Lie-Algebraic Structure: Vertical and Horizontal Emergence

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Abstract

Lie-algebraic structure can be examined through two complementary forms of organization that will be called vertical and horizontal emergence. These terms are used here as a conceptual framework rather than as standard classifications within Lie theory.

Vertical emergence refers to the construction of new algebraic objects from structures already present at a lower level—for example, the construction of curvature from a covariant derivative, or the appearance of higher multilinear operations in L_∞-algebras. Horizontal emergence refers to the integration of multiple algebraic structures into a larger structure in which interactions between the component algebras become essential. Semidirect products, matched pairs, and related constructions provide mathematically precise examples.

The distinction is useful because the familiar derived series

$$L^0 = L\,,\quad L^1 = [L, L], \qquad L^2 = [L^1, L^1], \ldots$$

does not by itself capture every way in which algebraic structure can become more elaborate. The derived series produces a nested sequence of subalgebras using the same underlying bracket. By contrast, extensions, deformations, higher brackets, curvature, and interacting combinations of algebras can introduce genuinely new structural relationships and new compatibility conditions.

This provides a possible mathematical analogy for Arthur Koestler's concept of the holon: a structure may retain a degree of autonomy at one level while simultaneously participating in a larger integrative structure. The analogy should not be mistaken for a theorem establishing that Lie algebras are literally holarchies. Rather, it suggests a productive correspondence between a well-defined mathematical phenomenon—structural integration across levels—and a broader philosophical account of organization.

Keywords: Coherence, Gestalt, Holon, Holarchy, Homeostasis, Lie Algebra, Organization, Two-sidedness.

1. Introduction

Lie theory offers one of the clearest mathematical settings in which relations can generate structure that is richer than the isolated elements with which one begins. From matrix groups and their infinitesimal generators to Lie algebras, representations, semidirect products, extensions, deformations, and the geometric structures associated with connections and curvature, the subject provides a remarkably varied account of how mathematical organization can be constructed, combined, and constrained. Standard introductions emphasize different aspects of this richness. Baker's (2006) develops the relation between matrix groups and their Lie-theoretic infinitesimal structure; Erdmann and Wildon's (2006) provide a systematic algebraic treatment of brackets, ideals, representations, and related constructions; and Gilmore's *(2008)* places Lie groups and Lie algebras within their broader geometric and physical applications. Taken together, these approaches reveal that Lie theory is not simply the study of a binary operation satisfying a small collection of axioms. It is a framework in which relations can be nested, transformed, coupled, extended, and incorporated into larger coherent structures.

The present paper develops this observation in a particular direction. It expands a pattern recognized in Smith (2026). That earlier work identified a recurring pre-space-time pattern of two-sided balancing: distinguishable poles or aspects become meaningful through their relation, while their relation can participate in a larger unity. The present paper asks what happens when this pattern is examined more closely within the internal richness of Lie-algebraic mathematics. The aim is not to claim that Lie theory proves an ontology of two-sidedness. Rather, Lie theory provides a rigorous mathematical laboratory in which one can investigate how distinguishable structures retain their identity while becoming components of increasingly integrated organizations.

This perspective is compatible with a broader conception of holonic organization, in which a structure may possess an identifiable organization at one level while simultaneously functioning as a component of a larger organization. Koestler's (1967) concept of the holon provides a useful philosophical expression of this principle: the part is neither merely an isolated entity nor simply dissolved into the whole. It retains a degree of autonomy while participating in a higher-order organization. The analogy developed here is deliberately modest. Lie algebras are not thereby shown to be holarchies, nor does Koestler's philosophical account follow as a theorem from Lie theory. What Lie theory does provide are precise mathematical examples in which this general organizational intuition can be examined without abandoning formal rigor.

The first distinction developed in this paper is between what will be called vertical emergence and horizontal emergence. These terms are introduced as conceptual descriptors rather than as standard classifications within Lie theory. Vertical emergence refers to the construction of new algebraic objects or operations from structures already present at a lower level. Horizontal emergence, by contrast, refers to the integration of distinct algebraic structures into a larger structure in which interactions between the components become essential. The distinction allows us to ask not merely how algebraic structures become more elaborate through nesting, but also how they become more complex through interaction, extension, and integration.

