# Odd Distinct Covering Systems Supported on Any Six Primes

Idan Hackmon
Independent researcher

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**Abstract**

We present an argument excluding finite covering systems with distinct odd moduli whose least common multiple has at most six distinct prime factors, with unrestricted prime exponents and integer residues. The argument refines capped reweighting by comparing the positive-part expectations of every auxiliary divisor count with those of a product of independent finite geometric-tail factors. For the first six odd primes the resulting rational budget is approximately $0.91015158 < 1$. A digit-embedding prime-replacement lemma transfers the fixed-prime exclusion to arbitrary prime supports. The probability argument is finite: truncation preserves the small product atoms needed for the calculation, and a finite geometric sum bounds the mean. Both full statements have been verified in Lean and Aristotle, with independent checks of the returned source and unrestricted exponents. Historical priority is not asserted.

## 1 Statement and context

A distinct covering system is a finite family $\{r_m \pmod m : m \in D\}$, where $D$ is a finite set of integers greater than one and the union of the residue classes equals $\mathbb{Z}$.

**Theorem 1.** *Let $P$ be a set of at most six odd primes. Let $D$ be any finite set of distinct integers greater than one, each of whose prime factors belongs to $P$. For every choice of integer residues $r_m$, there is an integer $z$ with $z \not\equiv r_m \pmod m$ for every $m \in D$.*

**Corollary 2.** *If an odd distinct covering system exists, the least common multiple of its moduli has at least seven distinct prime factors.*

The word "odd" is essential. No conclusion allowing the prime 2 is intended. The theorem also does not settle the unrestricted Erdős–Selfridge conjecture.

This continues the author's investigation of small prime supports [1]. We use the capped reweighting framework of Balister, Bollobás, Morris, Sahasrabudhe and Tiba [2, 3]. The refinement here retains the whole family of positive-part expectations of auxiliary counts, instead of bounding only their first and second moments. The argument below includes the comparison and the finite truncation needed for its use.

## 2 A general prime-replacement lemma

**Lemma 3.** *Let $2 \le q \le p$ and $E \ge 0$. If $x = \sum_{j=0}^{E-1} d_j q^j$ with $0 \le d_j < q$, put $F(x) = \sum_{j=0}^{E-1} d_j p^j$. For $0 \le a \le E$, the inverse image under $F$ of a class modulo $p^a$ is empty or a single class modulo $q^a$ in $\mathbb{Z}/q^E\mathbb{Z}$.*

*Proof.* Reduction of $F(x)$ modulo $p^a$ keeps its first $a$ base-$p$ digits. If the prescribed prefix has a digit at least $q$, the inverse image is empty. Otherwise that prefix specifies exactly the first $a$ base-$q$ digits of $x$. This also covers $a = 0$. □

**Theorem 4.** *Let $q_1, \dots, q_k$ and $p_1, \dots, p_k$ be lists of distinct primes with $q_i \le p_i$. If the distinct nontrivial divisors of $Q = \prod_i q_i^{E_i}$ cannot support a covering system, then neither can the distinct nontrivial divisors of $N = \prod_i p_i^{E_i}$.*

*Proof.* Use CRT and Lemma 3 coordinatewise. A class with modulus $\prod_i p_i^{a_i}$ pulls back either to the empty set or to one class with modulus $\prod_i q_i^{a_i}$. Replace each empty preimage by an arbitrary class with that latter modulus, thereby enlarging the union. Unique factorization preserves distinctness and nontriviality of the moduli. A covering of $\mathbb{Z}/N\mathbb{Z}$ would therefore produce a covering of $\mathbb{Z}/Q\mathbb{Z}$, contradicting the hypothesis. □

## 3 Finite capped reweighting

Fix an ordered list $p_1, \dots, p_k$ of odd primes and a period $N = \prod_i p_i^{e_i}$. Increase each $e_i$ to at least eight. If necessary, add arbitrary classes for all missing nontrivial divisors of $N$; this preserves distinctness and can only enlarge the covered set.

In each coordinate $\mathbb{Z}/p^e\mathbb{Z}$, remove the selected pure-power classes and let $S_p$ be the surviving set. Their total uniform measure is at most $\sum_{a=1}^{e} p^{-a} < 1/(p-1)$, so $S_p$ is nonempty. Its uniform probability law $\lambda_p$ satisfies

$$\lambda_p(r \bmod p^a) \le c_p p^{-a}, \qquad c_p = \frac{p-1}{p-2}. \tag{1}$$

Work on the product of these survivor sets. Starting with their product law, expose mixed-modulus classes in order of their last prime. At a step with new prime $p$, let $B$ be their union and let $\alpha(x)$ be its $\lambda_p$-measure in the fibre above an old point $x$.

