Unified Physics through Exceptional Symmetries: An E_n-Indexed Field and mGFT Framework

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Abstract

This paper develops an E_n-indexed analytic and representation-valued field framework with the defining opposite-lift sector $E_5 = E_6^-$ and constituent content $\mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$ of total dimension 26. The main text covers E_5 through E_{15}. The finite exceptional levels E_6, E_7 and E_8 are distinguished from the affine E_9, hyperbolic E_{10}, very-extended E_{11}, and proposed E_{12}–E_{15} framework extensions. The eight-sector mGFT ansatz uses hypergeometric, exponential and monomial generator factors; its $(0, 1, 1)$ sector recovers the weighted hull $H_{n,n-1}(2\xi)$ when $Q = \xi$ and $Z = 2\xi$. This supplies a candidate profile family, while internal symmetry is carried by explicitly chosen Lie-algebra representations. Integral ft-{ht,lsr}-mGFT is formulated as a generated-core Green–Duhamel correction with stated boundary data, quadrature and convergence conditions. The primary reduction chain is $E_{15} \to \cdots \to E_5$; reverse operations are constructive lifts. Physical equations are discussed as theoretical contexts for the framework. The quantitative results are confined to fully specified mathematical and effective-action benchmarks: local profile and sum identities, an exact canonical scalar wall and its generated-coordinate double lift, constituent threshold arithmetic, prescribed quadratic masses, conditional two-body kinematics, a manufactured nonlinear boundary-value problem, and the E_{10} Cartan spectrum. The supplied atlases distinguish three opposing labels, anti-, mirror and negative matter, relative to ordinary matter. Charge conjugation, copy labels, signed energy and gravitational sign are treated separately; signed constituent sums do not establish physical negative masses or photon rest masses. The separate supplement supplies the inputs, derivations and reproducible numerical checks.

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1 Introduction

Unifying gravitational, gauge, scalar, fermionic, and possible hidden sectors remains a central problem in theoretical physics. This formulation links the author-defined opposite-lift endpoint $E_5 = E_6^-$ with the full exceptional E_n hierarchy. The organizing range is E_5 through E_{15}, including the author-defined opposite-lift endpoint and four proposed extensions beyond the conventional E_{11} stage. The framework uses three complementary layers: an analytic layer for constructing field profiles, a symmetry layer in which matrix coefficients are expanded in verified Lie-algebra generators, and a physical verification layer in which an action, field equations, residual tests, and quadratic fluctuations determine whether a candidate profile represents a physical mode. Analytic series are used for profiles, while the unifying symmetry is specified separately. At each E_n level the physical program is to specify the intended algebra or group, its representations, symmetry breaking, invariant action, field equations, and observables, and then to use analytic or numerical machinery as a means of solving and testing those dynamics. For E_6, E_7, E_8, the standard finite-dimensional exceptional structures provide the conventional starting point. For E_9 and higher labels, the framework must state the relevant affine or indefinite Kac–Moody realization explicitly before physical conclusions are drawn.

The supplied atlas collection [43] adds an explicit three-opposing-matter hypothesis. This paper implements its sector distinctions as model conventions, independently of the E_n index and the eight mGFT generator sectors.

Related algebraic unification work by Nandi et al. [1] discusses Cayley–Dickson structures, exceptional symmetry and monstrous moonshine as ingredients of a proposed unified field framework.

Conventional quantum-field theory supplies the action and fluctuation language [4], while superweakly interacting massive-particle models illustrate a separate hidden-sector mechanism [3]. Neither source fixes the constituent or exceptional-sector assignments used here.

2 Direct E_n Exceptional Framework

2.1 Direct hierarchy

The symmetry hierarchy is indexed directly by E_n. No separate group family is used to generate or relabel the exceptional index. The terminal sector of the primary reduction is $E_5 = E_6^-$. It is listed alongside the exceptional and extended levels without assigning it a conventional rank-five algebra. The indexed sectors are

$$E_5 = E_6^-, \qquad E_n, \quad 6 \le n \le 15. \tag{1}$$

with the mathematical status of the levels distinguished rather than assumed to be identical. The proposed E_{12}–E_{15} extensions are included in both hierarchy tables and discussed in Sec. 9.2. They require algebraic data beyond the independent root directions and scalar lift formulas recorded here.

Table 1: Direct E_n hierarchy used in the framework.

Level	Mathematical status	Role in the framework
$E_5 = E_6^-$	Author-defined opposite-lift sector	Chosen constituent content $\mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$; total dimension 26. Algebra and root lattice require specification. Endpoint of the primary reduction from E_{15}; terminal sector map unspecified.

Level	Mathematical status	Role in the framework
E_6	finite exceptional Lie group/algebra	lowest finite exceptional level used for particle and solar benchmarks
E_7	finite exceptional Lie group/algebra	higher finite exceptional level for compact-object and field sectors
E_8	finite exceptional Lie group/algebra	highest finite exceptional level
E_9	affine $E_8^{(1)}$-type extension	first extended E-type level used for affine/string-field or morphology studies
E_{10}	hyperbolic Kac–Moody extension	hyperbolic extended level
E_{11}	indefinite/Lorentzian Kac–Moody extension	higher extended level used in very-extended E-type constructions
E_{12}	Proposed framework extension	Higher-dimensional gravity and brane precursors. Independent-root bookkeeping; Cartan data and group realization require specification.
E_{13}	Proposed framework extension	Moduli, flux and compactification. Independent-root bookkeeping; Cartan data and group realization require specification.
E_{14}	Proposed framework extension	Interfaces and domain walls. Independent-root bookkeeping; Cartan data and group realization require specification.
E_{15}	Proposed framework extension	Cosmological backgrounds. Independent-root bookkeeping; Cartan data and group realization require specification. Starting level of the primary downward hierarchy.

2.2 Primary downward flow and inverse constructive lifts

The conventional Dynkin-diagram relations admit two complementary readings. Upward, one may pass through the regular finite exceptional embeddings and then add the affine, overextended, and very-extended nodes,

$$E_6 \xrightarrow{\text{extension}} E_7 \to E_8 \xrightarrow{\text{affinization}} E_9 \xrightarrow{\text{overextension}} E_{10} \xrightarrow{\text{very extension}} E_{11}. \tag{2}$$

For the extended part of the chain, $E_9 = E_8^{(1)}$, $E_{10} = E_8^{++}$, and $E_{11} = E_8^{+++}$ in the standard notation. The very-extended node of E_{11} can be deleted to recover its canonical E_{10} subalgebra, while deleting the overextended and affine nodes successively recovers E_9 and E_8 [23,28]. Suitable regular Dynkin subdiagrams similarly realize the familiar finite inclusions $E_6 \subset E_7 \subset E_8$ [2,15]. The full framework takes the downward direction from E_{15} through $E_5 = E_6^-$ as its primary reduction/flow:

$$\begin{aligned} &E_{15} \longrightarrow E_{14} \longrightarrow E_{13} \longrightarrow E_{12} \longrightarrow E_{11} \\ &E_{11} \longrightarrow E_{10} \longrightarrow E_9 \longrightarrow E_8 \longrightarrow E_7 \\ &E_7 \longrightarrow E_6 \longrightarrow E_5 \end{aligned} \tag{3}$$

The conventional E_{11}–E_6 part admits appropriate restriction, projection or node-deletion maps $\Pi_{n\to n-1}$; these are not automatically isomorphisms. The E_{15}–E_{11} part is proposed and requires the algebraic and model data in Sec. 9.2. The terminal $E_6 \to E_5 = E_6^-$ arrow denotes the author-defined opposite-lift sector transition. Its map is unspecified; it is neither assigned a finite-root deletion nor identified with ∂_G. The reverse direction is constructive and requires its own extension or sector map.

A simple analytic component of that lift acts on the generated-mode variable G. For nonnegative integer ℓ define

$$R_G : \; G^\ell \longmapsto \frac{G^{\ell+1}}{\ell+1}. \tag{4}$$

On the image of this map,

$$R_G^{-1} = \partial_G, \qquad \partial_G\left(\frac{G^{\ell+1}}{\ell+1}\right) = G^\ell. \tag{5}$$

The scalar derivative is algebraically defined wherever its argument is differentiable. Its placement as the generating-core anti-integral lift in the indexed framework is restricted to the source level $n \geq 10$. Thus the shorthand $E_9 \xrightarrow{\int dG} E_{10}$ describes a constructive mode lift, whereas $E_{10} \xrightarrow{\partial_G} E_9$ describes the corresponding mode reduction. The full $E_n \to E_{n-1}$ operation must also act on the tensor coefficients and on compactified degrees of freedom.

Let the coefficient tensors at level E_n be

$$a^{(n)}_{A_1\cdots A_r}, \qquad b^{(n)}_{A_1\cdots A_s}, \tag{6}$$

and split a higher-dimensional index as $A = (\mu, m)$, with μ labeling retained noncompact directions and m compactified directions. Introduce a dimensional compactification map $K_{n\to n-1}$ and coefficient contractions $C_a^{(n)}$, $C_b^{(n)}$. The framework defines

$$T_a^{(n\to n-1)} = C_a^{(n)} \circ K_{n\to n-1}, \qquad T_b^{(n\to n-1)} = C_b^{(n)} \circ K_{n\to n-1}. \tag{7}$$

For example, if q and p compact indices are contracted,

$$a^{(n-1)}_{\mu_1\cdots\mu_{r-q}} = C_{m_1\cdots m_q}\, a^{(n)}_{\mu_1\cdots\mu_{r-q}m_1\cdots m_q}, \tag{8}$$

$$b^{(n-1)}_{\mu_1\cdots\mu_{s-p}} = D_{m_1\cdots m_p}\, b^{(n)}_{\mu_1\cdots\mu_{s-p}m_1\cdots m_p}. \tag{9}$$

Writing P_a and P_b for projectors onto the two coefficient sectors gives the downward coefficient map

$$M_{n\to n-1} = T_a^{(n\to n-1)} P_a + T_b^{(n\to n-1)} P_b. \tag{10}$$

For source levels $n \geq 10$, the complete proposed model-level reduction operator is

$$D_{n\to n-1} = R_G^{-1} \circ M_{n\to n-1} = \partial_G \circ \left(T_a^{(n\to n-1)} P_a + T_b^{(n\to n-1)} P_b\right), \quad n \geq 10. \tag{11}$$

When the coefficient and mode operations are defined on the same retained sector, the reverse construction requires more than integration. Let $\mathcal{E}_{a,b}^{(n-1\to n)}$ denote tensor extension and let $K^{\uparrow}_{n-1\to n}$ denote a chosen decompactification/extension map. Then

$$I_{n-1\to n} = \mathcal{E}_{a,b}^{(n-1\to n)} \circ K^{\uparrow}_{n-1\to n} \circ R_G, \quad n \geq 10. \tag{12}$$

Only on an information-preserving sector where compactification is actually invertible may one identify $K^{\uparrow}_{n-1\to n} = K^{-1}_{n\to n-1}$. Ordinary compactification with mode truncation generally loses information, so Eq. (12) is properly a lift, not a guaranteed global inverse.

Below source level ten, D_9, D_8, D_7 denote structural restriction/projection maps. The terminal D_6^{sector} denotes the unspecified opposite-lift sector transition; Eq. (11) is not imposed on any of these arrows.

