A Machine-Checked Thermodynamic Limit for Local Lattice Gauge Gibbs States

Lluis Eriksson

THE ERIKSSON PROGRAMME - Lean 4 artifact, branch `codex/thermodynamic-limit-kp`*, verified source checkpoint* `0be45284`

Abstract.

We formalize in Lean 4 the thermodynamic limit of bounded local Gibbs expectations for a periodic lattice gauge model in a uniform Kotecky-Preiss regime. The proof treats the complete finite-volume sequence: an exact one-volume marked expansion cancels the extensive far gas algebraically, common-window terms are transported exactly, and the remaining boundary contribution is bounded by the existing volume-uniform pinned cluster tail `connectedLattice_pinned_tail_volumeUniform`. The resulting explicit Cauchy modulus tends to zero, so completeness constructs an infinite-volume positive normalized real local state. On the intrinsic integer-coordinate local-observable algebra, the state carries a genuine additive action of Z^d and is invariant under every integer translation, including inverses. For SU(2), Haar probability measure, and the physical Wilson plaquette energy Re tr(U), the hypotheses are discharged throughout the explicit punctured intervals 0 < |beta| <= 10^-5 in d=2 and 0 < |beta| <= 10^-6 in the physical lattice dimension d=4. We also construct a genuine centered free-boundary exhaustion and prove that its complete cofinal sequence converges to the same state as periodic boundary conditions. Finally, the normalized finite-volume two-plaquette truncated-correlation bound passes unchanged to the state under explicit eventual realization and separation hypotheses. We do not claim arbitrary boundary conditions, a C*-algebraic state, a continuum limit, Osterwalder-Schrader reconstruction, or progress on the continuum Yang-Mills mass-gap problem.

1. Introduction

Cluster expansions are a standard route from finite-volume polymer representations to analytic and probabilistic control in regimes of small activities. The Kotecky-Preiss criterion gives a particularly useful volume-uniform convergence mechanism for abstract polymer models [1]. For lattice gauge systems, however, a thermodynamic-limit argument must still connect the abstract cluster estimates to genuine normalized Gibbs expectations. This connection is delicate because the numerator and partition function are both extensive; neither is expected to converge separately.

The underlying strong-coupling existence and clustering statement is classical; in particular, it belongs to the lattice gauge theory framework of Osterwalder and Seiler [2]. The contribution claimed here is not a new analytic thermodynamic-limit theorem, but a mechanically checked derivation from the finite Gibbs integral through every marked-expansion, cancellation, tail, and completeness bridge, together with an explicit proof that the packaged KP hypotheses are inhabited.

The formal development studied here closes that connection for local observables. Its central design rule is to cancel the far gas exactly before applying estimates. The proof then compares two finite volumes through a common non-wrapping window. Clusters that remain inside the window agree term by term, including their Ursell coefficients and activity monomials. Clusters that leave it are support-pinned boundary tails, and are controlled uniformly in volume.

The output is not an abstract compactness statement. It is a Cauchy theorem for the complete sequence of genuine Gibbs expectations, followed by a definition of the limit using completeness. This distinction is essential: a convergent subsequence would not establish a unique thermodynamic state or boundary-condition comparison.

2. Formal-design lessons

The main formal contribution is not the invention of a new cluster expansion, but the choice of interfaces that make every analytic bridge auditable. Several mathematically plausible shortcuts were rejected because they could support a formally correct endpoint while leaving the intended Gibbs statement unproved. This section records the four design decisions that most strongly shaped the development. They are reusable beyond this model because each concerns the boundary between an abstract convergence theorem and the concrete object that is supposed to converge.

2.1. Rejecting a free bulk hypothesis

A tempting interface is to assume that the normalized expectation has already been decomposed into a volume-independent bulk term plus a small boundary correction. Such a theorem can be useful after the decomposition has been proved, but it cannot itself establish the thermodynamic limit of the genuine Gibbs integral. If the bulk term is an unconstrained argument, the caller may instantiate it without any connection to the numerator or the partition function. The resulting Cauchy theorem is then conditional on exactly the identity that carries the mathematical content.