For $0 \le \delta < 1$, if $\alpha \le \delta$, delete the covered part and renormalize the uncovered part. If $\alpha > \delta$, multiply the uncovered weights by $1/(1-\delta)$ and the covered weights by $(\alpha-\delta)/(\alpha(1-\delta))$. Empty parts can be omitted. The construction preserves every old marginal. Each multiplier is at most $1/(1-\delta)$, hence its new conditional law $\kappa_x$ satisfies

$$\kappa_x(r \bmod p^a) \le C p^{-a}, \qquad C = \frac{c_p}{1-\delta}. \tag{2}$$

If an old point has zero mass, its conditional law can be chosen arbitrarily among laws satisfying this bound. Direct calculation gives

$$\mu_{\mathrm{new}}(B) = \frac{\mathbb{E}_{\mu_{\mathrm{old}}}(\alpha-\delta)_+}{1-\delta}, \qquad u_+ = \max(u, 0). \tag{3}$$

Later steps preserve this mass, since $B$ depends only on already exposed coordinates. Thus the sum of the step losses bounds the final covered mass. Pure-power classes have mass zero throughout.

## 4 Comparing auxiliary counts

**Lemma 5.** *Let $I_1, \dots, I_s$ be indicator random variables with marginal probabilities at most $b_1 \ge \cdots \ge b_s$, where $0 \le b_j \le 1$. Let $J_j = \mathbf{1}_{\{U \le b_j\}}$ for a common uniform random variable $U$. For every real $t$,*

$$\mathbb{E}\left(\sum_j I_j - t\right)_+ \le \mathbb{E}\left(\sum_j J_j - t\right)_+ .$$

*Proof.* For an integer $0 \le h \le s$, removing the first $h$ indicators gives $(\sum I_j - h)_+ \le \sum_{j>h} I_j$. Its expectation is at most $\sum_{j>h} b_j$, which equals the positive-part expectation of the nested family. For integer-valued counts the positive-part expectation is affine between consecutive integer thresholds. Negative thresholds follow from the first-moment inequality, and thresholds at least $s$ give zero on both sides. □

Assume $0 \le C \le p$. Define a finite random variable $A_e$ on $\{0, \dots, e\}$ by

$$\begin{aligned} &\mathbb{P}(A_e = 0) = 1 - C/p, \\ &\mathbb{P}(A_e = a) = C(p-1)/p^{a+1} && (1 \le a < e), \\ &\mathbb{P}(A_e = e) = C/p^e. \end{aligned}$$

Telescoping shows that these masses sum to one and $\mathbb{P}(A_e \ge a) = C/p^a$ for $1 \le a \le e$. Put $W_e = A_e + 1$.

**Lemma 6.** *Let $\mu_j$ be the measure after $j$ capped steps satisfying (2). Let $R_j = \prod_{i=1}^{j} W_{e_i}$, with independent comparison factors using the corresponding $p_i, C_i$. For every auxiliary choice of one class for each nontrivial divisor of the first $j$ coordinate periods, let $K$ count how many of those classes contain the sampled point. Then, for every real $t$,*

$$\mathbb{E}_{\mu_j}(K - t)_+ \le \mathbb{E}(R_j - 1 - t)_+.$$

*Proof.* For $j = 0$, both counts are zero. Suppose the assertion holds for the old coordinates. Decompose a new auxiliary count into its old part $K_0(x)$ and groups with new exponent $a \ge 1$. At fixed old point $x$, group $a$ has $1 + K_a(x)$ active old prefixes: the extra one is the pure $p^a$ modulus. Each corresponding new-coordinate indicator has conditional probability at most $C/p^a$.

Apply Lemma 5 conditionally, shifting the threshold by $K_0(x)$. The dominating nested count is

$$K_0(x) + \sum_{a=1}^{A_e} (1 + K_a(x)).$$

Here $A_e$ has the displayed finite law independently of $x$. For fixed $A_e = a$, subadditivity of the positive part gives

$$\left( a + \sum_{b=0}^{a} K_b - t \right)_+ \le \sum_{b=0}^{a} \left( K_b - \frac{t-a}{a+1} \right)_+ .$$

Every $K_b$ comes from one consistent auxiliary assignment on the old divisors. Apply the uniform induction hypothesis to each summand. The result is

$$(a+1)\mathbb{E}\left( R_{j-1} - 1 - \frac{t-a}{a+1} \right)_+ = \mathbb{E}((a+1)R_{j-1} - 1 - t)_+.$$

Averaging over $A_e$ completes the induction. Only the artificial comparison factors are independent; the proof assumes no independence of the actual processed coordinates. □

## 5 Step losses and finite evaluation

Choose an integer $h \ge 0$ with $h < p - 2$ and set $\delta = h/(p-2)$. The actual new mixed classes have nontrivial old prefixes, and their fibre union satisfies

$$\alpha(x) \le \sum_{a=1}^{e} c_p p^{-a} K_a(x), \qquad \sum_{a=1}^{e} c_p p^{-a} \le T := \frac{1}{p-2}.$$

These $K_a$ can be enlarged to full auxiliary counts if some classes are absent. Since $h \geq 0$ and $\delta = hT$,

$$(\alpha - \delta)_+ \leq \sum_{a=1}^{e} c_p p^{-a} (K_a - h)_+.$$

Combining Lemma 6 with (3) bounds this step by

$$\frac{\mathbb{E}(R_{j-1} - 1 - h)_+}{p - 2 - h}. \tag{4}$$

The mean of each comparison factor satisfies

$$\mathbb{E}W_e = 1 + C \sum_{a=1}^{e} p^{-a} \leq 1 + \frac{C}{p-1}.$$

For any positive integer-valued finite random variable $R$,

$$\mathbb{E}(R - 1 - h)_+ = \mathbb{E}R - (h+1) + \sum_{n=1}^{h} (h + 1 - n)\mathbb{P}(R = n). \tag{5}$$

This identity follows pointwise by separating $R \leq h$ from $R \geq h + 1$.

