Machine-Checked Positive-Area Evolution of the Infinite $\mathrm{SU}(2)$ Class Heat Kernel and Migdal Face Amplitudes

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Abstract

We give a kernel-checked positive-area calculus for the concrete SU(2) class heat kernel used in two-dimensional Yang–Mills theory. With irreducible label $n \in \mathbb{N}$, dimension $d_n = n + 1$, and Casimir $c_n = n(n + 2)/4$, the two-point class kernel is

$$K_t(g, h) = \sum_{n \geq 0} e^{-tc_n} \chi_n(g)\chi_n(h).$$

For every $t > 0$ and every derivative order $r \geq 0$, Lean verifies that the rth spectral jet converges and differentiates term by term to the $(r + 1)$st jet, and packages the family as a C^∞ map on the positive half-line. The proof supplies the explicit uniform majorant $(n + 1)^{2r+2} e^{-\varepsilon c_n}$ on $t > \varepsilon > 0$; thus the result is not an interface assumption about differentiability. We then differentiate the genuine normalized-Haar two-face Migdal integral, prove that its left and right area derivatives coincide with the Casimir-weighted derivative at the merged area, and show that the amplitude is infinitesimally invariant under the area transfer $(s, t) \mapsto (s + u, t - u)$. Finally, every normalized Wilson character is shown to satisfy its exact Casimir area ODE as an actual Haar integral against the infinite heat kernel. The artifact contains 25 new definitions and theorems, no local placeholders, and audited dependencies only on `propext`, `Classical.choice`, and `Quot.sound`. This is an analytic producer for positive-area heat evolution. It does not claim the four-face Makeenko–Migdal equation at a transverse crossing, whose remaining inputs are local Lie-group integration by parts and certified crossing geometry.

1 Result and claim boundary

Two-dimensional Yang–Mills theory has an exact heat-kernel description on finite planar graphs. A local Makeenko–Migdal equation differentiates a Wilson expectation with respect to the four face areas adjacent to a crossing and converts the alternating derivative into a contraction of the two cut strands [5, 4]. Three layers must not be conflated: the heat kernel must evolve differentiably in each positive area; the local area derivatives must be transferred through the Yang–Mills integral by an integration-by-parts argument; and the resulting Lie contraction must be evaluated in the chosen representation.

The finite-SU(2) contraction and trace-skein layer is treated in a companion formalization. The present paper addresses a different bottleneck: it constructs positive-time area derivatives of the *infinite* character kernel and transports them through the exact two-face Migdal integral. The distinction is summarized in Table 1.

The central sentence is therefore precise:

> *For the concrete infinite* SU(2) *character heat kernel, every positive-time spectral area jet exists with a kernel-checked uniform majorant, and the resulting derivative propagates exactly through the two-face Migdal Haar merge and Wilson-character observables.*

This result is stronger than a finite character cutoff and stronger than a package whose differentiability field is supplied by the user. It remains strictly weaker than the complete Makeenko–Migdal crossing theorem.

Table 1: Claim ledger. "Checked" means compiled in the accompanying Lean artifact; an open row is not imported as a hypothesis.

Statement	Status	Role
Positive-time termwise derivative of the infinite two-point class kernel	Checked	Analytic producer
All-order spectral jet recurrence with explicit uniform majorants	Checked	Positive-area regularity
Smooth C^∞ interface on the positive-time half-line	Checked	Reusable analytic interface
Differentiated left/right two-face Haar merge	Checked	Migdal face calculus
Zero derivative under $(s,t) \mapsto (s+u, t-u)$	Checked	Local area-transfer conservation
Casimir ODE for every normalized Wilson character integral	Checked	Observable consequence
Four-face alternating derivative at a transverse crossing	Open	Requires local integration by parts
PL embedding and crossing certification	Open	Geometric producer
Continuum projective-limit Yang–Mills field	Open	Outside this paper

