Machine-Checked Finite-Edge $\mathrm{SU}(2)$ Crossing Ward Identity: Pauli Generator Transfer and Single-Trace Closure

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Abstract

We formalize a finite-dimensional crossing Ward identity for fundamental SU(2) Wilson words. On every finite edge space $\mathrm{SU}(2)^{\mathcal{E}}$, two distinct coordinates are inserted into a concrete normalized trace word. Lean proves entrywise that right multiplication by each of three explicit SU(2) curves produces the normalized Pauli generators $i\sigma_a/2$, differentiates the trace word once at each selected coordinate, and identifies the Pauli-summed mixed derivative with the rank-two Fierz contraction. Two applications of Haar integration by parts transfer the corresponding mixed generators from a density to the Wilson word. The resulting integral closes exactly on the direct and reverse single-trace resolutions, with coefficients $-1/4$ and $-1/2$ in the chosen normalization. The producer contains 19 public declarations in 487 physical lines; all 13 new theorems depend only on `propext`, `Classical.choice`, and `Quot.sound`, with no local `sorry`, `admit`, or `axiom`. This is not a full Makeenko–Migdal area equation: the remaining physical input is a weak four-face identity against crossing-certified extended-gauge-invariant observables. In the program's heat time, generated by the Pauli Laplacian, its coefficient is $kappa = 2$.

1 Result and exact claim boundary

The rigorous planar Makeenko–Migdal equation is local at a transverse self-intersection. In edge variables, one first rewrites the alternating derivative in the four adjacent face areas as a mixed invariant derivative, and then transfers the group derivatives from the Yang–Mills density to the Wilson observable by Haar integration by parts [3]. The same local mechanism extends to compact surfaces [4]; a functional-integral derivation explains its continuum origin [5].

This paper closes the finite-edge generator side of that mechanism for the fundamental representation of SU(2). It connects two previously separate machine-checked layers: Haar generator transfer and finite-rank Pauli/trace–skein algebra. Its central statement is:

> *Under explicit dominated-flow data for a density on a finite* SU(2) *edge space, the sum of the three Pauli mixed density generators, paired with a crossing Wilson word, equals the density paired with a fixed linear combination of the two single-trace crossing resolutions.*

Table 1 separates this theorem from the area identity that is still open.

The result is a new sequel, not a replacement for either input paper. It is also distinct from the closed lattice loop equations used in the recent finite-N bootstrap program [6]: here the contribution is a kernel-checked local transfer and closure theorem with its regularity data made explicit.

2 The crossing word and its infinitesimal generators

Let $\mathcal{E}$ be a finite type, let $e_1, e_2 \in \mathcal{E}$ be distinct, and write $U = (U_e)_{e\in\mathcal{E}} \in \mathrm{SU}(2)^{\mathcal{E}}$. Fix $A, B \in \mathrm{SU}(2)$, representing the portions of the two strands between the selected local coordinates.

Table 1: Claim ledger. “Checked” means compiled in the accompanying Lean artifact. The open rows are not smuggled into a claimed area equation.

Statement	Status	Role
Concrete two-coordinate fundamental Wilson word on $\mathrm{SU}(2)^{\mathcal{E}}$	Checked	Crossing observable
First and ordered mixed Pauli derivatives at distinct edges	Checked	Generator normalization
Pauli sum equals the rank-two Fierz contraction	Checked	Finite-N algebra
Closure on direct and reverse single traces	Checked	SU(2)-specific output
Two Haar transfers for each Pauli direction	Checked	Analytic Ward step
Three-direction finite-edge crossing Ward identity	Checked	Headline theorem
Weak four-face identity with $\kappa = 2$	Open	Requires extended gauge invariance
Extended gauge invariance from embedded crossing data	Open	Geometric assembly
Full Makeenko–Migdal area equation	Open	Requires the two preceding rows

For a complex 2×2 matrix M, set

$$\operatorname{tr}_2(M) = \frac{1}{2}\operatorname{Tr}(M). \tag{1}$$

The crossing word is

$$\mathcal{W}_{A,B}(U) = \operatorname{tr}_2((AU_{e_1})(BU_{e_2}))\,. \tag{2}$$

For matrices $X, Y \in M_2(\mathbb{C})$ define the named first and ordered mixed generators

$$\mathcal{D}_{1,X}\mathcal{W}(U) = \operatorname{tr}_2((AU_{e_1})X(BU_{e_2}))\,, \tag{3}$$

$$\mathcal{D}_{2,Y}\mathcal{D}_{1,X}\mathcal{W}(U) = \operatorname{tr}_2((AU_{e_1})X(BU_{e_2})Y)\,. \tag{4}$$

These expressions are not merely declared to be derivatives. The producer first proves a matrix-word differentiation lemma.

