Zeta Functions on a Self-Similar Set:
What $-1/5$ Is, What It Isn't, and What Remains Open

Fabrizio Vassallo

vassalloff@gmail.com

September 2026

Abstract

A 2020 note [1] observed that the self-similar set F – the positive integers n for which $n-1$, in base 3, uses only the digits 0 and 2 – satisfies a telescoping identity that, read as a fixed-point equation, assigns the divergent sum $S = \sum_{n \in F} n$ the value $-1/5$. This paper reports what survives close scrutiny of that claim and what does not. We show rigorously, via the self-similarity $F = 3F \cup (3F-2)$ and the functional equation this induces on the theta function $\Theta(t) = \sum_{n \in F} e^{-nt}$, that the arithmetic zeta function $\zeta_F(s) = \sum_{n \in F} n^{-s}$ satisfies $\zeta_F(0) = 0$ and hence $\zeta_F(-1) = 0$: the zeta-regularized value of S is 0, not $-1/5$. We then construct the hierarchical graph G_k that the same self-similarity naturally generates, obtain its Laplacian spectrum in closed form by an exact backbone/localized-state decomposition (verified independently, both analytically and numerically, up to $k = 6$), and show that its spectral zeta function satisfies $\zeta_{G_k}(0) = N_k - 1$ identically for every connected graph on N_k vertices – a positive, divergent integer sequence of exact integer values, one per k – ruling out the specific identification made in the exploratory work, though not every conceivable regularization built from this same sequence (Section 3.3 discusses a different one that still produces $-1/5$, and why it is not the spectral zeta function of any single G_k). This directly falsifies an intermediate claim, made in the exploratory work this paper grew out of, that identified $-1/5$ with this spectral zeta function. We then examine, by an independent route (a Perron-type explicit formula expressing the counting and summatory functions of F as a sum over ζ_F's poles), whether a genuinely distinct summation functional – in Ramanujan's sense, not simply zeta regularization, and adapted to F's fractal structure rather than borrowed from the integers – might still recover $-1/5$. Conditional on a growth estimate we do not fully establish, this route shows that no constant of any kind survives in the asymptotics of F's counting or summatory function, confirming by a second, independent method that 0, not $-1/5$, is what F's own structure naturally produces; we verify the underlying machinery against an exact, parameter-free identity forced by $A(3^m) = 2^m$. One loophole is identified and left open – a Ramanujan-type summation of F's enumeration index, rather than its values, trivially recovers $-1/2$ but arguably encodes nothing about F specifically – and we explain why resurgence and alien calculus, sometimes proposed as a further alternative, have no natural point of contact with this problem. Finally, we discuss why the relevant physical motivation (zeta-regularized vacuum energies, and the empirically rich sign-dependence of Casimir–Lifshitz forces) is exactly that – motivation, not evidence.

Contents

1 Introduction

The starting point is a small, self-contained arithmetic fact, first recorded informally in [1]. Let

$$F = \{1, 3, 7, 9, 19, 21, 25, 27, 55, 57, 61, 63, \dots\}$$

be the set of positive integers n such that $n-1$, written in base 3, uses only the digits 0 and 2. F is self-similar: writing $3F = \{3n : n \in F\}$ and $3F - 2 = \{3n - 2 : n \in F\}$, one has the disjoint union $F = 3F \cup (3F - 2)$. Summing formally, and writing $S = \sum_{n \in F} n$, $N = \sum_{n \in F} 1$, this gives $S = 3S + (3S - 2N) = 6S - 2N$. If one is willing to assign N the value $-1/2$ – the value physicists routinely attach to $1 + 1 + 1 + \cdots$ – this becomes the fixed-point equation $S = 6S + 1$, whose unique solution is

$$S = -\frac{1}{5}.$$

This is a pleasant, throwaway curiosity as originally intended. Two extended sessions with other AI systems, starting from exactly this observation, tried to make it precise and, in the process, generated both real mathematics and an increasingly speculative chain of physical interpretations, built on an intermediate claim – that $-1/5$ is literally the spectral zeta function of a specific graph – that turns out not to survive rigorous checking. That speculative physical thread is set aside here, for now, in favor of reporting only what does survive scrutiny.

This paper keeps only what survives. Section 2 derives the arithmetic zeta function of F rigorously and shows its regularized value is 0, not $-1/5$. Section 3 constructs the hierarchical graph F's self-similarity naturally generates, solves its spectrum exactly, and shows in one line why its spectral zeta function can never equal $-1/5$ either – falsifying the specific claim that motivated much of the speculative material above. Section 4 takes the resulting puzzle seriously rather than declaring it closed: since neither natural regularization reproduces the original claim, we state precisely what would be needed for a different, legitimate summation method to do so, and why the standard physicists' shortcut for $N = -1/2$ does not simply transfer to this setting. Section 5 asks what a genuine *geometric*, rather than combinatorial, realization of the recursive structure would require. Section 6 closes with the one connection to the author's broader NATND program [2, 3] that appears genuinely promising, and is left as a stated future direction rather than a claimed result.

2 The Self-Similar Set F and its Arithmetic Zeta Function

2.1 Self-similarity and the fractal dimension [DERIVED]

Proposition 2.1 (Self-similarity)**.** $F = 3F \cup (3F - 2)$, *and the union is disjoint.*

This is immediate from the digit characterization: $n \in F$ iff $n - 1$ has base-3 digits in $\{0, 2\}$ only; prepending a digit 0 or 2 to $n-1$'s expansion gives $3(n-1) + 0 = 3n - 3$, i.e. the element $3n - 2 \in F$, or $3(n-1) + 2 = 3n - 1$, i.e. the element $3n \in F$ (indices shifted by the $+1$ convention), and every element of F other than 1 arises this way from exactly one $n \in F$. Independently of the digit characterization, Proposition 2.1 was verified computationally by generating F purely from the recursion (starting from $\{1\}$) and confirming it reproduces the digit-based set exactly (Listing 2).

Proposition 2.2 (Counting function and dimension)**.** *Let* $A(x) = \#\{n \in F : n \leq x\}$. *Then* $A(3^k) = 2^k$ *exactly for every* $k \geq 0$, *and consequently the abscissa of convergence of* $\zeta_F(s) = \sum_{n \in F} n^{-s}$ *is*

$$D = \frac{\ln 2}{\ln 3} = 0.6309297536\ldots$$

This follows directly from Proposition 2.1: each of the 2^{k-1} elements of F up to 3^{k-1} spawns exactly two elements of F up to 3^k (via $n \mapsto 3n - 2$ and $n \mapsto 3n$), giving $A(3^k) = 2A(3^{k-1})$ and $A(1) = 1$. Confirmed numerically (exactly, not asymptotically) in Listing 2 for $k = 5, 8, 11, 14$.

2.2 The theta function and its functional equation [DERIVED]

Proposition 2.3 (Exact functional equation)**.** *Let* $\Theta(t) = \sum_{n \in F} e^{-nt}$ *for* $t > 0$. *Then*

$$\Theta(t) = \left(1 + e^{2t}\right) \Theta(3t).$$

Proof. By Proposition 2.1, splitting the sum over $F = 3F \cup (3F-2)$ gives $\Theta(t) = \sum_{n\in F} e^{-3nt} + \sum_{n\in F} e^{-(3n-2)t} = \Theta(3t) + e^{2t}\Theta(3t) = (1+e^{2t})\Theta(3t)$. $\square$

This identity was checked independently to machine precision ($|\text{relative difference}| < 2\times 10^{-16}$) for $t \in \{1, 0.5, 0.2, 0.1, 0.05, 0.02, 0.01\}$ using F truncated at 14 generations ($|F| = 16384$ elements, up to $n \approx 4.8\times 10^6$), far beyond what any of the tested t values require for convergence (Listing 2).

2.3 The value at $s = 0$ [DERIVED]

Lemma 2.4 (Growth of Θ)**.** *$\Theta(t) \le 2\Gamma(D+1)\,t^{-D}$ for $0 < t \le 1$, and $\Theta(t) \le e^{-(t-1)}\Theta(1)$ for $t \ge 1$.*

Proof. For $t \ge 1$: since every $n \in F$ satisfies $n \ge 1$, $e^{-n(t-1)} \le e^{-(t-1)}$ for every such n, so $\Theta(t) = \sum_{n\in F} e^{-n}e^{-n(t-1)} \le e^{-(t-1)}\sum_{n\in F} e^{-n} = e^{-(t-1)}\Theta(1)$.

For $0 < t \le 1$: write $\Theta(t) = \int_{1^-}^{\infty} e^{-tx}\,dA(x)$ (a Riemann–Stieltjes integral; A jumps by 1 at each point of F, so this is literally $\sum_{n\in F} e^{-nt}$). Integrating by parts, using $A(1^-) = 0$ and $e^{-tX}A(X) \to 0$ as $X \to \infty$ (since $A(X) = O(X^D)$ by Proposition 2.2, while $e^{-tX} \to 0$ exponentially for fixed $t > 0$),

$$\Theta(t) = t\int_1^\infty e^{-tx}A(x)\,dx \le t\int_1^\infty e^{-tx}\cdot 2x^D\,dx \le 2t\int_0^\infty e^{-tx}x^D\,dx = 2\Gamma(D+1)\,t^{-D},$$

using $A(x) \le 2x^D$ for $x \ge 1$ (Proposition 2.2: if $3^k \le x < 3^{k+1}$ then $A(x) \le A(3^{k+1}) = 2^{k+1} \le 2x^D$, since $x^D \ge (3^k)^D = 2^k$) and the standard Gamma integral $\int_0^\infty e^{-tx}x^D\,dx = \Gamma(D+1)t^{-(D+1)}$. $\square$

Theorem 2.5. *$\zeta_F(-n) = 0$ for every integer $n \ge 0$. In particular $\zeta_F(0) = 0$ and $\zeta_F(-1) = 0$.*

Proof. Step 1: Mellin split. For $\Re(s) > D$, $\Gamma(s)\zeta_F(s) = \int_0^\infty t^{s-1}\Theta(t)\,dt = M(s) + I_2(s)$, where $M(s) = \int_0^1 t^{s-1}\Theta(t)\,dt$ and $I_2(s) = \int_1^\infty t^{s-1}\Theta(t)\,dt$. By Lemma 2.4's bound for $t \ge 1$, I_2 converges absolutely and locally uniformly for every $s \in \mathbb{C}$, so I_2 is entire. By the same lemma's bound for $t \le 1$, $M(s)$ is holomorphic on its stated domain $\Re(s) > D$ (the integral converges there, and only there, by comparison with $\int_0^1 u^{\Re(s)-D-1}du$).

Step 2: a self-consistent relation for M. Using Proposition 2.3 and substituting $u = 3t$,

$$M(s) = \int_0^1 t^{s-1}(1+e^{2t})\Theta(3t)\,dt = 3^{-s}\int_0^3 u^{s-1}(1+e^{2u/3})\Theta(u)\,du.$$

Split the last integral at $u = 1$: the piece over $(1,3)$, $J(s) := \int_1^3 u^{s-1}(1+e^{2u/3})\Theta(u)\,du$, is entire (finite interval, bounded continuous integrand). For the piece over $(0,1)$, expand $e^{2u/3} = \sum_{m\ge0}(2/3)^m u^m/m!$ – an everywhere-convergent power series, unlike the binomial series of Remark 2.8 – and integrate term by term (justified for $\Re(s) > D$ by absolute convergence, using Lemma 2.4 to bound each term):

$$\int_0^1 u^{s-1}(1+e^{2u/3})\Theta(u)\,du = 2M(s) + \sum_{m\ge1}\frac{(2/3)^m}{m!}M(s+m).$$

Collecting terms gives the functional relation

$$M(s)\big(1 - 2\cdot 3^{-s}\big) = 3^{-s}\Big[J(s) + \sum_{m\ge1}\frac{(2/3)^m}{m!}M(s+m)\Big], \tag{1}$$

valid, by this derivation, for $\Re(s) > D$, where every term $M(s+m)$ on the right is given directly by its defining convergent integral (Step 1), since $\Re(s+m) > D+1$.