The accepted interface therefore starts from `localGibbsExpectation` and proves a literal one-volume identity. The Boltzmann product is expanded into plaquette subsets, connected components disjoint from the insertion are factorized as the ordinary gas, and the unique sector touching the finite support is retained as a marked activity. Only after this exact factorization is the common far gas cancelled between numerator and denominator. Thus the later error term is not postulated: it is the restriction difference left by an algebraic equality. A terminal theorem is accepted only if its conclusion names the genuine Gibbs expectation and its hypotheses contain no free bulk or cancellation oracle.

2.2. Local coordinate charts instead of global volume embeddings

Comparing torus sizes N and M might suggest constructing a global map between their edge sets or gauge-configuration spaces. There is no canonical such map compatible with periodic wraparound: a coordinate that is local in one torus can cross a seam in another, and a global configuration map would impose choices irrelevant to a local observable. Building a universal-cover theory for entire configurations would also mix the finite product measure with geometry that is needed only on a bounded window.

Instead, an observable stores a finite coordinate chart independent of the ambient volume. A comparison chooses one non-wrapping window large enough for the support and for every cluster below the current size cutoff. Plaquettes, polymer activities, incompatibility, Ursell coefficients, and the marked integral are transported exactly inside that window; no map between complete gauge configurations is required. The residual terms are precisely clusters that leave the chart, so the geometric failure of transport becomes the event controlled by the pinned tail.

The same principle resolves inverse translations. Natural-number coordinates support positive shifts but do not form a translation group with a volume-independent guard. The final observable type uses coordinates in Z^d and realizes them periodically by integer remainders. A finite lower shift places any given observable in the old natural-coordinate chart, and the two realizations agree literally beyond an explicit bound. This is an eventual chart comparison, not a volume-dependent simulation of a negative shift by N-1 positive steps.

2.3. Keeping the rooted multiplicity

At cluster order n, an Ursell tuple has n+1 polymer coordinates. The boundary event says that some coordinate meets the marked region and some coordinate leaves the common window. Existing KP tails are rooted at coordinate zero. Reindexing an arbitrary witness to zero therefore introduces a factor n+1. Choosing the first witness does not remove the factor in general: once the mark is forgotten, as many as n+1 marked tuples can map to the same unmarked tuple. Without an additional uniqueness theorem for the touching polymer, the multiplicity is intrinsic.

The development exposes this combinatorial derivative rather than hiding it in a coarse constant. The factorial normalization obeys (n+1)/(n+1)! = 1/n!, and the remaining cardinality cost is absorbed by one unit of activity tilt. Consequently the honest hypothesis is the KP criterion at t+epsilon+1, not merely at t+epsilon. This stronger field propagates through the boundary theorem and is visible in the non-vacuity calculation. Recording the factor at the reindexing site prevents a downstream estimate from appearing stronger than the combinatorics justify.

2.4. Treating non-vacuity as part of the API

A packaged KP theorem may be internally consistent while having no proved interacting instance. In particular, beta=0 annihilates every Mayer activity and yields only the free product measure. It is a useful normalization check, but it does not demonstrate that the simultaneous radius, smallness, tilted, and marked-radius hypotheses can hold away from the trivial point. Declaring the abstract structure inhabited only at beta=0 would therefore make the advertised strong-coupling result mathematically empty.

For this reason the abstract regime and its physical constructor are separate audited endpoints. The constructor proves measurability, an explicit activity majorant, all tilted inequalities, and nonconstancy of the Wilson energy on punctured rational intervals. The d=2 and d=4 witnesses use conservative constants because their role is logical: they certify an open interacting range, not an optimized strong-coupling threshold. This separation also makes future constant improvements local; they can replace the constructor without changing the thermodynamic-limit proof.

2.5. A typed chain of evidence

The proof is deliberately not packaged as one theorem with a large record of opaque hypotheses. It is a chain of certificates whose codomain is the next layer's input. Equalities are established in the finite combinatorial layers; inequalities enter only when the terms that cross the common window have been isolated. Completeness is used only after a numerical Cauchy modulus has been produced. This ordering makes it impossible to replace a missing algebraic bridge by a later analytic estimate.

