pq}, \cdot)$ over the box. In Lemma 12, the hypothesis $X^{\sigma-2\theta} \leq PQ < X^\sigma$ and (2.10) give $j(PQ) \in \{0, 1\}$, and (2.11) gives

$$z(PQ) = \left(\frac{X^\tau}{PQ} \right)^{1/(j(PQ)+2)}.$$

Lemma 12 applies Buchstab's identity (2.7) to $S(\mathcal{A}_{pq}, q)$ at the level $z(PQ)$. This application subtracts the sum of $S(\mathcal{A}_{pqr}, r)$ over the primes $r \prec q$ with $|r|^2 \geq z(PQ)$, as in (6.18). Then (2.7) lowers the level of each term $S(\mathcal{A}_{pqr}, r)$ of the subtracted sum to $(X^\tau/(PQ|r|^2))^{1/(j(PQ)+1)}$. The proof of Lemma 12 shows in (6.22) that $j(|pqr|^2) \leq j(PQ) - 1$ for every product pqr of the subtracted sum, and deduces in (6.23) that the level of $S(\mathcal{A}_{pqr}, \cdot)$ in (6.12) is below $4z(|pqr|^2)$. Under the hypothesis $4\varepsilon < \tau - 3\sigma/2$, the bounds $Q \leq 2P$ and $PQ < X^\sigma$ give $|pqr|^2 \leq X^{\tau-3\varepsilon}$ for every product pqr in (6.12), as (6.17) shows. So the pieces of hypothesis (a) of Lemma 7 for the sum over (p, q, r) fall under (P4) of Lemma 8. The hypothesis $4\varepsilon < \tau - 3\sigma/2$ holds for small ε , since $\tau - 3\sigma/2 = 5\theta/2 - 1/4 > 0$.

Lemma 12 *Suppose that $X^\varepsilon \leq Q \leq 2P$ and $X^{\sigma-2\theta} \leq PQ < X^\sigma$, and that $4\varepsilon < \tau - 3\sigma/2$. Split*

$$\sum_{p \sim P, q \sim Q, q \prec p} S(\mathcal{A}_{pq}, q) = S_{<} + S_{\geq} \quad (6.11)$$

according as $|q|^2 < z(PQ)$ or $|q|^2 \geq z(PQ)$. Then $S_{<}$ has an asymptotic formula, and

$$\begin{aligned} S_{\geq} = & \sum_{p, q} S(\mathcal{A}_{pq}, z(PQ)) - \sum_{p, q} \sum_{\substack{r \prec q \\ |r|^2 \geq z(PQ)}} S\left(\mathcal{A}_{pqr}, \left(\frac{X^\tau}{PQ|r|^2}\right)^{1/(j(PQ)+1)}\right) \\ & + \sum_{p, q} \sum_{\substack{r \prec q \\ |r|^2 \geq z(PQ)}} \sum_{\substack{s \prec r \\ |s|^2 \geq z(PQ) + 1}} S(\mathcal{A}_{pqrs}, s), \end{aligned} \quad (6.12)$$

where each $\sum_{p, q}$ runs over $p \sim P$, $q \sim Q$, $q \prec p$, $|q|^2 \geq z(PQ)$. The sums over (p, q) and over (p, q, r) in (6.12) have asymptotic formulas.

Proof. In this proof, a sum or piece sifted at a constant level $y \geq 1$ has an asymptotic formula, by Lemmas 4 and 5 with $C_l = 4$ or by (P4) of Lemma 8, if the product n of the primes in each of its terms satisfies

$$y < 4z(|n|^2), \quad |n|^2 \leq X^{\tau-\varepsilon}. \quad (6.13)$$

For $|n|^2 < X^\sigma$, these bounds give (5.12) and the hypotheses of Lemma 5. For $|n|^2 \geq X^\sigma$, where $z(|n|^2) = X^\tau/|n|^2$ by (2.9) and (2.11), the same bounds give (5.11) and the hypotheses of Lemma 4.

By (2.10), every pair (p, q) of the box satisfies

$$X^{\sigma-(j(PQ)+1)\theta} \leq PQ \leq |pq|^2 < 4PQ < 4X^{\sigma-j(PQ)\theta}.$$

If $|pq|^2 < X^\sigma$, then the lower bound $X^{\sigma-(j(PQ)+1)\theta} \leq |pq|^2$ and (2.10) applied to $|pq|^2$ give

$$X^{\sigma-(j(PQ)+1)\theta} \leq |pq|^2 < X^{\sigma-j(|pq|^2)\theta},$$

so $j(|pq|^2) \leq j(PQ)$. If $j(PQ) = 1$, then $|pq|^2 < 4X^{\sigma-\theta} < X^\sigma$, so $|pq|^2 \geq X^\sigma$ occurs only for $j(PQ) = 0$. Since $PQ < X^\sigma$, also $X^\tau/|pq|^2 > X^\tau/(4PQ) > 1$ and $16PQ < 16X^\sigma \leq X^\tau$. With these bounds, (2.9) and (2.11) give

$$\begin{aligned} z(|pq|^2) &= \left(\frac{X^\tau}{|pq|^2} \right)^{1/(j(|pq|^2)+2)} \geq \left(\frac{X^\tau}{4PQ} \right)^{1/(j(PQ)+2)} > \frac{z(PQ)}{4} \quad (|pq|^2 < X^\sigma), \\ z(PQ) &= \left(\frac{X^\tau}{PQ} \right)^{1/2} \leq \frac{X^\tau}{4PQ} \leq \frac{X^\tau}{|pq|^2} = z(|pq|^2) \quad (|pq|^2 \geq X^\sigma). \end{aligned} \quad (6.14)$$

By (6.14) and the hypothesis $PQ < X^\sigma$, every pair (p, q) of the box satisfies

$$z(PQ) < 4z(|pq|^2), \quad |pq|^2 < 4PQ < 4X^\sigma \leq X^{\tau-\varepsilon}. \quad (6.15)$$

The bounds $Q \leq 2P$ and $PQ < X^\sigma$ give

$$PQ^2 = PQ \cdot Q \leq PQ(2PQ)^{1/2} = \sqrt{2}(PQ)^{3/2} < \sqrt{2}X^{3\sigma/2}. \quad (6.16)$$

For a prime ϖ with $|\varpi|^2 \leq 4Q$, the bounds $|p|^2 < 2P$ and $|q|^2 < 2Q$, (6.16) and the hypothesis $4\varepsilon < \tau - 3\sigma/2$ give

$$|pq\varpi|^2 < 16PQ^2 < 16\sqrt{2}X^{3\sigma/2} \leq X^{\tau-3\varepsilon} \quad (|\varpi|^2 \leq 4Q). \quad (6.17)$$

The sum $S_<$ of (6.11) is the configuration with variables p, q , ranges $[P, 2P)$ and $[Q, \min(2Q, z(PQ)))$, the constraint $q \prec p$, and the sieve condition of type (S) with i_S the index of q . By (5.6), a piece of hypothesis (a) of Lemma 7 has the level $y = \xi_{\lambda_q}$. The range $|q|^2 < z(PQ)$ of q and (6.15) give, for every pair (p, q) of the piece,

$$y \leq |q|^2 < z(PQ) < 4z(|pq|^2), \quad |pq|^2 < 4PQ \leq X^{\tau-\varepsilon}.$$

So a piece of hypothesis (a) satisfies (6.13) with $n = pq$ and falls under (P4). A straddling piece falls under (P1) by the bound for $|pq|^2$ in (6.15). By (5.7), the prime ϖ' of a window piece has $|\varpi'|^2 \in J_{\lambda_q}$, so $|\varpi'|^2 < (1 + 1/H)|q|^2 < 4Q$ by (5.1). The window piece therefore falls under (P1) by (6.17) with $\varpi = \varpi'$. Lemma 7 gives $S_<$ an asymptotic formula.

For $|q|^2 \geq z(PQ)$, Buchstab's identity (2.7) at the level $z(PQ)$ gives

$$S(\mathcal{A}_{pq}, q) = S(\mathcal{A}_{pq}, z(PQ)) - \sum_{\substack{r \prec q \\ |r|^2 \geq z(PQ)}} S(\mathcal{A}_{pqr}, r). \quad (6.18)$$

Every prime r of (6.18) has $|r|^2 \geq z(PQ)$, so (2.11) gives

$$|r|^{2(j(PQ)+2)} \geq z(PQ)^{j(PQ)+2} = \frac{X^\tau}{PQ}, \quad \text{that is,} \quad \frac{X^\tau}{PQ|r|^2} \leq |r|^{2(j(PQ)+1)}.$$

Hence the level $m_M(p, q, r)$ of $S(\mathcal{A}_{pqr}, \cdot)$ in (6.12) satisfies

$$m_M(p, q, r) = \left(\frac{X^\tau}{PQ|r|^2} \right)^{1/(j(PQ)+1)} \leq |r|^2. \quad (6.19)$$

Since $r \prec q$, every prime r of (6.18) also has $|r|^2 < 2Q$. With (6.19), (6.16), the exponent $1/(j(PQ)+1) \geq 1/2$ and the hypothesis $4\varepsilon < \tau - 3\sigma/2$, the bound $|r|^2 < 2Q$ gives

$$m_M(p, q, r) > \left(\frac{X^\tau}{2PQ^2} \right)^{1/(j(PQ)+1)} \geq \left(\frac{X^{\tau-3\sigma/2}}{2\sqrt{2}} \right)^{1/2} > \frac{X^{(\tau-3\sigma/2)/2}}{2} \geq X^{2\varepsilon} \geq \mu^4. \quad (6.20)$$

By (6.19) and (6.20), $\mu^4 \leq m_M(p, q, r) \leq |r|^2$, so (2.7) applies again, at the level $m_M(p, q, r)$:

$$S(\mathcal{A}_{pqr}, r) = S(\mathcal{A}_{pqr}, m_M(p, q, r)) - \sum_{\substack{s \prec r \\ |s|^{2(j(PQ)+1)} \geq X^\tau/(PQ|r|^2)}} S(\mathcal{A}_{pqrs}, s). \quad (6.21)$$

Inserting (6.21) into (6.18) and summing over the pairs of $S_{\geq}$ in (6.11) gives (6.12).

The sum over (p, q) in (6.12) is $\sum_{n \neq 0} a_n S(\mathcal{A}_n, z(PQ))$, where a_n is the number of pairs (p, q) in the range of $\sum_{p, q}$ with $pq = n$. Each a_n is 0 or 1, since $q \prec p$. By (6.15), every n with $a_n \neq 0$ satisfies (6.13) with $y = z(PQ)$. Lemma 5, applied to the terms with $|n|^2 < X^\sigma$, and Lemma 4, applied to the terms with $|n|^2 \geq X^\sigma$, give the sum over (p, q) in (6.12) an asymptotic formula.

The sum over (p, q, r) in (6.12) is the configuration with variables p, q, r , ranges $[P, 2P)$, $[\max(Q, z(PQ)), 2Q)$ and $[z(PQ), 2Q)$, the constraints $q \prec p$ and $r \prec q$, and the sieve condition of type (M) with the level $m_M(p, q, r)$ of (6.19). The level $m_M(p, q, r)$ has the form (5.2) required in type (M), with $Z_M = (X^\tau/(PQ))^{1/(j(PQ)+1)}$, with $f_i = -1/(j(PQ) + 1)$ for the index of r , and with $f_i = 0$ for the indices of p and q . The requirement $m_M(p, q, r) \geq \mu^4$ of Definition 5 is (6.20).

The pieces of hypothesis (a) and the window pieces of the sum over (p, q, r) need a lower bound for $|pqr|^2$. Every prime r of this sum has $|r|^2 \geq z(PQ)$, so (2.11), the bound $PQ \geq X^{\sigma-(j(PQ)+1)\theta}$ of (2.10) and the relation $\tau = \sigma + 2\theta$ of (2.1) give

$$|pqr|^2 \geq PQ z(PQ) = \begin{cases} (PQ X^\tau)^{1/2} \geq (X^{\sigma-\theta} X^\tau)^{1/2} = X^{\sigma+\theta/2} & \text{if } j(PQ) = 0, \\ (PQ)^{2/3} X^{\tau/3} \geq (X^{\sigma-2\theta})^{2/3} X^{\tau/3} = X^{\sigma-2\theta/3} & \text{if } j(PQ) = 1. \end{cases}$$

Every product pqr of the sum over (p, q, r) satisfies

$$|pqr|^2 \geq X^{\sigma-j(PQ)\theta}, \quad j(|pqr|^2) \leq j(PQ) - 1, \quad z(|pqr|^2) \geq \left(\frac{X^\tau}{|pqr|^2} \right)^{1/(j(PQ)+1)}. \quad (6.22)$$

In (6.22), the bound for $|pqr|^2$ follows from the lower bounds $X^{\sigma+\theta/2}$ and $X^{\sigma-2\theta/3}$ for $PQ z(PQ)$ in the cases $j(PQ) = 0$ and $j(PQ) = 1$. If $|pqr|^2 \geq X^\sigma$, then $j(|pqr|^2) = -1$ by (2.9). If $|pqr|^2 < X^\sigma$, then (2.10) and the bound for $|pqr|^2$ give

$$X^{\sigma-j(PQ)\theta} \leq |pqr|^2 < X^{\sigma-j(|pqr|^2)\theta},$$

so $j(|pqr|^2) < j(PQ)$. Since $|pqr|^2 < X^\tau$ by (6.17) with $\varpi = r$, (2.11) and the bound for $j(|pqr|^2)$ give the bound for $z(|pqr|^2)$:

$$z(|pqr|^2) = \left(\frac{X^\tau}{|pqr|^2} \right)^{1/(j(|pqr|^2)+2)} \geq \left(\frac{X^\tau}{|pqr|^2} \right)^{1/(j(PQ)+1)}.$$

By (5.6), a piece of hypothesis (a) of Lemma 7 has the level $y = \inf_{(p, q, r) \in \mathcal{T}(\lambda)} m_M(p, q, r)$. Every tuple (p, q, r) of the piece has $|pqr|^2 < 4PQ|r|^2$, so (6.19) and (6.22) give

$$y \leq m_M(p, q, r) < \left(\frac{4X^\tau}{|pqr|^2} \right)^{1/(j(PQ)+1)} \leq 4z(|pqr|^2). \quad (6.23)$$

With the bound $|pqr|^2 \leq X^{\tau-3\varepsilon}$ of (6.17) for $\varpi = r$, the piece satisfies (6.13) with $n = pqr$ and falls under (P4).

The straddling pieces fall under (P1) by (6.17) with $\varpi = r$. A window piece adds a prime ϖ' . On the piece, $|r|^2$ varies by at most the factor $1 + 1/H$, by (5.1). By (6.19), the level $m_M(p, q, r)$ depends on (p, q, r) only through the factor $|r|^{-2/(j(PQ)+1)}$, so $m_M(p, q, r)$ also varies by at most the factor $1 + 1/H$ on the piece. By (5.7) and the variation of $m_M(p, q, r)$ on the piece, $|\varpi'|^2$ lies in a short interval $J_{\lambda'}$ with

$$\xi_{\lambda'} < \sup_{(p, q, r) \in \mathcal{T}(\lambda)} m_M(p, q, r) \leq (1 + 1/H) m_M(p, q, r)$$

for every tuple (p, q, r) of the piece. Hence (5.1) and (6.23) give, for every tuple (p, q, r) of the piece,

$$|\varpi'|^2 < (1 + 1/H) \xi_{\lambda'} < (1 + 1/H)^2 m_M(p, q, r) < (1 + 1/H)^2 \left(\frac{4X^\tau}{|pqr|^2} \right)^{1/(j(PQ)+1)}. \quad (6.24)$$

If $j(PQ) = 0$, then (6.24) and (6.22) give

$$|pqr\varpi'|^2 < 4(1 + 1/H)^2 X^\tau \leq 5X^\tau, \quad |pqr|^2 \geq X^\sigma.$$

So the window piece falls under (P3) by (6.1), with $\mathcal{S} = \{p, q, r\}$ and $\varpi_{i_0} = \varpi'$. If $j(PQ) = 1$, then (6.24) and the bound $|pqr|^2 \leq X^{\tau-3\varepsilon}$ of (6.17) with $\varpi = r$ give

$$\begin{aligned} |pqr\varpi'|^2 &< (1 + 1/H)^2 |pqr|^2 \left(\frac{4X^\tau}{|pqr|^2} \right)^{1/2} = 2(1 + 1/H)^2 (X^\tau |pqr|^2)^{1/2} \\ &\leq 2(1 + 1/H)^2 X^{\tau-3\varepsilon/2} \leq X^{\tau-\varepsilon}, \end{aligned}$$

so the window piece falls under (P1). Lemma 7 gives the sum over (p, q, r) an asymptotic formula. $\square$

7 The ranges and the discarded regions

This section proves identity (3.5), defines the regions of Table 2, and checks the treatment of each range.

Lemma 13 *Let Π_1 and Π_2 be as in (3.4), and $\sum^*$ as in (3.5). Identity (3.5) holds for $\mathcal{E} \in \{\mathcal{A}, \mathcal{B}\}$. The sums $S(\mathcal{E}, X^\theta)$, $\sum_{p \in \Pi_1} S(\mathcal{E}_p, z(|p|^2))$ and $\sum_{p \in \Pi_2} S(\mathcal{E}_p, p)$ of (3.5) have asymptotic formulas. A prime p with $|p|^2 = X^\alpha$ lies in Π_2 exactly when $\theta \leq \alpha < \tau/5$ or $\sigma - 2\theta \leq \alpha < \tau/4$. The interval $[\sigma - 2\theta, \tau/4)$ is empty for $\theta \leq 3/26$.*

Proof. Buchstab's identity (2.7) for $S'(\mathcal{E}_1, X^{1/2})$, with $w = X^\theta$, gives

$$S(\mathcal{E}, X^{1/2}) = S(\mathcal{E}, X^\theta) - \sum_{X^\theta \leq |p|^2 < X^{1/2}} S(\mathcal{E}_p, p), \quad (7.1)$$

and $S(\mathcal{E}_p, p) = S'(\mathcal{E}_p, p)$ by (2.5) and (2.6). For $p \in \Pi_1$, the identity of (2.7) for $S'(\mathcal{E}_u, q)$, with p in place of both u and q and with $w = z(|p|^2) \leq |p|^2$, gives

$$S'(\mathcal{E}_p, p) = S(\mathcal{E}_p, z(|p|^2)) - \sum_{\substack{q \prec p \\ |q|^2 \geq z(|p|^2)}} S(\mathcal{E}_{pq}, q). \quad (7.2)$$

By (2.5) and (2.6), the functions $S(\mathcal{E}_p, z(|p|^2))$ and $S(\mathcal{E}_{pq}, q)$ in (7.2) equal the functions $S'(\mathcal{E}_p, z(|p|^2))$ and $S'(\mathcal{E}_{pq}, q)$ that (2.7) gives, since $z(|p|^2) \leq |p|^2$ by (3.4) and $q \prec p$. Inserting (7.2) into (7.1) for $p \in \Pi_1$ gives (3.5).

The term $S(\mathcal{E}, X^\theta)$ of (3.5) has an asymptotic formula by Corollary 6. In the sum of $S(\mathcal{E}_p, z(|p|^2)) = S'(\mathcal{E}_p, z(|p|^2))$ over $p \in \Pi_1$, for $|p|^2 \geq X^\sigma$ the level $z(|p|^2)$ is $X^\tau/|p|^2$ by (2.9) and (2.11), and $|p|^2 < X^{1/2} = X^{\tau-\theta}$, so Lemma 4 applies with $C_l = 1$. For $|p|^2 < X^\sigma$, Lemma 5 applies with $C_l = 1$ and the level $z(|p|^2)$.