Lemma 2.1 (Entrywise matrix-word derivative)**.** *Let $\gamma : \mathbb{R} \to \mathrm{SU}(2)$ and suppose that, for all $i, j \in \{0, 1\}$,*

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \gamma(s)_{ij} = X_{ij}.$$

Then for arbitrary $P, Q \in M_2(\mathbb{C})$,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=t} \operatorname{tr}_2(P\gamma(s)Q) = \operatorname{tr}_2(PXQ). \tag{5}$$

Proof. Lean expands the two diagonal entries of $P\gamma(s)Q$. The resulting scalar expression has four moving matrix entries, each with a supplied derivative. The proof applies the product and sum rules entrywise and divides the final trace by two. No differentiable-manifold instance for the subtype SU(2) is used. □

For a curve γ define the right action on one coordinate by

$$(R_{e,\gamma}(s,U))_j = \begin{cases} U_e\gamma(s), & j = e, \\ U_j, & j \neq e. \end{cases} \tag{6}$$

Lemma 2.1, associativity of matrix multiplication, and $e_1 \neq e_2$ give

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{W}(R_{e_1,\gamma}(s,U)) = \mathcal{D}_{1,X}\mathcal{W}(U), \tag{7}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathcal{D}_{1,X}\mathcal{W}(R_{e_2,\eta}(s,U)) = \mathcal{D}_{2,Y}\mathcal{D}_{1,X}\mathcal{W}(U), \tag{8}$$

whenever the entrywise tangents of γ and η at zero are X and Y respectively.

3 Concrete Pauli directions

Put $c(t) = \cos(t/2)$ and $s(t) = \sin(t/2)$. The imported Haar-IBP artifact constructs the following curves as elements of concrete SU(2):

$$\gamma_X(t) = \begin{pmatrix} c(t) & is(t) \\ is(t) & c(t) \end{pmatrix}, \qquad X = \frac{i\sigma_1}{2} = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}, \tag{9}$$

$$\gamma_Y(t) = \begin{pmatrix} c(t) & s(t) \\ -s(t) & c(t) \end{pmatrix}, \qquad Y = \frac{i\sigma_2}{2} = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \tag{10}$$

$$\gamma_Z(t) = \begin{pmatrix} c(t) + is(t) & 0 \\ 0 & c(t) - is(t) \end{pmatrix}, \qquad Z = \frac{i\sigma_3}{2} = \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}. \tag{11}$$

Each curve satisfies $\gamma_a(0) = 1$, preserves product Haar measure through (6), and has entrywise tangent $A_a \in \{X, Y, Z\}$ at zero.

Theorem 3.1 (Certified crossing derivatives). *For $a \in \{X, Y, Z\}$ and $e_1 \neq e_2$, the first derivative of $\mathcal{W}_{A,B}$ along R_{e_1,γ_a} is exactly $\mathcal{D}_{1,A_a}\mathcal{W}$. Differentiating that function along R_{e_2,γ_a} gives exactly $\mathcal{D}_{2,A_a}\mathcal{D}_{1,A_a}\mathcal{W}$.*

The producer contains six public specialization theorems, one first and one second step for each of the three directions. Thus the matrices used in the Fierz contraction below are proved to be the infinitesimal generators of the actual selected-edge actions.