2 The concrete $\mathrm{SU}(2)$ class heat kernel

Let μ_{H} be normalized Haar probability measure on SU(2). We use the irreducible characters χ_n, indexed by $n \in \mathbb{N}$, with

$$d_n = n+1, \qquad c_n = \frac{n(n+2)}{4}, \qquad |\chi_n(g)| \le n+1.$$

The Casimir convention gives $c_1 = 3/4$ in the fundamental representation. The concrete two-point class heat kernel is

$$K_t(g,h) = \sum_{n=0}^{\infty} e^{-tc_n} \chi_n(g)\chi_n(h), \qquad t > 0. \tag{1}$$

This is the two-point reproducing kernel on class functions. It is not the full group heat kernel written as a function of gh^{-1}. The concrete characters are real-valued—the substrate proves each one is the complex embedding of a real Chebyshev polynomial—so no conjugation is needed in (1). At $h = 1$, since $\chi_n(1) = d_n$, this agrees with the usual character density

$$K_t(g) = \sum_{n=0}^{\infty} d_n e^{-tc_n} \chi_n(g). \tag{2}$$

The formal substrate already proves the character bound, pointwise and uniform convergence, normalized Haar orthogonality, and the exact semigroup law

$$\int_{\mathrm{SU}(2)} K_s(g,x) K_t(k,x)\, \mu_{\mathrm{H}}(x) = K_{s+t}(g,k), \qquad s,t > 0. \tag{3}$$

Unlike a formal symbol for a heat kernel, equations (1)–(3) are built from concrete Chebyshev characters of Mathlib's special unitary matrix group and actual Bochner integrals.

3 All-order positive-area evolution

3.1 Spectral jets

For $r \in \mathbb{N}$, define the rth spectral area jet

$$J_r(t; g, h) = \sum_{n=0}^{\infty} (-c_n)^r e^{-tc_n} \chi_n(g)\chi_n(h). \tag{4}$$

The zeroth jet is $J_0 = K$. Formal differentiation suggests $\partial_t J_r = J_{r+1}$, but this conclusion requires locally uniform summability of the derivatives. The main analytic contribution is to discharge that requirement with an explicit family of majorants.

Lemma 3.1 (Uniform positive-time jet majorant)**.** *Fix $r \in \mathbb{N}$ and $\varepsilon > 0$. For every $t > \varepsilon$, $g, h \in \mathrm{SU}(2)$, and $n \in \mathbb{N}$,*

$$\left|(-c_n)^r e^{-tc_n} \chi_n(g)\chi_n(h)\right| \le (n+1)^{2r+2} e^{-\varepsilon c_n}. \tag{5}$$

The sequence on the right is summable.

Proof. The elementary bounds

$$0 \le c_n = \frac{n(n+2)}{4} \le (n+1)^2, \qquad |\chi_n(g)|, |\chi_n(h)| \le n+1$$

give the polynomial factor in (5); $t > \varepsilon$ gives $e^{-tc_n} \le e^{-\varepsilon c_n}$. Summability follows by comparison with a polynomial times a quadratic exponential. In Lean the comparison is reduced to the previously verified theorem `summable_pow_mul_exp_neg_casimir`; the exponent supplied to it is $2r+2$. □

Theorem 3.2 (Kernel-checked all-order area evolution)**.** *For every $r \in \mathbb{N}$, $t > 0$, and $g, h \in \mathrm{SU}(2)$, the series* (4) *converges and*

$$\frac{\partial}{\partial t} J_r(t; g, h) = J_{r+1}(t; g, h). \tag{6}$$

In particular, the concrete two-point class kernel is differentiable at every positive time and

$$\frac{\partial}{\partial t} K_t(g, h) = -\sum_{n=0}^{\infty} c_n e^{-tc_n} \chi_n(g)\chi_n(h). \tag{7}$$

