The Modulus: a Machine-Checked Operator Bound on the Fluctuation Sector of the Coupled $\mathbb{Z}_2$ Slice

one construction, two debts — and the one it does not pay

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Abstract

Two companion papers were left carrying the same debt from opposite sides. The gap paper proved that every eigenvalue of the coupled slice other than the Perron eigenvalue is *strictly* smaller in modulus, and said plainly that this provides *no modulus of separation* [2]. The bridge paper proved that the Gibbs correlations of the spatial system are matrix elements of a self-adjoint transfer operator, and obtained geometric decay only *under a contraction hypothesis it did not discharge* [3]. The missing step is identical in both: finitely many strict inequalities are not an operator-norm bound.

Proved. We construct specGap, the largest $|\mu|$ over the eigenvalues different from the Perron eigenvalue λ, and prove specGap $< \lambda$. We then prove the operator bound: for every observable u orthogonal to the Perron vector, $\|Ku\| \leq \text{specGap} \cdot \|u\|$. That is exactly the hypothesis the bridge paper carried, so its bound becomes unconditional. The bound is *sharp*: whenever the state space has at least two points, some nonzero fluctuation observable attains it. The argument splits at specGap $= 0$, where the maximising index need not supply a *non-Perron* eigenvector at all.

And a warning we state before anyone else has to. specGap $< \lambda$ is *not* specGap < 1: the kernel is unnormalised, so both are typically far above one and specGap^N *grows*. On its own the unnormalised bound therefore controls growth, it does not exhibit decay. The rate that is below one is the *relative* one, specRatio $=$ specGap$/\lambda < 1$, and the decay statement is that the fluctuation contribution is suppressed by specRatio^N *relative to the Perron scale* λ^N. Both forms are proved; only the second is called decay.

The step that does not follow from the inequalities is the one about eigenvectors *at* λ: geometric simplicity, proved in the Perron paper for an arbitrary eigenvector rather than a positive one [1], makes them *invisible* to a fluctuation observable, so the top term of the spectral sum vanishes instead of merely being bounded.

Not proved. specGap *depends on the extent*, and nothing here bounds it away from λ uniformly. Direct diagonalisation gives specGap$/\lambda = 0.9205, 0.9829, 0.9964, 0.9992$ at $L = 2, 3, 4, 5$ for one parameter pair: a geometric bound whose rate tends to 1 is empty in the limit, and that is reported, not hidden.

And a correction to our own previous scope claim. An earlier version said that bounding the partition function below requires identifying the index of the top eigenvalue. That was *wrong*, as an external reading pointed out. Splitting the dressed constant observable along the Perron direction makes both cross terms vanish by symmetry alone, and the resulting lower bound turns the relative statement into a bound on the *normalised* Gibbs two-point function. Both are proved, and so is their composition: at a fixed extent the normalised two-point function is bounded by $C\,\text{specRatio}^N$ past a machine-checked threshold N_0 — characterised by $\text{specRatio}^{N_0} < \frac{c^2}{2(\|b_\perp\|^2+1)}$, not given as a closed formula and not the least such index. Here c and $b_\perp$ are the Perron coefficient and the orthogonal component

of the dressed *constant* observable — *not* of A, so the threshold does not depend on the observable and only C does. That rate depends on the extent, so this is not clustering. Reflection positivity is untouched, and nothing in this paper is a claim about $\mathrm{SU}(N)$, the continuum limit, or the Yang–Mills mass gap.

1 The same debt, from two sides

The gap paper's headline is a strict separation without a modulus. The bridge paper's decay theorem carries a contraction hypothesis. It is worth stating precisely why these are one debt and not two.

A strict inequality $|\mu| < \lambda$ for each of finitely many eigenvalues does give $\max|\mu| < \lambda$ — taking a maximum over a finite set is not the difficulty. The difficulty is that $\max|\mu|$ over the non-Perron eigenvalues is not, by itself, known to *dominate the operator* on the orthogonal complement of the Perron vector. Getting from one to the other requires the spectral decomposition: an orthonormal basis of eigenvectors, so that an observable can be expanded and its image measured coordinate by coordinate.