Step 3: meromorphic continuation, strip by strip. (1) was derived assuming every $M(s+m)$, $m \geq 1$, comes from the original convergent integral – true only for $\Re(s) > D-1$. Rather than assert any global growth bound on M across all of $\mathbb{C}$ to get around this (which would presuppose the very continuation being constructed), we extend the domain of (1) one unit-width strip at a time. Suppose M has already been shown holomorphic on an open neighborhood of $\{\Re(s) > D - N\}$ for some integer $N \geq 0$ ($N = 0$ being Step 1's domain). For $\Re(s) > D-(N+1)$ and every $m \geq 1$, $\Re(s+m) > D-(N+1)+m \geq D-N$, so every term $M(s+m)$ on the right of (1) already lies in the previously-established domain. By Lemma 2.4, $|M(w)| \leq 2\Gamma(D+1)/(\Re(w)-D)$ for $\Re(w) > D$, giving, for s ranging over any compact $K \subset \{\Re(s) > D-(N+1)\}$ and m large enough that $\Re(s+m) > D$ throughout K, the explicit uniform bound $|M(s+m)| \leq 2\Gamma(D+1)/(\min_K \Re(s)+m-D) = O(1/m)$; only finitely many small m can fail this (those with $\Re(s+m) \leq D$ somewhere on K), and each such term is a fixed holomorphic function, hence bounded on the compact set K by continuity. Together with the coefficients $(2/3)^m/m!$, this dominates the series in (1) locally uniformly on $\{\Re(s) > D-(N+1)\}$, so the right-hand side is holomorphic there; dividing by the entire, nowhere-vanishing-except-at-s_k factor $1 - 2\cdot 3^{-s}$ extends M meromorphically to $\{\Re(s) > D-(N+1)\}$, holomorphic away from the points $s_k = D + 2\pi i k/\ln 3$ (Section 2.4). By induction on N, M admits a meromorphic continuation to all of $\mathbb{C}$, with possible poles only at the s_k – no assertion that these actually are poles is needed or made here.

Step 4: holomorphy at the nonpositive integers, by induction on n. We show M (hence $\Gamma(s)\zeta_F(s) = M(s) + I_2(s)$) is holomorphic on an open neighborhood of $s = -n$ for every integer $n \geq 0$.

$n = 0$: for s near 0 and every $m \geq 1$, $\Re(s+m)$ remains close to $m \geq 1 > D$, so every $M(s+m)$ in (1) comes directly from Step 1's domain, uniformly for s in a small neighborhood of 0; the series converges locally uniformly there by Lemma 2.4 as in Step 3. Since $1 - 2\cdot 3^0 = -1 \neq 0$, M is holomorphic on this neighborhood of $s = 0$.

$n \geq 1$: assume M is holomorphic on neighborhoods of $0, -1, \ldots, -(n-1)$. For s near $-n$ and $m = 1, \ldots, n$, $s+m$ is near $-n+m \in \{-(n-1), \ldots, 0\}$, where M is holomorphic by the induction hypothesis; for $m \geq n+1$, $\Re(s+m)$ remains close to $m-n \geq 1 > D$, the original domain. So, for s in a small enough neighborhood of $-n$, every term $M(s+m)$ is a well-defined holomorphic function of s there, and the series converges locally uniformly by the same argument as before. Since $1 - 2\cdot 3^{-(-n)} = 1 - 2\cdot 3^n \neq 0$ for every integer $n \geq 1$, M is holomorphic on this neighborhood of $s = -n$.

Step 5: the zeros. $\Gamma(s)$ has a simple pole at $s = -n$ ($n \geq 0$) with nonzero residue $(-1)^n/n!$, so $1/\Gamma(s)$ has a simple zero there. Since $\Gamma(s)\zeta_F(s) = M(s)+I_2(s)$ is holomorphic (finite) at $s = -n$ by Step 4, write $\Gamma(s)\zeta_F(s) = V_n + O(s+n)$ near $s = -n$; then

$$\zeta_F(s) = \big(\Gamma(s)\zeta_F(s)\big) \cdot \frac{1}{\Gamma(s)} = (V_n + O(s+n)) \cdot \big(c_n(s+n) + O((s+n)^2)\big) \xrightarrow[s\to -n]{} 0,$$

so $\zeta_F(-n) = 0$, *regardless of the value of V_n*. (The same argument fails for the ordinary Riemann zeta function precisely because its analogous Mellin transform, built from the Jacobi theta function, has a genuine pole at $s = 0$ – coming from a nonzero constant term in Jacobi's small-t theta expansion – so $\Gamma(s)\zeta(s)$ is *not* holomorphic there, and no cancellation forces $\zeta(0)$ to vanish. The mechanism here is that no such pole occurs for ζ_F at any nonpositive integer, which Step 4 establishes directly.) □

Corollary 2.6. *$\zeta_F(-1) = 0$, consistent with $\zeta_F(-1) = \frac{2}{5}\zeta_F(0)$.*

Remark 2.7 (Independent numerical check, not part of the proof)**.** *Theorem 2.5 gives $\zeta_F(-n) = 0$ directly, with no separate argument needed for $n \geq 1$. As an independent check, the same conclusion for $n = 1, \ldots, 8$ was also verified mechanically with exact rational arithmetic from the finite-truncation property of the binomial-form functional equation of Remark 2.8 at negative integers (Listing 2): at $s = -n$, that equation's right-hand side truncates to finitely many terms referencing only $\zeta_F(-(n-1)), \ldots, \zeta_F(0)$, all zero by (downward) induction. This is a consistency check on Theorem 2.5, not an alternative route to it – it presupposes the meromorphic continuation that Steps 3–4 establish.*

Remark 2.8 (What the shortcut alone does not prove)**.** *Substituting $s = 0$ into the binomial form $(1-2\cdot 3^{-s})\zeta_F(s) = \sum_{k\geq 1} \binom{s+k-1}{k} 2^k 3^{-s-k}\zeta_F(s+k)$ and noting $\binom{k-1}{k} = 0$ for every $k \geq 1$ gives the right answer,* but on its own*, before Theorem 2.5's Steps 3–4 have been established, this is not a proof: it is only legitimate to evaluate a functional equation at a boundary point of its original domain of derivation ($\Re(s) > D$) once that equation has independently been shown to persist as an identity of analytic functions in an actual neighborhood of that point – which is exactly the content of those steps, not something the termwise vanishing at the single point $s = 0$ establishes by itself. We flag this explicitly because the free-standing version of this shortcut is exactly what produces the unjustified step addressed in Section 4.*

2.4 Complex dimensions [OPEN] (partially resolved)

The functional equation forces $\zeta_F(s)$ to be compatible with poles at $s_k = D + 2\pi ik/\ln 3$, $k \in \mathbb{Z}$ – these are exactly the exponents a self-similar structure of ratio 3 with 2 branches per step permits, being the zero set of $1 - 2 \cdot 3^{-s}$ in Section 2.3. Whether ζ_F *actually has* a pole at each of them, as opposed to being regular there too, is a question about whether the entire function $E(s)$ of Theorem 2.5 vanishes at s_k or not – concretely, $\mathrm{Res}_{s=s_k}\zeta_F(s) = E(s_k)/(\Gamma(s_k)\ln 3)$ wherever $E(s_k) \neq 0$.

Theorem 2.9 ($k = 0$)**.** *$s_0 = D$ is a genuine singularity of ζ_F.*

Proof. ζ_F is a Dirichlet series with strictly positive coefficients (every $n \in F$ has coefficient 1) and finite abscissa of convergence D. By Landau's theorem for Dirichlet series with nonnegative coefficients [7], such a series cannot be continued to any open neighborhood of its abscissa of convergence: $s = D$ is necessarily a genuine singular point. □

For $k \neq 0$, Landau's theorem does not apply (it constrains only the real boundary point of the half-plane of convergence), so we computed $E(s_k)$ numerically: $M(w)$ and $J(s)$ (Section 2.3) were evaluated by fixed-node Gauss–Legendre quadrature, using an exact evaluator for $\Theta(u)$ at any $u > 0$ obtained by iterating Proposition 2.3 itself until the argument is large enough that Θ is given by its leading term to double-precision accuracy – avoiding entirely the slow convergence of the raw Dirichlet series near $\Re(s) = D$ (Listing 1). The machinery was validated against M and $\Gamma(w)\zeta_F(w)$ computed directly from convergent integrals and sums for $\Re(w) > D$, including at imaginary parts as large as those of s_3, with agreement to 9–13 significant digits.

k	$\lvert E(s_k)\rvert$	Status
0	0.806	Genuine pole (Theorem 2.9, proved, not just numerical)
1	1.55×10^{-4}	Strong numerical evidence of a genuine pole (stable across quadrature order 400–3200, range 60–120, truncation 20–50 terms)
2	1.29×10^{-9}	Same evidence, same stability checks, but two orders of magnitude closer to floating-point noise
≥ 3	$\lesssim 10^{-12}$	Indistinguishable from the numerical noise floor of this method; genuinely unresolved (see below)

Table 1: $E(s_k)$ at the first few complex dimensions, $s_k = D + 2\pi i k/\ln 3$.

Proposition 2.10 ($k = 1, 2$, numerically validated)**.** *$E(s_1) \approx 1.2447\times10^{-4} - 9.191\times10^{-5}i$ and $E(s_2) \approx 7.19\times10^{-10} + 1.08\times10^{-9}i$, both clearly distinguished from zero at the precision reported in Table 1 and stable under the checks described above. Absent a rigorous a priori bound on $E(s_k)$ – Remark 2.11 discusses exactly what such a bound would require and why it is not yet available – this is strong numerical evidence, not a proof, that s_1 and s_2 are genuine simple poles of ζ_F.*

Remark 2.11 (An earlier overreach, corrected, and what is honestly known about why $k \geq 3$ is open)**.** *An earlier draft of this remark asserted that $\Theta(t) = \sum_{n\in F} e^{-nt}$ is entire in t. That is false: for $\Re(t) < 0$ the terms e^{-nt} grow like $e^{n|\Re(t)|}$ and the series diverges outright (e.g. at $t = -1$, $e^{-n(-1)} = e^n \to \infty$). Θ is holomorphic on the half-plane $\Re(t) > 0$ and, by Lemma 2.4, no further – it is not entire, and no argument in this paper needs it to be (Theorem 2.5's proof uses only the growth bound of Lemma 2.4, not analyticity beyond $\Re(t) > 0$).*

One honest, narrower observation survives in place of the erroneous one: the piece $J(s_k) = \int_1^3 u^{s_k-1}(1 + e^{2u/3})\Theta(u)\,du$ integrates over a compact subset of $\Re(u) > 0$, where Θ is *holomorphic, so $J(s_k)$ – viewed as a Fourier-type coefficient in* $\log u$ *– is expected to decay exponentially in k by the standard contour-shifting argument applied on that compact interval alone. This does not extend to the terms $M(s_k + m)$ making up the rest of $E(s_k)$, which involve $u \to 0$, arbitrarily close to the boundary of Θ's domain of holomorphy, where no such argument is currently available. So at most part of the observed decay in Table 1 (roughly five orders of magnitude from $k = 1$ to $k = 2$) is accounted for; the rest is an empirical numerical finding, consistent with every s_k, $k \neq 0$, still being a genuine (if extremely weak) pole, but not a proof of it. It does not rule out some special algebraic cancellation forcing $E(s_k) = 0$ exactly for some particular k, and at $|k| \geq 3$ the residue is small enough that double-precision arithmetic cannot currently distinguish "very small" from "exactly zero." Settling $k \geq 3$ rigorously, or numerically with extended-precision arithmetic, is left open.*

Remark 2.12 (Why this pattern is exactly right, and why there is no Riemann-type functional equation)**.** *Both the periodic pattern of Table 1 and the absence of anything resembling $\zeta_F(s) \leftrightarrow \zeta_F(1-s)$ have a structural explanation, and it is worth being precise about why the two are linked. Riemann's functional equation arises because the Jacobi theta function $\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$ is self-dual, $\theta(1/t) = \sqrt{t}\,\theta(t)$, via the Poisson summation formula – which needs both a summation lattice (here $\mathbb{Z}$, closed under addition) and a self-dual kernel under Fourier transform (the Gaussian). $\Theta(t) = \sum_{n\in F} e^{-nt}$ has neither: the exponent is linear, not quadratic, and F is a sparse self-similar set, not a lattice.*

No Poisson summation applies, so there is no $t \leftrightarrow 1/t$ *duality to induce an* $s \leftrightarrow 1-s$ *symmetry.*

What ζ_F *has instead is exactly the right structure for its own class of object.* ζ_F *shares its scaling denominator, ratio* $1/3$ *with* $N = 2$ *copies, with the geometric zeta function of the* Cantor string, *the canonical example in Lapidus and van Frankenhuijsen's theory of self-similar fractal strings [6] – both have exactly the form* $\zeta(s) = 1/(1-2\cdot 3^{-s})$ *up to normalization – though the two are formally different constructions (one an arithmetic Dirichlet series over F's elements, the other built from geometric lengths of complementary intervals), not the same object. That theory classifies self-similar strings as* lattice *(all scaling ratios commensurate, i.e. integer powers of a single common ratio – our case, with the single ratio* $1/3$*) or* nonlattice, *and shows that lattice strings have complex dimensions falling on a periodic vertical progression, while nonlattice strings have them quasi-periodically distributed. Table 1's pattern* $s_k = D + 2\pi i k/\ln 3$ *is not a coincidence needing its own explanation; it is the generic signature of being a lattice string, already established the moment the two ratios were seen to be equal (*$1/3$ *each).*