Proof. Fix $t > 0$ and take $\varepsilon = t/2$. Each summand has the elementary derivative in (6). Lemma 3.1, applied at order $r+1$, supplies a summable bound on the derivatives throughout the connected open interval (ε, ∞). A uniform-limit differentiation theorem therefore identifies the derivative of the infinite sum with the sum of the derivatives. The formal proof uses `hasDerivAt_tsum_of_isPreconnected`; all its hypotheses are produced in the artifact. No differentiability axiom or opaque heat-equation field is introduced. □

Corollary 3.3 (Smooth positive-time interface)**.** *For every $r \in \mathbb{N}$ and $g, h \in \mathrm{SU}(2)$, the map $t \mapsto J_r(t; g, h)$ is C^∞ on $(0, \infty)$. Its ordinary derivative there is the next jet J_{r+1}.*

Proof. Lean first exposes the pointwise derivative as the declaration

`deriv_su2ClassHeatKernelAreaJet`.

A finite-order induction using the open-domain characterization of `ContDiffOn` then proves every C^m statement and packages them with `contDiffOn_infty`. The public endpoint is

`contDiffOn_su2ClassHeatKernelAreaJet`.

It introduces no new analytic hypothesis beyond Theorem 3.2. □

The theorem gives an explicit positive-time jet rather than merely an existence assertion. It also exposes the expected singular boundary: the majorants are uniform on $t > \varepsilon > 0$, not at $t = 0$. Nothing in the paper claims differentiability of the point-mass initial condition at zero.

4 Differentiated Migdal face merging

For two faces of areas $s, t > 0$ sharing an integrated holonomy x, define

$$D_{s,t}^{g,k}(x) = K_s(g, x) K_t(k, x). \tag{8}$$

Equation (3) is the exact two-face Migdal move. Combining it with Theorem 3.2 produces area derivatives of an actual Haar integral.

Theorem 4.1 (Differentiated two-face Migdal merge)**.** *For $s, t > 0$ and $g, k \in \mathrm{SU}(2)$,*

$$\frac{\partial}{\partial s} \int_{\mathrm{SU}(2)} D_{s,t}^{g,k}(x)\, \mu_{\mathrm{H}}(x) = J_1(s+t; g, k), \tag{9}$$

$$\frac{\partial}{\partial t} \int_{\mathrm{SU}(2)} D_{s,t}^{g,k}(x)\, \mu_{\mathrm{H}}(x) = J_1(s+t; g, k). \tag{10}$$

Proof. The semigroup theorem identifies both integrals in a positive neighborhood of the evaluation point with $K_{s+t}(g,k)$. Theorem 3.2 and the chain rule give (9) and (10). In the formal statement, the left sides retain the literal normalized Haar integrals; they are not replaced by a newly defined abbreviation before differentiation. $\square$

The local geometry of this theorem is shown in Figure 1. The internal edge is eliminated, and only the total area survives.

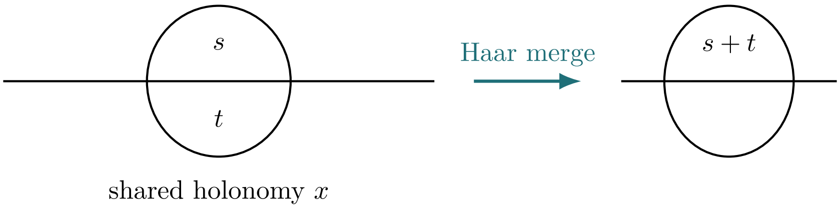

Figure 1: The two-face density integrates to a single heat kernel at the merged area. Differentiating either incident area produces the same spectral jet.

Corollary 4.2 (Infinitesimal area-transfer conservation)**.** *For $s, t > 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_{u=0} \int_{\mathrm{SU}(2)} D^{g,k}_{s+u,t-u}(x)\,\mu_{\mathrm{H}}(x) = 0. \tag{11}$$

Proof. For all sufficiently small u, both areas remain positive and the semigroup identity makes the integral equal to $K_{s+t}(g,k)$, a constant in u. Lean constructs the positive neighborhood explicitly and applies the derivative of a locally constant function. $\square$

Corollary 4.2 is a genuine local face identity, but it should not be confused with the four-face crossing equation. It is the conservation law for an eliminated internal edge. The crossing theorem requires a nontrivial observable whose edge derivatives survive the alternating area variation.