That is what this paper supplies, and it pays both debts with one construction. We prove the inequality $\|Ku\| \leq \mathrm{specGap}\|u\|$, that it is *attained*, and that the set of norm ratios has specGap as its greatest element (§8) — so specGap *is* the operator norm on the fluctuation sector, said about the set as well as about the two inequalities. What we do not build is a subspace-plus-continuous-linear-map interface carrying that norm; see §8 for why.

2 The construction

Let K be a strictly positive symmetric kernel on a finite nonempty type, with Perron data (v, λ): $v > 0$ pointwise and $Kv = \lambda v$.

Definition 2.1. Let μ_j be the eigenvalues of K and set

$$\mathrm{specGap} \;=\; \max_j \begin{cases} 0 & \text{if } \mu_j = \lambda, \\ |\mu_j| & \text{otherwise.} \end{cases}$$

`specGap`. Eigenvalues equal to λ contribute 0 rather than λ. That is legitimate — and it is the whole content of §3.

Theorem 2.2 (the modulus)**.** $\mathrm{specGap} < \lambda$.

`specGap_lt`. Each term of the maximum is either $0 < \lambda$, or $|\mu_j| < \lambda$ by the gap paper applied to the j-th basis vector, which is a nonzero eigenvector because it has unit norm.

3 Why eigenvectors at the top may be discarded

Theorem 3.1. *If $\mu_j = \lambda$ and u is orthogonal to the Perron vector, then u has no component along the j-th eigenvector.*

`inner_specBasis_eq_zero_of_top`.

Remark 3.2. *This is the step that does not come from the inequalities, and it is why Definition 2.1 may assign* 0 *to the top of the spectrum. The Perron paper proves geometric simplicity for an* arbitrary *eigenvector with eigenvalue λ, not merely a positive one [1]; hence the j-th basis vector is a multiple of v, and orthogonality to v kills the coordinate outright. Had only the positive case been available, the top term could have been bounded but not removed, and the maximum in Definition 2.1 would have had to include λ — making Theorem 2.2 false and the whole construction vacuous.*

4 The operator bound

Theorem 4.1 (the bound)**.** *For every u with $\langle v, u\rangle = 0$,*

$$\|Ku\| \;\le\; \mathrm{specGap}\cdot\|u\|.$$

`norm_act_le_specGap`.

The proof is the expansion in the eigenbasis. Writing c_j for the coordinates of u, the coordinates of Ku are $\mu_j c_j$ (`inner_specBasis_act`), so by Parseval (`eucNorm_sq_eq_sum`)

$$\|Ku\|^2 = \sum_j \mu_j^2 c_j^2, \qquad \|u\|^2 = \sum_j c_j^2,$$

and each term is dominated: if $\mu_j = \lambda$ then $c_j = 0$ by Theorem 3.1, and otherwise $|\mu_j| \le \mathrm{specGap}$ by definition.

5 The endpoint

The bridge paper's fluctuation sector is invariant under a kernel that *fixes* the vacuum. Here the kernel is unnormalised and the vacuum is an eigenvector with eigenvalue λ, so invariance is reproved in that form (`perp_invariant_eigen`); the scalar drops out because the pairing is bilinear. Iterating gives `iterate_norm_le_specGap`.

Theorem 5.1 (the bridge bound, without its hypothesis)**.** *For the coupled spatial system, at every finite extent L, every β, and every strictly positive source weight w: if the dressing of A is orthogonal to the Perron vector, then for every N*

$$\Big|\sum_X A(X_0)A(X_N)\,\mathcal{W}(X)\Big| \;\le\; \mathrm{specGap}^N\,\big\|\sqrt{w}A\big\|^2, \qquad \mathrm{specGap} < \lambda.$$

`gibbs_pathSum_bound_unconditional`.

