Machine-Checked Haar and Differential Descent at an $\mathrm{SU}(2)$ Crossing: A Four-Edge Compensated-Flow Ward Identity

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Abstract

The local Makeenko–Migdal mechanism is usually expressed in two languages: four holonomies on the edges incident to a crossing, and two effective group coordinates on which invariant derivatives act. We machine-check the exact measure and differential bridge between them for SU(2). For $a = (a_1, a_2, a_3, a_4) \in \mathrm{SU}(2)^4$ we prove that

$$\mathsf{r}(a) = (a_2 a_4^{-1}, a_1 a_3^{-1})$$

pushes normalized four-fold Haar measure to normalized two-fold Haar measure, and derive the corresponding integral identity for every strongly measurable complex observable. We then construct two gauge-compensated flows on the literal four-edge chart. They intertwine exactly with independent right multiplication of the two quotient coordinates. Consequently, first and mixed derivatives of the crossing Wilson word descend without a gauge choice. The three Pauli directions are proved individually, their mixed generators close pointwise into direct and reverse trace resolutions, and the previously established two-coordinate Ward identity is lifted to a four-edge compensated-flow integral theorem with coefficients $-1/4$ and $-1/2$. This operator is not identified with the ordinary-edge mixed operator of Driver–Hall–Kemp; the two have different reverse resolutions in general. The new Lean producer contains 26 public declarations in 426 physical lines; all 16 theorems have only the standard dependencies `propext`, `Classical.choice`, and `Quot.sound`. No weak four-face heat-kernel identity, alternating area derivative, or full Makeenko–Migdal equation is claimed.

1 Result and claim boundary

For planar Yang–Mills theory, the Makeenko–Migdal relation converts an alternating variation of the four face areas around a crossing into a local second-order differential operator on edge holonomies. The finite-dimensional form of this mechanism is rigorous for suitable observables with extended gauge invariance; see Driver–Hall–Kemp [2] and the compact-surface extension [3]. Formalization forces a useful separation of four interfaces:

1. the physical four-edge chart and its normalized Haar measure;
2. the two-coordinate differential Ward calculus;
3. comparison with the ordinary-edge mixed operator used by Driver–Hall–Kemp;
4. the analytic identification with four heat-kernel area derivatives.

The present paper closes the first-to-second interface for the compensated lift. The third and fourth remain outside the theorem surface. Table 1 is the claim ledger.

This is a new sequel, not a replacement. Its novelty is not another trace calculation: it proves that the probability measure, the gauge-compensated flows, and the differentiated Wilson observable commute with the same quotient map and therefore support a single four-edge integral endpoint.

Table 1: Claim ledger. “Checked” means compiled in the new producer; an imported row is a previously compiled theorem used by the endpoint.

Statement	Status	Mathematical role
Four-edge quotient pushes $\mu_{\mathrm{H}}^{\otimes 4}$ to $\mu_{\mathrm{H}}^{\otimes 2}$	Checked	Exact measure descent
Integral quotient formula for measurable complex observables	Checked	Transfers expectations
Two gauge-compensated four-edge flows and intertwining laws	Checked	Exact differential interface
Generic first and mixed derivative descent	Checked	Coordinate-free generator transport
Six concrete Pauli derivative theorems	Checked	Reusable public calculus
Pauli trace–skein closure on the four-edge chart	Checked	Direct/reverse resolutions
Two-coordinate dominated-flow Ward identity	Imported	Finite-dimensional analytic input
Four-edge quotient-lifted compensated-flow Ward endpoint	Checked	Main theorem
Comparison with the ordinary-edge DHK mixed operator	Open	Distinct differential interface
Weak four-face heat-density identity and area differentiation	Open	Must incorporate the comparison
Full Makeenko–Migdal equation on an arbitrary cellulation	Open	Requires both open interfaces