5 Wilson-character area ODE

Let

$$W_n(g) = \frac{1}{n+1}\chi_n(g)$$

be the normalized Wilson character. Orthogonality and the heat-kernel expansion give the exact simple-loop amplitude

$$\mathcal{W}_n(t) := \int_{\mathrm{SU}(2)} W_n(g) K_t(g)\,\mu_{\mathrm{H}}(g) = e^{-tc_n}. \tag{12}$$

Theorem 5.1 (Exact Casimir area ODE)**.** *For every $n \in \mathbb{N}$ and $t > 0$,*

$$\mathcal{W}_n'(t) = -c_n \mathcal{W}_n(t). \tag{13}$$

In the fundamental representation, $\mathcal{W}_1'(t) = -(3/4)\mathcal{W}_1(t)$.

Proof. The previously checked identity (12) holds at every positive time. It therefore holds throughout a neighborhood of the chosen $t > 0$. Differentiating the right side yields (13). The Lean conclusion is stated with `deriv` applied to the original Haar integral, and the final right side contains that same integral; the closed form is used as a proved bridge, not as a redefinition of the observable. $\square$

The ODE is classical. The contribution here is the exact mechanized chain from the concrete infinite character density and normalized Haar integral to the area derivative, with the $n(n+2)/4$ convention visible at every stage.

6 Relation to the Makeenko–Migdal crossing equation

Suppose a planar loop has a simple crossing with adjacent face areas t_1, t_2, t_3, t_4. In one common normalization the rigorous planar equation has the schematic form

$$(\partial_{t_1} - \partial_{t_2} + \partial_{t_3} - \partial_{t_4})\,\mathbb{E}[W(L)] = \mathbb{E}[W(L_1)W(L_2)], \tag{14}$$

with group- and normalization-dependent interpretation of the right side [5]. Theorem 3.2 supplies the positive-area derivatives of each heat factor from its convergent infinite series. Theorem 4.1 shows that this calculus survives a concrete Haar face elimination. A companion artifact evaluates the finite-SU(2) Lie contraction and closes its product of traces on two single-trace reconnections.

These results leave a sharply localized gap. To derive (14), one must still prove, for the edge-variable Yang–Mills density and an observable with extended gauge invariance at the crossing, that the alternating four-face derivative equals the appropriate mixed invariant-gradient contraction. In the classical proof this is the local integration-by-parts theorem of Levy and of Driver–Hall–Kemp [4, 5]. The present formalization neither assumes that theorem nor encodes it as a field.

Thus the certified dependency chain is

$$\boxed{\text{positive-area heat evolution}} \longrightarrow \boxed{\text{local integration by parts [open]}}$$
$$\longrightarrow \boxed{\text{SU(2) trace-skein closure}}.$$

The middle box, together with a certified embedded crossing, is the exact frontier for a complete machine-checked equation.

7 Formal architecture and audit

The new implementation is concentrated in two files. The producer module has 25 top-level definitions and theorems and 436 physical lines in the frozen source; the audit module checks the headline types and prints axiom dependencies. The proof reuses the pinned SU(2) substrate for characters, Haar measure, the infinite heat kernel, orthogonality, the semigroup law, and the exact simple-loop integral.