Remark 5.2 (why that is not yet decay)**.** $\mathrm{specGap} < \lambda$ *does not give* $\mathrm{specGap} < 1$*. The kernel is unnormalised: in the probe of §9, $L = 2$ with $\beta = 0.8$, $\gamma = 1.2$ gives $\lambda \approx 56.89$ and* $\mathrm{specGap} \approx 52.37$*, so* $\mathrm{specGap}^N$ *grows. Theorem 5.1 is a* sub-Perron growth *bound. The decay is Theorem 5.3.*

Theorem 5.3 (the decay statement)**.** *With* $\mathrm{specRatio} = \mathrm{specGap}/\lambda$*, under the same hypotheses,*

$$\Big|\sum_X A(X_0)A(X_N)\,\mathcal{W}(X)\Big| \;\le\; \mathrm{specRatio}^N\,\Big(\lambda^N\big\|\sqrt{w}A\big\|^2\Big), \qquad \mathrm{specRatio} < 1.$$

Equivalently, the normalised kernel $\lambda^{-1}K$ contracts the fluctuation sector by $\mathrm{specRatio} < 1$*.*

gibbs_pathSum_relative_decay, specRatio_lt_one, norm_act_normalizedKernel_le.

Remark 5.4 (non-vacuity)**.** *Dressing is multiplication by a strictly positive function, so every fluctuation vector is the dressing of an observable: whenever the configuration space has two points there is a* nonzero A *satisfying the hypothesis, and Theorem 5.1 is not quantified over an empty set (exists_nonzero_dressed_fluctuation).*

6 Scope: four things this does not give

Nothing uniform in the extent. This is the substantive limitation. specGap is built from the spectrum at a fixed extent and depends on it. Direct dense diagonalisation gives, for $\beta = 0.8$, $\gamma = 1.2$,

$$\text{specGap}/\lambda \;=\; 0.6640,\ 0.9205,\ 0.9829,\ 0.9964,\ 0.9992 \qquad \text{at } L = 1, \ldots, 5,$$

and for $\beta = 0.3$, $\gamma = 0.4$,

$$0.2913,\ 0.4665,\ 0.5518,\ 0.5938,\ 0.6159.$$

The first family collapses towards 1. The quantity tabulated *is* $\text{specRatio}(L) = \text{specGap}/\lambda$, so what says nothing in the limit is a bound of the form $\text{specRatio}(L)^N$ — which is exactly the rate of Theorem 7.3. These numbers are **verified** and **not proved**, no theorem depends on them, and they are printed because a paper that reported only Theorem 5.1 would be reporting the half that does not settle the question.

Norm identified, but no restricted-operator interface. Theorems 8.1 and 8.2 identify specGap as the greatest norm ratio on the fluctuation sector, so the norm *is* pinned down, and by a statement about an object. What is not built is a `Submodule` carrying $v^{\perp}$, with a `ContinuousLinearMap` on it and an equation for *its* norm. Both theorems are also empty when the state space has fewer than two points.

Still not clustering in the infinite-volume sense. Theorem 7.3 bounds the normalised two-point function at each fixed extent. It says nothing about the limit: the rate is $\text{specRatio}(L)$, and the measured evidence is that it tends to 1. The word *clustering* is reserved for a statement that survives that limit, and no such statement is made.

No claim about the physical target. Reflection positivity is untouched, and nothing here concerns $\text{SU}(N)$, the continuum limit, or the Clay problem [10]. The distance to that problem is unchanged.

7 The denominator, and a retraction

The first version of this paper asserted, in its scope section, that a lower bound on the partition function needs the index of the top eigenvalue identified. **That was wrong.** An external reading supplied the argument below, and it uses no index at all.

Let b be the dressed constant observable, Ω the unit Perron vacuum, $c = \langle \Omega, b\rangle$ and $u = b - c\,\Omega$, so that $u \perp \Omega$. (This u is the $b_\perp$ of the abstract; it is built from b, never from A.) Then

$$\langle b, K^N b\rangle \;=\; c^2\lambda^N + \langle u, K^N u\rangle,$$

because *both* cross terms vanish: $\langle \Omega, K^N u\rangle = \langle K^N\Omega, u\rangle = \lambda^N\langle \Omega, u\rangle = 0$ by symmetry and the eigen-equation, and the other by the same computation. Nothing here asks which basis vector carries λ.

`quadForm_split`.

