Global Ratio Monotonicity for a Killed von Mises Bridge

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Abstract

For $\beta > 0$ let $I_m = I_m(\beta)$ denote modified Bessel functions of the first kind, and set $a_m = I_m^2\,[(m-1)I_{m-1}^2 + (m+1)I_{m+1}^2]$, $b_m = m\,I_m^4$, $F_A(t) = \sum_{m\geq 1} a_m \sin(mt)$, $F_B(t) = \sum_{m\geq 1} b_m \sin(mt)$, and $E = F_A/(2F_B)$. The global ratio-monotonicity problem for the surface expansion of a two-dimensional $SU(2)$ lattice gauge observable is: (i) $F_B > 0$ on $(0, \pi)$, and (ii) $E' < 0$ on $(0, \pi)$, for every $\beta > 0$. Both statements are proved. Positivity of F_B has two exact proofs. Ratio monotonicity is reduced to exact algebraic identities and outward-rounded interval certificates: small and compact β are handled by pair identities and interval Taylor models; $20 \leq \beta \leq 1000/9$ by a direct Wronskian cover; and $\beta \geq 1000/9$ has three certified moving-edge λ-lanes. The remaining $\lambda \geq 3$ lane is closed by the exact identity $E'/(-\sin(t/2)/2) = \mathcal{Q} + X_{\text{full}}$, where $\mathcal{Q} > 19/20$, an exact main–mirror–rest decomposition, and a division-free covariance certificate proving $X_{\text{main}} > -1/20$ on two adjacent rectangles that cover the full angular interval. The exact near and far relay margins are positive. All load-bearing production and independent replay transcripts are checked for exact rational coverage, dependency hashes, strict outward-rounded decision endpoints, and byte equality. The structural core is exact: E is, as an algebraic identity, the mean of $\cos\psi$ under the midpoint law of a four-step killed von Mises bridge; the generating kernels reduce, via the Neumann addition theorem, to two-dimensional integrals of a *single* Bessel function whose saddle deficit is an exact sum of two squares; and exact saddle cancellations yield the coefficients of the (verified) closed second-order law $E = \cos(t/2)(1 - c(t)/\beta) + O(\beta^{-2})$, $c(t) = (4\cos^2(t/4) - 1)/(2\cos(t/4)\cos(t/2))$. Three certified negative results (interval arithmetic, two implementations, nested enclosures) kill every monotone full-path coupling, with an exact mechanism at threshold $\beta|\cos t| = 3/2$. At the π-endpoint we also give exact identities for the cubic coefficient c_3 (telescoped alternating form, integral form, parity) together with its verified prefactor law. Every claim is labelled exact / certified / verified; the machine-checked lemmas are Lean 4/Mathlib, machine-checked modulo classical Bessel inputs carried as named hypotheses.

1 Introduction

This paper consolidates the resolution status of a two-part positivity problem that arises as the last analytic gap in a surface (character) expansion for two-dimensional $SU(2)$ lattice gauge theory [5, 6, 7]. The problem is stated entirely in terms of modified Bessel functions and trigonometric series, and we treat it as such; no claim of physical consequence beyond the stated expansion is made, and importance is not inherited by thematic proximity.

This Wilson/von Mises problem is distinct from the exact heat-kernel theory. The latter now has a machine-checked two-dimensional $SU(2)$ construction through Weyl integration, character convolution and Migdal subdivision, culminating in the exact simple-loop area law [9]. That result neither supplies nor requires the present ratio monotonicity: the plaquette weights, coefficient families and target inequality here are different. We cite it to fix the boundary between an exactly solved heat-kernel model and the Wilson-action surface ordering studied below, not to transfer a theorem by thematic proximity.

Throughout, $\beta > 0$, $I_m = I_m(\beta)$, and

$$a_m = I_m^2\big[(m-1)I_{m-1}^2 + (m+1)I_{m+1}^2\big], \qquad b_m = m\, I_m^4, \qquad m \geq 1. \tag{1}$$

Both sequences decay super-exponentially, so

$$F_A(t) = \sum_{m\geq 1} a_m \sin(mt), \qquad F_B(t) = \sum_{m\geq 1} b_m \sin(mt), \qquad E(t) = \frac{F_A(t)}{2F_B(t)}, \tag{2}$$

are real-analytic on $\mathbb{R}$, and the Wronskian $W = 2(F_A' F_B - F_A F_B')$ satisfies $W = 4F_B^2\,(E)'$ wherever $F_B \neq 0$ [7].

The global statement is Theorem 2. Its positivity part is proved below twice. Its monotonicity part combines the exact skeleton of Theorem 40 with independently archived interval certificates and the terminal relay of Section 9. The question was born inside this programme as the global form of an ε-window positivity required by the surface expansion; no external pedigree is claimed.

Convention 1 (tricotomy). Every assertion below carries one of three labels. *Exact*: a proved identity or inequality. *Certified*: a numerical statement rigorously enclosed by interval arithmetic with outward rounding, together with complete analytic bounds for domains and truncations. Production/replay equality and independent implementations are *software audits*, recorded separately with their declared coverage. The certified negatives `certify_time3_negative` and `certify_bridge_matrix` carry a *full twin audit* (nested re-computation plus a complete independent Arb implementation, both transcripts archived); for `certify_thmB` has two completed independent implementations, with the mpmath transcript (53 adaptive boxes, stability pass) and the Arb transcript (86 boxes, stability pass) both archived. Every load-bearing finite- and high-β union has a hash-bound production/replay audit. *Verified*: high-precision numerics with stated grids and precision policies, not a proof. No verified statement is used as a hypothesis of an exact one.

2 The theorem and the relay

Theorem 2 (the Surface Theorem)**.** *For every $\beta > 0$:* (i) $F_B > 0$ *on* $(0, \pi)$*, and* (ii) $E' < 0$ *on* $(0, \pi)$*.*

Part (i) is Theorem 37. The following table records the closed relay for part (ii). Every certified row is an outward-rounded interval certificate with an executable coverage validator.

range	method	artifact	transcript	state
$(0, \frac{1}{20}]$	small-β lemma	Lem. 41	—	complete
$[\frac{1}{20}, 3]$, all t	pair skeleton	`certify_thmB`	Arb + mpmath	complete
$[3, 6]$, $t{\in}[0.6, \pi{-}\frac{1.5}{\beta}]$	series intervals	`certify_bulk_arb`	Arb	complete
$[6, 15]$, $t{\in}[0.6, \pi{-}\frac{1.5}{\beta}]$	bivariate Taylor series	`certify_bulk_beta_taylor`	Arb	complete
$[15, 20]$, $t{\in}[0.6, \pi{-}\frac{1.5}{\beta}]$	bivariate Taylor extension	`certify_bulk_beta_taylor`	Arb	complete
$t \in (0, 0.6]$, $3 \le \beta \le 20$	W/t^3 + Taylor splice	`certify_left_edge`	Arb	complete
$3 \le \beta \le 20$, $t \in [\pi{-}\frac{1.5}{\beta}, \pi)$	endpoint quotient + Taylor splice	`right-edge union`	Arb	complete
$[20, 1000/9]$, all t	finite Wronskian union	`finite role audit`	Arb + replay	complete
$\beta \ge 1000/9$, $0 < \lambda \le 3/2$	finite/half-line G5	two G5 unions	Arb + replay	complete
$\beta \ge 1000/9$, $3/2 \le \lambda \le 2$	ratio G5	`G5 lambda2`	Arb + replay	complete
$\beta \ge 1000/9$, $2 \le \lambda \le 3$	ratio-side4 G5	`G5 lambda3`	Arb + replay	complete
$\beta \ge 1000/9$, $\lambda \ge 3$	weak-main three-block relay	Prop. 29	Arb + replay	complete

The compact and high-β rows are owned by their canonical Arb transcripts and executable coverage validators, with adaptive partitions and all script/output hashes recorded in the accompanying run manifests.

For the finite row the logical relay is direct. With $W = 2(F_A' F_B - F_A F_B')$ and $E = F_A/(2F_B)$, exact differentiation gives

$$W = 4F_B^2 E'.$$

The algebra audit `verify_surface_finite_w_sign_relay.py` also checks $W^J = s^2 W$ under the common positive scale used by the finite certificates. The three adjacent witnesses cover $0 < t \le 3/5$, $3/5 \le t \le \pi - 3/(2\beta)$, and $\pi - 3/(2\beta) \le t < \pi$ for the full closed interval $20 \le \beta \le 1000/9$. Their union validator checks strict negativity, exact rational adjacency, dependency hashes, and production/replay equality.

For the half-line put $\delta = 1/\beta$ and $\lambda = \beta(\pi - t)$. The three G5 lanes meet at $\lambda = 3/2$ and 2, and the last meets the analytic interior lane at $\lambda = 3$; the first G5 lane itself meets its finite and half-line archives at $\delta = 1/125$ ($\beta = 125$). The remaining lane $\lambda \ge 3$ is Proposition 29. Its near/far covariance rectangles and the three G5 lanes meet in closed overlaps, so the global relay has no seam gap. The moving-boundary constant is always written $C_{\rm win} = 3/2$; the bare letter C below means $\cos(t/2)$.

Remark 3 (supersession of the e_4 route). An earlier plan closed the block $t \in (0, 0.6]$, $\beta \ge 3$ by an e_4-coefficient lemma with a fixed radius; it is superseded, not silently deleted. A feasibility check showed the sinc criterion below dominating with ratios from 15 up to 10^{80} across $\beta \in [3, 15]$ for $t \le 0.3$ (*verified*, design-grade), while the apparent infeasibility of $(0, 0.6]$ belonged to the cancellation of the series tool, which the exp-factored integrator does not suffer. The compact row is now closed directly through $\beta = 20$, and the weak-main covariance certificate covers the remaining high-β interior. On the finite interval between them the registered route is the exact common scaling $J_m = e^{-\beta} I_m$, which multiplies W/t^3 by $e^{-8\beta} > 0$ before interval evaluation. The sinc lemma below remains an exact structural result and an independent diagnostic, but no longer carries the terminal left-edge load. The e_4 lemma is retired.

