Machine-Checked Finite-SU(2) Trace-Skein Closure for Makeenko–Migdal Crossing Terms

From Normalized Pauli Contraction to All-Order Single-Trace Resolution

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Abstract

Finite-rank Makeenko–Migdal equations generate products of Wilson traces at self-intersections. For SU(2), this apparent multitrace obstruction closes exactly on single traces, but the statement is normalization-sensitive: the traceless Lie algebra contributes a finite-rank correction that disappears for $U(2)$ and must not be dropped. We give a Lean 4/Mathlib formalization of the complete group-algebraic closure mechanism on Mathlib's concrete special unitary matrix group. With normalized trace $\tau(A) = \mathrm{Tr}(A)/2$ and normalized anti-Hermitian Pauli directions $X_j = \mathrm{i}\sigma_j/2$, the kernel checks the Casimir identity $\sum_j X_j^2 = -3I/4$, the Fierz identity

$$\sum_{j=1}^{3} X_j B X_j = -\tau(B)I + \tfrac{1}{4}B,$$

the induced crossing contraction, and the rank-two skein identity

$$\tau(g)\tau(h) = \tfrac{1}{2}\big(\tau(gh) + \tau(gh^{-1})\big).$$

Consequently the finite-SU(2) crossing term $\tau(g)\tau(h) - \frac{1}{4}\tau(gh)$ is the linear combination $\frac{1}{4}\tau(gh) + \frac{1}{2}\tau(gh^{-1})$. We then formalize a universal local interface with four cyclically ordered branch holonomies, an independent orientation on each branch, the two opposite-strand words, and precisely the two direct/reversed reconnections. Its corrected crossing term closes on those two reconnections for every branch assignment and orientation choice. A recursive theorem also extends the reduction to products of arbitrarily many fundamental traces. The identities are classical; the contribution is a concrete, kernel-checked normalization bridge from Pauli contraction to the single-trace closure used in finite-rank loop equations. We do not claim a formal derivation of the Yang–Mills area derivative, planar loop geometry, or the full Makeenko–Migdal equation.

1 Introduction

The Makeenko–Migdal equation replaces a local variation of a Wilson loop at a crossing by a contraction of the two strands created by cutting the loop. Rigorous two-dimensional versions were proved on the plane and on compact surfaces by Driver, Hall, Kemp, and Gabriel, building on work of Lévy and others [3, 4, 2]. At finite rank, the contraction generally produces a product of traces rather than a closed equation for one Wilson trace. The large-N limit removes this obstruction only after a separate factorization argument.

Rank two is different. The Cayley–Hamilton identity for determinant-one 2×2 matrices turns a product of two fundamental traces into two single traces. Kazakov and Zheng exploit precisely this fact to obtain linear finite-$SU(2)$ lattice and continuum loop equations [5]. The reduction is elementary, but three convention choices interact:

1. normalized versus unnormalized trace;
2. $U(2)$ versus $SU(2)$ Lie algebra contraction;
3. the scale of the invariant inner product, hence the fundamental Casimir and heat-kernel time.

The purpose of this paper is to make that interface exact and machine checked. The key finite-rank term is $1/4$. It is the N^{-2} correction left after removing the central $u(1)$ direction from $u(2)$. Dropping it silently turns the $SU(2)$ equation into the $U(2)$ equation.

1.1 Central result

Let $\tau(A) = \mathrm{Tr}(A)/2$ and let

$$X_1 = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_3 = \frac{\mathrm{i}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For arbitrary $A, B \in M_2(\mathbb{C})$, the formalized contraction theorem is

$$\sum_{j=1}^{3} \tau(AX_jBX_j) = -\tau(A)\tau(B) + \frac{1}{4}\tau(AB). \tag{1}$$

For $g, h \in \mathrm{SU}(2)$, the formalized skein theorem is

$$\tau(g)\tau(h) = \frac{1}{2}\tau(gh) + \frac{1}{2}\tau(gh^{-1}). \tag{2}$$

Combining them yields a linear single-trace expression for the negative of (1), which is the algebraic right-hand side delivered by the local Makeenko–Migdal integration-by-parts step. Section 4 upgrades this binary identity to a quantified four-branch oriented crossing interface.

1.2 Claim boundary

Table 1 is part of the theorem statement, not a disclaimer to be read after it.