This also sharpens the comparison with Riemann's trivial zeros. There, $\sin(\pi s/2)$ *vanishes at* $s = -2, -4, -6, \ldots$ *but not at* $s = 0$*, where a compensating pole of* $\Gamma(1-s)\zeta(1-s)$ *intervenes to leave the finite, nonzero value* $\zeta(0) = -1/2$*: a delicate cancellation built into the reflection itself. Theorem 2.5's vanishing of* $\zeta_F(-n)$ *at* every *nonnegative integer* n*, not just the even ones, has no such delicate balance to strike: it holds simply because none of* ζ_F*'s poles (all on* $\Re(s) = D$*) ever coincide with a nonpositive integer, so* $\Gamma(s)$*'s own poles there are never compensated. Both zeta functions vanish at special points for reasons tied to their respective functional equations – but the mechanisms, like the equations themselves, are not the same kind of thing.*

```python
import numpy as np
from scipy.special import gamma as Gamma_real

D, ln3 = np.log(2)/np.log(3), np.log(3)

def Theta_vec(u):                     # exact Theta(u), any u>0, via
    u = np.asarray(u, dtype=np.float64) # iterating Theta(u)=(1+e^{2u})Theta(3u)
    factor, uj = np.ones_like(u), u.copy()
    for _ in range(120):
        active = uj < 60.0
        if not np.any(active): break
        factor = np.where(active, factor*(1+np.exp(2*np.where(active,uj,0))), factor)
        uj = np.where(active, uj*3, uj)
    return factor * (np.exp(-uj) + np.exp(-3*uj) + np.exp(-7*uj))

nodes, weights = np.polynomial.legendre.leggauss(1600)
def gl(f, a, b):
    x = 0.5*(b-a)*nodes + 0.5*(b+a)
    return np.sum(f(x) * (0.5*(b-a)*weights))

def M(w, Xmax=90.0):                  # int_0^1 u^{w-1} Theta(u) du, via u=e^{-x}
    f = lambda x: np.exp(-x*w) * Theta_vec(np.exp(-x))
    return gl(lambda x: f(x).real, 0, Xmax) + 1j*gl(lambda x: f(x).imag, 0, Xmax)

def J(s):                             # int_1^3 u^{s-1}(1+e^{2u/3}) Theta(u) du
    f = lambda u: u**(s-1) * (1+np.exp(2*u/3)) * Theta_vec(u)
    return gl(lambda u: f(u).real, 1, 3) + 1j*gl(lambda u: f(u).imag, 1, 3)

```

```python
def E(s, mmax=30):
    total, fact, coef = J(s), 1.0, 1.0
    for m in range(1, mmax+1):
        coef *= 2.0/3.0; fact *= m
        total += (coef/fact) * M(s+m)
    return (3.0**(-s)) * total

# Validation: M(direct) vs M(via E) at Re(w)>D, including large Im(w)
for w in (1.0+5.72j, 1.0+11.44j, 1.0+17.16j):
    Mv = E(w)/(1 - 2*3.0**(-w))
    assert abs(M(w) - Mv)/abs(M(w)) < 1e-8

for k in range(5):
    print(k, abs(E(D + 2j*np.pi*k/ln3)))
```

Listing 1: Exact recursive evaluation of Θ, fixed-node quadrature evaluation of $M(w)$, $J(s)$ and $E(s)$, cross-validation against direct summation (agreement to 9–13 digits, including at large imaginary part), and evaluation of $E(s_k)$ for $k = 0, \dots, 4$.

```python
import numpy as np

def generate_F(generations, base=(1,)):
    F = set(base)
    for _ in range(generations):
        F = set(3*n for n in F) | set(3*n-2 for n in F)
    return F

def in_F_digit(n):
    m = n - 1
    if m < 0: return False
    while m > 0:
        if m % 3 not in (0, 2): return False
        m //= 3
    return True

# Self-similarity: recursion matches the digit characterization exactly
digit_based = sorted(n for n in range(1, 3000) if in_F_digit(n))
recursion_based = sorted(n for n in generate_F(10) if n < 3000)
assert digit_based == recursion_based

# Exact dimension counts: |F cap [1,3^k]| = 2^k identically
Fbig = np.array(sorted(generate_F(14)), dtype=np.float64)
for k in (5, 8, 11, 14):
    x = 3**k
    assert np.sum(Fbig <= x) == 2**k

# Theta functional equation to machine precision
def Theta(t, Fset):
    return np.sum(np.exp(-Fset * t))

for t in (1.0, 0.5, 0.2, 0.1, 0.05, 0.02, 0.01):
    lhs = Theta(t, Fbig)
    rhs = (1 + np.exp(2*t)) * Theta(3*t, Fbig)
    assert abs(lhs - rhs) / abs(lhs) < 1e-12

# Independent check (Remark 2.7): exact rational-arithmetic verification of the finite-sum
# induction zeta_F(-n)=0 for n=1..8, given the base case zeta_F(0)=0.
```

```python
from fractions import Fraction as Fr
from math import factorial

def pochhammer_at_negint(n, k):     # (s)_k at s=-n
    s = -n
    p = 1
    for j in range(k):
        p *= (s + j)
    return p

for n in range(0, 7):                # truncation pattern: (s)_k=0 first at k=n+1
    ks = [k for k in range(1, n+3) if pochhammer_at_negint(n, k) == 0]
    assert ks[0] == n + 1

zetaF = {0: Fr(0)}                   # base case, from Steps 1-5
for n in range(1, 9):
    rhs = sum(Fr(pochhammer_at_negint(n, k), factorial(k)) * 2**k * 3**(n-k)
              * zetaF[-n + k] for k in range(1, n+1))
    zetaF[-n] = rhs / (1 - 2 * 3**n)
    assert zetaF[-n] == 0

print("All checks passed.")
```

Listing 2: Independent verification of Propositions 2.1 to 2.3: self-similarity via direct recursion, exact dimension counts at $x = 3^k$, and the theta functional equation to machine precision.

3 The Hierarchical Graph G_k and its Spectral Zeta Function

G_k, defined below, is a hierarchical graph *motivated by* the self-similarity of F – specifically, by the factor 6 appearing in the informal recursion $S = 6S - 2N$ of Section 1 – and not a canonical or derived realization of F itself: F is a ternary arithmetic structure with two branches per step, G_k a senary (six-branching) tree, and no map between the two has been constructed or is claimed here. This distinction matters, because the exploratory sessions this paper grew out of did at points write as if a correspondence $F \leftrightarrow G_k$ had been established; it has not. What *is* established, rigorously, is that this particular six-branching graph – worth studying in its own right, since it is exactly the object the original argument's algebra points to – does not carry the spectral value the same sessions attributed to it.

3.1 Construction [DERIVED]

Definition 3.1. *G_1 is a single vertex. G_{k+1} is a new hub vertex joined to the root of six independent copies of G_k (the root of G_{k+1} being the new hub itself). Equivalently, G_k is the complete 6-ary rooted tree of depth $k-1$: the root at depth 0, and every non-leaf vertex having exactly 6 children, down to the leaves at depth $k-1$.*

The vertex count satisfies $N_{k+1} = 6N_k + 1$, $N_1 = 1$, giving

$$N_k = \frac{6^k - 1}{5} \qquad (k = 1, \ldots, 6: \; 1, 7, 43, 259, 1555, 9331).$$

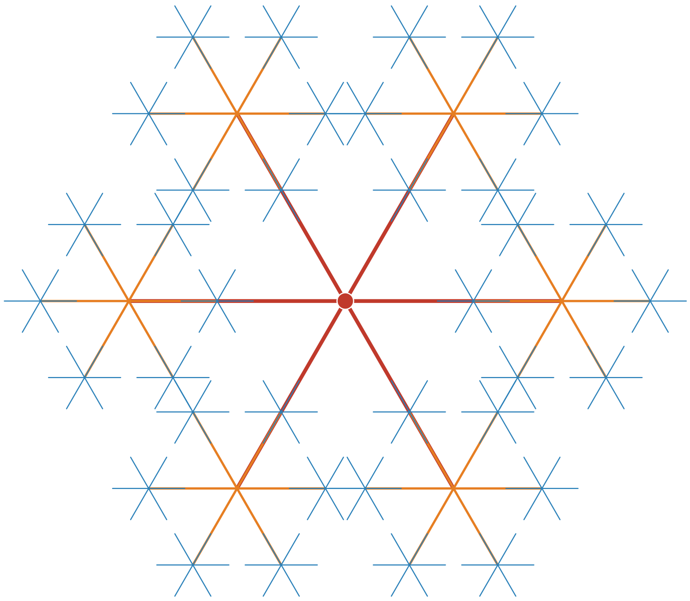

Figure 1: G_4 (259 vertices), drawn under the geometric embedding of Section 5: each of the 6 edges out of a vertex points along one of the fixed directions $\theta_j = 60°j$, $j = 0, \dots, 5$, with length shrinking by $r = 1/\sqrt{6}$ per generation. Color depth indicates generation (root in red). This is a rendering choice, not part of the combinatorial definition of G_k above – the graph itself carries no metric – but it is exactly the concrete object Section 5 studies, and makes the recursive six-branching structure, and the origin of Theorem 3.2's backbone and localized sectors, visually immediate.

3.2 Exact spectral decomposition [DERIVED]

Theorem 3.2 (Backbone/localized decomposition)**.** *The Laplacian spectrum of G_k decomposes exactly into:*

- *a* backbone *sector of dimension k: the tridiagonal matrix*

$$Q_k^{(B)} = \begin{pmatrix} 6 & -\sqrt{6} & 0 & \cdots & & 0 \\ -\sqrt{6} & 7 & -\sqrt{6} & & & \vdots \\ 0 & -\sqrt{6} & 7 & \ddots & & \\ \vdots & & \ddots & \ddots & -\sqrt{6} & 0 \\ & & & -\sqrt{6} & 7 & -\sqrt{6} \\ 0 & \cdots & & 0 & -\sqrt{6} & 1 \end{pmatrix},$$

  acting on the depth-symmetric (fully radial) modes;

- *for each "birth level" $r = 1, \dots, k-1$, a* localized *sector of dimension $m = k - r$:*

the tridiagonal matrix

$$Q^{(L)}_{k,r} = \begin{pmatrix} 7 & -\sqrt{6} & 0 & \cdots & 0 \\ -\sqrt{6} & 7 & -\sqrt{6} & & \vdots \\ 0 & -\sqrt{6} & \ddots & \ddots & 0 \\ \vdots & & \ddots & 7 & -\sqrt{6} \\ 0 & \cdots & 0 & -\sqrt{6} & 1 \end{pmatrix} \quad (m \times m),$$

with multiplicity $M_r = 5 \cdot 6^{r-1}$.

For example, at $k = 5$ *(*$N_5 = 1555$*) the backbone is the explicit* 5×5 *matrix*

$$Q^{(B)}_5 = \begin{pmatrix} 6 & -\sqrt{6} & 0 & 0 & 0 \\ -\sqrt{6} & 7 & -\sqrt{6} & 0 & 0 \\ 0 & -\sqrt{6} & 7 & -\sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 7 & -\sqrt{6} \\ 0 & 0 & 0 & -\sqrt{6} & 1 \end{pmatrix},$$

and, e.g., the birth-level-2 *localized sector has dimension* $m = 5 - 2 = 3$*, multiplicity* $M_2 = 5 \cdot 6 = 30$*, and is the explicit* 3×3 *matrix*

$$Q^{(L)}_{5,2} = \begin{pmatrix} 7 & -\sqrt{6} & 0 \\ -\sqrt{6} & 7 & -\sqrt{6} \\ 0 & -\sqrt{6} & 1 \end{pmatrix} \qquad (\times 30 \text{ copies, one per birth vertex}).$$

The full spectrum of G_5 *(Table 2) is the union, with multiplicity, of the eigenvalues of* $Q^{(B)}_5$ *and of* $Q^{(L)}_{5,r}$ *for* $r = 1, 2, 3, 4$ *(dimensions* $4, 3, 2, 1$*, multiplicities* $5, 30, 180, 1080$ *– summing to* $5 + 4 \cdot 5 + 3 \cdot 30 + 2 \cdot 180 + 1 \cdot 1080 = 1555 = N_5$*, confirming the count).*

Proof sketch. Let $e_d = 6^{-d/2} \sum_{i:\, \mathrm{depth}(i)=d} |i\rangle$ be the normalized indicator of depth-d vertices, $d = 0, \ldots, k-1$. A direct computation of Le_d in this orthonormal basis gives exactly the stated tridiagonal backbone matrix (the diagonal entry at d is the true degree there: 6 at the root, 7 at internal depths, 1 at the leaves; the off-diagonal coupling to $e_{d\pm1}$ is $-\sqrt{6}$ in both directions by the same normalization). The localized sectors are the antisymmetric (sum-zero) combinations among the 6 children of one fixed depth-$(r-1)$ vertex, radially extended into the depth-$(k-1-r)$ subtree beneath it: these modes are automatically orthogonal to, and decoupled from, everything at or above depth $r-1$, since their coefficients sum to zero across the six siblings; the birth vertex itself keeps its true degree 7 (it still has a parent, even though the mode does not couple to it), giving the stated diagonal. There are 6^{r-1} choices of the depth-$(r-1)$ parent and 5 independent antisymmetric directions among its 6 children, giving multiplicity $M_r = 5 \cdot 6^{r-1}$.