Lemma 4 (exact; sinc positivity)**.** *Let $1 \le m < n$ and $T_{mn}(t) = (m-n)\sin((m+n)t) + (m+n)\sin((n-m)t)$ as in Section 4. Then $T_{mn}(t) > 0$ for $0 < t \le \pi/(m+n)$.*

Proof. Write $s = m+n$, $p = n-m \ge 1$, so $T_{mn}(t) = s\sin(pt) - p\sin(st)$. Since $\sin(x)/x$ is strictly decreasing on $(0,\pi]$ and $0 < pt < st \le \pi$, we get $\sin(pt)/(pt) > \sin(st)/(st)$, i.e. $st\sin(pt) > pt\sin(st)$; dividing by $t > 0$ gives $s\sin(pt) > p\sin(st)$. □

Lemma 5 (relay criterion for $\beta \in [6, 15]$)**.** *Let $\langle f\rangle = \iint_{[-\pi,\pi]^2} Kf\,ds\,d\alpha$ with $K = I_1(2\beta R)/R > 0$, and let $\bar{E}$ be any rational constant. Then $E' < 0$ at (t,β) if and only if $\langle D\rangle > 0$ and*

$$W_c := \langle N_t\rangle\langle D\rangle + \langle (N - \bar{E}D)\,\partial_t \log K\rangle\langle D\rangle - (\langle N\rangle - \bar{E}\langle D\rangle)\langle D\,\partial_t\log K\rangle < 0$$

(the $\bar{E}$-terms cancel identically). The certificate encloses the five integrals by exp-factored interval cells: $z = 2\beta R$ is linearized at the exact rational cell center with interval-gradient remainder; the linear exponential integrates exactly; and the two scaled Bessel factors are $A(z) = e^{-z}I_1(z)/z$ — strictly decreasing, since $(\log A)' = I_0/I_1 - 1 - 2/z < 0$ follows from the classical lower bound at $\nu = 0$ (Amos [1]; [2], Thm. 1) and $\sqrt{1+z^2} \le z+1$ — enclosed by pure endpoint evaluation, and $B(z) = e^{-z}H_B(z)$, where

$$H_B(z) := \frac{I_0(z) - 2I_1(z)/z}{z^2} = \frac{I_2(z)}{z^2}, \qquad H_B(0) = \tfrac{1}{8}$$

(*the second equality is the recurrence [4, 10.29.1] at $\nu = 1$; positive-coefficient entire series; the subscript distinguishes it from the entry-law kernel $H(\psi)$*)*, enclosed unconditionally by $[e^{-z_{\rm hi}}H_B(z_{\rm lo}), e^{-z_{\rm lo}}H_B(z_{\rm hi})]$; the identity $K\,\partial_t\log K = 32\beta^3 c\,c'\,(1-P-Q)H_B(z)$, $c = \cos(t/4)$, removes $I_1(z)/z$ from the covariance integrands exactly. The (t,β)-box layer is Proposition 30. The independent global $[6, 15]$ coverage is Proposition 32.*

Lemma 6 (exact; the S-free relative criterion)**.** *Let $S = \sin(t/2)$, $E = \langle N\rangle/\langle D\rangle$, and suppose $\langle D\rangle > 0$. Then for all $t \in (0,\pi)$ and $\beta > 0$,*

$$\frac{E'(t,\beta)}{-\frac{1}{2}S} = \frac{\langle \cos 2s + \cos\alpha\cos s\rangle + 8\beta^3 \iint (N - E\,D)(1-P-Q)\,H_B(z)\,ds\,d\alpha}{\langle D\rangle},$$

and consequently

$$E' < -\tfrac{1}{4}S \iff \mathcal{C} := \langle \cos 2s + \cos\alpha\cos s\rangle + 8\beta^3 \iint (N - E\,D)(1-P-Q)H_B - \tfrac{1}{2}\langle D\rangle > 0.$$

Proof. Since $D_t = 0$, $E' = [\langle N_t\rangle + \iint (N - ED)\,\partial_t K]/\langle D\rangle$. The first term factors S exactly: $N_t = -\frac{S}{2}(\cos 2s + \cos\alpha\cos s)$. So does the second: $\partial_t K = K\,\partial_t\log K = 32\beta^3 c\,c'\,(1-P-Q)H_B(z)$ with $c = \cos(t/4)$, and $c\,c' = \cos(t/4)\cdot(-\frac{1}{4}\sin(t/4)) = -\frac{S}{8}$ by the double angle, so $\partial_t K = -4\beta^3 S\,(1-P-Q)H_B$. Dividing by $-\frac{1}{2}S$ cancels S identically and yields the display. For the equivalence: the denominator $-\frac{1}{2}S$ is negative on $(0,\pi)$, so $E' < -\frac{1}{4}S$ is the *lower* bound $E'/(-\frac{1}{2}S) > \frac{1}{2}$, i.e. $\mathcal{C} > 0$. □

Corollary 7 (exact; the bilinear quotient-free form)**.** *With $\Phi := \cos 2s + \cos\alpha\cos s$ (the letter A stays reserved for the scaled Bessel factor $e^{-z}I_1(z)/z$), $J_N := \iint N(1-P-Q)H_B\,ds\,d\alpha$, $J_D := \iint D(1-P-Q)H_B\,ds\,d\alpha$, and $\langle D\rangle > 0$,*

$$E' < -\tfrac{1}{4}\sin(t/2) \iff \widehat{\mathcal{C}} := \langle D\rangle\langle\Phi\rangle + 8\beta^3(\langle D\rangle J_N - \langle N\rangle J_D) - \tfrac{1}{2}\langle D\rangle^2 > 0.$$

Proof. Multiply $\mathcal{C}$ of Lemma 6 by $\langle D\rangle > 0$ and substitute $E = \langle N\rangle/\langle D\rangle$: the quotient disappears by bilinearity, and the inequality direction is preserved. $\widehat{\mathcal{C}}$ is a bilinear combination of five integrals with no quotient anywhere; it is the assembly's sole target. □

Remark 8 (why $\widehat{\mathcal{C}}$ is the assembly's object). In $\widehat{\mathcal{C}}$ the variable t enters only through $c = \cos(t/4) \in (\frac{\sqrt{2}}{2}, 1)$: no kinematically forced vanishing factor and no singularity anywhere in the relay range — the common linear zero of margin and correction at $t \to 0$ has been cancelled identically. The endpoint degeneracy is thus removed structurally; uniform positivity still requires explicit uniform estimates, which are the assembly's content. The first-order calibration constant is $a(t) = [\frac{1}{2} + \frac{1}{8c^2}]/c \le \frac{3\sqrt{2}}{4} \approx 1.061$ (supremum at $c = \sqrt{2}/2$), so the first-order threshold is $2\sup a \approx 2.12$; the target of the analytic relay is $\widehat{\mathcal{C}} > 0$ uniformly for $\beta \ge \beta_0$ with the remainder included. At the calibration point $(t, \beta) = (1.5, 8)$ the measured value is $\mathcal{C}/\langle D\rangle \approx 0.417$ (*verified*), against the threshold 0. The built-in unit test: the $1/\beta$ expansion of $\mathcal{C}/\langle D\rangle + \frac{1}{2}$ must reproduce $1 - a(t)/\beta$; the audited quantity is $\beta \cdot$ (deviation from first order), which must converge (an $O(\beta^{-2})$ remainder halves the β-*scaled* deviation when β doubles, and divides the unscaled deviation by four).

Lemma 9 (exact; far-region pointwise and kernel bounds)**.** (a) $\min_{s,\alpha} \Phi = -\frac{9}{8}$ *exactly: with* $u = |\cos s|$ *the minimum of* $2u^2 - u - 1$ *on* $[0, 1]$ *is attained at* $u = \frac{1}{4}$ *and equals* $-\frac{9}{8}$. (b) $|D| \le 2$ *and, by Corollary 44* ($|E| < 1$), $|N - ED| \le |N| + |D| \le (2|C| + 1) + 2 \le 5$ *and* $|1 - P - Q| \le 1$. (c) *On the deficit region* $\{(s, \alpha) : 2c - R \ge \Delta_0\}$ *one has* $z = 2\beta R \le z_s - 2\beta\Delta_0$ *with* $z_s = 4\beta c$, *and since* $A(z) = e^{-z} I_1(z)/z$ *is strictly decreasing with* $A(0^+) = \frac{1}{2}$,

$$K = 2\beta\, e^{z} A(z) \;\le\; \beta\, e^{z_s}\, e^{-2\beta\Delta_0} \quad \textit{on the region.}$$

The far-region contributions of all five integrals of Corollary 7 are then bounded by the two tail families of Lemma 10 below.

Proof. (a) $\cos\alpha\cos s \ge -|\cos s|$ with equality attainable, and $\cos 2s = 2u^2 - 1$; minimizing $g(u) = 2u^2 - u - 1$ gives $g(\frac{1}{4}) = -\frac{9}{8}$. (b) Triangle inequalities; $|N| \le |C|\,|\cos 2s| + |\cos\alpha|\,(|C\cos s| + \sin^2 s) \le 2|C| + 1$. (c) $R \le 2c - \Delta_0$ gives $z \le z_s - 2\beta\Delta_0$; write $K = I_1(z)/R = 2\beta I_1(z)/z = 2\beta e^z A(z)$ and use the monotonicity of A (Section on the two scaled functions) with $A(z) \le A(0^+) = \frac{1}{2}$ uniformly. □

Lemma 10 (exact; the two tail families)**.** $0 < H_B(z) \le \frac{1}{4} I_1(z)/z = \frac{1}{4} e^z A(z) \le \frac{1}{8} e^z$ *(term by term: the coefficient of* $(z/2)^{2k}$ *in* H_B *is* $\frac{1}{4}\frac{1}{(k+2)\,k!\,(k+1)!} \le \frac{1}{8\,k!\,(k+1)!}$, *the coefficient in* $\frac{1}{4} I_1(z)/z$ *being* $\frac{1}{8\,k!\,(k+1)!}$). *Consequently, writing* δX *for the far-region part of an integral* X *over* $\{2c - R \ge \Delta_0\}$,

$$|\delta\langle f\rangle| \le 4\pi^2\, |f|_\infty\, \beta\, e^{z_s - 2\beta\Delta_0}, \qquad |\delta J_f| \le \tfrac{\pi^2}{2}\, |f(1 - P - Q)|_\infty\, e^{z_s - 2\beta\Delta_0},$$

for the three K*-integrals and the two* H_B*-integrals respectively; on the same region* $z \le z_s - 2\beta\Delta_0$ *also bounds* e^z *in* $H_B \le \frac{1}{8} e^z$.

Proposition 11 (exact; Region III propagation)**.** *Let* $\widehat{\mathcal{C}}_{\text{near}}$ *denote* $\widehat{\mathcal{C}}$ *with every integral restricted to* $\{2c - R < \Delta_0\}$. *Then for every* $\Delta_0 \in (0, 2c)$ *and* $\beta \ge 1$,

$$\big|\widehat{\mathcal{C}} - \widehat{\mathcal{C}}_{\text{near}}\big| \;\le\; 576\,\pi^4 \beta^4\, e^{2z_s}\, e^{-2\beta\Delta_0}.$$

Proof. Global masses: $\iint K \le 4\pi^2\beta e^{z_s} =: M$ and $\iint H_B \le \frac{\pi^2}{2}e^{z_s} =: M_H$ (the pointwise bounds $K \le \beta e^{z_s}$, $H_B \le \frac{1}{8}e^{z_s}$ hold on all of $[-\pi,\pi]^2$ since $z \le z_s$). With the pointwise constants of Lemma 9 ($|\Phi|_\infty \le 2$, $|D|_\infty \le 2$, $|N|_\infty \le 3$, $|N(1-P-Q)|_\infty \le 3$, $|D(1-P-Q)|_\infty \le 2$), Lemma 10 gives $|\delta\langle D\rangle|, |\delta\langle\Phi\rangle| \le 2a$, $|\delta\langle N\rangle| \le 3a$, $|\delta J_N| \le 3b$, $|\delta J_D| \le 2b$, where $a := Me^{-2\beta\Delta_0}$, $b := M_H e^{-2\beta\Delta_0}$ (note $4\pi^2\beta e^{z_s-2\beta\Delta_0} = a$ and $\frac{\pi^2}{2}e^{z_s-2\beta\Delta_0} = b$). For any product, $|XY - X_{\mathrm{n}}Y_{\mathrm{n}}| \le |\delta X|\,|Y|_{\max} + |X|_{\max}|\delta Y|$ with $|\langle f\rangle| \le |f|_\infty M$, $|J_f| \le |f(1-P-Q)|_\infty M_H$. Summing: $\langle D\rangle\langle\Phi\rangle$ contributes $\le 8aM$; $\langle D\rangle J_N$ and $\langle N\rangle J_D$ contribute $\le 12MM_He^{-2\beta\Delta_0}$ each (using $aM_H = bM = MM_He^{-2\beta\Delta_0}$); $\frac{1}{2}\langle D\rangle^2$ contributes $\le 4aM$. Total: $12aM + 192\beta^3MM_He^{-2\beta\Delta_0} = (192\pi^4\beta^2 + 384\pi^4\beta^4)e^{2z_s}e^{-2\beta\Delta_0} \le 576\pi^4\beta^4e^{2z_s}e^{-2\beta\Delta_0}$ for $\beta \ge 1$. ☐

Remark 12 (the choice of Δ_0 belongs to the assembly). With these deliberately crude (but fully named) constants, at $\beta = 15$ the bound forces $\Delta_0 \approx 1.2$ rather than the design value 0.3: the natural saddle-mass scale is $\langle D\rangle \asymp e^{z_s}\beta^{-3/2}$ (saddle height $\asymp e^{z_s}\beta^{-1/2}$ times a two-dimensional Gaussian area $\asymp \beta^{-1}$), so $\langle D\rangle^2 \asymp e^{2z_s}\beta^{-3}$ and the comparison quotient carries $\beta^7e^{-2\beta\Delta_0}$, consuming ≈ 34 of exponent at $\beta = 15$. By Lemma 18 ($m_* \ge \frac{1}{2}$), $\Delta_0 = 1.2$ suffices with margin (Corollary 13). The enlargement changes no leading bound, subject to the forthcoming geometric shell estimate: the shell technique that will bound Region II is insensitive to its outer edge — the telescoping sum over deficit shells is controlled by the kernel mass beyond the *inner* radius Δ_1 (up to its geometric closing factor, part of that estimate). Since $\Delta_0 < 2c$ is free, the constant hunt stays honest.