Claim	Status	Evidence or boundary
Concrete normalized Pauli matrices and $-3I/4$ Casimir	kernel checked	`su2Pauli_casimir`
Rank-two Fierz and crossing contraction	kernel checked	`su2Pauli_fierz`, `su2PauliCrossingContraction_eq`
$SU(2)$ trace-skein identity	kernel checked	`su2_matrix_trace_skein`, `su2_fundamentalWilson_skein`
Finite-$SU(2)$ corrected crossing closure	kernel checked	`su2_makeenkoMigdal_crossingTerm_closed`
Four oriented branches and both local reconnections	kernel checked	`su2_fourBranchCrossing_term_closed`
All-order multitrace-to-single-trace recursion	kernel checked	`su2WilsonProductExpansion_eq`
Yang–Mills area-derivative identity	not claimed	requires heat-equation differentiation and integration by parts
Geometric certification of an embedded transverse crossing	not claimed	the four-branch datum is algebraic, not an embedding object

Table 1: Claim ledger. "Kernel checked" refers to the pinned Lean artifact and the audit described in Section 7.

2 Conventions and the finite-rank correction

We work in $M_2(\mathbb{C})$ with normalized trace $\tau(I) = 1$. The three matrices X_1, X_2, X_3 are anti-Hermitian and traceless. With invariant inner product

$$\langle X, Y \rangle = 2 \operatorname{Re} \operatorname{Tr}(X^* Y),$$

they are the standard normalized Pauli directions. Their quadratic Casimir in the defining representation is

$$X_1^2 + X_2^2 + X_3^2 = -\frac{3}{4} I. \tag{3}$$

The same value appears as $n(n+2)/4 = 3/4$ at irreducible label $n = 1$ in the heat-kernel convention of the preceding exact two-dimensional formalization [8].

For $u(2)$ with the central direction included, the corresponding magic formula is

$$\sum_X XBX = -\tau(B)I.$$

Passing to the traceless algebra removes the central basis vector and leaves

$$\sum_{j=1}^{3} X_j B X_j = -\tau(B)I + \frac{1}{4}B. \tag{4}$$

Thus the distinction between $U(2)$ and $SU(2)$ is not cosmetic. It creates the term that later appears as $-\frac{1}{4}\mathcal{W}(L)$ in the finite-rank crossing equation.

Theorem 2.1 (Kernel-checked Pauli contraction)**.** *For all $A, B \in M_2(\mathbb{C})$, equations* (3), (4), *and* (1) *hold.*

Proof. Equation (4) is a four-entry matrix identity. Multiplying it on the left by A, applying τ, and using cyclicity and linearity gives (1). The Lean proof does not invoke representation theory or a prepackaged Casimir theorem: it expands the concrete matrices, reduces the four entries, uses $\mathrm{i}^2 = -1$, and normalizes the resulting polynomial identities in $\mathbb{C}$. ☐

The main declaration is deliberately stronger in one direction than required: A and B are arbitrary complex matrices. Special unitarity is needed only for the skein step.

3 The rank-two trace skein

For a 2×2 matrix h with determinant one, Cayley–Hamilton gives

$$h + h^{-1} = \operatorname{Tr}(h)I.$$

Multiplication by g and application of the trace yield

$$\operatorname{Tr}(gh) + \operatorname{Tr}(gh^{-1}) = \operatorname{Tr}(g)\operatorname{Tr}(h). \tag{5}$$

Dividing by four gives (2). The factor $1/2$ on its right-hand side is forced by normalized trace; it disappears if one writes the same identity with the unnormalized trace, as in much of the physics literature.

Theorem 3.1 (Kernel-checked $SU(2)$ skein)**.** *For all $g, h \in \mathrm{SU}(2)$, equation* (2) *holds in* $\mathbb{C}$.

Proof. Mathlib represents SU(2) as the special unitary subgroup of complex 2×2 matrices. The formal proof expands both matrix traces into their four entries, uses the special-unitary entry relations

$$h_{11} = \overline{h_{00}}, \qquad h_{10} = -\overline{h_{01}},$$

and cancels the off-diagonal terms. This route avoids importing an abstract trace-algebra package and tests the exact coercions and inverse convention used by the downstream formalization. $\square$

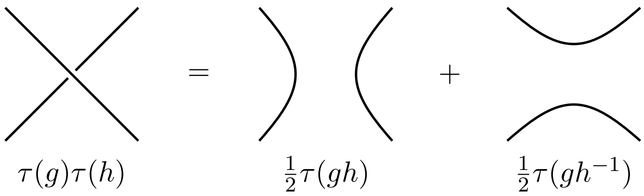

Figure 1: Algebraic skein resolution. The drawing is mnemonic; the formal input consists only of two group elements and contains no planar embedding.

