Fourier Transverse Modes Obstruct Volume-Uniform Critical Coercivity in a Flat Lattice Gauge Block Form

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Abstract

Let a fine periodic lattice have side LN' and let Q_L be the L^{-d}-normalized block average of length-L line integrals. In four dimensions, the rescaling $Q_L \mapsto LQ_L$ repairs the elementary constant-field scaling obstruction to a Poincare estimate. We prove that it cannot yield coercivity on the full one-cochain space with a constant uniform in the block side. For every $L \geq 2$, every fixed $N' \geq 1$, and $N_c \geq 2$, we embed the first within-block Fourier phase $\zeta_L = \exp(2\pi i/L)$ in a real two-plane of the internal coordinate space and construct a transverse one-cochain A_L. It satisfies

$$Q_L A_L = 0, \qquad \operatorname{div} A_L = 0,$$

and, for every dimension $d \geq 2$,

$$\|A_L\|^2 = (LN')^d, \qquad \langle A_L, K_0 A_L\rangle = (LN')^d \lambda_L, \qquad \lambda_L = 4\sin^2\left(\frac{\pi}{L}\right) \leq \frac{4\pi^2}{L^2}.$$

Thus the Rayleigh quotient of $K_0 + (s_L Q_L)^*(s_L Q_L)$ is exactly λ_L for every scalar rescaling s_L. Every admissible full-space Poincare constant obeys $C_{\mathrm{P}}(L) \geq 1/\lambda_L \geq L^2/(4\pi^2)$, so no constant chosen before L can work. Here "volume-uniform" means uniform while the block/fine side L varies at an arbitrary fixed positive coarse side N'; it is not a claim about $N' \to \infty$ at fixed L.

The construction, kernel identities, exact Hodge energy, Rayleigh identity, and quantified no-go theorem are formalized in Lean 4. The focused axiom audit reports only `propext`, `Classical.choice`, and `Quot.sound`. The theorem concerns the stated flat full-domain form; it does not contradict gauge-restricted propagator constructions and makes no infinite-volume, continuum, or Yang–Mills mass-gap claim.

1 Introduction

Finite-range positive operators admit Combes–Thomas localization estimates whose constants depend on the lower spectral edge [4]. In a block-spin treatment of lattice gauge fields, a natural finite-dimensional precision is a flat Hodge operator plus a positive block penalty. The question addressed here is sharply limited: can the form

$$\langle A, K_0 A\rangle + \|LQ_L A\|^2 \tag{1}$$

control the full fine one-cochain norm with a constant independent of the block side L?

The factor L is not arbitrary. With the line-integral block map used below, it exactly balances direction-wise constant fields in dimension four. That necessary scaling test passes, but it probes only the constant sector. The block map also has mean-zero transverse modes in its exact kernel.

The first Fourier mode on the within-block cycle makes this obstruction both local to every block and spectrally sharp at low momentum.

The main result has four components.

(i) The witness is defined on the exact finite periodic lattice and works for every $L \geq 2$, with no parity restriction.

(ii) Its cancellation under Q_L and under the flat divergence is exact in every block and at every positive coarse side N'.

(iii) Its Hodge Rayleigh quotient is the first-cycle eigenvalue $4\sin^2(\pi/L)$, hence is $O(L^{-2})$.

(iv) The quantifier order is explicit and machine checked: one proposed constant C_{P} is chosen before all block sides L.

An earlier square-wave witness gives the weaker even-scale quotient $8/L$. It already disproves the same gate, but the Fourier construction identifies the actual low-frequency mechanism and strengthens the quantitative lower bound on Poincare constants from linear to quadratic growth.

The conclusion is a domain diagnosis, not a general no-go for lattice gauge renormalization. Classical constructions use gauge restrictions, quotient structures, or background-dependent operators. The present theorem instead tests the unrestricted flat form in (1). A repaired theorem may still hold on a physically justified sector or after adding a positive interacting Hessian that lifts the explicit transverse family.

2 Finite lattice, block map, and quantifiers

Fix integers $d \geq 2$, $L \geq 1$, $N' \geq 1$, and $N_c \geq 2$. The fine site set is

$$\Lambda_{L,N'} = \left(\mathbb{Z}/(LN')\mathbb{Z}\right)^d,$$

and a positive bond is (x, μ) with $x \in \Lambda_{L,N'}$ and $\mu \in \{0, \ldots, d-1\}$. The real internal coordinate space is $E_{N_c} = \mathbb{R}^{N_c^2-1}$. For a one-cochain $A(x, \mu) \in E_{N_c}$, set

$$\|A\|^2 = \sum_{x \in \Lambda_{L,N'}} \sum_{\mu=0}^{d-1} \|A(x, \mu)\|^2.$$

At the trivial background, let D_1 be the discrete exterior derivative and δ the discrete divergence. The flat Hodge operator K_0 satisfies

$$\langle A, K_0 A\rangle = \|D_1 A\|^2 + \|\delta A\|^2. \tag{2}$$

In the formal development this is derived from the adjoint-defined operator; it is not introduced as an axiom.