4 Pointwise Pauli and trace–skein closure

Set

$$G = AU_{e_1}, \qquad H = BU_{e_2}.$$

The Pauli-summed mixed generator is

$$\mathcal{C}(G, H) = \operatorname{tr}_2(GXHX) + \operatorname{tr}_2(GYHY) + \operatorname{tr}_2(GZHZ). \tag{12}$$

The rank-two Fierz identity in the chosen normalization gives

$$\mathcal{C}(G, H) = -\operatorname{tr}_2(G)\operatorname{tr}_2(H) + \frac{1}{4}\operatorname{tr}_2(GH). \tag{13}$$

For $G, H \in \mathrm{SU}(2)$, the normalized trace–skein identity reads

$$\operatorname{tr}_2(G)\operatorname{tr}_2(H) = \frac{1}{2}\big(\operatorname{tr}_2(GH) + \operatorname{tr}_2(GH^{-1})\big). \tag{14}$$

Combining (13) and (14) yields

$$\mathcal{C}(G, H) = -\frac{1}{4}\operatorname{tr}_2(GH) - \frac{1}{2}\operatorname{tr}_2(GH^{-1}). \tag{15}$$

Define the direct and reverse crossing resolutions by

$$\mathcal{W}_{\mathrm{dir}}(U) = \operatorname{tr}_2((AU_{e_1})(BU_{e_2}))\,, \tag{16}$$

$$\mathcal{W}_{\mathrm{rev}}(U) = \operatorname{tr}_2\Big((AU_{e_1})(BU_{e_2})^{-1}\Big)\,. \tag{17}$$

Theorem 4.1 (Pointwise single-trace closure). *For every finite edge configuration U,*

$$\sum_{a \in \{X,Y,Z\}} \mathcal{D}_{2,A_a}\mathcal{D}_{1,A_a}\mathcal{W}(U) = -\frac{1}{4}\mathcal{W}_{\mathrm{dir}}(U) - \frac{1}{2}\mathcal{W}_{\mathrm{rev}}(U). \tag{18}$$

The two terms on the right are single normalized traces. This is the SU(2)-specific linear closure that makes the finite-N loop equations especially tractable [6]. Equation (18) itself is purely pointwise matrix algebra; no measure, heat kernel, or area parameter occurs in it.

5 Haar transfer and the crossing Ward identity

Equip $\mathrm{SU}(2)^{\mathcal{E}}$ with normalized product Haar measure $\mu_{\mathrm{H}}^{\mathcal{E}}$. The actions (6) preserve this measure. For each Pauli direction a, let

$$D_{2,a}\rho, \qquad D_{1,a}D_{2,a}\rho$$

be named density generators. The formal assumptions are not a black-box "Ward identity": they are two instances of an explicit dominated-flow record. Each record contains a positive local radius, measurability, constant norm bounds along the selected flow, and the relevant pointwise derivative identities. These data justify differentiation under the integral and guarantee integrability of all products below.

The imported two-generator Haar theorem then gives, direction by direction,

$$\int_{\mathrm{SU}(2)^{\mathcal{E}}} (D_{1,a}D_{2,a}\rho)(U)\mathcal{W}(U)\,\mathrm{d}\mu_{\mathrm{H}}^{\mathcal{E}}(U) = \int_{\mathrm{SU}(2)^{\mathcal{E}}} \rho(U)(\mathcal{D}_{2,A_a}\mathcal{D}_{1,A_a}\mathcal{W})(U)\,\mathrm{d}\mu_{\mathrm{H}}^{\mathcal{E}}(U). \tag{19}$$

The positive sign records the cancellation of the two minus signs from integration by parts.

Theorem 5.1 (Finite-edge SU(2) crossing Ward identity)**.** *Under the six explicit dominated-flow records needed for the two transfers in each of the X, Y, Z directions,*

$$\begin{aligned}\int_{\mathrm{SU}(2)^{\mathcal{E}}} &\left(\sum_{a\in\{X,Y,Z\}} D_{1,a}D_{2,a}\rho(U)\right)\mathcal{W}(U)\,\mathrm{d}\mu_{\mathrm{H}}^{\mathcal{E}}(U)\\ &= \int_{\mathrm{SU}(2)^{\mathcal{E}}} \rho(U)\left(-\frac{1}{4}\mathcal{W}_{\mathrm{dir}}(U) - \frac{1}{2}\mathcal{W}_{\mathrm{rev}}(U)\right)\,\mathrm{d}\mu_{\mathrm{H}}^{\mathcal{E}}(U).\end{aligned} \tag{20}$$

Proof. Apply (19) for $a = X, Y, Z$. The constant bounds in the six records prove integrability of the three density-side products and the three observable-side products, so the Bochner integral may be distributed over the finite sum. Apply the pointwise identity (18) inside the right-hand integral. $\square$

Theorem 5.1 is the headline compiled statement. Its regularity assumptions are ordinary sufficient conditions for the two integrations by parts; they do not assume the conclusion or any area derivative.