By (3.4) and (3.7), a prime p with $|p|^2 = X^\alpha$ and $\theta \leq \alpha < 1/2$ lies in Π_2 exactly when $\beta_0(\alpha) > \alpha$. For $\alpha \geq \sigma$, $\beta_0(\alpha) = \tau - \alpha \leq \alpha$, since $\sigma > \tau/2$ by Table 5. For $\alpha < \sigma$, (2.10) applied to X^α and (3.7) give

$$\sigma - (j(X^\alpha) + 1)\theta \leq \alpha < \sigma - j(X^\alpha)\theta, \quad \beta_0(\alpha) = \frac{\tau - \alpha}{j(X^\alpha) + 2} > \alpha \iff \alpha < \frac{\tau}{j(X^\alpha) + 3}.$$

For $j(X^\alpha) = 0$ this is impossible, because then $\alpha \geq \sigma - \theta > \tau/3$ by Table 5. For $j(X^\alpha) = 1$ it gives $\alpha \in [\sigma - 2\theta, \tau/4)$, since $\tau/4 < \tau/3 < \sigma - \theta$, and $\tau/4 \leq \sigma - 2\theta$ for $\theta \leq 3/26$. For $j(X^\alpha) = 2$ it gives $\theta \leq \alpha < \tau/5$, since $\theta > \sigma - 3\theta$ and $\tau/5 < \sigma - 2\theta$ by Table 5. For $j(X^\alpha) \geq 3$, $\alpha < \sigma - 3\theta < \theta$, which is excluded.

In the sum over $p \in \Pi_2$, (2.5) and (2.6) show that the term $S(\mathcal{E}_p, p)$ differs from $S'(\mathcal{E}_p, |p|^2)$ only by the cofactors divisible by the other prime p' of norm $|p|^2$, when it exists and $p' \prec p$. So $0 \leq S'(\mathcal{E}_p, |p|^2) - S(\mathcal{E}_p, p) \leq |\mathcal{E}_{pp'}|$. For $p \in \Pi_2$, $|p|^2 < z(|p|^2)$ by (3.4), and $|p|^2 < X^{\tau/4} < X^\sigma$ by the description of Π_2 , so Lemma 5 gives the sum of $S'(\mathcal{E}_p, |p|^2)$ over $p \in \Pi_2$ an asymptotic formula. Since $|pp'|^2 = |p|^4 < X^{\tau/2}$, Lemmas 1(a) and 18(a) give, with p restricted to the primes of Π_2 for which p' exists and $p' \prec p$,

$$\sum_{p \in \Pi_2} |\mathcal{A}_{pp'}| = \frac{2}{\pi X^\sigma} \sum_{p \in \Pi_2} |\mathcal{B}_{pp'}| + \mathcal{O}(X^\tau (\log X)^{-A}), \quad \sum_{p \in \Pi_2} |\mathcal{B}_{pp'}| \ll X \sum_{|p|^2 \geq X^\theta} |p|^{-4} \ll X^{1-\theta}.$$

So the sums of $|\mathcal{A}_{pp'}|$ and of $2|\mathcal{B}_{pp'}|/(\pi X^\sigma)$ over $p \in \Pi_2$ are $\mathcal{O}(X^\tau(\log X)^{-A})$, and the sum of $S(\mathcal{E}_p, p)$ over $p \in \Pi_2$ has an asymptotic formula. $\square$

At $\theta = 0.1129$ the primes of Π_2 have $\alpha \in [0.1129, 0.12258]$, so D begins at $\alpha = \tau/5 = 0.12258$ in Figure 1.

In the sum $\sum^*$, every element counted by a term $S(\mathcal{E}_{pq}, q)$ with $|p|^2|q|^4 \geq X$ has cofactor $t = 1$, because a cofactor $t \neq 1$ has $|t|^2 \geq |q|^2$ and would give $|pqt|^2 \geq |p|^2|q|^4 \geq X$. Then $pq \in \mathcal{E}$, $q \prec p$ and $|p|^2 < X^{1/2}$ force $X^{1/2}/2 < |q|^2 \leq |p|^2 < X^{1/2}$, and Lemma 17(d) bounds the number of such pairs with $pq \in \mathcal{B}$ by $\mathcal{O}(X/(\log X)^2)$. These terms are non-negative and enter (3.5) with the sign +, so we discard them. Their contribution tends to 0 after division by $S(\mathcal{B}, X^{1/2})$, which is of order $X/\log X$ by (9.1). The remaining pairs have exponents in the closure of the domain D of (3.8).

On each part of Ranges 1 to 6, the hypotheses of the lemmas used there are linear inequalities in α , β and θ . For Lemmas 11 and 12, the hypotheses on P and Q read, in exponents and up to the margin ε ,

$$\begin{aligned} \text{Lemma 11: } \quad & \varepsilon \leq \beta \leq \alpha, \quad \sigma \leq \alpha + \beta \leq \tau, \quad \text{and (a) or (b) of (6.4):} \\ & \text{(a) } \alpha + \beta \leq \zeta, \quad \beta \leq \rho, \quad \alpha + 2\beta \leq \nu; \quad \text{(b) } \alpha \leq \zeta, \quad 2\beta \leq \rho, \quad \alpha + 2\beta \leq \nu; \end{aligned} \quad (7.3)$$

$$\text{Lemma 12: } \quad \varepsilon \leq \beta \leq \alpha, \quad \sigma - 2\theta \leq \alpha + \beta < \sigma.$$

In Lemma 11, (a) or (b) is needed only for the sum over (p, q, r) in (6.7). On D, $\beta < \alpha$ by (3.8), and $\beta \geq c(\theta) > \varepsilon$ with $c(\theta)$ of (7.10), so $\varepsilon \leq \beta \leq \alpha$ holds on every range. Where Lemma 11 is applied below, $\alpha + \beta \leq \tau$ holds: in case (a), since $\alpha + \beta \leq \zeta < \tau$; on R, since $\alpha + \beta < \sigma + \rho/2 < \tau$; and where $\beta < (\tau - \alpha)/2$, since then $\alpha + \beta < (\alpha + \tau)/2 < \tau$. Appendix D lists the inequalities between the parameters that these hypotheses use, with their values at $\theta = 0.1129$. On each range, the discarded region consists of the pairs on which no lemma of Section 6 applies.

Range 1 consists of the pairs with $\sigma \leq \alpha < 1/2$. Here $\beta_0(\alpha) = \tau - \alpha$ by (3.7), and $\beta < (1 - \alpha)/2 < \alpha$, since $\alpha \geq \sigma > 1/3$ by Table 5. The hypotheses $X^\sigma \leq P \leq X^{\zeta - \varepsilon}$, $Q \geq PX^{-2\theta}$ and $PQ \leq X^{\nu - \varepsilon}$ of Lemma 9 read $\sigma \leq \alpha \leq \zeta$, $\beta \geq \alpha - 2\theta$ and $\alpha + \beta \leq \nu$ in exponents. So Lemma 9 gives the sum $\sum^*$ over a box an asymptotic formula where these inequalities hold. Lemmas 11 and 12 do not apply in Range 1, since $\beta \geq \tau - \alpha$ gives $\alpha + \beta \geq \tau$ there. The discarded region is

$$D_1 := \{\sigma \leq \alpha < 1/2, \tau - \alpha \leq \beta < (1 - \alpha)/2\} \setminus \{\alpha - 2\theta \leq \beta \leq \nu - \alpha\}. \quad (7.4)$$

The set subtracted in D_1 is empty for $\alpha > (\nu + 2\theta)/2$, since $\alpha - 2\theta \leq \nu - \alpha$ means $\alpha \leq (\nu + 2\theta)/2$. Since $(\nu + 2\theta)/2 < \zeta$ by Table 5, the condition $\alpha \leq \zeta$ of Lemma 9 holds wherever this set is non-empty.

Range 2 consists of the pairs with $1/3 \leq \alpha < \sigma$. Here $\beta_0(\alpha) = (\tau - \alpha)/2$, as Section 3.3 shows, and $\beta < (1 - \alpha)/2 \leq \alpha$, since $\alpha \geq 1/3$. On the whole range, $\beta \geq (\tau - \alpha)/2$ and $\alpha \geq 1/3$ give $\alpha + \beta \geq (\alpha + \tau)/2 \geq (1/3 + \tau)/2 > \sigma$ by Table 5, so $PQ \geq X^\sigma$. The bound $\beta \geq (\tau - \alpha)/2$ is $|q|^2 \geq X^\tau/|pq|^2$ in exponents, so $S_{<} = 0$ in (6.6) on every box at distance at least ε from the boundary of D. Where $\alpha + \beta \leq \zeta$, case (a) of (7.3) holds, since $\beta \leq \zeta - 1/3 < \rho$ and $\alpha + 2\beta \leq 2\zeta - 1/3 < \nu$ by Table 5. On the region R of (3.10), case (b) holds by Section 3.3. The discarded region is

$$D_2 := \{1/3 \leq \alpha < \sigma, \max(\zeta - \alpha, (\tau - \alpha)/2) \leq \beta < (1 - \alpha)/2, \beta > \rho/2\}. \quad (7.5)$$

Range 3 consists of the pairs with $\rho \leq \alpha < 1/3$. Here (2.10) gives $j(X^\alpha) = 0$, since $\rho > \sigma - \theta$ by Table 5, so $\beta_0(\alpha) = (\tau - \alpha)/2$ by (3.7), and $\beta < \alpha$. Lemma 11 applies in case (a) of (7.3) where $\alpha + \beta \leq \zeta$. Indeed, there, by (2.1) and Table 5,

$$\beta \leq \zeta - \rho < \rho, \quad \alpha + 2\beta \leq 2\zeta - \rho = \nu, \quad \alpha + \beta \geq \frac{\alpha + \tau}{2} \geq \frac{\rho + \tau}{2} > \sigma.$$

Where $\alpha + \beta > \zeta$, case (b) fails, since $2\beta > 2(\zeta - 1/3) = 8\theta/3 > \rho$ by Table 5. The discarded region is

$$D_4 := \{\rho \leq \alpha < 1/3, \zeta - \alpha \leq \beta < \alpha\}. \quad (7.6)$$

Its lower limit $\zeta - \alpha$ of β exceeds $\beta_0(\alpha) = (\tau - \alpha)/2$, since $\alpha < 1/3 < 2\zeta - \tau$ by Table 5.

Range 4 consists of the pairs with $\sigma - \theta \leq \alpha < \rho$. Here (2.10) gives $j(X^\alpha) = 0$, since $\sigma - \theta \leq \alpha < \rho < \sigma$ by Table 5, so $\beta_0(\alpha) = (\tau - \alpha)/2$ by (3.7). Lemma 11 applies in case (a) of (7.3) where $\alpha + 2\beta \leq \nu$. Indeed, there, by (2.1),

$$\alpha + \beta \leq \frac{\alpha + \nu}{2} < \frac{\rho + \nu}{2} = \zeta, \quad \beta \leq \frac{\nu - \sigma + \theta}{2} = \frac{\rho + \theta}{2} < \rho,$$

and $\alpha + \beta \geq (\alpha + \tau)/2 \geq (1 - \theta)/2 > \sigma$. Where $\beta \geq (\tau - \alpha)/2$ in Ranges 4 and 5, case (b) fails, since $\alpha < \rho$ gives $2\beta \geq \tau - \alpha > \tau - \rho > \rho$ by Table 5. The discarded region is

$$D_6 := \{\sigma - \theta \leq \alpha < \rho, (\nu - \alpha)/2 \leq \beta < \alpha\}. \quad (7.7)$$

Its lower limit $(\nu - \alpha)/2$ of β exceeds $\beta_0(\alpha) = (\tau - \alpha)/2$, since $\tau < \nu$ by Table 5.

Range 5 consists of the pairs with $\alpha < \sigma - \theta$ and $\alpha + \beta \geq \sigma - \theta$. Here (2.10) gives $j(X^\alpha) \geq 1$, so $\beta_0(\alpha) \leq (\tau - \alpha)/3$ by (3.7), and pairs with $\beta < (\tau - \alpha)/2$ occur. Where $\alpha + \beta \geq \sigma$ and $\beta < (\tau - \alpha)/2$, the sum of $S(\mathcal{A}_{pq}, q)$ over a box is $S_<$ of (6.6), and Lemma 11 gives it an asymptotic formula. Where $\alpha + \beta \geq \sigma$, $\beta \geq (\tau - \alpha)/2$ and $\alpha + 2\beta \leq \nu$, Lemma 11 applies in case (a) of (7.3), since there $\beta < \alpha < \sigma - \theta < \rho$ and $\alpha + \beta \leq (\alpha + \nu)/2 < (\sigma - \theta + \nu)/2 < \zeta$ by Table 5. Where $\alpha + \beta < \sigma$, (2.10) gives $j(X^{\alpha+\beta}) = 0$, and Lemma 12 applies with $j(PQ) = 0$. For $j(PQ) = 0$, the condition $|q|^2 \geq z(PQ)$ of $S_>$ in (6.11) reads, by (2.11) and in exponents, $\beta \geq (\tau - \alpha - \beta)/2$, that is, $\beta \geq (\tau - \alpha)/3$. Since $\beta < \alpha$, the pairs of $S_>$ have $\alpha > \tau/4$. The discarded region is

$$D_8 := \{\nu/3 \leq \alpha < \sigma - \theta, (\nu - \alpha)/2 \leq \beta < \alpha\}. \quad (7.8)$$

On D_8 , $\beta \geq (\nu - \alpha)/2 > (\tau - \alpha)/2$, and $\alpha + \beta \geq (\alpha + \nu)/2 \geq 2\nu/3 > \sigma$, since $\alpha \geq \nu/3$ and $\sigma < 2\nu/3$ by Table 5. So the conditions $\beta \geq (\tau - \alpha)/2$ and $\alpha + \beta \geq \sigma$ of the part of Range 5 that D_8 covers hold on all of D_8 .

Range 6 consists of the pairs with $\alpha + \beta < \sigma - \theta$. Since $p \in \Pi_1$, Lemma 13 gives $\alpha \geq \tau/5 > \sigma - 3\theta$, by Table 5. So (2.10) gives $j(X^\alpha) \leq 2$, and (3.7) gives $\beta \geq \beta_0(\alpha) \geq (\tau - \alpha)/4$. Hence

$$\alpha + \beta \geq \frac{3\alpha + \tau}{4} \geq \frac{2\tau}{5} > \sigma - 2\theta,$$

by Table 5, so (2.10) gives $j(X^{\alpha+\beta}) = 1$, and Lemma 12 applies with $j(PQ) = 1$. For $j(PQ) = 1$, the condition $|q|^2 \geq z(PQ)$ of $S_>$ in (6.11) reads, by (2.11) and in exponents, $\beta \geq (\tau - \alpha - \beta)/3$, that is, $\beta \geq (\tau - \alpha)/4$. So $S_< = 0$ in (6.11) on every box at distance at least ε from the boundary of D . With $\alpha + \beta < \sigma - \theta$ and (2.1), the bound $\beta \geq (\tau - \alpha)/4$ gives $\alpha < (4\sigma - 4\theta - \tau)/3 = \sigma - 2\theta$.

In the third sum of (6.7), the exponents satisfy $\tau - \alpha - \beta \leq \delta < \gamma < \beta$. In that of (6.12), with $j(X^{\alpha+\beta})$ in place of $j(PQ)$, they satisfy

$$\frac{\tau - \alpha - \beta}{j(X^{\alpha+\beta}) + 2} \leq \gamma < \beta, \quad \frac{\tau - \alpha - \beta - \gamma}{j(X^{\alpha+\beta}) + 1} \leq \delta < \gamma.$$

With these limits, the third sums are discarded over the following regions of $(\alpha, \beta, \gamma, \delta)$:

$$\begin{aligned} E_3 : & 1/3 \leq \alpha < \sigma, (\tau - \alpha)/2 \leq \beta \leq \zeta - \alpha, \tau - \alpha - \beta \leq \delta < \gamma < \beta; \\ E_R : & (\alpha, \beta) \in R, \tau - \alpha - \beta \leq \delta < \gamma < \beta; \\ E_5 : & \rho \leq \alpha < 1/3, (\tau - \alpha)/2 \leq \beta \leq \zeta - \alpha, \tau - \alpha - \beta \leq \delta < \gamma < \beta; \\ E_7 : & \sigma - \theta \leq \alpha < \rho, (\tau - \alpha)/2 \leq \beta \leq (\nu - \alpha)/2, \tau - \alpha - \beta \leq \delta < \gamma < \beta; \\ E_9 : & \alpha < \sigma - \theta, (\tau - \alpha)/2 \leq \beta \leq \min((\nu - \alpha)/2, \alpha), \tau - \alpha - \beta \leq \delta < \gamma < \beta; \\ E_{10} : & \tau/4 \leq \alpha \leq \sigma - \theta, (\tau - \alpha)/3 \leq \beta < \min(\alpha, \sigma - \alpha), \\ & (\tau - \alpha - \beta)/2 \leq \gamma < \beta, \tau - \alpha - \beta - \gamma \leq \delta < \gamma; \\ E_{11} : & \tau/5 \leq \alpha \leq \sigma - 2\theta, (\tau - \alpha)/4 \leq \beta < \min(\alpha, \sigma - \theta - \alpha), \\ & (\tau - \alpha - \beta)/3 \leq \gamma < \beta, (\tau - \alpha - \beta - \gamma)/2 \leq \delta < \gamma. \end{aligned} \quad (7.9)$$

In E_9 , $(\tau - \alpha)/2 \leq \beta \leq \alpha$ gives $\alpha \geq \tau/3$, so $\alpha + \beta \geq (\alpha + \tau)/2 \geq 2\tau/3 > \sigma$ by Table 5, and the pairs of E_9 lie in the part of Range 5 where Lemma 11 applies. In E_{10} , $\alpha \geq \tau/4$ and $\beta \geq (\tau - \alpha)/3$ give $\alpha + \beta \geq (2\alpha + \tau)/3 \geq \tau/2 > \sigma - \theta$, so the pairs of E_{10} lie in Range 5. The region E_{11} contains the set of exponents over which the third sum is discarded in Range 6, since $\beta_0(\alpha) \geq (\tau - \alpha)/4$. For $\theta < 3/26$, the primes with $\alpha \in [\tau/4, \sigma - 2\theta]$ have $j(X^\alpha) = 2$ by (2.10), since $\sigma - 3\theta < \tau/5 < \tau/4$ by Table 5, and $\beta_0(\alpha) = (\tau - \alpha)/4$ by (3.7). Their pairs have $\alpha + \beta < 2\alpha < 2(\sigma - 2\theta) < \sigma$ by Table 5, so Lemma 12 treats

them, with $j(PQ) = 0$ in Range 5 and $j(PQ) = 1$ in Range 6. Lemma 12 does not depend on $j(X^\alpha)$, so the discarded third sums of these pairs lie in E_{10} and E_{11} .

Lemma 16 needs a positive lower bound for the exponents and an upper bound for their sum. All exponents on D and on the regions (7.9) are at least

$$c(\theta) := \min\left(4\theta^2, \frac{5\theta}{2} - \frac{1}{4}, \frac{26\theta - 1}{24}, \frac{1 - 2\theta}{6}\right) > 0, \quad (7.10)$$

and their sums satisfy

$$\alpha + \beta < 1 - c(\theta) \quad \text{on D}, \quad \alpha + \beta + \gamma + \delta < 1 - c(\theta) \quad \text{on the regions (7.9)}. \quad (7.11)$$

The minimum in (7.10) is positive, since $4\theta^2 > 0$ and $5\theta/2 - 1/4$, $(26\theta - 1)/24$ and $(1 - 2\theta)/6$ are quantities of Table 5. On D, $\alpha \geq \theta$ and $\beta \geq \beta_0(\alpha)$. For $\alpha \geq \sigma$, (3.7) gives $\beta_0(\alpha) = \tau - \alpha > \theta$. For $\alpha < \sigma$, (2.10) gives $\alpha < \sigma - j(X^\alpha)\theta$. Then $\alpha \geq \theta$ gives $(j(X^\alpha) + 1)\theta < \sigma$, that is, $j(X^\alpha) + 2 < (\sigma + \theta)/\theta = 1/(2\theta)$, and $\tau - \alpha > \tau - \sigma = 2\theta$, so (3.7) gives

$$\beta_0(\alpha) = \frac{\tau - \alpha}{j(X^\alpha) + 2} > \frac{2\theta}{j(X^\alpha) + 2} > 4\theta^2.$$

Also $\alpha + \beta < 1 - \beta \leq 1 - c(\theta)$ on D, since $\beta < (1 - \alpha)/2$ and $\beta \geq c(\theta)$.