2 The physical Haar quotient

Let $a = (a_1, a_2, a_3, a_4) \in \mathrm{SU}(2)^4$ list the four distinct outgoing edge variables in cyclic order. The local geometric word is

$$\mathcal{W}_4^{\alpha,\beta}(a) = \mathrm{tr}_2(a_3^{-1}\beta a_2 a_4^{-1}\alpha a_1), \qquad \alpha, \beta \in \mathrm{SU}(2), \tag{1}$$

where $\mathrm{tr}_2 = \frac{1}{2}\,\mathrm{Tr}$ is the normalized fundamental trace. The quotient coordinates are

$$\mathsf{r}(a) = (v_0, v_1) = (a_2 a_4^{-1}, a_1 a_3^{-1}). \tag{2}$$

They are the two physical entries of the full gauge-fixing chart

$$\Psi(a) = ((a_2 a_4^{-1}, a_4), (a_1 a_3^{-1}, a_3)). \tag{3}$$

The prior chart theorem proves that Ψ preserves literal four-fold Haar. Projection onto the first coordinate of each pair preserves the associated product measure. Composing these two measure-preserving maps yields the main measure statement directly.

Theorem 2.1 (Haar quotient). *The map $\mathsf{r} : \mathrm{SU}(2)^4 \to \mathrm{SU}(2)^2$ is measure preserving from $\mu_{\mathrm{H}}^{\otimes 4}$ to $\mu_{\mathrm{H}}^{\otimes 2}$; equivalently,*

$$\mathsf{r}_*\mu_{\mathrm{H}}^{\otimes 4} = \mu_{\mathrm{H}}^{\otimes 2}.$$

The statement includes normalized measures on both sides; no hidden gauge volume or proportionality constant is introduced. It immediately produces an integration theorem in the exact generality supported by the Bochner integral.

Corollary 2.2 (Exact quotient integration). *If $F : \mathrm{SU}(2)^2 \to \mathbb{C}$ is almost-everywhere strongly measurable with respect to $\mu_{\mathrm{H}}^{\otimes 2}$, then*

$$\int_{\mathrm{SU}(2)^4} F(\mathsf{r}(a))\, d\mu_{\mathrm{H}}^{\otimes 4}(a) = \int_{\mathrm{SU}(2)^2} F(v)\, d\mu_{\mathrm{H}}^{\otimes 2}(v). \tag{4}$$

The result is stronger than checking the formula only for the Wilson word. It is a reusable transport theorem for any physical density, derivative, or resolved observable meeting the measurability hypothesis.

3 Gauge-compensated physical flows

Right multiplication of $v_0 = a_2 a_4^{-1}$ by a curve $\gamma(t)$ cannot be realized by the naive change $a_2 \mapsto a_2\gamma(t)$: the factor must pass the return edge a_4^{-1}. The correct physical insertion is its conjugate by a_4. Similarly, the second coordinate uses a_3. Define

$$T^{\gamma}_{0,t}(a) = (a_1,\ a_2(a_4^{-1}\gamma(t)a_4),\ a_3,\ a_4), \tag{5}$$

$$T^{\gamma}_{1,t}(a) = (a_1(a_3^{-1}\gamma(t)a_3),\ a_2,\ a_3,\ a_4). \tag{6}$$

For $R^{\gamma}_{i,t}$ denoting right multiplication of quotient coordinate i by $\gamma(t)$, cancellation gives the following exact identities.

Theorem 3.1 (Intertwining)**.** *For every curve $\gamma : \mathbb{R} \to \mathrm{SU}(2)$, time t, and configuration a,*

$$\mathsf{r}(T^{\gamma}_{0,t}a) = R^{\gamma}_{0,t}(\mathsf{r}a), \qquad \mathsf{r}(T^{\gamma}_{1,t}a) = R^{\gamma}_{1,t}(\mathsf{r}a). \tag{7}$$

$$\begin{tikzcd}
\mathrm{SU}(2)^4 \arrow[r, "T^{\gamma}_{i,t}"] \arrow[d, "\mathsf{r}"'] & \mathrm{SU}(2)^4 \arrow[d, "\mathsf{r}"] \\
\mathrm{SU}(2)^2 \arrow[r, "R^{\gamma}_{i,t}"'] & \mathrm{SU}(2)^2
\end{tikzcd}$$

The square commutes for $i = 0, 1$. The upper flow modifies one physical edge by a return-edge conjugate; the lower flow is ordinary right multiplication.