Corollary 13 (Region III numeric tooth)**.** *By Lemma 18 ($m_* \ge \frac{1}{2}$), for $\beta \ge 15$ and $t \in (0, \pi - C_{\mathrm{win}}/\beta]$, with $\Delta_0 = 1.2$,*

$$\frac{|\widehat{\mathcal{C}} - \widehat{\mathcal{C}}_{\mathrm{near}}|}{\langle D\rangle^2} \;\le\; \frac{576\pi^4}{m_*^2}\,\beta^7e^{-2.4\beta} \;\le\; 2304\,\pi^4\,\beta^7e^{-2.4\beta},$$

and the right side is decreasing in β on $[15,\infty)$ ($\frac{d}{d\beta}\log(\beta^7e^{-2.4\beta}) = \frac{7}{\beta} - 2.4 < 0$ for $\beta \ge 15$), with value $\le 8.9\times10^{-3}$ at $\beta = 15$: a single evaluation certifies the whole half-line. (An earlier draft hypothesized $m_0 = 1$, consistent with every measurement but not reached by the proven floor; it is superseded by Lemma 18, not falsified — constant four times uglier, foundation harder.)

Convention 14 (cited constants travel with their normalization). All double integrals in this section are over $[-\pi,\pi]^2$ with no normalization: $\langle f\rangle := \iint Kf\,ds\,d\alpha$. Every numerically measured constant quoted in this manuscript travels with its triplet (normalization, t, β); the saddle-mass lemma is stated as an explicit function $m_0(c)$, never as a loose scalar, since the saddle prefactor carries a c-dependence and measurements confirm it ($m_0 \approx 1.492$ at $(t,\beta) = (1.5, 15)$, 2.110 at $(2.5, 15)$, 1.353 at $(1.0, 40)$ under this convention). The one historical discrepancy is *closed exactly*: via $F_B = \frac{\beta\sin(t/2)}{16\pi^2}\langle D\rangle$, i.e. $\langle D\rangle = 16\pi^2F_B/(\beta\sin(t/2))$, the F_B-series reproduces all three readings; an earlier report of 1.05 was the $(2.5, 15)$ value computed with the bracket $D/2$ in place of D — a factor of exactly 2, identified, not a numerical uncertainty. Uniform floors are written $m_* := \inf_{c\in[\sqrt{2}/2,1]} m_0(c)$.

Lemma 15 (exact; analytic lower bound for I_1)**.** *For $z \ge 2$, $I_1(z) \ge \frac{e^z}{\sqrt{2\pi z}}(1 - \frac{1}{2z}) - \frac{3}{2}$; consequently $I_1(z) \ge \frac{e^z}{\sqrt{2\pi z}}(1 - \frac{0.6}{z})$ for $z \ge 10$.*

Proof. From $I_1(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\theta\,d\theta$ and $u = \sin(\theta/2)$, $I_1(z) = \frac{2e^z}{\pi}\int_0^1 e^{-2zu^2}\frac{1-2u^2}{\sqrt{1-u^2}}\,du$. On $[0, \frac{1}{\sqrt{2}}]$ drop $\frac{1}{\sqrt{1-u^2}} \ge 1$ (there $1-2u^2 \ge 0$) and complete the Gaussian: $\int_0^\infty e^{-2zu^2}(1-2u^2)\,du =$

$\frac{1}{2}\sqrt{\frac{\pi}{2z}}(1-\frac{1}{2z})$, at the cost of the Gaussian residue, derived by $u^2=\frac{1}{2}+v$ (so $2u^2-1=2v$, $du\le dv/\sqrt{2}$): $\int_{1/\sqrt{2}}^{\infty}e^{-2zu^2}(2u^2-1)\,du \le \sqrt{2}\,e^{-z}\int_0^\infty ve^{-2zv}dv = \frac{e^{-z}}{2\sqrt{2}\,z^2}\le\frac{3}{z}e^{-z}$. On $[\frac{1}{\sqrt{2}},1]$, $|1-2u^2|\le 1$ and $e^{-2zu^2}\le e^{-z}$ give at most $\frac{\pi}{4}e^{-z}$. Multiplying by $\frac{2e^z}{\pi}$: the main term is $\frac{e^z}{\sqrt{2\pi z}}(1-\frac{1}{2z})$, the two tails contribute at most $\frac{1}{2}+\frac{6}{\pi z}\le\frac{3}{2}$ for $z\ge 2$. For the corollary: at $z\ge 10$, $\frac{e^z}{\sqrt{2\pi z}}\cdot\frac{0.1}{z}\ge\frac{e^{10}}{\sqrt{20\pi}}\cdot\frac{1}{100}\approx 27.8\ge\frac{3}{2}$, and the left side increases in z. □

Lemma 16 (exact; global upper companion for I_1)**.** *For every $z>0$, $I_1(z)\le e^z/\sqrt{2\pi z}$.*

Proof. In the representation of Lemma 15, $I_1(z)=\frac{2e^z}{\pi}\int_0^1 e^{-2zu^2}\frac{1-2u^2}{\sqrt{1-u^2}}\,du$, the integrand is ≤ 0 on $[\frac{1}{\sqrt{2}},1]$; on $[0,\frac{1}{\sqrt{2}}]$ one has $\frac{1-2u^2}{\sqrt{1-u^2}}\le 1$: squaring is legitimate there ($1-2u^2\ge 0$) and reduces the claim to $u^2(4u^2-3)\le 0$, true for $u^2\le\frac{3}{4}$. Hence $I_1(z)\le\frac{2e^z}{\pi}\int_0^\infty e^{-2zu^2}\,du=\frac{2e^z}{\pi}\cdot\frac{1}{2}\sqrt{\frac{\pi}{2z}}=\frac{e^z}{\sqrt{2\pi z}}$. (The companions of Lemma 22 sharpen this for $z\ge 20$; the point of the global form is the range $z<20$, which the rest bound of Lemma 18 genuinely visits.) □

Lemma 17 (exact; the area lemma)**.** *For $0\le u<1$,*

$$\mathrm{Area}\{(s,\alpha)\in[-\pi,\pi]^2:\ P+Q\le u\}\ \le\ \frac{4\pi u}{\sqrt{1-u}},$$

and the same bound holds for the mirror set $\{\tilde P+\tilde Q\le u\}$, $\tilde P:=1-P$, $\tilde Q:=1-Q$. Consequently, for $c^2\ge\frac{1}{2}$ (so $w\ge g:=P+Q-2PQ$),

$$\mathrm{Area}\{w\le v\}\ \le\ \bar A(v):=\min\Big(\frac{16\pi v}{\sqrt{1-2v}}\Big|_{v<\frac{1}{2}},\ 4\pi^3 v\Big),$$

with $\bar A$ nondecreasing (a minimum of two increasing branches; they cross at $v_c=\frac{1}{2}(1-16/\pi^4)=0.41787\ldots$).

Proof. On $[0,\pi]^2$ the substitution $dP=\sqrt{P(1-P)}\,ds$ gives

$$\mathrm{Area}_{[0,\pi]^2}\{P+Q\le u\}=\iint_{P+Q\le u}\frac{dP\,dQ}{\sqrt{PQ(1-P)(1-Q)}}\le\frac{1}{\sqrt{1-u}}\iint_{P+Q\le u}\frac{dP\,dQ}{\sqrt{PQ}}=\frac{\pi u}{\sqrt{1-u}}$$

($(1-P)(1-Q)\ge 1-u$ on the region; the last integral is u times the exact Beta integral $\Gamma(\frac{1}{2})^2/\Gamma(2)=\pi$); the full square is four copies, and the mirror set is the image under $(s,\alpha)\mapsto(\pi-s,\pi-\alpha)$. For the deficit sublevels: g is invariant under the mirror substitution and $g\ge\frac{1}{2}(P+Q)$ where $P+Q\le 1$, so $\{g\le v\}\subset\{P+Q\le 2v\}\cup\{\tilde P+\tilde Q\le 2v\}$ and the first branch follows; globally $P\ge(s/\pi)^2$ (concavity of sin) puts each component inside a region of area $2\pi^3 v$. □

Lemma 18 (chains exact, floor certified; the signed minoration: uniform saddle mass)**.** *Define $m(\beta,c):=\beta^{3/2}e^{-4\beta c}\langle D\rangle$ and the window-uniform mass $m_*:=\inf\{m(\beta,c):\beta\ge 15,\ t\in(0,\pi-C_{\mathrm{win}}/\beta]\}$. For every $\beta\ge 15$ and $t\in(0,\pi-C_{\mathrm{win}}/\beta]$,*

$$m(\beta,c)\ \ge\ m_{\mathrm{low}}(\beta,c):=T_1-T_G-T_M-T_R\ \ge\ \tfrac{1}{2},$$

with the four explicit pieces displayed in the proof; in particular $m_*\ge\frac{1}{2}$, and every statement of this manuscript carried on the candidate floor $m_*\ge\frac{1}{2}$ is unconditional from here on. The window is not cosmetic: at $t=\pi$ one has $c=s_4$, the involution of Remark 19 becomes a symmetry of K, and $\langle D\rangle=0$ identically.

Remark 19 (the sign gap, exposed by the involution). A restricted-domain bound $\int_B KD \geq (\ldots)$ on a rectangle B around the saddle cannot by itself minorize $\langle D \rangle = \iint KD$, because D changes sign on the torus: under the exact involution $T(s,\alpha) = (\pi - s, \pi - \alpha)$ one has $D \circ T = -D$ and $K_c \circ T = K_{s_4}$ with $s_4 = \sin(t/4)$, so the mirror ball contributes $\int_{T(B)} K_c D = -\int_B K_{s_4} D < 0$ — a negative part the restricted-domain argument omitted. The owed repair is the *signed minoration* $\langle D \rangle \geq L_{\mathrm{main}}(c) - U_{\mathrm{mirror}}(s_4) - U_{\mathrm{rest}}$, where the mirror term carries the exact relative suppression $e^{-4\beta(c-s_4)} \leq e^{-4C_{\mathrm{win}}/\pi} \approx 0.148$ at the moving boundary (via $c - s_4 = \sqrt{2}\sin\frac{\pi-t}{4} \geq \frac{\pi-t}{\pi}$ from $\sin y \geq \frac{2\sqrt{2}}{\pi} y$ on $[0, \frac{\pi}{4}]$) — order one, not negligible; note also $s_4 \in (0, \frac{\sqrt{2}}{2})$ falls *outside* the main range, so the mirror-ball upper bound needs its own (upper) chain — supplied by step (iii) of the proof of Lemma 18. The measured $m_0 \in [1.35, 2.11]$ now calibrates the lemma's floor (slack 2.2–4.2×: a floor, not an asymptote).

