Machine-Checked Haar Integration by Parts on Finite $\mathrm{SU}(2)$ Edge Spaces: Pauli Flows and Two-Generator Transfer

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Abstract

We formalize the finite-dimensional integration-by-parts mechanism used in local proofs of the Makeenko–Migdal equation. For a measure-preserving real flow T_t on a finite measure space, Lean verifies differentiation under the integral from an explicit locally uniform integrable majorant and proves $\int Df \, \mathrm{d}\mu = 0$. A bounded dominated product interface then yields

$$\int (Df)g \, \mathrm{d}\mu = -\int f(Dg) \, \mathrm{d}\mu,$$

and two successive applications transfer a mixed pair of generators from a density to an observable with positive sign. These results are instantiated on normalized Haar probability measure of concrete $\mathrm{SU}(2)$ and on every finite product $\mathrm{SU}(2)^{\mathcal{E}}$, for left and right multiplication of one selected edge. To fix the representation normalization, we construct three explicit trigonometric curves in $\mathrm{SU}(2)$ and prove entrywise that their tangents at the identity are $i\sigma_1/2, i\sigma_2/2, i\sigma_3/2$. The producer contains 30 public definitions, structures, and theorems in 519 physical lines; its audit contains no local `sorry`, `admit`, or `axiom`, and the headline results depend only on `propext`, `Classical.choice`, and `Quot.sound`. This closes the Haar integration-by-parts layer. It does not claim the full four-area Makeenko–Migdal identity: the remaining formal inputs are the heat-density directional identity, extended gauge invariance at a crossing, and their geometric assembly.

1 Result and claim boundary

Rigorous local proofs of the planar Makeenko–Migdal equation reduce the loop equation to finite-dimensional operations on edge holonomies. In the edge-variable proof of Driver, Hall, and Kemp, an alternating derivative in the four adjacent face areas is first converted into a mixed invariant gradient, and Haar integration by parts moves the relevant derivatives from the Yang–Mills density to the observable [1]. Their compact-surface extension uses the same local right-multiplication convention $a_i \mapsto a_i e^{sX}$ [2].

This paper isolates and machine-checks the analytic transfer layer. It starts above the level of a particular heat kernel: the central theorem applies to any genuinely measure-preserving flow for which differentiation is justified by a supplied majorant. It then descends to the concrete normalized Haar measure of Mathlib's matrix group $\mathrm{SU}(2)$ and to arbitrary finite edge sets. Table 1 records the exact boundary.

The central result can be stated without reference to a formalization:

> *On a finite* $\mathrm{SU}(2)$ *edge space with normalized product Haar measure, any right- or left-invariant edge generator satisfying the explicit dominated flow interface is skew-symmetric for the bilinear integral pairing $B(f,g) = \int fg \, \mathrm{d}\mu$; two ordered generators can therefore be transferred from the density to the observable with positive sign.*

No invariance of an integral is postulated. Measure preservation is proved for the concrete translations, and the equality of integrals is derived from the pushforward theorem.

2 A dominated measure-preserving flow theorem

Let (A, μ) be a measurable space with finite measure and let $T : \mathbb{R} \times A \to A$. Write $T_t(x) = T(t,x)$. We do not require a global flow law $T_{s+t} = T_s \circ T_t$; the argument needs only

$$(T_t)_*\mu = \mu \quad (t \in \mathbb{R}), \qquad T_0 = \mathrm{id}\,. \tag{1}$$

Table 1: Claim ledger. “Checked” means compiled in the accompanying Lean artifact; open rows are not introduced as hypotheses of a claimed loop equation.

Statement	Status	Role
Zero integral of a dominated generator along a measure-preserving flow	Checked	Abstract analytic producer
Product-rule Haar integration by parts	Checked	One-derivative transfer
Positive-sign transfer of two ordered generators	Checked	Mixed-gradient interface
Left/right single-edge flows on $\mathrm{SU}(2)^{\mathcal{E}}$ preserve product Haar measure	Checked	Edge-variable specialization
Three identity-based curves tangent to $i\sigma_a/2$	Checked	Normalization certificate
Alternating four-area derivative equals a density mixed derivative	Open	Heat-density local identity
Extended gauge invariance at a certified crossing	Open	Crossing-specific cancellation
Complete Makeenko–Migdal equation	Open	Requires the two preceding rows

