Machine-Checked Extended Gauge Invariance at an $\mathrm{SU}(2)$ Crossing: Four-Edge Wilson Holonomy, Haar-Preserving Quotient Coordinates, and Reduction to the Ward Chart

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Abstract

We formalize extended gauge invariance at a simple four-edge SU(2) crossing and connect the geometric edge chart to the two-coordinate chart used by a machine-checked crossing Ward identity. On $\mathrm{SU}(2)^4$ we define the two opposite-edge right actions from the abstract Makeenko–Migdal theorem, prove that they are commuting product-Haar-preserving actions, and show that their common parameter composes to ordinary vertex gauge invariance. The four-edge Wilson word

$$\mathrm{tr}_2(a_3^{-1}\beta a_2 a_4^{-1}\alpha a_1)$$

is proved invariant under both half-actions: one transformation cancels inside the word, while the other becomes conjugation inside the normalized trace. We construct the explicit quotient $\mathsf{r}(a) = (a_2 a_4^{-1}, a_1 a_3^{-1})$, a canonical section, and prove a universal factorization theorem for every extended-gauge-invariant complex function, including uniqueness of the descended function. The complete map from the cyclic four-edge chart to physical and gauge coordinates is proved to preserve literal four-fold Haar measure in one public endpoint. Finally, the four-edge Wilson word is identified exactly with the prior two-coordinate crossing word evaluated on $\mathsf{r}(a)$. The Lean producer has 56 public declarations in 488 physical lines; all 36 theorems depend only on `propext`, `Classical.choice`, and `Quot.sound`, with no local proof escape. No heat-kernel area derivative or full Makeenko–Migdal equation is claimed.

1 Result and claim boundary

At a simple crossing, Driver–Hall–Kemp isolate a symmetry stronger than ordinary gauge invariance. If the four distinct outgoing edge variables are a_1, a_2, a_3, a_4 in cyclic order, a function f has extended gauge invariance when, for every x in the structure group,

$$f(a_1, a_2, a_3, a_4) = f(a_1 x, a_2, a_3 x, a_4) = f(a_1, a_2 x, a_3, a_4 x). \tag{1}$$

This is Definition 2.4 of [3]. It is the symmetry used in the abstract Makeenko–Migdal equation to convert the alternating four-face area variation into one mixed edge generator. Ordinary vertex gauge invariance, which multiplies all four outgoing variables by the same x, is weaker.

The present paper machine-checks the finite-dimensional geometry behind (1) for the fundamental SU(2) Wilson word and proves the quotient structure that it induces. Table 1 states the precise boundary.

This is a new sequel, not a replacement for the preceding crossing Ward, Haar integration-by-parts, or trace–skein papers. Its contribution is the missing local four-edge symmetry and quotient interface. It does not infer an area derivative from that symmetry.

Table 1: Claim ledger. "Checked" means compiled in the new Lean producer.

Statement	Status	Role
Odd and even half-gauge actions on four outgoing edges	Checked	Primary-source definition
Action laws, commutation, and ordinary vertex action	Checked	Symmetry algebra
Product-Haar preservation of both half-actions	Checked	Measure compatibility
Extended gauge invariance of the four-edge Wilson word	Checked	Geometric observable
Explicit quotient, section, canonical representative	Checked	Orbit coordinates
Existence and uniqueness of factorization of every invariant function	Checked	Quotient theorem
Full cyclic-chart physical/gauge map preserves four-fold Haar	Checked	Gauge-volume separation
Exact reduction to the two-coordinate Ward word	Checked	Interface to prior theorem
Embedding an arbitrary planar cellulation into this local chart	Open	Combinatorial geometry
Weak four-face heat-density identity with $\kappa = 2$	Open	Analytic Makeenko–Migdal input
Full Makeenko–Migdal area equation	Open	Requires the two open rows

2 The four-edge actions

Write a local configuration as $a = (a_1, a_2, a_3, a_4) \in \mathrm{SU}(2)^4$. We use zero-based indices in Lean and one-based indices in the paper. Define

$$\mathsf{O}_x(a) = (a_1x, a_2, a_3x, a_4), \tag{2}$$
$$\mathsf{E}_x(a) = (a_1, a_2x, a_3, a_4x), \tag{3}$$
$$\mathsf{V}_x(a) = (a_1x, a_2x, a_3x, a_4x). \tag{4}$$

The producer realizes these maps as right multiplication in the product group $\mathrm{SU}(2)^4$ by $(x, 1, x, 1)$, $(1, x, 1, x)$, and (x, x, x, x) respectively.