The discussion therefore begins with the derived series as a simple example of vertical nesting. This establishes a baseline for distinguishing mere repetition of an existing operation from the emergence of genuinely new structure. The paper then turns to horizontal integration, where direct sums provide a non-interacting baseline and semidirect products demonstrate how relations between component algebras become part of the organization of the whole. It subsequently considers central extensions, deformations, L_∞-algebras, curvature, and the Bianchi identities as examples in which new structures and compatibility conditions arise. The final sections return to two-sidedness and the proposed middle term, asking how these mathematical constructions may illuminate a broader conception of holonic organization while maintaining a clear distinction between what the mathematics establishes and what remains interpretive.

2. The Derived Series: Vertical Nesting Without New Operations

Let L be a Lie algebra over a field of characteristic zero. Its derived series is defined recursively by $L^0 = L$, $L^1 = [L, L]$, and $L^{n+1} = [L^n, L^n]$.

Thus

$$L \supseteq L^1 \supseteq L^2 \supseteq \cdots$$

Each L^n is a Lie subalgebra of L. The bracket at each stage is not a new bracket; it is the original Lie bracket restricted to the smaller subspace.

This makes the derived series a useful baseline for what may be called vertical nesting. There is genuine structural organization across levels, but the operation responsible for that organization has not itself changed.

The Jacobi identity,

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0,$$

continues to hold because each derived algebra inherits the Lie-algebra structure from L.

Consequently, the derived series by itself does not exhibit the type of emergence considered here. No additional operation is introduced, and no new compatibility principle is required to establish that the successive derived algebras are Lie algebras.

It is therefore useful to distinguish nesting from emergence.

A nested sequence may simply repeat the same structural rule at increasingly restricted levels. Emergence, in the stronger sense used in this essay, occurs when the organization of one level gives rise to a new mathematical object, a new operation, or a new compatibility condition that cannot be identified merely with the original object restricted to a smaller domain.

This distinction is the starting point for the broader framework.

3. Horizontal Integration: From Direct Sums to Interacting Structures

Suppose L_1 and L_2 are Lie algebras. Their direct sum $L_1 \oplus L_2$

is itself a Lie algebra with bracket $[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])$.

The two components are algebraically independent in the sense that

$$[(x_1, 0), (0, y_2)] = 0 \ \ and \ [(x_1, 0), (0, y_2)] = 0.$$

The direct sum therefore provides an example of horizontal aggregation without interaction. Two complete algebraic structures are placed side by side, but the whole contains no additional interaction between them.

The situation becomes more interesting when the two structures are coupled.

For example, if L_1 acts on L_2 by derivations, one can construct the semidirect product

$L_1 \ltimes L_2$. Its bracket has the schematic form $[(x, a), (y, b)] = ([x, y], x \cdot b - y \cdot a + [a, b])$.

The cross terms $x \cdot b - y \cdot a$ are not present in the direct sum. They express the interaction of the two component algebras.

But an important mathematical qualification is necessary: cross terms cannot simply be added arbitrarily. For the resulting operation to define a Lie algebra, antisymmetry and, crucially, the Jacobi identity must be satisfied.

Thus, the Jacobi identity functions as a structural compatibility condition. It determines which proposed interactions are admissible.

This provides a precise mathematical basis for the idea of a horizontal holon. The whole is not merely the juxtaposition of two parts. It contains relations between the parts that belong to the structure of the whole itself.

More sophisticated examples include matched pairs of Lie algebras and Lie bialgebras. In such constructions, the interaction between component structures is itself mathematically organized rather than being an arbitrary addition.

The conceptual point is therefore not that every cross-term produces emergence. Rather: Horizontal emergence occurs when independently identifiable structures become components of a larger algebraic structure whose defining relations include interactions between those components.

The whole remains analyzable in terms of its parts, but it is not exhausted by their independent algebraic descriptions.

4. Vertical Emergence: Extensions, Deformations, and Higher Operations

A different kind of structural development occurs when a new mathematical object is constructed from an existing algebra.