Figure 1: Exact flow descent. Differentiation may be performed before or after applying the quotient map.

The conjugating factors are forced by (2); they are not a gauge choice. This distinction matters because the final integral remains over all four edge variables with their literal product Haar law.

4 Differential descent of the crossing word

Set

$$\mathcal{W}^{\beta,\alpha}_2(v_0, v_1) = \mathrm{tr}_2(\beta v_0 \alpha v_1).$$

Cyclicity of the trace and (2) give the already checked identity

$$\mathcal{W}^{\alpha,\beta}_4(a) = \mathcal{W}^{\beta,\alpha}_2(\mathsf{r}a). \tag{8}$$

For a matrix tangent X, let $\mathcal{D}^X_0\mathcal{W}_2$ be the first right-invariant generator, and for X, Y let $\mathcal{D}^Y_1\mathcal{D}^X_0\mathcal{W}_2$ be the ordered mixed generator. The four-edge generators are defined by pullback:

$$\mathcal{D}^X_{4,0}\mathcal{W}_4(a) = (\mathcal{D}^X_0\mathcal{W}_2)(\mathsf{r}a), \tag{9}$$

$$\mathcal{D}^Y_{4,1}\mathcal{D}^X_{4,0}\mathcal{W}_4(a) = (\mathcal{D}^Y_1\mathcal{D}^X_0\mathcal{W}_2)(\mathsf{r}a). \tag{10}$$

Theorem 4.1 (Generic derivative descent)**.** *Suppose γ has entrywise derivative X at zero. Then*

$$\left.\frac{d}{dt}\right|_0 \mathcal{W}_4(T^{\gamma}_{0,t}a) = \mathcal{D}^X_{4,0}\mathcal{W}_4(a).$$

If the entrywise derivative is Y, then

$$\left.\frac{d}{dt}\right|_0 (\mathcal{D}^X_{4,0}\mathcal{W}_4)(T^{\gamma}_{1,t}a) = \mathcal{D}^Y_{4,1}\mathcal{D}^X_{4,0}\mathcal{W}_4(a).$$

Both statements are formalized as `HasDerivAt` *theorems.*

The proof is a composition of (7), (8), and the earlier two-coordinate derivative theorem. The producer then specializes both stages to the three anti-Hermitian Pauli directions X, Y, Z, yielding six concrete public derivative theorems. Thus the generator language used in the Ward identity has a verified realization as differentiation of the original four-edge holonomy.

5 Compensated and ordinary-edge operators

The preceding descent must not be confused with the ordinary-edge mixed operator in Driver–Hall–Kemp. Their operator differentiates the original variables by

$$a_1 \longmapsto a_1 e^{sA}, \qquad a_2 \longmapsto a_2 e^{tA},$$

using the same Lie-algebra basis element A at both edges. On (1), the corresponding Pauli contraction is

$$C_{\mathrm{DHK}}(a) = \sum_{A\in\{X,Y,Z\}} \mathrm{tr}_2\Big(a_3^{-1}\beta a_2 A a_4^{-1}\alpha a_1 A\Big)\,. \tag{11}$$

By contrast, the quotient-lifted operator compiled here is

$$C_{\mathrm{q}}(a) = \sum_{A\in\{X,Y,Z\}} \mathrm{tr}_2\Big(\beta a_2 a_4^{-1} A \alpha a_1 a_3^{-1} A\Big)\,. \tag{12}$$

Equations (5)–(6) show why: in the original edge coordinates the two insertions are conjugated independently by a_4^{-1} and a_3^{-1}. Invariance of a Pauli contraction permits one common orthogonal change of basis, not two unrelated adjoint rotations. Hence C_{q} and C_{DHK} are not equal in general.