For $f, D : A \to \mathbb{C}$, assume that f is integrable, D is almost everywhere strongly measurable, and there exist $\varepsilon > 0$ and an integrable $b : A \to \mathbb{R}$ such that for every x and $|t| < \varepsilon$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} f(T_s x) = D(T_t x), \tag{2}$$

$$\|D(T_t x)\| \le b(x). \tag{3}$$

Theorem 2.1 (Generator integral vanishes)**.** *Under* (1)–(3)*,*

$$\int_A D(x)\ \mathrm{d}\mu(x) = 0. \tag{4}$$

Proof. Set $F(t,x) = f(T_t x)$ and $F'(t,x) = D(T_t x)$. The hypotheses of the dominated local differentiation theorem for Bochner integrals follow from (2) and (3). Thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \int_A F(t,x)\ \mathrm{d}\mu(x) = \int_A D(x)\ \mathrm{d}\mu(x).$$

On the other hand, pushforward invariance gives

$$\int_A f(T_t x)\ \mathrm{d}\mu(x) = \int_A f(x)\ \mathrm{d}((T_t)_*\mu)(x) = \int_A f(x)\ \mathrm{d}\mu(x).$$

The left side is constant in t, so its derivative is zero. Lean verifies both derivatives and identifies them using uniqueness of the derivative. $\square$

The formal theorem is stronger than an “invariant integral” interface: its input is a `MeasurePreserving` map, and the invariant integral is an internal consequence. It also makes the analytic burden visible. A merely pointwise derivative is not enough; the majorant in (3) is part of the theorem’s public data.

3 Product-rule integration by parts

For reusable applications, the artifact packages the assumptions for two functions in a structure

$$\mathcal{D}(\mu, T; f, g, Df, Dg).$$

It contains a common positive radius, four nonnegative constant bounds, almost-everywhere strong measurability of f, g, Df, Dg, local uniform norm bounds along T_t, and the two pointwise derivative identities. Constant bounds suffice because the measure is finite; in the concrete applications it is a probability measure.

Theorem 3.1 (Dominated integration by parts)**.** *Assume* (1) *and* $\mathcal{D}(\mu, T; f, g, Df, Dg)$*. Then*

$$\int_A Df(x) g(x)\ \mathrm{d}\mu(x) = -\int_A f(x) Dg(x)\ \mathrm{d}\mu(x). \tag{5}$$

Proof. Apply Theorem 2.1 to $h = fg$. The product rule gives $Dh = (Df)g + f(Dg)$. The four constant bounds in $\mathcal{D}$ produce the integrable majorant

$$C = B_{Df}B_g + B_f B_{Dg}.$$

The same data separately prove integrability of $(Df)g$ and $f(Dg)$, so the Bochner integral splits over the sum. Rearranging the resulting zero-sum identity gives (5). $\square$

The ordering in (5) is retained even though the scalar codomain is $\mathbb{C}$; this makes the interface compatible with later noncommutative extensions. Equation (5) is a skew-symmetry statement for the bilinear pairing $B(f, g) = \int fg \, \mathrm{d}\mu$, not a claim about adjointness for the Hermitian pairing $\int f\overline{g} \, \mathrm{d}\mu$.

3.1 Two ordered generators

Let $T^{(1)}, T^{(2)}$ be two measure-preserving flows. Name the derivatives of a density ρ by $D_2\rho, D_1D_2\rho$, and those of an observable f by $D_1 f, D_2 D_1 f$. No commutation theorem is assumed between the flows.

Theorem 3.2 (Two-generator transfer)**.** *If the dominated pair interfaces needed for the two indicated applications of Theorem 3.1 hold, then*

$$\int_A (D_1 D_2 \rho) f \, \mathrm{d}\mu = \int_A \rho(D_2 D_1 f) \, \mathrm{d}\mu. \tag{6}$$

Proof. The first integration by parts gives

$$\int (D_1 D_2 \rho) f = -\int (D_2\rho)(D_1 f).$$

The second gives

$$\int (D_2\rho)(D_1 f) = -\int \rho(D_2 D_1 f).$$

The signs cancel. The formal statement names $D_1D_2\rho$ and D_2D_1f separately, so it cannot silently use commutativity of mixed derivatives. $\square$

This two-step theorem is the exact sign pattern used when a mixed derivative is transferred from a Yang–Mills density to a crossing observable.