Proof of Lemma 18. Split the torus into the main rectangle $B = [-r, r]^2$, $r = \frac{6}{5}$, the mirror rectangle $B' = \{|s| \geq \pi - r,\ |\alpha| \geq \pi - r\}$, and the rest. On B the identity $KD = 2K - 2K(P+Q)$ is exact ($D = 2(1 - P - Q)$); off B we pay $|D| \leq 2$:

$$\langle D \rangle \ \geq\ 2\int_B K - 2\int_B K(P+Q) - 2\int_{B'} K - 2\int_{\mathrm{rest}} K.$$

On B, $w \leq P + Q \leq 2\sin^2\frac{r}{2} = w_{\max} = 0.6377$, so $z = z_s\sqrt{1-w} \geq 0.6019\, z_s \geq 25.5$.

(i) The main mass, from below. By the lower companion of Lemma 22; the frozen-prefactor monotonicity ($x^{-3/2}(1 - a/x - b/x^2)$ decreases on $x \in [0.6, 1]$ for $a \leq \frac{3}{160}$, $b \leq \frac{0.6}{400}$); mini-lemma (a) ($z \leq z_s$, so the bracket freezes at z_s); the root from below $\sqrt{1-w} \geq 1 - \frac{w}{2} - \frac{w^2}{4}$ (valid to $w = 2(\sqrt{2}-1)$); the factorization $w^2 \leq (P+Q)^2 \leq 2(P^2+Q^2)$ with $P \leq s^2/4$, which splits the quartic penalty per variable; $e^{-x} \geq 1 - x$; $\mathrm{erfc}(x) \leq e^{-x^2}$; and exact Gaussian moments:

$$2\beta^{3/2}e^{-z_s}\int_B K \ \geq\ T_1 := \frac{\sqrt{2\pi}}{2}\, c^{-5/2}\Big(1 - \frac{3}{8z_s} - \frac{0.6}{z_s^2}\Big)\Big(1 - e^{-18\beta c/25} - \frac{3}{8\beta c}\Big)^2.$$

(ii) The $K(P+Q)$ correction, from above. With the upper companion; the exact prefactor identity $z^{-3/2} = z_s^{-3/2}(1-w)^{-3/4}$; $e^{z - z_s} \leq e^{-2\beta c w}$; the convexity-endpoint absorption $(1 - w)^{-3/4}e^{-\beta_1 c_1 w/10} \leq 1.089$ on $[0, w_{\max}]$ ($\beta_1 c_1 = 15/\sqrt{2}$); the rate floor $w \geq 0.6811\,(P+Q)$ on B ($c^2 \geq \frac{1}{2}$); the concavity floor $P \geq 0.2214\, s^2$ on $|s| \leq r$; and the exact Gamma moment:

$$2\beta^{3/2}e^{-z_s}\int_B K(P+Q) \ \leq\ T_G := \frac{\beta}{2\sqrt{2\pi}\, c^{3/2}} \cdot \frac{1.089\,(\pi/4)}{(0.2214 \cdot 1.9 \cdot 0.6811\,\beta c)^2} \ \leq\ \frac{2.0784}{\beta c^{7/2}}.$$

(iii) The mirror, from above. The chart identity $1 - w = (s_4/c)^2(1 - w')$, $w' = P' + Q' - P'Q'/s_4^2$, $P' = \cos^2(s/2)$, is exact, so on B' one has $z = z_s(s_4)\sqrt{1-w'}$, $z_s(s_4) = 4\beta s_4$, with the exact relative suppression $e^{-4\beta\delta_4}$. Three zones. For $s_4 \geq 0.58$ (where $w' \geq 0$ on B' needs only $s_4 \geq \sin(0.6)/\sqrt{2} = 0.3993$, and $z \geq 20.9$ at the worst corner) the machinery of (ii) transplants verbatim to the chart:

$$2\beta^{3/2}e^{-z_s}\int_{B'} K \ \leq\ T_M := \frac{\beta}{2\sqrt{2\pi}\, s_4^{3/2}} \cdot \frac{1.230\,\pi}{0.2214 \cdot 1.9\,(1 - \frac{w_{\max}}{4s_4^2})\beta s_4}\ e^{-4\beta\delta_4}.$$

For $s_4 \in [0.40, 0.58)$: $z \leq z_s(s_4)$ on B' and $K = 2\beta A(z)e^z \leq \beta e^{z_s(s_4)}$ give $T_M = 8r^2\beta^{5/2}e^{-4\beta\delta_4}$ with $\delta_4 \geq 0.2346$ — dead. For $s_4 < 0.40$: $R^2 \leq 4c^2\sin^4(0.6) + 4s_4^2$ on B' gives $T_M = 8r^2\beta^{5/2}e^{-4\beta\Delta_{\mathrm{far}}}$, $\Delta_{\mathrm{far}} \geq 0.4211$ — dead.

(iv) The rest, from above. Outside $B \cup B'$ the case analysis on $g = P + Q - 2PQ$ gives the floor $w \geq \sin^2(0.6) > 0.318$. The kernel majorant $K \leq 2\beta e^z/(\sqrt{2\pi}z^{3/2})$ is witnessed on *all* of $z > 0$ by Lemma 16 (the layers below reach $z \approx 13.4$ at the window corner, outside the companions' range). With Lemma 17 sharpened by the disk $s^2 + \alpha^2 \leq 4\sin^2(0.6)$ (it lies inside B with $P + Q \leq \sin^2(0.6) \leq v$, so its area $4\pi \sin^2(0.6) \geq 4.006$ is never rest-mass: $\bar{A}_r := \bar{A} - 4.006$), the Abel-corrected layer sum on the fixed grid $v_0 = 0.318$, $\delta v = 0.02$, $V = 0.9$, with $\varphi(v) = (1-v)^{-3/4}e^{-2\beta cv}$ (decreasing there for $\beta c \geq 3.75$):

$$T_R := \frac{\beta}{2\sqrt{2\pi}\, c^{3/2}}\Big[\varphi(v_0)\bar{A}_r(v_1) + \sum_{k\geq 1}\varphi(v_k)\big(\bar{A}_r(v_{k+1}) - \bar{A}_r(v_k)\big)\Big] + 2(2\pi)^2\beta^{5/2}e^{-(1-\sqrt{0.1})z_s}.$$

The bottom-mass term is not optional — the first layer's mass may be all of $\bar{A}_r(v_1)$; the shard covers $\{w > 0.9\}$ via $z \leq z_s\sqrt{0.1}$ and $K \leq \beta e^z$.

(v) The floor. Every piece of m_{low} is monotone in β at fixed t in the favorable direction (the T_1 brackets increase; $T_G \sim \beta^{-1}$; the zone-L mirror is a β-free coefficient times $e^{-4\beta\delta_4}$; the crude zones and the shard are $\beta^{5/2}e^{-\beta\cdot\text{const}}$ with constants far above the $5/(2\beta)$ threshold; each layer term $\beta\varphi(v_k)$ decreases for $\beta \geq 1/(2cv_0)$), so the infimum sits on $\beta_{\min}(t) = \max(15, C_{\text{win}}/(\pi - t))$, and the resulting one-dimensional floor is *certified* by interval arithmetic (python-flint arb, 120 bits; transcript `cascade1_floor_arb_transcript.txt`, committed): 800 intervals plus a stub on $t \in (0, \pi - 0.1]$ at $\beta = 15$, worst enclosure lower bound 1.03611; 60 overlapping hull boxes along the moving boundary, worst 0.811835; and the deep tail $\pi - t \leq 10^{-3}$ by hand-monotone reductions, enclosure $[2.20442, 2.21398]$ — all $> \frac{1}{2}$, ball-and-boolean. Design calibration (verified, bearing no load): $m_{\text{true}}/m_{\text{low}} = 1.10$–$1.95$ across seven bench cells with every chain piece dominating its true counterpart, and the bench's independent $m_{\text{true}}(1.5, 15) = 1.4918$ reproduces the 1.492 of Convention 14. The chain constants of (i)–(iv) are exact consequences of the named lemmas; the elementary links are design-checked in `cascade1_signed_minoration.py` (committed with transcript; five independent audit rounds on the fabrication, their transcripts committed). $\square$

Lemma 20 (exact; two global mini-lemmas for Step 0)**.** (a) *If $c^2 \geq \frac{1}{2}$ (the whole relay range) then $R \leq 2c$, i.e. $z \leq z_s$, on all of $[-\pi, \pi]^2$: indeed $4c^2 - R^2 = 4c^2(P+Q) - 4PQ \geq 8c^2\sqrt{PQ} - 4PQ = 4\sqrt{PQ}\,(2c^2 - \sqrt{PQ}) \geq 0$ since $\sqrt{PQ} \leq 1 \leq 2c^2$.* (b) *The fluctuation $F := N - CD$ satisfies the global pointwise bound $|F| \leq 12P$: from* $\cos 2s - \cos s = -2\sin(3s/2)\sin(s/2)$ *and* $1 - \cos s = 2\sin^2(s/2)$,

$$F = -2C\sin(3s/2)\sin(s/2) - \cos\alpha\big[2C\sin^2(s/2) + \sin^2 s\big],$$

and $|\sin(3s/2)| \leq 3|\sin(s/2)|$, $\sin^2 s = 4P(1-P) \leq 4P$, $|C| \leq 1$ *give* $|F| \leq 6P + (2P + 4P) = 12P$.

Remark 21 (certified; Step 0 of the assembly: the E-window)*.* Lemma 20 makes the first assembly step a single moment estimate: $|E - C| = |\langle F\rangle|/\langle D\rangle \leq 12\langle P\rangle_K/\langle D\rangle$, with no case split (the F-bound is global, and (a) orients every kernel bound in one direction on the whole torus). The moment bound is *certified*, in the additive-mirror form the truth demands (the mirror mass sits at $P \approx 1$ and contributes $O(e^{-4\beta\delta_4}) = O(1)$ on the moving boundary, so no purely multiplicative window constant can exist near $t = \pi$): for $\beta \geq 15$, $t \in (0, \pi - C_{\text{win}}/\beta]$,

$$\frac{\langle P\rangle_K}{\langle D\rangle} \;\leq\; \frac{1+\varepsilon_b}{4\beta c} \;+\; 3\,e^{-2\beta\delta_4}, \qquad \varepsilon_b = \begin{cases} 4.0, & t \leq 2.89,\\ 8.8, & t > 2.89,\end{cases}$$

whence $|E-C| \le \frac{3}{\beta c}(1+\varepsilon_b) + 36\, e^{-2\beta\delta_4}$. The constants ride Lemma 18's chains ($\langle P\rangle \le \frac{1}{2}\int_B K(P+Q)$ plus the mirror and rest masses, $\langle D\rangle \ge m_{\mathrm{low}}$) and were certified by the same interval sweep (transcript `cascade2_step0_eps_transcript.txt`, committed; the certificate's bulk zone extends to the binary64 value 2.8999999999999999112, and 2.89 is stated so the claim stays strictly inside it). Against the exact first-order $\kappa(c) = (4c^2-1)/(2c) \le \frac{3}{2}$ the bulk constant carries a factor ≈ 10 of slack — fat, said plainly; its clients are bounded, not flattered: the $\times 2$-class μ_F input for the mirror clause rides the extracted leading coefficients, not this window. The eight-cell calibration table ($\beta|E-C| \nearrow \kappa(c)$ from below) is the step's unit test.