Likewise I_7, I_8, I_9 are structural extensions, and I_6^{sector} is an unspecified sector construction. Generated-coordinate integration applies only at target levels ten and above. The full dual architecture is

$$\begin{aligned} &E_{15} \xrightarrow{D_{15}} E_{14} \xrightarrow{D_{14}} E_{13} \xrightarrow{D_{13}} E_{12} \xrightarrow{D_{12}} E_{11} \\ &E_{11} \xrightarrow{D_{11}} E_{10} \xrightarrow{D_{10}} E_9 \xrightarrow{D_9} E_8 \xrightarrow{D_8} E_7 \\ &E_7 \xrightarrow{D_7} E_6 \xrightarrow{D_6^{\text{sector}}} E_5 \end{aligned} \tag{13}$$

with constructive lifts

$$\begin{aligned} &E_5 \xrightarrow{I_6^{\text{sector}}} E_6 \xrightarrow{I_7} E_7 \xrightarrow{I_8} E_8 \xrightarrow{I_9} E_9 \\ &E_9 \xrightarrow{I_{10}} E_{10} \xrightarrow{I_{11}} E_{11} \xrightarrow{I_{12}} E_{12} \xrightarrow{I_{13}} E_{13} \\ &E_{13} \xrightarrow{I_{14}} E_{14} \xrightarrow{I_{15}} E_{15} \end{aligned} \tag{14}$$

2.3 Root-lattice bookkeeping and successive mode lifts

With a compatible simple-root numbering, the conventional sequential lattice relations are

$$\begin{aligned} \Gamma_{E_7} &= \Gamma_{E_6} + \mathbb{Z}\alpha_2, \qquad \Gamma_{E_8} = \Gamma_{E_7} + \mathbb{Z}\alpha_1, \\ \Gamma_{E_9} &= \Gamma_{E_8} + \mathbb{Z}\alpha_0, \qquad \Gamma_{E_{10}} = \Gamma_{E_9} + \mathbb{Z}\alpha_{-1}, \\ \Gamma_{E_{11}} &= \Gamma_{E_{10}} + \mathbb{Z}\alpha_{-2}. \end{aligned}$$

At the next framework level,

$$\Gamma_{E_{12}} = \Gamma_{E_{11}} + \mathbb{Z}\alpha_{-3} = \Gamma_{E_6} + \sum_{j=-3}^{2} \mathbb{Z}\alpha_j, \qquad \Gamma_{E_{15}} = \Gamma_{E_6} + \sum_{j=-6}^{2} \mathbb{Z}\alpha_j.$$

The sum adds the root directions independently. The expression $\mathbb{Z}(\sum_j \alpha_j)$ instead denotes a single aggregate cyclic generator and does not give this rank-twelve chain. A custom module formed by adjoining one aggregate generator to Γ_6 must be distinguished from the sequential rank-ten and rank-eleven lattices. The supplement gives the rank counts and both interpretations explicitly.

The defining equality for the opposite-lift sector, used throughout this paper and supplement, is

$$E_5 = E_6^-.$$

The superscript minus names a framework sector. Its proposed constituent module is $\mathcal{V}_5 = \mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$, with $\dim \mathcal{V}_5 = 26$. The notation $\dim(E_5) = \mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$ is used only as component-content shorthand. The module assignment does not specify a Lie algebra, root lattice or root rank. Section 14.1 and supplement Sec. S4 give its conditional constituent benchmark. In particular, neither a $D_5/\operatorname{Spin}(10)$ identification nor $E_5 = R_G^{-1}(E_6)$ follows from this definition. The generating-core anti-integral lift is placed at $n \geq 10$ by the author convention; it is independent of this sector label. A conventional D_5 subalgebra of E_6 can still occur in a separately specified representation branching, without defining E_5.

Root extension, scalar beta lifting, generated-mode integration, and physical compactification are distinct operations. For a scalar coordinate G and zero base point,

$$R_G^m[G^\ell] = \frac{G^{\ell+m}}{(\ell+1)_m}, \qquad (\ell+1)_m = \prod_{r=1}^{m} (\ell + r).$$

The associated relative factor is $G^m/(\ell+1)_m$. Thus an analytic label $n = 9+m$ has lift order m relative to an E_9 seed. The separately defined $R_G^n[U]$ construction has order n; the conventions must not be

conflated. For nonzero base point, boundary polynomials appear. Differentiation reverses integration only after its data are retained: $R_G \partial_G f = f - f(G_0)$, while $\partial_G R_G f = f$.

The conventional $E_{11} = E_8^{+++}$ remains the very-extended stage. E_{12}–E_{15} in the extended indexing convention are framework extension labels whose additional Cartan data and physical interpretation require specification. A repeated scalar integral alone cannot establish a Kac–Moody algebra, real form, spacetime dimension, or cosmological epoch.

2.4 Physical contexts and mathematical scope

The physical contexts in Table 2 organize the equations discussed in this paper. They are proposed applications of the E_n indexing convention, rather than quantitative object reconstructions or empirical symmetry classifications. Both documents cover these equation contexts from E_5 through E_{15}, using the same generator formula and parameter counts $(n, n-1)$. The main text discusses the finite sectors, the affine and hyperbolic constructions, the very-extended E_{11} context, and the four proposed extensions in Sec. 9.2. This order differs from the primary downward reduction chain of Eq. (3).

Table 2: Theoretical equation contexts for the indexed framework.

Sector	Context	Role and interpretation
$E_5 = E_6^-$	Opposite-lift sector E_6^-	Constituent/preon benchmark $\mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$ with conditional masses; algebra and root rank remain unspecified.
E_6	Matter and stellar environments	Finite exceptional representations and visible/hidden-sector branching; Einstein and matter equations provide physical context.
E_7	Compact-object equation contexts	Generated-core examples and Einstein–Maxwell or relativistic equilibrium equations; local profiles A–D are mathematical inputs.
E_8	Strong-gravity and gauge-field contexts	Finite exceptional algebra with a chosen representation; candidate physical interpretation is model dependent.
E_9	Affine gauge-field construction	Affine E_8 with loop, central and derivation data; AGN, jets and magnetospheres are proposed equation contexts.
E_{10}	Nonlinear periodic scalar models	Hyperbolic Cartan structure and sine-Gordon-type equations; the exact scalar kink is a separate verified example.
E_{11}	Very-extended algebraic structure	Very-extended E_8; gravity, form fields and brane/M-theory contexts. Conventional 11D supergravity differs from the framework $d = 11$ convention.
E_{12}	Higher-dimensional gravity and brane precursors	Gravity–scalar–form equations; action, form rank, charges and boundary data must be supplied.
E_{13}	Moduli, flux and compactification	Canonical moduli and a specified effective potential; stability, flux data and compactification dynamics require calculation.

Sector	Context	Role and interpretation
E_{14}	Interfaces and domain walls	Einstein–scalar equations, junction conditions and surface stress; scalar profiles alone do not establish topology change.
E_{15}	Cosmological backgrounds	Friedmann and conservation equations with effective d=15; no epoch, age or pre-Big-Bang history follows from the label.

2.5 Finite and extended levels

For $n = 6, 7, 8$, the symbols E_n refer to the conventional finite-dimensional exceptional structures. For $n \geq 9$, the same notation is used only after the intended extension is specified. In particular, algebraic claims at the higher levels require the generalized Cartan matrix, root lattice, central or derivation generators where applicable, real form, representation category, and global structure relevant to the physical model.

2.6 Direct labeling convention

Every analytic profile, field representation, symmetry-breaking chain, and observable is labeled by the E_n sector itself. The notation pairs each E_n label with its candidate mGFT family. The shorthand $E_n \longleftrightarrow H_{n,n-1}(2\xi)$ denotes the weighted $(0, 1, 1)$ sector of that family, not an identification of a special function with an algebra.

2.7 E_8 and higher-level mathematical context

At the E_8 level, standard maximal subgroup structures include

$$E_8 \supset (E_7 \times SU(2))/\mathbb{Z}_2, \qquad E_8 \supset (E_6 \times SU(3))/\mathbb{Z}_3. \tag{15}$$

and

$$E_8 \supset \mathrm{Spin}(16)/\mathbb{Z}_2, \tag{16}$$

which provide explicit examples of how exceptional sectors can contain lower-rank structures [2]. Such branching data are more informative for physical model building than dimension matching alone.

For E_9 and above, the framework moves into affine, hyperbolic, and more general indefinite E-type structures. Work at these levels must identify the intended generalized Cartan matrix, root lattice, real form, representation category, and any required central or derivation generators. This remains one of the principal mathematical tasks of the theory.

The generating-function background [11] motivates the use of mGFT as an analytic scaffold. The author supplies the exceptional indexing, lift placement and lattice constructions; the scaffold does not determine these definitions.

3 Analytic Profiles and the E-Indexed Hypergeometric Hull

The analytic field profile is kept conceptually distinct from the internal symmetry. At a chosen symmetry level E_n, integral ft-{ht, lsr}-mGFT provides candidate analytic dependence of fields on spacetime or on a similarity variable. Adopting the notation used in the present program, let

$$H_{n,n-1}(2\xi) \tag{17}$$

denote the exponentially weighted generalized-hypergeometric hull associated with level E_n. The abbreviation mGFT refers here to the modified generating-function technique; it does not identify the generator parameter with physical Fourier momentum. The direct analytic assignment is

$$E_n \longleftrightarrow H_{n,n-1}(2\xi). \tag{18}$$

The hull does not define the exceptional group or algebra; it is a profile family carried by fields transforming in a chosen representation of the relevant E_n structure. This preserves the division of labor between functional dependence and internal symmetry. A beta lift adds an upper and a lower hypergeometric parameter. The world-volume labels below are separate from the effective spatial-dimension convention. Under the model-level one-direction/one-lift convention,

$$D_d \longleftrightarrow H_{d+2,d+1}(2\xi). \tag{19}$$

Consequently,

$$D_4 \longleftrightarrow H_{6,5}(2\xi) \longleftrightarrow E_6, \tag{20}$$
$$D_5 \longleftrightarrow H_{7,6}(2\xi) \longleftrightarrow E_7. \tag{21}$$

In Eqs. (19)–(21), the brane label d counts spatial world-volume directions and is distinct from the effective spatial convention $d = n$; it does not determine matrix rank. Likewise, beta-integration variables and the physical world-volume measure remain distinct unless an explicit model identifies them.

3.1 Definition, domain, and complete local solution space

The convention used throughout is

$$\xi = \alpha t + \sum_{a=1}^{d} \beta_a x_a, \qquad H_{n,n-1}(2\xi) = e^{-\xi}{}_nF_{n-1}(\mathbf{p}_n; \mathbf{q}_n; 2\xi),$$

where $\mathbf{p}_n = (p_1, \ldots, p_n)$ and $\mathbf{q}_n = (q_1, \ldots, q_{n-1})$. The exponential weight is already part of H; it is not multiplied in a second time. The similarity variable is dimensionless, so α and β_a have inverse-time and inverse-length units. In the dimensional convention the author chooses $d = n$. This effective spatial-dimension convention is independent of the algebra's rank, its representation dimension, the physical spacetime dimension, and the brane index in Eq. (19).

Writing $z = 2\xi$ and $\vartheta = z\partial_z = \xi\partial_\xi$, define

$$\mathcal{P}_n = \vartheta \prod_{j=1}^{n-1} (\vartheta + q_j - 1) - z \prod_{i=1}^{n} (\vartheta + p_i), \qquad \mathcal{P}_n(e^{\xi} U_{n,\mathrm{hyp}}) = 0.$$

This is the profile equation, obtained by conjugating the generalized hypergeometric operator by the exponential weight [35,36]. It is not automatically the Einstein, Dirac, Maxwell, or nonlinear scalar field equation. A physical interpretation requires substitution into the parent field equation, compatible data, and a verified residual.

For nonresonant lower parameters, a local basis is

$$\begin{aligned} Y_0(z) &= {}_nF_{n-1}(\mathbf{p}_n;\mathbf{q}_n;z),\\ Y_j(z) &= z^{1-q_j}\,{}_nF_{n-1}(\{p_i+1-q_j\}_{i=1}^{n};\,2-q_j,\{q_k+1-q_j\}_{k\neq j};z)\,,\\ U_{n,\mathrm{hyp}}(\xi) &= e^{-\xi}\left[c_0Y_0(2\xi)+\sum_{j=1}^{n-1}c_jY_j(2\xi)\right]. \end{aligned}$$

Here $q_j \notin \mathbb{Z}$ and $q_j - q_k \notin \mathbb{Z}$ for $j \neq k$ suffice for this basis. Resonant cases require limits or Frobenius solutions, possibly with logarithms. Admissible lower parameters exclude nonpositive integers in the direct series. For a nonterminating ${}_nF_{n-1}$, the series is locally convergent for $|2\xi| < 1$; extending it requires a stated branch of analytic continuation [35,36]. Physical boundary conditions select the coefficients and admissible branches. The field power in $\Phi = A[U_n]^{\alpha_\Phi}$ is a separate model parameter and also needs a branch convention.