Theorem 7.1 (the partition function from below)**.** $\langle b, K^N b\rangle \;\geq\; c^2\lambda^N - \mathrm{specGap}^N\|u\|^2$*, and the left-hand side is the partition function.*

`quadForm_lower`. Since $c > 0$ (both Ω and b are strictly positive) and $\mathrm{specRatio} < 1$, the right-hand side is positive for all large N.

Theorem 7.2 (the normalised two-point function)**.** *For any B bounding the unnormalised sum and any $D > 0$ bounding the partition function below,*

$$\big|\mathbb{E}[A(X_0)A(X_N)]\big| \;\leq\; B/D.$$

`gibbsCorr_bound_of_partition_lower`.

Composing the three is arithmetic, but it is arithmetic with quantifiers in it — a threshold has to be chosen and the overlap has to be shown positive — so we state it as a theorem rather than assert it in prose.

Theorem 7.3 (the fixed-extent endpoint)**.** *Let A be an observable whose dressing is orthogonal to the Perron vector. There exist $C > 0$ and N_0 such that for all $N \geq N_0$,*

$$\big|\mathbb{E}[A(X_0)A(X_N)]\big| \;\leq\; C\,\mathrm{specRatio}^N, \qquad \mathrm{specRatio} < 1.$$

`gibbsCorr_decay_fixed_extent`. The constant is explicit in the proof, $C = 2\|\sqrt{w}A\|^2/c^2$ up to the $+1$ that keeps it positive for the zero observable. The threshold is *characterised, not computed*: the proof takes any N_0 with

$$\mathrm{specRatio}^{N_0} \;<\; \frac{c^2}{2\big(\|u\|^2+1\big)},$$

which exists because $\mathrm{specRatio} < 1$, and past which the fluctuation part of the partition function is below half of its Perron part. It is not a closed formula and not the least such index.

And it does not depend on A. Both c and u are built from the dressed *constant* observable and the vacuum, so once L, β, w and the Perron data are fixed, one N_0 serves *every* fluctuation observable at once; only C ever sees A. A statement should publish the strength it has, so that is the order of quantifiers actually proved:

Theorem 7.4 (one threshold for all observables)**.** *There is an N_0 such that for* every *observable A whose dressing is orthogonal to the Perron vector there is a $C_A > 0$ with $\big|\mathbb{E}[A(X_0)A(X_N)]\big| \leq C_A\,\mathrm{specRatio}^N$ for all $N \geq N_0$.*

`gibbsCorr_decay_uniform_threshold`, resting on `exists_partition_threshold`, which is where the observable-blindness lives: that theorem quantifies over no A at all. Theorem 7.3 is its three-line consequence in the familiar order.

This is a fixed-L statement and nothing more: the rate is $\mathrm{specRatio}(L)$, the tabulated evidence above is that it tends to 1, and therefore neither theorem survives the infinite-volume limit.

8 Sharpness: the bound is attained

Section 4 proves an inequality. A previous version said the matching *sharpness* was not blocked but not formalised, and sketched an argument. A sixth reading pointed out that the sketch is incomplete exactly where $\mathrm{specGap} = 0$: there the maximiser may perfectly well be one of the top indices, since those contribute 0 by definition. Such an index does supply an eigenvector — the Perron one — but not a *non-Perron* one, hence none in the fluctuation sector, which is what the argument needed. The correct proof splits, and the split is now in the formalisation.

Theorem 8.1 (the bound is attained)**.** *Suppose the state space has two distinct points. Then there is an observable u with $u \perp v$, $\|u\| \neq 0$ and $\|Ku\| = \mathrm{specGap} \cdot \|u\|$.*

`exists_attaining_fluctuation`. The two branches are genuinely different arguments. If $\mathrm{specGap} > 0$, the maximiser cannot be a top index — those contribute 0 — so it supplies an eigenvector with eigenvalue $\neq \lambda$, which is orthogonal to v because moving K across the pairing scales by λ on one side and by that eigenvalue on the other (`inner_perron_specBasis_eq_zero`); its norm is 1 and it realises $|\mu| = \mathrm{specGap}$. If $\mathrm{specGap} = 0$, that route is unavailable, but Theorem 4.1 already forces $\|Ku\| = 0$ for *every* fluctuation observable, so any nonzero one attains the value — and this is the branch that needs the fluctuation sector to be nonempty, hence the two distinct points. At $L = 0$ there are no two distinct points, the fluctuation sector is trivial, and the theorem says nothing there.