Interface	*Exported certificate*	*Downstream role*
Gibbs integral / polymer gas	*Exact marked one-volume expansion*	*Cancels the far gas without a bulk oracle*
Two torus volumes	*Common-window equality of local terms*	*Identifies every contribution below the cutoff*
Cluster leaves the window	*Rooted tail at t+epsilon+1*	*Gives a bound uniform in both volumes*
Uniform numerical bound	*Explicit Cauchy modulus tending to zero*	*Constructs the complete-sequence limit*
Signed local coordinates	*Literal Z^d action and eventual chart equality*	*Transfers finite covariance, including inverses*
Centered seam deletion	*Gas restriction plus inclusion-exclusion*	*Compares periodic with the proved free exhaustion*
Finite correlation estimate	*Eventual realization and separation bridge*	*Passes the quantitative bound to the state*

This layering is also a debugging protocol. If a proposed endpoint mentions a new hypothesis that already states the next row's certificate, the bridge has been skipped. If a reindexing changes the summation domain, its multiplicity must appear before a norm bound is taken. If a cross-volume equality depends on a global configuration map, the window interface has leaked. These checks forced two early prototypes to be replaced even though their conditional endpoints elaborated.

The organization keeps claim scope in the types as well. The free boundary object is one concrete restriction, not a parameter ranging over unspecified boundary conditions. The infinite functional is bundled with linearity, order, a unit, and the Z^d action, but not with a C*-norm that has not been constructed. The correlation consumer asks for eventual plaquette realization and separation rather than silently assigning a distance to every abstract observable. The final paper ledger is therefore derived from available interfaces, not from the informal theorem one might hope to state.

Interface discipline.
Every terminal conclusion is connected to the genuine finite Gibbs integral by exact equalities before estimates are applied; every multiplicity introduced by rooting is charged explicitly; every cross-volume comparison is local to a common chart; and the abstract uniform KP package is accompanied by a nonzero physical instance.

3. Finite-volume setting

Fix a dimension d, a measurable group G with measurable multiplication and inversion, a probability measure mu on G, a bounded measurable plaquette energy pe, and a real coupling beta. At torus extent N, a gauge configuration assigns an element of G to each positively oriented edge. The Wilson action is the sum of pe evaluated on oriented plaquette holonomies, and the Gibbs measure is the normalized tilt of the product gauge measure.

A compatible local observable records a finite support with coordinates independent of the ambient volume, a measurable bounded kernel, and a minimum admissible volume. Its realization in each sufficiently large torus reads exactly those edge coordinates. The finite Gibbs expectation E_n(O) is the genuine normalized integral of that realization at extent n+1.

> Finite local substrate.
> For every compatible local observable O, its realization is measurable and bounded; E_n is positive and normalized. Addition, multiplication, and real scalar multiplication are realized exactly at jointly admissible volumes.

4. Exact marked expansion and cancellation

Expanding each plaquette Boltzmann factor into 1 plus its Mayer activity gives a finite sum over plaquette subsets. For a local insertion O, components disjoint from the observable support factor as the ordinary polymer gas. Components touching the support form a marked activity. Mayer-Ursell inversion is then applied at one fixed volume.

```
E_n(O)
  = sum_{marked S0} markedIntegral_n(O,S0)
      * exp( clusterSum(P_n restricted to far(S0))
             - clusterSum(P_n) ).
```

This identity is exact. The extensive contribution has already cancelled between numerator and partition function. No estimate of the full partition function, and no free bulk hypothesis, appears in the thermodynamic comparison.

5. Uniform pinned boundary control

For a fixed marked set, the remaining exponent is a restriction difference. It is reindexed as a series of Ursell tuples that meet the marked region and leave the common window. Pinning an arbitrary tuple coordinate at position zero costs its cardinality n+1. The formal development records this factor explicitly and absorbs it by the unit-cardinality tilt t+epsilon+1; it is not silently discarded.

> Rooted boundary tail.
> The local correction tail is summable uniformly in the torus volume. Its proof invokes `connectedLattice_pinned_tail_volumeUniform` literally, after the exact rooted reindexing and the honest unit tilt.