Several standard constructions illustrate this possibility.

4.1 Central Extensions

Given a Lie algebra L, a central extension introduces an additional central direction. At the vector-space level one may write

$$\tilde{L} = L \oplus \mathbb{F}z$$

where z is central. A modified bracket can take the form

$$[x, y]_{\tilde{L}} = [x, y]_L + \omega(x, y)z.$$

Here ω is a bilinear alternating map. For the new bracket to satisfy Jacobi, ω must satisfy a cocycle condition.

This is an important example because the additional structure is not obtained simply by restricting the old algebra. A new direction has been introduced, while the consistency of that extension is constrained by an additional mathematical condition.

The appropriate interpretation, therefore, is not that the Jacobi identity itself has somehow appeared from nowhere. Rather, the extension introduces a new compatibility problem whose solution is governed by Lie-algebra cohomology.

4.2 Deformations

A deformation modifies the bracket continuously. Formally one may write

$$[x, y]_t = [x, y]_0 + t\varphi_1(x, y) + t^2\varphi_2(x, y) + \cdots .$$

The requirement that $[\cdot,\cdot]_t$ remain a Lie bracket imposes the Jacobi identity at every order in t.

At first order, this leads to a cocycle condition on φ_1. At higher orders, additional compatibility conditions arise.

Deformation theory therefore illustrates a particularly clear form of structural emergence:

$$existing\ bracket \longrightarrow modified\ bracket \longrightarrow compatibility\ conditions.$$

The new structure is constrained by the algebraic consistency of the whole deformation.

4.3 L_∞-Algebras

An even more striking generalization is provided by L_∞-algebras.

Instead of having only one binary operation

$$l_2(x, y),$$

an L_∞-algebra contains a sequence of multilinear operations

$$l_1,\ l_2,\ l_3, \ldots$$

subject to a hierarchy of generalized Jacobi, or homotopy-Jacobi, identities.

These operations should not be regarded simply as successive copies of the original Lie bracket. They represent a genuinely richer algebraic structure in which the failure of one identity to hold strictly can itself be controlled by a higher operation.

This makes L_∞-algebras especially suggestive for the present framework. There is a hierarchy of operations together with a hierarchy of compatibility relations.

Nevertheless, the phrase **“**each level regulates the next**”** should be understood metaphorically rather than as a standard theorem about L_∞-algebras. The precise statement is that the higher operations participate in a coherent system of generalized Jacobi identities.

5. Curvature: A Geometric Example of Vertical Construction

The most important geometric example is the Riemann curvature tensor.

Let ∇ be a connection on a manifold. The curvature operator is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z \; - \nabla_{[X,Y]} Z.$$

Equivalently,

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

This formula deserves careful interpretation.

The expression

$$[\nabla X, \nabla Y]$$

is a commutator of differential operators, whereas

$$[X,Y]$$

is the Lie bracket of vector fields. The curvature operator combines these structures with the connection in a way that measures the failure of covariant differentiation to commute in the manner determined solely by the Lie bracket of vector fields.

Curvature is a tensor derived from the connection. This can be interpreted as constructed from lower-level geometric structure, namely the manifold, vector fields, and connection. In this limited and precise sense, it provides a good example of what this essay calls vertical emergence.

It would be too strong, however, to say that curvature is literally a “new Lie bracket.” The curvature tensor is a tensorial operator derived from a connection; it is not itself simply another Lie bracket on the original vector-field space.

The distinction is important because it allows the conceptual argument to remain ambitious without making a mathematically incorrect identification.

6. The Bianchi Identities as Higher-Level Compatibility Relations

For a torsion-free connection, the first Bianchi identity is

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0.$$

This cyclic relation expresses an important algebraic constraint on the curvature tensor.

The second Bianchi identity is

$$(\nabla_X R)(Y,Z) + (\nabla_Y R)(Z,X) + (\nabla_Z R)(X,Y) = 0,$$

in the appropriate tensorial notation.

These identities are not additional arbitrary assumptions appended to curvature. They arise from the underlying differential-geometric structure and provide powerful consistency relations.