6 Relation to the Makeenko–Migdal equation

Let t_1, t_2, t_3, t_4 denote the areas of the four faces adjacent to a crossing. The local Makeenko–Migdal operator is

$$\mathcal{A}_{\mathbf{t}} := \partial_{t_1} - \partial_{t_2} + \partial_{t_3} - \partial_{t_4}. \tag{21}$$

The rigorous planar edge-variable proofs show, under extended gauge invariance, how (21) becomes a mixed invariant derivative of the local density [3]. Levy's finite-N and large-N analysis places this equation in the recursive area system for Wilson loops [2]. The present theorem begins immediately after that conversion. Its dependency boundary is shown in Figure 1.

There are two normalization issues at this boundary. Let $\Delta_P = \sum_{a\in\{X,Y,Z\}} D_a^2$ for the Pauli directions in Section 3. Their fundamental Casimir is $3/4$, and the program kernel uses $c_n = n(n+2)/4$ and weights e^{-tc_n}; thus its time satisfies $\partial_t K_t = \Delta_P K_t$. Driver–Hall–Kemp use a time s satisfying $\partial_s R_s = \frac{1}{2}\Delta_P R_s$ [3]. Hence $K_t = R_{2t}$, so the alternating program-area derivative is twice the Driver–Hall–Kemp derivative.

More importantly, the missing input is not a pointwise equality of densities. For a crossing observable $\mathcal{W}$ with certified extended gauge invariance and the required dominated differentiability, the needed weak identity is

$$\int_{\mathrm{SU}(2)^{\mathcal{E}}} (\mathcal{A}_{\mathbf{t}}\rho_{\mathbf{t}})(U)\,\mathcal{W}(U)\,d\mu_{\mathrm{H}}^{\mathcal{E}}(U) = -2\int_{\mathrm{SU}(2)^{\mathcal{E}}} \left(\sum_{a\in\{X,Y,Z\}} D_{1,a}D_{2,a}\rho_{\mathbf{t}}\right)(U)\mathcal{W}(U)\,d\mu_{\mathrm{H}}^{\mathcal{E}}(U). \tag{22}$$

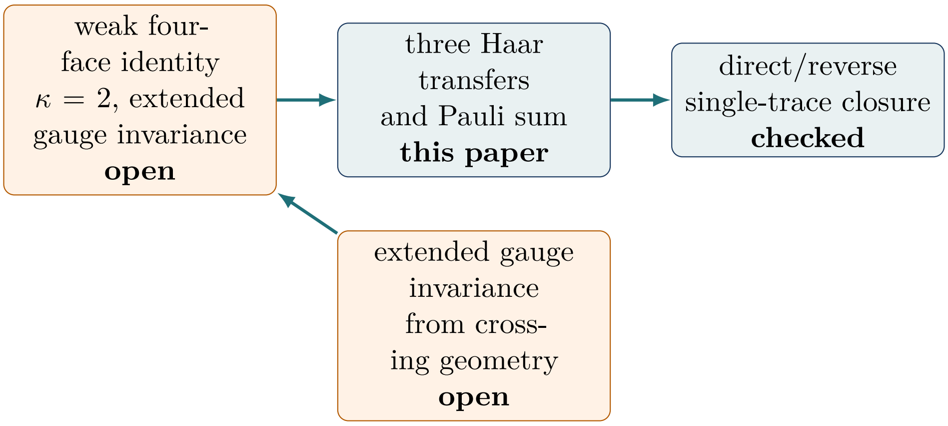

Figure 1: The machine-checked crossing Ward chain and the remaining physical input. The area relation is weak, not a pointwise density identity. The open boxes are not fields of the theorem proved here.