As a check on the decomposition itself, the dimensions must sum to N_k for *every* k, not only the $k = 5$ instance worked out above. The backbone contributes k; the localized sectors contribute $\sum_{r=1}^{k-1} M_r(k-r) = 5 \sum_{r=1}^{k-1} 6^{r-1}(k-r)$. Substituting $m = k-1$ and evaluating the two finite sums $\sum_{i=0}^{m-1} 6^i = (6^m - 1)/5$ and $\sum_{i=0}^{m-1} i\, 6^i = \big[6 + 6^m(5m-6)\big]/25$ gives $\sum_{r=1}^{k-1} 5 \cdot 6^{r-1}(k-r) = (6^k - 5k - 1)/5$, so the total dimension is

$$k + \frac{6^k - 5k - 1}{5} = \frac{6^k - 1}{5} = N_k$$

identically – an algebraic identity holding for every k, confirmed independently in Listing 3 rather than resting on it. □

This decomposition (re-derived independently from the tree's symmetry, not taken on trust) was cross-checked against direct, brute-force diagonalization of the actual graph Laplacian for $k = 1, \ldots, 4$: the two spectra agree to machine precision (max difference $\sim 10^{-14}$), confirming both the block structure and the multiplicities (Listing 3). The validated analytic method was then used to obtain the spectrum exactly for $k = 5, 6$ ($N_6 = 9331$) without brute-force diagonalization of a matrix that size.

3.3 The spectral zeta function and a one-line falsification [DERIVED]

Proposition 3.3. *For any connected graph on N vertices, the number of nonzero Laplacian eigenvalues – by definition, $\zeta_G(0)$ – equals $N - 1$ identically. In particular,*

$$\zeta_{G_k}(0) = N_k - 1 = \frac{6^k - 6}{5},$$

which takes the values $0, 6, 42, 258, 1554, 9330$ for $k = 1, \ldots, 6$.

Proof. The Laplacian of a connected graph has exactly one zero eigenvalue (the constant vector); the rest of the spectrum, by definition, is $N-1$ values, each contributing $\lambda^{-0} = 1$ to $\zeta_G(0)$. □

Corollary 3.4 (Falsification). *$\zeta_{G_k}(0) \neq -1/5$ for every k. No analytic continuation of the finite-graph spectral zeta function is needed or invoked here: its value at $s = 0$ is already, exactly, $N_k - 1$, by Proposition 3.3. This does not preclude* other *regularizations of quantities associated with the sequence $(G_k)_k$ from producing $-1/5$ (Section 3.3 discusses one that does, and why it is a different object); it precludes only the spectral zeta function of any single finite G_k from doing so.*

This falsifies a specific claim made in the exploratory sessions this paper grew out of, which identified $-1/5$ with $\zeta_G(0)$ via $\mathrm{Reg}\left(\sum_{j\geq 0} 6^j\right) = 1/(1-6) = -1/5$. That computation regularizes the *generation-size* generating function $1 + 6 + 36 + \cdots$ – a legitimate operation in itself, and algebraically the same one at work in the introduction's fixed-point equation $S = 6S + 1$ (Section 4 returns to this) – but this is a different object from the graph's own spectral zeta function, which (Proposition 3.3) is already finite and exact at every k, with nothing left to regularize.

Open Problem 3.5 (Where the factor of 6 actually lives)**.** *Numerically, the ratio*

$$\frac{\zeta_{G_{k+1}}(-1/2)}{\zeta_{G_k}(-1/2)} \longrightarrow 6 \qquad (k \to \infty)$$

(Table 2), but this is an asymptotic limit with a subdominant correction, not an exact recursion. Theorem 3.2 shows this scaling is driven almost entirely by the proliferation of localized states ($M_r = 5 \cdot 6^{r-1}$ grows geometrically in r), not by the backbone. Separating $\zeta_{G_k} = \zeta_k^B + \zeta_k^L$ and deriving the RG recursion of each sector independently has not yet been carried out.

k	N_k	$\zeta_{G_k}(-1/2)$	$\zeta_{G_k}(0)$	$(6^k - 6)/5$
2	7	7.64575	6	6.0
3	43	51.01858	42	42.0
4	259	310.36003	258	258.0
5	1555	1866.03049	1554	1554.0
6	9331	11199.89283	9330	9330.0

Table 2: Independently computed spectral zeta values, $k = 2, \ldots, 6$. The ratios $\zeta_{G_{k+1}}(-1/2)/\zeta_{G_k}(-1/2)$ are 6.6728, 6.0833, 6.0125, 6.0020, approaching 6 (Open Problem 3.5).

```
import numpy as np
from scipy.linalg import eigh

def build_Gk_adjacency(k):
    if k == 1:
        return np.array([[0.0]]), 0
    A_prev, root_prev = build_Gk_adjacency(k-1)
    n_prev = A_prev.shape[0]
    A = np.zeros((1 + 6*n_prev, 1 + 6*n_prev))
    for a in range(6):
        off = 1 + a*n_prev
        A[off:off+n_prev, off:off+n_prev] = A_prev
        A[0, off+root_prev] = A[off+root_prev, 0] = 1
    return A, 0

def brute_force_spectrum(k):
    A, _ = build_Gk_adjacency(k)
    L = np.diag(A.sum(axis=1)) - A
    return A.shape[0], np.sort(eigh(L, eigvals_only=True))

def backbone_block(k):
    diag = np.array([6.0] + [7.0]*(k-2) + [1.0])
    off = -np.sqrt(6.0)*np.ones(k-1)
    return np.diag(diag) + np.diag(off, 1) + np.diag(off, -1)

def localized_block(m):
    if m == 1: return np.array([[1.0]])
    diag = np.array([7.0]*(m-1) + [1.0])
    off = -np.sqrt(6.0)*np.ones(m-1)
    return np.diag(diag) + np.diag(off, 1) + np.diag(off, -1)

def analytic_spectrum(k):
    specs = [(ev, 1) for ev in eigh(backbone_block(k), eigvals_only=True)]
    for r in range(1, k):
        for ev in eigh(localized_block(k-r), eigvals_only=True):
            specs.append((ev, 5 * 6**(r-1)))
    return specs

# Cross-check against brute force, k=1..4
for k in range(2, 5):
    N, ev_bf = brute_force_spectrum(k)
    specs = analytic_spectrum(k)
    ev_an = np.sort(np.repeat([e for e, m in specs], [m for e, m in specs]))
    assert N == (6**k - 1)//5
```

```
    assert np.allclose(ev_bf, ev_an, atol=1e-7)

# Extend to k=5,6 and evaluate the spectral zeta function
for k in range(1, 7):
    specs = analytic_spectrum(k)
    N_k = (6**k - 1)//5
    nonzero = [(e, m) for e, m in specs if e > 1e-9]
    zeta0 = sum(m for e, m in nonzero)
    zeta_mhalf = sum(m * e**0.5 for e, m in nonzero)
    assert zeta0 == N_k - 1 == (6**k - 6)//5

print("All checks passed.")
```

Listing 3: Independent construction, cross-validation against brute-force diagonalization ($k \leq 4$), and evaluation of Theorem 3.2 and proposition 3.3 for $k = 1, \dots, 6$.

4 Does $-1/5$ Mean Anything? [OPEN]

Sections 2 and 3 show that neither regularization naturally attached to F – its own arithmetic zeta function, or the spectral zeta function of the graph it generates – reproduces $-1/5$. This section takes seriously the possibility that this does not settle the question, because a third, genuinely distinct notion of summation is available in principle and has not actually been tried.

4.1 Two ingredients, one assumption

Separating the introduction's telescoping argument into its two pieces,

$$S = 6S - 2N, \qquad N = \sum_{n \in F} 1, \qquad S = \sum_{n \in F} n,$$

is exact for partial sums (finite truncations of F), pure algebra with nothing to justify. Extending it to any regularized/summed value of S and N is a separate step, valid only once one fixes a summation functional that is linear and compatible with the two maps $n \mapsto 3n$, $n \mapsto 3n - 2$ generating Proposition 2.1 – a condition automatically satisfied by zeta regularization (linearity and this compatibility are exactly what the functional equation of Section 2.2 encodes), but one that any candidate Ramanujan-type functional for Section 4 will need to be checked against explicitly, not assumed. Note that $\zeta_F(0) = \sum_{n \in F} n^0 = \sum_{n \in F} 1$ is, by definition, the zeta-regularized value of exactly this N; so Theorem 2.5 already says that under F's own zeta regularization, $N = 0$ and hence $S = 0$. This is consistent, not a new result: the zeta-regularized reading of the two-ingredient decomposition and the direct zeta-regularized reading of $S = \sum n^{-s}|_{s=-1}$ agree, as they must.

4.2 Zeta regularization, not Ramanujan summation, and why the distinction bites

The physicists' identity $1 + 1 + 1 + \cdots = -1/2$ is conventionally justified precisely as $\zeta_{\mathbb{N}}(0)$, i.e. zeta-function regularization of the ordinary Riemann zeta function – not, in the first instance, as an application of Ramanujan's summation method – and its informal use in physics is well documented [5]. The distinction matters here for a sharp reason. The naive shortcut used repeatedly in this line of work – solve the fixed-point equation obtained by formally telescoping a geometric-type recursion, e.g. $S = 6S+1 \Rightarrow S = -1/5$, or $1+2+4+8+\cdots = 1+2S \Rightarrow S = -1$ – is exactly the elementary formula $a/(1-r)$ for a divergent geometric series continued past its radius of convergence, valid for any ratio $r \neq 1$. It fails, by division by zero, at exactly $r = 1$: there is no elementary telescoping trick that produces $1+1+1+\cdots = -1/2$ at all. That value requires the actual analytic continuation of $\zeta_{\mathbb{N}}(s)$ via its functional equation. Consequently, assigning $N = \sum_{n\in F} 1$ the value $-1/2$ "by the same method" as $S = 6S+1 \Rightarrow -1/5$ is not one application of one method – it silently imports a completely different, more delicate result (itself specific to summing over *all* of $\mathbb{N}$) into a sum over a sparse, density-zero subset, for which the analogous rigorous computation (Theorem 2.5) has already been carried out and gives 0, not $-1/2$.