The mismatch is also visible after trace–skein closure. Both direct resolutions equal the original crossing word, but the reverse resolutions are different:

$$\mathcal{W}_{\mathrm{q,rev}}(a) = \mathrm{tr}_2\Big(\beta a_2 a_4^{-1} a_3 a_1^{-1}\alpha^{-1}\Big)\,, \tag{13}$$

$$\mathcal{W}_{\mathrm{DHK,rev}}(a) = \mathrm{tr}_2\Big(a_3^{-1}\beta a_2 a_1^{-1}\alpha^{-1} a_4\Big)\,. \tag{14}$$

The first is the pullback of the reverse resolution on the quotient. The second is formed from the two cut-loop holonomies $a_3^{-1}\beta a_2$ and $a_4^{-1}\alpha a_1$. The Lean theorem below concerns (12) and (13) only. Establishing a relation to (11), most plausibly together with the concrete four-face thermal density and its gauge-coordinate integration, is an explicit open interface.

6 Trace–skein closure and the quotient-lifted Ward theorem

Let $\mathcal{W}_{4,\mathrm{dir}}$ and $\mathcal{W}_{4,\mathrm{rev}}$ be the direct and reverse resolutions, obtained by pulling the corresponding two-coordinate normalized-trace pairings back along r. The fundamental SU(2) completeness relation gives a pointwise closure.

Theorem 6.1 (Pauli closure)**.** *For every $a \in \mathrm{SU}(2)^4$,*

$$\sum_{A\in\{X,Y,Z\}} \mathcal{D}^A_{4,1}\mathcal{D}^A_{4,0}\mathcal{W}_4(a) = -\frac{1}{4}\mathcal{W}_{4,\mathrm{dir}}(a) - \frac{1}{2}\mathcal{W}_{4,\mathrm{rev}}(a). \tag{15}$$

The coefficients reflect the chosen Pauli normalization and normalized fundamental trace. They are compiled constants, not inferred from numerical experiments.

For the integral theorem, write $\rho : \mathrm{SU}(2)^2 \to \mathbb{C}$ for a physical density and $D_1D_2\rho_A$ for its mixed derivative in Pauli direction A. The Lean structure `SU2CrossingPauliWardData` packages ρ, the six derivative functions, and exactly six `DominatedFlowPair` certificates. These certificates provide the measurability, integrability, product rule, and dominated differentiation required by two integrations by parts. They are assumptions of the endpoint, not assertions about a heat-kernel density.

Theorem 6.2 (Four-edge quotient-lifted compensated-flow Ward identity)**.** *For $\alpha, \beta \in \mathrm{SU}(2)$ and any certified Ward data D,*

$$\begin{aligned}\int_{\mathrm{SU}(2)^4} \left(\sum_{A=X,Y,Z} D_1D_2\rho_A(\mathsf{r}a)\right)\mathcal{W}_4^{\alpha,\beta}(a)\, d\mu_{\mathrm{H}}^{\otimes 4}(a)&\\ = \int_{\mathrm{SU}(2)^4} \rho(\mathsf{r}a)\left[-\frac{1}{4}\mathcal{W}^{\alpha,\beta}_{4,\mathrm{dir}}(a) - \frac{1}{2}\mathcal{W}^{\alpha,\beta}_{4,\mathrm{rev}}(a)\right] d\mu_{\mathrm{H}}^{\otimes 4}(a).&\end{aligned} \tag{16}$$

The proof transports the left integrand to $\mathrm{SU}(2)^2$ using Corollary 2.2, applies the machine-checked two-coordinate Ward identity, uses Theorem 6.1, and transports the result back. The two uses of quotient integration require separate measurability proofs; both are discharged from the fields of the dominated-flow certificates.