This makes the Bianchi identities useful analogues of higher-level regulators in the present conceptual vocabulary (interpretive and not standard differential-geometric language).

Again, precision matters. The Bianchi identities should not be described simply as new versions of the Jacobi identity. There are, however, deep structural relationships between the Jacobi identity for commutators of differential operators and the identities satisfied by curvature.

The useful conceptual correspondence is therefore:

$$connection \longrightarrow curvature \longrightarrow Bianchi\ compatibility.$$

The later structure depends upon the earlier structure, while introducing mathematical relations that become visible only after the new object—curvature—has been constructed.

7. A Unified Structural Picture

The preceding examples suggest two complementary directions in which algebraic complexity can develop.

Horizontal structure

Horizontal organization combines distinct structures:

$$L_1,\ L_2 \longrightarrow L_1 \oplus L_2$$

and, when interactions are introduced,

$$L_1,\ L_2 \longrightarrow L_1 \ltimes L_2,$$

or more general interacting constructions.

The significant feature is interaction between components.

Examples include:

- direct sums as the non-interacting baseline;
- semidirect products;
- matched pairs;
- Lie bialgebras;
- other coupled algebraic structures.

Vertical structure

Vertical organization constructs new mathematical objects from existing ones:

$$algebraic\ structure \longrightarrow new\ operation/object \longrightarrow new\ compatibility\ relations.$$

Examples include:

- central extensions;
- deformations;
- L_∞-algebras;
- connections and curvature;

- curvature and Bianchi identities.

The two directions need not be independent. A sophisticated mathematical structure may exhibit both simultaneously.

One can therefore represent the conceptual picture schematically as

$$\boxed{\text{parts} \xrightarrow{\text{horizontal interaction}} \text{integrated whole}}$$

And

$$\boxed{\text{existing structure} \xrightarrow{\text{vertical construction}} \text{new structure} \xrightarrow{\text{compatibility}} \text{higher organization.}}$$

This distinction provides a more precise version of the intuition that mathematical organization can develop both across structures and through levels of structure.

8. Relation to Koestler's Holarchy

Arthur Koestler's concept of the holon describes entities that possess a degree of autonomy while simultaneously functioning as components of larger organizations.

The mathematical constructions considered here provide useful analogies.

A direct sum illustrates components that remain largely independent. A semidirect product demonstrates that the components can retain identifiable structure while becoming dynamically related to one another. A central extension introduces additional structure while preserving the original algebra as a recognizable component. Curvature is constructed from a connection but is not reducible to the connection considered merely as a list of its local coefficients. L_∞-algebras extend the idea further by organizing several levels of multilinear operation within a single coherent structure.

These examples suggest a mathematical vocabulary for discussing a principle that is central to holarchic thinking: A whole may preserve the identity of its components while possessing relations and properties that belong specifically to the organization of the whole.

This should be regarded as a structural analogy, not as a proof that Koestler's philosophical theory follows from Lie theory.

The value of the correspondence lies instead in the fact that Lie theory supplies precise examples in which local algebraic identities, interactions, extensions, and higher compatibility conditions coexist within a larger organization.

9. Two-Sidedness and the Middle Term

The relation is important because two arbitrary sides do not become meaningful as a system merely by being placed next to one another.

Whenever relational structure becomes expressible, two-sidedness emerges as a necessary feature of the relation, even though the two sides may be indistinguishable at the non-relational ontological limit. The apparent mystery arises when two-sidedness is interpreted as a claim about two fundamentally distinct substances. Ontological two-sidedness proposes something different: at the non-relational limit, the two sides are not fundamentally distinguishable. Distinguishability arises with relation itself. Thus, two-sidedness is not proposed as a catalogue of fundamental entities but as a mode of inquiry into how relational structure becomes organized. Hence, the interpretation grounds an understanding of distinguishability (or discernment) that is brought on by relation that underpins a holonic organization.

$$two\ distinguishable\ sides + relation \longrightarrow organized\ whole$$

$$non - relational\ identity \longrightarrow relational\ distiction \longrightarrow holonic\ organization$$

In standard physics and cosmology, the prevailing assumption is that something must have happened to all the antimatter in the early universe, because today we observe only matter. The usual explanation is that some process tipped the balance, causing the primordial antimatter to annihilate and leaving behind the matter-dominated cosmos we see.