Theorem 2.1 (Half-action algebra)**.** *For all $x, y \in \mathrm{SU}(2)$ and $a \in \mathrm{SU}(2)^4$,*

$$\mathsf{O}_1(a) = a, \qquad \mathsf{O}_y(\mathsf{O}_x(a)) = \mathsf{O}_{xy}(a),$$
$$\mathsf{E}_1(a) = a, \qquad \mathsf{E}_y(\mathsf{E}_x(a)) = \mathsf{E}_{xy}(a),$$
$$\mathsf{O}_x(\mathsf{E}_y(a)) = \mathsf{E}_y(\mathsf{O}_x(a)), \qquad \mathsf{V}_x(a) = \mathsf{E}_x(\mathsf{O}_x(a)).$$

Consequently, extended gauge invariance implies ordinary vertex gauge invariance.

The proof is coordinatewise group algebra. The last conclusion is not installed as a field of the definition: Lean derives it by two applications of the two half-invariance hypotheses.

Let $\mu_{\mathrm{H}}^{\otimes 4}$ be normalized product Haar measure.

Theorem 2.2 (Haar preservation)**.** *Each O_x and E_x preserves $\mu_{\mathrm{H}}^{\otimes 4}$.*

This follows from right invariance of Haar on the concrete product group; the proof term uses Mathlib's measure-preserving right translation theorem. Thus the symmetries in (1) are compatible with the actual integration measure used by the finite-edge Ward theorem.

3 The Wilson word has extended gauge invariance

Fix $\alpha, \beta \in \mathrm{SU}(2)$, representing the portions of the two loop strands away from the four local outgoing edges. In the order-reversing parallel-transport convention of [3], define

$$\mathcal{W}_{\alpha,\beta}(a) = \mathrm{tr}_2(a_3^{-1}\beta a_2 a_4^{-1}\alpha a_1), \qquad \mathrm{tr}_2(M) = \tfrac{1}{2}\,\mathrm{Tr}(M). \tag{5}$$

This is the concrete $\mathrm{SU}(2)$ specialization of the primary-source word $\mathrm{tr}(a_3^{-1}\beta a_2 a_4^{-1}\alpha a_1)$.

Theorem 3.1 (Four-edge Wilson invariance)**.** *For all $a \in \mathrm{SU}(2)^4$ and $x \in \mathrm{SU}(2)$,*

$$\mathcal{W}_{\alpha,\beta}(\mathsf{O}_x a) = \mathcal{W}_{\alpha,\beta}(a) = \mathcal{W}_{\alpha,\beta}(\mathsf{E}_x a).$$

Hence $\mathcal{W}_{\alpha,\beta}$ has extended gauge invariance.

The two equalities have distinct mechanisms. For the odd action,

$$\begin{aligned}&(a_3 x)^{-1}\beta a_2 a_4^{-1}\alpha(a_1 x)\\ &\qquad = x^{-1}(a_3^{-1}\beta a_2 a_4^{-1}\alpha a_1)x,\end{aligned}$$

so invariance is cyclicity of the normalized trace. For the even action, the adjacent factors x and x^{-1} cancel:

$$a_3^{-1}\beta(a_2 x)(a_4 x)^{-1}\alpha a_1 = a_3^{-1}\beta a_2 a_4^{-1}\alpha a_1.$$

Lean verifies both identities in the concrete matrix subgroup. It separately proves normalized-trace conjugation invariance and two-factor cyclicity, so neither cancellation is hidden behind an assumed "Wilson invariance" axiom.

4 Explicit quotient and universal factorization

The two invariant relative coordinates are

$$\mathsf{r}(a) = \left(a_2 a_4^{-1},\ a_1 a_3^{-1}\right) \in \mathrm{SU}(2)^2. \tag{6}$$

They are unchanged under both half-actions. Define the section

$$s(u,v) = (v,u,1,1) \tag{7}$$

and the canonical representative

$$c(a) = (a_1 a_3^{-1}, a_2 a_4^{-1}, 1, 1) = s(\mathsf{r}(a)). \tag{8}$$

Theorem 4.1 (Constructive quotient chart)**.** *The maps $\mathsf{r} : \mathrm{SU}(2)^4 \to \mathrm{SU}(2)^2$ and $s : \mathrm{SU}(2)^2 \to \mathrm{SU}(2)^4$ satisfy*

$$\mathsf{r} \circ s = \mathrm{id}_{\mathrm{SU}(2)^2}.$$

Moreover,

$$\mathsf{E}_{a_4^{-1}}(\mathsf{O}_{a_3^{-1}}(a)) = c(a).$$

If $f : \mathrm{SU}(2)^4 \to \mathbb{C}$ has extended gauge invariance and $\bar{f}(u) = f(s(u))$, then

$$f(a) = f(c(a)) = \bar{f}(\mathsf{r}(a)) \tag{9}$$

for every $a \in \mathrm{SU}(2)^4$. Moreover, if $g : \mathrm{SU}(2)^2 \to \mathbb{C}$ satisfies $f = g \circ \mathsf{r}$, then $g = \bar{f}$.