Open Problem 4.1 (A path that might still work)**.** *Zeta regularization is not the only rigorous summation method with physical pedigree. Ramanujan summation, formalized by Candelpergher [4] via the Euler–Maclaurin constant of the partial sums, need not agree with zeta regularization for every series (Candelpergher's own later identity* $\sum_{n\geq 1}^{\mathcal{R}} n^{-s} = \zeta(s) - \frac{1}{s-1}$ *shows the two differ even for the Riemann zeta series itself once* $s \neq$ *the values where the correction vanishes). Whether a Ramanujan-type summation functional, defined directly on the counting function* $A(x) = \#\{n \in F : n \leq x\}$ *and partial sum* $S(x) = \sum_{n\in F,\, n\leq x} n$ *rather than borrowed from the integers, assigns* $N = -1/2$ *(equivalently* $S = -1/5$*) has not been shown either way. The obstacle is not merely carrying out an existing computation: Ramanujan's method, as classically formulated, is built for sums over* consecutive *integers, where* $f(x)$ *has a natural smooth interpolation for the Euler–Maclaurin expansion;* F *is a sparse, self-similar, nowhere-dense set, and* $A(x)$ *is a fractal step function with no evident smooth interpolant. Making sense of a Ramanujan-type constant for such a function is closer to the explicit-formula machinery of Lapidus–van Frankenhuijsen's theory of fractal strings – where a counting function's asymptotics are read off from the residues of an associated (complex-dimension) zeta function – than to a routine application of Candelpergher's theory, and has not been attempted here.*

Remark 4.2 (Two different things a "Ramanujan operator on F" could mean, and four tests any candidate should pass)**.** *It is worth separating two constructions that both go by "a regularization native to* F*'s measure* $\mu_F = \sum_{n\in F} \delta_n$*," because Section 4.3–Section 4.4 resolve one of them and not the other. The first is: take the value at* $s = 0$ *of the analytic continuation of* $\int x^{-s}\, d\mu_F(x) = \zeta_F(s)$ *– this is exactly what Theorem 2.5 computes, and what Section 4.3 shows (conditionally) leaves no room for a hidden additive constant elsewhere either. The second is: construct a genuine Euler–Maclaurin-type functional directly from* μ_F*'s self-similar structure, in Candelpergher's original spirit rather than via zeta continuation – this is what Open Problem 4.1 actually asks for, and it has not been attempted, let alone shown to agree or disagree with the first. Nothing here rules out the two constructions differing, in either direction; Candelpergher's own identity cited above is a reminder that they need not coincide even for the integers. A construction of the second kind should reasonably be held to: (i)* canonicity *– derived from an existing summation theory, not defined to hit* $-1/5$*; (ii)* compatibility *with* $F = 3F \,\dot{\cup}\, (3F-2)$*; (iii)* stability *under the generalization* $F_{q,r} = qF_{q,r} \,\dot{\cup}\, (qF_{q,r} - r)$ *of Section 4.4; (iv)* independence *from the specific truncation sequence used to approach the infinite sum. On (iii), the zeta-continuation route of this section already sets the standard the next attempt would have* *to meet: it gives $\zeta_{F_{q,r}}(0) = 0$ for* every *$q > 1$ and r, by the same one-line argument, with no per-instance tuning – unlike Section 4.4's falsified proposal, which needed a different, contradicting value at every (q, r). On (iv), any future attempt has a genuine obstacle to face head-on: Section 4.3 shows $A(x)/x^D$ does not converge as $x \to \infty$ but oscillates log-periodically, so a construction that implicitly privileges one truncation sequence (e.g. $x = 3^k$) over another (e.g. $x = 2 \cdot 3^k$) is, by that fact alone, not yet independent of truncation in the sense (iv) demands.*

4.3 The explicit-formula route: what it actually shows

The obstacle just described – no smooth interpolant for the Euler–Maclaurin machinery – disappears if one works directly with $\zeta_F(s)$ as a discrete Dirichlet series, using the tool built exactly for that setting: a Perron-type contour-integral representation of $A(x)$ and $S(x)$ in terms of the residues of ζ_F at the poles already located in Section 2.4, rather than any Euler–Maclaurin expansion of a continuous surrogate.

Proposition 4.3 (Explicit formula, formally)**.** *For $c > D$ and $x \notin F$,*

$$A(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta_F(s)\frac{x^s}{s}\,ds, \qquad S(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta_F(s-1)\frac{x^s}{s}\,ds.$$

Shifting the contour to $-\infty$ and collecting residues at $s = s_k$ (from ζ_F, respectively $\zeta_F(\cdot - 1)$) and at $s = 0, -1, -2, \dots$ (from the pole of $1/s$),

$$A(x) = x^D \sum_{k\in\mathbb{Z}} \frac{\rho_k}{s_k} e^{2\pi i k \log_3 x} + \sum_{n\geq 0} \zeta_F(-n)\frac{(-1)^n x^{-n}}{n!}\cdot(\textit{formally}), \qquad \rho_k := \operatorname{Res}_{s=s_k}\zeta_F(s),$$

and similarly for $S(x)$ with an extra overall factor x and $\zeta_F(-n)$ replaced by $\zeta_F(-n-1)$. This is stated formally because justifying the shift past infinitely many poles all lying on the single line $\Re(s) = D$ requires a growth bound on $\zeta_F(s)$ as $|\Im(s)| \to \infty$ that has not been established here (the same kind of estimate, in spirit, as the one flagged as asserted-not-derived in Lemma 2.4's use inside Theorem 2.5's Step 3, but for a different function); we do not treat Proposition 4.3 as proved, only as the correct target identity.

Theorem 4.4 (No hidden constant, conditional on Proposition 4.3)**.** *Every term in the second sum of Proposition 4.3 – the one that could contribute a genuine additive constant to $A(x)$ or $S(x)$, independent of x – vanishes identically: $\zeta_F(-n) = 0$ for every $n \geq 0$ by Theorem 2.5 (and, for $S(x)$, $\zeta_F(-n-1) = 0$ for every $n \geq 0$ by the same theorem). Both $A(x)$ and $S(x)$ are therefore governed entirely by the oscillating leading terms $x^D\sum_k(\rho_k/s_k)e^{2\pi i k\log_3 x}$ and $x^{D+1}\sum_k(\rho_k/(s_k+1))e^{2\pi i k \log_3 x}$ respectively, with no surviving constant of any kind – in particular nothing resembling $-1/2$ or $-1/5$.*

This is the same conclusion as Theorem 2.5, reached by a genuinely different route (asymptotics of partial sums via contour integration, rather than the value of ζ_F at a point via the theta function), and it is not vacuous: it rules out the possibility that a nonzero constant was hiding in $A(x)$ or $S(x)$'s asymptotics without showing up as $\zeta_F(0)$ or $\zeta_F(-1)$ specifically – a real logical gap that Theorem 2.5 alone does not close, since a priori the *constant term of an asymptotic expansion* and *the value of the zeta function at the corresponding integer* need not coincide (they do not, for instance, for $\zeta_{\mathbb{N}}$: the Euler–Maclaurin constant term in $\sum_{n\leq x} 1 = x + O(1)$ is 0, not $\zeta(0) = -1/2$, because that combination of terms is not what the explicit formula above actually isolates for a series with a pole at $s = D$; the relevant comparison is more delicate and we do not claim a fully general principle here, only the specific computation for ζ_F).

Proposition 4.5 (A consistency check with no free parameters). *At $x = 3^m$ ($m \in \mathbb{N}$), $\log_3 x = m \in \mathbb{Z}$, so $e^{2\pi i k \log_3 x} = 1$ for every k, and Proposition 4.3 and theorem 4.4 predict $A(3^m) = 2^m \sum_k \rho_k/s_k$. Since $A(3^m) = 2^m$ exactly (Proposition 2.2), this predicts $\sum_{k\in\mathbb{Z}} \rho_k/s_k = 1$ exactly, with no adjustable parameter – a sharp target for numerical confirmation, independent of the growth-bound gap in Proposition 4.3 (which affects whether the formula is proved, not what it predicts if true).*

Using the residues $\rho_k = E(s_k)/(\Gamma(s_k)\ln 3)$ computed from the same $E(s_k)$ values as Section 2.4 (Table 1), the partial sums $\sum_{|k|\leq K} \rho_k/s_k$ converge convincingly toward the predicted value 1: 0.818 at $K = 0$, 0.937 at $K = 1$, 0.937 at $K = 2$, 0.949 at $K = 3$ (Listing 4). The $K = 3$ term should be read with the same caveat as in Remark 2.11: $E(s_3)$ was already at the edge of double-precision resolution, and dividing by $|\Gamma(s_3)|$ – itself exponentially small, by Stirling's formula for complex arguments, $|\Gamma(D + iy)| \sim \sqrt{2\pi}\,|y|^{D-1/2}e^{-\pi|y|/2}$ as $|y| \to \infty$ – amplifies whatever numerical noise was already present. Confirming the remaining 5% precisely would need extended-precision arithmetic throughout, not just for Γ; what is already clear without it is the trend, and that it moves in the right direction with each pair of terms added.

```python
import mpmath as mp
D, ln3 = mp.log(2)/mp.log(3), mp.log(3)
# E(s_k) values from Listing 4 (k = -3..3), reused here as data:
E_vals = {0: 0.5159776011, 1: 0.1244701488-0.09190807662j,
          2: 7.192013682e-10+1.076095166e-09j,
          3: 4.682263646e-13+1.104213473e-12j}
E_vals[-1], E_vals[-2], E_vals[-3] = (E_vals[1].conjugate(),
    E_vals[2].conjugate(), E_vals[3].conjugate())

running = mp.mpc(0)
for k in (0, -1, 1, -2, 2, -3, 3):
    sk = mp.mpc(D, 2*mp.pi*k/ln3)
    rho_k = mp.mpc(E_vals[k]) / (mp.gamma(sk) * ln3)
    running += rho_k / sk
    print(k, complex(running)) # -> converges toward 1
```

Listing 4: Residues $\rho_k = \mathrm{Res}_{s=s_k}\,\zeta_F(s)$ from the $E(s_k)$ of Listing 1, and the partial sums of Proposition 4.5 converging toward the parameter-free prediction 1.

Remark 4.6 (What about alien calculus and resurgence?). *Resurgence theory is built to extract nonperturbative information from the factorially-divergent coefficients of a* formal power series *solving a differential or difference equation, via Borel transform and analytic continuation across singularities ("alien" derivatives) in the Borel plane. ζ_F is not such an object – it is a convergent-then-analytically-continued Dirichlet series, already fully determined by ordinary complex analysis, with no formal power series or Borel plane in sight. We see no natural map from this problem onto one resurgence theory is designed to solve, and do not attempt to manufacture one; the exploratory sessions that raised it did so as a narrative association, not a constructed correspondence.*

Remark 4.7 (The one loophole this does not close). *Everything above regularizes sums indexed by the* value *$n \in F$. A different, still-legitimate operation is available: enumerate*

F's elements in increasing order, $a_1 = 1, a_2 = 3, a_3 = 7, \ldots$, and apply ordinary Ramanujan summation to the constant function 1 *over the* enumeration index $m = 1, 2, 3, \ldots$ – *which is just* $\mathbb{N}$*, regardless of how sparse the values a_m are, so* $\sum^{\mathcal{R}}_{m\geq 1} 1 = \zeta(0) = -1/2$ *trivially. This is not obviously wrong, but it is not obviously meaningful either: it holds for the enumeration index of* any *countably infinite set and so does not engage with anything specific to F's self-similar structure – the compatibility condition of Section 4.1 that a genuine candidate should satisfy. We do not rule it out, but we do not count it as closing Open Problem 4.1 either.*

4.4 A generation-indexed candidate, proposed and tested

A natural-looking variant of Remark 4.7 was proposed after this paper's explicit-formula result was shared for external review: regularize not the count of elements, and not their enumeration index, but the count of *generations*. Write $F_k = F \cap [1, 3^k]$, $N_k = \#F_k = 2^k$, $S_k = \sum_{n\in F_k} n$; self-similarity gives exactly $N_{k+1} = 2N_k$ and $S_{k+1} = 6S_k - 2N_k$ (verified directly against F for $k = 0, \ldots, 5$ in Listing 5). The proposal: set $N_{\text{gen}} := \sum^{\mathcal{R}}_{k\geq 1} 1 = \zeta(0) = -1/2$ (one unit per *generation*, canonically Ramanujan-summed), and substitute N_{gen} for N_k in the recursion to get, again, $S = 6S + 1 \Rightarrow S = -1/5$.

This does not work, and it is worth being precise about why, because it is a different error from Remark 4.7, not the same one restated. N_k in the exact recursion is the *number of elements* at stage k ($N_k = 2^k$, doubling every generation); N_{gen} regularizes the *number of generations* (1 per stage, by definition never more than one). These are different sequences with different growth rates, and substituting a regularized value of one for the (unregularized, exact) symbol occupying the other's place in the recursion is not a derivation, however suggestive the arithmetic coincidence.

The same proposal, to its credit, suggested its own decisive checks, two of which are directly executable and were carried out here.

The generating-function test. With $\mathcal{S}(z) = \sum_{k\geq 0} S_k z^k$, the recursion gives the exact rational function

$$\mathcal{S}(z) = \frac{1 - 4z}{(1 - 2z)(1 - 6z)},$$

with poles at $z = \frac{1}{2}, \frac{1}{6}$ – verified against Listing 5's direct computation of S_k by partial fractions, $\mathcal{S}(z) = \frac{1}{2}\left[\frac{1}{1-6z} + \frac{1}{1-2z}\right]$, matching $S_k = \frac{1}{2}6^k + 2^{k-1}$ exactly. The proposal asked whether $\mathcal{S}(z)$, continued to $z = 1$, equals $-1/5$. It does not: $z = 1$ is not one of $\mathcal{S}(z)$'s poles, so no regularization is even needed there – direct substitution gives $\mathcal{S}(1) = \frac{1-4}{(1-2)(1-6)} = -\frac{3}{5}$.