For a coarse site $y \in (\mathbb{Z}/N'\mathbb{Z})^d$, write

$$B_y = \{Ly + r : 0 \leq r_\nu < L \text{ for every } \nu\}.$$

For a coarse positive bond (y, μ), define the linearized line-integral average

$$(Q_L A)(y, \mu) = L^{-d} \sum_{x \in B_y} \sum_{k=0}^{L-1} A(x + ke_\mu, \mu). \tag{3}$$

For a scalar sequence s_L, the tested precision is

$$H_{L,s} = K_0 + (s_L Q_L)^*(s_L Q_L). \tag{4}$$

Definition 2.1 (critical full-space gate)**.** In $d = 4$, fix $N' \geq 1$, $N_c \geq 2$, and the adjoint-model parameter defining the flat maps. The gate asserts that there is a number $C_{\mathrm{P}} > 0$ such that, for every integer $L \geq 1$ and every fine one-cochain A,

$$\|A\|^2 \leq C_{\mathrm{P}} \left(\langle A, K_0 A \rangle + \|L Q_L A\|^2 \right). \tag{5}$$

Remark 2.2 (meaning of volume-uniform)**.** *The coarse side N' is arbitrary but fixed inside one instance of the gate. The quantified side L then varies, so the fine volume $(LN')^4$ varies and the same C_{P} must work for all of it. This paper does not call a bound "volume-uniform" merely because it is uniform as $N' \to \infty$ at fixed L; that is a different quantifier statement.*

3 The block-periodic Fourier witness

Let

$$\zeta_L = \exp(2\pi i/L), \qquad L \geq 2.$$

Then $\zeta_L^L = 1$, $\zeta_L \neq 1$, and the finite geometric sum gives

$$\sum_{r=0}^{L-1} \zeta_L^r = 0. \tag{6}$$

Choose orthonormal vectors $u_0, u_1 \in E_{N_c}$, possible because $N_c^2 - 1 \geq 3$, and define the real-linear isometry

$$\iota : \mathbb{C} \longrightarrow E_{N_c}, \qquad \iota(a + ib) = a u_0 + b u_1.$$

For $m \in \mathbb{Z}/(LN')\mathbb{Z}$, define the block-periodic internal profile

$$\phi_L(m) = \iota\Big(\zeta_L^{m \bmod L}\Big). \tag{7}$$

It has unit norm and exact zero mean in every length-L block.

Choose distinct directions $i \neq j$ and set

$$A_L(x, \mu) = \begin{cases} \phi_L(x_j), & \mu = i, \\ 0, & \mu \neq i. \end{cases} \tag{8}$$

The field points in direction i and varies only in the transverse direction j.

Lemma 3.1 (norm and block kernel)**.** *For every $L \geq 2$ and $N' \geq 1$,*

$$\|A_L\|^2 = (LN')^d, \qquad Q_L A_L = 0. \tag{9}$$

Consequently $(s_L Q_L) A_L = 0$ for every real scalar s_L.

Proof. Only direction-i bonds contribute to the norm, one at each fine site, and $\|\phi_L(m)\| = 1$. For the block map, a coarse bond outside direction i sees only zero components. In direction i, line shifts preserve x_j. After the line sum is factored out, the sum over the within-block j-offset is (6). This cancellation occurs separately in every source block, hence for every N'. □

Lemma 3.2 (transverse divergence)**.** *At the trivial background,*

$$\delta A_L = 0.$$

Proof. Only the direction-i component enters. That component depends only on x_j and is invariant under forward and backward shifts in direction i. ☐

The witness therefore lies simultaneously in the exact kernel of the block map and the flat gauge-constraint kernel. Its remaining energy is purely curvature energy.