7.1 Exact public declarations

The all-order analytic core appears as follows (line breaks shortened for print; the smoothness order is rendered as the ASCII word `infinity`):

```
theorem hasDerivAt_su2ClassHeatKernelAreaJet
    (r : Nat) {t : Real} (ht : 0 < t) (g h : SU2) :
    HasDerivAt
      (fun s : Real => su2ClassHeatKernelAreaJet r s g h)
      (su2ClassHeatKernelAreaJet (r + 1) t g h) t

theorem contDiffOn_su2ClassHeatKernelAreaJet
    (r : Nat) (g h : SU2) :
    ContDiffOn Real infinity
      (fun t : Real => su2ClassHeatKernelAreaJet r t g h)
      (Set.Ioi 0)

theorem hasDerivAt_integral_su2TwoFaceClassDensity_left
    {s t : Real} (hs : 0 < s) (ht : 0 < t) (g k : SU2) :
    HasDerivAt
      (fun u : Real => integral
        (fun h : SU2 => su2TwoFaceClassDensity u t g k h)
        su2HaarProb)
      (su2ClassHeatKernelAreaJet 1 (s + t) g k) s

theorem hasDerivAt_integral_su2TwoFace_areaTransfer_zero
    {s t : Real} (hs : 0 < s) (ht : 0 < t) (g k : SU2) :
    HasDerivAt
      (fun u : Real => integral
        (fun h : SU2 =>
          su2TwoFaceClassDensity (s + u) (t - u) g k h)
        su2HaarProb)
      0 0
```

The Wilson endpoint is:

```
theorem deriv_integral_su2NormalizedWilson_mul_heatKernel
    {t : Real} (ht : 0 < t) (n : Nat) :
    deriv (fun s : Real => integral
      (fun g : SU2 => su2NormalizedWilsonCharacter n g *
        heatKernelCharacterSeries su2CharacterTable s g)
      su2HaarProb) t =
    -(su2CasimirEigenvalue n : Complex) *
      integral (fun g : SU2 =>
        su2NormalizedWilsonCharacter n g *
          heatKernelCharacterSeries su2CharacterTable t g)
        su2HaarProb
```

These declarations make the quantifiers, positivity hypotheses, integral measure, derivative point, and Casimir coefficient independently inspectable.

7.2 Dependency audit

The audit prints

```
'...hasDerivAt_su2ClassHeatKernelAreaJet' depends on axioms:
  [propext, Classical.choice, Quot.sound]
'...contDiffOn_su2ClassHeatKernelAreaJet' depends on axioms:
  [propext, Classical.choice, Quot.sound]
'...hasDerivAt_integral_su2TwoFaceClassDensity_left' depends on axioms:
  [propext, Classical.choice, Quot.sound]
'...hasDerivAt_integral_su2TwoFace_areaTransfer_zero' depends on axioms:
  [propext, Classical.choice, Quot.sound]
'...deriv_integral_su2NormalizedWilson_mul_heatKernel' depends on axioms:
  [propext, Classical.choice, Quot.sound]
```

These are the standard logical dependencies inherited from Mathlib; the new modules contain no local `axiom`, `sorry`, or `admit`. The verification script compiles the producer and audit modules separately, then runs a placeholder scan.

8 Prior art and exact delta

Heat-kernel formulations and exact two-dimensional Yang–Mills amplitudes are classical. Migdal introduced recursion equations [1]; Driver proved continuum expectations and lattice convergence on the plane [2]; Levy developed Markovian holonomy fields and the master field [3, 4]; and Driver, Hall, and Kemp gave three local proofs of the planar Makeenko–Migdal equation [5]. The compact-surface extension is due to Driver, Gabriel, Hall, and Kemp [6]. Recent work also analyzes convergence of discrete Makeenko–Migdal equations to the continuum [7].

No new classical heat equation or loop equation is claimed here. The delta is a proof-assistant artifact with all of the following simultaneously:

1. the concrete infinite SU(2) character kernel rather than a finite cutoff;
2. a uniform positive-time majorant at every derivative order;
3. an all-order termwise spectral-jet theorem;
4. a public C^∞ interface on the positive-time half-line;
5. derivatives of literal normalized-Haar two-face integrals;
6. a local area-transfer conservation theorem; and
7. an observable Casimir ODE retaining the original Haar integral in its formal conclusion.

