The Collapse: Machine-Checked Bond Reflection Positivity and a Site Reflection Form for the Coupled $\mathbb{Z}_2$ Slice

complex observables of a whole half — one bridge closed to the measure, one still open

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Abstract

Bond reflection, and this one is about the measure. A whole path X of $2m+2$ slices has a past half past X and a future half; write rev X for the future half *read backwards from the far end*, which is the reflected copy. Let F be a complex observable of an entire half. Then for every L, every m, every strictly positive source weight w and every $\beta \geq 0$,

$$\sum_X \overline{F(\text{past } X)}\, F(\text{rev } X)\, W(X) \;\geq\; 0,$$

the sum running over all whole paths and W being the ordinary Gibbs weight. This is the finite-volume Osterwalder–Schrader reflection-positivity inequality for this model at odd separation. The sum is *unnormalised*, which is the form the axiom is about; dividing by the partition function is division by a positive number and preserves the sign. The Gram matrix of a finite family of such observables, with complex coefficients, satisfies it as well. It is obtained by proving that assembling a past half, a crossing bond and a reversed future half is a *bijection* onto paths, and that the weights multiply with exactly one crossing factor.

Site reflection, and this one is not, yet. Through a site the two halves *share* the middle slice, so the assembly is not a product of two independent halves and its bijection is a different statement. What is proved there is the half-chain form, its collapse identity and its Gram positivity — at *every* β, negative coupling included — but **not** its identification with the path measure. For that geometry the object remains a candidate, and §8 says so row by row.

On the β hypothesis. $\beta \geq 0$ is proved *sufficient* and is not proved necessary here; at $L = 0$ it could not be, since the bare kernel is then the scalar 1. The witness that would make the boundary exact from one site upwards is authorised by a gate and is not written.

The mechanism, in one sentence. Summing out the interior of a half sends F to a vector indexed by its boundary slice alone, after which the bond case reduces to positive semidefiniteness of the kernel — machine-checked in the companion paper [5] — and the site case to a weighted squared norm, which uses no property of the kernel at all.

Where the two reflections differ, and it is not cosmetic. Through a bond the two halves are disjoint and meet through one kernel factor, so positivity needs the kernel itself positive semidefinite — which, for $L \geq 1$, holds exactly when $\beta \geq 0$ [5]. Through a site the halves *share* the middle slice; the form is then a sum of squared moduli divided by the weight of that slice, so it is non-negative for *every* β, negative coupling included, and no property of the kernel is used at all.

What the source weight does. In the companion paper the weight was handled by congruence: conjugation by $\sqrt{w}$ cannot change the sign of a quadratic form. Here it is

not conjugated away, it is *summed* away — and what is left over differs between the two geometries. In the bond case every source weight is absorbed into the collapse and the bare kernel remains; in the site case the shared boundary slice survives as a factor $1/w(\sigma)$, because each half carries that slice's weight and the product would count it twice. Different reasons, same conclusion: nothing in the hypotheses depends on w beyond positivity.

What this is not. No reconstruction: the physical Hilbert space as the quotient of the past algebra by the null space of this form is not built. Nothing here concerns uniformity in the extent, SU(N), the continuum limit, or the Yang–Mills mass gap [7].

On the pre-registration. Four gates were committed before a line of the module was written; §7 reports what happened to each. Two of them read a minimum eigenvalue rather than sampling observables, which is the instrument the previous campaign's autopsy said was needed.

1 The gap this paper attacks

The companion paper proves the reflected two-point form of the Gibbs measure non-negative — but for *real* observables of a *single* slice, evaluated at the two *ends* of a path [5]. The Osterwalder–Schrader axiom asks for something wider in three directions at once: the observable should be a function of the whole past half-chain, it should be complex, and the statement should be about a matrix, not a number [1, 2]. Reflection positivity is one axiom among several, and on a lattice it is the one that yields a self-adjoint transfer operator by the standard reconstruction [3, 4].

That paper also said, in its scope section, that the missing step is a *construction* and not a further inequality: collapse each half to a boundary vector and the two inequalities you need are the ones already proved. This paper carries out that construction and reports how far it gets.