4 Finite $\mathrm{SU}(2)$ edge configurations

Let $\mathcal{E}$ be a finite type of oriented edge coordinates and equip $\mathrm{SU}(2)^{\mathcal{E}}$ with normalized product Haar measure $\mu_{\mathrm{H}}^{\mathcal{E}}$. For an edge $e \in \mathcal{E}$ and $h \in \mathrm{SU}(2)$, define the single-edge multiplier

$$m_{e,h}(j) = \begin{cases} h, & j = e, \\ 1, & j \neq e. \end{cases} \tag{7}$$

Given a curve $\gamma : \mathbb{R} \to \mathrm{SU}(2)$, define

$$L_{e,\gamma}(t, U) = m_{e,\gamma(t)} U, \tag{8}$$

$$R_{e,\gamma}(t, U) = U m_{e,\gamma(t)}. \tag{9}$$

Multiplication is pointwise on $\mathrm{SU}(2)^{\mathcal{E}}$.

Theorem 4.1 (Single-edge Haar invariance)**.** *For every finite $\mathcal{E}$, every $e \in \mathcal{E}$, every curve γ, and every $t \in \mathbb{R}$, both maps in (8)–(9) preserve $\mu_{\mathrm{H}}^{\mathcal{E}}$. If $\gamma(0) = 1$, both maps at $t = 0$ are the identity.*

Proof. The finite product of compact groups is a compact group, and its normalized Haar probability measure is invariant under left and right multiplication. Apply those library theorems to the concrete group element $m_{e,\gamma(t)}$. The identity statement follows coordinatewise from (7). $\square$

Combining Theorems 3.1, 3.2, and 4.1 gives compiled left- and right-generator integration by parts on every finite edge space, plus the concrete right-right two-generator transfer

$$\int_{\mathrm{SU}(2)^{\mathcal{E}}} (D_1 D_2 \rho)(U) f(U) \, \mathrm{d}\mu_{\mathrm{H}}^{\mathcal{E}}(U) = \int_{\mathrm{SU}(2)^{\mathcal{E}}} \rho(U)(D_2 D_1 f)(U) \, \mathrm{d}\mu_{\mathrm{H}}^{\mathcal{E}}(U). \tag{10}$$

The right convention matches the invariant derivatives used in the compact surface formulation of Driver–Gabriel–Hall–Kemp [2].

5 Pauli normalization certificates

An abstract curve would leave the normalization of the Lie algebra generator external. The artifact therefore constructs three explicit curves. Put $c(t) = \cos(t/2)$ and $s(t) = \sin(t/2)$. In the fundamental representation,

$$\gamma_X(t) = \begin{pmatrix} c(t) & is(t) \\ is(t) & c(t) \end{pmatrix}, \qquad X = \frac{i\sigma_1}{2} = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}, \tag{11}$$

$$\gamma_Y(t) = \begin{pmatrix} c(t) & s(t) \\ -s(t) & c(t) \end{pmatrix}, \qquad Y = \frac{i\sigma_2}{2} = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \tag{12}$$

$$\gamma_Z(t) = \begin{pmatrix} c(t) + is(t) & 0 \\ 0 & c(t) - is(t) \end{pmatrix}, \qquad Z = \frac{i\sigma_3}{2} = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}. \tag{13}$$

Theorem 5.1 (Certified Pauli tangents)**.** *Each $\gamma_a(t)$ lies in concrete* SU(2)*, satisfies $\gamma_a(0) = 1$, and for all $i, j \in \{0, 1\}$,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\gamma_a(t))_{ij} = (A_a)_{ij}, \qquad (A_X, A_Y, A_Z) = (X, Y, Z). \tag{14}$$

Proof. Each curve is built through the previously checked equivalence between SU(2) and the unit sphere in $\mathbb{C}^2$. The identity $\cos^2(t/2) + \sin^2(t/2) = 1$ discharges membership in the sphere. Lean then expands the four matrix entries, performs all twelve scalar derivative calculations, and verifies the conjugation signs in the lower row. In particular, the factor 1/2 is proved rather than inferred from notation. □

The same Pauli matrices were used in the companion trace–skein artifact, where their quadratic Casimir and Fierz contraction were checked. Theorem 5.1 now proves that those matrices are the actual infinitesimal generators of concrete Haar-preserving edge flows.