The minus sign and the weak scope follow from the local abstract Makeenko–Migdal equation: extended gauge invariance cancels extra terms only after integration [3, Eq. (50)]. The factor 2 is the time conversion above. Combining (22) with Theorem 5.1, and passing the area derivative through the integral, gives the conditional consequence

$$\mathcal{A}_{\mathbf{t}} \int \rho_{\mathbf{t}} \mathcal{W} \, d\mu_{\mathrm{H}}^{\mathcal{E}} = \int \rho_{\mathbf{t}} \left(\tfrac{1}{2} \mathcal{W}_{\mathrm{dir}} + \mathcal{W}_{\mathrm{rev}} \right) d\mu_{\mathrm{H}}^{\mathcal{E}}. \tag{23}$$

Neither (22) nor (23) is a theorem of this paper. The compiled contribution supplies the generator transfer and finite-SU(2) single-trace closure once the weak geometric input is proved.

7 Formal architecture and audit

The new producer is `Lean2dYangMills/SU2CrossingWard.lean`. It has 487 physical lines and 19 public declarations: one abbreviation, five definitions, and thirteen theorems. The dedicated audit imports the producer, checks all six named objects, and prints the kernel dependencies of all thirteen theorems. Table 2 gives the paper–Lean map.

Table 2: Headline paper–Lean map. All names are checked by the dedicated audit module.

Paper statement	Lean declaration
Matrix-word derivative, Lemma 2.1	`hasDerivAt_normalizedTrace_const_mul_curve_mul_const`
First selected-edge derivative	`hasDerivAt_su2CrossingWilsonWord_edge1_zero`
Second selected-edge derivative	`hasDerivAt_su2CrossingFirstGenerator_edge2_zero`
Concrete Pauli derivative family	`hasDerivAt_su2CrossingWilsonWord_pauliX/Y/Z`
Concrete mixed derivative family	`hasDerivAt_su2CrossingFirstGenerator_pauliX/Y/Z`
Pauli sum equals contraction	`su2CrossingMixedGenerator_pauli_sum_eq_contraction`
Pointwise closure, Theorem 4.1	`su2CrossingMixedGenerator_pauli_sum_closed`
One-direction transfer, (19)	`integral_su2Crossing_oneDirection_transfer`
Crossing Ward identity, Theorem 5.1	`integral_su2Crossing_pauliWard_closed`

The environment is Lean `4.29.0-rc6`, pinned to Mathlib commit

`07642720480157414db592fa85b626dafb71355b`.

A clean source build completes 2913 jobs. A source scan reports no local `sorry`, `admit`, or `axiom` in either new module. Every new theorem has the same printed dependency set:

{`propext`, `Classical.choice`, `Quot.sound`}.

These are standard logical dependencies inherited from Mathlib, not project-local physical assumptions.

8 Reproduction protocol

The direct-review archive is a source package rather than a dump of compiled objects. From its root on Windows PowerShell:

```
lake update
lake exe cache get
.\verify-su2-crossing-ward.ps1
```

The wrapper performs four checks: it verifies the pinned toolchain and Mathlib revision, builds the producer and its source dependency closure, compiles the dedicated axiom audit, and rejects local proof escapes before checking the exact count of 19 public declarations. The archive excludes `.lake`, `.olean`, and `.ilean` files so that a reviewer cannot accidentally accept the author's development cache as evidence.

9 Conclusion

The finite-edge SU(2) crossing mechanism now has a single compiled theorem from density generators to single-trace resolutions. The proof verifies the actual Pauli derivatives of the Wilson word, performs two Haar transfers in each direction, retains the finite-N Fierz correction, and closes the result with the exact SU(2) trace–skein identity. This removes a genuine interface gap between the earlier Haar-IBP and trace–skein artifacts.

The remaining bottleneck is sharply smaller but still substantive: (22) must be proved as a weak identity for the concrete heat density against crossing-certified extended-gauge-invariant observables, with the program-time factor $\kappa = 2$. It must not be promoted to a pointwise density equality. Until then, the present result is a finite-dimensional crossing Ward identity, not a complete Makeenko–Migdal area equation.

References

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