All three directions are closed, and for the bond so is the identification. The observables are complex, they depend on the entire half-path, and the statement is about a Gram matrix (Theorem 5.3). For the bond geometry the form is moreover *proved equal* to the reflected Gibbs sum over whole paths (§6), so there the theorems are about the measure. For the site geometry that identification is still open, and the two are not conflated anywhere below.

2 The objects

A slice is $\sigma : \{1, \dots, L\} \to \mathbb{Z}_2$ and the decoupled kernel is

$$K(\sigma, \tau) \;=\; \prod_j e^{\beta\, s(\sigma_j, \tau_j)}, \qquad s = +1 \text{ if equal, } -1 \text{ otherwise.}$$

A *past half-path* is a map $a : \{0, \dots, m\} \to \{\text{slices}\}$, and an observable of the past half-chain is a function F of one, valued in $\mathbb{C}$. Its weight is the ordinary Gibbs weight of that half,

$$W(a) \;=\; \Big(\prod_{t=0}^{m} w(a_t)\Big)\Big(\prod_{t=0}^{m-1} K(a_t, a_{t+1})\Big),$$

which is the same function the bridge module already uses for whole paths [6]; **no new notion of weight is introduced anywhere in this work**. Reflection means reading a half backwards from the far end.

A convention, stated where the notation appears. We write $\langle \Theta F, G \rangle$ for the reflected pairing built from the pair (past, rev) of §6. It is notation: no global involution of paths is constructed anywhere in this work, and none is used.

Definition 2.1 (the collapse)**.** $(\Phi F)(\sigma) \;=\; \sum_{a\,:\,a_m = \sigma} W(a)\, F(a).$

`collapse`, with its linearity at `collapse_add` and `collapse_smul`. The regrouping of a sum over halves by boundary slice — the only combinatorial fact either factorisation needs — is `sum_halves_by_edge`.

3 Through a site: every β

With the plane through slice m the two halves share that slice. Its weight is carried by both halves, so the product of the two half weights counts it *twice*, and it is divided out once so that the assembled path carries it exactly once. That division is the $1/w(\sigma)$ in Theorem 3.1, and it is also why the site assembly is a fibred product rather than a product.

Theorem 3.1 (site factorisation)**.** $\langle \Theta F, F \rangle_{\text{site}} \;=\; \sum_{\sigma} \dfrac{|(\Phi F)(\sigma)|^2}{w(\sigma)}.$

`osPairingSite_eq`. Nothing about K enters: the halves communicate only through the shared slice.

Corollary 3.2. *For every strictly positive w and* every *β — negative coupling included — the form is a non-negative real.*

`osPairingSite_nonneg`.

4 Real to complex, once and generically

Lemma 4.1. *For a real* symmetric *kernel, $\sum_{i,j} \overline{z_i} K_{ij} z_j \;=\; \sum_{i,j} K_{ij}\, x_i x_j + \sum_{i,j} K_{ij}\, y_i y_j$ with $z = x + iy$.*

`complexQuad_eq`, resting on the cancellation of the cross terms at `sum_cross_antisymm`. This is the entire content of "real observables to complex ones", and it costs exactly symmetry. It is stated for an arbitrary finite index type, so it is reusable outside this lane.

5 Through a bond: sufficiency of $\beta \geq 0$

Theorem 5.1 (bond factorisation)**.** $\langle \Theta F, F \rangle_{\text{bond}} \;=\; \sum_{\sigma,\tau} \overline{(\Phi F)(\sigma)}\, K(\sigma,\tau)\, (\Phi F)(\tau).$

`osPairingBond_eq`. The source weight is not conjugated away; it is summed away, into Φ. What is left at the boundary is the bare kernel.

Theorem 5.2 (the form, for $\beta \geq 0$)**.** *For every strictly positive w and every $\beta \geq 0$, $\langle \Theta F, F \rangle_{\text{bond}}$ is a non-negative real, for every complex observable F of the whole past half-chain.*

`osPairingBond_nonneg`, which is Theorem 5.1 composed with Lemma 4.1 and the companion paper's positive semidefiniteness of K at $\beta \geq 0$ [5].

The matrix, not one diagonal entry. The axiom is about a Gram matrix. The cross form $\langle \Theta F, G\rangle_{\mathrm{bond}}$ is defined separately, factors through the collapse by the same argument, and the matrix is assembled from it.