On the regions (7.9), $\delta < \gamma < \beta$, and $\beta \leq \alpha$: on E_9 , E_{10} and E_{11} by their definitions, and on the other regions by Table 5. On E_3 and E_5 , $\alpha \geq \rho$, since $\rho < 1/3$, and $\zeta - \alpha \leq \alpha$ for $\alpha \geq \rho$ by $\zeta < 2\rho$. On E_R , $\beta \leq \rho/2 < 1/3 \leq \alpha$. On E_7 , $(\nu - \alpha)/2 \leq \alpha$ for $\alpha \geq \sigma - \theta$ by $\nu/3 < \sigma - \theta$. So it suffices to bound δ from below and $\alpha + 3\beta$ from above. On E_3 , E_R , E_5 , E_7 and E_9 , the lower limit of δ is $\tau - \alpha - \beta$, so an upper bound for $\alpha + \beta$ bounds δ from below. On E_3 , E_5 , E_7 and E_9 , the pairs lie in the parts of Ranges 2 to 5 where case (a) of (7.3) holds, so $\alpha + \beta \leq \zeta$. On E_R , $\alpha + \beta < \sigma + \rho/2$ by (3.10). Hence, on these regions,

$$\delta \geq \tau - \alpha - \beta \geq \min\left(\tau - \zeta, \tau - \sigma - \frac{\rho}{2}\right) = \min\left(\frac{1 - 2\theta}{6}, \frac{26\theta - 1}{24}\right).$$

On E_{10} , $\alpha + \beta < \sigma$ and $\gamma < \beta < \sigma/2$, so $\delta \geq \tau - \alpha - \beta - \gamma > \tau - 3\sigma/2 = 5\theta/2 - 1/4$. On E_{11} , $\alpha + \beta < \sigma - \theta$ and $\gamma < \beta < (\sigma - \theta)/2$, so

$$\delta \geq \frac{\tau - \alpha - \beta - \gamma}{2} > \frac{\tau - 3(\sigma - \theta)/2}{2} = 2\theta - \frac{1}{8} \geq \frac{5\theta}{2} - \frac{1}{4}.$$

For the sums of the exponents in (7.11), $\alpha + \beta + \gamma + \delta \leq \alpha + 3\beta$, and

$$\alpha + 3\beta \leq \begin{cases} 3\zeta - 2/3 & \text{on } E_3, \\ \sigma + 3\rho/2 & \text{on } E_R, \\ 3\zeta - 2\rho & \text{on } E_5, \\ (3\nu - \sigma + \theta)/2 & \text{on } E_7, \\ 4\nu/3 & \text{on } E_9, \\ 2\sigma & \text{on } E_{10}, \\ 2\sigma - 2\theta & \text{on } E_{11}. \end{cases}$$

On E_9 , the bound $4\nu/3$ follows from $\alpha + 2\beta \leq \nu$ and $\beta \leq \min((\nu - \alpha)/2, \alpha) \leq \nu/3$. By (2.1), the bounds for $\alpha + 3\beta$ are $1/3 + 4\theta$, $5/8 + 7\theta/4$, $5/6 + \theta/3$, $(5 + 18\theta)/8$, $(7 + 10\theta)/9$, $1 - 2\theta$ and $1 - 4\theta$, so each is below $11/12$ for $1/9 < \theta < 1/8$. Also $11/12 < 1 - c(\theta)$, since $c(\theta) \leq 4\theta^2 < 1/16$.

8 Asymptotic formulas on the regions G and L

This section proves that the third sums of (6.7) and (6.12) have asymptotic formulas on the regions G and L of (3.11). Let $p \sim P$, $q \sim Q$ be a box on which Lemma 11 or Lemma 12 gives identity (6.7) or (6.12). For powers R and R' of 2, the quadruples (p, q, r, s) of the third sum of that identity with $r \sim R$ and $s \sim R'$ form a *four-dimensional box*. Its *exponent box* is the set of the $(\alpha, \beta, \gamma, \delta)$ with $X^\alpha \in [P, 2P)$, $X^\beta \in [Q, 2Q)$, $X^\gamma \in [R, 2R)$ and $X^\delta \in [R', 2R')$. Let G_ε and L_ε be the regions defined by the inequalities of (3.11) with margin ε .

Lemma 14 *If the exponent box of a four-dimensional box lies in G_ε or in L_ε , then the sum of $S(\mathcal{A}_{pqrs}, s)$ over the box has an asymptotic formula.*

Proof. All ranges lie in $[\mu^4, X]$, as Definition 5 requires. Indeed, all norms are below $2P < X$, and $|p|^2 \geq |q|^2 \geq Q \geq X^\varepsilon$ by the hypotheses of Lemmas 11 and 12. In (6.7), $|r|^2$ and $|s|^2$ are at least $X^\tau/(PQ) \geq X^\varepsilon$, since $PQ \leq X^{\tau-\varepsilon}$. In (6.12), $|r|^2 \geq z(PQ) > X^\theta$ by (2.12), and $|s|^2 \geq m_M(p, q, r) > X^{2\varepsilon}$ by (6.19) and (6.20). Since $s \prec r \prec q \prec p$, (2.5) and (2.6) give $S(\mathcal{A}_{pqrs}, s) = S'(\mathcal{A}_{pqrs}, s)$. The sum of $S'(\mathcal{A}_{pqrs}, s)$ over the box is the configuration with variables p, q, r, s and the box ranges. Its constraints are $q \prec p$, $r \prec q$ and $s \prec r$, and the lower limits of $|q|^2$, $|r|^2$ and $|s|^2$ in (6.7) or (6.12). Its sieve condition is of type (S), with i_S the index of s . The lower limits of $|q|^2$ and $|r|^2$ are constants, and so is that of $|s|^2$ for (6.7). For (6.12), the lower limit of $|s|^2$ is the monomial constraint $|r|^2|s|^{2(j(PQ)+1)} \geq X^\tau/(PQ)$, with $j(PQ)$ of (2.9). Its exponent $j(PQ) + 1$ is at most 2, as Definition 5 requires, since $j(PQ) \in \{0, 1\}$. By Lemma 7, it suffices to check the pieces.

Suppose the exponent box lies in G_ε . At one point of the exponent box, a non-empty set $\mathcal{S} \subset \{p, q, r\}$ satisfies $\sigma \leq x_S \leq \tau - \delta$ with margin ε . The exponent box has diameter $\mathcal{O}(1/\log X)$, so $\mathcal{S}$ satisfies these inequalities with margin $\varepsilon/2$ on the whole exponent box. Every piece of hypotheses (a), (b) and (c) of Lemma 7 falls under (P3) of Lemma 8 with this $\mathcal{S}$ and $\varpi_{i_0} = s$. Indeed, $x_S \geq \sigma + \varepsilon/2$ and $x_S + \delta \leq \tau - \varepsilon/2$ on the exponent box, and $1 - \tau = \sigma$ by (2.1), so

$$\left| \prod_{i \in \mathcal{S}} \varpi_i \right|^2 \geq X^{\sigma+\varepsilon/2} \geq X^\sigma, \quad \frac{X/2}{\sup |s \prod_{i \in \mathcal{S}} \varpi_i|^2} \geq \frac{X^{1-\tau+\varepsilon/2}}{2} \geq X^\sigma.$$

Suppose the exponent box lies in L_ε . Since $|s|^2 \geq \mu^4$, Buchstab's identity (2.7) at the level μ^4 , with s' running over primes, gives

$$S'(\mathcal{A}_{pqrs}, s) = S'(\mathcal{A}_{pqrs}, \mu^4) - \sum_{s' \prec s, |s'|^2 \geq \mu^4} S'(\mathcal{A}_{pqrs s'}, s'). \quad (8.1)$$

Summed over the box, the term $S'(\mathcal{A}_{pqrs}, \mu^4)$ of (8.1) is the configuration with the same variables and constraints, and with the sieve condition of type (C). As for G , one of the alternatives that define l in (3.11) holds with margin $\varepsilon/2$ on the whole exponent box, so the pieces of hypothesis (a) fall under (P2), and the straddling pieces under (P1), with $\varepsilon/4$ in place of ε . There are no window pieces, since the window (5.7) is empty in type (C). The sum of $S'(\mathcal{A}_{pqrs s'}, s')$ in (8.1), over the box, is the configuration with the further prime variable s' of range $[\mu^4, 2R')$, the further constraint $s' \prec s$, and the sieve condition of type (S) with i_S the index of s' . In the same way, a non-empty set $\mathcal{S} \subset \{p, q, r, s\}$ satisfies $\sigma \leq x_S \leq \tau - \delta$ with margin $\varepsilon/2$ on the whole exponent box. On every tuple of a piece, $|s'|^2 < 2R' \leq 2|s|^2$, since s' has range $[\mu^4, 2R')$ and $s \sim R'$. So the computation for G gives

$$\frac{X/2}{\sup |s' \prod_{i \in \mathcal{S}} \varpi_i|^2} \geq \frac{X/4}{\sup |s \prod_{i \in \mathcal{S}} \varpi_i|^2} \geq \frac{X^{1-\tau+\varepsilon/2}}{4} \geq X^\sigma.$$

So every piece, the window pieces included, falls under (P3) with this $\mathcal{S}$ and $\varpi_{i_0} = s'$. Lemma 7 gives the sums of $S'(\mathcal{A}_{pqrs}, \mu^4)$ and of $S'(\mathcal{A}_{pqrs s'}, s')$ in (8.1) over the box asymptotic formulas. By (8.1), the sum of $S'(\mathcal{A}_{pqrs}, s)$ over the box then has an asymptotic formula. In both cases, the asymptotic formula holds uniformly in P , Q , R and R' by the convention of Section 2 on implied constants. $\square$

9 The discarded sums and the lower bound

This section proves Lemma 16, the lower bound announced in Section 3. An element of $\mathcal{B}$ has norm in $[X/2, X)$, so it has no prime factor of norm below $X^{1/2}$ exactly when it is a Gaussian prime. So Lemma 17(d), with the counting function π_G of that lemma, gives

$$S(\mathcal{B}, X^{1/2}) = \pi_G(X) - \pi_G(X/2) = (1 + o(1)) \frac{X}{2 \log X}. \quad (9.1)$$

The next lemma evaluates the sums over $\mathcal{B}$ with exponents in a given region, which bound the discarded sums.

Lemma 15 Let $K \geq 1$ and $c_1 > 0$, and let $\mathcal{R} \subset \mathbb{R}^K$ be a bounded set whose boundary has Lebesgue measure 0, and whose points $\mathbf{y} = (y_1, \dots, y_K)$ satisfy $y_i \geq c_1$ and $y_K \leq y_i$ for all i , and $\sum_{i=1}^K y_i \leq 1 - c_1$. With $\mathbf{y}(\varpi) := (\log |\varpi_i|^2 / \log X)_{i \leq K}$ for a tuple of primes $\varpi = (\varpi_1, \dots, \varpi_K)$,

$$\sum_{\substack{\mathbf{y}(\varpi) \in \mathcal{R} \\ \varpi_K \prec \varpi_i \ (i < K)}} S(\mathcal{B}_{\varpi_1 \dots \varpi_K}, \varpi_K) = S(\mathcal{B}, X^{1/2}) \left(\int_{\mathcal{R}} \omega \left(\frac{1 - \sum_{i=1}^K y_i}{y_K} \right) \frac{d\mathbf{y}}{y_1 \dots y_{K-1} y_K^2} + o(1) \right). \quad (9.2)$$

(Proof in Appendix C.)

Lemma 16 For every small $\varepsilon > 0$ there is a number $\Phi_\varepsilon(\theta)$, with $\Phi_\varepsilon(\theta) \rightarrow \Phi(\theta)$ as $\varepsilon \rightarrow 0$, such that

$$S(\mathcal{A}, X^{1/2}) \geq \frac{2}{\pi X^\sigma} S(\mathcal{B}, X^{1/2}) (1 - \Phi_\varepsilon(\theta) - o(1)). \quad (9.3)$$

The same holds with Φ_P in place of Φ . In particular, if $\Phi(\theta) < 1$ or $\Phi_P(\theta) < 1$, then $S(\mathcal{A}, X^{1/2}) \gg X^\tau / \log X$.

Proof. Identity (3.5) holds for $\mathcal{A}$ and for $\mathcal{B}$, and its terms other than $\sum^*$ have asymptotic formulas by Lemma 13. The domain D of (3.8) is divided by Ranges 1 to 6 into finitely many parts, which are polytopes, and Section 7 gives the treatment of each part. The hypotheses of the lemma applied on a part are finitely many linear inequalities in (α, β) that hold on the closed part (Section 7 and Appendix D). Each of them has integer coefficients, not all 0, so it holds with margin ε at the points of the part at distance at least ε from its boundary.

Split $\sum^*$ into the $\mathcal{O}((\log X)^2)$ dyadic boxes $p \sim P$, $q \sim Q$, whose exponent rectangles have side $\log 2 / \log X$. Call a box *good* if its exponent rectangle lies inside a part whose treatment is not a discard, at distance at least ε from the boundary of that part. Then all its pairs belong to $\sum^*$, and the lemma of that part applies with margin. All other boxes are discarded, and so is the third sum of every good box on which (6.7) or (6.12) is used. Each discard leaves a lower bound for $S(\mathcal{A}, X^{1/2})$, since every term of $\sum^*$ is non-negative and enters (3.5) with the sign $+$, and every term of a third sum is non-negative and enters (6.7) or (6.12) with the sign $+$. For a discarded box, let $\Delta_{P,Q}(\mathcal{B})$ be the sum of $S(\mathcal{B}_{pq}, q)$ over the pairs of $\sum^*$ in the box. For a good box on which (6.7) or (6.12) is used, let $\Delta_{P,Q}(\mathcal{B})$ be the third sum of that identity for $\mathcal{E} = \mathcal{B}$, and for the other good boxes let $\Delta_{P,Q}(\mathcal{B}) := 0$. Summing the error terms of (2.8) over the $\mathcal{O}((\log X)^2)$ good boxes, each $\mathcal{O}(X^\tau (\log X)^{-A})$ uniformly in the boxes by Section 6, and using (3.5) for $\mathcal{B}$, we obtain

$$S(\mathcal{A}, X^{1/2}) \geq \frac{2}{\pi X^\sigma} \left(S(\mathcal{B}, X^{1/2}) - \sum_{P,Q} \Delta_{P,Q}(\mathcal{B}) \right) - \mathcal{O}(X^\tau (\log X)^{2-A}), \quad (9.4)$$

where $\sum_{P,Q}$ runs over the dyadic boxes $p \sim P$, $q \sim Q$ of $\sum^*$. In the third sums, split r and s into dyadic ranges, which gives the four-dimensional boxes of Section 8. Call a four-dimensional box *good* if its exponent box lies in G_ε or in L_ε . Lemma 14 gives the sum over every good four-dimensional box an asymptotic formula, uniformly in the boxes. Every other four-dimensional box stays discarded. The number of four-dimensional boxes is $\mathcal{O}((\log X)^4)$. So (9.4) holds with the sums over $\mathcal{B}$ of the good four-dimensional boxes removed from $\Delta_{P,Q}(\mathcal{B})$, and with $(\log X)^{4-A}$ in place of $(\log X)^{2-A}$.

The pairs of $\sum^*$ with $|p|^2 |q|^4 \geq X$ contribute $\mathcal{O}(X / (\log X)^2)$ to the discarded sums over $\mathcal{B}$ that remain in (9.4) (Section 7), and this is $o(S(\mathcal{B}, X^{1/2}))$ by (9.1). Every other pair of a discarded box has exponents in the closure of D , within distance 2ε of a part whose treatment is a discard, of a boundary between parts, or of ∂D , since the box is not good and its exponent rectangle has diameter $\mathcal{O}(1 / \log X)$. The third sums use the quantities $X^\tau / (PQ)$ and $z(PQ)$, which are constant on a box, in place of $X^\tau / |pq|^2$ and $z(|pq|^2)$. Since $PQ \leq |pq|^2$, this raises the lower limits of γ and δ in (7.9), so the exponents of every quadruple of a third sum of a good box lie in the closure of one of the regions (7.9). If the exponents of such a quadruple lie in G and its four-dimensional box is not good, then a point of the exponent box, at distance $\mathcal{O}(1 / \log X)$ from them, lies outside G_ε . The same holds for L . So the exponents of every quadruple of a discarded four-dimensional box lie within distance 2ε of the points of the regions (7.9) outside $G \cup L$, or of the boundary of one of the polytopes whose unions define G , L and L in (3.11).

Let $\Phi_\varepsilon(\theta)$ be the sum of the losses (3.13) of the following sets. One is the set of the points of the closure of D within distance 2ε of a part whose treatment is a discard, of a boundary between parts, or of ∂D . For

each region E_k of (7.9), another is the set of the points of the closure of E_k within distance 2ε of $E_k \setminus (G \cup L)$ or of the boundary of one of the polytopes whose unions define G , L and L . Lemma 15, with $K = 2$ and $K = 4$, bounds the other discarded sums over $\mathcal{B}$ by $S(\mathcal{B}, X^{1/2})(\Phi_\varepsilon(\theta) + o(1))$. Its hypotheses hold for these sets with $c_1 = c(\theta)$. The sets are bounded, and their boundaries have measure 0, since each is the intersection of the closure of a finite union of polytopes with the set of the points within distance 2ε of a finite union of convex polytopes. On them, $y_i \geq c_1$ and $\sum_{i=1}^K y_i \leq 1 - c_1$ by (7.10) and (7.11), and $y_K \leq y_i$, since $\beta \leq \alpha$ on the closure of D and $\delta \leq \gamma \leq \beta \leq \alpha$ on the closures of the regions (7.9).

As $\varepsilon \rightarrow 0$, these sets decrease to the closures of the regions in (3.15), together with finitely many boundaries. The parts of D and the sets G , L , L and P are finite unions of polytopes, so these boundaries are null sets. The integrands are bounded, so $\Phi_\varepsilon(\theta) \rightarrow \Phi(\theta)$. Finally, $X^\tau (\log X)^{4-A} = o(X^{-\sigma} S(\mathcal{B}, X^{1/2}))$ for $A > 5$, by (9.1). So, in (9.4) with the good four-dimensional boxes removed and with $(\log X)^{4-A}$ in place of $(\log X)^{2-A}$, the discarded sums over $\mathcal{B}$ are at most $S(\mathcal{B}, X^{1/2})(\Phi_\varepsilon(\theta) + o(1))$ and the error term is $o(X^{-\sigma} S(\mathcal{B}, X^{1/2}))$, which gives (9.3). The same argument applies with P in place of L , since the inequalities of (3.12) with margin ε imply those of L with margin ε , by the computation after (3.12). If $\Phi(\theta) < 1$, fix ε with $\Phi_\varepsilon(\theta) < 1$. Then (9.3) and (9.1) give $S(\mathcal{A}, X^{1/2}) \gg X^\tau / \log X$. The same holds if $\Phi_P(\theta) < 1$. $\square$

10 Evaluation of the losses and proof of Theorem 2

The losses (3.13) are computed numerically by two methods, which differ in the quadrature and in the way they remove $G \cup L$ from the regions E_k , so that each checks the other. Both are implemented in the folder `Code` (see the data availability statement) and use the same limits of the regions and the same table of Buchstab's function ω of (3.14). The first method also uses a table of its integral $\Omega(u) := \int_1^u \omega(t) dt$. On each interval $[h, h+1]$, ω is a Chebyshev series of degree 40 and Ω its integral, of degree 41, built for $h = 1, 2, \dots, 39$ in turn from (3.14), which determines ω on $[h, h+1]$ from its values on $[1, h]$. They are tabulated on a grid of step 10^{-5} and evaluated between grid points by cubic Hermite interpolation, with the derivative ω of Ω and the derivative $\omega'(u) = (\omega(u-1) - \omega(u))/u$ of ω , which (3.14) gives on each interval $[h, h+1]$. By (7.10), every argument of ω in the losses is below $1/c(\theta)$, which is less than 40, since each quantity in (7.10) exceeds $1/36$ for $1/9 < \theta < 1/8$. On $[1, 3]$, ω has the closed forms

$$\omega(u) = \frac{1}{u} \quad (1 \leq u \leq 2), \quad \omega(u) = \frac{1 + \log(u-1)}{u} \quad (2 \leq u \leq 3),$$

and the tables agree with these formulas and their integrals to 10^{-14} . On $[2, 40)$, they satisfy (3.14) to 1.5×10^{-13} .