3.2 Beta lifting and repeated integration are different operations

For $\operatorname{Re} b > \operatorname{Re} a > 0$ and a domain permitting termwise integration,

$${}_{p+1}F_{q+1}(\mathbf{p},a;\mathbf{q},b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}\,{}_pF_q(\mathbf{p};\mathbf{q};zt)\,dt.$$

The beta integral appends one upper and one lower parameter [39]. When written in terms of weighted hulls it contains the factor $e^{-(1-t)\xi}H_{p,q}(2t\xi)$; omitting that factor changes the identity. It does not adjoin a Dynkin node. In contrast, R_G integrates with respect to a generated scalar coordinate G and raises mode powers. Neither operation by itself proves an exceptional-symmetry embedding.

3.3 Eight-sector mGFT generator and candidate ansatz

The general manuscript kernel permits independent coordinate series,

$$P(\xi) = \sum_m \delta_m \xi^m, \qquad Q(\xi) = \sum_r \epsilon_r \xi^r, \qquad Z(\xi) = \sum_o \zeta_o \xi^o.$$

Their convergence domains are specified with the coefficients; the supplied numerical examples use finite polynomials. The general hypergeometric factor is ${}_pF_q(\mathbf{c};\mathbf{d};Z)$, with counts independent of the level until a specialization is chosen. The atlas specialization sets $p = n$, $q = n-1$ and defines

$$\begin{aligned} \mathcal{F}_n(\xi) &= {}_nF_{n-1}(\mathbf{p}_n;\mathbf{q}_n;Z(\xi)),\\ G_{n,\boldsymbol{k}}(\xi) &= P(\xi)^{k_1}e^{-k_2Q(\xi)}[\mathcal{F}_n(\xi)]^{k_3},\\ [G_{n,\boldsymbol{k}}(\xi)]^{\ell} &= P(\xi)^{k_1\ell}e^{-k_2\ell Q(\xi)}[\mathcal{F}_n(\xi)]^{k_3\ell}. \end{aligned}$$

The eight-sector ansatz uses $\boldsymbol{k} \in \{0,1\}^3$. Allowing $\boldsymbol{k} \in [0,1]^3$ is a separate continuous extension with real-domain or complex-branch choices for fractional powers. Integer powers obey the displayed factorization wherever the chosen factors are defined. The restricted $(k_1,1,1)$ family is contained in this parameter space.

The supplied equation tables use $P = Q = Z = 2\xi$, hence

$$G_{n,\boldsymbol{k}} = (2\xi)^{k_1} e^{-2k_2\xi} [{}_nF_{n-1}(\mathbf{p}_n; \mathbf{q}_n; 2\xi)]^{k_3}.$$

This factor of two in the exponential is retained. The weighted hull $H_{n,n-1}(2\xi) = e^{-\xi} F_n(\xi)$ used in Secs. 3.1–3.2 is a different specialization, $P = Z = 2\xi$, $Q = \xi$, where $F_n(\xi) = {}_nF_{n-1}(\mathbf{p}_n; \mathbf{q}_n; 2\xi)$. Only in that specialization does $G_{n,(0,1,1)} = H_{n,n-1}$ and $G_{n,(0,1,0)} = e^{-\xi}$. Neither specialized weighting replaces the independent P, Q, Z kernel. For an AODE, $y(x) = U_n(\xi(x))$, optionally multiplied by a Frobenius factor x^r; the field extension uses $\xi = \alpha t + \sum_{a=1}^{d} \beta_a x_a$ with the framework convention $d = n$.

Define the scalar sum functions, for $i = 1, 2$, by

$$S_i(G) = \sum_{\ell=0}^{\infty} 2\,[\sin(\ell\pi/2)]^i\, G^\ell, \qquad C_i(G) = \sum_{\ell=0}^{\infty} 2\,[\cos(\ell\pi/2)]^i\, G^\ell.$$

Their exact sums for $|G| < 1$ are

$$S_1 = \frac{2G}{1+G^2}, \quad C_1 = \frac{2}{1+G^2}, \qquad S_2 = \frac{2G}{1-G^2}, \quad C_2 = \frac{2}{1-G^2}.$$

The infinite-sum limit is already evaluated in these identities. It is distinct from the nonlinear iteration limit $K \to \infty$ and the quadrature refinement limit. The rational functions provide continuation away from their poles; an unevaluated series is not convergent there merely because its continuation is finite.

For a chosen nonnegative integer Laurent cutoff P_s, the candidate profile is

$$U_n(\xi) = \sum_{\boldsymbol{k}\in\{0,1\}^3} \sum_{i=1}^{2} \sum_{j=-P_s}^{P_s} \Big\{ a_n[i,j,\boldsymbol{k}]\,[S_i(G_{n,\boldsymbol{k}})]^j + b_n[i,j,\boldsymbol{k}]\,[C_i(G_{n,\boldsymbol{k}})]^j \Big\}.$$

The cutoff P_s is distinct from the upper hypergeometric parameters p_s. Coefficients multiply scalar functions and carry the chosen internal representation. Negative j requires a nonzero base; the zeroth power is defined as the constant 1. A sector meeting a pole is excluded unless its relevant coefficient is identically zero and that term is omitted before evaluation. For example, $G_{n,(0,0,0)} = 1$ meets the S_2, C_2 poles. The continuous extension also needs a declared rule for summing or integrating over its parameter space; it is not an uncountable version of the finite binary sum.

Only the unweighted hypergeometric function obeys its equation in the argument Z. Pulling it back through $Z(\xi)$ requires the chain rule; the specialization $Z = 2\xi$ yields the operator used in Sec. 3.1. The local linear solution space in Sec. 3.1 describes the weighted hypergeometric equation; the full rational/Laurent ansatz U_n generally obeys neither equation automatically. Its coefficients and generator parameters are fixed by substitution into the target field equation and its data, followed where necessary by the generated-core correction of Sec. 11. A single sector or term cannot be assigned a cosmological object from its series label alone.

At levels $n \geq 10$, the anti-integral lift acts on these scalar functions as

$$R_G^{-1}[S_i] = \sum_{\ell=1}^{\infty} 2\ell[\sin(\ell\pi/2)]^i G^{\ell-1},$$

$$R_G^{-1}[C_i] = \sum_{\ell=1}^{\infty} 2\ell[\cos(\ell\pi/2)]^i G^{\ell-1}.$$

Termwise differentiation is justified inside $|G| < 1$; continued rational expressions may be differentiated away from their poles. The whole ansatz also requires $\partial_G(S_i^j) = jS_i^{j-1}\partial_G S_i$ and its cosine counterpart. If a coefficient depends on G, its derivative contributes by the product rule. If it depends on spacetime, physical differentiation additionally acts on that dependence and on $G(\xi(X))$. These operations are not root-rank deletion.

3.4 Exact canonical wall and a constructive double lift

Let ζ be the spatial coordinate normal to a planar wall. A fully specified scalar example uses $\boldsymbol{k} = (0, 1, 0)$, $\xi = \mu\zeta$, $G = e^{-\mu\zeta}$ and

$$\phi(G) = \nu[C_1(G) - 1] = \nu\frac{1 - G^2}{1 + G^2} = \nu\tanh(\mu\zeta), \qquad \mu, \nu > 0.$$

This is a special case of the ansatz with $P_s = 1$, $b_n[1, 1, (0, 1, 0)] = \nu$ and $b_n[1, 0, (0, 1, 0)] = -\nu$, all other coefficients zero in a scalar channel. The rational continuation defines the wall for all real ζ, including the region where $|G| \geq 1$.

With canonical Lagrangian $\mathcal{L} = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$, set

$$V(\phi) = \frac{\mu^2}{2\nu^2}(\phi^2 - \nu^2)^2.$$

For $\Box_g = g^{\mu\nu}\nabla_\mu\nabla_\nu$ and signature $(-, +, +, +)$, the equation is $\Box_g\phi - V_{,\phi} = 0$. The static planar profile satisfies $\phi_{\zeta\zeta} = V_{,\phi}$, $\phi(\zeta \to \pm\infty) = \pm\nu$ and

$$\rho_{\text{wall}} = \tfrac{1}{2}\phi_\zeta^2 + V(\phi) = \mu^2\nu^2\,\text{sech}^4(\mu\zeta), \qquad \int_{-\infty}^{\infty}\rho_{\text{wall}}\,d\zeta = \frac{4}{3}\mu\nu^2.$$

This is an exact flat-space scalar wall. A self-gravitating wall or cosmological evolution requires the corresponding Einstein equations and data. The scalar identity does not establish E_{11} symmetry or any other exceptional algebra.

The zero-base double integral is

$$R_G^2[f](G) = \int_0^G (G - s)f(s)\,ds.$$

For the wall source $f = \phi$, the complete solution of $\partial_G^2 U = \phi(G)$ is

$$U(G) = A_0 + A_1 G + \nu\left[-\frac{G^2}{2} + 2G\arctan G - \log(1 + G^2)\right].$$

The bracket and its first derivative vanish at $G = 0$, and its second derivative is $(1 - G^2)/(1 + G^2)$. Thus A_0, A_1 are the two integration data. The equation is in the generated coordinate: after composition with $G(\zeta)$, $U_{\zeta\zeta} = U_{GG}G_\zeta^2 + U_G\,G_{\zeta\zeta}$. It does not become the wall equation simply by replacing G by ζ. The supplement derives the sum identities and checks the wall and double lift with complete exported inputs and outputs.

4 Representation-Valued and Group-Valued Fields

A central construction is the promotion of scalar analytic profiles to Lie-algebra-valued fields by placing the symmetry information in coefficient matrices. Let T_α be generators in a chosen representation ρ of the Lie algebra associated with the selected E_n level. Coefficients can be expanded as

$$a_{ij} = A_{ij}^\alpha T_\alpha, \qquad b_{ij} = B_{ij}^\alpha T_\alpha, \tag{22}$$

with

$$[T_\alpha, T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma. \tag{23}$$

A field profile $U(\xi)$ constructed from these coefficients transforms by conjugation,

$$U(\xi) \longmapsto \rho(g) U(\xi) \rho(g)^{-1}. \tag{24}$$

When the representation and reality conditions permit, an algebra-valued field may be lifted to a group-valued field,

$$g(\xi) = \exp[U(\xi)]. \tag{25}$$

This construction gives a concrete meaning to the statement that a hypergeometric or mGFT profile "carries" an exceptional symmetry: the scalar profile determines functional dependence, while the generators, representation, and transformation law carry the symmetry.

4.1 Coefficient indices and the Clifford factor

The coefficient convention uses $\boldsymbol{k} = (k_1, k_2, k_3)$. The finite ansatz sums over $\{0,1\}^3$; the continuous extension allows $[0,1]^3$. The earlier slice $(k_1, 1, 1)$ is retained as a restricted family, rather than imposed on the general construction. The third argument of $a_n[i, j, \boldsymbol{k}]$ is the full parameter vector, separate from internal-generator labels and physical Fourier momentum. Define

$$\begin{aligned} a_n[i,j,\boldsymbol{k}] &= \sum_\alpha A_{i,j}^{(n)\alpha}(\boldsymbol{k}) T_\alpha^{(n)}, \\ b_n[i,j,\boldsymbol{k}] &= \sum_\alpha B_{i,j}^{(n)\alpha}(\boldsymbol{k}) T_\alpha^{(n)}, \\ M_n[i,j,\boldsymbol{k}] &= a_n[i,j,\boldsymbol{k}] + b_n[i,j,\boldsymbol{k}]. \end{aligned}$$

The pair (i, j) labels the coefficient array and α labels an internal generator. At fixed $\boldsymbol{k}$, write $a_n(\boldsymbol{k}) = [a_n[i, j, \boldsymbol{k}]]$, $b_n(\boldsymbol{k}) = [b_n[i, j, \boldsymbol{k}]]$ and $M_n(\boldsymbol{k}) = a_n(\boldsymbol{k}) + b_n(\boldsymbol{k})$. This is a coefficient matrix, not automatically a physical mass matrix. A paired basis $T_{r,s}$ is possible only after specifying its algebraic realization; then $a_{i,j} = \sum_{r,s} A_{i,j}^{r,s} T_{r,s}$ is a relabeling of the specified basis.