But this interpretation overlooks a deeper possibility: what we observe may be only a relation, not a literal inventory of separate substances. The apparent “absence” of antimatter could instead reflect the fact that the two sides of a deeper two-sided structure present themselves as a unified whole.[1]

[1] Boyle, Finn and Turok's (2018) proposed a physical cosmological model in which CPT symmetry, and that motivates a second cosmic sheet (see Boyle 2022). However, ontological two-sidedness uses the two-sided structure at a more fundamental philosophical/variational level: the two sides are not primarily two cosmic histories, but two aspects whose relation and subsequent sublation into a unified whole is itself ontologically significant (e.g., Smith 2021).

In this interpretation, experimentally produced antimatter need not be regarded as evidence for a second, ontologically separate inventory of substance. It may instead represent a locally distinguishable expression of the same deeper relational structure whose two-sidedness becomes apparent only through interaction. Therefore, the empirical distinction between matter and antimatter does not by itself establish that they are fundamentally separate ontological substances. Moreover, when observed antimatter encounters observed matter, the two can annihilate, suggesting—within this interpretation—that a locally inverted state tends to return to the underlying relation from which the distinction emerged.

If this alternative perspective is correct, then the matter-dominated universe is simply the unified expression of both sides, and the rich structure we observe becomes a natural consequence of that underlying relational unity.

"*My Wife and My Mother-in-Law***"** becomes a compact visual metaphor for the theory of two-sidedness because the drawing forces perception to oscillate between two incompatible yet co-present interpretations. The young woman and the old woman are not blended; they are *structurally entangled*, each emerging only when the viewer's attention shifts its stance. This is precisely the mode of inquiry that the theory reduces to: a disciplined ability to hold two ontic presentations at once, letting the mind toggle between them until a deeper gestalt order reveals itself. In this sense, the illusion models how a homeostatic balance between perspectives can generate coherence — not by collapsing difference, but by sustaining the tension that allows hidden structure to surface.

In the algebraic examples above, the middle term is represented by different mathematical structures depending upon the context:

- an action in a semidirect product;
- a cocycle in a central extension;
- deformation terms in a deformed bracket;
- higher operations in an L_∞-algebra;
- a connection in differential geometry;
- curvature as an expression of the relation between covariant differentiations.

Calling such a relation a middle term is philosophical terminology rather than standard Lie-algebra terminology. Nevertheless, it points toward something mathematically concrete: the organization of a whole is often encoded not simply in its components but in the relations and compatibility conditions connecting them.

This is perhaps the most defensible mathematical interpretation of the proposed homeostatic[2] idea.

10. What the Mathematics Does—and Does Not—Establish

The mathematical examples establish several things with reasonable confidence.

First, Lie theory contains many constructions in which a new structure is generated from an existing one.

Second, algebraic integration can produce structures that contain identifiable component algebras together with additional interaction terms.

Third, the consistency of these constructions is often controlled by identities or compatibility conditions such as Jacobi, cocycle conditions, Maurer–Cartan-type equations, and Bianchi identities.

Fourth, richer algebraic systems such as L_∞-algebras demonstrate that mathematical organization need not terminate with a single binary operation.

What the mathematics does not establish is that these structures prove an ontological theory of two-sidedness, consciousness, gravitation, or cosmic holarchy. Nor does it establish that Jacobi identities or Bianchi identities are literally physical homeostats.

Those are further interpretive hypotheses.

The appropriate claim is therefore more modest and, arguably, more useful:

Lie theory provides a mathematically rigorous laboratory in which one can study how structure is preserved, coupled, extended, transformed, and regulated across multiple levels. These phenomena offer a possible formal analogy for a broader theory in which two-sided organization and an integrative middle term play fundamental roles.