For $h \leq 7$ and $e_i \geq 8$, each needed atom $W_{e_i} = n \leq 7$ is unaffected by truncation. Define

$$g_{p,C}(1) = 1 - C/p, \qquad g_{p,C}(n) = C(p-1)/p^n \quad (n \geq 2).$$

The small product atoms are computed by the finite divisor convolution

$$a_{\varnothing}(n) = \mathbf{1}_{\{n=1\}}, \qquad a_{L \cup (p,C)}(n) = \sum_{d \mid n} g_{p,C}(d) a_L(n/d).$$

Meanwhile $\mathbb{E}R \leq M_L := \prod_{(p,C) \in L} (1 + C/(p-1))$. Thus the numerator in (4) is at most

$$M_L - (h+1) + \sum_{n=1}^{h} (h + 1 - n) a_L(n).$$

No infinite expectation or limiting argument is used.

## 6 Six primes and arbitrary supports

Use the following processing parameters:

| $p$ | $h$ | $\delta$ | $C = (p-1)/(p-2-h)$ |
|---|---|---|---|
| 3 | 0 | 0 | 2 |
| 7 | 1 | 1/5 | 3/2 |
| 5 | 1 | 1/3 | 2 |
| 11 | 3 | 1/3 | 5/3 |
| 13 | 5 | 5/11 | 2 |
| 17 | 7 | 7/15 | 2 |

All satisfy $0 \le \delta < 1$ and $0 \le C \le p$. The finite rational calculation above gives the total loss bound

$$\frac{2366560668680684606075303063}{260018300309514869222100000000} = 0.9101515800430254\ldots < 1.$$

For example the first three losses are 0, 1/12, and 8/21. The rational convolution is supplied in `CountProfileArithmetic.lean`. The profile used by the formal covering argument is evaluated directly in `Six/ProfileValues.lean` and `Six/Budget.lean`. An uncovered point therefore exists in the product of pure-power survivor sets. CRT gives an integer avoiding every class. This proves the fixed-prime exclusion for $\{3, 5, 7, 11, 13, 17\}$.

*Proof of Theorem 1.* Pad $P$ to six distinct odd primes if necessary and sort them increasingly as $p_1 < \cdots < p_6$. Choose finite exponents so every modulus in $D$ divides their product of powers. The first six odd primes satisfy $q_i \le p_i$ coordinatewise. Apply Theorem 4 to the fixed-prime exclusion. The auxiliary enlargement of exponents to at least eight imposes no restriction on the original exponents. □

## 7 Formalization status and attribution

The complete statements are proved in `Six/Final.lean`: six arbitrary odd primes with unrestricted natural exponents, and supports of at most six odd primes specified directly by prime-factor containment. Independent restatements pass in `Six/Check.lean` and in two separately generated check files. Their axiom closures contain only `propext`, `Classical.choice`, and `Quot.sound`.

Aristotle completed the proof project. Its final returned sources were compared byte for byte and rebuilt locally, with all 8,083 build jobs passing. The formal argument uses a rational closed-form profile and a finite mixture inequality; its budget is proved equal to the supplied rational calculation. Replacing exponents by $\max(e, 1)$ removes the positive-exponent assumption of intermediate lemmas. The release includes the source, independent checks, axiom reports, and reproduction commands.

The development uses Lean 4.28.0 with Mathlib revision

`8f9d9cff6bd728b17a24e163c9402775d9e6a365`.

This work was developed with AI assistance, including Codex and Aristotle. The underlying reweighting framework is credited to [2, 3]. Historical priority of the six-prime conclusion or of the comparison method is not asserted.

## References

[1] I. Hackmon, *On Odd Covering Systems with Few Prime Factors*, viXra:2604.0035v1 (2026), https://www.vixra.org/abs/2604.0035.

[2] P. Balister, B. Bollobás, R. Morris, J. Sahasrabudhe and M. Tiba, *On the Erdős Covering Problem: the density of the uncovered set*, arXiv:1811.03547 (2018), https://arxiv.org/abs/1811.03547.

[3] P. Balister, B. Bollobás, R. Morris, J. Sahasrabudhe and M. Tiba, *The Erdős–Selfridge problem with square-free moduli*, Algebra & Number Theory 15 (2021), 609–626, https://arxiv.org/abs/1901.11465.