6 How this enters the local loop equation

For a crossing with four adjacent faces, the local Makeenko–Migdal operator is the alternating area derivative

$$\partial_{t_1} - \partial_{t_2} + \partial_{t_3} - \partial_{t_4}. \tag{15}$$

Driver–Hall–Kemp prove that, under extended gauge invariance, applying (15) to the local Yang–Mills integral yields the negative of a mixed invariant-gradient integral [1, Theorem 2.6]. Their local proof then uses Haar integration by parts to move derivatives from the density to the observable. The certified dependency chain is shown in Figure 1.

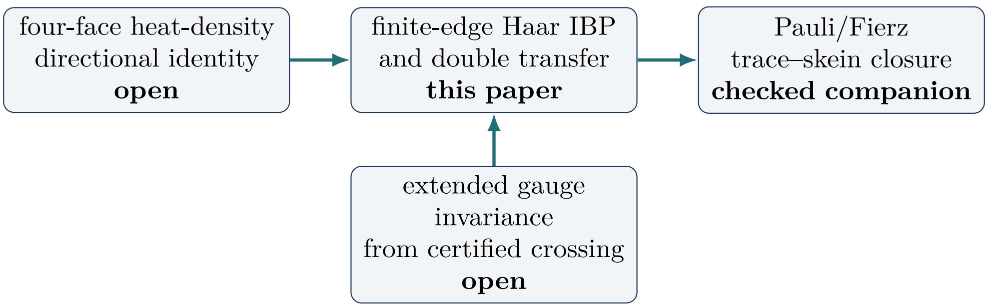

Figure 1: The present result closes the analytic transfer layer but does not collapse the open producers on its left into a claimed theorem.

Theorem 3.2 exactly certifies the two minus signs contributed by the two integrations by parts. Theorem 5.1 fixes the invariant directions that subsequently enter the Fierz contraction. What remains is not "integration by parts" in disguise: one must still prove that the alternating area derivative of the concrete four-face density is the named mixed generator and that extended gauge invariance combines the incident edges with the required signs. Neither statement is assumed here.

7 Formal architecture and audit

The new artifact consists of one 519-line producer and one 44-line audit module. The producer exposes 30 top-level declarations: one structure, 11 definitions, and 18 theorems. Table 2 lists the headline map.

Table 2: Paper statements and exact Lean declarations.

Paper statement	Lean declaration
Theorem 2.1	`integral_generator_eq_zero_of_measurePreserving_flow`
Dominated pair interface	`DominatedFlowPair`
Theorem 3.1	`integral_generator_mul_eq_neg_integral_mul_generator`
Theorem 3.2	`integral_two_generators_transfer`
Theorem 4.1	`measurePreserving_su2EdgeLeftFlow`, `measurePreserving_su2EdgeRightFlow`
Equation (10)	`integral_su2Edge_two_rightGenerators_transfer`
Theorem 5.1	`hasDerivAt_su2PauliXCurve_entry`, `...Y...`, `...Z...`

The public shape of the central declarations is:

```
theorem integral_generator_mul_eq_neg_integral_mul_generator
    {mu : Measure Alpha} [IsFiniteMeasure mu]
    (T : Real -> Alpha -> Alpha)
    (hT : forall t, MeasurePreserving (T t) mu mu)
    (hT_zero : forall x, T 0 x = x)
    (f g Df Dg : Alpha -> Complex)
    (P : DominatedFlowPair mu T f g Df Dg) :
    (integral fun x => Df x * g x) =
      -(integral fun x => f x * Dg x)

theorem integral_two_generators_transfer ... :
    (integral fun x => D1D2rho x * f x) =
      integral fun x => rho x * D2D1f x
```

The displayed code shortens measure annotations and binders only for print; the audit transcript records the complete elaborated types.

The axiom audit prints, for all headline results,

```
depends on axioms: [propext, Classical.choice, Quot.sound]
```

These are standard logical dependencies inherited from Mathlib. A separate placeholder scan covers both new modules and rejects any occurrence of `sorry`, `admit`, or `axiom`.