Theorem 5.3 (the Gram identity)**.** *For every finite family $F_1, \dots, F_k$ of complex half-chain observables and every complex $c_1, \dots, c_k$,*

$$\sum_{i,j} \overline{c_i}\, c_j \, \langle \Theta F_i, F_j\rangle_{\mathrm{bond}} \;=\; \Big\langle \Theta\Big(\sum_i c_i F_i\Big), \sum_i c_i F_i \Big\rangle_{\mathrm{bond}},$$

and for $\beta \geq 0$ both sides are the same non-negative real.

`osPairingBond_gram` and `osPairingBond_gram_nonneg`, with the cross form at `osPairingBondCross`, its factorisation at `osPairingBondCross_eq`, the linearity of the collapse at `collapse_sum`, and the four-fold reindexing the assembly needs stated on its own at `sum_comm4`. Positivity is *transported* to the matrix rather than re-proved: the identity says the Gram sum *is* the form of the combined observable.

And the same for the other reflection. A subtitle that says *two* reflections should not carry a Gram matrix for one of them. The site cross form is $\langle \Theta F, G\rangle_{\mathrm{site}} = \sum_\sigma \overline{(\Phi F)(\sigma)}\,(\Phi G)(\sigma)/w(\sigma)$ (`osPairingSiteCross_eq`), and its Gram matrix is positive semidefinite at *every* β, since only $w > 0$ enters: `osPairingSite_gram` and `osPairingSite_gram_nonneg`, using the three-fold reindexing at `sum_comm3`.

6 The bond bridge

Everything above is about a form defined on pairs of halves. This section says that for the bond geometry that form *is* the reflected Gibbs sum, so the positivity statements are about the measure.

Lemma 6.1 (assembly is a bijection)**.** *Sending a pair of halves (a, b) to the path $a_0, \dots, a_m, b_m, \dots, b_0$ is a bijection onto paths of $2m{+}2$ slices, with inverse $X \mapsto (\mathrm{past}\, X,\ \mathrm{rev}\, X)$.*

`bondEquiv`, built from `joinBond`, `pastOf` and `futRevOf`, together with `joinBond_pastOf_futRevOf`. A word on what `futRevOf` is and is not: it sends a *path* to a *half*, namely the reflected future half, so it is not an involution on paths and this paper does not call it Θ. The reflection enters through the pair $(\mathrm{past}, \mathrm{rev})$, which is what the pairing is built from; a global involutive Θ on paths can be built from the same assembly and is not needed for any statement here.

Lemma 6.2 (the weights multiply)**.** $W(\mathrm{assemble}(a, b)) = W(a)\, K(a_m, b_m)\, W(b)$.

`gibbsWeight_joinBond`. The kernel factors split into the bonds inside the past half, *one* bond crossing the plane, and the bonds inside the future half read backwards — which the symmetry of the kernel restores (`prod_K_joinBond`); the source weights split with no remainder (`prod_w_joinBond`). That exactly one factor crosses is the reason the collapse works at all.

Theorem 6.3 (the bond bridge)**.** $\displaystyle\sum_X \overline{F(\mathrm{past}\, X)}\, F(\mathrm{rev}\, X)\, W(X) \;=\; \langle \Theta F, F\rangle_{\mathrm{bond}}$, *the sum over whole paths of $2m + 2$ slices.*

`osPairingBond_eq_gibbsSum`. Composing it with Theorem 5.2 gives the inequality this paper is named for:

Theorem 6.4 (Osterwalder–Schrader positivity at odd separation)**.** *Let $L, m \in \mathbb{N}$, let w be a source weight with $w(\sigma) > 0$ for every slice σ, and let $\beta \geq 0$. Then for every complex observable F of a whole half-chain,*

$$\sum_X \overline{F(\text{past } X)}\, F(\text{rev } X)\, W(X) \;\geq\; 0,$$

and for every finite family $F_1, \ldots, F_k$ of such observables and every $c_1, \ldots, c_k \in \mathbb{C}$,

$$\sum_{i,j} \overline{c_i}\, c_j \sum_X \overline{F_i(\text{past } X)}\, F_j(\text{rev } X)\, W(X) \;\geq\; 0.$$

Both sums are unnormalised.