In the first method, the integral over β in the two-dimensional losses (3.13) is evaluated exactly through Ω , since

$$\omega\left(\frac{1-\alpha-\beta}{\beta}\right) \frac{1}{\beta^2} = -\frac{1}{1-\alpha} \frac{d}{d\beta} \Omega\left(\frac{1-\alpha}{\beta} - 1\right). \quad (10.1)$$

With $\alpha + \beta + \gamma$ in place of α and δ in place of β , (10.1) gives the integral over δ in the four-dimensional losses. For fixed (α, β, γ) , the part of the δ -range outside $G \cup L$ is a finite union of intervals, and (10.1) is applied on each of them. The remaining integrals are computed by composite Gauss–Legendre quadrature. For the regions D_k , the α -axis is cut into 256 equal panels. For the regions E_k , each of the α -, β - and γ -axes is cut into 16 equal panels. The panels are cut further at the points where a limit of integration changes its formula, and at the points where a sum of some of α , β and γ equals σ , ρ or ζ , at which the part of the δ -range in $G \cup L$ jumps. On the γ -axis, they are also cut where the δ -range becomes non-empty and where a limit of δ meets $(1 - \alpha - \beta - \gamma)/2$, the value at which the argument of ω equals 1, since the integrand in γ has a kink at these points. Each resulting subinterval carries 10 Gauss–Legendre nodes.

In the second method, the two-dimensional losses are computed by nested adaptive Gauss–Kronrod quadrature of the integrand itself [19], and their error estimate is that of the outer adaptive rule. The four-dimensional losses are computed by scrambled Sobol quasi-Monte Carlo integration [17, 22], with ω read from its table instead of through Ω and with the sets G and L tested pointwise. Each of these losses is the mean of 8 estimates, from independent scramblings of 2^{20} points each. The error estimate, which measures the spread of the 8 estimates, is the square root of the sum of the squared deviations of the 8 estimates from their mean, divided by $(8 \cdot 7)^{1/2}$.

Table 3 lists the losses at $\theta = 0.1129$. The first method gives $\Phi(0.1129) = 0.9990033$, and the second gives 0.9990044. At $\theta = 0.119$, the decomposition of Harman and Lewis, with the regions derived from the lemmas (Remark 27) and with the cut by which Harman and Lewis exclude a subset of G from the region of φ_7 (Appendix E), gives the total 0.9425585, below the bound 0.943 of Harman and Lewis [10, p. 134]. Table 7 compares the losses with the bounds of Harman and Lewis region by region. Table 4 lists the values of θ at which the total loss equals 1 as the groupings are added. Figure 2 shows, as a function of θ , the total loss of the decomposition of each row of Table 4. At each of the 61 values of θ in the figure, the total loss decreases down the rows of Table 4.

Region	First method	Second method	Error estimate
D_1	0.3201605	0.3201605	$< 10^{-12}$
D_2	0.3437028	0.3437028	$< 10^{-12}$
D_4	0.2132922	0.2132922	$< 10^{-12}$
D_6	0.0480512	0.0480512	$< 10^{-12}$
D_8	0.0684437	0.0684437	$< 10^{-12}$
$E_3 \setminus (G \cup L)$	0.0000098	0.0000098	8.8×10^{-10}
$E_R \setminus (G \cup L)$	0.0010206	0.0010207	6.8×10^{-8}
$E_5 \setminus (G \cup L)$	0.0006199	0.0006199	4.4×10^{-8}
$E_7 \setminus (G \cup L)$	0.0004451	0.0004451	8.5×10^{-9}
$E_9 \setminus (G \cup L)$	0.0002727	0.0002726	5.6×10^{-8}
$E_{10} \setminus (G \cup L)$	0.0028858	0.0028870	1.0×10^{-6}
$E_{11} \setminus (G \cup L)$	0.0000988	0.0000988	1.1×10^{-8}
$\Phi(0.1129)$	0.9990033	0.9990044	1.0×10^{-6}

Table 3: The losses (3.13) at $\theta = 0.1129$ of the regions of Table 2, with $G \cup L$ removed from the regions E_k , computed by the methods of Section 10 and rounded to 7 decimal places, and the error estimate of the second method. The row $\Phi(0.1129)$ gives the sums of the unrounded losses and the square root of the sum of the squared error estimates.

Decomposition	θ with total loss 1	$1/2 - 2\theta$
Harman and Lewis	0.1167825	0.2664351
with R	0.1141073	0.2717854
with R and G	0.1139312	0.2721376
with R, G and P	0.1132049	0.2735902
with R, G and L	0.1128554	0.2742892

Table 4: The value of θ at which the total loss equals 1, for the decomposition of Harman and Lewis with the regions derived from the lemmas (Remark 27) and with the cut by which Harman and Lewis exclude a subset of G from the region of φ_7 (Appendix E), computed by the first method of Section 10. Each row labelled “with” adds to this decomposition the groupings on the regions it lists. The column $1/2 - 2\theta$ gives the exponent in $\{\sqrt{p}\} < p^{-(1/2-2\theta)}$.

The computation is not carried out in interval arithmetic, so its accuracy is checked by refining the quadrature of the first method and by comparing the methods. The finest rule tested for the first method has 512 panels on the α -axis for the regions D_k , 32 panels per axis for the regions E_k , and 12 nodes per subinterval. Between the rule used for Table 3 and this finest rule, the losses at $\theta = 0.1129$ change by at most 2.5×10^{-9} , and their total by 2.6×10^{-10} . The 8 coarser rules tested have 6 to 12 nodes per subinterval and 2 to 16 panels per axis for the regions E_k , with 16 times as many panels on the α -axis for the regions D_k . As the number of nodes per subinterval times the number of panels per axis grows from 12 to 192, the total differs from that of the finest rule by amounts that decrease from 8.7×10^{-6} to 6.8×10^{-11} . The unrounded totals of the methods differ by 1.2×10^{-6} , which is 1.1 times the unrounded error estimate of the second method in Table 3. Both quantities come almost entirely from E_{10} . The gap $1 - 0.9990033 = 9.967 \times 10^{-4}$

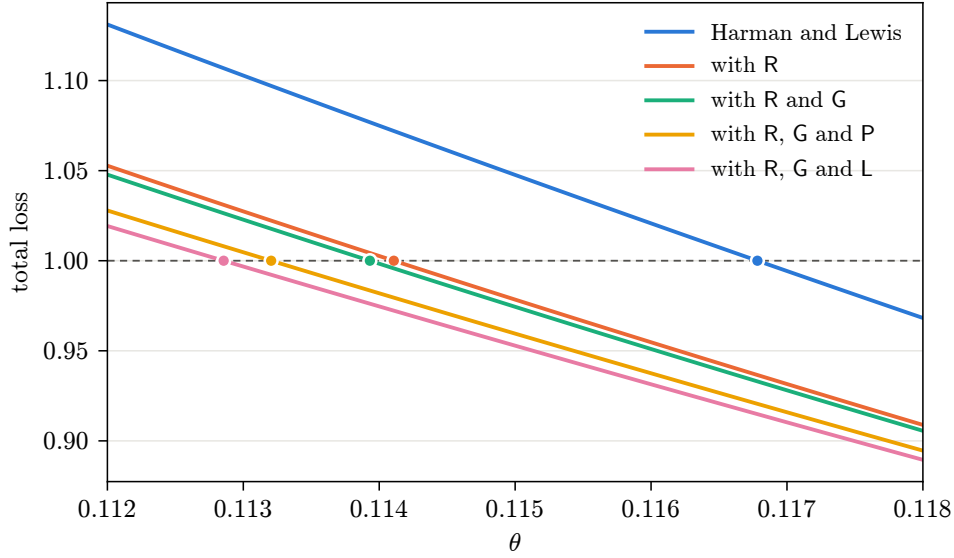


Figure 2: The total loss of each decomposition of Table 4, computed by the first method of Section 10 at 61 values of θ from 0.112 to 0.118. The dashed line marks the total loss 1, and the dots mark the values of θ in Table 4.

exceeds this difference, this error estimate and the change 2.6×10^{-10} of the total from the rule of Table 3 to the finest rule by a factor of more than 800.

Proof of Theorem 2. Let $\theta = 0.1129$. By Table 3, $\Phi(\theta) < 1$, so Lemma 16 gives $S(\mathcal{A}, X^{1/2}) \gg X^\tau / \log X = X^{1/2+\theta} / \log X$ by (2.1), uniformly in ψ . By (2.2) and (2.5), $S(\mathcal{A}, X^{1/2})$ is the number of Gaussian primes in $\mathcal{A}$, so this is (1.3). $\square$

With the polytope P of (3.12) in place of L (Definition 4), the first and second methods give $\Phi_P(0.1133) = 0.9978243$ and 0.9978250 , so the same argument gives (1.3) with $\theta = 0.1133$. As in the proof of Theorem 1, (1.2) then gives $\{\sqrt{p}\} < p^{-0.2734}$ for infinitely many primes.

11 Conclusion

Of the total loss 0.999 at $\theta = 0.1129$, the regions D_1 , D_2 and D_4 account for 0.877. On these regions, after Buchstab’s identity (2.7) introduces a third prime $r \prec q$, no factorization of pqr satisfies the conditions (3.2) of the linear estimate once $|r|^2$ is close to $|q|^2$. We have not found another grouping that gives the sums over these regions asymptotic formulas. A further improvement may need new arithmetic information for the Hecke characters λ^m of (A.1). Harman and Lewis [10, p. 134] call the improvement of their arithmetic information “the more interesting challenge”. The numerical evaluation of the losses could be made rigorous by interval arithmetic.

Data availability

The code that computes every number in Section 10, Table 7 and Appendix D, and the data of Figures 1 and 2, are in the folder Code of the repository <https://github.com/xuda-ye-math/Gaussian-Prime>. The code is written in Julia 1.13, except for a Python script that draws the figures, and the file Manifest.toml of the folder fixes the versions of its Julia packages. The file README.md of the folder describes each computation and its output.

A Proofs of the linear and bilinear estimates

This appendix derives Lemmas 1 and 2 of Section 4 from the arithmetic information. It begins with standard facts, which Sections 5, 7 and 9 and Appendices B and C also use. Lemma 17 bounds sums over Gaussian primes, and Lemma 18 counts Gaussian integers and their divisors. Lemma 19 is the truncated Perron formula used in the proof of Lemma 2.

Lemma 17 *The number $\pi_G(y)$ of Gaussian primes of norm less than y satisfies $\pi_G(y) = \text{Li}(y) + \mathcal{O}(y \exp(-c(\log y)^{1/2}))$ for some $c > 0$, where $\text{Li}(y) := \int_2^y dt/\log t$. Consequently, for fixed $B > 0$ and with H of (5.1), the following hold once X is large.*

(a) For fixed $\kappa \geq 1$ and every $Y \geq \mu$,

$$\sum_{Y \leq |\varpi|^2 < Y(1+1/H)^\kappa} |\varpi|^{-2} \leq \frac{2\kappa}{H \log Y}.$$

(b) $\sum_{|\varpi|^2 < y} |\varpi|^{-2} \leq \log \log y + C$ for $y \geq 3$.

(c) For fixed $a_0 > 0$, uniformly for $a_0 \leq a < b \leq 1$,

$$\sum_{X^a \leq |\varpi|^2 < X^b} |\varpi|^{-2} = \log \frac{b}{a} + \mathcal{O}\left(\frac{1}{\log X}\right).$$

(d) $\pi_G(y) - \pi_G(y/2) = (1 + \mathcal{O}(1/\log y)) y/(2 \log y)$.

Proof. For $a \in \{1, 3\}$, let $\pi(y; 4, a)$ be the number of rational primes $\ell < y$ with $\ell \equiv a \pmod{4}$. A Gaussian prime is $1 + i$, a prime of norm ℓ for a rational prime $\ell \equiv 1 \pmod{4}$, or a rational prime $\ell \equiv 3 \pmod{4}$, of norm ℓ^2 . Hence $\pi_G(y) = 2\pi(y; 4, 1) + \pi(y^{1/2}; 4, 3) + \mathcal{O}(1)$. The formula for $\pi_G(y)$ follows by partial summation from the prime number theorem in progressions modulo 4 [23, Section 1.2], in the form

$$\sum_{\ell \leq x, \ell \equiv a \pmod{4}} \log \ell = \frac{x}{2} \left(1 + \mathcal{O}(\exp(-c'(\log x)^{3/5}(\log \log x)^{-1/5})) \right), \quad a \in \{1, 3\},$$

with a constant $c' > 0$. Statements (b) and (c) follow by partial summation, and (d) follows from the formula for $\pi_G(y)$ and

$$\text{Li}(y) - \text{Li}(y/2) = \int_{y/2}^y \frac{dt}{\log t} = \left(1 + \mathcal{O}\left(\frac{1}{\log y}\right) \right) \frac{y}{2 \log y},$$

since $1/\log t = (1 + \mathcal{O}(1/\log y))/\log y$ for $y/2 \leq t \leq y$. In (a), the main term contributes

$$\log \left(1 + \frac{\kappa \log(1 + 1/H)}{\log Y} \right) \leq \frac{\kappa}{H \log Y},$$

and the error term contributes $\mathcal{O}(\exp(-c(\log Y)^{1/2}))$. Since $Y \geq \mu$, (2.1) gives $\log Y \geq (\log X)^{4/5}$, so $H = (\log X)^B \leq (\log Y)^{5B/4}$. Hence the error term is smaller than $1/(H \log Y)$ for all $Y \geq \mu$ once X is large, and (a) follows from $\kappa \geq 1$. $\square$

Lemma 18 (a) For $M \geq 1$, the number of Gaussian integers v with $X/(2M) \leq |v|^2 < X/M$ is at most CX/M . In particular $|\mathcal{B}_u| \leq CX/|u|^2$, since the cofactor n/u of each $n \in \mathcal{B}_u$ satisfies $X/(2|u|^2) \leq |n/u|^2 < X/|u|^2$.

(b) For $k \geq 1$ and $Y \geq 2$,

$$\sum_{|n|^2 \leq Y} d(n)^k \ll_k Y (\log Y)^{2^k}, \quad \sum_{|n|^2 \leq Y} d(n)^k |n|^{-2} \ll_k (\log Y)^{2^k}.$$

(c) A Gaussian integer n has at most $d(n)^{h-1}$ ordered factorizations into h factors.

Proof. For (a), the count is 0 if $X/M < 1$, and $\pi X/(8M) + \mathcal{O}((X/M)^{1/2} + 1)$ otherwise. For (b), d is multiplicative with $d(\varpi^e) = e + 1$. So, by Lemma 17(b), the sum of $d(n)^k |n|^{-2}$ is at most

$$\prod_{|\varpi|^2 \leq Y} \sum_{e \geq 0} (e+1)^k |\varpi|^{-2e} \leq \exp \left(\sum_{|\varpi|^2 \leq Y} (2^k |\varpi|^{-2} + C_k |\varpi|^{-4}) \right) \ll_k (\log Y)^{2^k},$$

and the sum of $d(n)^k$ is at most Y times the sum of $d(n)^k |n|^{-2}$. For (c), the number $d_h(n)$ of ordered factorizations satisfies, by induction on h ,

$$d_h(n) = \sum_{u|n} d_{h-1}(n/u) \leq d(n)^{h-1}.$$

□

Lemma 19 For $y > 0$, $c > 0$ and $T_0 \geq c$, let

$$I(y) := \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} y^s \frac{ds}{s}.$$

Then $|I(y)| \leq y^c \log(3T_0/c)$, and $|I(y) - \mathbf{1}[y > 1]| \leq y^c / (\pi T_0 |\log y|)$ for $y \neq 1$.

Proof. The bound $|I(y)| \leq y^c \log(3T_0/c)$ follows from

$$|I(y)| \leq \frac{y^c}{2\pi} \int_{-T_0}^{T_0} \frac{dt}{(c^2 + t^2)^{1/2}} = \frac{y^c}{\pi} \log \frac{T_0 + (c^2 + T_0^2)^{1/2}}{c} \leq \frac{y^c}{\pi} \log \frac{3T_0}{c},$$

since $T_0 \geq c$ gives $T_0 + (c^2 + T_0^2)^{1/2} \leq (1 + \sqrt{2}) T_0 < 3T_0$. For $y > 1$, move the segment to $\operatorname{Re} s = -K$ and let $K \rightarrow \infty$. The residue at 0 is 1, the integral over $\operatorname{Re} s = -K$ tends to 0, and each horizontal side is at most

$$\frac{1}{2\pi T_0} \int_{-\infty}^c y^u du = \frac{y^c}{2\pi T_0 \log y}.$$

For $y < 1$, move the segment to $\operatorname{Re} s = K$, where there is no pole, and obtain the same bound with $|\log y|$. □

For an integer m and a Gaussian integer $n \neq 0$ put, as in Harman and Lewis [10, p. 122],

$$\lambda^m(n) := \left(\frac{n}{|n|} \right)^{4m}. \quad (\text{A.1})$$

This is a Hecke character. It is unchanged when n is multiplied by a unit, so it is well defined under the identification of associates fixed in Section 1, and it is completely multiplicative. Since $\lambda^m(n) = e^{4im \arg n}$, it detects the argument of n modulo $\pi/2$. Lemmas 20 to 23 below are quoted without proof from Harman and Lewis [10]. The linear estimate, Lemma 1, rests on Lemmas 20 and 23, and the bilinear estimate, Lemma 2, on Lemmas 20, 21 and 22.

Lemma 20 (Harman and Lewis [10], (12)) For every $T \geq 1$ and all non-negative weights a_n ,

$$\sum_{n \in \mathcal{A}} a_n = \frac{2}{\pi X^\sigma} \sum_{n \in \mathcal{B}} a_n + \mathcal{O} \left(\frac{1}{T} \sum_{n \in \mathcal{B}} a_n \right) + \mathcal{O} \left(X^{-\sigma} \sum_{m=1}^T \left| \sum_{n \in \mathcal{B}} a_n \lambda^m(n) \right| \right).$$

Harman and Lewis state Lemma 20 for any weights and note that it follows from Baker [2, Chapter 2]. It is quoted here for non-negative weights, the only ones used.

Lemma 21 (Harman and Lewis [10], Lemma 1) For $T \geq 1$, $N \geq 1$ and all coefficients $f(n)$,

$$\sum_{m=1}^T \left| \sum_{|n|^2 \leq N} f(n) \lambda^m(n) \right|^2 \ll (T+N) \sum'_{|n|^2 \leq N} |f_1(n)|^2, \quad f_1(n) := \sum_{\arg u = \arg n} f(u),$$

where $\sum'$ runs over the primitive Gaussian integers $n = x + iy$, $\gcd(x, y) = 1$.

Lemma 22 (Harman and Lewis [10], Lemma 4) Let $R, T_0 \geq 2$, $1 \leq m \leq T$ and $|t| \leq T_0$, and suppose that

$$T \geq S \geq R + \exp((\log(TT_0))^{4/5}).$$

Then

$$\left| \sum_{R \leq |\varpi|^2 < S} |\varpi|^{2it} \lambda^m(\varpi) \right| \ll S \exp\left(-\frac{\log S}{2(\log(TT_0))^{7/10}}\right).$$

Lemma 21 is Coleman [5, Lemma 6], and Lemma 22 rests on the zero-free region of Coleman [4].

Lemma 23 (Harman and Lewis [10], Lemmas 6 and 7) Let $A > 0$, $T := X^\sigma (\log X)^A$, and let a_u, b_v be divisor-bounded.

(a) If $U \leq X^\tau \mu^{-1}$, then

$$\sum_{m=1}^T \left| \sum_{u \sim U} a_u \sum_{ut \in \mathcal{B}} \lambda^m(ut) \right| \ll X (\log X)^{-A}.$$

(b) If $U \leq X^\zeta \mu^{-1}$, $V \leq X^\rho \mu^{-1}$ and $UV \leq X^\nu \mu^{-1}$, then

$$\sum_{m=1}^T \left| \sum_{u \sim U} a_u \sum_{v \sim V} b_v \sum_{uvt \in \mathcal{B}} \lambda^m(uvt) \right| \ll X (\log X)^{-A}.$$

Here t runs over all Gaussian integers with the stated product in $\mathcal{B}$.

Remark 24 in Appendix E explains the conditions $ut \in \mathcal{B}$ and $uvt \in \mathcal{B}$ of Lemma 23.