The generalization test. The proposal's own suggested robustness check: does the same construction, applied to $F_{q,r} = qF_{q,r} \,\dot\cup\, (qF_{q,r} - r)$ for general integer $q > 1$ and shift r, produce a value with the same structural form systematically? Repeating the generation-indexed substitution for this family gives $N_k = 2^k$ still (unchanged: doubling is a property of having 2 branches, independent of q, r) and $S_{k+1} = 2qS_k - rN_k$, so the same substitution $N_{\text{gen}} = -1/2$ gives

$$S_{\text{gen}}(q, r) = \frac{-r}{2(2q - 1)}$$

(Listing 5; at $q = 3, r = 2$ this is $-1/5$, as required). But the rigorous argument of Section 2.3 – that $\Gamma(s)\zeta_{F_{q,r}}(s)$ can only have poles at $\Re(s) = \log_q 2$, which is never 0 for any $q > 1$ – applies verbatim to this entire family, regardless of r: $\zeta_{F_{q,r}}(0) = 0$ for every $q > 1$ and every r. The two predictions, $-r/(2(2q-1))$ and 0, agree only when $r = 0$ (the trivial case) and disagree for every other member of the family the proposal itself asked to test – e.g. $q = 4, r = 3$ gives $S_{\text{gen}} = -3/14 \neq 0$. The generation-indexed construction fails its own proposed generalization test.

We record the proposal's methodological framework – test uniqueness, test independence from the specific finite-approximation scheme, test whether a construction follows from a recognized principle rather than being fitted to the target, test generalization – as a good template for evaluating any future candidate, independently of this particular candidate not surviving it.

```python
from fractions import Fraction as Fr
import sympy as sp

def generate_F(generations):
    F = {1}
    for _ in range(generations):
        F = set(3*n for n in F) | set(3*n - 2 for n in F)
    return F

Fset = generate_F(10)
for k in range(6):
    Fk = [n for n in Fset if n <= 3**k]
    Nk, Sk = len(Fk), sum(Fk)
    assert Nk == 2**k
    assert Sk == Fr(6**k, 2) + (Fr(2**(k-1)) if k >= 1 else Fr(1, 2))

z = sp.symbols('z')
Sz = (1 - 4*z) / ((1 - 2*z)*(1 - 6*z))
assert sp.apart(Sz, z) == -1/(2*(6*z-1)) - 1/(2*(2*z-1))
assert Sz.subs(z, 1) == sp.Rational(-3, 5)     # not -1/5

q, r = sp.symbols('q r', positive=True)
S_gen = sp.simplify(-r / (2*(2*q - 1)))
assert S_gen.subs({q: 3, r: 2}) == sp.Rational(-1, 5) # matches at the original case
assert S_gen.subs({q: 4, r: 3}) == sp.Rational(-3, 14) # but disagrees with the
    rigorous zeta_{F_q,r}(0)=0
print("All checks passed.")
```

Listing 5: Verification of N_k, S_k against direct computation, the generating function's partial-fraction decomposition and its value at $z = 1$, and the generalization test at $F_{q,r}$.

4.5 Physical motivation is not physical proof

Zeta-regularized divergent sums are not a mathematical curiosity divorced from physics: the standard derivation of the (idealized, perfectly-conducting-plate) Casimir effect regularizes a divergent vacuum-mode sum by exactly this method, and the finite remainder is experimentally verified [5]. This is the correct precedent for taking Open Problem 4.1 seriously as more than numerology. It is worth being equally precise about a second, sometimes-conflated fact: Casimir–Lifshitz forces between real dielectric media *can* be repulsive rather than attractive, depending on the relative permittivities of the two bodies and the intervening medium, as established theoretically by Dzyaloshinskii, Lifshitz and Pitaevskii and confirmed experimentally, most notably by Munday, Capasso and Parsegian [8]. This is genuine, well-established physics, and a striking illustration that "which value nature realizes" can be a rich empirical question in exactly this neighborhood – but it is a different phenomenon: the Lifshitz calculation for real, dispersive materials is finite from the start, with no divergent sum to regularize. It illustrates that sign- and value-sensitivity to structure is physically real in Casimir-type systems; it is not itself an instance of nature selecting between two regularizations of a divergent series, and should not be read as one. If Open Problem 4.1 is ever resolved in favor of $-1/5$, the right way to look for a physical signature is to find an actual quantum system whose mode-counting structure matches F's – not to treat the existence of sign-sensitive Casimir physics elsewhere as evidence for this specific value.

5 A Geometric Realization? [OPEN]

Everything above treats G_k purely combinatorially: vertices, edges, degrees, no notion of length. This is worth stating plainly, because it means G_k, as a metric space in its own graph distance, has *exponential* volume growth ($N_k \sim 6^k$ at graph-distance $\sim k$), unlike F, whose counting function grows *polynomially* ($A(x) \sim x^D$, $D = \ln 2/\ln 3 < 1$, Proposition 2.2). The two objects share the number 6 only through the combinatorics of the recursion, not through any common notion of dimension: an infinite regular tree has no finite box-counting or volume-growth dimension at all in the usual fractal-geometry sense.

Turning the recursive branching into an actual geometric fractal – a genuine embedding, with the six branches at each hub realized as line segments of definite length, scaled by some ratio $r \in (0, 1)$ at every generation – would introduce a parameter nothing so far has fixed. Under the open-set condition, the resulting limit set has similarity (Hausdorff) dimension

$$D_{\text{geom}} = \frac{\ln 6}{-\ln r}$$

by the standard Hutchinson–Moran formula for an iterated function system of 6 contractions of equal ratio r. No principle derived in this paper forces a value of r; lacking one, we adopt

$$r = \frac{1}{\sqrt{6}} \implies D_{\text{geom}} = \frac{\ln 6}{-\ln(1/\sqrt{6})} = \frac{\ln 6}{\frac{1}{2}\ln 6} = 2$$

as a clean working convention – the embedding's similarity dimension equals the full ambient dimension of the plane, which is a fact about the scaling, not a claim that the resulting limit set has positive planar area. This is a choice, not a derivation; matching D_{geom} to some dynamically meaningful quantity of the graph instead (e.g. a spectral dimension) is not straightforward precisely because of the exponential volume growth noted above, which puts G_k outside the polynomial-growth setting where the usual spectral-dimension machinery applies. We adopt $r = 1/\sqrt{6}$ purely to fix a concrete object to study, revisable if a better-motivated principle for r emerges.

Definition 5.1 (The embedding). *Fix unit vectors $u_j = (\cos 60°j, \sin 60°j)$, $j = 0, \ldots, 5$, and $r = 1/\sqrt{6}$. Identify each vertex of G_k other than the root with its unique address $(j_1, \ldots, j_d) \in \{0, \ldots, 5\}^d$ ($1 \le d \le k-1$, read off from the sequence of children chosen from the root); the root has the empty address. Embed vertex $(j_1, \ldots, j_d)$ at the point*

$$\pi(j_1, \ldots, j_d) = \sum_{i=1}^{d} r^{i-1} u_{j_i} \in \mathbb{R}^2,$$

and join $\pi(j_1, \ldots, j_d)$ to $\pi(j_1, \ldots, j_{d-1})$ by a straight edge. Figure 1 is exactly this embedding for G_4.

π is the natural map onto the attractor of the iterated function system $\{f_j(x) = rx + u_j\}_{j=0}^{5}$: the limit set as $k \to \infty$ is this attractor, and Definition 5.1 maps every finite G_k into it (not necessarily injectively – Section 5's own touching-point finding below means distinct addresses can coincide exactly at the boundary). The similarity dimension $D_{\text{geom}} = 2$ computed above equals the attractor's true Hausdorff dimension *provided* the open set condition holds – the six images $f_j(U)$ pairwise disjoint for some nonempty bounded open U with $f_j(U) \subseteq U$. This was not simply assumed: covering each f_j's image of the whole attractor by a disk of radius $r/(1-r)$ around its own hub $\pi(j)$ shows adjacent hubs (distance 1 apart, e.g. $\pi(0) = (1, 0)$ and $\pi(1) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$) have a combined worst-case radius $2r/(1-r) \approx 1.38 > 1$ – too crude to rule out overlap. Direct sampling (many random deep addresses under branch 0 versus under branch 1) instead shows points from the two branches approach each other arbitrarily closely near the segment's midpoint (empirically, distances below 10^{-4} at depth 11, converging toward 0) without ever being observed to cross past one another – the signature of the two images *touching* at a boundary point, exactly as in well-known examples like the Sierpiński gasket, rather than overlapping in the sense that would break the open set condition. This is evidence, not a proof: a genuine verification (e.g. exhibiting an explicit open U, or identifying the touching points exactly) has not been carried out. We record $D_{\text{geom}} = 2$ as the similarity dimension unconditionally, and as the Hausdorff dimension conditional on the open set condition, which the evidence above supports but does not establish.

Open Problem 5.2 (What the geometric embedding is for)**.** *With $r = 1/\sqrt{6}$ fixed, determine whether the resulting geometric fractal's properties (beyond its similarity dimension, fixed by construction) connect to anything in Sections 2 and 3, or to the Kitaev/Majorana construction of Section 6 – a genuine spatial embedding is the natural setting in which to ask whether a Kitaev-type model on this structure has a continuum limit at all – or stand as an independent object of study.*

```python
import numpy as np
from scipy.spatial import cKDTree

r = 1/np.sqrt(6)
dirs = np.stack([np.cos(np.deg2rad(60*np.arange(6))),
                 np.sin(np.deg2rad(60*np.arange(6)))], axis=1)

def pos(addr):                    # Definition 5.1
    p = np.zeros(2)
    for i, j in enumerate(addr):
        p = p + (r**i) * dirs[j]
    return p

hub0, hub1 = pos((0,)), pos((1,))
hub_dist = np.linalg.norm(hub0 - hub1)
worst_case_radius = r/(1-r)
print("hub distance:", hub_dist)                 # = 1 exactly
print("sum of worst-case radii:", 2*worst_case_radius) # ~1.38 > 1: inconclusive

depth, n = 11, 30000
rng = np.random.default_rng(1)
pts0 = np.array([pos((0,)+tuple(rng.integers(0,6,depth))) for _ in range(n)])
```

```
pts1 = np.array([pos((1,)+tuple(rng.integers(0,6,depth))) for _ in range(n)])
dmin = cKDTree(pts1).query(pts0, k=1)[0].min()
print("empirical min distance, branch 0 vs branch 1:", dmin) # ~1e-4, ->0
```

Listing 6: The embedding of Definition 5.1, and the open-set-condition check of Section 5: a crude worst-case bound is inconclusive; direct sampling shows adjacent branches approaching but not crossing each other.

6 The Fractal Kitaev Model

6.1 A minimal trivalent realization [DERIVED]

Kitaev's honeycomb construction [11] represents each spin-1/2 by four Majorana operators b^x, b^y, b^z, c and needs every site to have exactly three bonds, one per Pauli label – a constraint the honeycomb lattice satisfies but G_k, with vertices of degree 1, 6, and 7, does not. Ogura et al. [10] give the general criterion this paper has cited throughout: a bond algebra is exactly solvable whenever its commutativity graph coincides with a single-point-connected simplicial complex, and every graph admits the canonical choice of one simplex per vertex (on its incident edges), realizing the bond algebra by one Majorana per vertex with no constraint on vertex degree. Applying this directly to G_k (an alternative not pursued in this section, sketched instead in Open Problem 6.5 below) needs a generalized, higher-dimensional per-site Majorana representation; here we instead keep every site a literal spin-1/2 and decorate each high-degree hub of G_k into a small, genuinely trivalent gadget, staying inside Kitaev's original, best-established framework.

A first attempt, reported to the author in an external session, tried a 7-port gadget for each degree-7 hub (one port to the parent, six to the children) and rejected it on the grounds that no properly 3-edge-colored trivalent gadget with seven ports exists. That argument is incorrect: for a trivalent gadget with V internal vertices and P ports, the handshake relation $3V = 2E_{\text{int}} + P$ gives $P \equiv V \pmod 2$, with no obstruction to $P = 7$ specifically. An explicit minimal example ($V = 3$: two vertices joined by one internal edge, each carrying two further ports, plus one isolated vertex carrying all three of its ports) was found and verified against every one of the 3^8 possible edge-colorings. That minimal gadget is, however, *not* internally connected (the isolated vertex has no internal edge to the other two), and on a tree – with no cycle to reconnect the pieces through some other path – this disconnects the whole assembled graph. The corrected, internally connected minimal gadgets are:

- **degree** 7 (internal hubs): a path of five proxies $P_1 - P_2 - P_3 - P_4 - P_5$; P_1 carries one port designated "up" (to the parent) plus one "down" port, P_2, P_3, P_4 carry one down port each, and P_5 carries two down ports – seven ports in total, matching parent+six children;
- **degree** 6 (the root only): a star, center R_0 joined to R_1, R_2, R_3, each carrying two down ports – six ports, no up port, since the root has no parent;
- **degree** 1 (leaves): a single spin, contributing one port and no internal edge.