4 Single-crossing closure

Define the normalized fundamental Wilson function

$$\mathcal{W}(g) := \tau(g), \qquad g \in \mathrm{SU}(2).$$

The negative Pauli contraction in (1) is

$$\mathcal{W}(g)\mathcal{W}(h) - \frac{1}{4}\mathcal{W}(gh). \tag{6}$$

The first term is the cut-loop product familiar from Makeenko–Migdal; the second is the finite-$SU(2)$ correction.

Theorem 4.1 (Finite-$SU(2)$ crossing closure)**.** *For all $g, h \in \mathrm{SU}(2)$,*

$$\mathcal{W}(g)\mathcal{W}(h) - \frac{1}{4}\mathcal{W}(gh) = \frac{1}{4}\mathcal{W}(gh) + \frac{1}{2}\mathcal{W}(gh^{-1}). \tag{7}$$

Equivalently,

$$\sum_{j=1}^{3} \tau(gX_jhX_j) = -\frac{1}{4}\mathcal{W}(gh) - \frac{1}{2}\mathcal{W}(gh^{-1}).$$

Proof. Substitute Theorem 3.1 into (6) and collect coefficients. The second formula is Theorem 2.1 with the sign reversed. $\square$

Equation (7) is the exact reason the fundamental $SU(2)$ loop equation is linear at finite rank. No probabilistic factorization is used. Since the identity holds pointwise in (g, h), any linear expectation or integral may be applied after the reduction, provided the group-valued loop words are measurable and integrable.

Remark 4.2 (Generator conventions). An area-derivative equation may carry an overall factor depending on whether the heat semigroup is generated by Δ, $\Delta/2$, or a coupling-scaled operator. The relative coefficients in (7) are fixed by normalized trace and the Pauli convention above. This paper formalizes those relative coefficients, not an unstated choice of Yang–Mills time.

4.1 Four oriented branches and two reconnections

The binary variables (g, h) can hide the local word bookkeeping. We therefore formalize it explicitly. Let

$$C = (u_0, u_1, u_2, u_3; \epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3), \qquad u_j \in \mathrm{SU}(2), \quad \epsilon_j \in \{+1, -1\},$$

be four cyclically ordered branch holonomies after transport to the crossing base point. Write $H_j = u_j$ for $\epsilon_j = +1$ and $H_j = u_j^{-1}$ for $\epsilon_j = -1$. Opposite positions form the two cut strand words

$$g_C = H_0 H_2, \qquad h_C = H_1 H_3.$$

The two local reconnections are then

$$r_+(C) = g_C h_C, \qquad r_-(C) = g_C h_C^{-1}.$$

Thus path reversal is represented by group inversion and concatenation by group multiplication, with no convention left implicit.

Theorem 4.3 (Four-branch oriented closure)**.** *For every assignment of the four branch holonomies and all sixteen orientation choices,*

$$\mathcal{W}(g_C)\mathcal{W}(h_C) - \frac{1}{4}\mathcal{W}(r_+(C)) = \frac{1}{4}\mathcal{W}(r_+(C)) + \frac{1}{2}\mathcal{W}(r_-(C)). \tag{8}$$

The Pauli-contracted word closes on the same two reconnections with coefficients $-1/4$ and $-1/2$.

Proof. The Lean structure stores all four group elements and four orientations. Reduction of the orientation map produces g_C and h_C; then Theorem 4.1 applies uniformly. The contracted statement follows from Theorem 2.1. No case split over the sixteen orientations is needed because inversion is internal to the definitions. □

This is the promised closure theorem rather than a list of low-order examples. It consumes certified local holonomies; it does not claim that a planar embedding, by itself, has already produced or certified those inputs.

5 All-order single-trace resolution

The two-factor skein relation iterates. Define recursively, for $g, h_1, \ldots, h_m \in \mathrm{SU}(2)$,

$$\begin{aligned} R(g; \varnothing) &= \mathcal{W}(g), \\ R(g; h_1, h_2, \ldots, h_m) &= \frac{1}{2}\Big(R(gh_1; h_2, \ldots, h_m) + R(gh_1^{-1}; h_2, \ldots, h_m)\Big). \end{aligned}$$

Every leaf of this recursion contains one trace of one group word. There are 2^m leaves before identical words are collected.