Proof of Lemma 1. With $c'_n := \sum_{v|n} c_v \geq 0$,

$$\sum_v c_v |\mathcal{A}_v| = \sum_{n \in \mathcal{A}} c'_n, \quad \sum_{n \in \mathcal{B}} c'_n = \sum_v c_v |\mathcal{B}_v|.$$

Apply Lemma 20 with $a_n = c'_n$ and $T = X^\sigma (\log X)^A$. By Lemma 18(a) and (b), its error term $\mathcal{O}(T^{-1} \sum_{n \in \mathcal{B}} c'_n)$ is at most

$$X^{-\sigma} (\log X)^{-A} \cdot CX \sum_v c_v |v|^{-2} \ll X^\tau (\log X)^{C-A}.$$

The error term of Lemma 20 with the sums over λ^m is

$$X^{-\sigma} \sum_{m \leq T} \left| \sum_v c_v \sum_{vt \in \mathcal{B}} \lambda^m(vt) \right|.$$

Under (a), after the range of v is split into $\mathcal{O}(\log X)$ dyadic ranges, it is $\ll X^\tau (\log X)^{1-A}$ by Lemma 23(a). Under (b) it is $\ll X^\tau (\log X)^{-A}$ by Lemma 23(b), with u' as its variable v and U' as its V . Since A is arbitrary, the lemma follows. $\square$

Proof of Lemma 2. The proof separates the variables by Lemma 19 and bounds the resulting sums by Lemmas 21 and 22.

Fix A , let $T = X^\sigma (\log X)^A$ be as in the proof of Lemma 1, and let E_m be the sum of $a_u b_v \lambda^m(uv\varpi)$ over the triples counted in (4.1) with $\mathcal{E} = \mathcal{B}$. As in the proof of Lemma 1, it suffices to show that $\sum_{m \leq T} |E_m| \ll$

$X(\log X)^{-A}$. A triple occurs only if $X/(8UV) \leq |\varpi|^2 < X/(UV)$. Put $Z'_1 := \max(Z_1, X/(8UV))$ and $Z'_2 := \min(Z_2, X/(UV))$, and assume that $Z'_1 < Z'_2$, since otherwise $E_m = 0$. Then

$$E_m = \sum_{u \sim U, v \sim V} \sum_{Z'_1 \leq |\varpi|^2 < Z'_2} a_u b_v \lambda^m(uv\varpi) \mathbf{1}[X/2 \leq |uv\varpi|^2 < X],$$

and each triple in this sum has $|uv\varpi|^2 < 4X$.

For $N = |uv\varpi|^2 < 4X$, write

$$\mathbf{1}[X/2 \leq N < X] = \mathbf{1}[X/N > 1] - \mathbf{1}[(X/2)/N > 1],$$

and apply Lemma 19 to $y = Y/N$ for $Y \in \{X, X/2\}$, with $c = 1/\log X$ and $T_0 = X^3$. Since $N \geq 1$ and $Y \leq X$, $y^c \leq X^c = e$. If $|N - Y| \geq 1/2$, then $|\log y| \geq 1/(16X)$ by the mean value theorem, since $N, Y < 4X$. For these N , Lemma 19 bounds the error $|I(y) - \mathbf{1}[y > 1]|$ by $16eX/(\pi T_0)$. At most 2 integers N have $|N - Y| < 1/2$, and for them the error is at most $|I(y)| + 1 \leq e \log(3T_0/c) + 1$.

Each $N < 4X$ is the norm of at most $d(N)$ Gaussian integers, since each of them divides N , and by Lemma 18(c) each of these is the product of at most $d(N)^2$ triples, each with weight $a_u b_v \leq d(N)^{2C_0}$. As a Gaussian integer, N has norm $N^2 < 16X^2$, so Lemma 18(b) with $Y = 16X^2$ gives $d(N)^h \ll_h X^2(\log X)^{2h}$, that is, $d(N) \ll_h X^{3/h}$, for every fixed $h \geq 1$. So all these quantities are $X^{o(1)}$. Summed with these weights over the at most $4X$ integers N with $|N - Y| \geq 1/2$ and the at most 2 others, the errors of Lemma 19 add up to

$$4X \cdot X^{o(1)} \cdot \frac{16eX}{\pi T_0} + 2X^{o(1)}(e \log(3T_0/c) + 1) = \mathcal{O}(X^{o(1)}) \quad (T_0 = X^3).$$

Since λ^m is completely multiplicative, Lemma 19 then gives

$$E_m = \frac{1}{2\pi i} \int_{c-iT_0}^{c+iT_0} (X^s - (X/2)^s) F_m(s) G_m(s) P_m(s) \frac{ds}{s} + \mathcal{O}(X^{o(1)}), \quad (\text{A.2})$$

where

$$F_m(s) := \sum_{u \sim U} a_u \lambda^m(u) |u|^{-2s}, \quad G_m(s) := \sum_{v \sim V} b_v \lambda^m(v) |v|^{-2s}, \quad P_m(s) := \sum_{Z'_1 \leq |\varpi|^2 < Z'_2} \lambda^m(\varpi) |\varpi|^{-2s}.$$

On the line $\operatorname{Re} s = c$, $|X^s - (X/2)^s| \leq 2e$, and $\int_{-T_0}^{T_0} |c+it|^{-1} dt \ll \log X$. So, summing (A.2) over $m \leq T$ and using $TX^{o(1)} = X^{\sigma+o(1)}$, we obtain

$$\sum_{m \leq T} |E_m| \ll \log X \cdot \sup_{|t| \leq T_0} \sum_{m \leq T} |F_m(c+it) G_m(c+it) P_m(c+it)| + X^{\sigma+o(1)}. \quad (\text{A.3})$$

By Cauchy's inequality, the sum over m in (A.3) satisfies

$$\sum_{m \leq T} |F_m(c+it) G_m(c+it) P_m(c+it)| \leq \max_{m \leq T} |P_m(c+it)| \left(\sum_{m \leq T} |F_m(c+it)|^2 \right)^{1/2} \left(\sum_{m \leq T} |G_m(c+it)|^2 \right)^{1/2}.$$

To bound $\sum_{m \leq T} |F_m(c+it)|^2$, apply Lemma 21 with the norm bound $2U$ and with the coefficients $f(u) = a_u |u|^{-2s}$ for $u \sim U$ and $f(u) = 0$ otherwise. A Gaussian integer u with $\arg u = \arg n$, n primitive, is $u = kn$ with k a positive rational integer. So $|f_1(n)| \leq \sum_k d(kn)^{C_0}$, where $k \geq 1$ runs over the integers with $k^2 |n|^2 \in [U, 2U)$. This sum has at most $(2U)^{1/2}/|n|$ terms. By Cauchy's inequality and $d(kn) \leq d(k)d(n)$,

$$\sum'_n |f_1(n)|^2 \leq (2U)^{1/2} \sum_{k^2 \leq 2U} d(k)^{2C_0} \sum_{|n|^2 < 2U/k^2} d(n)^{2C_0} |n|^{-1}. \quad (\text{A.4})$$

The sum of $d(n)^{2C_0} |n|^{-1}$ in (A.4) is $\ll (U/k^2)^{1/2} (\log X)^C$, by Lemma 18(b) and partial summation. Inserting this bound in (A.4) gives

$$\sum'_n |f_1(n)|^2 \ll U (\log X)^C \sum_{k \leq (2U)^{1/2}} \frac{d(k)^{2C_0}}{k}.$$

The sum over the rational integers k is at most a product over the rational primes $\ell \leq (2U)^{1/2}$:

$$\prod_{\ell \leq (2U)^{1/2}} \sum_{g \geq 0} (g+1)^{4C_0} \ell^{-g} \ll (\log X)^{2^{4C_0}}.$$

Here $d(\ell^g) \leq (g+1)^2$ for every rational prime ℓ , and $\sum_{\ell \leq x} 1/\ell \leq \log \log x + \mathcal{O}(1)$ follows from the prime number theorem modulo 4. So $\sum'_n |f_1(n)|^2 \ll U(\log X)^C$, and, since $T \leq C_1 U(\log X)^A$, uniformly in $|t| \leq T_0$,

$$\sum_{m \leq T} |F_m(c+it)|^2 \ll (T+U)U(\log X)^C \ll U^2(\log X)^{A+C}.$$

The same bound holds for $\sum_{m \leq T} |G_m(c+it)|^2$ with V in place of U .

It remains to bound $P_m(c+it)$. The weight $|\varpi|^{-2c}$ in its terms is decreasing in $|\varpi|^2$ and is at most 1. By partial summation, $|P_m(c+it)|$ is at most 2 times the largest value, over $Z \in [Z'_1, Z'_2]$, of

$$\left| \sum_{Z'_1 \leq |\varpi|^2 < Z} \lambda^m(\varpi) |\varpi|^{-2it} \right|.$$

Each of these sums is the difference of the sums over $2 \leq |\varpi|^2 < S$ for $S = Z$ and for $S = Z'_1$. Lemma 22, with $-t$ in place of t , applies to them with $R = 2$, $|t| \leq T_0$ and $1 \leq m \leq T$, once

$$T \geq S \geq 2 + \exp((\log(TT_0))^{4/5}).$$

The condition $S \leq T$ holds, since $S \leq Z'_2$, and the hypothesis $U, V \geq X^\sigma/C_1$ of Lemma 2, (2.1) and $\theta < 1/6$ give

$$S \leq \frac{X}{UV} \leq C_1^2 X^{2\theta} \leq X^\sigma (\log X)^A = T.$$

For the lower condition on S , the bounds $S \geq Z'_1 \geq Z_1$, the hypothesis $Z_1 \geq \mu^4/2$ of Lemma 2, (2.1), $\log(TT_0) \leq 4 \log X$ and $4^{4/5} < 3.1$ give

$$S \geq \frac{\mu^4}{2} > 2 + \exp((\log(TT_0))^{4/5}).$$

The function $S \exp(-\log S / (2(\log(TT_0))^{7/10}))$ increases with S , and $Z'_2 \leq X/(UV)$. Since $\log Z'_2 \geq \log(\mu^4/2)$, $\log(TT_0) \leq 4 \log X$ and $4^{7/10} < 8/3$, once X is large

$$\frac{\log Z'_2}{2(\log(TT_0))^{7/10}} \geq \frac{4(\log X)^{4/5} - \log 2}{2 \cdot 4^{7/10}(\log X)^{7/10}} \geq \frac{3}{4}(\log X)^{1/10}.$$

So Lemma 22 gives, uniformly in $|t| \leq T_0$,

$$\max_{m \leq T} |P_m(c+it)| \ll \frac{X}{UV} \exp\left(-\frac{3}{4}(\log X)^{1/10}\right).$$

Collecting the bounds in (A.3), we obtain

$$\sum_{m \leq T} |E_m| \ll X(\log X)^{A+C+1} \exp\left(-\frac{3}{4}(\log X)^{1/10}\right) + X^{\sigma+o(1)} \ll X(\log X)^{-A}.$$

□

B Proofs of the sieve lemmas

This appendix proves Lemmas 3, 4 and 5 of Section 4 and Lemma 7 of Section 5. In the proofs of Lemmas 4 and 5, Lemma 3 treats the sifting by the primes of norm below μ^4 , and Lemma 2 is applied only to primes of norm at least $\mu^4/2$. As in Section 5, short intervals remove the conditions that couple the variables.

Proof of Lemma 3. The proof is Brun's pure sieve: the inclusion–exclusion over the primes of norm below $z_0(u)$ that divide the cofactor is truncated after the divisors with M prime factors, and the divisors with $M + 1$ prime factors bound the error of the truncation. Here $M := \lfloor (\varepsilon_0/4)(\log X)^{1/5} \rfloor - 1$. Once X is large,

$$\frac{\varepsilon_0}{5}(\log X)^{1/5} \leq M + 1 \leq \frac{\varepsilon_0}{4}(\log X)^{1/5},$$

so that $\mu^{4(M+1)} \leq X^{\varepsilon_0}$ by (2.1) and $M + 1 \geq e^2 \log \log X$. For $z \geq 1$, let $\mathcal{D}^+(z)$ and $\mathcal{D}^-(z)$ be the sets of products of an even and of an odd number, at most M , of distinct primes of norm less than z , with the empty product 1 in $\mathcal{D}^+(z)$, and let $\mathcal{D}^0(z)$ be the set of products of $M + 1$ distinct primes of norm less than z . We first show that $\sum_u c_u S'(\mathcal{E}_u, z_0(u))$ has an asymptotic formula if, for each $\mathcal{D} \in \{\mathcal{D}^+, \mathcal{D}^-, \mathcal{D}^0\}$, the sum of $c_u |\mathcal{E}_{uD}|$ over u and over $D \in \mathcal{D}(z_0(u))$ has one.

For $u \mid n$, let k be the number of distinct primes of norm less than $z_0(u)$ that divide n/u . Exactly $\binom{k}{i}$ products of i distinct primes of norm less than $z_0(u)$ divide n/u . So every $n \in \mathcal{E}_u$ satisfies

$$\begin{aligned} \#\{D \in \mathcal{D}^+(z_0(u)) : D \mid n/u\} - \#\{D \in \mathcal{D}^-(z_0(u)) : D \mid n/u\} &= \sum_{i=0}^M (-1)^i \binom{k}{i}, \\ \#\{D \in \mathcal{D}^0(z_0(u)) : D \mid n/u\} &= \binom{k}{M+1}. \end{aligned}$$

The alternating sum is 1 for $k = 0$, that is, for the n counted by $S'(\mathcal{E}_u, z_0(u))$. For $k \geq 1$,

$$\sum_{i=0}^M (-1)^i \binom{k}{i} = (-1)^M \binom{k-1}{M},$$

and this is at most $\binom{k}{M+1}$ in absolute value, since $\binom{k}{M+1} = \frac{k}{M+1} \binom{k-1}{M}$ and $\binom{k-1}{M} = 0$ for $k \leq M$. Summing over $n \in \mathcal{E}_u$, we obtain

$$S'(\mathcal{E}_u, z_0(u)) = \sum_{D \in \mathcal{D}^+(z_0(u))} |\mathcal{E}_{uD}| - \sum_{D \in \mathcal{D}^-(z_0(u))} |\mathcal{E}_{uD}| + \Delta_u(\mathcal{E}), \quad |\Delta_u(\mathcal{E})| \leq \sum_{D \in \mathcal{D}^0(z_0(u))} |\mathcal{E}_{uD}|. \quad (\text{B.1})$$

Multiply (B.1) by c_u and sum over u . By assumption, the sums of $c_u |\mathcal{A}_{uD}|$ over u and over $D \in \mathcal{D}^\pm(z_0(u))$ are $2/(\pi X^\sigma)$ times those for $\mathcal{B}$, up to $\mathcal{O}(X^\tau (\log X)^{-A})$. So

$$\begin{aligned} \sum_u c_u S'(\mathcal{A}_u, z_0(u)) - \frac{2}{\pi X^\sigma} \sum_u c_u S'(\mathcal{B}_u, z_0(u)) \\ = \sum_u c_u \left(\Delta_u(\mathcal{A}) - \frac{2}{\pi X^\sigma} \Delta_u(\mathcal{B}) \right) + \mathcal{O}(X^\tau (\log X)^{-A}). \end{aligned}$$

The bound for $|\Delta_u(\mathcal{E})|$ in (B.1) and the assumption for $\mathcal{D}^0$ give

$$\begin{aligned} \left| \sum_u c_u \left(\Delta_u(\mathcal{A}) - \frac{2}{\pi X^\sigma} \Delta_u(\mathcal{B}) \right) \right| &\leq \sum_u c_u \sum_{D \in \mathcal{D}^0(z_0(u))} \left(|\mathcal{A}_{uD}| + \frac{2}{\pi X^\sigma} |\mathcal{B}_{uD}| \right) \\ &= \frac{4}{\pi X^\sigma} \sum_u c_u \sum_{D \in \mathcal{D}^0(z_0(u))} |\mathcal{B}_{uD}| + \mathcal{O}(X^\tau (\log X)^{-A}). \end{aligned}$$

Every $D \in \mathcal{D}^0(z_0(u))$ is a product of $M + 1$ distinct primes of norm below μ^4 , and, since $\log \log \mu^4 = (4/5) \log \log X + \log 4$ by (2.1), Lemma 17(b) gives $\sum_{|\varpi|^2 < \mu^4} |\varpi|^{-2} \leq \log \log X$ once X is large, so

$$\sum_{D \in \mathcal{D}^0(z_0(u))} |D|^{-2} \leq \frac{(\log \log X)^{M+1}}{(M+1)!}.$$

By Lemma 18(a) and (b),

$$\sum_u c_u \sum_{D \in \mathcal{D}^0(z_0(u))} |\mathcal{B}_{uD}| \leq CX \sum_u c_u |u|^{-2} \frac{(\log \log X)^{M+1}}{(M+1)!} \ll X(\log X)^C \left(\frac{e \log \log X}{M+1} \right)^{M+1} \leq X(\log X)^C e^{-(M+1)},$$

since $M+1 \geq e^2 \log \log X$. Finally $e^{-(M+1)} \leq \exp(-(\varepsilon_0/5)(\log X)^{1/5})$ is smaller than every power of $1/\log X$. Hence

$$\sum_u c_u S'(\mathcal{A}_u, z_0(u)) = \frac{2}{\pi X^\sigma} \sum_u c_u S'(\mathcal{B}_u, z_0(u)) + \mathcal{O}(X^\tau (\log X)^{-A}).$$

It remains to show that the assumption holds in cases (a) and (b). Every $D \in \mathcal{D}(z_0(u))$ has $|D|^2 < \mu^{4(M+1)} \leq X^{\varepsilon_0}$. In case (a), the coefficient $\sum_{v=uD, D \in \mathcal{D}(z_0(u))} c_u$ of v is non-negative, at most $d(v)^{C_0+1}$, and supported on $|v|^2 \leq X^\tau \mu^{-1}$, so Lemma 1(a) applies. In case (b), z_0 is constant. The products $u' = u_2 D$ with $b_{u_2} \neq 0$ and $D \in \mathcal{D}(z_0)$ have $|u'|^2 < 2U_2 X^{\varepsilon_0}$, and the coefficient $\sum_{u'=u_2 D, D \in \mathcal{D}(z_0)} b_{u_2}$ of u' is at most $d(u')^{C_0+1}$. So u' lies in one of the $\mathcal{O}(\log X)$ dyadic ranges $u' \sim U'$ with U' among $U_2 X^{\varepsilon_0}, U_2 X^{\varepsilon_0}/2, U_2 X^{\varepsilon_0}/4, \dots$. On each of them, the bounds on U_2 and $U_1 U_2$ in (b) give $U' \leq X^\rho \mu^{-1}$ and $U_1 U' \leq X^\nu \mu^{-1}$, and Lemma 1(b) applies with u_1 and u' as its variables u and u' . $\square$

The proofs of Lemmas 4 and 5 confine primes to the short intervals J_λ of (5.1), with the indices λ_q defined after it. The following estimates bound the terms in which the norm of a prime lies in the short interval of a level or of another prime. For an index λ put

$$w_\lambda := \sum_{|\varpi|^2 \in J_\lambda} |\varpi|^{-2}. \quad (\text{B.2})$$

If $\xi_{\lambda+1} > \mu^4$, then (5.1) gives $\xi_\lambda \geq \mu^4/2 \geq \mu$, so Lemma 17(a) gives

$$w_\lambda \leq \frac{2}{H \log(\mu^4/2)}. \quad (\text{B.3})$$

In sums over λ , the index runs over the λ with $J_\lambda \cap [1, 2X] \neq \emptyset$. By Lemma 17(b), $\sum_\lambda w_\lambda \leq 2 \log \log X$, so (B.3) gives

$$\sum_{\lambda: \xi_{\lambda+1} > \mu^4} w_\lambda^2 \leq \frac{4 \log \log X}{H \log(\mu^4/2)}. \quad (\text{B.4})$$