`gibbsSum_reflected_nonneg` and `gibbsSum_reflected_gram_nonneg`. The matrix form is a cheap consequence of the diagonal one applied to $\sum_i c_i F_i$, and is written down for exactly that reason: otherwise the prose would claim a matrix statement about the measure while the module exhibited a matrix for the intermediate form and a diagonal identity for the path sum.

No such theorem exists here for the site geometry.

7 The four gates

`scripts/judge_spatial_os.py`, committed before a line of the module existed.

The plan for this campaign announced *three* gates. Writing them down showed that the site collapse and the bond collapse are two different identities licensing two different theorems, and this lane has twice been punished for putting two claims behind one verdict. The count was moved to four, in public, before anything was run. Raising a pre-registration before it executes is legitimate; discovering afterwards that one gate was really two is not.

- **A1 (site), A2 (bond)** — the two collapse identities, as *matrix* identities to 10^{-12}, against a pairing matrix built by brute-force enumeration of every path, so that they do not presuppose the factorisation they test. Both PASS.

  Their status has since diverged, and the section should not pretend otherwise. What each gate tests is the composite of two things: the factorisation through the collapse, and the assembly of halves into paths. For A2 both are now theorems (§6), so that gate has been superseded by proof. For A1 the factorisation is a theorem and the assembly is not, so that gate is still the only evidence for the site assembly, and it remains a verification.

- **B** — the *minimum eigenvalue* of that matrix at $\beta \geq 0$. The previous campaign's autopsy measured that drawing random observables can miss a violating direction carrying 10^{-3} of the weight; an eigenvalue cannot miss it. PASS.

- **C** — below zero at odd separation the minimum eigenvalue must be strictly negative, and in the cell $(L = 1, m = 0, w \equiv 1)$ it must equal $e^{\beta} - e^{-\beta}$ to 10^{-12}. Measured agreement: $\leq 2.2 \times 10^{-16}$. PASS.

Gate C licenses a sharpness witness that this paper does not yet contain (§8). A passed gate is a permission, not a theorem.

8 What is open, and which part is load-bearing

Status, one row per claim. The previous version called the identification "the link", as though it were one problem. It is two, and one of them is now closed.

Claim	Status
Bond assembly is a bijection onto paths	proved
Bond weight identity	proved
Bond form = reflected Gibbs sum	proved
Bond positivity of the Gibbs sum, $\beta \geq 0$	proved
Bond Gram matrix on the Gibbs sum	proved
Site collapse identity and Gram positivity	proved
Site assembly bijection (shared middle slice)	**open**
Site weight identity	**open**
Site form = reflected Gibbs sum	**open**
Necessity of $\beta \geq 0$ for $L \geq 1$	**open**
Reconstruction: null quotient, transfer operator	**open**

The site geometry is combinatorially different and not a variation on the bond one: the halves share the middle slice, so pairs of halves are not a product but a fibred product over that slice, and the weight of the shared slice appears twice and must be divided out. Gate A1 verifies the site identity to 10^{-12} against brute-force enumeration; that is why the site rows say *open* and not *doubtful*. Gate A2, which verified the bond identity, has been superseded by proof.

The sharpness witness. Gate C authorises it and it is not written. Consequently $\beta \geq 0$ appears above as a sufficient hypothesis only, and the module's own docstring now says exactly that — the previous version's docstring still opened with EXACTLY, and the sentence added to correct it pointed at "the separate witness below", which does not exist. Retiring an overclaim by installing a dangling reference is worse than the overclaim: a reader can check a claim and cannot check a pointer to nothing.

The route to the witness is identified: at $m = 0$ the collapse is multiplication by w, hence surjective when $w > 0$, so an observable can be chosen whose boundary vector is the sign observable that the companion paper already drives negative below zero [5].

No reconstruction. The quotient by the null space of this form, with the transfer operator descending to a self-adjoint contraction, is not built. Nothing here concerns uniformity in the extent, SU(N), the continuum limit, or the Clay problem [7].