Theorem 5.1 (All-order trace closure)**.** *For every $m \geq 0$ and $g, h_1, \ldots, h_m \in \mathrm{SU}(2)$,*

$$R(g; h_1, \ldots, h_m) = \mathcal{W}(g) \prod_{r=1}^{m} \mathcal{W}(h_r). \tag{9}$$

Proof. Induct on m. The empty case is the definition. In the successor case, apply the induction hypothesis to both branches and factor out the common tail product. The remaining bracket is exactly the two-factor identity (2). □

The Lean theorem quantifies over a `List SU2` and generalizes the accumulated word during induction. It therefore proves one reusable theorem, not a family of separately expanded low-order examples.

6 Formalization architecture

The new module is `Lean2dYangMills/SU2TraceSkein.lean`. It extends the concrete matrix and character layer of the exact two-dimensional project; it does not depend on the cellulation, gauge-fixing, or heat-kernel integration modules. This low dependency position is intentional: the trace closure is a group-algebraic consumer that should be reusable by several future analytic routes.

The principal definitions are:

```
def su2NormalizedMatrixTrace
    (A : Matrix (Fin 2) (Fin 2) Complex) : Complex :=
  Matrix.trace A / 2

def su2PauliCrossingContraction
    (A B : Matrix (Fin 2) (Fin 2) Complex) : Complex :=
  su2NormalizedMatrixTrace (A * su2PauliX * B * su2PauliX) +
  su2NormalizedMatrixTrace (A * su2PauliY * B * su2PauliY) +
  su2NormalizedMatrixTrace (A * su2PauliZ * B * su2PauliZ)
```

The matrix-trace definition is shown with its exact compiled identifiers; the group observable $\mathcal{W}$ is not applied to arbitrary matrices. The compiled crossing endpoint is:

```
theorem su2_makeenkoMigdal_crossingTerm_closed (g h : SU2) :
    su2FundamentalWilson g * su2FundamentalWilson h -
        (1 / 4 : Complex) * su2FundamentalWilson (g * h) =
      (1 / 4 : Complex) * su2FundamentalWilson (g * h) +
        (1 / 2 : Complex) * su2FundamentalWilson (g * h^-1)
```

The all-order endpoint is:

```
theorem su2WilsonProductExpansion_eq (g : SU2) (hs : List SU2) :
    su2WilsonProductExpansion g hs =
      su2FundamentalWilson g *
        (hs.map su2FundamentalWilson).prod
```

The local crossing interface is a separate compiled layer:

```
structure SU2FourBranchCrossing where
  branch0 branch1 branch2 branch3 : SU2
  orientation0 orientation1 : SU2BranchOrientation
  orientation2 orientation3 : SU2BranchOrientation

theorem su2_fourBranchCrossing_term_closed
    (crossing : SU2FourBranchCrossing) :
  crossing.correctedCrossingTerm =
    (1 / 4 : Complex) *
        su2FundamentalWilson crossing.directResolution +
      (1 / 2 : Complex) *
        su2FundamentalWilson crossing.reverseResolution
```

6.1 Dependency separation

The module proves no theorem whose premise is named "Makeenko–Migdal". This is a feature. If a later file formalizes the local heat-equation integration by parts, it can import this module and obtain the $SU(2)$ closure without re-proving matrix algebra. Conversely, the present theorems remain valid for any probability law, lattice action, or deterministic assignment of group elements.

7 Audit and reproducibility

The artifact pins Lean `v4.29.0-rc6` and Mathlib commit `07642720480157414db592fa85b626dafb71355b`. Verification requires an explicit module build, followed by an oracle query for the headline declarations. The companion review package includes the source module, an audit module, the exact build transcript, a theorem manifest, and SHA-256 hashes. Its source archive is `lean-su2-trace-skein-makeenko-migdal-source.zip`, SHA-256 `387D69A029D4718EEDB5AE5203C06540594066EA349820EF14376A86B6150F44`.

The minimum independent check is:

```
lake env lean Lean2dYangMills/SU2TraceSkein.lean
lake env lean Lean2dYangMills/AuditSU2TraceSkein.lean
```

The audit prints the axioms of the eleven headline declarations. The expected trust boundary is the standard one for this project: Lean's kernel, the pinned toolchain, Mathlib, and the integrity of the cited source snapshot. A clean kernel check does not establish novelty and is not a substitute for peer review.

Availability boundary. No permanent public source URL is claimed in this version. The only authorized publication route currently accepts the manuscript PDF but not a supplementary ZIP. Consequently an independent source audit requires receiving the exact content-addressed companion archive named above. A local filesystem path is not evidence of public availability, and the paper does not represent it as one.