Equation (16) is the strongest unconditional conclusion of this paper. Calling its mixed operator the ordinary-edge operator (11), or calling its left side an alternating four-area derivative, would require further comparison and heat-kernel theorems. No such identification appears in the Lean statement or manuscript.

7 Proof design and reusable consequences

The endpoint is assembled from four independently reusable transports. This factorization identifies exactly where a future heat-kernel theorem must enter.

Measure transport. The proof of Theorem 2.1 does not expand a Haar integral by hand. It composes the verified full chart Ψ with the two projections $(v, g) \mapsto v$. Projection from a product probability measure is measure preserving, and the equivalence between pairs and functions on `Fin 2` supplies the last representation change. The argument is therefore stable under replacement of the integrand and independent of trace identities.

Observable transport. Equation (8) was established before the new module, but the current proof uses it at the integral endpoint rather than only as a pointwise simplification. The left integrand in (16) is first identified almost everywhere with a composite $F_L \circ \mathsf{r}$ and transferred by Corollary 2.2. After the two-coordinate Ward theorem, a separately constructed $F_R \circ \mathsf{r}$ is transferred back. Keeping F_L and F_R distinct prevents a hidden assumption that the closed resolution is measurable merely because the unclosed derivative sum is measurable; the proof establishes this through the pointwise Pauli identity.

Differential transport. The flows (5) and (6) literally realize quotient differentiation on the original edges. For example,

$$a_2(a_4^{-1}\gamma a_4)a_4^{-1} = (a_2 a_4^{-1})\gamma.$$

The return-edge conjugation is forced by the quotient formula. Once Theorem 3.1 is available, the generic `HasDerivAt` endpoints follow by rewriting the complete curve rather than redoing matrix differentiation in four variables. The six Pauli theorems instantiate this common interface without extra assumptions.

Hypothesis transport. The structure D deliberately lives on $\mathrm{SU}(2)^2$. Its six dominated-flow certificates are exactly those consumed by the earlier integration-by-parts theorem: two coordinate directions for each of three Pauli axes. The new four-edge theorem neither duplicates these certificates nor postulates that a four-variable density has unexplained derivatives. It integrates the pullback $\rho \circ \mathsf{r}$ against $\mu_{\mathrm{H}}^{\otimes 4}$. Any future construction of heat-kernel Ward data on the quotient can therefore be inserted unchanged into Theorem 6.2.

Two reusable consequences are immediate. Any physical observable factoring through r inherits the exact Haar quotient formula. Also, any matrix tangent curve with an entrywise derivative may use the generic differential endpoints; the Pauli basis is required only for the completeness relation (15). These are public interfaces for subsequent formalization, not auxiliary steps hidden inside the headline proof.

8 Formal architecture and audit

The new producer imports the prior extended-gauge chart, which itself imports the concrete SU(2) Haar, derivative, trace–skein, and Ward layers. The resulting proof path is shown in Figure 2. The dashed boxes separate the ordinary-edge operator comparison from the heat-density step; neither is silently absorbed into the compiled path.

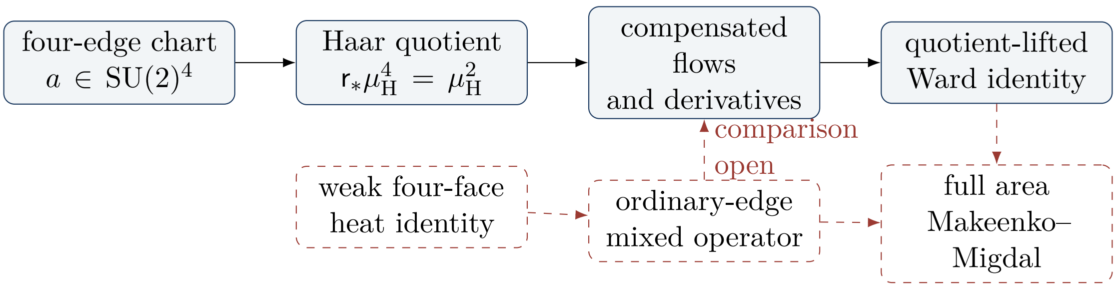

Figure 2: Compiled compensated-flow path (solid). The ordinary-edge comparison and heat-density extension remain open (dashed).