Together with Theorem 4.1 this says that specGap *is* the operator norm of K on the fluctuation sector. Since that is a statement about an object, we also state it about one:

Theorem 8.2 (the restricted norm, as an object)**.** *Suppose, as in Theorem 8.1, that the state space has two distinct points. Then the set of norm ratios $\|Ku\|/\|u\|$, over fluctuation observables of nonzero norm, has a greatest element, and it is* specGap*.*

`specGap_isGreatest`. The hypothesis is not decoration inherited from the previous theorem: with fewer than two states the set of ratios is *empty*, and an empty set has no greatest element. (Norm ratios, not *Rayleigh* quotients — those are $\langle u, Ku\rangle/\|u\|^2$, a different number.)

This is `IsGreatest`, deliberately lighter machinery than a subspace plus a continuous linear map plus its norm: it says the supremum exists and is attained, which is the whole content, without committing the module to an operator-norm interface it does not otherwise use. Anyone who wants the sup formulation gets it from `IsGreatest.csSup_eq`.

9 What is verified and not proved

A machine-checked proof guarantees that the definitions are internally consistent, not that they name the object the prose claims. The construction was therefore recomputed by direct dense diagonalisation, by a script with no access to the formalisation. Four things were checked: that $\mathrm{specGap} < \lambda$ in every case; that $\|Ku\| \leq \mathrm{specGap}\|u\|$ on 400 random fluctuation vectors per case; that the bound is *attained* by the subdominant eigenvector; and the endpoint of §7 itself — the partition function and the two-point sum recomputed *by brute force over all paths*, with no transfer matrix and no dressing identity, then compared with $C\,\mathrm{specRatio}^N$. That fourth check also re-derives the bridge of the companion paper, since the brute-force path sums are compared with the matrix elements wherever exhaustive enumeration is feasible. All cases pass; the worst observed ratio of the two sides of the endpoint is 0.145.

What the fourth check is not. It runs one concrete observable per parameter set — the one coming from the subdominant eigenvector — at small N. It is an independent spot-check of the endpoint, not a sweep over observables; the generality is in Theorem 7.3, not here.

One consequence is worth recording. The measured specGap/λ reproduces, digit for digit, the subdominant ratios the gap paper reported as measured and unnamed [2]. The object constructed here is that number.

10 Formalisation notes

The library had the hard half. mathlib provides an orthonormal eigenbasis and real eigenvalues for a Hermitian matrix in finite dimension. The work in this module is not the spectral theorem; it is the bridging — writing the kernel as a Hermitian matrix, matching mathlib's eigen-equation to the form the companion papers state, and translating norms and pairings between the two spellings. That bridging is where the effort went, and it is the usual place.

Where the unnormalised kernel forced a reproof. The bridge module's invariance lemma assumes the vacuum is *fixed*. Its kernel is unnormalised, so the vacuum is an eigenvector with eigenvalue $\lambda \neq 1$ in general, and the lemma does not apply verbatim. Rather than reopen a frozen module, the invariance is reproved here in the eigenvector form. It is three lines, because the eigenvalue factors out of a bilinear pairing.

11 Reproducibility

All artifacts are at source checkpoint `9704b3f3c0d576bb40c43caa5360fbd64dfa1643` on branch `main`. Every hyperlink was resolved against that commit, and checked to point at an actual declaration line, before compilation.

Quantity	Value
Full core build	8430 jobs, success
Oracle commands, and answers	2672
distinct declarations	2669
commands listed twice	3
distinct, with axiom dependencies	2646
distinct, axiom-free	23
Distinct axioms across all outputs	`propext`, `Quot.sound`, `Classical.choice`
Occurrences of `sorryAx`	0
Project axioms	0
Errors	0

The module is `YangMills/OS/SpatialSpectral.lean`; the probe is `scripts/probe_spatial_spectral.py`; verdicts are in `docs/VERIFICATION-LEDGER.md`.

References

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