The outer marked-set sum is controlled separately by decomposing a marked plaquette set into connected components. Each component is charged to the finite observable support, and a volume-uniform lattice animal bound resums the possibilities. Combining the outer tail with the normalization-correction tail yields the explicit error

```
Delta_O(q)
  = 2 * markedOuterTailBound_O(q)
    + markedSmallLayerCauchyBound_O(q,q),
Delta_O(q) -> 0 as q -> infinity.
```

6. Whole-sequence thermodynamic state

Complete-sequence Cauchy theorem.
Under a packaged uniform local KP regime, the sequence n |-> E_n(O) is Cauchy for every compatible local observable O. For each tolerance, the proof selects one cutoff q with Delta_O(q) below that tolerance and one explicit volume threshold above which every pair of volumes is bounded by Delta_O(q).

Completeness of the complex numbers supplies the limit. Since every finite expectation is real, the limit is real. Exact finite-volume algebra, positivity, normalization, and translation covariance pass to the limit by uniqueness of limits.

To obtain an honest translation group rather than a semigroup of positive shifts, we introduce local observables whose edge coordinates lie in Z^d. Periodic realization uses integer remainders and is defined at every volume. A canonical finite lower shift moves each observable into the existing natural-coordinate chart; beyond its explicit coordinate bound the two realizations agree literally. This eventual equality transfers the complete-sequence Cauchy theorem and its cofinal uniqueness without introducing a subsequence or a compactness argument.

Infinite local Gibbs state.
The constructed functional omega_infinity is real linear, positive, and normalized on the ordered integer-coordinate local-observable space with unit. For every z in Z^d it satisfies omega_infinity(z + O)=omega_infinity(O). The acting object is an additive group, so inverse translations are part of the same literal action rather than volume-dependent positive words.

The domain has an observable product, but no C*-norm or C*-completion is constructed here. Accordingly the result is described as a positive normalized local state, not as a C*-algebraic state.

7. Genuine free boundary and uniqueness at the proved scope

The periodic torus is cut along a centered seam. Plaquettes crossing that seam receive zero activity; the resulting polymer gas is proved literally equal to a restriction of the periodic gas. The observable is centered so that its support lies in the interior rather than on the cut.

Inclusion-exclusion cancels all clusters except those that meet both the local marked support and the deleted seam. Short clusters cannot do so once the common window fits. Long clusters are bounded by the same support-pinned tail. At a common volume,

```
| E_n^periodic(O) - E_n^free(centered O) |
  <= 2 * markedOuterTailBound_O(q)
     + markedSmallLayerCauchyBound_O(q,q).
```

Define the free volume index by the explicit threshold plus q. These indices are cofinal, so the periodic expectations along them converge to omega_infinity(O). The displayed difference tends to zero by squeezing. Therefore the complete explicitly indexed free sequence converges to the same value.

Periodic versus centered free boundary.
The full cofinal sequence of genuine centered free-box expectations converges to the same infinite value as periodic boundary conditions. This is the boundary independence formalized here; no type or theorem for arbitrary boundary conditions is asserted.

8. Truncated correlations

For local observables O and P, define the finite truncated correlation as E_n(OP)-E_n(O)E_n(P). The complete finite-volume sequence converges to the analogous expression in omega_infinity. Hence every eventual volume-uniform finite bound passes unchanged to the limit.

The formalization additionally bridges the existing normalized two-plaquette theorem to this local-observable definition. If, in all sufficiently large volumes, O and P realize a bounded measurable holonomy observable f at distinct plaquettes p_n and q_n with touching distance at least 2k, the finite normalized estimate gives

```
| omega_infinity(OP) - omega_infinity(O) omega_infinity(P) |
  <= C(d,B,beta,s,t,epsilon) * exp(-epsilon*k).
```

The eventual realization and separation data remain explicit hypotheses of this bridge. The theorem does not claim that every pair of abstract local observables is a separated two-plaquette pair.

9. Explicit non-vacuous SU(2) intervals

A conditional KP theorem can be mathematically empty if its hypotheses are never instantiated, and the point beta=0 would describe only the free product measure. The development therefore constructs a concrete nonzero interval. For d=2 and SU(2), the physical plaquette energy Re tr(U) is measurable, bounded by 2, and formally proved nonconstant.