Lemma 22 (exact; two-term two-sided Bessel companions)**.** *For $z \ge 20$,*

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}}\Big(1 - \frac{3}{8z} + \varepsilon_1(z)\Big), \qquad I_0(z) = \frac{e^z}{\sqrt{2\pi z}}\Big(1 + \frac{1}{8z} + \varepsilon_0(z)\Big),$$

with $|\varepsilon_1(z)| \le 0.6/z^2$ and $|\varepsilon_0(z)| \le 0.4/z^2$; and, one order deeper by the same chain,

$$I_1(z) = \frac{e^z}{\sqrt{2\pi z}}\Big(1 - \frac{3}{8z} - \frac{15}{128z^2} + \varepsilon_1^{(3)}(z)\Big), \qquad I_0(z) = \frac{e^z}{\sqrt{2\pi z}}\Big(1 + \frac{1}{8z} + \frac{9}{128z^2} + \varepsilon_0^{(3)}(z)\Big),$$

with $|\varepsilon_1^{(3)}(z)| \le 1.02/z^3$ and $|\varepsilon_0^{(3)}(z)| \le 0.84/z^3$.

Proof. Both start from the representations in the proof of Lemma 15: with $u = \sin(\theta/2)$,

$$I_1(z) = \frac{2e^z}{\pi}\int_0^1 e^{-2zu^2}\,\frac{1-2u^2}{\sqrt{1-u^2}}\,du, \qquad I_0(z) = \frac{2e^z}{\pi}\int_0^1 \frac{e^{-2zu^2}}{\sqrt{1-u^2}}\,du.$$

On $[0, \frac{1}{\sqrt{2}}]$ expand the algebraic factor one order past Lemma 15: $\frac{1}{\sqrt{1-u^2}} = 1 + \frac{u^2}{2} + R_2(u)$ with the Lagrange form $R_2 = \frac{3}{8}u^4(1-\xi)^{-5/2}$, $\xi \in (0, u^2)$, so $0 \le R_2 \le \frac{3}{8}2^{5/2}u^4 \le 2.13\,u^4$ there. The completed Gaussian moments are exact, $\int_0^\infty u^{2k}e^{-2zu^2}du = \frac{(2k-1)!!}{2(4z)^k}\sqrt{\frac{\pi}{2z}}$, and produce the displayed polynomial parts: for I_1, $(1-2u^2)(1+\frac{u^2}{2}) = 1-\frac{3}{2}u^2-u^4$ gives $1-\frac{3}{8z}-\frac{3}{16z^2}$; for I_0, $1+\frac{u^2}{2}$ gives $1+\frac{1}{8z}$. Four error sources, each explicit: (i) the R_2 term, at most $\frac{3}{8}2^{5/2}\cdot\frac{3}{16z^2} = \frac{0.398}{z^2}$ relative to the $k=0$ moment (for I_1 multiplied further by $|1-2u^2| \le 1$); (ii) the exact $-u^4$ moment of I_1, $-\frac{3}{16z^2}$, absorbed into ε_1 with its sign; (iii) the Gaussian completion residues, bounded as in Lemma 15 by $\frac{3}{z}e^{-z}$-scale terms; (iv) the segment $[\frac{1}{\sqrt{2}}, 1]$, bounded by $\sqrt{2}\,e^{-z}$ for I_0 ($\frac{1}{\sqrt{1-u^2}} \le \frac{\sqrt{2}}{\sqrt{2(1-u)}}$, integrable) and by $\frac{\pi}{4}e^{-z}$ for I_1. At $z \ge 20$ the exponential sources (iii)–(iv) are below $10^{-3}/z^2$ against the prefactor $e^z/\sqrt{2\pi z}$; collecting, $|\varepsilon_1| \le \big(0.398 + 0.188 + 10^{-3}\big)/z^2 \le 0.6/z^2$ and $|\varepsilon_0| \le \big(0.398 + 10^{-3}\big)/z^2 \le 0.4/z^2$. Both truncations carry the correct next-order signatures ($-\frac{15}{128}z^{-2}$ for $\nu = 1$, $+\frac{9}{128}z^{-2}$ for $\nu = 0$; measured $\sup z^2|\varepsilon_1| = 0.1227$, $\sup z^2|\varepsilon_0| = 0.0743$): the inked constants are the derived chain's, the measurements are calibration only. For the three-term forms, one more order of the same expansion: $\frac{1}{\sqrt{1-u^2}} = 1 + \frac{u^2}{2} + \frac{3}{8}u^4 + R_3(u)$ with $R_3 = \frac{5}{16}u^6(1-\xi)^{-7/2} \le \frac{5}{16}2^{7/2}u^6 \le 3.54\,u^6$ on $[0, \frac{1}{\sqrt{2}}]$; for I_1, $(1-2u^2)(1+\frac{u^2}{2}+\frac{3}{8}u^4) = 1-\frac{3}{2}u^2-\frac{5}{8}u^4-\frac{3}{4}u^6$ gives $1 - \frac{3}{8z} - \frac{15}{128z^2} - \frac{45}{256z^3}$; the R_3 term contributes at most $3.54\cdot\frac{15}{64z^3} \le 0.83/z^3$, the exact $-\frac{45}{256}z^{-3}$ term $0.176/z^3$ with its sign, the exponential sources $10^{-2}/z^3$: $|\varepsilon_1^{(3)}| \le 1.02/z^3$. For I_0, $1 + \frac{u^2}{2} + \frac{3}{8}u^4$ gives $1 + \frac{1}{8z} + \frac{9}{128z^2}$ and $|\varepsilon_0^{(3)}| \le (0.83 + 0.01)/z^3 \le 0.84/z^3$. (Calibration: measured $z^3|\varepsilon_1^{(3)}| \le 0.111$, $z^3|\varepsilon_0^{(3)}| \le 0.080$ on $[20, 112]$ — factor 9 of chain headroom; bench transcript committed.) $\square$

Lemma 23 (exact analytic; certified constants)**.** *For $z \geq 20$, put $A(z) = e^{-z} I_1(z)/z$ and $B(z) = e^{-z} I_2(z)/z^2$. Then*

$$A(z) = \frac{z^{-3/2}}{\sqrt{2\pi}} \left(1 - \frac{3}{8z} - \frac{15}{128z^2} - \frac{105}{1024z^3} - \frac{4725}{32768z^4} + \varepsilon_A(z)\right),$$
$$B(z) = \frac{z^{-5/2}}{\sqrt{2\pi}} \left(1 - \frac{15}{8z} + \frac{105}{128z^2} + \frac{315}{1024z^3} + \frac{10395}{32768z^4} + \varepsilon_B(z)\right),$$

where

$$|\varepsilon_A(z)| \leq \frac{0.635}{z^5}, \qquad |\varepsilon_B(z)| \leq \frac{2.427}{z^5}.$$

Consequently

$$\left| z\frac{B(z)}{A(z)} - \left(1 - \frac{3}{2z} + \frac{3}{8z^2} + \frac{3}{8z^3} + \frac{63}{128z^4}\right)\right| \leq \frac{3.181}{z^5}.$$

Proof. The standard positive integral representation, followed by $u = z(1-x)$, gives

$$A(z) = \frac{\sqrt{2}}{\pi z^{3/2}} \int_0^{2z} e^{-u} u^{1/2} \left(1 - \frac{u}{2z}\right)^{1/2} du,$$
$$B(z) = \frac{2^{3/2}}{3\pi z^{5/2}} \int_0^{2z} e^{-u} u^{3/2} \left(1 - \frac{u}{2z}\right)^{3/2} du.$$

Split at $u = z$. On the first part, if $c_k = (-1)^k \binom{\alpha}{k}$ and $\alpha \in \{\frac{1}{2}, \frac{3}{2}\}$, the absolute ratio of consecutive omitted binomial terms is below $1/2$; hence the degree-four remainder is at most $2|c_5|(u/(2z))^5$. Complete the retained moments to infinity. The resulting error, with $p = \alpha$, is bounded by

$$\frac{2|c_5|\Gamma(p+6)}{(2z)^5} + \Gamma(p+1, z) + \sum_{k=0}^{4} \frac{|c_k|\Gamma(p+k+1, z)}{(2z)^k}.$$

Finally $\Gamma(a, z) \leq e^{-z} z^a/(z - a + 1)$ for $z > a - 1$: after $u = z + v$, use $(1 + v/z)^{a-1} \leq e^{(a-1)v/z}$. After multiplication by z^5, each exponential term is decreasing for $z \geq 20$, so outward-rounded evaluation at $z = 20$ gives respectively 0.634032 and 2.426710; the displayed decimals round upward. Formal division of the two degree-four polynomials gives $1, -3/2, 3/8, 3/8, 63/128$. Writing $P_B - P_A Q_4 = z^{-5} R(1/z)$, interval evaluation on $0 \leq 1/z \leq 1/20$, together with $P_A - 0.635z^{-5} > 0$, gives 3.180121 for the quotient remainder, again rounded upward. All rational coefficients, monotonicity checks, and outward-rounded endpoint evaluations are reproduced by `surface_bessel_integral_remainder.py`. □

Corollary 24 (exact; the measure ratio)**.** *Define $r(z) := \dfrac{I_2(z)}{z\, I_1(z)}$, so that $H_B(z) = r(z)\, \dfrac{I_1(z)}{z} = r(z)\, \dfrac{K}{2\beta}$ pointwise. Then for $z \geq 20$,*

$$r(z) = \frac{1}{z} - \frac{3}{2z^2} + \eta(z), \qquad |\eta(z)| \leq \frac{1.23}{z^3};$$

and, from the three-term companions,

$$\eta(z) = \frac{3}{8z^3} + \eta^{(4)}(z), \qquad |\eta^{(4)}(z)| \leq \frac{2.5}{z^4}.$$

Moreover the convergent fifth-order companion is

$$r(z) = \frac{1}{z} - \frac{3}{2z^2} + \frac{3}{8z^3} + \frac{3}{8z^4} + \frac{63}{128z^5} + \eta^{(6)}(z), \qquad |\eta^{(6)}(z)| \le \frac{3.181}{z^6}.$$

If $z = z_s\sqrt{1-w}$, $z_s \ge 40$, *and* $0 \le w \le \frac{1}{2}$, *then, with* $(q_0, \dots, q_4) = (1, -\frac{3}{2}, \frac{3}{8}, \frac{3}{8}, \frac{63}{128})$,

$$r(z) - r(z_s) = \sum_{n=0}^{4} \frac{q_n}{z_s^{n+1}} \Big((1-w)^{-(n+1)/2} - 1\Big) + \eta_\Delta, \qquad |\eta_\Delta| \le \frac{28.622}{z_s^6}.$$

Proof. $r = \frac{1}{z}\frac{I_0}{I_1} - \frac{2}{z^2}$ by $I_2 = I_0 - 2I_1/z$ [4, 10.29.1]. By Lemma 22, with $a = \frac{1}{8z} + \varepsilon_0$ and $b = -\frac{3}{8z} + \varepsilon_1$, $\frac{I_0}{I_1} - 1 = \frac{a-b}{1+b}$, and $1 + b \ge 1 - \frac{3}{160} - \frac{0.6}{400} \ge 0.9797$ at $z \ge 20$; hence $\frac{I_0}{I_1} = 1 + \frac{1}{2z} + \eta_2$ with

$$|\eta_2| = \Big|\frac{(\varepsilon_0 - \varepsilon_1) - \frac{1}{2z}\, b}{1+b}\Big| \le \frac{\frac{1.0}{z^2} + \frac{3}{16z^2} + \frac{0.3}{z^3}}{0.9797} \le \frac{1.23}{z^2},$$

and $r = \frac{1}{z}(1+\frac{1}{2z}+\eta_2) - \frac{2}{z^2} = \frac{1}{z} - \frac{3}{2z^2} + \frac{\eta_2}{z}$. For the refined form, divide the three-term companions: $\frac{I_0}{I_1} = 1 + \frac{1}{2z} + \frac{3}{8z^2} + \eta_3'$ with $|\eta_3'| \le (1.02+0.84+0.2)/(0.97\, z^3) \le 2.13/z^3$ (the 0.2 collects the cubic cross-products of the two-term coefficients), whence $\eta = \frac{3}{8z^3} + \frac{\eta_3'}{z}$ and $|\eta^{(4)}| \le 2.13/z^4 \le 2.5/z^4$. (Calibration: measured $\sup_{z\in[20,112]} z^3|\eta| = 0.395$ and $z^3\eta \to \frac{3}{8} = 0.375$ exactly as derived; measured $z^4|\eta^{(4)}| \le 0.41$ — factor 6 of chain headroom.) The last display is Lemma 23, since $r = B/A$. For the deficit form, apply its remainder at z and z_s: $z^{-6} = z_s^{-6}(1-w)^{-3} \le 8z_s^{-6}$, so their sum is at most $9(3.180121)z_s^{-6} < 28.622z_s^{-6}$, using the outward-rounded constant before its coarser three-decimal display; the five polynomial terms subtract exactly as displayed. □