Equation (9) is the universal factorization statement of the paper: it applies to every complex-valued extended-gauge-invariant function, not only the Wilson word, and the public uniqueness theorem follows by evaluating a proposed factorization on the section s. The proof supplies the actual group elements that move a configuration to its canonical representative.

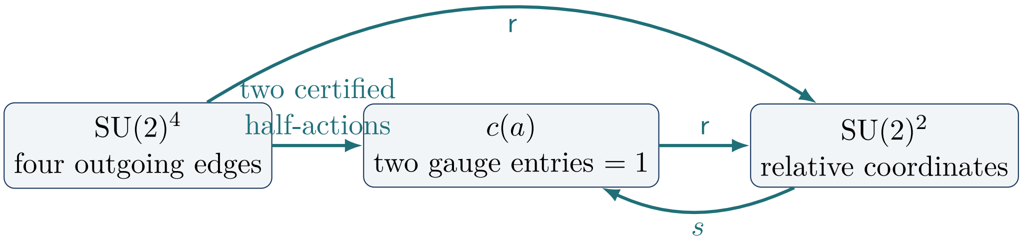

Figure 1: The explicit orbit reduction. Extended-gauge-invariant functions factor through the lower-dimensional chart; no quotient type is postulated.

5 Haar-preserving physical/gauge coordinates

Regroup the four variables into the opposite pairs

$$((a_2, a_4), (a_1, a_3)) \in (\mathrm{SU}(2) \times \mathrm{SU}(2))^2.$$

The simultaneous pair gauge fix is

$$\Phi((p_0, g_0), (p_1, g_1)) = ((p_0 g_0^{-1}, g_0), (p_1 g_1^{-1}, g_1)). \tag{10}$$

Its first entries are precisely the two components of (6); the second entries retain the gauge coordinates. Writing

$$\Psi(a) = \Phi((a_2, a_4), (a_1, a_3)) = ((a_2 a_4^{-1}, a_4), (a_1 a_3^{-1}, a_3)),$$

the formal statement starts directly from the original cyclic function type `Fin 4 -> SU2`; the regrouping is itself implemented as a measurable equivalence and is not left implicit.

Theorem 5.1 (Four-fold Haar-preserving split)**.** *The map* $\Psi : \mathrm{SU}(2)^4 \to (\mathrm{SU}(2) \times \mathrm{SU}(2))^2$ *sends* $\mu_{\mathrm{H}}^{\otimes 4}$ *to* $(\mu_{\mathrm{H}} \otimes \mu_{\mathrm{H}}) \otimes (\mu_{\mathrm{H}} \otimes \mu_{\mathrm{H}})$*. Its two physical outputs equal* $\mathsf{r}(a)$*.*

The proof composes the product-Haar-preserving coordinate permutation with the product of two instances of the Haar shear $(p, g) \mapsto (pg^{-1}, g)$. This matters analytically: the quotient variables are not merely orbit labels; they arise from a measure-preserving change of variables that retains the discarded gauge volume as two independent Haar coordinates.

6 Exact interface to the two-edge Ward chart

The preceding crossing Ward artifact uses

$$\mathcal{W}^{(2)}_{A,B}(U) = \mathrm{tr}_2((AU_1)(BU_2)), \qquad U \in \mathrm{SU}(2)^2. \tag{11}$$

The connection to the four-edge geometry is now a theorem.

Theorem 6.1 (Four-edge to two-edge reduction)**.** *For every* $a \in \mathrm{SU}(2)^4$*,*

$$\mathcal{W}_{\alpha,\beta}(a) = \mathcal{W}^{(2)}_{\beta,\alpha}(\mathsf{r}(a)). \tag{12}$$

Indeed, cyclicity moves a_3^{-1} from the beginning of (5) to the end, giving

$$\mathrm{tr}_2(\beta(a_2 a_4^{-1})\alpha(a_1 a_3^{-1})),$$

which is exactly (11). Thus the word differentiated and closed in the earlier Ward theorem is not an unrelated two-variable ansatz: it is the explicit quotient of the primary four-edge Wilson holonomy.

Figure 2 records what this closes and what remains open.