4 Exact first-mode energy

Define

$$\lambda_L = |\zeta_L - 1|^2. \tag{10}$$

For every r, multiplication by the unit complex number ζ_L^r gives

$$\left\|\iota(\zeta_L^r) - \iota(\zeta_L^{r+1})\right\|^2 = |1 - \zeta_L|^2 = \lambda_L, \tag{11}$$

including the periodic jump $r = L - 1$ because $\zeta_L^L = 1$. Elementary trigonometry yields

$$\lambda_L = 2 - 2\cos(2\pi/L) = 4\sin^2(\pi/L) \le \frac{4\pi^2}{L^2}. \tag{12}$$

Proposition 4.1 (exact flat Hodge energy)**.** *For the transverse witness* (8)*,*

$$\langle A_L, K_0 A_L\rangle = (LN')^d \lambda_L. \tag{13}$$

Proof. Lemma 3.2 removes the divergence term in (2). In the plaquette curl, every direction pair other than the pair containing i and j vanishes: either the field component is zero or its profile is unchanged by the shift. Each (i, j) plaquette contributes the common value λ_L from (11). There is one contributing geometric plaquette at each of the $(LN')^d$ fine sites under the convention used by D_1. Summation gives (13). ☐

Theorem 4.2 (exact Rayleigh quotient)**.** *For every scalar rescaling s_L,*

$$\frac{\langle A_L, H_{L,s} A_L\rangle}{\|A_L\|^2} = \lambda_L = 4\sin^2(\pi/L). \tag{14}$$

Thus the best full-space coercivity constant c_L of $H_{L,s}$ obeys

$$c_L \le \lambda_L \le \frac{4\pi^2}{L^2}. \tag{15}$$

Proof. Lemma 3.1 removes the block term from (4). Divide Proposition 4.1 by the positive norm in Lemma 3.1; the entire volume factor cancels. ☐

5 No uniform critical coercivity

Theorem 5.1 (all-scales, all-coarse-side no-go). *Fix $N' \geq 1$, $N_c \geq 2$, and the adjoint-model parameter defining the flat operators. The critical full-space gate in Definition 2.1 is false. More quantitatively, at every $L \geq 2$ any constant valid at that scale must satisfy*

$$C_{\mathrm{P}}(L) \geq \frac{1}{\lambda_L} \geq \frac{L^2}{4\pi^2}. \tag{16}$$

Proof. Insert A_L into (5). By Lemma 3.1 and Proposition 4.1,

$$(LN')^4 \leq C_{\mathrm{P}}(LN')^4 \lambda_L.$$

The volume factor is positive, so $1 \leq C_{\mathrm{P}}\lambda_L$, proving the first inequality in (16); the second follows from (12). If one C_{P} worked for all L, it would force $L^2 \leq 4\pi^2 C_{\mathrm{P}}$ for every L, a contradiction. $\square$

Corollary 5.2 (scalar normalization is irrelevant). *No real scalar sequence s_L can restore full-space uniform coercivity for $K_0 + (s_L Q_L)^*(s_L Q_L)$.*

Proof. The same A_L belongs to $\ker Q_L$, hence to $\ker(s_L Q_L)$, at every scale. $\square$

The theorem isolates the lower spectral edge as the obstruction. Separate finite-range, kernel-amplitude, ball-count, and block-metric localization estimates are not invalidated. What fails is the positive scale-independent coercivity input required to turn those estimates into a uniform inverse or inverse-square-root bound on the full domain.

6 Relation to prior gauge-RG constructions

Bałaban's propagator papers study the Gaussian vector-field theory underlying lattice gauge renormalization and then extend the analysis to operators with restrictions at several scales and to iterated renormalization transformations [1, 2]. In that program, block averages are not used in isolation: gauge conditions, restricted domains, minimizers, and background-dependent propagators are part of the construction. Modern expositions of covariant axial gauge likewise emphasize an axial restriction adapted to block averaging [3].

There is therefore no conflict between classical positivity statements on a gauge-fixed or otherwise restricted subspace and Theorem 5.1. The claims have different domains. The exact delta established here is:

> For the explicit finite periodic line-integral map (3), the flat full-space form $K_0 + (s_L Q_L)^*(s_L Q_L)$ has block-periodic transverse Fourier modes in the simultaneous kernels of Q_L and the flat divergence, with Rayleigh quotient $4\sin^2(\pi/L)$ at every $L \geq 2$ and every $N' \geq 1$.

This is a no-go for that unrestricted coercivity proposal, not a priority claim over the general observation that gauge fields require gauge fixing, and not a no-go for Bałaban's restricted propagators. Indeed, the result explains why a domain restriction or quotient is mathematically load-bearing rather than cosmetic.

7 Machine-checked realization

The Lean 4 development is a seven-file chain headed by

`YangMills/RG/PhysicalCriticalRescalingFourierNoGoAllScales.lean`.

Table 1 maps the analytic statements to selected declarations.