Table 2 maps the paper statements to the public Lean surface. The producer has 426 physical lines and 26 public declarations: nine definitions, one structure, and 16 theorems.

The dedicated audit invokes `#print axioms` on every new theorem. Each report contains exactly the standard logical dependencies `propext`, `Classical.choice`, and `Quot.sound`. A clean-source wrapper pins Lean `v4.29.0-rc6` and Mathlib commit `07642720480157414db592fa85b626d afb71355b`, builds the full 2915-job dependency closure, compiles the audit, rejects `sorry`, `admit`, and local `axiom` in the two new modules, checks the line and declaration counts, and records SHA-256 hashes.

Table 2: Paper-to-Lean map for the new producer.

Paper result	Lean declaration	Kind
Haar quotient, Thm. 2.1	`su2CrossingReducedConfiguration_measurePreserving`	theorem
Integral quotient, Cor. 2.2	`integral_su2CrossingReducedConfiguration`	theorem
Physical flows, (5)–(6)	`su2CrossingPhysicalFlow0`, `...Flow1`	definitions
Intertwining, Thm. 3.1	`su2CrossingReducedConfiguration_physicalFlow0/1`	2 theorems
Generic differential descent, Thm. 4.1	`hasDerivAt_su2FourEdgeCrossing..._zero`	2 theorems
Concrete Pauli calculus	`hasDerivAt_su2FourEdgeCrossing..._pauliX/Y/Z`	6 theorems
Pauli closure, Thm. 6.1	`su2FourEdgeCrossingMixedGenerator_pauli_sum_closed`	theorem
Ward hypotheses	`SU2CrossingPauliWardData`	structure
Quotient-lifted endpoint, Thm. 6.2	`integral_su2FourEdgeCrossing_pauliWard_closed`	theorem

9 Conclusion

The four-edge crossing chart and the two-coordinate Ward chart are now linked by more than an algebraic equality of Wilson words. Their probability laws agree under an exact Haar pushforward; quotient right-invariant directions have explicit gauge-compensated lifts; and the Pauli-summed Ward identity returns to a literal four-edge integral with its quotient direct/reverse coefficients intact. This closes the compensated quotient-lift and no more. The comparison with the ordinary-edge Driver–Hall–Kemp operator, the heat-kernel four-area identity, arbitrary-cellulation geometry, and the full Makeenko–Migdal equation remain future work.

References

[1] Y. Makeenko and A. Migdal, Exact equation for the loop average in multicolor QCD, *Physics Letters B* **88** (1979), 135–137.

[2] B. K. Driver, B. C. Hall, and T. Kemp, Three proofs of the Makeenko–Migdal equation for Yang–Mills theory on the plane, *Communications in Mathematical Physics* **351** (2017), 741–774; arXiv:1601.06283.

[3] B. K. Driver, F. Gabriel, B. C. Hall, and T. Kemp, The Makeenko–Migdal equation for Yang–Mills theory on compact surfaces, *Communications in Mathematical Physics* **352** (2017), 967–978; arXiv:1602.03905.

[4] T. Lévy, The master field on the plane, *Astérisque* **388** (2017).

[5] L. de Moura and S. Ullrich, The Lean 4 theorem prover and programming language, in *Automated Deduction – CADE 28*, LNCS 12699, Springer, 2021, 625–635.

[6] The Mathlib Community, The Lean mathematical library, in *Proceedings of CPP 2020*, ACM, 2020, 367–381.