8 Relation to prior work

Makeenko and Migdal introduced loop equations in the large-colour setting [1]. Rigorous planar and compact-surface area equations were later obtained in a finite-dimensional heat-kernel formulation [3, 4, 2]. Those works contain results far more general than the matrix algebra proved here, including the analytic integration-by-parts step that this paper explicitly does not claim.

Kazakov and Zheng state the determinant-one rank-two identity $\mathrm{Tr}(U)\,\mathrm{Tr}(V) = \mathrm{Tr}(UV) + \mathrm{Tr}(UV^\dagger)$ and use it to close finite-$SU(2)$ lattice and continuum loop equations on single traces [5]. Thus the mathematical linearity is not new. The delta of the present work is:

- a kernel-checked derivation on the concrete Mathlib $SU(2)$ type;
- simultaneous verification of the Pauli/Casimir and trace-skein normalizations;
- explicit isolation of the $1/4$ special-unitary correction;
- a quantified four-branch interface with independent orientations and both local reconnections;
- an all-order recursive closure theorem rather than only the binary trace identity;
- a reusable interface to the earlier formalized heat-kernel model.

A targeted search on 1 August 2026 across GitHub, Mathlib, the Lean community index, and the sources cited above did not locate an existing Lean formalization of this exact Pauli-to-skein chain. This is a search report, not a categorical priority claim.

9 Limitations and next producer

The current artifact now defines the complete algebraic target of local crossing geometry: four oriented branch holonomies, opposite-strand words, and both reconnections. It does not define an embedded planar graph, a regular loop, a certified transverse crossing, face areas, or the alternating derivative

$$\partial_{t_1} - \partial_{t_2} + \partial_{t_3} - \partial_{t_4}.$$

It does not prove that differentiating a Yang–Mills expectation gives the negative Pauli contraction. It also does not construct a continuum holonomy field or treat arbitrary representations.

The next producer is therefore sharply specified: formalize the local abstract Makeenko–Migdal theorem for four adjacent heat kernels, with extended gauge invariance and all generator conventions explicit. Once that producer yields the contraction in (1), Theorem 4.1 closes its $SU(2)$ fundamental-trace output immediately through Theorem 4.3. A separate geometric layer must still prove that the transported paths of an embedded transverse crossing instantiate the four-branch structure with the required cyclic order.

This separation also provides a kill criterion. If the analytic local theorem cannot be formalized without importing hypotheses equivalent to its conclusion, the present paper remains valid as a group-algebraic result, but no future manuscript should advertise a machine-checked Makeenko–Migdal area equation.

10 Conclusion

The finite-$SU(2)$ loop equation is linear for an exact algebraic reason. The traceless Pauli contraction contributes a $1/4$ correction, and the determinant-one trace skein converts the remaining product of traces into two single traces. Lean checks this chain on concrete complex matrices, packages all four oriented local branches and both reconnections in a universal closure theorem, and extends the binary relation to products of arbitrary finite length. The result is a small but load-bearing bridge: it neither assumes nor overstates the analytic Makeenko–Migdal theorem, yet it removes the complete finite-rank multitrace obstruction once that theorem is available.

A Theorem-to-artifact map

Paper statement	Lean declaration
Equation (3)	`su2Pauli_casimir`
Equation (4)	`su2Pauli_fierz`
Equation (1)	`su2PauliCrossingContraction_eq`
Equation (5)	`su2_matrix_trace_skein`
Theorem 3.1	`su2_fundamentalWilson_skein`
Theorem 4.1	`su2_makeenkoMigdal_crossingTerm_closed`
Equivalent contracted form	`su2PauliCrossingContraction_eq_closedResolutions`
Theorem 4.3	`su2_fourBranchCrossing_term_closed`
Four-branch skein form	`su2_fourBranchCrossing_trace_skein`
Four-branch contracted form	`su2_fourBranchCrossing_pauliContraction_closed`
Theorem 5.1	`su2WilsonProductExpansion_eq`

Table 2: Exact correspondence between the mathematical claims and compiled declarations.

B Migration note

This is a new companion paper, not a replacement for ai.viXra:2607.0039. That paper proves the exact heat-kernel evaluation for certified finite disk cellulations with a simple retained exterior boundary. The present work adds a different independently citable theorem: finite-$SU(2)$ trace closure at a crossing. It reuses the concrete $SU(2)$ matrix layer but does not duplicate the conditioned-edge theorem, the gauge-fixing construction, or the exact simple-loop area law. The two papers should cite each other asymmetrically: the present paper cites 2607.0039 as its formal substrate, while a future version of the substrate may list this module as the beginning of the intersecting-loop frontier.

References

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