Mathematical content	Lean declaration
Root-of-unity block cancellation	`sum_blockFourierRoot_pow_eq_zero`
Lie-valued cancellation in every block	`sum_blockFourierProfile_block_eq_zero`
Eigenvalue bound $\lambda_L \le (2\pi/L)^2$	`blockFourierEigenvalue_le`
Exact block-map kernel	`flatBlockConstraintQCLM_blockFourierMode_eq_zero`
Zero transverse divergence	`gaugeConstraintQCLM_blockFourierModeCochain_eq_zero_of_ne`
Exact Hodge quadratic form	`flatGaugeHodgeK0_inner_blockFourierModeCochain_of_ne`
Exact Rayleigh quotient	`blockFourierMode_rayleigh_eq`
Necessary bound $1 \le C_{\mathrm{P}}\lambda_L$	`criticalRescaledFlatPoincare_fourier_lower_bound`
All-scales no-go	`volumeUniformCriticalRescaledFlatPoincareGate_fourier_false`

Table 1: Theorem-to-artifact map.

The formal result uses the exact norm definition $\lambda_L = \|\zeta_L - 1\|^2$ and proves the bound $\lambda_L \le (2\pi/L)^2$. Equation (12) is the elementary closed-form identification of that norm. The focused oracle

`YangMills/RG/PhysicalCriticalRescalingFourierNoGoOracle.lean`

reports exactly

[`propext`, `Classical.choice`, `Quot.sound`]

for all nine audited declarations, with no project axiom and no `sorryAx`.

Immutable source and reproduction

The theorem source is fixed at repository commit

`f21539ed0bb880a04078de369bf5cbf063f7b101`.

The SHA-256 of the head theorem file is

`5F0890D14EA6981CBE6459605A634386ED2C844F6D03A6947E33B34250E2F5AE`.

The companion source packet `critical_rescaling_fourier_no_go_sources_v1.1.zip` has SHA-256

`AEF0AEAB6841FE4B26F6A5160305793646083EB72166BC5C751C8E2A5AA88132`.

The packet contains the seven theorem/oracle files, the manifest, the Lean toolchain, and the pinned Lake manifest. The toolchain is Lean `v4.29.0-rc6`. The pinned Mathlib revision is

`07642720480157414db592fa85b626dafb71355b`.

From the repository root:

```
lake build YangMills.RG.PhysicalCriticalRescalingFourierNoGoAllScales
lake env lean YangMills/RG/PhysicalCriticalRescalingFourierNoGoOracle.lean
lake build YangMillsCore
```

The verified runs completed successfully with 8184 jobs for the focused theorem build and 8481 jobs for the aggregate core. The oracle dependencies are exactly those displayed above. A direct search found no `sorry`, `admit`, or project `axiom` in the seven source files.

8 What remains open

Restricted or quotient sector. A repaired domain may remove or quotient the obstructive transverse family, but the restriction must be physically justified and stable under the renormalization operations. Coercivity on that new domain remains to be proved.

Additional positive Hessian. An interacting Wilson Hessian or another positive term V_L could lift the family if

$$\langle A_L, V_L A_L \rangle \geq c_* \|A_L\|^2$$

with $c_* > 0$ independent of L. This is not a consequence of the flat calculation and is not assumed here.

No continuum conclusion. The theorem is finite-dimensional. It proves no positive coercivity for an interacting theory, no infinite-volume measure, no continuum limit, no Osterwalder–Schrader reconstruction, and no Yang–Mills mass gap.

9 Conclusion

Critical scalar rescaling can correct a normalization mismatch but cannot remove an exact kernel. The first within-block Fourier phase produces, at every $L \geq 2$ and every positive coarse side, a transverse divergence-free cochain annihilated by Q_L. Its exact Rayleigh quotient is $4\sin^2(\pi/L)$, so the best full-space spectral lower bound is at most $4\pi^2/L^2$. The current flat full-domain critical gate is therefore impossible for any scalar normalization of the block map.

The negative result is operational: it identifies the precise input that must change. A successful continuation must justify a different domain or add an operator that supplies uniform energy to the explicitly exhibited Fourier sector.

A The square-wave precursor

For even $L = 2M$, the real profile that equals $+1$ on the first half of each block and -1 on the second half also has zero block mean. Its two jumps per block have squared magnitude four, giving total one-dimensional energy $8N'$ and Rayleigh quotient $8/L$. This witness is useful because it requires only a single internal vector and elementary interface counting. The Fourier witness is stronger in three ways: it works for every $L \geq 2$, it identifies the first periodic eigenmode, and it improves the quantitative obstruction from $8/L$ to $4\sin^2(\pi/L) = O(L^{-2})$.

References

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