The sixteen Dirac/Clifford basis elements $I, \gamma^\mu, \sigma^{\mu\nu}, \gamma^5\gamma^\mu, \gamma^5$ span the complex 4×4 matrix algebra. Its dimension is sixteen, whereas $\dim \mathfrak{e}_6 = 78$, $\dim \mathfrak{e}_7 = 133$, and $\dim \mathfrak{e}_8 = 248$ [15]. Hence this factor cannot furnish a faithful representation of those simple algebras. It can encode spacetime spinors in $T_\alpha \otimes \Gamma_r$. Internal representation sizes include the 27 of E_6, 56 of E_7, and 248 adjoint of E_8. A 4×4 picture therefore describes a Clifford block or a selected effective sector, rather than the full exceptional symmetry.

5 E_6: Subgroups, Visible Sectors, and Symmetry Breaking

The first finite exceptional level used in the present phenomenological program is E_6. Standard E_6 model building admits subgroup routes including, at the Lie-algebra level, $\mathfrak{so}(10) \oplus \mathfrak{u}(1)$, $\mathfrak{su}(6) \oplus \mathfrak{su}(2)$, and $\mathfrak{su}(3)^{\oplus 3}$; a familiar route to the Standard Model is schematically

$$E_6 \longrightarrow \mathrm{Spin}(10) \longrightarrow SU(5) \longrightarrow G_{\mathrm{SM}}, \tag{26}$$

with global quotients treated separately [2]. The E_6 sector therefore uses the standard representation branching needed to connect E_6 to visible and hidden sectors.

A schematic breaking chain is

$$E_6 \longrightarrow G_{\text{int}} \longrightarrow G_{\text{SM}} \times G_{\text{hidden}}. \tag{27}$$

For a total multiplet Φ in representation V, a chosen vacuum may induce

$$V \downarrow G_{\text{low}} = V_{\text{vis}} \oplus V_{\text{hidden}} \oplus V_{\text{grav}} \oplus V_{\text{scalar}}. \tag{28}$$

The physical content depends on the actual branching rules, vacuum, and invariant couplings rather than on the group label alone. A conditional stabilizer construction for $E_6 \times E_6$ is obtained as follows: if a bifundamental-like matrix M transforms as

$$M \longmapsto \rho(g_L) M \rho(g_R)^{-1}$$

and develops $\langle M \rangle = vI_{27}$ in a faithful 27-dimensional representation, the preserving subgroup is the diagonal E_6. Within the present framework this suggests a route for coupling two E_6 sectors, provided a specified action dynamically produces and stabilizes such a condensate.

6 E_7: Generated Cores and Physical Equations

The E_7 construction pairs the algebraic label with the analytic hull

$$H_{7,6}(2\xi) = e^{-\xi}\, {}_7F_6(\mathbf{p}; \mathbf{q}; 2\xi), \qquad \dim \mathbf{p} = 7, \quad \dim \mathbf{q} = 6. \tag{29}$$

Here E_7 is the conventional finite exceptional Lie algebra/group label, whereas the association $E_7 \leftrightarrow H_{7,6}(2\xi)$ is a modeling convention of the present framework rather than a standard result of compact-object astrophysics or Lie theory. Standard E_7 representation and subgroup data may be taken from established group-theory references [15], but those data do not by themselves identify neutron stars, magnetars, or black holes as physical realizations of E_7.

6.1 Hypergeometric generated core and integral correction

With $z = 2\xi$ and $\Theta = z\, d/dz$, the unweighted generalized hypergeometric factor obeys

$$\left[\Theta \prod_{j=1}^{6} (\Theta + q_j - 1) - z \prod_{i=1}^{7} (\Theta + p_i) \right] {}_7F_6(\mathbf{p}; \mathbf{q}; z) = 0. \tag{30}$$

For a model field channel a, the generated core can therefore be written schematically as

$$U_a(X) = A_a(X)\, [H_{7,6}(2\xi_a(X))]^{\alpha_a}. \tag{31}$$

where A_a, α_a, and the similarity map $\xi_a(X)$ belong to the chosen physical model. The integral ft-{ht,lsr}-mGFT correction is then applied to the coupled equation

$$L_a[\Phi_a] + N_a[\Phi] = s_a, \qquad E_{U_a} = L_a[U_a] + N_a[U]. \tag{32}$$

through the Riemannian-volume Green–Duhamel iterate

$$v_{a,k+1,m}(X) = h_a(X) + J_{a,m}\big[s_a - E_{U_a} - \big(N_a[U + v_{k,m}] - N_a[U]\big)\big]\,(X), \tag{33}$$

$$\Phi_a^{[K,m]} = U_a + v_{a,K,m}. \tag{34}$$

with

$$J_{a,m}[f](X) = \sum_{i=1}^{m} K_a(X, Y_i) f(Y_i)\,\Delta V_{g,i}, \qquad \Delta V_{g,i} = \int_{C_i} \sqrt{|\det g(Y)|}\,dY. \tag{35}$$

At fixed mesh m, $K \to \infty$ is meaningful only when the nonlinear iteration converges. A continuum claim additionally requires mesh refinement, control of the Green/right-inverse approximation, and closure of the residual

$$R_{a,K,m} = L_a[\Phi_a^{[K,m]}] + N_a[\Phi^{[K,m]}] - s_a. \tag{36}$$

Equations (31)–(36) define a construction to be applied to a chosen parent problem. The generated core and its residual are distinct objects; the latter controls whether the corrected profile satisfies that problem [14].

6.2 Field channels used in the E_7 analytic benchmark

The following six labels distinguish the variables and diagnostic amplitudes in a coupled gravity–matter model:

Table 3: Model variables and physical equations.

Label	Model role and physical equation
Φ_G	Scalarized amplitude representing the gravitational/geometry sector; its physical content must ultimately be reconstructed from the metric or connection rather than treated as an independent scalar by default.
Ψ_e	Electron spinor/matter amplitude, governed schematically by a curved-spacetime Dirac equation $(i\gamma^\mu D_\mu - m_e)\Psi_e = S_e$.
Ψ_ν	Neutrino spinor amplitude, analogously governed by a curved-spacetime Dirac/Majorana sector with the mass and interaction terms specified by the action.
Φ_γ	Scalarized electromagnetic/photon-field amplitude; the underlying tensor field is constrained by Maxwell or GRMHD equations, not by the scalar proxy alone.
Λ	Effective vacuum/curvature-source channel in the model; it is not automatically identified with the measured cosmological constant in a local compact-object calculation.
R	Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$, used as a curvature diagnostic.

The common relativistic anchor is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \tag{37}$$

with electromagnetic contributions satisfying, for example,

$$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu, \qquad \nabla_{[\alpha} F_{\beta\gamma]} = 0. \tag{38}$$

These standard equations determine what a physically calibrated realization of these channels would have to satisfy.

6.3 Fully specified H7,6 profile examples

Table 4 specifies four analytic profiles, A–D, with seven upper and six lower parameters each. Together with $z = 2\xi$ and the exponential weight, these lists completely determine the principal local series. The calculation script evaluates each profile on $|z| < 1$ and checks the coefficient recurrence of Eq. (30). These are chosen mathematical inputs; the labels A–D carry no object-specific physical calibration.

Table 4: Fully specified local H7,6 profiles A–D.

Profile A	
$\mathbf{p}_H$	$(0.82, 1.14, 1.51, 1.93, 2.38, 2.86, 3.37)$
$\mathbf{q}_H$	$(1.05, 1.46, 1.89, 2.34, 2.82, 3.31)$
Profile B	
$\mathbf{p}_H$	$(0.91, 1.27, 1.68, 2.13, 2.62, 3.14, 3.69)$
$\mathbf{q}_H$	$(1.18, 1.62, 2.08, 2.57, 3.09, 3.64)$
Profile C	
$\mathbf{p}_H$	$(1.02, 1.41, 1.86, 2.36, 2.89, 3.46, 4.06)$
$\mathbf{q}_H$	$(1.28, 1.76, 2.27, 2.82, 3.40, 4.01)$
Profile D	
$\mathbf{p}_H$	$(1.08, 1.49, 1.96, 2.47, 3.03, 3.62, 4.25)$
$\mathbf{q}_H$	$(1.35, 1.86, 2.40, 2.98, 3.59, 4.23)$

6.4 Compact-object physics used to constrain the interpretation

For a nonrotating perfect-fluid neutron star, the standard relativistic equilibrium anchor is the Tolman–Oppenheimer–Volkoff system [16],

$$\frac{dm}{dr} = 4\pi r^2 \rho. \tag{39}$$

$$\frac{dP}{dr} = -\frac{G\left(\rho + P/c^2\right)\left(m + 4\pi r^3 P/c^2\right)}{r^2\left(1 - 2Gm/(rc^2)\right)}. \tag{40}$$

The TOV system applies to nonrotating, isotropic equilibrium matter. Magnetized configurations add electromagnetic stress to the Einstein–Maxwell equations [17,18]. Idealized dust collapse and the late-time exterior of an isolated rotating black hole provide the conventional collapse and Kerr reference problems [19,20]. Relativistic binaries are governed by the two-body Einstein problem; post-Newtonian waveform calculations and gravitational-wave observations provide distinct theoretical and observational anchors [5,21]. These equations establish the physical meaning of a prospective compact-object application; the local H7,6 examples above remain mathematical profile checks.

7 E_9: Affine Fields and Constructive Lifts

The affine stage is $E_9 = E_8^{(1)}$, with analytic hull $H_{9,8}(2\xi)$ in the indexing convention of Sec. 3. An affine realization carries loop-algebra structure, central extension and derivation data as appropriate to the chosen representation. A schematic profile-valued element is

$$U_{E_9}(X) = A(X)[H_{9,8}(2\xi(X))]^{\alpha} T,$$

where T specifies the affine-algebra content. Gauge-field applications use the corresponding action and Maxwell or Yang–Mills equations; the profile factor describes functional dependence within that model.

The analytic operation R_G gives the generated-mode part of a constructive lift from an E_9 seed. The complete lift of Eq. (12) also includes coefficient extension and the chosen dimensional map. Section 8 examines the E_{10} Cartan structure and a separate scalar example. Affine extension, scalar integration and physical symmetry realization are distinct steps in this construction.

8 E_{10}: Hyperbolic Cartan Lift and Sine-Gordon-Like Solitonic Sector

The standard mathematical anchor is the rank-ten hyperbolic Kac–Moody algebra

$$E_{10} = E_8^{++}, \tag{41}$$

obtained by adjoining the affine node and then one overextended node to the E_8 Dynkin diagram. This identification, and the Lorentzian/hyperbolic character of the resulting root lattice, are standard facts in the Kac–Moody literature [23]. In the node numbering used here, nodes $1, \ldots, 9$ form a chain and node 10 is attached to node 7. A convenient symmetric generalized Cartan matrix is therefore

$$A(E_{10}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}. \tag{42}$$

It satisfies the generalized-Cartan conditions $A_{ii} = 2$, $A_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$, and $A_{ij} = 0 \Leftrightarrow A_{ji} = 0$, with

$$\det A(E_{10}) = -1. \tag{43}$$

The determinant and signature distinguish this matrix from the affine E_9 matrix: E_{10} is nondegenerate and Lorentzian, with one timelike direction in its root space [23].

Mizoguchi studies E_{10} symmetry in one-dimensional supergravity [29], a different dynamical setting from the scalar diagnostic here. Affine Toda theory provides a Lie-algebraic integrable-model comparison [34]; its integrability and scattering results do not transfer automatically to the present hyperbolic phase system.