Proof of Lemma 4. Buchstab's identity (2.7) lowers the level $l(n)$ to μ^4 , where Lemma 3 applies. Fix A , let C be such that $\sum_{|n|^2 \leq X} a_n |n|^{-2} \leq (\log X)^C$ (Lemma 18(b)), and take $B = A + C + 3$. Split the support of (a_n) into $\mathcal{N}_1 := \{n : l(n) \leq \mu^4\}$ and $\mathcal{N}_2 := \{n : l(n) > \mu^4\}$. On $\mathcal{N}_1$, Lemma 3(a) applies with $c_n = a_n \mathbf{1}[n \in \mathcal{N}_1]$, the level $z_0 = l$ and $\varepsilon_0 = \varepsilon/2$, since $X^{\tau-\varepsilon} \leq X^{\tau-\varepsilon/2} \mu^{-1}$. On $\mathcal{N}_2$, Buchstab's identity (2.7) gives

$$S'(\mathcal{E}_n, l(n)) = S'(\mathcal{E}_n, \mu^4) - \sum_{\mu^4 \leq |q|^2 < l(n)} S'(\mathcal{E}_{nq}, q), \quad (\text{B.5})$$

and the term $S'(\mathcal{E}_n, \mu^4)$ of (B.5), summed with a_n , has an asymptotic formula by Lemma 3(a) with the constant level μ^4 and $\varepsilon_0 = \varepsilon/2$. It remains to treat

$$\Sigma_0(\mathcal{E}) := \sum_{n \in \mathcal{N}_2} a_n \sum_{\mu^4 \leq |q|^2 < l(n)} S'(\mathcal{E}_{nq}, q). \quad (\text{B.6})$$

Two conditions couple n, q and the cofactor: $|q|^2 < l(n)$, and the condition that every prime factor of the cofactor is $\succcurlyeq q$. For $n \in \mathcal{N}_2$ let $\lambda(n)$ be the index with $l(n) \in J_{\lambda(n)}$. Then

$$\mathbf{1}[|q|^2 < l(n)] = \mathbf{1}[\lambda_q < \lambda(n)] + \mathbf{1}[\lambda_q = \lambda(n)] \mathbf{1}[|q|^2 < l(n)].$$

Moreover, a prime factor $\varpi' \prec q$ of the cofactor with norm at least ξ_{λ_q} has its norm in J_{λ_q} , so

$$0 \leq S'(\mathcal{E}_{nq}, \xi_{\lambda_q}) - S'(\mathcal{E}_{nq}, q) \leq \sum_{|\varpi'|^2 \in J_{\lambda_q}} |\mathcal{E}_{nq\varpi'}|.$$

By the splitting of $\mathbf{1}[|q|^2 < l(n)]$, $\Sigma_0(\mathcal{E})$ consists of its terms with $\lambda_q < \lambda(n)$ and its terms with $\lambda_q = \lambda(n)$. In each term with $\lambda_q < \lambda(n)$, the bound for $S'(\mathcal{E}_{nq}, \xi_{\lambda_q}) - S'(\mathcal{E}_{nq}, q)$ replaces $S'(\mathcal{E}_{nq}, q)$ by $S'(\mathcal{E}_{nq}, \xi_{\lambda_q})$ with an error of at most $a_n \sum_{|\varpi'|^2 \in J_{\lambda_q}} |\mathcal{E}_{nq\varpi'}|$. Each term with $\lambda_q = \lambda(n)$ lies between 0 and $a_n |\mathcal{E}_{nq}|$, since $S'(\mathcal{E}_{nq}, q) \leq |\mathcal{E}_{nq}|$. Grouped by $\lambda = \lambda_q$, these bounds give

$$\left| \Sigma_0(\mathcal{E}) - \sum_{\lambda} M_{\lambda}(\mathcal{E}) \right| \leq \sum_{\lambda} (R_{\lambda}(\mathcal{E}) + R'_{\lambda}(\mathcal{E})), \quad (\text{B.7})$$

where

$$\begin{aligned} M_{\lambda}(\mathcal{E}) &:= \sum_{\lambda(n) > \lambda} a_n \sum_{|q|^2 \in J_{\lambda}, |q|^2 \geq \mu^4} S'(\mathcal{E}_{nq}, \xi_{\lambda}), \\ R_{\lambda}(\mathcal{E}) &:= \sum_{\lambda(n) = \lambda} a_n \sum_{|q|^2 \in J_{\lambda}, |q|^2 \geq \mu^4} |\mathcal{E}_{nq}|, \\ R'_{\lambda}(\mathcal{E}) &:= \sum_{\lambda(n) \geq \lambda} a_n \sum_{|q|^2 \in J_{\lambda}, |q|^2 \geq \mu^4} \sum_{|\varpi'|^2 \in J_{\lambda}} |\mathcal{E}_{nq\varpi'}|. \end{aligned}$$

Each of these sums has the form

$$\sum_{n,v} a_n b_v \# \{q : |q|^2 \in J_{\lambda}, |q|^2 \geq \mu^4, nvq \in \mathcal{E}\},$$

with n restricted to $\lambda(n) > \lambda$, $\lambda(n) = \lambda$ or $\lambda(n) \geq \lambda$, and with v the cofactor of nq . For M_{λ} , the coefficient b_v is the indicator that every prime factor of v has norm at least ξ_{λ} . For R_{λ} it is 1, and for R'_{λ} it is the number of primes of norm in J_{λ} dividing v , which is at most $d(v)$. If $\lambda(n) \geq \lambda$ and $|q|^2 \in J_{\lambda}$, then

$$|q|^2 < \xi_{\lambda+1} \leq (1 + 1/H) l(n) \leq \frac{2C_l X^{\tau}}{|n|^2},$$

so (2.2) and (2.1) give

$$|v|^2 \geq \frac{X/2}{|nq|^2} > \frac{X/2}{2C_l X^{\tau}} = \frac{X^{\sigma}}{4C_l}.$$

In addition, $|n|^2 \geq X^{\sigma}/C_l$ and $|q|^2 \geq \mu^4$. After n and v are split into dyadic ranges $n \sim U$ and $v \sim V$, which have $U \geq X^{\sigma}/(2C_l)$ and $V \geq X^{\sigma}/(8C_l)$, Lemma 2 with n as its variable u , $C_1 = 8C_l$, $Z_1 = \max(\xi_{\lambda}, \mu^4)$ and $Z_2 = \xi_{\lambda+1}$ gives M_{λ} , R_{λ} and R'_{λ} asymptotic formulas, uniformly in λ . Here $Z_1 < Z_2$ for the λ with $\xi_{\lambda+1} > \mu^4$, and for the other λ these sums vanish. At most $2H \log X$ indices and $\mathcal{O}((\log X)^2)$ pairs of dyadic ranges occur, so Lemma 2 is applied $\mathcal{O}(H(\log X)^3)$ times. With $A + B + 5$ in place of A in each application, the errors add up to $\mathcal{O}(X^{\tau}(\log X)^{-A})$.

By (B.7) and the asymptotic formulas for the sums M_{λ} ,

$$\Sigma_0(\mathcal{A}) - \frac{2}{\pi X^{\sigma}} \Sigma_0(\mathcal{B}) \ll \sum_{\lambda} (R_{\lambda} + R'_{\lambda})(\mathcal{A}) + \frac{2}{\pi X^{\sigma}} \sum_{\lambda} (R_{\lambda} + R'_{\lambda})(\mathcal{B}) + X^{\tau}(\log X)^{-A},$$

and the sums over $\mathcal{A}$ equal $\frac{2}{\pi X^{\sigma}}$ times those over $\mathcal{B}$ up to the same error. Lemma 18(a) and (B.2) give

$$\sum_{\lambda} R_{\lambda}(\mathcal{B}) \ll X \sum_n a_n |n|^{-2} w_{\lambda(n)}, \quad \sum_{\lambda} R'_{\lambda}(\mathcal{B}) \ll X \sum_n a_n |n|^{-2} \sum_{\lambda: \xi_{\lambda+1} > \mu^4} w_{\lambda}^2,$$

since $\mu^4 \leq |q|^2 < \xi_{\lambda+1}$ in R'_{λ} . For $n \in \mathcal{N}_2$, $\xi_{\lambda(n)+1} > l(n) > \mu^4$. So, by (B.3) and (B.4),

$$\sum_{\lambda} R_{\lambda}(\mathcal{B}) + \sum_{\lambda} R'_{\lambda}(\mathcal{B}) \ll \frac{X(\log X)^C \log \log X}{H(\log X)^{4/5}} \ll X(\log X)^{-A}.$$

Hence Σ_0 has an asymptotic formula, and so does $\sum_n a_n S'(\mathcal{E}_n, l(n))$. $\square$

Proof of Lemma 5. Buchstab's identity (2.7) is iterated until the product of n and the primes it introduces has norm at least X^σ , and Lemma 4 is applied to that product. Fix A , let C be as in the proof of Lemma 4, and take $B = A + C + 4$. With $\mathcal{N}_1$ and $\mathcal{N}_2$ as in the proof of Lemma 4, Lemma 3(a) applies on $\mathcal{N}_1$ with the level l and $\varepsilon_0 = \theta$, since (2.1) gives

$$|n|^2 < X^\sigma = X^{\tau-2\theta} \leq X^{\tau-\theta} \mu^{-1}.$$

On $\mathcal{N}_2$, (B.5) holds, and the term $S'(\mathcal{E}_n, \mu^4)$ of (B.5) again has an asymptotic formula by Lemma 3(a), with $\varepsilon_0 = \theta$. It remains to treat the sum $\Sigma_0(\mathcal{E})$ of (B.6).

A *chain of length* $i \geq 0$ is a tuple $\mathbf{p} = (n, p_1, \dots, p_i, q)$ with $n \in \mathcal{N}_2$ and primes $p_1 \succ \dots \succ p_i \succ q$ of norm at least μ^4 , such that the largest of these primes (p_1 , or q if $i = 0$) has norm less than $l(n)$, and $|np_1 \dots p_{i'}|^2 < X^\sigma$ for $1 \leq i' \leq i$. Write $N(\mathbf{p}) := np_1 \dots p_i q$ and $q(\mathbf{p}) := q$. Let $\mathcal{K}_i$ be the set of chains of length i , split into the exits $\mathcal{K}_i^{\geq}$, with $|N(\mathbf{p})|^2 \geq X^\sigma$, and the continuations $\mathcal{K}_i^<$, with $|N(\mathbf{p})|^2 < X^\sigma$, and put

$$\Sigma_i(\mathcal{E}) := \sum_{\mathbf{p} \in \mathcal{K}_i} a_n S'(\mathcal{E}_{N(\mathbf{p})}, q(\mathbf{p})) \quad (i \geq 1).$$

The same sum over $\mathcal{K}_0$ is $\Sigma_0(\mathcal{E})$ of (B.6). For $\mathbf{p} \in \mathcal{K}_i^<$, Buchstab's identity (2.7) gives

$$S'(\mathcal{E}_{N(\mathbf{p})}, q) = S'(\mathcal{E}_{N(\mathbf{p})}, \mu^4) - \sum_{q' \prec q, |q'|^2 \geq \mu^4} S'(\mathcal{E}_{N(\mathbf{p})q'}, q').$$

The map $(\mathbf{p}, q') \mapsto (n, p_1, \dots, p_i, q, q')$, on the pairs that occur in these sums, is a bijection onto $\mathcal{K}_{i+1}$. Hence $\Sigma_i = \Sigma_i^{\geq} + F_i - \Sigma_{i+1}$, where

$$\Sigma_i^{\geq}(\mathcal{E}) := \sum_{\mathbf{p} \in \mathcal{K}_i^{\geq}} a_n S'(\mathcal{E}_{N(\mathbf{p})}, q(\mathbf{p})), \quad F_i(\mathcal{E}) := \sum_{\mathbf{p} \in \mathcal{K}_i^<} a_n S'(\mathcal{E}_{N(\mathbf{p})}, \mu^4).$$

A chain of length i has $\mu^{4i} \leq |p_1 \dots p_i|^2 < X^\sigma$, so $\mathcal{K}_i$ is empty for $i > i_{\max} := \lfloor \sigma(\log X)^{1/5}/4 \rfloor$. Hence, for $\mathcal{E} = \mathcal{A}$ and for $\mathcal{E} = \mathcal{B}$,

$$\Sigma_0 = \sum_{i=0}^{i_{\max}} (-1)^i (\Sigma_i^{\geq} + F_i). \quad (\text{B.8})$$

The primes of a chain are distinct and their order is forced, so a chain is determined by its product and the set of its primes. The number of chains with a given product N is therefore at most $d(N)$, and every coefficient defined from chains in this proof is at most $d(N)^{C_0+1}$, uniformly in i . The sum F_i is $\sum_N c_i(N) S'(\mathcal{E}_N, \mu^4)$, where $c_i(N)$ is the sum of a_n over the chains $\mathbf{p} \in \mathcal{K}_i^<$ with $N(\mathbf{p}) = N$. This $c_i(N)$ is supported on $|N|^2 < X^\sigma$, so Lemma 3(a) with $\varepsilon_0 = \theta$ gives F_i an asymptotic formula, uniformly in i .

Let $C_E := 2C_l^{\lceil \sigma/\theta \rceil + 2}$. We claim that every $\mathbf{p} \in \mathcal{K}_i^{\geq}$ satisfies

$$X^\sigma \leq |N(\mathbf{p})|^2 \leq C_E X^{\tau-\theta/2}, \quad |N(\mathbf{p})|^2 |q(\mathbf{p})|^2 \leq C_E X^\tau. \quad (\text{B.9})$$

Every exit has $|np_1 \dots p_i|^2 < X^\sigma \leq |N(\mathbf{p})|^2$, by the definition of a chain for $i \geq 1$ and by $|n|^2 < X^\sigma$ for $i = 0$. Since $1 \leq |n|^2 < X^\sigma$, (2.9) and (2.10) give $0 \leq j(|n|^2) < \sigma/\theta$. By (2.10), $|n|^2 \geq X^{\sigma-(j(|n|^2)+1)\theta}$, so (2.11) and the relation $\tau = \sigma + 2\theta$ of (2.1) give

$$z(|n|^2) \leq \left(\frac{X^\tau}{X^{\sigma-(j(|n|^2)+1)\theta}} \right)^{1/(j(|n|^2)+2)} = X^{(j(|n|^2)+3)\theta/(j(|n|^2)+2)} \leq X^{3\theta/2}.$$

The hypothesis $l(n) \leq C_l z(|n|^2)$ and (2.11) give

$$|n|^2 l(n)^{j(|n|^2)+2} \leq C_l^{j(|n|^2)+2} X^\tau.$$

If $|q|^2 \leq X^\theta$, then

$$|N(\mathbf{p})|^2 |q|^2 = |np_1 \dots p_i|^2 |q|^4 < X^{\sigma+2\theta} = X^\tau.$$

If $|q|^2 > X^\theta$, then every prime of $\mathbf{p}$ has norm greater than X^θ , so

$$|n|^2 X^{i\theta} \leq |np_1 \cdots p_i|^2 < X^\sigma \leq |n|^2 X^{(j(|n|^2)+1)\theta}.$$

This gives $i \leq j(|n|^2)$. Every prime of $\mathbf{p}$ has norm less than $l(n)$, so

$$|N(\mathbf{p})|^2 |q|^2 < |n|^2 l(n)^{i+2} \leq |n|^2 l(n)^{j(|n|^2)+2} \leq C_l^{j(|n|^2)+2} X^\tau.$$

It remains to bound $|N(\mathbf{p})|^2$. Since $|np_1 \cdots p_i|^2 < X^\sigma$ and $|q|^2 < l(n) \leq C_l X^{3\theta/2}$,

$$|N(\mathbf{p})|^2 < X^\sigma |q|^2 < C_l X^{\sigma+3\theta/2} = C_l X^{\tau-\theta/2}.$$

Now confine the prime $q(\mathbf{p})$ to short intervals. For $\mathbf{p} \in \mathcal{K}_i^{\geq}$ with $\lambda_{q(\mathbf{p})} = \lambda$,

$$0 \leq S'(\mathcal{E}_{N(\mathbf{p})}, \xi_\lambda) - S'(\mathcal{E}_{N(\mathbf{p})}, q(\mathbf{p})) \leq \sum_{|\varpi'|^2 \in J_\lambda} |\mathcal{E}_{N(\mathbf{p})\varpi'}|.$$

Hence

$$\left| \Sigma_i^{\geq}(\mathcal{E}) - \sum_\lambda \sum_N c_{i,\lambda}(N) S'(\mathcal{E}_N, \xi_\lambda) \right| \leq R_i(\mathcal{E}), \quad R_i(\mathcal{E}) := \sum_\lambda \sum_N c_{i,\lambda}(N) \sum_{|\varpi'|^2 \in J_\lambda} |\mathcal{E}_{N\varpi'}|, \quad (\text{B.10})$$

where $c_{i,\lambda}(N)$ is the sum of a_n over the chains $\mathbf{p} \in \mathcal{K}_i^{\geq}$ with $N(\mathbf{p}) = N$ and $\lambda_{q(\mathbf{p})} = \lambda$. By (B.9), every N in the support of $c_{i,\lambda}$ and every chain $\mathbf{p}$ counted in $c_{i,\lambda}(N)$ satisfy

$$X^\sigma \leq |N|^2 \leq C_E X^{\tau-\theta/2} \leq X^{\tau-\theta/4}, \quad \xi_\lambda \leq |q(\mathbf{p})|^2 \leq C_E X^\tau / |N|^2.$$

So Lemma 4, with the constant level ξ_λ , with $\theta/4$ in place of ε and with C_E in place of C_l , gives each sum $\sum_N c_{i,\lambda}(N) S'(\mathcal{E}_N, \xi_\lambda)$ in (B.10) an asymptotic formula.

For each λ , the sum over N and ϖ' in R_i is a sum (4.1) with N as its variable u , ϖ' as its prime ϖ and the cofactor v of $N\varpi'$ as its variable v . The norm of ϖ' lies in J_λ , and $\xi_\lambda \geq \mu^4/2$ since $\xi_{\lambda+1} > |q(\mathbf{p})|^2 \geq \mu^4$. Since $\xi_{\lambda+1} = (1 + 1/H) \xi_\lambda$ by (5.1), $\xi_\lambda |N|^2 \leq C_E X^\tau$ and $\sigma + \tau = 1$ by (2.1), the cofactor has

$$|v|^2 > \frac{X/2}{|N|^2 \xi_{\lambda+1}} \geq \frac{X/2}{(1 + 1/H) C_E X^\tau} \geq \frac{X^\sigma}{3C_E}.$$

After N and v are split into dyadic ranges, Lemma 2 with $C_1 = 6C_E$, $Z_1 = \xi_\lambda$ and $Z_2 = \xi_{\lambda+1}$ gives this sum an asymptotic formula, uniformly in λ .

The stage i consists of the sums $\Sigma_i^{\geq}$ and F_i . Each stage has at most $2H \log X$ indices λ , at most $\log X$ dyadic ranges of N and at most $\log X$ dyadic ranges of v , since $X^\sigma \leq |N|^2 \leq X^{\tau-\theta/4}$ and $X^\sigma/(3C_E) \leq |v|^2 < X^\tau$. So (B.8) and (B.10) express Σ_0 through at most $2H(\log X)^2$ applications of Lemma 4, at most $2H(\log X)^3$ of Lemma 2 and 1 of Lemma 3 per stage, over at most $(\log X)^{1/5}$ stages. These are $\mathcal{O}(H(\log X)^{16/5})$ sums in all. With $A + B + 5$ in place of A in each, their errors add up to $\mathcal{O}(X^\tau (\log X)^{-A})$.