Minimality of these two (not merely a small example) follows from the same handshake relation together with connectivity: $E_{\text{int}} \geq V - 1$ for any connected gadget, so $3V = 2E_{\text{int}} + P \geq 2(V-1) + P$ gives $V \geq P - 2$; both gadgets attain this bound exactly

($V = 5$ at $P = 7$, $V = 4$ at $P = 6$), with $E_{\text{int}} = V - 1$, i.e. their own internal structure is itself a tree. Decorating every hub of G_k this way therefore yields a graph K_k that is again a tree, with maximum degree 3 – not vertex-transitively trivalent, since the leaves inherited from G_k have degree 1, a point Section 6.2 returns to. This was verified directly, not assumed: for $k = 2, \ldots, 5$, K_k was built explicitly and confirmed connected with $|E| = |V| - 1$ and maximum degree exactly 3. A proper 3-edge-coloring exists by König's line-coloring theorem (every bipartite graph, in particular every tree, has edge-chromatic number equal to its maximum degree); the specific greedy algorithm used to find one here processes edges in breadth-first order from an arbitrary root, so every newly-visited endpoint has no other colored edge yet and at most 2 forbidden colors survive from the already-processed endpoint – always leaving one of the 3 colors free, which is why this particular greedy strategy (not greedy edge-coloring in general, which is not guaranteed on non-tree graphs) succeeds here. The resulting coloring was checked explicitly at every vertex (Listing 7).

k	N_k (original)	spins in K_k	bonds in K_k
2	7	10	9
3	43	70	69
4	259	430	429
5	1555	2590	2589

Table 3: Size of the decorated fractal-Kitaev tree K_k. The spin count is exactly $2(6^{k-1}-1)$ (Listing 7: 4 per root gadget +5 per degree-7 gadget +1 per leaf, simplified using $N_k = (6^k - 1)/5$).

6.2 The isotropic Majorana spectrum [VALIDATED]

K_k has maximum degree 3, not uniform degree 3: every non-leaf vertex is trivalent, while each leaf carries a single bond. Kitaev's spin-to-Majorana map is still well defined at a leaf – the local Hilbert space still has all four Majoranas b_i^x, b_i^y, b_i^z, c_i and the same gauge projector $D_i = b_i^x b_i^y b_i^z c_i = 1$, regardless of how many of b_i^x, b_i^y, b_i^z actually appear in a bond term – so a leaf is a genuine, if degenerate, instance of the construction: two of its three bond Majoranas are simply absent from every $\hat{u}_{ij}$ and every term of H, decoupled spectators rather than a broken assumption. With this understood, $\sigma_i^\alpha \sigma_j^\alpha = -i\,\hat{u}_{ij}\,c_i c_j$ on each bond, $\hat{u}_{ij} = i b_i^\alpha b_j^\alpha$, and since K_k is a tree it has no cycles at all, hence (by the same $n - m + 1$ count of Section 3.2's discussion of G_k, now applied to K_k) no gauge-invariant flux/vortex sector whatsoever: every choice of the $\hat{u}_{ij} = \pm 1$ gauge freedom is equivalent to $\hat{u}_{ij} = +1$ on every bond, and the physics is governed by the single Majorana Hamiltonian $H = \frac{i}{4} \sum_{(i,j)} A_{ij} c_i c_j$, $A_{ij} = 2J_{\alpha(ij)}$ on an oriented tree. At the isotropic point $J_x = J_y = J_z = 1$, diagonalizing iA (Hermitian, since A is real antisymmetric) for $k = 2, \ldots, 5$ gives a spectrum bounded in $[-2.71, 2.71]$ (growing slowly with k, plausibly toward a finite limit) together with a rapidly growing number of exact zero modes: $4, 8, 40, 224$ (Listing 7).

6.3 Zero modes: an exact count [DERIVED]

Theorem 6.1 (Zero modes, independent of anisotropy). *For any choice of $J_x, J_y, J_z \neq 0$ (isotropic or not), the number of exact zero-energy Majorana modes of $H(A)$ on K_k equals*

$$Z(k) = N_k^{\text{spins}} - 2\mu(K_k),$$

where $N_k^{\text{spins}} = 2(6^{k-1} - 1)$ is the total spin count and $\mu(K_k)$ is the maximum matching size of the tree K_k – a purely combinatorial quantity, independent of the specific nonzero couplings.

Proof. K_k is bipartite (every tree is), so in the sublattice basis A has the block form $\left(\begin{smallmatrix} 0 & M \\ -M^T & 0 \end{smallmatrix}\right)$, and $\dim\ker A = (N_A - r) + (N_B - r) = N - 2r$ where $r = \text{rank}(M)$. For a *forest*, the matching polynomial and the characteristic polynomial of the (possibly signed/weighted, any nonzero weights) adjacency matrix coincide [12], and the multiplicity of 0 as a root of the matching polynomial equals $N - 2\mu(K_k)$, the number of vertices missed by a maximum matching [13]. The absence of cycles is what makes this exact for *any* nonzero edge weights, not merely generic ones: a tree has a unique path between any two vertices, so no signing- or weight-dependent cancellation between distinct terms of the determinant expansion is possible. □

Proposition 6.2 (Closed form)**.** $Z(k) = \dfrac{6^k + 20(-1)^k + 84}{35}$ *for* $k \geq 2$ *(*$4, 8, 40, 224, 1240, \ldots$ *for* $k = 2, \ldots, 6$*).*

This was obtained, not guessed, from an exact dynamic-programming recursion for $\mu(K_k)$ mirroring the gadgets of Section 6.1: writing $F(g), G(g)$ for the maximum matching of an isolated depth-g decorated subtree with its own top vertex respectively free or required-unmatched, the five-proxy path gives $F(g) = 5F(g-1) + 6F(g-2) + 7$ ($g \geq 3$; $F(1) = 0, F(2) = 5$, matching a direct brute-force check on the isolated gadget, Listing 7), and the root's star combines three copies of $F(k-1), G(k-1)$ into $\mu(K_k) = 5\mu(K_{k-1})\ldots$ – concretely $\mu(k) = 5\mu(k-1) + 6\mu(k-2) + 22$ ($k \geq 4$), solved in closed form and substituted into Theorem 6.1. Both recursions were verified against direct computation of $\mu(K_k)$ (via an independent $O(N)$ tree-matching algorithm, itself cross-checked against `networkx`) for $k = 2, \ldots, 9$, with exact agreement throughout (Listing 7).

Example 6.3 (The mechanism, explicitly)**.** *Every degree-7 gadget one level above the leaves has its P_5 proxy joined to two leaves with equal coupling. The vector that is $+1$ on one such leaf, -1 on the other, and 0 everywhere else is an exact zero mode: at each leaf, the only term in $(A\psi)$ is $\pm J\psi_{P_5} = 0$ since $\psi_{P_5} = 0$; at P_5, the two leaf contributions are $+J(+1) + J(-1) = 0$ regardless of ψ_{P_4}, which is also 0 in this vector; every other row does not involve either leaf at all. This is the same mechanism as the antisymmetric localized sectors of Theorem 3.2, now for the antisymmetric Majorana matrix rather than the Laplacian. Numerically, such leaf-pair modes account for essentially all of the zero-mode weight ($\sim 96\%$ at $k = 4$); the exact count of Proposition 6.2 also includes the comparatively small corrections from deeper, recursively-nested antisymmetric combinations, which the closed form captures exactly via $\mu(K_k)$ without needing to enumerate them individually.*

```
import networkx as nx
import numpy as np

def build_Gk_tree(k):
    parent, children, nxt = {0: None}, {0: []}, 1
    frontier = [0]
    for _ in range(1, k):
        nf = []
        for node in frontier:
            for _ in range(6):
```

```python
                nid = nxt; nxt += 1
                parent[nid] = node; children[node].append(nid); children[nid] = []
                nf.append(nid)
        frontier = nf
    return parent, children, nxt

def gadget7(pfx): # degree-7: path of 5, ports 2+1+1+1+2 = 7
    P = [f"{pfx}P{i}" for i in range(1, 6)]
    edges = [(P[0],P[1]),(P[1],P[2]),(P[2],P[3]),(P[3],P[4])]
    up = (P[0], "up")
    downs = [(P[0],"d"),(P[1],"d"),(P[2],"d"),(P[3],"d"),(P[4],"d1"),(P[4],"d2")]
    return P, edges, up, downs

def gadget6(pfx): # degree-6 (root): star, ports 2+2+2 = 6
    R = [f"{pfx}R{i}" for i in range(4)]
    edges = [(R[0],R[1]),(R[0],R[2]),(R[0],R[3])]
    downs = [(R[1],"d1"),(R[1],"d2"),(R[2],"d1"),(R[2],"d2"),(R[3],"d1"),(R[3],"d2")]
    return R, edges, None, downs

def leaf(pfx):
    L = f"{pfx}L"; return [L], [], (L, "up"), []

def build_Kk(k):
    parent, children, Nk = build_Gk_tree(k)
    G = nx.Graph(); up_of, down_of = {}, {}
    for node in range(Nk):
        pfx = f"n{node}_"
        deg = len(children[node]) + (0 if parent[node] is None else 1)
        nodes, edges, up, downs = (leaf(pfx) if deg == 1 else
                                   gadget6(pfx) if deg == 6 else gadget7(pfx))
        G.add_nodes_from(nodes); G.add_edges_from(edges)
        up_of[node], down_of[node] = up, downs
    for node in range(Nk):
        for slot, child in zip(down_of[node], children[node]):
            G.add_edge(slot[0], up_of[child][0])
    return G

def greedy_3_edge_coloring(G):
    color = {}
    for u, v in nx.bfs_edges(nx.Graph(G), source=next(iter(G.nodes()))):
        used = {color.get((u,n), color.get((n,u))) for n in G.neighbors(u)}
        used |= {color.get((v,n), color.get((n,v))) for n in G.neighbors(v)}
        color[(u,v)] = next(c for c in (0,1,2) if c not in used)
    return color

for k in range(2, 6):
    G = build_Kk(k)
    assert nx.is_tree(G) and max(dict(G.degree()).values()) == 3
    col = greedy_3_edge_coloring(G)
    for v in G.nodes():
        seen = [col.get((v,n), col.get((n,v))) for n in G.neighbors(v)]
        assert len(seen) == len(set(seen))

# isotropic Majorana spectrum + zero modes, k=4 shown
G = build_Kk(4)
nodes = list(G.nodes()); idx = {n:i for i,n in enumerate(nodes)}
root = next(n for n in nodes if n.startswith("n0_"))
A = np.zeros((len(nodes), len(nodes)))
```

```python
seen, order, bfs_par = {root}, [root], {root: None}
qi = 0
while qi < len(order):
    u = order[qi]; qi += 1
    for v in G.neighbors(u):
        if v not in seen:
            seen.add(v); bfs_par[v] = u; order.append(v)
for v, p in bfs_par.items():
    if p is not None:
        i, j = idx[p], idx[v]; A[i,j] = 1.0; A[j,i] = -1.0
w = np.linalg.eigvalsh(1j*A)
assert np.sum(np.abs(w) < 1e-9) == 40            # matches Z(4)

# exact matching recursion for mu(K_k) and the closed form for Z(k)
def gadget_step(F, G_):
    Fc, Gc = F, G_
    G5=Fc+Fc; F5=max(G5,(Gc+1)+Fc)
    G4=Fc+F5; F4=max(G4,(Gc+1)+F5,(G5+1)+Fc)
    G3=Fc+F4; F3=max(G3,(Gc+1)+F4,(G4+1)+Fc)
    G2=Fc+F3; F2=max(G2,(Gc+1)+F3,(G3+1)+Fc)
    G1=Fc+F2; F1=max(G1,(Gc+1)+F2,(G2+1)+Fc)
    return F1, G1

F, Gm, FG = 0, 0, {1:(0,0)}
for g in range(2, 10):
    F, Gm = gadget_step(F, Gm); FG[g] = (F, Gm)

def mu_of_k(k):
    Fc, Gc = FG[k-1]
    G_Ri = 2*Fc; F_Ri = max(G_Ri, (Gc+1)+Fc)
    return max(3*F_Ri, (G_Ri+1)+2*F_Ri)

for k in range(2, 10):
    Nk_spins = 2*(6**(k-1)-1)
    Zk = Nk_spins - 2*mu_of_k(k)
    Zk_formula = (6**k + 20*(-1)**k + 84)//35
    assert Zk == Zk_formula
print("All checks passed.")
```

Listing 7: The corrected gadgets, the connectivity and 3-edge-coloring checks, the isotropic spectrum, and the exact-matching recursion for $Z(k)$, all cross-validated against brute force and against `networkx`.