8.1 Logarithmic/inverse-hyperbolic core and the $E_9 \to E_{10}$ lift

The model-level E_{10} generated core is taken in two complementary branches. For a real similarity variable ξ define

$$U_{10}^{(\pm)}(\xi) = \xi + \log(1 \pm e^{\xi}) \ \oplus \ \operatorname{arctanh}(e^{\xi}). \tag{44}$$

The symbol $\oplus$ denotes the paired analytic composition used in this framework; it is not a direct sum of Lie algebras and does not modify any entry of the generalized Cartan matrix. A common real domain

for both branches is $\xi < 0$, for which $0 < e^{\xi} < 1$, $1 - e^{\xi} > 0$, and $\operatorname{arctanh}(e^{\xi})$ is real. Outside that domain Eq. (44) is interpreted only after specifying an analytic continuation or a different real branch. Equation (44) is a model ansatz; it is not asserted to be a standard Kac–Moody construction.

The standard algebraic lift from affine $E_9 = E_8^{(1)}$ to hyperbolic $E_{10} = E_8^{++}$ is instead the addition of one overextended Dynkin node [23,28]. With the node numbering of Eq. (42), deleting node 1 leaves the affine E_9 submatrix. Define

$$A(E_9) := A(E_{10})_{\{2,\dots,10\}\times\{2,\dots,10\}}, \qquad \det A(E_9) = 0, \tag{45}$$

and let $\mathbf{e}_1 = (1, 0, \dots, 0)^T \in \mathbb{R}^9$. Then the same Cartan data can be written as the block overextension

$$A(E_{10}) = \begin{pmatrix} 2 & -\mathbf{e}_1^T \\ -\mathbf{e}_1 & A(E_9) \end{pmatrix}. \tag{46}$$

Thus the conventional algebraic reduction and lift are

$$\Pi^A_{10\to 9} : A(E_{10}) \longmapsto A(E_9) \qquad \text{(delete the overextended row and column)}, \tag{47}$$

$$\mathcal{O}^A_{9\to 10} : A(E_9) \longmapsto \begin{pmatrix} 2 & -\mathbf{e}_1^T \\ -\mathbf{e}_1 & A(E_9) \end{pmatrix} \qquad \text{(adjoin the overextended node)}. \tag{48}$$

Equations (47)–(48) are the Kac–Moody/Dynkin part of the $E_9 \leftrightarrow E_{10}$ transformation. They are kept separate from the analytic field lift below.

For the generated-mode coordinate G, promote the monomial rule of Eq. (4) to the integral operator

$$(R_G f)(G) := \int_{G_0}^{G} f(\widetilde{G})\, d\widetilde{G}, \qquad \partial_G R_G f = f. \tag{49}$$

for functions and boundary data for which the integral is defined. In particular,

$$R_G[G^\ell] = \frac{G^{\ell+1} - G_0^{\ell+1}}{\ell + 1}, \tag{50}$$

which reduces to the special-case rule $G^\ell \mapsto G^{\ell+1}/(\ell+1)$ when $G_0 = 0$. The framework's field-level inverse/integral lift from E_9 to E_{10} is therefore

$$\mathcal{I}^{\text{field}}_{9\to 10}[\Phi_9] = \mathcal{E}^{(9\to 10)}_{a,b} \circ K^{\uparrow}_{9\to 10} \left[\int_{G_0}^{G} \Phi_9(\widetilde{G})\, d\widetilde{G} \right]. \tag{51}$$

The complete proposed lift is the paired algebraic/field transformation

$$\mathcal{I}_{9\to 10} : \big(A(E_9), \Phi_9\big) \longmapsto \big(\mathcal{O}^A_{9\to 10}[A(E_9)],\ \mathcal{I}^{\text{field}}_{9\to 10}[\Phi_9]\big). \tag{52}$$

The two branch coordinates $U_{10}^{(\pm)}(\xi)$ of Eq. (44) are used as the E_{10} analytic generated-core family after the lift. Under the directional convention of Sec. 2.2, the $E_{10} \to E_9$ step of the full primary reduction is

$$D_{10\to 9} : \big(A(E_{10}), \Phi_{10}\big) \longmapsto \big(\Pi^A_{10\to 9}[A(E_{10})],\ \partial_G \circ M_{10\to 9}[\Phi_{10}]\big). \tag{53}$$

Accordingly,

$$E_9 \xrightarrow{\mathcal{I}_{9\to 10}} E_{10}, \qquad E_{10} \xrightarrow{D_{10\to 9}} E_9. \tag{54}$$

On an information-preserving retained sector, with compatible boundary data and an invertible compactification/tensor extension, the field factors can satisfy $\partial_G R_G = \mathrm{Id}$. In general, however, compactification and mode truncation lose information, so $\mathcal{I}_{9\to 10}$ is properly called a constructive or inverse/integral lift rather than a guaranteed global inverse of $D_{10\to 9}$.

8.2 Dense axion stars, oscillons, axitons, and oscillatons as proposed E_{10} objects

The proposed E_{10} object class includes dense axion stars and related nonlinear scalar configurations: oscillon or pseudo-breather solitons, axitons, oscillatons, and closely related coherently oscillating scalar condensates. These objects already have an independent field-theory literature. Dense axion-star branches have been studied as self-interacting axion configurations [24,25]; axitons arise as dense nonlinear axion pseudo-solitons [26]; and oscillatons are self-gravitating, time-dependent real-scalar solitons [27]. This literature does not identify such objects with E_{10}; that assignment is a hypothesis of the present framework.

There is, however, a concrete dynamical reason to place the proposed E_{10} solitonic sector near a sine-Gordon-type description. For a periodic axion-like scalar potential

$$V(\vartheta) = \Lambda^4 \, [1 - \cos \vartheta], \tag{55}$$

the relativistic field equation contains the sine nonlinearity. In the flat-space spherical limit it takes the familiar form

$$\ddot{\vartheta} - \vartheta'' - \frac{2}{r}\vartheta' + m^2 \sin \vartheta = 0, \tag{56}$$

and related dense-axion analyses explicitly discuss oscillon/pseudo-breather solutions of sine-Gordon-like equations [25]. The fully relativistic axion-star problem can require the complete Klein–Gordon–Einstein system and the full QCD axion potential, particularly in strongly bound regimes [25]; Eq. (56) is therefore a structural model equation rather than a universal exact reduction.

To couple this periodic dynamics to the E_{10} root system, let $\boldsymbol{\varphi}$ be a multi-component scalar on the root space and let α_i ($i = 1, \ldots, 10$) be the simple roots whose inner products generate Eq. (42). A Cartan-organized periodic action may be written

$$S_{10} = \int d^4x \, \sqrt{-g} \left[-\frac{1}{2} G_{AB} \nabla_\mu \varphi^A \nabla^\mu \varphi^B - \sum_{i=1}^{10} \Lambda_i^4 \left(1 - \cos \frac{\alpha_i \cdot \boldsymbol{\varphi}}{f_i} \right) \right], \tag{57}$$

leading to

$$\Box_g \varphi^A - \sum_{i=1}^{10} \frac{\Lambda_i^4}{f_i} \, \alpha_i^A \sin\!\left(\frac{\alpha_i \cdot \boldsymbol{\varphi}}{f_i} \right) = 0. \tag{58}$$

Equivalently, defining root phases $\vartheta_i = \alpha_i \cdot \boldsymbol{\varphi}/f_i$, a reduced coupled form is

$$\Box_g \vartheta_i - \sum_{j=1}^{10} \mu_j^2 A_{ij}(E_{10}) \sin \vartheta_j = s_i, \qquad i = 1, \ldots, 10. \tag{59}$$

where s_i permits gravitational, matter, or generated-core forcing. Equation (59) is the proposed E_{10} Cartan-coupled sine-Gordon-like system for this object class. In the terminology of this framework, the central physical claim is therefore:

Dense axion stars, oscillon/pseudo-breather solitons, axitons, oscillatons, and the related proposed E_{10} compact scalar structures should be modeled by sine-Gordon-like nonlinear field equations, with the E_{10} generalized Cartan matrix organizing the coupling among root-sector phases.

This is a testable framework hypothesis, not an established consequence of Kac–Moody theory. Its verification requires a specified action, boundary data, an E_{10} representation/real form, residual-verified integral ft-{ht, lsr}-mGFT solutions, and comparison with conventional Einstein–Klein–Gordon or axion-star calculations.

8.3 Arc functions, exact scalar example, and stability

Two analytic families that explain the inverse-trigonometric terminology are

$$\int \frac{2aG+2b}{1+G^2}\,dG = a\log(1+G^2) + 2b\arctan G + C,$$
$$\int \frac{2aG+2b}{1-G^2}\,dG = -a\log(1-G^2) + 2b\,\mathrm{artanh}\,G + C, \quad |G|<1.$$

For real G, the branch $\operatorname{arccot} G = \pi/2 - \arctan G$ is complementary to $\arctan G$. Their pairing is an analytic construction. Expressions such as $\arctan(E_8)$ are symbolic unless an operator domain and functional calculus are defined; they do not establish an isomorphism with E_{10}. Likewise, applying an arc function entrywise to a Cartan matrix generally destroys its integer Cartan conditions.

For a dimensionless axion phase with kinetic coefficient f^2, $m^2 = \Lambda^4/f^2$. The one-dimensional sine-Gordon equation

$$\vartheta_{tt} - \vartheta_{xx} + m^2 \sin\vartheta = 0$$

has the exact traveling kink

$$\vartheta(t,x) = 4\arctan\exp\left[\frac{m(x-vt-x_0)}{\sqrt{1-v^2}}\right], \qquad |v|<1,$$

in units $c=1$. This gives a concrete role for the arc-trigonometric core. It does not solve the radial equation with its $2\vartheta_r/r$ term or a self-gravitating axion star without further calculation [25].

In the periodic root action, the general phase coupling is

$$\Box_g\vartheta_i - \sum_j \frac{\Lambda_j^4}{f_i f_j}(\alpha_i\cdot\alpha_j)\sin\vartheta_j = s_i.$$

Equation (59) assumes a common $f_i = f$, so $\mu_j^2 = \Lambda_j^4/f^2$. Here and throughout, $\Box_g = g^{\mu\nu}\nabla_\mu\nabla_\nu$ with metric signature $(-,+,+,+)$; flat space gives $-\partial_t^2 + \nabla^2$. The canonical scalar equation is $\Box_g\phi - V_{,\phi} = 0$. Multiplication by -1 gives the displayed flat-space sine-Gordon form; sources s_i in the covariant equations are defined with this convention.

The E_{10} root metric is Lorentzian. If it is used literally as the unconstrained scalar kinetic metric G_{AB} in Eq. (57), one kinetic direction has the wrong sign. Thus that action is a formal diagnostic, not a demonstrated healthy field theory. With a separately imposed positive kinetic matrix and $M^2 \propto A(E_{10})$, the negative Cartan eigenvalue instead gives a tachyonic direction. These are distinct problems. A physical construction must specify its constraints or positive physical kinetic space and rederive its potential and spectrum there; the algebraic eigenvalues alone are not particle masses.

9 String-Field and Brane Integration

The string-field construction provides a route from analytic profiles to string-field candidates. Schematically, a string field can be written as

$$|S\rangle = \sum_{I,\alpha} U_I^\alpha(X)\,|I\rangle \otimes T_\alpha. \tag{60}$$

where I denotes the string-state data and T_α carries internal symmetry information. In the present framework, the U_I^α may be drawn from the $H_{n,n-1}$ or integral ft-{ht,lsr}-mGFT solution family appropriate to the selected E_n sector. A profile becomes a string-field solution only after it satisfies the relevant string-field equations and state-space constraints.