It remains to bound the remainders R_i . By Lemma 18(a) and (B.2),

$$\sum_{i=0}^{i_{\max}} R_i(\mathcal{B}) \ll X \sum_n a_n |n|^{-2} \sum_{i \geq 0} \sum_{p_1 \succ \cdots \succ p_i} \prod_{h=1}^i |p_h|^{-2} \sum_{\lambda: \xi_{\lambda+1} > \mu^4} w_\lambda^2,$$

where the factor w_λ^2 comes from the primes q and ϖ' , whose norms both lie in J_λ , and $|q|^2 \geq \mu^4$ gives $\xi_{\lambda+1} > \mu^4$. The sum of $\prod_{h=1}^i |p_h|^{-2}$ over the tuples $p_1 \succ \cdots \succ p_i$ of primes of norm at most X is at most $(2 \log \log X)^i / i!$ by Lemma 17(b), so the sum over i is at most $(\log X)^2$. With (B.4),

$$\sum_{i=0}^{i_{\max}} R_i(\mathcal{B}) \ll \frac{X(\log X)^{C+2} \log \log X}{H(\log X)^{4/5}} \ll X(\log X)^{-A},$$

and the asymptotic formulas for the sums R_i give

$$\sum_{i=0}^{i_{\max}} R_i(\mathcal{A}) \ll X^\tau (\log X)^{-A}.$$

Summing (B.8) term by term gives

$$\Sigma_0(\mathcal{A}) - \frac{2}{\pi X^\sigma} \Sigma_0(\mathcal{B}) = \mathcal{O}(X^\tau (\log X)^{-A}).$$

Hence Σ_0 has an asymptotic formula, and so does $\sum_n a_n S'(\mathcal{E}_n, l(n))$. $\square$

Proof of Lemma 7. The proof compares $\Sigma_{\mathcal{C}}$, tuple by tuple, with the sum of the pieces $\mathcal{P}_{\lambda}(\mathcal{E}; y_{\lambda})$ of (5.5) over the index vectors that satisfy all constraints. Fix A and take $B = A + 1$. For a tuple ϖ with $|\varpi_i|^2 \in \mathcal{I}_i$ for $1 \leq i \leq k$, let $\lambda(\varpi) := (\lambda_{\varpi_1}, \dots, \lambda_{\varpi_k})$ be its index vector, so that $\varpi \in \mathcal{T}(\lambda(\varpi))$. Let $g_c(\lambda) := 1$ if λ satisfies a constraint c and $h_c(\lambda) := 1$ if λ straddles c , and let both be 0 otherwise. An index vector that neither satisfies nor straddles c has no tuple satisfying c , so

$$\mathbf{1}_c(\varpi) = g_c(\lambda(\varpi)) + \mathbf{1}_c(\varpi) h_c(\lambda(\varpi)), \quad g_c h_c = 0.$$

Let $\Xi_{\lambda}(t)$ be the indicator that every prime factor of t has norm at least the level y_{λ} of (5.6). For $\lambda = \lambda(\varpi)$ we have $\Xi(\varpi, t) \leq \Xi_{\lambda}(t)$, and, by the paragraph after (5.8), the difference is at most the number of prime factors of t with norm in $J_{\lambda'}$ for some λ' in the window $\mathcal{W}(\lambda)$ of (5.7). All these indicators take values in $\{0, 1\}$. If $\prod_c g_c(\lambda) = 1$, then $h_c(\lambda) = 0$ and $\mathbf{1}_c(\varpi) = 1$ for every c . Otherwise $g_c(\lambda) = 0$ for some c , and $\prod_c \mathbf{1}_c(\varpi) = 1$ forces $h_c(\lambda) = 1$ for this c , by $\mathbf{1}_c = g_c + \mathbf{1}_c h_c$. So, for $\lambda = \lambda(\varpi)$,

$$\left| \prod_c \mathbf{1}_c(\varpi) \Xi(\varpi, t) - \prod_c g_c(\lambda) \Xi_{\lambda}(t) \right| \leq \sum_c h_c(\lambda) + \prod_c g_c(\lambda) (\Xi_{\lambda}(t) - \Xi(\varpi, t)). \quad (\text{B.11})$$

Let $\mathcal{L}_0$ be the set of index vectors that satisfy all constraints, and $\mathcal{L}_c$ the set of those that straddle c , so that $\prod_c g_c$ is the indicator of $\mathcal{L}_0$ and h_c that of $\mathcal{L}_c$. Multiply (B.11) by the indicator that $\varpi_1 \cdots \varpi_k t \in \mathcal{E}$ and $|\varpi_i|^2 \in \mathcal{I}_i$ for $1 \leq i \leq k$, and sum over the tuples ϖ and over t . By (5.3) and (5.5), this gives

$$\left| \Sigma_{\mathcal{C}}(\mathcal{E}) - \sum_{\lambda \in \mathcal{L}_0} \mathcal{P}_{\lambda}(\mathcal{E}; y_{\lambda}) \right| \leq E^+(\mathcal{E}), \quad (\text{B.12})$$

where

$$E^+(\mathcal{E}) := \sum_c \sum_{\lambda \in \mathcal{L}_c} \mathcal{P}_{\lambda}(\mathcal{E}; 1) + \sum_{\lambda \in \mathcal{L}_0} \sum_{\lambda' \in \mathcal{W}(\lambda)} \mathcal{P}'_{\lambda, \lambda'}(\mathcal{E}).$$

Since $B = A + 1$, at most $(2H \log X)^k = 2^k (\log X)^{k(A+2)}$ index vectors have $\mathcal{T}(\lambda) \neq \emptyset$ by (5.1), and each window (5.7) has at most $2k + 2$ elements. Then (B.12) for $\mathcal{A}$ and for $\mathcal{B}$, and the hypotheses with $A + (k + 1)(A + 3)$ in place of A , give

$$\begin{aligned} \Sigma_{\mathcal{C}}(\mathcal{A}) - \frac{2}{\pi X^\sigma} \Sigma_{\mathcal{C}}(\mathcal{B}) &= \mathcal{O}\left(E^+(\mathcal{A}) + \frac{2}{\pi X^\sigma} E^+(\mathcal{B})\right) + \mathcal{O}(X^\tau (\log X)^{-A}), \\ E^+(\mathcal{A}) &= \frac{2}{\pi X^\sigma} E^+(\mathcal{B}) + \mathcal{O}(X^\tau (\log X)^{-A}). \end{aligned}$$

Since $H = (\log X)^{A+1}$ and $(\log \log X)^k \leq (\log X)^{4/5}$ once X is large, it suffices to show

$$E^+(\mathcal{B}) \ll \frac{X(\log \log X)^k}{H(\log X)^{4/5}}. \quad (\text{B.13})$$

By Lemma 18(a), a tuple ϖ contributes at most $CX/|\varpi_1 \cdots \varpi_k|^2$ to $\mathcal{P}_{\lambda}(\mathcal{B}; 1)$, and a tuple (ϖ, ϖ') at most $CX/|\varpi_1 \cdots \varpi_k \varpi'|^2$ to $\mathcal{P}'_{\lambda, \lambda'}(\mathcal{B})$. If λ straddles c , then $\mathcal{T}(\lambda)$ contains tuples with $m_c(\varpi) \leq Z_c$ and tuples with $m_c(\varpi) \geq Z_c$. Since each $|\varpi_i|^2$ varies by at most the factor $1 + 1/H$ on $\mathcal{T}(\lambda)$ and $|e_{c,i}| \leq 2$,

$m_c(\varpi)$ varies there by at most the factor $(1 + 1/H)^{2k}$. So every tuple of $\mathcal{T}(\lambda)$ has $m_c(\varpi)$ within the factor $(1 + 1/H)^{2k}$ of Z_c . The sets $\mathcal{T}(\lambda)$ are disjoint, so the sum of $\mathcal{P}_\lambda(\mathcal{B}; 1)$ over $\lambda \in \mathcal{L}_c$ runs over distinct tuples, each with $m_c(\varpi)$ within the factor $(1 + 1/H)^{2k}$ of Z_c . Choose i_1 with $e_{c,i_1} \neq 0$. For fixed ϖ_i , $i \neq i_1$, the factor $(1 + 1/H)^{2k}$ on each side of Z_c and $|e_{c,i_1}| \geq 1$ confine the norm $|\varpi_{i_1}|^2$ to an interval $[Y, Y(1 + 1/H)^{4k}]$, with $Y \geq \mu^4$ since $\mathcal{I}_{i_1} \subset [\mu^4, X]$, and Lemma 17(a) bounds the sum of $|\varpi_{i_1}|^{-2}$ over it by $8k/(H \log \mu^4)$. Each other variable contributes at most $\sum_{|\varpi|^2 \leq X} |\varpi|^{-2} \leq 2 \log \log X$, by Lemma 17(b). Since $\log \mu^4 = 4(\log X)^{4/5}$ by (2.1), the sums over $\mathcal{L}_c$ in $E^+(\mathcal{B})$ satisfy

$$\sum_c \sum_{\lambda \in \mathcal{L}_c} \mathcal{P}_\lambda(\mathcal{B}; 1) \leq CX \sum_c \frac{8k(2 \log \log X)^{k-1}}{H \log \mu^4} \ll \frac{X(\log \log X)^k}{H(\log X)^{4/5}}.$$

In the sum over $\mathcal{L}_0$, for fixed λ , the norm of ϖ' runs over an interval $[Y', Y'(1 + 1/H)^{2k+2}]$ with $Y' \geq \mu^4/2$ by (5.8), and Lemma 17(a) bounds the sum of $|\varpi'|^{-2}$ over it by $(4k+4)/(H \log(\mu^4/2))$. The remaining sums over $\mathcal{T}(\lambda)$ add up, over all λ , to at most $(2 \log \log X)^k$. So

$$\sum_{\lambda \in \mathcal{L}_0} \sum_{\lambda' \in \mathcal{W}(\lambda)} \mathcal{P}'_{\lambda, \lambda'}(\mathcal{B}) \leq CX \frac{4k+4}{H \log(\mu^4/2)} (2 \log \log X)^k \ll \frac{X(\log \log X)^k}{H(\log X)^{4/5}},$$

and with the bound for the sums over $\mathcal{L}_c$ this gives (B.13). $\square$

C Discarded sums as integrals

This appendix proves Lemma 15, which the proof of Lemma 16 uses to bound the discarded sums over $\mathcal{B}$.

Proof of Lemma 15. We first count tuples $(\varpi_0, \dots, \varpi_{K'})$ with a *free prime* ϖ_0 , restricted only by $\varpi_0 \cdots \varpi_{K'} \in \mathcal{B}$. Let $K' \geq 1$, and let $\mathcal{R}' \subset \mathbb{R}^{K'}$ be bounded, with boundary of measure 0, and with $y_i \geq c_1$ and $\sum_{i=1}^{K'} y_i \leq 1 - c_1$ on $\mathcal{R}'$. We claim

$$\#\{(\varpi_0, \dots, \varpi_{K'}) : \mathbf{y}(\varpi_1, \dots, \varpi_{K'}) \in \mathcal{R}', \varpi_0 \cdots \varpi_{K'} \in \mathcal{B}\} = \frac{X}{2 \log X} \left(\int_{\mathcal{R}'} \frac{d\mathbf{y}}{y_1 \cdots y_{K'} (1 - \sum_{i=1}^{K'} y_i)} + o(1) \right). \quad (\text{C.1})$$

For fixed $\varpi_1, \dots, \varpi_{K'}$ with $\prod_{i=1}^{K'} |\varpi_i|^2 \leq X^{1-c_1}$, Lemma 17(d) gives the number of ϖ_0 as

$$\pi_G \left(\frac{X}{\prod_{i=1}^{K'} |\varpi_i|^2} \right) - \pi_G \left(\frac{X}{2 \prod_{i=1}^{K'} |\varpi_i|^2} \right) = (1 + o(1)) \frac{X}{2 \prod_{i=1}^{K'} |\varpi_i|^2 \log(X / \prod_{i=1}^{K'} |\varpi_i|^2)}.$$

So the number of tuples in (C.1) is

$$(1 + o(1)) \frac{X}{2 \log X} \sum_{\varpi_1, \dots, \varpi_{K'}} F(\mathbf{y}(\varpi_1, \dots, \varpi_{K'})) \prod_{i=1}^{K'} |\varpi_i|^{-2}, \quad F(\mathbf{y}) := \frac{1[\mathbf{y} \in \mathcal{R}']}{1 - \sum_{i=1}^{K'} y_i}.$$

Let m_X be the measure on $[c_1, 1]$ that gives mass $|\varpi|^{-2}$ to the point $\log |\varpi|^2 / \log X$ for each prime ϖ with $X^{c_1} \leq |\varpi|^2 \leq X$, so that

$$\sum_{\varpi_1, \dots, \varpi_{K'}} F(\mathbf{y}(\varpi_1, \dots, \varpi_{K'})) \prod_{i=1}^{K'} |\varpi_i|^{-2} = \int F dm_X^{\otimes K'}.$$

By Lemma 17(c) with $a_0 = c_1$, the $m_X^{\otimes K'}$ -measure of every product of intervals $\prod_{i=1}^{K'} [a_i, b_i)$ with $c_1 \leq a_i < b_i \leq 1$ tends to $\prod_{i=1}^{K'} \log(b_i/a_i)$, its mass under $\prod_{i=1}^{K'} dy_i/y_i$. The function F is bounded and continuous outside the null set $\partial \mathcal{R}'$, hence Riemann integrable: there are step functions $F_- \leq F \leq F_+$ on finite grids

of such products with $\int (F_+ - F_-) \prod_{i=1}^{K'} dy_i/y_i$ arbitrarily small. The integrals of F_- and F_+ against $m_X^{\otimes K'}$ tend to their integrals against $\prod_{i=1}^{K'} dy_i/y_i$, and

$$\int F_- dm_X^{\otimes K'} \leq \int F dm_X^{\otimes K'} \leq \int F_+ dm_X^{\otimes K'}.$$

So $\int F dm_X^{\otimes K'}$ tends to $\int F \prod_{i=1}^{K'} dy_i/y_i$, the integral in (C.1), and (C.1) follows.

Now consider the sum in (9.2). For each tuple ϖ in its range, $S(\mathcal{B}_{\varpi_1 \dots \varpi_K}, \varpi_K)$ counts the cofactors $t = \varpi'_1 \dots \varpi'_h$ with $\varpi_K \preceq \varpi'_1 \preceq \dots \preceq \varpi'_h$ and $\varpi_1 \dots \varpi_K t \in \mathcal{B}$. Here $h \leq 1/c_1$, and $t \neq 1$ because

$$\prod_{i=1}^K |\varpi_i|^2 \leq X^{1-c_1} < X/2.$$

Every element of $\mathcal{B}$ is the product of $\mathcal{O}(1)$ tuples $(\varpi, \varpi'_1, \dots, \varpi'_h)$, since all their primes have norm at least X^{c_1} . So the number of tuples in which a prime p is repeated is at most a constant times the number of elements of $\mathcal{B}$ divisible by p^2 for some prime p with $|p|^2 \geq X^{c_1}$, and, by Lemma 18(a),

$$\sum_{|p|^2 \geq X^{c_1}} |\mathcal{B}_{p^2}| \leq \sum_{|p|^2 \geq X^{c_1}} \frac{CX}{|p|^4} \ll X^{1-c_1}.$$

A prime distinct from a prime p and of the same norm is the complex conjugate of p , and their product has norm $|p|^4$, so the same bound holds for the tuples that contain distinct primes of equal norm. The remaining tuples have distinct norms. For them, $y_K \leq y_i$ on $\mathcal{R}$ gives $|\varpi_K|^2 < |\varpi_i|^2$, so the condition $\varpi_K \prec \varpi_i$ ($i < K$) of (9.2) holds. Their number is $1/h!$ times the number of such tuples in which $\varpi'_1, \dots, \varpi'_h$ are each $\succ \varpi_K$ and in any order.

Apply (C.1) with $K' = K + h - 1$ and the free prime $\varpi_0 = \varpi'_h$ to the region

$$\mathcal{R}'_h := \left\{ (y, x_1, \dots, x_{h-1}) : y \in \mathcal{R}, x_i \geq y_K \ (1 \leq i \leq h-1), 1 - \sum_{i=1}^K y_i - \sum_{i=1}^{h-1} x_i \geq y_K \right\},$$

where $x_i := \log |\varpi'_i|^2 / \log X$ is the exponent of ϖ'_i for $1 \leq i \leq h-1$, and put

$$x_h := 1 - \sum_{i=1}^K y_i - \sum_{i=1}^{h-1} x_i.$$

Since $|\varpi_1 \dots \varpi_K t|^2 \in [X/2, X)$, the condition $x_h \geq y_K$ of $\mathcal{R}'_h$ is the condition $\varpi'_h \succ \varpi_K$ in exponents, up to $\mathcal{O}(1/\log X)$. So the number of the tuples with $\varpi'_1, \dots, \varpi'_h$ in any order lies between the values of (C.1) for the regions obtained from $\mathcal{R}'_h$ by moving the boundary of this condition by a fixed amount below $c_1/2$ in either direction. These regions satisfy the hypotheses of (C.1) with $c_1/2$ in place of c_1 . The integrals over these regions tend to the integral over $\mathcal{R}'_h$ as the amount tends to 0.

The integrand of (C.1) on $\mathcal{R}'_h$ is $(y_1 \dots y_K)^{-1} \prod_{i=1}^h 1/x_i$, and $\mathcal{R}'_h$ imposes $x_i \geq y_K$ for $1 \leq i \leq h$. Both are symmetric in $x_1, \dots, x_h$, and so is the measure $dx_1 \dots dx_{h-1}$, since exchanging x_h and x_i for $i < h$ changes $(x_1, \dots, x_{h-1})$ by an affine map of determinant -1 . So $1/h!$ of the integral over $\mathcal{R}'_h$ is the integral over the subset with $x_1 \leq \dots \leq x_h$. Hence the sum in (9.2) is

$$\frac{X}{2 \log X} \left(\int_{\mathcal{R}} g \left(1 - \sum_{i=1}^K y_i; y_K \right) \frac{d\mathbf{y}}{y_1 \dots y_K} + o(1) \right),$$

where

$$g(w; b) := \sum_{h \geq 1} \int_{b \leq x_1 \leq \dots \leq x_{h-1} \leq w - \sum_{i < h} x_i} \prod_{i < h} \frac{dx_i}{x_i} \cdot \left(w - \sum_{i < h} x_i \right)^{-1}.$$

It remains to show that $g(w; b) = \omega(w/b)/b$, since then the integrand $g(1 - \sum_{i=1}^K y_i; y_K)/(y_1 \dots y_K)$ is that of (9.2). The term $h = 1$ of $g(w; b)$ is $1/w$ for $w \geq b$. For $h \geq 2$, the smallest coordinate x_1 lies in

$[b, w/2]$, and the other coordinates $x_2 \leq \dots \leq x_h$ are at least x_1 and sum to $w - x_1$, so they give the term $h - 1$ of $g(w - x_1; x_1)$. Hence

$$g(w; b) = \frac{1}{w} + \int_b^{w/2} g(w - x; x) \frac{dx}{x} \quad (w \geq b), \quad g(w; b) = 0 \quad (w < b). \quad (\text{C.2})$$

For $w \geq b$, put $\omega((w - x)/x)/x$ in place of $g(w - x; x)$ in (C.2). The substitution $x = w/(v + 1)$ and then (3.14) at $u = w/b$ give

$$\frac{1}{w} + \int_b^{w/2} \omega\left(\frac{w - x}{x}\right) \frac{dx}{x^2} = \frac{1}{w} \left(1 + \int_1^{w/b-1} \omega(v) dv\right) = \frac{1}{b} \omega\left(\frac{w}{b}\right),$$

so $\omega(w/b)/b$ satisfies (C.2). The functions $g(w; b)$ and $\omega(w/b)/b$ both vanish for $w < b$, and (C.2) determines them by induction on $\lceil w/b \rceil$, since $(w - x)/x \leq w/b - 1$. So $g(w; b) = \omega(w/b)/b$, and (9.2) follows from (9.1). $\square$

D Inequalities used in the decomposition

Table 5 lists the inequalities between the parameters (2.1) that the decomposition uses, directly or through the lemmas of Sections 4–6. Each inequality of a row is equivalent to the positivity of the quantity of that row. The quantities are linear in θ and non-negative at $\theta = 1/9$ and $\theta = 1/8$, hence positive for $1/9 < \theta < 1/8$.

E Comparison with the argument of Harman and Lewis

Table 6 lists the points where this paper adds to, or differs from, the argument of Harman and Lewis [10]. The paragraphs and remarks after the table give the details, in the order of the table.

Remark 24 *Harman and Lewis print, in their (27) and (28), the conditions $ut \in \mathcal{A}$ and $uvt \in \mathcal{A}$, here written in the notation of Lemma 23. The proof of their Lemma 6 selects the norm range by the factor $(X/|u|^2)^s(1 - 2^{-s})$ in a Perron integral of the Hecke L -function $L(s, \chi^m)$ of Harman and Lewis [10, (19)], with no condition on the argument, and the proof of their Lemma 7 proceeds in the same way. So the conditions are $ut \in \mathcal{B}$ and $uvt \in \mathcal{B}$, which is also the form needed in Lemma 20. Their stated bound is $X\eta$, where $\eta = (\log X)^{-A}$ with A arbitrarily large [10, p. 120]. This bound holds for every A , since the proofs of their Lemmas 6 and 7 save a power of μ . The hypothesis $\theta > 1/14$ of their Lemma 7 holds throughout this paper.*

Remark 25 *Harman and Lewis [10, Lemma 5 and (30)] state the bilinear estimate for $U, V \geq X^\sigma$ and $UV \leq X\mu^{-1}$, with the prime ϖ restricted only through the condition $uv\varpi \in \mathcal{E}$. After Perron’s formula, the prime in their proof [10, p. 126] runs over the norms $|\varpi|^2 \leq 4X/|uv|^2$, and the proof applies their Lemma 4, which is Lemma 22, with $|t| \leq X$ and $S \leq 4X/(UV)$. As stated, Lemma 22 then requires*

$$S \geq \exp(((1 + \sigma) \log X + A \log \log X)^{4/5}),$$

and this lower bound is about $\mu^{1.3}$. This requirement is not met when UV is close to $X\mu^{-1}$. Lemma 2 stays within the range of Lemma 22: its prime has norm at least $\mu^4/2$.