Lemma 25 (conditional historical saddle extraction; not used in Theorem 2)**.** *Let* $T(c) := \dfrac{4c^2-1}{8c^3}$ $(= [\frac{1}{2} - \frac{1}{8c^2}]/c > 0$ *on the relay range) and* $s_4 := \sin(t/4)$, $\delta_4 := c - s_4 = \sqrt{2}\sin\frac{\pi - t}{4}$. Assume ($\mathrm{H_{tail}}$) (*the* β^{-4}*-tail hypothesis, stated in step* (iv)(e) *below*)*. Then for* $\beta \ge \beta_1 = 15$ *and* $t \in (0, \pi - C_{\mathrm{win}}/\beta]$,

$$\Big|\frac{4\beta^3\,(\mu_D\nu_N - \mu_N\nu_D)}{\mu_D^2} - \frac{T(c)}{\beta}\Big| \;\le\; \frac{R_1(c)}{\beta^2} \;+\; M(c,\beta),$$

where $\mu_f = \iint K f$, $\nu_f = \iint D\, H_B\, f$ (*Convention 14*), M *is the explicit mirror bound of step* (v) *below, and*

$$R_1(c) = |r_2(c)| + \frac{2\,\Theta_3(c)}{\beta_1}, \qquad r_2(c) = \frac{-8c^4 + 15c^2 - 4}{32\,c^6},$$

with Θ_3 *the explicit* β^{-3}*-bucket constant of step* (iv)*: one* Θ_3/β_1 *absorbs the extracted-chain remainder* ρ_3*, the other is exactly the budget* ($\mathrm{H_{tail}}$) *grants the unextracted tail — the two are additive and are counted separately* (*accounting repaired after external review; a joint bound* $|\rho_3| + |\tau| \le \Theta_3/(\beta_1\beta^2)$*, which would restore the sharp* R_1*, is the* β^{-4}*-bucket work that will also discharge* ($\mathrm{H_{tail}}$))*.*

Proof. First line (the half-angle identity, cited). With $C = 2c^2 - 1$ (the half-angle law), the fluctuation $F = N - CD$ of Lemma 20(b) vanishes at the saddle $s = \alpha = 0$: $N(0,0) = 2C = C\,D(0,0)$. The bilinear form is invariant under $N \mapsto N - aD$ for constant a, so N is replaced by F throughout; every leading-order product then dies at the saddle, and the first surviving order carries the lemma.

(0) Single measure. By Corollary 24, $\nu_f = \frac{1}{2\beta}\langle D\, f\, r(z)\rangle$. Freezing $r_s := r(z_s)$ and writing $\Delta r := r(z) - r_s$ splits

$$\frac{4\beta^3(\mu_D\nu_F - \mu_F\nu_D)}{\mu_D^2} = \underbrace{\frac{2\beta^2 r_s\big[\langle D\rangle\langle DF\rangle - \langle F\rangle\langle D^2\rangle\big]}{\langle D\rangle^2}}_{X_1} + \underbrace{\frac{2\beta^2\big[\langle D\rangle\langle DF\,\Delta r\rangle - \langle F\rangle\langle D^2\,\Delta r\rangle\big]}{\langle D\rangle^2}}_{X_2},$$

five moments of polynomial weights and one deficit weight, all under the single measure K.

(i) The deficit chart. $R^2 = 4c^2(1-w)$ with $w := P+Q-PQ/c^2$ *exactly*, so $z = z_s\sqrt{1-w}$; on the relay range $0 \le w \le 1$ (Lemma 20(a)), and on the saddle ball $\{w \le \frac{1}{2}\}$ every kernel prefactor is a $(1-w)$-power with two-sided binomial bounds (the bilateral root floor: for $0 \le w \le \frac{1}{2}$, $1 - \frac{w}{2} - \frac{w^2}{4} \le \sqrt{1-w} \le 1 - \frac{w}{2}$, and the $-\frac{3}{4}$, $-\frac{1}{2}$, -1 powers likewise, each with its Lagrange constant at $w = \frac{1}{2}$).

(ii) Saddle chart and extraction. With $s = \sigma/\sqrt{\beta}$, $\alpha = \tau/\sqrt{\beta}$, the exponent $z - z_s + \frac{c}{2}(\sigma^2+\tau^2)$ has *identically vanishing* constant term (the Gaussian $e^{-c\rho^2/2}$ is exact at leading order), and each of the five moments expands in $1/\beta$ with polynomial-Gaussian coefficients whose remainder constants come from three named sources only: Lemma 22 (kernel correction and ε_1), Corollary 24 (r_s and η), and the root floor of (i). The Gaussian moments are exact. The assembled coefficients were extracted symbolically and machine-verified twice with independent truncation depths (scripts `derive_page_pass2.py` and the audit's independent rebuild, archived with transcripts; the extraction is protected not by a blanket truncation ceiling but by the *asserted structural zeros* — the exponent's vanishing constant term, Gaussian parity, the bracket zeros at orders β^0 and $\beta^{-1/2}$ — together with the structural minima $F = O(\beta^{-1})$, $\Delta r = O(\beta^{-2})$):

$$\beta X_1 \;\longrightarrow\; 2\,T(c), \qquad \beta X_2 \;\longrightarrow\; -\,T(c) \qquad \text{symbolically},$$

with next coefficients $r_{2,1}(c) = \frac{-44c^4+29c^2-6}{32c^6}$, $r_{2,2}(c) = \frac{18c^4-7c^2+1}{16c^6}$, and the sum telescopes to

$$\frac{4\beta^3(\mu_D\nu_N - \mu_N\nu_D)}{\mu_D^2} = \frac{T(c)}{\beta} + \frac{r_2(c)}{\beta^2} + \rho_3, \qquad r_2 = r_{2,1} + r_{2,2},$$

on the saddle ball, with ρ_3 collected in (iv).

(iii) Why the split is not cosmetic. X_1 carries the frozen ratio: its bracket is the pure covariance whose leading cancellation is the identity of the first line — no absolute-value route survives it (the measured $\times 500$ loss of the crude covariance bound). X_2 carries the ratio variation in the deficit weight Δr, two-sidedly controlled by the root floor. The two targets $2T$ and $-T$ were pre-registered as intermediate judges with closed forms and verified independently *before* this page was written; their integer proportion is the fingerprint of the single carrier.

(iv) The remainder ρ_3, by named source — every term of order β^{-3}. (a) ε_1 enters every K-moment through the common factor $1 - \frac{3}{8z} + \varepsilon_1$; its *constant* part $\varepsilon_1(z_s)$ cancels exactly in both quotients (numerators and denominators are homogeneous of degree two in the moments), so only its variation over the ball survives — and the variation now has its *witness*: with $g_1 := z^2\varepsilon_1 = -\frac{15}{128} + z^2\varepsilon_1^{(3)}$ (three-term companion),

$$|\varepsilon_1(z) - \varepsilon_1(z_s)| \le \frac{|g_1(z) - g_1(z_s)|}{z^2} + |g_1(z_s)|\Big|\frac{1}{z^2} - \frac{1}{z_s^2}\Big| \le \frac{4.94}{z_s^3} + \frac{0.34\,w}{z_s^2}$$

on $\{w \le \frac{1}{2}\}$ ($|g_1 + \frac{15}{128}| \le 1.02/z$, $(1-w)^{-1} - 1 \le 2w$, $1/z^2 \le 2/z_s^2$): a β^{-3} perturbation of βX with constant $\le 0.5/c^3$ (both pieces). (b) the frozen tail $\eta(z_s)$ is a relative perturbation

$|\eta_s| z_s \le 1.23/z_s^2$ of r_s, hence perturbs $\beta X_1 \approx 2T$ by at most $\frac{0.154\,T(c)}{c^2}\beta^{-2}$ *on* βX, i.e. at β^{-3} on the normalized object. (c) the deficit tail, witnessed by the refined ratio of Corollary 24:

$$\Delta\eta = \frac{3}{8z_s^3}\left[(1-w)^{-3/2} - 1\right] + \Delta\eta^{(4)}, \qquad |\Delta\eta| \le \frac{1.37\,w}{z_s^3} + \frac{12.5}{z_s^4}$$

($(1-w)^{-3/2} - 1 \le 3.66\,w$ and $(1-w)^{-2} \le 4$ on $[0, \frac{1}{2}]$): at most $\frac{0.95\,T(c)}{c^2}\beta^{-2}$ on βX_2, the two pieces at the Gaussian w-moment and absolutely. The stronger convergent companion of Corollary 24 now replaces the unexpanded part of this particular source by the explicit z_s^{-4} and z_s^{-5} terms plus $28.622 z_s^{-6}$; the older display is retained as a valid coarse client bound. This removes no other exponent, geometry, or completion tail from $(\mathrm{H}_{\mathrm{tail}})$. (d) the ball's Gaussian tails and the ring and far regions are the two templated families of Lemma 10, exponentially below every polynomial source. (e) the saddle chain's own next polynomial coefficient, extracted by the same machinery one order deeper (with the kernel's next companion order $-\frac{15}{128}z^{-2}$ carried; the β^0 and β^{-1} coefficients are unchanged by it — re-verified):

$$r_3(c) = \frac{-12c^6 - 485c^4 + 796c^2 - 224}{1024\,c^9},$$

whose judge-cell values $+0.0844$, $+0.1762$, $+0.3264$, $+0.5601$ independently match the audit's measured Richardson drifts (0.112, 0.177, —, 0.572 at the three audited t's). The independent regular-coordinate engine now carries one further exact order:

$$r_4(c) = \frac{28c^8 + 41c^6 - 1464c^4 + 1856c^2 - 500}{1024\,c^{12}},$$

the coefficient of δ^3 in Y. One further manifested symbolic order gives

$$r_5(c) = \frac{12940c^{10} + 16077c^8 + 173288c^6 - 1300912c^4 + 1358400c^2 - 346112}{262144\,c^{15}},$$

the coefficient of δ^4. A coefficientwise exact-series engine, after reproducing this r_5, gives one more manifested head,

$$r_6(c) = \frac{8148c^{12} + 17095c^{10} + 10768c^8 + 634576c^6 - 2557408c^4 + 2283296c^2 - 549376}{131072\,c^{18}},$$

the coefficient of δ^5. The r_4 run reproduces T, r_2, r_3 before asserting this identity; its transcript and executed blob are owned by the manifested validator `surface_remainder_delta0_fourth_coefficient`; the next run and validator independently own r_5, while `surface_remainder_delta0_sixth_coefficient` owns r_6 together with the hash of its coefficientwise dependency. These extra heads do not alter the registered Θ_3 below and, without a uniform δ^6 remainder, carry no relay load. Collecting,

$$\Theta_3(c) \;=\; |r_3(c)| \;+\; \frac{1.10\,T(c)}{c^2} \;+\; \frac{0.5}{c^3} \;+\; 0.05,$$

and $R_1 = |r_2| + 2\Theta_3/\beta_1$ evaluates on the four judge cells ($c = 0.99,\ 0.93,\ 0.87,\ 0.81$) to 0.244, 0.322, 0.426, 0.557 — still inside the pre-registered acceptance band [residual, $3\times$ residual] at every cell (residuals 0.101–0.292, bands $[0.101, 0.303]$, $[0.144, 0.432]$, $[0.200, 0.599]$, $[0.292, 0.876]$), while r_2 alone reproduces the measured residual table: $+0.1001$, $+0.1444$, $+0.1996$, $+0.2653$.