The parameters t=epsilon=eta=1/100 are fixed. On |beta|<=10^-5, the activity is bounded using the quadratic exponential remainder:

```
exp(2*|beta|) - 1 <= 2.1*|beta|.
```

Elementary rational upper bounds on the remaining exponentials then discharge the radius, smallness, unit-tilt, and marked-radius fields of the uniform regime. These constants are deliberately unoptimized: the certified radii below are far smaller than the order-one strong-coupling domains customarily obtained by analytic treatments. They certify non-vacuity of this formal package, not a competitive estimate of the strong-coupling boundary.

> Concrete two-dimensional strong-coupling regime.
> For every real beta with 0<|beta|<=10^-5, the d=2 SU(2) Wilson theory has the constructed infinite local Gibbs state. Positivity, normalization, full Z^d-translation invariance, and periodic-versus-centered-free convergence are therefore non-vacuous physical statements on an interval.

The same rational proof is instantiated separately in the physical lattice dimension d=4. Here the animal constant grows from (16*2+1)^2=1089 to (16*4+1)^2=4225 and the smallness prefactor from 32 to 64. A deliberately conservative exponential majorant then discharges all five fields on 0<|beta|<=10^-6.

> Concrete four-dimensional lattice regime.
> For every real beta with 0<|beta|<=10^-6, the d=4 SU(2) Wilson lattice theory has the constructed infinite local Gibbs state, with the same positivity, normalization, full Z^d-translation invariance, and periodic-versus-centered-free convergence conclusions. This is a four-dimensional lattice statement, not a continuum limit.

10. Formal verification record

Endpoint	*Role*
connectedLattice_pinned_tail_volumeUniform	*Banked volume-uniform rooted KP tail*
cauchySeq_localGibbsExpectation_kpUniform	*Complete-sequence Cauchy theorem*
infiniteLocalGibbsState	*Positive normalized local state*
integerLocalGibbsExpectation_vadd	*Exact finite invariance for every z in Z^d*
integerInfiniteLocalGibbsState	*Positive normalized state with the full translation action*
su2InfiniteLocalGibbsStateOnPuncturedInterval	*Physical nonzero interval*
su2D4InfiniteLocalGibbsStateOnPuncturedInterval	*Physical d=4 lattice interval*
norm_localGibbsExpectation_sub_freeBoundary_le_kpUniform	*Same-volume periodic/free bound*
tendsto_freeBoundaryThermodynamicExpectation	*Complete free sequence has the same limit*
abs_infiniteLocalGibbsTruncatedCorrelation_le_twoPlaquette	*Finite normalized bound passes to the state*

Direct focal elaboration of all terminal modules and direct elaboration of the root `YangMillsCore.lean` terminated with exit code 0 under Lean 4.29.0-rc6. Twenty headline `#print axioms` checks reported exactly `[propext, Classical.choice, Quot.sound]`.
All nine dependency HEADs matched the manifest. After supplying a process-local safe.directory setting for the ownership-mismatched checkout, the fixed-toolchain canonical root build completed with empty stderr and the literal final line `Build completed successfully (8460 jobs).` No dependency was deleted or updated and no global Git configuration was changed.

11. Scope and open directions

Proved here	*Not proved here*
Complete periodic expectation sequence is Cauchy	*Continuum or lattice-spacing limit*
Positive normalized real local state	*C*-completion or C*-state*
Genuine Z^d action, including inverse translations	*Finite-cutoff covariance of the centered free exhaustion under every z*
Periodic and one genuine centered free exhaustion	*Arbitrary boundary conditions*
Two-plaquette bound passes under explicit geometry	*Geometry-free clustering for arbitrary local observables*
d=2 and d=4 SU(2) lattice instances	*Four-dimensional continuum Yang-Mills or OS reconstruction*

The next mathematical extensions are to instantiate the eventual separated-plaquette geometry for explicit signed translated observables, and formulate additional physical boundary conditions as concrete polymer restrictions. These are separate theorems; none is required for the whole-sequence periodic and centered-free limits established above.

References

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