A targeted search found Mathlib support for Haar measure, calculus of series, and Lie-group structures, but did not locate a prior Lean artifact containing this exact SU(2) heat-kernel area calculus. We do not make a priority claim; an exhaustive cross-prover search would be required for one.

9 Limitations and migration note

Limitations. The restriction $t > 0$ is essential to the present majorant argument. The paper does not establish estimates uniform as $t \downarrow 0$. It treats the class heat kernel and the exact two-face Haar merge, not arbitrary smooth observables on $\mathrm{SU}(2)^E$. The local invariant vector fields, their Laplacian, and the integration-by-parts identity at a crossing remain to be formalized. There is no PL embedding, isotopy theorem, or continuum projective limit in the artifact.

Migration note. The finite-cellulation heat-kernel substrate remains the reference for exact disk reduction and the simple-loop area law. The submitted trace-skein paper remains the reference for the finite-SU(2) contraction and algebraic closure of crossing terms. The present paper is a new sequel: it reuses those proved definitions, adds positive-area analytic evolution, and neither replaces nor duplicates their main theorems. Until a public identifier exists for the trace-skein companion, it should be cited as a submitted companion preprint, not as a published record.

10 Conclusion

The infinite SU(2) class heat kernel now has a machine-checked positive-area differential calculus. Every spectral jet differentiates to the next under a uniform, explicit, summable Casimir majorant, and the entire hierarchy is exported as a C^∞ map on positive time. That derivative survives the genuine two-face Migdal Haar integral, detects identical left and right face evolution, vanishes under area-preserving transfer across the eliminated edge, and yields the exact Wilson-character Casimir ODE.

This closes a real analytic producer that the trace-skein formalization did not contain. It also leaves the next theorem honest and sharply stated: a complete formal Makeenko–Migdal crossing equation requires a local Lie-group integration-by-parts producer and certified planar crossing data. Neither is hidden in the present package.

A Theorem-to-artifact map

Table 2: Main mathematical statements and exact Lean declarations.

Paper statement	Lean declaration
Lemma 3.1	`norm_su2ClassHeatKernelAreaJetTerm_le_majorant`, `summable_su2AreaJetMajorant`
Theorem 3.2	`hasDerivAt_su2ClassHeatKernelAreaJet`, `su2ClassHeatKernelAreaJet_zero`
Smooth interface	`contDiffOn_su2ClassHeatKernelAreaJet`
First derivative (7)	`hasDerivAt_su2ClassHeatKernel_time`
Theorem 4.1	`hasDerivAt_integral_su2TwoFaceClassDensity_left`, `..._right`
Corollary 4.2	`hasDerivAt_integral_su2TwoFace_areaTransfer_zero`
Theorem 5.1	`deriv_integral_su2NormalizedWilson_mul_heatKernel`

B Reproduction protocol

The source archive pins Lean `v4.29.0-rc6`. The pinned Mathlib commit is

`07642720480157414db592fa85b626dafb71355b`.

From the archive root:

```
lake update
lake env lean Lean2dYangMills/SU2HeatKernelAreaEvolution.lean
lake env lean Lean2dYangMills/AuditSU2HeatKernelAreaEvolution.lean
```

The supplied PowerShell wrapper `verify-su2-area-evolution.ps1` performs the same two compilations and the local-placeholder scan. The release manifest records SHA-256 values for the PDF, TeX source, source ZIP, audit transcript, claim boundary, and verification script. Because the authorized publication route accepts only a manuscript PDF, the companion archive must be supplied directly to an independent reviewer unless a separate public repository is later authorized. No local filesystem path is presented as a public artifact locator.

References

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[4] T. Levy, *The master field on the plane*, Astérisque 388 (2017); arXiv:1112.2452.

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[8] L. Eriksson, *A machine-checked exact evaluation of the two-dimensional* SU(2) *heat-kernel lattice model on certified finite combinatorial disk cellulations*, ai.viXra:2607.0039 (2026), with version update pending.