One debt, declared and carried as the hypothesis the statement assumes: writing $\tau(\beta, c)$ for the β^{-4}-and-beyond chain tail (not separately extracted; it depends on c through the same chains), the display holds *unconditionally* with R_1/β^2 replaced by $R_1/\beta^2 + |\tau(\beta, c)|$, and the statement's assumption is exactly

$$(\mathrm{H_{tail}}): \qquad |\tau(\beta, c)| \;\le\; \frac{\Theta_3(c)}{\beta_1\,\beta^2} \quad \text{for } \beta \ge \beta_1,\ t \in (0, \pi - C_{\mathrm{win}}/\beta]$$

— i.e. the tail is absorbed by the same Θ_3 headroom its Lagrange forms ride. No inference from the direct K2 certificate to this separately defined tail is made. The superseded sharp chain instead used the joint carrier of the superseded sharp Target 27 directly.

(v) The mirror clause. Under the involution of Remark 19, $T(s, \alpha) = (\pi - s, \pi - \alpha)$: $D \circ T = -D$, $N \circ T = C(2x^2 - 1) + y(Cx + 1 - x^2)$ (its own object: only the $-1 + x^2$ block flips sign), $|F \circ T| \le 5$, and $R^2(\cdot; c) \circ T = R^2(\cdot; s_4)$ *exactly*: every mirror-region moment is a saddle-chart moment at parameter s_4, with exact relative suppression $e^{-4\beta\delta_4}$. Moreover

$$C = 2c^2 - 1 = (c - s_4)(c + s_4) = \delta_4\,(c + s_4)$$

— the half-angle key once more — so the mirror terms carrying C are uniformly tamed by $\sup_{x>0} xe^{-4x} = \frac{1}{4e}$. For $s_4 \ge \frac{1}{2}$ ($t \ge \frac{2\pi}{3}$; this zone contains the stress cell) the s_4-saddle mass obeys the derived Gaussian upper bound $\Lambda(s_4)\beta^{-3/2}e^{4\beta s_4}$ with $\Lambda(s_4) = 1.34\, s_4^{-5/2}$ (the chain constant $1.08/q(s_4)$ with $q = 1 - \frac{0.0492}{s_4^2}$ evaluated at the zone edge $s_4 = \frac{1}{2}$; at the stress cell the local value is 1.21) [chain: $K \le 2\beta e^z/(\sqrt{2\pi}z^{3/2})$ for $z \ge 20$ from Lemma 22; $z \ge z_s(s_4)/\sqrt{2} \ge 21$ on the ball at $\beta \ge 15$ by the root floor; $e^{z - z_s(s_4)} \le e^{-2\beta s_4 \tilde{w}}$ with $\tilde{w} \ge (1 - \frac{0.0492}{s_4^2})(P' + Q')$ and $P' \ge 0.2458\, s'^2$ on the ball — bench: dominates the verified-quadrature mirror-ball mass by $\times 2.7$ at the stress cell], and the Gaussian ρ'^2-moment ratio is $\le 4.7/(\beta s_4)$. With the corner expansion $|F \circ T - 4C| \le 6\rho'^2$ (Hessian sup), $r(z(\cdot; s_4)) \le \sqrt{2}/(4\beta s_4)$ on the ball, and the main-side bounds $\nu_D \le 0.715\,\mu_D/(\beta^2 c)$, $|\nu_F| \le 12.09\,\mu_D/(\beta^3 c^2)$, $|\mu_F| \le 34.2\,\mu_D/(\beta c)$ (all from the $\langle P\rangle$-chain $P \le 1.11\,w$ on the ball, $D \ge 1.8$ there, and Lemma 10 off it), the four cross terms and the second-order term assemble to the *fully derived* bound

$$\begin{aligned} M(c, \beta) = \frac{\Lambda(s_4)}{m_*}\Big[\big(\tfrac{4\sqrt{2}}{s_4} + \tfrac{11.44}{c}\big)(c + s_4)\beta\delta_4 + \tfrac{39.9}{s_4^2} + \tfrac{96.7}{c^2} + \tfrac{96.7}{c\,s_4} + \tfrac{80.7}{c\,s_4}\Big]e^{-4\beta\delta_4} \\ + \frac{10\sqrt{2}}{s_4}\Big(\frac{\Lambda(s_4)}{m_*}\Big)^2 \beta\, e^{-8\beta\delta_4} \end{aligned}$$

for $s_4 \ge \frac{1}{2}$, while for $s_4 < \frac{1}{2}$ (where $\delta_4 \ge 0.366$) the crude global bounds $K \le \beta e^{z(\cdot; s_4)}$ ($A \le \frac{1}{2}$) and $H_B \le K/(8\beta)$ (Lemma 10) give $M \le \frac{92}{m_*^2}\,\beta^{9/2}e^{-4\beta\delta_4} \le 2.1 \times 10^{-2}$, decreasing on $\beta \ge 15$. Throughout, $m_* \ge \frac{1}{2}$ is Lemma 18 (the signed minoration, discharged): M is unconditional, as is Corollary 13. □

Remark 26 (status and tricotomy of the extraction)*.* The closed forms $2T$, $-T$, $r_{2,1}$, $r_{2,2}$, r_3, r_4, r_5, and r_6 are *exact* (symbolic extraction with proved-zero asserts, machine-verified twice at independent truncation depths; trust base `sympy`, the `derive_holonomic` precedent). These head coefficients now have a third, structurally different derivation. On the regular coordinate $\delta = 1/\beta$, write $P = \sin^2(\sigma\sqrt{\delta}/2)/\delta$ and rationalize $(\sqrt{1 - w} - 1)/\delta = -(w/\delta)/(1 + \sqrt{1 - w})$ before expanding. The resulting pointwise carrier is analytic at $\delta = 0$; exact two-dimensional Gaussian moments first prove the bilinear constant term zero and then give, successively, $Y(0, t) = T(c)$,

$\partial_\delta Y(0,t) = r_2(c)$, the δ^2 coefficient $r_3(c)$, the δ^3 coefficient $r_4(c)$, the δ^4 coefficient $r_5(c)$, and the δ^5 coefficient $r_6(c)$. The independent executable derivations are `surface_remainder_delta0_first_coefficient.py` and `surface_remainder_delta0_fourth_coefficient.py`, followed by the manifested fifth- and sixth-coefficient derivators; they use the convergent companions of Lemma 23, not the older formal ε engine. This agreement strengthens the exact head; it does not provide a separate uniform bound for the historical split tail. The remainder constants of (iv)–(v) are *exact* consequences of Lemmas 22, 24 (including the three-term witnesses for the variation constants) and the root floor; M's normalization by the saddle-mass floor is Lemma 18 (unconditional); the β^{-4} chain tail is the explicit hypothesis ($\mathrm{H}_{\mathrm{tail}}$) of step (iv)(e). This historical split-tail lemma carries no load in Theorem 2; the superseded sharp Target 27 used the assembled joint remainder, whereas the theorem uses the weak-main covariance route and the finite row is certified directly. An optional sharpening of the already unconditional mirror bound is likewise omitted because it carries no relay load. The residual table, the stress-cell pieces, and the independent Richardson verification (targets hit to relative 10^{-6} at three t's, drift-halving ratios ≈ 2.00) are *verified* numerics, quoted as calibration of the pre-registered judges, bearing no load in the bound.

Target 27 (superseded sharp K2 research statement)**.** *Let $\delta = 1/\beta$, $c = \cos(t/4)$, $F = N - CD$, and, on the principal saddle square, put*

$$X_{\mathrm{main}} := \frac{4\beta^3(\mu_D\nu_F - \mu_F\nu_D)}{\mu_D^2}, \qquad Y := \beta X_{\mathrm{main}},$$

where $\mu_f = \iint Kf$ and $\nu_f = \iint DH_B f$. On the certified K2 union,

$$\left|Y(\delta,t) - T(c) - r_2(c)\delta\right| \le \Theta_3(c)\delta^2.$$

Moreover $X_{\mathrm{main}} > 0$ whenever $0 < \delta \le 9/1000$ and $\lambda = (\pi - t)/\delta \ge 3$.

Remark 28 (superseded sharp K2 certificate chain). The archived sharp-positive chain is not evidence for the displayed target. Its endpoint assembler and full-moment error propagators applied the exact leading ratio $H_0/K_0 = 1/(8\cos(t/4))$ twice. The affected values and uniform charges therefore had the wrong normalization even though their internal production/replay bookkeeping agreed. Those transcripts are superseded and no numerical margin from them is quoted here.

K4 and the S1"'/S2"' weighted-remainder judges belong only to a possible future proof of this stronger research statement. They remain unresolved and carry no load in Theorem 2, which instead uses the division-free weak-main certificate below.

Proposition 29 (certified terminal high-β relay)**.** *For every $\beta \ge 1000/9$ and every $t \in (0,\pi)$ one has $E'(t,\beta) < 0$.*

Proof. Put $\delta = 1/\beta$ and $\lambda = \beta(\pi - t)$, and write $E_t = \partial_t E = E'$. If $0 < \lambda \le 3$, Lemma 61 reduces the sign to the positivity of its cancellation-free five-family quotient. Three adjacent certified λ-lanes cover

$$[0, 3/2], \qquad [3/2, 2], \qquad [2, 3].$$

The first lane is the union of a 375-cell finite archive on $1/125 \le \delta \le 9/1000$ and a 600-cell half-line archive on $0 \le \delta \le 1/125$; the next two lanes contain 225 and 450 cells. Each validator checks the exact rational union, strict positivity of the denominator family and quotient, dependency hashes, and an independent byte-identical replay. Since $\lambda > 0$ for $t < \pi$, $-E_t/\lambda > 0$ implies $E_t < 0$.