8 Prior art and formalization delta

Finite-dimensional integration by parts on compact groups is classical. The novel mathematical identity is not claimed here. Driver, Hall, and Kemp give three rigorous local proofs of the planar Makeenko–Migdal equation and state the mixed-gradient formula under extended gauge invariance [1]. Driver, Gabriel, Hall, and Kemp extend the local equation to compact surfaces and explicitly formulate invariant derivatives through right multiplication of edge variables [2]. Driver's functional-integral account also emphasizes that the previous rigorous proofs are finite-dimensional integration-by-parts arguments [3]. Recent work studies passage from discrete loop equations to the continuum [4].

The exact formalization delta is the simultaneous presence of:

1. differentiation of an invariant integral derived from `MeasurePreserving`, not supplied as an equality field;
2. an explicit locally uniform integrable majorant for the generator;
3. a reusable bounded product interface and proved integrability of both terms in the product rule;
4. ordered two-generator transfer without assuming commutation;
5. concrete left and right translations on arbitrary finite SU(2) edge products; and
6. concrete SU(2) curves with all three Pauli tangents checked entrywise.

Mathlib supplies the measure theory, Bochner integration, compact-group Haar measure, and differentiation infrastructure. We do not make a priority claim for proof assistants; an exhaustive cross-prover survey would be needed for that.

9 Limitations and migration note

Limitations. The theorem uses explicit bounded dominated-flow data. It does not yet derive those bounds automatically from smoothness and compactness, nor does it build a general Lie derivative or Laplace–Beltrami operator. The Pauli curves are certified entrywise at zero; the artifact does not need or claim a global matrix-exponential theorem. Most importantly, it does not prove the heat-density identity that turns (15) into the appropriate mixed edge derivative, nor the extended gauge invariance arising from a crossing. Consequently this paper must not be cited as a complete formal proof of the Makeenko–Migdal equation.

Migration note. This is a new sequel, not a replacement. The previously submitted positive-area evolution paper remains the analytic source for differentiating the infinite class heat kernel. The trace–skein paper remains the algebraic source for the Pauli/Fierz contraction. The present paper supplies the integration-by-parts bridge between those layers. Until public ai.viXra identifiers are assigned, the two companions should be described as submitted manuscripts, not published records.

10 Conclusion

The finite-edge Haar transfer mechanism needed by local loop equations is now machine checked. Measure preservation plus a dominated derivative gives a zero integrated generator; the product rule gives skew-symmetry for the bilinear integral pairing; a second application transfers two ordered derivatives with positive sign. The result holds on every finite product of concrete SU(2) Haar spaces, on either side of a selected edge, and the three Pauli normalizations are certified by explicit group-valued curves.

This closes a genuine missing layer without overstating the endpoint. The remaining route to a complete formal local Makeenko–Migdal equation is now sharply localized: connect the alternating four-area heat derivative to the mixed edge generator, prove the required extended gauge invariance from crossing geometry, and compose those producers with the present transfer and the existing trace–skein closure.

A Reproduction protocol

The exact source archive pins Lean `v4.29.0-rc6` and Mathlib commit

`07642720480157414db592fa85b626dafb71355b`.

From the archive root:

```
lake update
lake exe cache get
lake build Lean2dYangMills.SU2HaarIntegrationByParts
lake env lean Lean2dYangMills/AuditSU2HaarIntegrationByParts.lean
```

The supplied wrapper `verify-su2-haar-ibp.ps1` performs both compilations, the two-file placeholder scan, and the exact public-declaration count. The release manifest records SHA-256 values for the manuscript, source, archive, audit transcript, and verification script. The authorized publication route accepts only the PDF; the source archive is therefore a direct-review artifact and is not represented as public until a separate public route exists.

References

[1] B. K. Driver, B. C. Hall, and T. Kemp, *Three proofs of the Makeenko–Migdal equation for Yang–Mills theory on the plane*, Communications in Mathematical Physics 351 (2017), 741–774; arXiv:1601.06283.

[2] B. K. Driver, F. Gabriel, B. C. Hall, and T. Kemp, *The Makeenko–Migdal equation for Yang–Mills theory on compact surfaces*, Communications in Mathematical Physics 352 (2017), 967–978; arXiv:1602.03905.

[3] B. K. Driver, *A Functional Integral Approaches to the Makeenko–Migdal Equations*, arXiv:1709.04041 (2017; rev. 2019).

[4] H. Shen, S. A. Smith, and R. Zhu, *Makeenko–Migdal equations for 2D Yang–Mills: from lattice to continuum*, arXiv:2412.15422 (2024).