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[2] Ashby's (1947) homeostat provides a cybernetic precedent for treating organization as a relation that regulates distinguishable states of a larger system. My use of "two-sided balancing" extends this structural intuition: rather than treating the two sides as independent substances, they are understood as distinguishable poles whose relation is constitutive of the organized whole. The homeostat therefore functions as a middle term—not necessarily as a third component, but as the organizing relation through which the two sides remain distinct while participating in a coherent system.

This leaves open the possibility that the analogy could eventually become more than an analogy, but it does not assume that conclusion in advance.

11. From Two-Sidedness to Gestalt and the Missing Middle Term

Consider a relation that can be extended recursively, both horizontally across distinct elements and vertically through successive levels of nesting. The first genuinely non-trivial case occurs when three elements are nested into a binary relational expression:

$$[x,[y,z]],[y,[z,x]],[z,[x,y]],$$

At this level, the relation is no longer merely dyadic. The three possible nestings must find a common coherence, expressed mathematically by the Jacobi identity,

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0.$$

Ordinarily, the Jacobi identity is presented as a formal axiom of a Lie algebra. In the framework proposed here, however, it can be interpreted differently. It is the local trace of a more general requirement for global coherence, that can only be described qualitatively rather than formally as the following will demonstrate.

The crucial distinction is between the two levels. At the local level, we encounter the individual binary interactions. But the three nested possibilities cannot be understood adequately in isolation. They belong to a larger relational configuration—the first non-trivial gestalt in which the parts must simultaneously accommodate one another.

The Lie bracket, $[x,y]$, represents the interaction of two generators, x and y, and is itself a generator of the algebra. The broader interactions therefore retain a mirror-like template expressed by antisymmetry, $[x,y]=-[y,x]$. The significance of this relation is not simply that the bracket changes sign when the order of its arguments is reversed. It establishes a template for relational consistency: the interaction viewed from one side and the interaction viewed from the other are not independent, but are constrained by one another. In this limited sense, the bracket can be regarded as an error-checking or error-correcting template—not in the technical sense of an error-correcting code, but in the sense that the structure itself constrains which relational expressions can be mutually consistent. Analogous principles of constrained relational organization can be found in other domains, including, by analogy, DNA-based error correction.

The antisymmetry of the bracket also implies $[x,x]=0$, so that when the two arguments are indistinguishable, their antisymmetric interaction vanishes. Bilinearity extends this relational symmetry through sign reversal: $-[x,y]=[-x,y]=[x,-y]$.

Thus, the bracket establishes a two-sided relational template in which distinction and opposition are inseparable from the relation itself.

When three generators x, y, and z interact, however, this binary template is no longer sufficient by itself. The interaction of x with y can itself enter into a further interaction with z, producing expressions such as $[x,[y,z]]$, along with the corresponding permutations $[y,[z,x]], [z,[x,y]]$.

At this level, a greater coherence is required. The three-way organization cannot be obtained merely by considering each binary interaction independently. The Jacobi identity imposes a constraint upon the possible combinations of these nested relations. The important point is that this higher-order balance does not replace the binary template. It constrains it. The original two-sided structure expressed by the antisymmetric bracket remains present at the lower level, while the Jacobi identity expresses a coherence that emerges when those relations participate in a larger configuration.

In this sense, the balance remains fundamentally two-sided even when it appears at a higher level of organization. The higher-order relation does not introduce a third substance that stands between two others. Rather, it regulates the way the two-sided relational template can participate in a larger whole.

This is also where the formal mathematical description reaches an important boundary. The bracket and the Jacobi identity can be stated precisely, but the question of why the larger configuration should be understood as a coherent whole rather than merely as a collection of binary relations is no longer answered by adding another algebraic component. The Lie algebra must take the bracket and its Jacobi constraint together as part of its foundational structure.

The qualitative interpretation proposed here therefore does not claim that some additional mathematical object has been discovered beyond the Jacobi identity. Rather, it asks whether the formal relationship between bracket and Jacobi identity provides a precise mathematical trace of a more general organizing principle: one in which local two-sided relations remain identifiable while being constrained by the coherence of the larger structure.

The point is not that the whole can be obtained by simply adding another part to the parts. It is that the organization of the parts imposes constraints upon the relations through which those parts become a whole. In that sense, the “middle term” is better understood not as a third component, but as the organizing relation that remains implicit in the transition from local interaction to global coherence.