9 Reproducibility

All *code and verification* artifacts — the module, the oracle, the judges, the ledger — are at source checkpoint `2d0346b2cc75d1cdf2e0ea0cefea4a8ef3be9c84` on branch `main`. This manuscript and its PDF are not: they are committed on top of that checkpoint, in the commit whose parent it is, since a document cannot cite the commit that introduces it. Every hyperlink was resolved against `2d0346b2cc75d1cdf2e0ea0cefea4a8ef3be9c84`, and checked to point at an actual declaration line, before compilation, again inside the compiled PDF, and again live over HTTP.

Where the counters were measured, and why not at this commit. They were measured on commit `e232819d`, in a window with no other elaborator running on the host. Between that commit and this checkpoint `git diff --name-only` returns exactly `DASHBOARD.md`, `docs/VERIFICATION-LEDGER.md`, `papers/spatial-os/spatial_os.pdf`, `papers/spatial-os/spatial_os.tex`, `scripts/check_code_identical.py` and `scripts/gen_changed_file_list.py` together with `YangMills/OS/SpatialOS.lean`. Of those, only `YangMills/OS/SpatialOS.lean` is elaborated, and it changed *only in comments*; every other file listed is a non-Lean artefact that the elaborator never sees. Both the list and this sentence are produced from the command's output rather than typed, because twice the list was typed and twice it was wrong.

That is a checkable claim rather than a promise: `scripts/check_code_identical.py` takes a repository and *two revisions*, reads both from git, strips every comment from each and compares what remains. It keeps all other whitespace, because indentation is syntax in Lean and a comparator that collapsed it could call two files identical when they parse differently; it drops only lines left entirely blank and trims trailing space, on the lines where neither carries meaning; it replaces each comment by *spaces of that comment's own width*, one per character; and it treats string literals as data.

Each of those is there because its absence was a defect. Deleting a comment outright maps `foo/- c -/bar` and `foobar` to the same text although the first is two identifiers and the second one. Collapsing a comment to a single space moves the column of any code following an inline comment, and in Lean a column decides whether a token belongs to a nested block. Trimming trailing space and blank lines *inside* a string identifies `"a  ...` with `"a ...`, and those characters are program data — so that trimming is applied only to lines holding no string character.

Scope, stated because a certificate that overstates its reach is worse than none. The scanner models ordinary string literals with backslash escapes. It does not model raw strings or interpolation and it is not a Lean lexer. Within the lexical fragment `SpatialOS.lean` actually uses, it errs only towards reporting a difference that is not one. On `e232819d` against the code checkpoint it reports IDENTICAL, 25600 code characters on both sides. The counters were not re-measured because the host's Lean dependency tree was destroyed between the two versions (§10), and re-running them requires a full restore.

Quantity	Value
Full core build	8464 jobs, success
checkpoint before this module	8463 jobs
Oracle commands, and answers	2794
distinct declarations	2791
commands listed twice	3
distinct, with axiom dependencies	2765
distinct, axiom-free	26
Declarations in this module	42
of them in the oracle	42
Distinct axioms across all outputs	`propext`, `Quot.sound`, `Classical.choice`
Occurrences of `sorryAx`	0
Project axioms	0
Errors	0

The job count incremented from 8463 to 8464 when the module joined the core, which is this

repository's standing check that a new module is really being elaborated. It did not move again when declarations were added to an existing module, which is the correct reading of that check rather than a failure of it. The oracle counters close: $2765 + 26 = 2791$ and $2791 + 3 = 2794$, the three repeats being long-standing duplicate lines elsewhere in the file. Every declaration of this module appears in the oracle, reconciled by name.

The module is `YangMills/OS/SpatialOS.lean` and the companion is `YangMills/OS/SpatialReflection.lean`.

10 A note on the host

While this version was being prepared, the machine's Lean dependency tree was destroyed: the `mathlib` working copy lost all 8429 of its files, and every other package directory disappeared, leaving only the git metadata. The pinned commit object survived, so the sources were restored by resetting to the revision the manifest names; the compiled artifacts did not survive, and the remaining dependencies must be re-fetched before anything can be elaborated again.

This is recorded rather than hidden, because it is the reason the counters in §9 carry a commit that is not this one. Nothing about the mathematics changed, and the check named there is what makes that statement verifiable instead of merely asserted.

References

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