Harman and Lewis invoke a fundamental lemma of sieve theory in the proofs of their Lemmas 8, 9, 11 and 12. Lemma 3 states the form used in this paper, at levels up to μ^4 , and Appendix B proves it by Brun’s pure sieve.

Lemmas 8, 9 and 10 of Harman and Lewis [10] give the sums $\sum_n a_n S(\mathcal{E}_n, l(n))$ asymptotic formulas, and Lemmas 4 and 5 and Corollary 6 prove these formulas. The proofs of Harman and Lewis apply the bilinear estimate with a prime of norm down to μ , while Lemma 2 keeps the prime at norm at least $\mu^4/2$. For the prime q of the first Buchstab step, they remove the condition $|q|^2 < l(n)$ by Perron’s formula [10, p. 128], and the proof of Lemma 4 removes it by short intervals. Their Lemma 9 iterates Buchstab’s identity (2.7)

Quantity	Inequality and use	Value at $\theta = 0.1129$
$\theta - 1/10$	$3\sigma/2 < \tau, \sigma - 3\theta < \theta$; Lemmas 12 and 13	0.0129000
$1/8 - \theta$	$\tau/5 < \sigma - 2\theta$; Lemma 13	0.0121000
$(1 - 8\theta)/3$	$\nu + \rho < 1, 2\rho < \tau$; Lemma 9, case (b) fails in Ranges 4 and 5	0.0322667
$1/6 - \theta$	$2\theta < \sigma, \tau < 2\sigma, 1/3 < \sigma$; Lemmas 2 and 9 to 13, Range 1	0.0537667
$2\theta - 1/6$	$\sigma - \theta < 1/3$; $j(X^\alpha) = 0$ in Range 2	0.0591333
$3\theta/2 - 1/12$	$\sigma < (1/3 + \tau)/2$; $PQ \geq X^\sigma$ in Range 2 and on R	0.0860167
$(3 - 22\theta)/12$	$2\zeta - 1/3 < \nu, \rho < 1/3$; $PQ^2 \leq X^\nu$ in Range 2, $\beta \leq \alpha$ on E_3	0.0430167
$(1 + 6\theta)/12$	$\zeta - 1/3 < \rho$; $Q \leq X^\rho$ in Range 2	0.1397833
$(1 - 2\theta)/6$	$\zeta < \tau$; $PQ \leq X^{\tau-\varepsilon}$ where $\alpha + \beta \leq \zeta$	0.1290333
$(14\theta - 1)/6$	$\sigma < \zeta, \zeta < 2\rho$; $P \leq X^\zeta$ on R, $Q \leq X^\rho$ in Range 3, $\beta \leq \alpha$ on E_3 and E_5	0.0967667
$(5 - 34\theta)/24$	$\zeta - \sigma < \rho/2, \rho < \sigma$; $R \neq \emptyset$, Lemma 9	0.0483917
$(26\theta - 1)/24$	$\sigma + \rho/2 < \tau$; $PQ \leq X^{\tau-\varepsilon}$ on R	0.0806417
$(7 - 22\theta)/24$	$\rho/2 < 1/3$; $\beta < \alpha$ on E_R	0.1881750
$(1 - 7\theta)/3$	$\tau/3 < \sigma - \theta$; Lemma 13	0.0699000
$(46\theta - 5)/12$	$\sigma - \theta < \rho$; $j(X^\alpha) = 0$ in Range 3, $Q \leq X^\rho$ in Range 5	0.0161167
$(58\theta - 5)/24$	$\sigma < (\rho + \tau)/2$; $PQ \geq X^\sigma$ in Range 3	0.0645083
$(10\theta - 1)/12$	$\rho < 8\theta/3$; case (b) fails in Range 3	0.0107500
$(10\theta - 1)/6$	$1/3 < 2\zeta - \tau, \sigma < 2\tau/3$; $\beta_0(\alpha) < \zeta - \alpha$ on D_4 , $PQ \geq X^\sigma$ on E_9	0.0215000
$(1 - 2\theta)/24$	$(\nu + 2\theta)/2 < \zeta$; Lemma 9 in Range 1	0.0322583
$(11 - 82\theta)/36$	$\nu/3 < \sigma - \theta$; $D_8 \neq \emptyset, \beta \leq \alpha$ on E_7	0.0483944
$(1 - 2\theta)/12$	$\tau < \nu$; $\beta_0(\alpha) < (\nu - \alpha)/2$ on D_6 and D_8	0.0645167
$(14\theta - 1)/9$	$\sigma < 2\nu/3$; $\alpha + \beta > \sigma$ on D_8	0.0645111
$(46\theta - 5)/24$	$(\nu + \sigma - \theta)/2 < \zeta$; $PQ \leq X^\zeta$ in Range 5	0.0080583
$5\theta/2 - 1/4$	$\sigma - \theta < \tau/2$; E_{10} lies in Range 5	0.0322500
$5\theta - 1/2$	$\sigma < 4\theta$; $\alpha + \beta < \sigma$ for $\alpha < \sigma - 2\theta$	0.0645000
$17\theta/5 - 3/10$	$\sigma - 2\theta < 2\tau/5$; $ pq ^2 \geq X^{\sigma-2\theta}$ in Range 6	0.0838600
$21\theta/5 - 2/5$	$\sigma - 3\theta < \tau/5$; $j(X^\alpha) \leq 2$ in Range 6	0.0741800

Table 5: Inequalities between the parameters (2.1), written as quantities that must be positive, with the place where each is used and its value at $\theta = 0.1129$.

up to $(\log X)^{1/5}$ times. The proof of Lemma 5 organizes this iteration into stages, with coefficients bounded uniformly in the stage.

Lemmas 8 and 9 of Harman and Lewis [10] concern S and coefficients a_n bounded by a divisor function, while Lemmas 4 and 5 are stated for S' and for $a_n \geq 0$. The forms with S and with complex a_n follow. For a divisor-bounded $a_n \geq 0$ and a function l ,

$$\sum_n a_n S(\mathcal{E}_n, l(n)) = \sum_n a'_n S'(\mathcal{E}_n, l(n)),$$

where $a'_n := a_n$ if every prime factor of n has norm at least $l(n)$, and $a'_n := 0$ otherwise. A divisor-bounded complex sequence is a combination of 4 non-negative divisor-bounded sequences.

Lemma 8 of Harman and Lewis [10] is the case of Lemma 4 with $n \sim Y$ and $l(n) = R|n|^k$, where $R > 0$ is a constant, $-2 \leq k \leq 2$ and $\mu \leq l(n)$.

Lemma 9 of Harman and Lewis [10], for $n \sim Y$ with $X^\theta \ll Y \ll X^\sigma$ and $l(n) = R|n|^k \ll z(Y)$, follows from Lemmas 5 and 4. For the n with $|n|^2 < X^\sigma$, Lemma 5 applies: $Y \leq |n|^2 < 2Y$ gives $j(|n|^2) \leq j(Y)$ by (2.9), and then (2.11) gives

$$z(|n|^2) \geq \left(\frac{X^\tau}{2Y}\right)^{1/(j(Y)+2)} \geq \frac{z(Y)}{2}.$$

For the n with $|n|^2 \geq X^\sigma$, Lemma 4 applies, since $z(Y) \leq X^\tau/Y \leq 2X^\tau/|n|^2$. Lemma 10 of Harman and Lewis [10] is Corollary 6.

In the proofs of their Lemmas 11 and 12, Harman and Lewis remove the conditions that couple the variables before applying their bilinear estimate (30), by “applying Perron’s formula to disentangle the

Item of Harman and Lewis [10]	Difference	Treated in
(27), (28)	conditions $ut \in \mathcal{A}$, $uvt \in \mathcal{A}$ read as $\mathcal{B}$	Remark 24
Lemma 5 and (30)	prime restricted to norms at least $\mu^4/2$, within the range of their Lemma 4 (Lemma 22)	Remark 25
fundamental lemma in Lemmas 8, 9, 11, 12	stated and proved here	Lemma 3
Lemmas 8–10	proofs given in full	Lemmas 4, 5, Corollary 6
Lemmas 11, 12, (30)	conditions coupling variables removed by short intervals	Lemma 7
Lemma 11	lower bound X^τ/P for Q not needed	Lemma 9
Lemma 12	hypothesis $R \geq X^\theta$ not needed	Lemma 10
Lemma 13	a level not of the form $R n ^k$; case (b) added	Lemma 11
part 6 of §5	step stated and proved here	Lemma 12
(37), sum of $S(\mathcal{E}_p, z(p ^2))$	level above the prime for small $ p ^2$	Lemma 13, Remark 26
§5	decomposition extended to $\theta < 3/26$	Section 7
regions of φ_2 and φ_{10} , value of χ	printed limits and value made precise	Remark 27
cuts in $\varphi_3, \varphi_7, \varphi_9$	the cut in φ_7 lies in $\mathbb{G}$; that in φ_3 and φ_9 follows from the limits of δ	Section 8; (7.9) and (3.13)
bounds for $\varphi_2, \varphi_5, \varphi_9$ at $\theta = 0.119$	below the losses	Table 7
(17)	factor $1 + \mathcal{O}(\eta)$ replaced by the term $o(1)$	Lemma 15

Table 6: Points where this paper adds to, or differs from, the argument of Harman and Lewis [10], and the place where each is treated. The column “Item of Harman and Lewis [10]” uses the numbering of lemmas and displays of Harman and Lewis. The symbols η , φ_k and χ are those of Harman and Lewis [10]. In the row of their Lemma 13, $R|n|^k$, with a constant $R > 0$, is the form of the levels in their Lemmas 8 and 9; in the row of their Lemma 12, R is the dyadic size of their prime r . The cuts are the inequalities by which Harman and Lewis remove subregions of the regions of φ_3 , φ_7 and φ_9 .

relationship between r and s ” and by “removing the interdependence between s and t ” [10, p. 130]. Lemma 7 removes them by short intervals.

In their Lemma 11, Harman and Lewis assume

$$X^\sigma \leq P \ll X^\chi, \quad \max(X^\tau/P, PX^{-2\theta}) \ll Q \ll X^\nu/(P\mu).$$

where $\chi = (34\theta + 7)/24$ [10, p. 120], which equals $(\nu + 2\theta)/2$. Since $\chi < \zeta$ by Table 5, their hypotheses imply those of Lemma 9 on boxes at distance at least ε from the boundary of the region that these hypotheses define. Their lower bound X^τ/P for Q is not used.

Lemma 12 of Harman and Lewis [10] assumes $X^\sigma \leq PQ \leq X^\zeta \mu^{-1}$, $PQR \leq X^\nu \mu^{-1}$ and $P \geq Q \geq R \geq X^\theta$, and concerns the sum of $S(\mathcal{A}_{pqr}, X^\tau/(PQ))$ over $p \sim P$, $q \sim Q$ and $r \sim R$. The terms of this sum with $|r|^2 < X^\tau/(PQ)$ vanish, since $S(\mathcal{A}_{pqr}, X^\tau/(PQ))$ then sifts out r . So, for $Q \geq X^\tau/(PQ)$, this sum is the sum (6.3) with $\mathcal{I}_q = [Q, 2Q]$ and $\mathcal{I}_r = [\max(R, X^\tau/(PQ)), 2R]$, without the constraints $q \prec p$ and $r \prec q$. On boxes at distance at least ε from the boundary of the region that the hypotheses of their Lemma 12 define, the proof of case (a) of Lemma 10 gives this sum an asymptotic formula, with $|r|^2 < 2R$ and $|pqr|^2 < 8PQR$ in place of $|r|^2 < 2Q$ and $|pqr|^2 < 8PQ^2$. The bound $Q \leq X^{\rho-\varepsilon}$ of case (a) holds, since $Q^2 \leq PQ \leq X^\zeta$ and $\zeta < 2\rho$ by Table 5. The proof of Lemma 10 applies without the constraints $r \prec q$ and $q \prec p$. The bound $|p|^2 \geq P \geq Q = \inf \mathcal{I}_q$ replaces $|p|^2 \geq |q|^2$ where that proof shows that every prime factor of pqr has norm at least $X^\tau/(PQ)$. Dropping a constraint changes none of its other checks.

Their hypothesis $R \geq X^\theta$ is not used in the proof of Lemma 10. In their Lemma 13, where their Lemma 12 is applied, $Q^2 P \geq X^\tau$ and $PQ \leq X^\zeta \mu^{-1}$, and every r has $|r|^2 \geq X^\tau/(PQ)$. Since $r \sim R$ and

$\tau - \zeta - \theta = (1 - 8\theta)/6$ is positive, the bound $PQ \leq X^\zeta \mu^{-1}$ gives

$$R > \frac{|r|^2}{2} \geq \frac{X^\tau}{2PQ} \geq \frac{X^{\tau-\zeta}\mu}{2} > X^\theta.$$

The decomposition of Section 7 applies Lemma 10, through Lemma 11, on boxes of R with $PQ > X^{1/2}$, where $X^\tau/(PQ) < X^\theta$, so primes r with $|r|^2 < X^\theta$ occur.

Lemma 13 of Harman and Lewis [10] is case (a) of Lemma 11. Since $|q|^2 < 2Q$, the sum $S_{\geq}$ of (6.6) vanishes when $2PQ^2 \leq X^\tau$, and their condition $Q^2P < X^\tau$ is this case up to the factor 2. Case (a) of Lemma 11 also needs $Q \leq X^{\rho-\varepsilon}$, which is not among their hypotheses. Their hypotheses imply it for small ε , as for their Lemma 12. The proof of their Lemma 13 cites their Lemmas 8 and 12. The sum over the pairs with $|q|^2 < X^\tau/(PQ)$ has the prime level q , which is not a level $R|pq|^k$ of their Lemma 8. Lemmas 4 and 7 of this paper allow this level. Case (b) of Lemma 11 has no counterpart in their Lemma 13. Section 3 describes its use on R .

Part 6 of Harman and Lewis [10, §5] uses the case $j(PQ) = 1$ of Lemma 12, which is stated and proved here. For $j(PQ) = 0$, Lemma 12 is Lemma 14 and (36) of Harman and Lewis [10]. Then $z(PQ) = (X^\tau/(PQ))^{1/2}$ by (2.11), and, since $|q|^2 < 2Q$, the sum $S_{\geq}$ of (6.11) vanishes when $4PQ^3 \leq X^\tau$. Their condition $PQ^3 < X^\tau$ is this case up to the factor 4.

Remark 26 Harman and Lewis [10, (37)] sum $S(\mathcal{E}_p, z(|p|^2))$ over all p with $X^\theta \leq |p|^2 < X^{1/2}$. For $p \in \Pi_2$, (3.4) gives $z(|p|^2) > |p|^2$, so $S(\mathcal{E}_p, z(|p|^2))$ sifts p itself and vanishes, while the term $S(\mathcal{E}_p, p)$ of the first Buchstab step remains. Identity (3.5) keeps the sum of $S(\mathcal{E}_p, p)$ over $p \in \Pi_2$, and Lemma 13 gives it an asymptotic formula.

The primes with $\tau/4 \leq \alpha < \sigma - 2\theta$ exist only for $\theta < 3/26$, so they do not occur in the decomposition of Harman and Lewis [10, §5] at $\theta = 0.119$. Section 7 treats them in the paragraph after (7.9).

Remark 27 The regions of Section 7 are derived from the lemmas. Of these regions, those for φ_2 and φ_{10} differ from the regions that Harman and Lewis print. Harman and Lewis [10, p. 132] print the lower limit $\zeta - \alpha$ for β in φ_2 . For $\alpha > (1 + 10\theta)/6$, this limit lies below $\beta_0(\alpha) = (\tau - \alpha)/2$ of (3.7), and the strip between these limits contains no pair of Σ^* . Harman and Lewis print the upper limit $1/2 - 2\theta - \alpha$ for β in φ_{10} , which makes the region empty for every $\alpha \geq \tau/4$. The condition $|pq|^2 < X^\sigma$ of Lemma 12 with $j(PQ) = 0$ gives the upper limit $\sigma - \alpha$. They also print $\chi = 0.46205$ at $\theta = 0.119$ [10, p. 120], while $(34\theta + 7)/24 = 0.46025$.

Harman and Lewis discard the third sums of (6.7) and (6.12), except on a few subregions of the regions of φ_3 , φ_7 and φ_9 that they remove by further inequalities, the cuts of Table 6. They print a grouping of the kind that defines G and L only in their φ_7 [10, p. 133]. There, the condition $\delta > 1 - \alpha - \gamma - \sigma$ when $\alpha + \gamma \geq \sigma$ excludes from the region of φ_7 the subset of G given by $\mathcal{S} = \{p, r\}$. This exclusion changes no loss once G is added, and the evaluation of the decomposition of Harman and Lewis in Section 10 keeps it. Their cut $\gamma \leq \alpha + \beta - 2\theta$ in φ_3 and φ_9 is implied by the limits of δ . On E_3 and E_9 , $\delta \geq \tau - \alpha - \beta$ by (7.9), and the integrand of $\Phi(F')$ in (3.13) vanishes unless $\delta \leq (1 - \alpha - \beta - \gamma)/2$. These bounds are compatible only if

$$\gamma \leq 1 - 2\tau + \alpha + \beta = \alpha + \beta - 2\theta.$$

At $\theta = 0.119$, the bounds that Harman and Lewis print for φ_2 , φ_5 and φ_9 lie below the losses of Table 7. For φ_9 they “omit here a couple of extra conditions on the integral ranges, corresponding to other ways of combining variables to get asymptotic formulae” [10, p. 133], and the loss of E_9 in Table 7 is that of the region they print.

Harman and Lewis [10, (17)] state the case $K = 2$ of Lemma 15 with a factor $1 + \mathcal{O}(\eta)$ in place of the term $o(1)$ of (9.2), where $\eta = (\log X)^{-A}$ as in Remark 24. For $K = 4$, Lemma 15 is the form they use for φ_3 , φ_5 , φ_7 , φ_9 , φ_{10} and φ_{11} . The proof of Lemma 15 counts the free prime by the main term $y/(2 \log y)$ of Lemma 17(d), and

$$\text{Li}(y) - \text{Li}(y/2) = \left(1 + \frac{1 - \log 2}{\log y} + \mathcal{O}((\log y)^{-2})\right) \frac{y}{2 \log y}.$$

So this main term has a relative error of exact order $1/\log y$, and the proof gives the term $o(1)$, not a factor $1 + \mathcal{O}(\eta)$.

Loss	Region	First method	Second method	Bound of Harman and Lewis
φ_1	D_1	0.2747838	0.2747838	0.275
φ_2	$D_2 \cup R$	0.3456458	0.3456458	0.341
φ_3	E_3	0.0000918	0.0000918	0.001
φ_4	D_4	0.1554561	0.1554561	0.156
φ_5	E_5	0.0011859	0.0011859	0.001
φ_6	D_6	0.1116255	0.1116255	0.112
φ_7	E_7	0.0008847	0.0008847	0.001
φ_8	D_8	0.0356394	0.0356394	0.036
φ_9	E_9	0.0010752	0.0010752	0.001
φ_{10}	E_{10}	0.0161420	0.0161419	0.017
φ_{11}	E_{11}	0.0000285	0.0000285	0.001
total		0.9425585	0.9425585	0.943

Table 7: The losses (3.13) of the decomposition of Harman and Lewis at $\theta = 0.119$, computed by the methods of Section 10 with the regions derived from the lemmas (Remark 27) and with the cut of Harman and Lewis in φ_7 , rounded to 7 decimal places. The column “Bound of Harman and Lewis” gives the bounds of Harman and Lewis [10, pp. 132–134].

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