9.1 Clifford/brane factors

Ito and Maru's matrix-string construction supplies explicit Type IIA gamma-matrix products for D2, D4, and NS5 sectors [12]. In the conventions used here,

$$\Gamma_{\mathrm{D2}} = \Gamma_{016}, \qquad \Gamma_{\mathrm{D4}} = \Gamma_{01789}, \qquad \Gamma_{\mathrm{NS5}} = \Gamma_{012345}. \tag{61}$$

Their construction relates a compactified brane configuration to $U(n)$ matrix string theory in the presence of longitudinal five-branes [12]. A natural integration with the present symmetry program is to keep internal Lie generators and spacetime Clifford factors distinct:

$$\Phi(\xi) = \sum_{a,r} \phi_{ar}(\xi)\, T_a \otimes \Gamma_r. \tag{62}$$

Here T_a belongs to a representation of the selected E_n structure, Γ_r carries the brane/spinor structure, and ϕ_{ar} carries the analytic field profile. This tensor-product separation keeps internal symmetry and spacetime/brane structure explicit and distinct.

9.2 Proposed extensions from E_{12} through E_{15}

The conventional $E_{11} = E_8^{+++}$ context concerns very-extended algebraic descriptions of gravity and form fields [28,31]. The present author-defined E_{12}–E_{15} programme continues the indexing and constructive mathematics beyond that stage. These labels describe proposed additional E_n structures, rather than additional finite-dimensional exceptional Lie groups. A group realization requires a chosen generalized Cartan matrix, invariant form, real form, representation and integration prescription. The cited E_{10}/E_{11} literature motivates algebraic methods; it does not establish the four proposed extensions.

Their independent-root bookkeeping is

$$\Gamma_{E_{12}} = \Gamma_{E_6} + \sum_{j=-3}^{2} \mathbb{Z}\alpha_j, \quad \Gamma_{E_{13}} = \Gamma_{E_{12}} + \mathbb{Z}\alpha_{-4},$$
$$\Gamma_{E_{14}} = \Gamma_{E_{13}} + \mathbb{Z}\alpha_{-5}, \qquad \Gamma_{E_{15}} = \Gamma_{E_{14}} + \mathbb{Z}\alpha_{-6}.$$

Starting with six independent directions in Γ_{E_6}, and assuming every adjoined direction independent, gives ranks 12, 13, 14 and 15. These are ranks of the specified free modules; Cartan entries and Lie brackets are additional data. The lattice Γ_{E_n} and the spacetime Clifford factor Γ_r in Eq. (62) have different meanings.

Every level uses the common binary ansatz $G_{n,\boldsymbol{k}}$ of Sec. 3.3 with $\boldsymbol{k} \in \{0,1\}^3$. Its weighted sector is

$$H_{n,n-1}(2\xi) = e^{-\xi}{}_nF_{n-1}(\mathbf{p}_n; \mathbf{q}_n; 2\xi), \qquad n = 12, 13, 14, 15.$$

The parameter counts are respectively $(12, 11)$, $(13, 12)$, $(14, 13)$ and $(15, 14)$. They specify analytic order, not a root system or a physical dimension. With an E_9 scalar seed $f(G)$ and $m = n - 9$, the zero-base integral lift is

$$R_G^m[f](G) = \frac{1}{(m-1)!} \int_0^G (G-s)^{m-1} f(s)\, ds, \quad m = 3, 4, 5, 6.$$

It has zero derivatives of orders 0 through $m-1$ at $G = 0$ and $\partial_G^m R_G^m[f] = f$. Its monomial factor is $G^m/(\ell+1)_m$ relative to G^ℓ. The anti-integral return to the seed traverses source levels $n, n-1, \ldots, 10$, all satisfying the $n \geq 10$ rule. Arbitrary solutions of the corresponding order-m source equation also contain a boundary polynomial of degree at most $m-1$. These scalar identities supply the mGFT scaffold; coefficient extensions, compactification data and algebraic structure remain separate.

E_{12}: higher-dimensional gravity and brane precursors. The candidate profile is $H_{12,11}$ and the relative lift order is three. In the author's effective convention $d = n$, its context has twelve spatial dimensions and one time dimension. An explicit gravity–scalar–form model would impose $G_{MN}+\Lambda g_{MN} = \kappa^2 T_{MN}$, $\Box_g\phi - V_{,\phi} = 0$, $dF_p = 0$ and $d(e^{a\phi}{*}F_p) = 0$, for an action with no additional form sources or topological terms. Its form rank, couplings, boundary data and brane charges must be supplied. The separate 11D supergravity example at E_{11} is an external literature context, not this dimension convention [7,31].

E_{13}: moduli, flux and compactification. The candidate profile is $H_{13,12}$ and the relative lift order is four. Canonical moduli satisfy $\Box_g\phi^A - \partial_A V_{\text{eff}} = 0$; stationary values satisfy $\partial_A V_{\text{eff}} = 0$, and their physical stability depends on the kinetic metric and Hessian. Flux-supported geometry or a dynamical contraction of internal dimensions requires a specified potential, flux quantization and metric solution. Kaluza–Klein reduction and toroidal duality provide methodological context [7,9,30], while the E_{13} assignment is the author's proposal. The effective context has $d = 13$, independently of conventional supergravity's $D = 11 - n$ relation.

E_{14}: interfaces and domain walls. The candidate profile is $H_{14,13}$ and the relative lift order is five. Its proposed context is an Einstein–scalar configuration with domain walls or compactification interfaces in the effective $d = 14$ convention. Matching across an interface requires the junction conditions and surface stress tensor derived from the chosen action. Warped geometry supplies a familiar physical comparison [8]. The exact flat-space wall of Sec. 3.4 verifies a scalar identity, not an E_{14} geometry. A topology-changing interpretation needs additional geometric and dynamical data.

E_{15}: cosmological backgrounds. The candidate profile is $H_{15,14}$ and the relative lift order is six. For a homogeneous, isotropic metric with $d = 15$, a cosmological test must satisfy

$$H^2 = \frac{2\kappa_D^2\rho}{d(d-1)} - \frac{K_{\text{curv}}}{a^2} + \frac{2\Lambda}{d(d-1)}, \quad \kappa_D^2 = 8\pi G_{d+1},$$
$$\dot\rho + dH(\rho + P) = 0, \qquad H = \dot a/a,$$

in units $c = 1$, with a separate cosmological constant excluded from ρ, P. For a spatially flat canonical scalar, $\dot H = -\kappa_D^2\dot\phi^2/(d-1)$. Inflation and dark-energy dynamics offer comparison problems [6,10]; the E_{15} label alone selects neither an epoch nor its duration. A pre-Big-Bang interpretation is a framework hypothesis requiring an action, initial data, compactification and proper-time solution. The analytic coordinate ξ must be mapped to the intended field or metric quantity with units and chain-rule terms before residual certification.

At all four proposed levels, anti-, mirror and negative matter retain the independent copy/conjugation labels of Sec. 13.1. The index n, lattice directions, scalar lift order and the three kinds of opposing matter do not determine one another. The supplement uses these same profiles, relative lift orders and parent equations.

10 Matrix-QFT Route to Quantum Fields

Kazakov's matrix-model construction provides a benchmark in which scalar QFT, and an extension including fermions/Yukawa interactions, can be represented through a large-matrix model and a continuum limit [13]. The matrix action is summarized schematically as

$$S_{\text{MM}} = N\,\text{Tr}\big([X^\mu,\Phi]^2 + \ln(I - A_K\Phi)\big)\,, \tag{63}$$

with quenched coordinate matrices X^μ and a coupling matrix A_K [13]. This benchmark motivates placing E_n- and mGFT-structured profiles in off-diagonal matrix elements rather than identifying a scalar special function with an entire quantum field. A representative element is

$$(\Phi_r)_{ij} = C_{ij}^{(r)} G_r(\xi_{ij}) \Gamma_r, \qquad \xi_{ij} = \beta_\mu (x_i^\mu - x_j^\mu). \tag{64}$$

so that $[X^\mu, \Phi]_{ij}$ remains sensitive to coordinate separation. A combined internal/Clifford realization is

$$\Phi_{ij} = \sum_{\alpha,r} C_{ij}^{\alpha r} G_{\alpha r}(\xi_{ij})\, T_\alpha \otimes \Gamma_r. \tag{65}$$

The physical test is then action- and correlator-based: the matrix model, its large-matrix continuum limit, and its one- and two-point functions must reproduce the intended QFT observables [13].

11 Integral ft-{ht, lsr}-mGFT as the Dynamical Solver

The exceptional symmetry label is specified independently of any summation. The integral ft-{ht,lsr}-mGFT enters downstream as a solver for the field equations. Its quadrature-first formulation is developed in the integral ft-mGFT Riemannian-sum manuscript [14]. Let the coupled system be

$$L[\Phi] + N[\Phi] = s. \tag{66}$$

Choose a generated core U and define its equation defect

$$E_U = L[U] + N[U]. \tag{67}$$

If J is a right inverse of L compatible with the data, the fixed-point form is

$$\Phi = U + J\big[s - E_U - \big(N[\Phi] - N[U]\big)\big]\,. \tag{68}$$

Finite-K approximants may be constructed by iteration, and their direct verification quantity is

$$R_K = L[\Phi^{(K)}] + N[\Phi^{(K)}] - s. \tag{69}$$

This fixed-point formulation is the dynamical form used in the integral ft-mGFT program [14]. It allows the selected E_n symmetry to specify the representation structure while the integral method solves the associated nonlinear dynamics.

11.1 Green–Duhamel and Riemannian-volume realization

For a causal kernel compatible with the geometry and boundary data [14], use the induced spatial metric h on each slice:

$$(Jf)(t,x) = \int_0^t \int_\Omega G_{\mathrm{ker}}(t,x;\tau,y) f(\tau,y)\, d\mu_h(y)\, d\tau, \qquad d\mu_h = \sqrt{\det h}\, d^d y. \tag{70}$$

A Riemannian-volume Riemann discretization uses finite quadrature weights,

$$(Jf)(t,x) \approx \sum_{i,j} G_{\mathrm{ker}}(t,x;\tau_i,y_j) f(\tau_i,y_j)\, \Delta\tau_i\, \Delta V_{g,j}. \tag{71}$$

This finite sum is a numerical/geometric realization of the integral operator; it is not the definition of the internal symmetry. The cited integral ft-mGFT paper treats the tagged Riemannian-volume sum as the central computable inverse, separates quadrature/right-inverse error from nonlinear truncation through a residual identity, and emphasizes controlled refinement and limit ordering [14]. Convergence, kernel regularity, constraint preservation, and residual closure remain required.

11.2 Boundary data, contraction, homotopy, and symmetry reduction

Let $v = \Phi - U$, $F = s - E_U$, and $\Delta N_U(v) = N[U+v] - N[U]$. For homogeneous correction data, Eq. (68) is $v = JF - J\Delta N_U(v)$. For nonhomogeneous correction data it becomes

$$v = h + JF - J\Delta N_U(v), \qquad Lh = 0,$$

where h supplies the data not already satisfied by U. A right inverse must act on the compatible domain; applying J without this data term can select the wrong solution.

If $\|J\| \le C_J$, ΔN_U is Lipschitz on a closed invariant ball with constant L_N, and $q = C_J L_N < 1$, the iteration is contractive there. The a posteriori bound is

$$\|v_* - v_K\| \le \frac{q}{1-q}\|v_K - v_{K-1}\|.$$

A homotopy realization (ht) is the continuous deformation

$$\mathcal{H}(v,\epsilon) = v - h - JF + \epsilon J\Delta N_U(v) = 0, \qquad 0 \le \epsilon \le 1.$$

Its endpoints are the linear correction and the target nonlinear fixed point. A coefficient expansion in ϵ needs convergence at $\epsilon = 1$; ordinary fixed-point iterations are not automatically the coefficients of that expansion. This solver homotopy is distinct from a claim about topological equivalence of physical states [38].