The remarkable possibility is therefore that Jacobi is simultaneously local and global. Locally, it appears as an algebraic constraint on three nested binary relations. Globally, it can be interpreted as the residue—the mathematical trace—of a balancing process in which two sides have been sublated into a coherent unity.

This provides a possible bridge between reductionism and holism. Reductionism is not rejected: the local relations remain real and mathematically describable. But they are no longer treated as ontologically self-sufficient. The local structure is understood as the manifestation of a larger organization. Conversely, holism need not become an appeal to an inexplicable whole: its influence appears in the local mathematics as a constraint on how relations can coherently combine.

The resulting picture is therefore neither parts first nor whole first, but a recursive relation between them:

$$\boxed{\text{two-sided relation} \to \text{nested relation} \to \text{tension} \to \text{balancing} \to \text{coherent unity}}$$

and then, recursively,

$$\boxed{\text{coherent unity} \to \text{new relational level} \to \text{new two-sidedness} \to \text{new balancing.}}$$

The middle term is consequently not another component that can simply be added to the parts. It is the regulative relation that allows the parts to become a whole without ceasing to be distinct. The whole is more than the sum of its parts precisely because the relation that organizes the parts is not itself reducible to any one of them.

In this interpretation, the Jacobi identity becomes especially significant. It is not being proposed merely as another mathematical rule, but as a candidate example of how global coherence can leave a precise mathematical footprint at the local level. The mathematics stops where formal relations have been established; what remains is the question of the balancing relation that makes those formal relations coherent as a gestalt.

That is the missing link in a formalism that demands closure with the greater whole:

$$\boxed{\text{local mathematics} \;\longleftrightarrow\; \text{homeostatic middle term} \;\longleftrightarrow\; \text{global gestalt.}}$$

The proposal is therefore not that mathematics fails, but that mathematics may describe the trace of coherence without being identical to the source of coherence itself.

12. Conclusion

The classical derived series provides a simple model of algebraic nesting:

$$L \supseteq L^1 \supseteq L^2 \supseteq \cdots.$$

It demonstrates organization across levels, but it does not introduce a fundamentally new operation at each stage.

Other constructions reveal a richer picture. Horizontal integration can combine distinct algebras through nontrivial actions and cross-relations. Vertical construction can produce extensions, deformations, higher brackets, curvature, and associated compatibility conditions. The Jacobi identity, cocycle conditions, higher homotopy identities, and Bianchi identities illustrate different forms of mathematical constraint.

The resulting picture can be summarized as follows:

$$\boxed{\text{Derived series nested structure}} \quad \longrightarrow \quad \boxed{\text{Vertical construction new mathematical structure}}$$

While

$$\boxed{L_1,\ L_2,\dots} \quad \longrightarrow \quad \boxed{\text{interacting whole}}.$$

Together these provide a useful conceptual framework for thinking about holonic emergence in mathematical structure.

The broader philosophical proposal is that what appears empirically as a collection of structures may represent only the visible trace of a richer organization whose relational or “middle-term” regulated structure is less immediately apparent. Such a proposal is not speculative once a broader evidential base is recognized in the qualitative sense that the whole cannot be explained by its parts. However, the value of a mathematical formality depends upon whether the conceptual correspondence developed here can eventually be translated into explicit mathematical models producing testable consequences; e.g., might eventually provide a route toward connecting Lie brackets with Friston’s (2010) Markov blanket that supports a template that permits active inference while regulating itself according to the *free energy principle* (Friston et al. 2023). Otherwise, the formality ends with the Jacobi identity, and a more qualitative analysis begins.

For the present, the mathematics supports a more limited but still significant conclusion: structure in Lie theory is not exhausted by simple nesting or aggregation. There are rigorous mechanisms by which new relations, operations, compatibility conditions, and higher levels of organization arise. These mechanisms provide a legitimate mathematical setting in which the intuition behind vertical and horizontal holonic emergence can be explored without requiring the speculative interpretation to be accepted in advance.

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