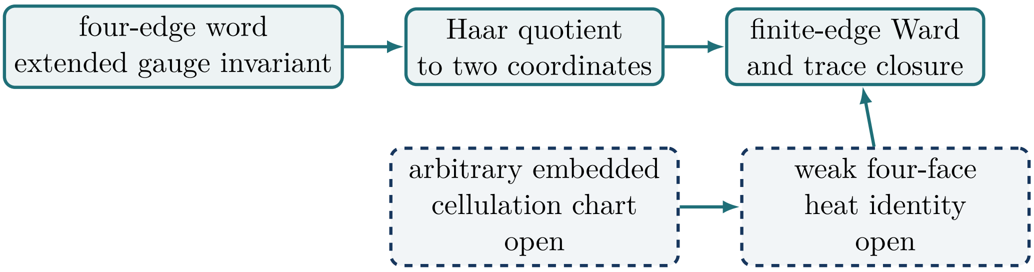

Figure 2: Compiled interfaces (solid) and remaining inputs (dashed). This paper closes the local four-edge symmetry and quotient, not the area equation.

7 Lean architecture and audit

The new producer is `Lean2dYangMills/SU2CrossingExtendedGauge.lean`. It contains 56 public declarations: two abbreviations, seventeen definitions, one proposition structure, and 36 theorems. Table 2 maps the headline statements to exact declaration names.

Table 2: Headline paper–Lean map.

Paper statement	Lean declaration
Extended gauge invariance predicate	`SU2CrossingExtendedGaugeInvariant`
Half-action Haar preservation, Theorem 2.2	`measurePreserving_su2CrossingOdd/EvenAction`
Action algebra, Theorem 2.1	`su2CrossingOddAction_mul`, `su2CrossingOddAction_evenAction_commute`
Ordinary invariance from extended invariance	`SU2CrossingExtendedGaugeInvariant.vertex`
Four-edge Wilson invariance, Theorem 3.1	`su2FourEdgeCrossingWilsonWord_extendedGaugeInvariant`
Canonical orbit representative	`su2Crossing_toCanonical`
Universal factorization, Theorem 4.1	`SU2CrossingExtendedGaugeInvariant.factor_through_reduced`, `SU2CrossingExtendedGaugeInvariant.descended_unique`
Haar-preserving split, Theorem 5.1	`su2CrossingFullGaugeFix_measurePreserving`
Four-edge/two-edge bridge, Theorem 6.1	`su2FourEdgeCrossingWilsonWord_eq_twoEdgeWord`

The environment is Lean `4.29.0-rc6`, pinned to Mathlib commit

`07642720480157414db592fa85b626dafb71355b`.

The dedicated audit module prints the axiom dependencies of all 36 theorems. Every theorem has precisely the standard dependency set

{`propext`, `Classical.choice`, `Quot.sound`}.

The verification wrapper builds the producer, compiles the audit, scans both new modules for local proof escapes, checks the exact public-declaration count, and records immutable hashes. The release archive excludes `.lake`, `.olean`, and `.ilean` objects.

8 Conclusion

The local geometry preceding the SU(2) crossing Ward identity is now machine checked. The primary four-edge symmetry is implemented as two commuting Haar-preserving actions; the fundamental Wilson holonomy is proved invariant under both; every invariant function factors through an explicit two-coordinate quotient with a unique descended function; and the complete cyclic-chart physical/gauge split preserves four-fold Haar measure. The quotient Wilson word is exactly the two-coordinate word used by the compiled Pauli-generator Ward theorem.

Two boundaries remain. First, an arbitrary embedded planar cellulation must be shown to supply the ordered four distinct outgoing edges and the word decomposition assumed by the local chart. Second, the concrete heat-kernel density must satisfy the weak alternating four-area identity, in the program normalization with $\kappa = 2$. Neither statement is a theorem of this paper. The result is therefore a rigorous geometric and measure-theoretic interface, not a full Makeenko–Migdal area equation.

References

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[2] T. Lévy, The master field on the plane, *Astérisque* **388** (2017), arXiv:1112.2452.

[3] B. K. Driver, B. C. Hall, and T. Kemp, Three proofs of the Makeenko–Migdal equation for Yang–Mills theory on the plane, *Communications in Mathematical Physics* **351** (2017), 741–774; arXiv:1601.06283.

[4] B. K. Driver, F. Gabriel, B. C. Hall, and T. Kemp, The Makeenko–Migdal equation for Yang–Mills theory on compact surfaces, *Communications in Mathematical Physics* **352** (2017), 967–978; arXiv:1602.03905.

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[6] The Mathlib Community, The Lean mathematical library, in *Proceedings of CPP 2020*, ACM (2020), 367–381.