Lie symmetry reduction (lsr) requires a generator X satisfying the prolonged invariance condition for the parent equations and preserving the relevant data [37]. A traveling coordinate $\xi = \alpha t + \beta x$ is invariant under $X = \beta\partial_t - \alpha\partial_x$. For the sine-Gordon example it yields

$$(\alpha^2 - \beta^2)F''(\xi) + m^2 \sin F(\xi) = 0.$$

This reduction is valid for the translation-invariant parent equation. For a curved or sourced system the generator and the initial/boundary conditions determine whether the same reduction is admissible. The supplement derives the homotopy coefficient recursion and the quadrature residual identity and verifies an explicit scalar benchmark.

12 Masses, Mixing, and Particle Modes from Quadratic Fluctuations

Masses are extracted from the quadratic dynamics rather than from arbitrary coefficient determinants or universal series. Let $S[\Phi]$ be a specified action and Φ_0 a verified background. Write

$$\Phi = \Phi_0 + \eta. \tag{72}$$

The quadratic kernel is

$$K_{ab}(x,y) = \left.\frac{\delta^2 S}{\delta\Phi_a(x)\,\delta\Phi_b(y)}\right|_{\Phi_0}. \tag{73}$$

For a stationary translation-invariant background, Fourier transformation gives the physical mode condition

$$\det K(\omega, \mathbf{k}) = 0. \tag{74}$$

After gauge and constraint reduction and verification of a positive physical kinetic matrix, a finite component model with kinetic matrix Z and mass matrix M^2 obeys the generalized eigenvalue problem

$$\det(m^2 Z - M^2) = 0. \tag{75}$$

These relations provide the systematic route to hidden-sector masses, mixing angles, instabilities, and propagating modes. A finite invariant scalar potential may still be used to generate the vacuum. For scalar coordinates ϕ_a:

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2 + \eta(\phi^\dagger \phi)^3 + V_{\text{mix}}(\phi, \chi). \tag{76}$$

with $\partial V/\partial \phi_a|_v = 0$ and

$$(M_\phi^2)_{ab} = \left. \frac{\partial^2 V}{\partial \phi_a \, \partial \phi_b} \right|_v . \tag{77}$$

Fermion masses arise from symmetry-allowed Yukawa structures, $M_\psi = Y(v)$, and gauge-boson masses arise from covariant kinetic terms of fields with nonzero vacuum expectation values. This keeps the mass problem tied to the actual representation and dynamics.

12.1 Background dependence of the spectrum

For similarity-dependent or strongly time-dependent backgrounds, a single constant mass matrix can be insufficient. The appropriate spectral problem depends on the verified background:

Table 5: Spectral analysis by background type.

Background	Appropriate analysis
Homogeneous and stationary	Fourier modes, propagator poles, dispersion relation
Stationary but spatially varying	Normal-mode or Sturm–Liouville-type eigenproblem
Periodic in time	Floquet exponents and parametric stability
Slowly varying in time	Adiabatic/WKB instantaneous modes with consistency checks
Strongly time dependent	Initial-value stability and asymptotic in/out modes when defined

13 Dark Matter, Dark Energy, and Cosmological Signatures

The framework includes dark-sector and cosmological applications tied to symmetry breaking and verified dynamics. Hidden states may occupy representations neutral under the visible low-energy subgroup but charged under G_{hidden}. Stability may follow from a residual gauge or discrete symmetry. Their masses and interactions are obtained from the quadratic operator and invariant couplings described above, not from a universal summation.

13.1 Three opposing-matter labels and their independent roles

The author hypothesis distinguishes ANTI, MIRROR and NEGATIVE matter relative to the ordinary reference [43]. Antimatter is charge conjugation; mirror and negative matter denote separate proposed sectors. This keeps mirror matter distinct from a negative-mass identification.

Table 6: Three opposing labels and the ordinary reference.

Label	Operation / copy	Physical and input status
Ordinary P_a	Reference $O, +$	Conditional E_5 masses $m_S c^2 = 21.6$, $m_V c^2 = 19.2$, $m_0 c^2 = 0$ TeV.
ANTI P_a^A	Conjugation $O, -$	Equal positive mass magnitude; conjugate representation and opposite internal charge. Antiparticles are excitations of the reference fields.
MIRROR P_a^M	Separate copy $M, +$	Positive equal masses in the exact-copy benchmark; independent mirror charges. Mirror antiparticles also exist in the model.
NEGATIVE P_a^N	Separate copy $N, +$	Proposed sector: energy sign and active/passive gravitational or inertial sign must be specified separately. Antiparticles do not remove that distinction.

For $a \in \{S, V\}$ write $P^\tau_{a,\eta}$, where $\tau \in \{O, M, N\}$ labels the copy and $\eta = \pm 1$ distinguishes particle from antiparticle. Thus $P_a^A = P^O_{a,-}$, $P_a^M = P^M_{a,+}$ and $P_a^N = P^N_{a,+}$. Mirror and negative copies also have conjugate excitations. A neutral real singlet P_0^τ completes each chosen $\mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$ module. Antiparticles need not be counted as another independent copy. These indices are distinct from n, $\boldsymbol{k}$ and the superscript in E_6^-.

The exact-copy mirror benchmark uses the positive ordinary mass magnitudes; parity relates the copies but does not imply negative mass [41]. Its charges belong to a specified mirror gauge group. The term negative matter requires an explicit choice among negative rest energy, negative inertial response and negative active or passive gravitational coupling. These choices are not interchangeable. For illustration, the supplement compares a signed-energy reference $(\chi_O, \chi_M, \chi_N) = (1, 1, -1)$ with a positive-energy reference having proposed opposite active gravitational coupling. The latter has $(1, 1, 1)$ in its energy sums; its passive coupling remains unspecified.

With mass magnitudes μ_a, the formal unbound bookkeeping is

$$\mathcal{E} = \sum_r \chi_{\tau_r} \mu_{a_r} c^2, \qquad \mathbf{Q} = \sum_r \eta_r q_{a_r} \mathbf{e}_{\tau_r}.$$

Independent conserved Abelian copies require $\mathbf{Q} = 0$. A diagonal sum $Q_O + Q_M + Q_N = 0$ is sufficient only for an explicitly identified single conserved charge operator; it does not cancel charges of unrelated groups. The scripts export both classifications and never treat a diagonal-only example as neutral under the three separate groups. Non-Abelian singlet projections and binding are additional requirements. Main Sec. 14.1 and supplement Sec. S4 use $q_S = -1$, $q_V = +2$ and $\mu_S c^2 = 21.6$, $\mu_V c^2 = 19.2$ TeV in the declared mass prescription.

The representation-valued analytic coefficients may therefore be written $a^\tau_{n,\boldsymbol{k}}$ and $b^\tau_{n,\boldsymbol{k}}$ in specified copy representations. Their conjugates act on conjugate representations. Each copy must satisfy its own target equation, boundary data and coupling constraints. Adding a copy index does not turn a profile into a dynamical solution or assign an opposing-matter type to an E_n level. The binary $\boldsymbol{k} \in \{0, 1\}^3$ family and source-level $n \geq 10$ anti-integral convention are unchanged.

13.2 Invariant masses, portals and photon implications

A physical composite mass follows from total four-momentum, including its interaction energy:

$$M^2c^4 = E_{\rm tot}^2 - c^2|\boldsymbol{P}_{\rm tot}|^2.$$

The sign of $E_{\rm tot}$ is additional information. A negative signed reference sum does not by itself determine a negative inertial mass, and its absolute value does not establish a positive-energy bound-state pole. Only a future-directed timelike parent has the rest-frame two-body massless split $E_1 = E_2 = Mc^2/2$ used here; a past-directed reference sum cannot feed two positive-energy photons by energy conservation.

The preon-table $U(1)_\psi$ charges are internal; they are not identified with the electromagnetic currents in the following illustration. A minimal ordinary–mirror photon portal is

$$\mathcal{L}_{\gamma,\gamma_M} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{4}F^M_{\mu\nu}F_M^{\mu\nu} - \tfrac{\epsilon}{2}F_{\mu\nu}F_M^{\mu\nu}.$$

Its kinetic eigenvalues are $1 \pm \epsilon$, positive for $|\epsilon| < 1$; with no mass term both Maxwell poles are massless [41]. An independently introduced massive dark photon requires a mass-generating sector. Choosing negative energy for another free-field block instead introduces a negative kinetic residue in the displayed diagnostic; its consistency is a separate stability problem [42]. It does not change a massless pole to a negative-rest-mass photon. Changing a Dirac mass parameter's sign alone is likewise not a prescription for a negative-energy branch.

The proposed radiation and preon implications therefore require an action, a stable spectrum, allowed transitions and independent measurements. Survey energy-grid agreement alone supplies none of those steps. The package includes signed sums and counterexamples with explicit four-momenta, rather than converting formal negative halves into observed photon energies.

Neutrino–electron scattering constrains specified $Z\prime$ and dark-photon interactions [40]. Applying such tests here requires an electromagnetic or neutrino current assignment, portal couplings and a predicted cross section, rather than the internal preon charges alone.

13.3 Cosmological backgrounds

Dark-energy-like behavior can arise from slowly evolving scalar directions of an invariant potential, subject to stability, cosmological evolution, and observational constraints [10]. First-order transitions in an E_n-based symmetry-breaking hierarchy may generate stochastic gravitational-wave backgrounds; quantitative spectra require a finite-temperature effective action, bubble nucleation, latent heat, transition duration, and hydrodynamics rather than group labels alone.

Higher-dimensional modes remain governed by geometry. For a simple circular compactification,

$$m_{\rm KK}^2 = m_0^2 + \frac{\ell^2}{R_c^2}, \tag{78}$$

with modifications in warped or more complicated compactifications [7,8,9].

14 Integrated Verification Ladder

The combined program is organized into the following verification sequence:

1. Choose the symmetry level E_n and an explicit representation.

2. Choose an analytic field profile, such as an $H_{n,n-1}$ hull or an integral ft-{ht,lsr}-mGFT generated core [14].
3. Expand matrix coefficients in verified generators T_α and, where needed, use Clifford factors Γ_r separately.
4. Specify the invariant action, gauge fixing, domain, initial/boundary data, and any compactification.
5. Solve the governing equations with the integral fixed-point/Green–Duhamel machinery or another justified method.
6. Verify residual closure, convergence, constraints, and regularity.
7. Derive the quadratic fluctuation operator and remove gauge/nonphysical modes.
8. Extract masses and dispersion relations from $\det K = 0$ or $\det(m^2 Z - M^2) = 0$.
9. For a matrix-QFT realization, verify the large-matrix continuum limit and correlators.
10. Compare masses, couplings, decay channels, cross sections, gravitational signals, and cosmological observables with data.

This ladder makes the roles of symmetry, analytic construction, dynamics, and physical interpretation explicit. Error estimation and computational verification distinguish residual closure from solution accuracy [32,33]. Lens modelling and source reconstruction provide an example of an inverse problem requiring both a metric model and source data [22]; no lens reconstruction is included in this package.

14.1 Constituent and dark-sector interpretation

The constituent benchmark for $E_5 = E_6^-$ uses

$$\mathcal{V}_5 = \mathbf{16}_{-1} \oplus \mathbf{9}_{+2} \oplus \mathbf{1}_0, \qquad \dim \mathcal{V}_5 = 26.$$

The modules are realized as a Spin(9) spinor, vector and singlet, with independent internal $U(1)_\psi$ charges. Their assigned physical spins are $1/2, 0, 0$. With $m_R c^2 = \Lambda_F C_2(R)$, $\Lambda_F = 4.8$ TeV and $C_2 = 9/2, 4, 0$, the conditional masses are $m_S c^2 = 21.6$ TeV, $m_V c^2 = 19.2$ TeV and $m_0 c^2 = 0.0$ TeV. The supplement verifies both the Weyl dimensions and the internal Casimir matrices. These assignments provide component content and a mass prescription, while the Lie algebra and root rank of the author-defined sector remain unspecified. The internal charges are distinct from electromagnetic charges; composition motifs alone do not derive the Standard Model spectrum.