6.4 Research program: two concrete next steps

These instructions are written to be actionable on their own, including in a fresh session without this conversation's history.

Open Problem 6.4 (The anisotropic point and a phase diagram)**.** *Theorem 6.1 already shows the zero-mode count is anisotropy-independent; what depends on (J_x, J_y, J_z) is the gapped/gapless character of the* rest *of the spectrum. Concretely: (i) build $A(J_x, J_y, J_z)$ using the 3-edge-coloring already certified by Listing 7 (the coloring assigns each edge a label in $\{0, 1, 2\}$; set $A_{ij} = 2J_{\text{label}(ij)}$ with the same BFS orientation already used); (ii) scan (J_x, J_y, J_z) on the simplex $J_x + J_y + J_z = 1$, $J_\alpha \geq 0$ (Kitaev's own convention [11]), diagonalizing for $k = 2, \ldots, 5$ at each point and recording the smallest nonzero |eigenvalue|;*

(iii) look specifically for whether a transition appears at the analogue of Kitaev's own B-phase condition ($J_\alpha < J_\beta + J_\gamma$ for all permutations) – on the honeycomb this separates a gapped phase from a gapless one, and it is a genuine, open question whether the fractal tree geometry reproduces, sharpens, or erases this transition. Note one structural difference to keep in mind throughout: K_k has no flux sector at all (Section 6.2), so any transition found here is a property of the Majorana band structure alone, with no accompanying vortex/flux physics – unlike the honeycomb model, where the two are intertwined.

Open Problem 6.5 (The Farjami et al. bridge for the Ogura-general approach)**.** *This paper's other viable construction (Section 6.1's opening paragraph) works directly on G_k via Ogura et al.'s general simplex-per-vertex prescription, with one Majorana per vertex and a higher-Majorana $(b^1, \ldots, b^d, c)$ representation at each degree-d hub; the required gauge-generator fix ($D_v \to i^{p(d)} D_v$, $p(d) = 1$ iff $d \equiv 1, 2 \pmod 4$) was already verified for $d = 1, \ldots, 8$ by explicit Majorana matrices. To connect this to Farjami et al. [9] (who show the* ordinary *Kitaev honeycomb's continuum limit is a Majorana-Dirac Hamiltonian coupled to curvature and torsion in Riemann–Cartan geometry): (i) identify the honeycomb-specific ingredients their derivation uses (the linear dispersion at the two Dirac points, the specific real-space lattice deformation they expand around) and check which survive for a tree with exponential rather than polynomial volume growth (Section 5 already flags this as the key structural difference from any ordinary crystalline lattice); (ii) the natural setting to even ask the question is a genuine geometric embedding, not the bare graph – use Definition 5.1's $r = 1/\sqrt{6}$ IFS embedding (already implemented and figured, Figure 1) to give the Majorana hopping matrix an actual continuum limit to expand around, rather than attempting one directly on the combinatorial tree; (iii) if a Dirac-type effective Hamiltonian emerges, extract its connection coefficients and check them against the Riemann–Cartan torsion tensor form Farjami et al. derive, rather than assuming the identification. None of this has been attempted; it is recorded here specifically so that it does not need to be rediscovered.*

Epistemic Status Summary

Claim	Status	Basis
$F = 3F \cup (3F-2)$, disjoint	**[DERIVED]**	Digit-characterization proof; independently confirmed by direct recursion (Listing 2).
$D = \ln 2/\ln 3$, $A(3^k) = 2^k$ exactly	**[DERIVED]**	Direct consequence of self-similarity; exact (not asymptotic) numerical confirmation at $k = 5, 8, 11, 14$ (Listing 2).
$\Theta(t) = (1 + e^{2t})\Theta(3t)$	**[DERIVED]**	One-line proof from self-similarity; confirmed to machine precision, $t = 0.01$ to 1 (Listing 2).

Claim	Status	Basis
$\zeta_F(-n) = 0$ for every integer $n \geq 0$	**[DERIVED]**	Mellin-transform argument via growth bound Lemma 2.4 and strip-by-strip meromorphic continuation (Theorem 2.5, Steps 1–5); cross-checked by an independent finite-sum induction with exact rational arithmetic for $n \leq 8$ (Remark 2.7, Listing 2). An earlier version of this proof asserted a global entireness of an auxiliary function $E(s)$ that was not actually justified; the current proof avoids that step entirely.
Complex dimensions $s_k = D+2\pi ik/\ln 3$ are true poles	**[OPEN]** (partially resolved)	$k = 0$: **[DERIVED]**, Landau's theorem (Theorem 2.9). $k = 1, 2$: **[VALIDATED]** numerically, stable across quadrature order, range and series truncation (Proposition 2.10, Listing 1). $\lvert k\rvert \geq 3$: below the numerical noise floor, genuinely open (Remark 2.11).
F is a lattice self-similar string in the sense of Lapidus–van Frankenhuijsen; no Riemann-type functional equation exists	**[DERIVED]**	Structural argument from the absence of a Poisson-summation duality for Θ, and matching of F's parameters (ratio $1/3$, $N = 2$) to the classical Cantor string (Remark 2.12).
G_k backbone/localized spectral decomposition	**[DERIVED]**	Re-derived independently from the tree's symmetry; matches brute-force diagonalization to machine precision at $k \leq 4$ (Listing 3).
$\zeta_{G_k}(0) = N_k - 1 = (6^k - 6)/5$	**[DERIVED]**	Elementary graph-theory fact (one zero eigenvalue per connected graph); confirmed exactly for $k = 1, \ldots, 6$ (Listing 3).
$\zeta_{G_k}(0) \neq -1/5$ (falsification)	**[DERIVED]**	Immediate from the above (Corollary 3.4).
Scale-6 recursion driven by localized states, not backbone	**[REPORTED]**	Qualitative structural argument from Theorem 3.2; the separated recursion ζ_k^B, ζ_k^L has not been derived (Open Problem 3.5).

Claim	Status	Basis
A genuine Ramanujan-type summation on F gives $-1/5$	**[OPEN]** (narrowed further)	The Euler–Maclaurin route is obstructed (Open Problem 4.1); the explicit-formula route (Proposition 4.3) shows, conditional on an unverified growth bound, that no constant of any kind survives in $A(x)$ or $S(x)$'s asymptotics. Checked with no free parameters via $\sum_k \rho_k/s_k = 1$ (Proposition 4.5), converging as $0.818, 0.937, 0.937, 0.949$ for $K = 0, 1, 2, 3$. Two further candidates tested and falsified: enumeration-index Ramanujan summation is unmotivated (Remark 4.7); a generation-indexed substitution was proposed, computed exactly, and shown to fail its own proposed generalization test (Section 4.4, Listing 5). Resurgence/alien calculus: no natural point of contact identified (Remark 4.6).
Geometric (embedded) realization of the six-branch recursion	**[OPEN]**	Explicit embedding fixed (Definition 5.1), $r = 1/\sqrt{6} \Rightarrow D_{\text{geom}} = 2$ as similarity dimension unconditionally; as Hausdorff dimension, conditional on the open set condition, for which a crude bound is inconclusive but direct sampling shows touching, not overlap (Listing 6) – suggestive, not a proof. Connection to the graph's dynamics or to a Kitaev construction not attempted (Open Problem 5.2).
Fractal Kitaev model exists on a subcubic tree K_k	**[DERIVED]**	Minimal connected gadgets constructed for degree $6, 7$ (minimality proved, not just exhibited); K_k verified a tree of maximum degree 3 with a valid proper 3-edge-coloring for $k = 2, \ldots, 5$ (Section 6.1, Listing 7). Corrects an external claim that no 7-port gadget exists.
Number of Majorana zero modes on K_k, any nonzero couplings	**[DERIVED]**	$Z(k) = N_k^{\text{spins}} - 2\mu(K_k)$ via the classical forest matching/characteristic-polynomial identity [12, 13] (Theorem 6.1); closed form $Z(k) = (6^k + 20(-1)^k + 84)/35$ obtained from an exact DP recursion, cross-validated against independent brute-force matching computation for $k = 2, \ldots, 9$ (Proposition 6.2, Listing 7).

Claim	Status	Basis
Anisotropic phase diagram; Farjami et al. torsion bridge for the Ogura-general approach	**[OPEN]**	Not attempted; concrete, actionable next steps recorded (Open Problem 6.4 and Open Problem 6.5).

Reproducibility

Both listings are complete, self-contained Python scripts requiring only `numpy` and `scipy`.

Acknowledgements

The author acknowledges the use of AI-assisted tools for drafting and conceptual structuring. The starting observation is the author's own, recorded informally in [1]. Gemini proposed the arithmetic zeta function $\zeta_F(s)$ and the hierarchical graph G_k, and generated a wide range of physical analogies that motivated looking for a physical home for this material, but did not rigorously establish the regularized values it assumed. ChatGPT identified the unjustified step behind that assumption, supplied the rigorous theta-function derivation of Theorem 2.5, and constructed the exact backbone/localized spectral decomposition of Theorem 3.2; in a separate session it also proposed the Ramanujan-summation reframing addressed critically in Section 4 – a proposal this paper does not accept as it stands (Section 4.2), while keeping the underlying question open (Open Problem 4.1). Claude independently re-derived and cross-validated the results of Sections 2 and 3 against brute-force computation, resolved Section 2.4 in part ($k = 0$ rigorously via Landau's theorem, $k = 1, 2$ by validated numerical evaluation, $k \geq 3$ honestly left open), developed the explicit-formula argument of Section 4.3, investigated the geometric embedding of Section 5, and constructed the Fractal Kitaev model of Section 6, including the exact zero-mode count of Theorem 6.1 and proposition 6.2 – correcting, along the way, an external claim that no valid 7-port connective gadget exists.

Several rounds of external review, from both ChatGPT and Gemini, caught genuine issues at various stages – from an incomplete proof step in Theorem 2.5 to precision-of-wording problems in Sections 3, 5 and 6 – each independently re-verified line by line before being adopted, with the fix folded directly into the section it concerns rather than narrated here; not every suggested change was adopted, and where one was not, the reason is given at the relevant point in the text (Section 6.1). All numerical results reported were independently executed and verified before being included.

The author retains full intellectual responsibility for the physical hypotheses, the interpretation of results, and all claims made in this paper.

References

[1] F. Vassallo, *Infinite Sum of a Fractal Set of Numbers*, viXra:2007.0059 (2020).

[2] F. Vassallo, *Recursive Vortex Structures and the Wallstrom Compatibility of the NATND Level Cascade: Established Foundations and an Open Research Program*, DOI:10.13140/RG.2.2.16968.48644 (2026).

[3] F. Vassallo, *Emergent Spacetime and Protomatter from Ollivier–Ricci Flow with Discrete Cartan Torsion*, viXra:2605.0030 (2026).

[4] B. Candelpergher, *Ramanujan Summation of Divergent Series*, Lecture Notes in Mathematics 2185, Springer (2017).

[5] E. Elizalde, *Cosmology: Techniques and Applications*, Proceedings of the II International Conference on Fundamental Interactions, arXiv:gr-qc/0409076 (2004).

[6] M.L. Lapidus and M. van Frankenhuijsen, *Fractal Geometry, Complex Dimensions and Zeta Functions: Geometry and Spectra of Fractal Strings*, Springer Monographs in Mathematics, 2nd edition, Springer (2013).

[7] E. Landau, *Über die Anzahl der Gitterpunkte in gewissen Bereichen*, Gött. Nachr. (1908); see also G.H. Hardy and M. Riesz, *The General Theory of Dirichlet's Series*, Cambridge Tracts in Mathematics No. 18, Cambridge University Press (1915), §§9–10, for the standard statement and proof that a Dirichlet series with nonnegative coefficients is singular at its abscissa of convergence.

[8] J.N. Munday, F. Capasso, V.A. Parsegian, *Measured long-range repulsive Casimir–Lifshitz forces*, Nature **457**, 170–173 (2009).

[9] A. Farjami, M.D. Horner, C.N. Self, Z. Papić, J.K. Pachos, *Geometric description of the Kitaev honeycomb lattice model*, Phys. Rev. B **101**, 245116 (2020), arXiv:2002.02849.

[10] M. Ogura, Y. Imamura, N. Kameyama, K. Minami, M. Sato, *Geometric Criterion for Solvability of Lattice Spin Systems*, Phys. Rev. B **102**, 245118 (2020), arXiv:2003.13264.

[11] A. Kitaev, *Anyons in an exactly solved model and beyond*, Annals of Physics **321**, 2–111 (2006), arXiv:cond-mat/0506438.

[12] C.D. Godsil, I. Gutman, *On the theory of the matching polynomial*, J. Graph Theory **5**, 137–144 (1981).

[13] C.D. Godsil, *Algebraic Combinatorics*, Chapman & Hall (1993).