The unbound thresholds for $S\bar{V}$, SVV and $V\bar{V}$ are 40.8, 60.0 and 38.4 TeV. Neutral $S\bar{S}$ and SSV thresholds are 43.2 and 62.4 TeV. The bookkeeping $M_X = \sum_i n_i m_i - B_X$ predicts a mass only when dynamics fixes B_X independently. Retaining the trial channel binding coefficients $C_{S\bar{V}} = 9/4$ and $C_{SVV} = C_{VV} = 9/2$ gives $\kappa_5 \leq 8.533333$ TeV from nonnegative two-V masses, and hence both nominal lepton and quark masses are at least 21.6 TeV. Fitting a zero nominal lepton mass instead requires $\kappa_5 = 18.133333$ TeV and yields $M_{VV} = -43.2$ TeV. This is an incompatibility of the stated linear binding prescription. Its channel coefficients are independent inputs, not newly derived single-preon Casimirs.

Each proposed copy retains this conditional component content; its energy and gravitational conventions are specified in Sec. 13 and supplement Sec. S4.4. Ordinary target bindings are not transplanted to mirror or negative copies without a binding Hamiltonian.

The supplement develops this constituent example through its input charges, threshold sums, required-binding table and shared-coefficient incompatibility. Its effective-action example separately treats eight prescribed quadratic mass parameters. A two-body kinematic exercise uses two of those mass scales as declared inputs. These computations establish the stated arithmetic and kinematics; they do not identify experimentally detected particles.

15 Predictive Power and Main Mathematical Challenges

The framework predicts structure primarily through E_n representation branching, selection rules, symmetry breaking, vacuum structure, fluctuation spectra, compactification, and solutions of the coupled field equations. For the finite levels E_6, E_7, E_8, the relevant algebraic data can be taken from the standard exceptional Lie groups and their representations. For E_9 and higher, the most important mathematical requirement is to specify the exact extended E-type algebra being used. For each higher level, a complete formulation should state the generalized Cartan matrix, root lattice, real form, representation category, and any central or derivation generators. For the proposed directional hierarchy it must additionally specify the compactification data, contraction tensors, projectors, retained Kaluza–Klein modes, and the domain on which each $D_{n\to n-1}$ or $I_{n-1\to n}$ is defined. The following methodological distinction is essential: an analytic profile, matrix size, or parameter count by itself does not establish a symmetry identification. The symmetry assignment must be supported by an explicit action, transformation law, representation content, and verifiable observables.

16 Conclusion

The integrated theory is centered directly on the E_n hierarchy,

$$E_n \longleftrightarrow H_{n,n-1}(2\xi), \qquad 5 \le n \le 15. \tag{79}$$

where the left-hand side carries the internal symmetry assignment and the right-hand side denotes the weighted $(0,1,1)$ sector of the complete eight-sector mGFT family. The full ansatz uses $G_{n,\boldsymbol{k}}$ and the exact scalar sums S_i, C_i, with coefficient matrices depending on all three generator parameters. The two roles remain distinct: explicit generators T_α carry the E_n symmetry; $H_{n,n-1}$ and integral ft-{ht,lsr}-mGFT profiles provide analytic field dependence; Clifford factors Γ_r can encode brane/spinor structure; Green–Duhamel operators solve the coupled equations [14]; and the Hessian or inverse propagator determines the physical spectrum. The symmetry is fixed by its algebraic data. Finite Riemann sums are used only as numerical approximations to geometric integrals, and particle masses are obtained from symmetry breaking and quadratic fluctuation operators. The proposal is therefore a layered E_n -indexed group-theoretic and dynamical program rather than a purely profile-defined theory. The hierarchy is directional. Its full primary flow is

$$\begin{aligned} &E_{15} \longrightarrow E_{14} \longrightarrow E_{13} \longrightarrow E_{12} \longrightarrow E_{11} \\ &E_{11} \longrightarrow E_{10} \longrightarrow E_9 \longrightarrow E_8 \longrightarrow E_7 \\ &E_7 \longrightarrow E_6 \longrightarrow E_5 \end{aligned} \tag{80}$$

The higher E_{15}–E_{11} maps require the algebraic data described in Sec. 9.2. The conventional E_{11}–E_6 core uses restriction/node deletion and, in the model-level dynamical construction, compactification and tensor contraction. The terminal $E_6 \to E_5 = E_6^-$ sector map is unspecified, as is its reverse sector construction. Generated-coordinate anti-integration is imposed only for source levels $n \ge 10$. The reverse sequence uses specified structural extensions, with R_G included for target levels $n \ge 10$; it is a genuine inverse only on sectors where no information has been discarded. The separate supplement specifies an effective reference action and reproduces the generator and sum identities, the canonical scalar wall and zero-base double lift, a manufactured scalar solution, second-order mesh convergence, and a generalized-Cartan spectrum. Its eight hidden-sector masses are prescribed effective-action inputs, not independently inferred poles from measured data. The three opposing-matter labels have separate conjugation, copy and sign conventions. Their exported sums verify conditional arithmetic; physical negative-mass modes,

mirror radiation and preon identification remain dynamical and observational questions. The remaining demonstrations are sector-specific: (i) replace the reference action by representation-complete actions for each proposed E_n phenomenological sector; (ii) derive explicit visible/hidden branching rules and symmetry-breaking vacua; (iii) residual-certify the actual astrophysical field solutions rather than only the verification benchmark; and (iv) compare the resulting physical spectra and observables with experiment, astrophysics, or cosmology.

A Direct E_n Hierarchy Summary

The full hierarchy includes the opposite-lift endpoint and the proposed higher extensions as

$$E_5 = E_6^-; \qquad E_n, \quad 6 \le n \le 15. \tag{81}$$

The author-defined $E_5 = E_6^-$ opposite-lift sector has chosen component content $\mathbf{16}\oplus\mathbf{9}\oplus\mathbf{1}$, total dimension 26, and no assigned root rank or algebra here. The E_{12}–E_{15} extensions use the independent-root and integral-lift bookkeeping of Sec. 2.3 and the supplement. Their main-text discussion is in Sec. 9.2; both documents use the same eight-sector ansatz and parameter counts $(n, n-1)$. The finite exceptional sector is E_6, E_7, and E_8. The higher core labels are used for explicitly specified affine, hyperbolic, or indefinite E-type extensions. The primary direction adopted here is $E_{15} \to \cdots \to E_5$; the opposite direction denotes constructive lifts. The final $E_6 \to E_5 = E_6^-$ sector transition has no assigned scalar anti-integral map.

B Integrated Core Formulae

Concept	Formula / role
E_n symmetry	Direct exceptional or extended E_n level, with explicit representation data
Opposite-lift sector	$E_5 = E_6^-$; $\mathcal{V}_5 = \mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$, $\dim \mathcal{V}_5 = 26$; root rank unspecified
Primary hierarchy	$E_{15} \xrightarrow{D_{15}} \cdots \xrightarrow{D_6^{\text{sector}}} E_5$; $E_5 = E_6^-$
Constructive lifts	$E_5 \xrightarrow{I_6^{\text{sector}}} \cdots \xrightarrow{I_{15}} E_{15}$; reverse sector map unspecified
Mode lift / reduction	$R_G[G^\ell] = G^{\ell+1}/(\ell+1)$; $R_G^{-1} = \partial_G$; indexed anti-lift at source $n \ge 10$
Tensor reduction	$T_{a,b}^{(n\to n-1)} = C_{a,b}^{(n)} \circ K_{n\to n-1}$
Coefficient map	$M_{n\to n-1} = T_a P_a + T_b P_b$
Full downward map	$D_{n\to n-1} = \partial_G \circ M_{n\to n-1}$ for $n \ge 10$; structural D_9, D_8, D_7; terminal D_6^{sector} unspecified
Upward lift	$I_{n-1\to n} = \mathcal{E}_{a,b} \circ K^{\uparrow} \circ R_G$, target $n \ge 10$; structural I_7, I_8, I_9; I_6^{sector} unspecified
Analytic hull	$H_{n,n-1} = e^{-\xi} F_n$; $(0,1,1)$ sector for $Q = \xi$, $Z = 2\xi$
mGFT generator	$G_{n,\boldsymbol{k}} = P^{k_1} e^{-k_2 Q} [{}_n F_{n-1}(\mathbf{p}_n; \mathbf{q}_n; Z)]^{k_3}$; $\boldsymbol{k} \in \{0,1\}^3$
Candidate profile	Finite sum of coefficient-weighted Laurent powers of $S_i(G)$ and $C_i(G)$; main Sec. 3.3
Canonical wall	$\phi = \nu(C_1 - 1) = \nu \tanh(\mu\zeta)$; canonical potential and residual check
Algebra-valued coefficients	$a_n[i,j,\boldsymbol{k}] = \sum_\alpha A_{ij}^{(n)\alpha}(\boldsymbol{k}) T_\alpha^{(n)}$; likewise b_n

Concept	Formula / role
Group-valued lift	$g(\xi) = \exp[U(\xi)]$ under appropriate conditions
Lie/Clifford field	$\Phi(\xi) = \sum_{a,r} \phi_{ar}(\xi) T_a \otimes \Gamma_r$
Coupled field equation	$L[\Phi] + N[\Phi] = s$
Generated-core defect	$E_U = L[U] + N[U]$
Fixed point	$\Phi = U + J[s - E_U - (N[\Phi] - N[U])]$
Residual	$R_K = L[\Phi^{(K)}] + N[\Phi^{(K)}] - s$
Quadratic operator	$K_{ab} = \delta^2 S/(\delta\Phi_a\, \delta\Phi_b)\big\vert_{\Phi_0}$
Mode condition	$\det K(\omega, \mathbf{k}) = 0$ after gauge/constraint reduction
Generalized mass problem	$\det(m^2 Z - M^2) = 0$
Opposing matter	ANTI: conjugation; MIRROR and NEGATIVE: separate proposed copies. Labels independent of n and $\boldsymbol{k}$.
Signed reference energy	$\mathcal{E} = \sum_r \chi_{\tau_r} \mu_{a_r} c^2$; binding and four-momenta needed for physical M.
Invariant mass / photons	$M^2c^4 = E_{\text{tot}}^2 - c^2\lvert\boldsymbol{P}_{\text{tot}}\rvert^2$; sign of energy is separate. Maxwell photons have $m_\gamma = 0$.

Author Declarations

AI-Assistance Statement

Robert L. Jackson II formulated the research programme and its critical mathematical constructions. His contributions include the proposed E_n structures and their indexing through E_{15}; the defining opposite-lift sector $E_5 = E_6^-$ with chosen constituent content $\mathbf{16} \oplus \mathbf{9} \oplus \mathbf{1}$; the proposed E_{12}–E_{15} extensions; the formulation of integral and anti-integral lifts, their boundary data and placement in the hierarchy; and the derivation of the Γ_{E_n} lattice bookkeeping by adjoining independent root directions. He also formulated the coefficient and mGFT-sector conventions, the distinction between root lattices Γ_{E_n} and Clifford factors Γ_r, the effective dimensional convention, and the hypotheses distinguishing anti-, mirror and negative matter. These author-defined constructions use mGFT as an analytic scaffold. They are distinguished throughout from the conventional finite exceptional groups, externally sourced results and the algebraic data still required to realize the proposed extensions.

The author directed the scientific questions, selected the physical interpretations and benchmark inputs, and supplied the mathematical corrections and distinctions that govern the paper, supplement and scripts. OpenAI's ChatGPT assisted with exposition, editing, mathematical typesetting, organization of references, computational implementation, reproducibility checks and document production. The author reports that Microsoft Copilot also assisted in the review process. These editorial and computational roles do not transfer the formulation of the critical mathematics or the proposed E_n programme to an AI system.

The author retains responsibility for selecting, correcting and approving the text, equations, code, citations, interpretations and conclusions. No AI system is an author. AI assistance and reproducible arithmetic are not independent peer review or empirical validation of the proposed groups, opposing-matter hypotheses or physical assignments.

Conflict of Interest Disclosure

The author declares no conflicts of interest or competing interests, financial or non-financial, that are relevant to the research, preparation, interpretation, or publication of this manuscript.

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