Suppose now that $\lambda \geq 3$. Set

$$\Phi = \cos 2s + \cos\alpha\cos s, \qquad J_N = \iint N(1-P-Q)H_B, \qquad J_D = \iint D(1-P-Q)H_B,$$

and

$$\mathcal{Q} := \frac{\langle \Phi \rangle}{\langle D \rangle}, \qquad X_{\text{full}} := \frac{8\beta^3(\langle D\rangle J_N - \langle N\rangle J_D)}{\langle D\rangle^2}.$$

Lemma 6 gives the exact identity

$$\frac{E'}{-\frac{1}{2}\sin(t/2)} = \mathcal{Q} + X_{\text{full}}. \tag{3}$$

The signed-mass lemma makes every displayed denominator strictly positive. A complete high-β certificate proves

$$\mathcal{Q} > \frac{19}{20} \quad \text{on} \quad \beta \geq 1000/9, \quad 0 < t \leq \pi - \frac{3}{2\beta}.$$

For the bilinear term split the torus into the principal square B, its involutive mirror B', and the rest R. For a measurable set U write

$$\mu_h(U) := \iint_U Kh, \qquad \nu_h(U) := \iint_U DH_B h, \qquad \kappa := 4\beta^3,$$

and define the block moments

$$\begin{aligned}
(a, f, u, w) &= (\mu_D, \mu_F, \nu_D, \nu_F)(B),\\
(b, g, v, x) &= (\mu_D, \mu_F, \nu_D, \nu_F)(B'),\\
(e, k, y, z) &= (\mu_D, \mu_F, \nu_D, \nu_F)(R).
\end{aligned}$$

Thus $d_1 = a + b$, $d = d_1 + e$, $\rho = -b/a$, and

$$\begin{aligned}
C_{\text{mirror}} &:= \frac{\kappa}{d_1}\left(x - \frac{f}{a}v\right) + \frac{\kappa(u+v)}{d_1^2}\left(\frac{bf}{a} - g\right),\\
C_{\text{rest}} &:= \frac{\kappa}{d}\left(z - \frac{f+g}{d_1}y\right) + \frac{\kappa(u+v+y)}{d^2}\left(\frac{e(f+g)}{d_1} - k\right).
\end{aligned}$$

These definitions make the exact two-stage assembly

$$X_1 = \frac{X_{\text{main}}}{1-\rho} + C_{\text{mirror}}, \qquad X_{\text{full}} = \frac{d_1}{d}X_1 + C_{\text{rest}} = \frac{a}{d}X_{\text{main}} + \frac{d_1}{d}C_{\text{mirror}} + C_{\text{rest}},$$

where indeed $d_1 = a(1-\rho)$. Equivalently, if $\mathbb{P}$ is the probability measure on B with density KD/a, the division-free centred identity is

$$X_{\text{main}} = 4\beta^3 \operatorname{Cov}_{\mathbb{P}}\left(\frac{F}{D}, \frac{H_B}{K}D\right).$$

is evaluated without dividing interval balls by a parameter interval. Two adjacent 576-row Arb production/replay pairs certify

$$X_{\text{main}} > -\frac{1}{20}$$

on

$$[0, 9/1000] \times [0, 21/10] \quad \text{and} \quad [0, 9/1000] \times [21/10, 31415927/10^7].$$

Every row prints and validates explicit outward lower endpoints for the positive denominator and for X_{main}; production and replay are byte-identical. Since $\pi < 31415927/10^7$, the two rectangles cover all physical $0 < t < \pi$.

It remains only to propagate this weak premise. Put $p = \sin(t/4)$. On the common-variable lane $p \geq 101/200$, the certified bounds

$$0 \leq \rho < \frac{7}{200}, \qquad |C_{\text{mirror}}| < \frac{43}{50}, \qquad |C_{\text{rest}}| < \frac{1}{100000}, \qquad 0 < \frac{d_1}{d} < 1 + 10^{-15}$$

and the exact assembly give

$$\mathcal{Q} + X_{\text{full}} > \frac{368403499999991201}{965000000000000000} > 0.$$

On the fixed-gap lane $p \leq 101/200$, the mirror changes X_{main} by less than 10^{-30} and the rest by less than $1/100000$, so

$$\mathcal{Q} + X_{\text{full}} > \frac{19}{20} - \frac{1}{20} - 10^{-30} - \frac{1}{100000} > \frac{899}{1000} > 0.$$

The p lanes meet at $101/200$. The weak-main rectangles overlap at $t = 21/10$, and the four λ lanes meet at $3/2$, 2, and 3. Thus there is no domain gap. Because $-\sin(t/2)/2 < 0$ on $(0, \pi)$, (3) yields $E' < 0$. The executable composition `audit_surface_g2_weak_terminal_cover.py` reruns all identities, input validators, and seams.

The adaptive grid counts and the smallest printed adaptive lower endpoint are execution diagnostics of the first-passing-grid rule; they are not used as analytic margins. □

Proposition 30 (certified local box)**.** *For every $(t, \beta) \in [1.5, 1.51] \times [8, 8.05]$ one has $\langle D \rangle > 0$ and $W_c < 0$; hence, by Lemma 5, $E'(t, \beta) < 0$ on this box.*

Certificate. Ball-arithmetic machine certificate (python-flint 0.9.0 / Arb). The executed script is identified by SHA-256 prefix `1d888e99` and is archived byte-exact in the repository, together with the self-contained transcript (both file names carry the hash; the transcript's provenance header records the full hash, library versions, and stage parameters). Three stages, one transcript. *Point* $(1.5, 8)$: 1,951,141 cells, $W_c/\langle D \rangle^2 \subset [-0.4546, -0.1752]$. *Stability* (precision 120, $\Delta z \leq 0.12$): 2,500,336 cells, ratio $\subset [-0.4304, -0.1978]$, nested inside the point enclosure. *Box*: union of nine parametric sub-boxes ($\Delta\beta \leq 0.02$, $\Delta t \leq 0.005$), each with strictly negative W_c enclosure and $\langle D \rangle > 0$ (worst W_c upper end $\leq -2.8 \times 10^{21}$), 22,986,387 cells in total, 7558 s. All comparisons are strict ball comparisons; the displayed intervals are outward-rounded supersets of the printed enclosures. This proposition validates the exp-factored mechanism and calibrates its coverage design. The full row is closed independently by Proposition 32. □

Proposition 31 (certified; compact left edge)**.** *For every $3 \leq \beta \leq 20$ and $0 < t \leq 0.6$, one has $W(t, \beta) < 0$.*

Certificate. The odd function W has an exact zero of order at least three at $t = 0$. The Arb program `certify_left_edge_beta_taylor_arb.py` therefore encloses the analytic quotient W/t^3 on $[0, 0.2]$ by mixed Taylor derivatives at zero; it bounds the order-13 beta remainder through $\partial_t^3 \partial_\beta^{13} W$, so no interval containing zero is ever divided. On $[0.2, 0.6]$ it splices to the ordinary bivariate-Taylor certificate. The exhaustive 140-bit run covers 170 contiguous beta boxes, 170 normalized boxes and 883 regular boxes, with no beta-step reduction. The transcript records script hash `1ea85d53...19d6ddc` and commit `643c62d6...`; its manifest is `left-edge-beta-taylor-20260712T093905Z.json` and its validator checks the two box counts, every beta adjacency, the splice configuration, and the terminal verdict. □

Proposition 32 (certified; compact relay on $[6, 15]$)**.** *For every* $6 \le \beta \le 15$ *and* $0.6 \le t \le \pi - 1.5/\beta$*, one has* $W(t, \beta) < 0$*.*

Certificate. The outward-rounded Arb program `certify_bulk_beta_taylor_arb.py` partitions $[6, 15]$ into 90 contiguous rational intervals of width 0.1. On each interval it expands W at the exact rational midpoint to order 12 in β and order 9 in t. Central mixed derivatives are obtained from $I_n^{(q)} = 2^{-q} \sum_{j=0}^{q} \binom{q}{j} I_{n-q+2j}$, with $I_{-n} = I_n$; Lagrange remainders are bounded absolutely at the upper β endpoint. The positive coefficient tails use $I_{n+1}/I_n < x/(2(n+1))$ and $I_{n-1}/I_n = I_{n+1}/I_n + 2n/x$, giving a common geometric tail and the derivative majorant $[4(1 + 2(m + 1)/\beta_{\mathrm{lo}})]^q$. The moving upper edge is covered from outside using the rational upper bound $3.1415927 > \pi$. The run at 130-bit precision certifies 3,222 t-boxes, with no reduced β step, and terminates `CERTIFIED`. Its transcript records script SHA–256 `f69cfaad...f97b9aa`, Git commit `e60cac9e...`, Python 3.12.6 and python-flint 0.9.0. The manifest `bulk-beta-taylor-20260712T064219Z.json` owns the transcript; an executable validator checks all 90 adjacencies, the total box count, the terminal verdict, and equality of the executed script with the recorded commit blob. This proposition concerns the compact bulk row only: it does not certify the separate moving-edge constant $C_{\mathrm{win}} = 1.5$. □

Proposition 33 (certified; compact extension to $\beta = 20$)**.** *For every* $15 \le \beta \le 20$ *and* $0.6 \le t \le \pi - 1.5/\beta$*, one has* $W(t, \beta) < 0$*.*

Certificate. The same bivariate-Taylor Arb machine and tail contracts as in Proposition 32 cover 89 contiguous β-boxes and 4,736 strict t-boxes. The initial width 0.1 is reduced once, at $\beta = 16.1$, to 0.05; the transcript records that refinement and the executable validator checks the resulting mixed-width adjacency through $\beta = 20$. The run uses 130-bit Arb arithmetic, script hash `f69cfaad...f97b9aa`, and commit `668ed6ac...`. Its manifest is `bulk-beta-taylor-15-20-20260712T080953Z.json`. □

Remark 34 (what certifies, and what audits). The mathematical certificate for each sub-box is its *single* rigorous Arb enclosure: outward-rounded ball arithmetic makes one honest enclosure a proof of the strict inequalities, per sub-box. The precision-120 stability stage (with its nesting inside the point enclosure) and the *planned* pre-registered sampled `mpmath.iv` reproduction are *implementation audits*: they test the software that produced the enclosures, not the inequalities themselves, and they are declared as such in the relay table. This fixes the convention for the $[6, 15]$ coverage: one certificate-grade Arb enclosure per sub-box; stability and sampling audit the pipeline.

Lemma 35 (safe square roots in the archived run)**.** *In the archived script (*SHA-256 *`1d888e99`), the square-root guard clamps a negative lower endpoint to* 0 *and, when the* upper *endpoint ball straddles* 0*, returns the dyadic upper bound* $2^{8-p/2}$ *at working precision* $p \ge 90$*. This bound is valid throughout the run. Indeed, the guarded inputs are true squares:* $R^2 = 4[c_0^2(1-P-Q)+PQ]$ *with* $c_0^2 \le 1$*,* $P, Q \in [0, 1]$*, and* $z^2 = 4\beta^2 R^2$ *with* $\beta \le 15$*, so* $|R^2| \le 16$*,* $|z^2| \le 14400$*; the corresponding* balls *are built by at most twelve correctly rounded operations on factors of unit* *size, hence* $|R^2_{\rm ball}| < 16.01$ *and* $|z^2_{\rm ball}| < 14401 < 2^{14}$ *(magnitude means* $|\mathrm{mid}| + \mathrm{rad}$*). The upper-endpoint ball* $u = \mathrm{mid} + \mathrm{rad}$ *involves two exact arf-to-ball conversions and one correctly rounded addition, so* $\mathrm{rad}(u) \le \mathrm{ulp}(2^{14}) = 2^{15-p}$*. If* u *straddles* 0 *then* $|\mathrm{mid}(u)| \le \mathrm{rad}(u)$*, hence the true upper endpoint is at most* $2\,\mathrm{rad}(u) \le 2^{16-p}$ *and its square root is at most* $2^{8-p/2}$ *— exactly the coded constant. (In the definitive script the branch is replaced by* `abs_upper()`*, an automatic outer bound with no ad-hoc constant; the present lemma covers the archived run.)*

The two scaled functions

The certificate's precision rests on three one-line facts. *(a)* $A(z) = e^{-z} I_1(z)/z$ is strictly decreasing on $(0,\infty)$: by the classical lower bound $I_{\nu+1}(z)/I_\nu(z) \ge z/(\nu + 1 + \sqrt{(\nu+1)^2 + z^2})$, valid for $z > 0$ and $\nu \ge 0$ (Amos [1]; both members of the classical two-sided pair are proved in [2], Theorem 1), taken at $\nu = 0$, together with $\sqrt{1+z^2} \le z + 1$ (square both sides), $I_1/I_0 > z/(z+2)$, which is exactly $(\log A)' = I_0/I_1 - 1 - 2/z < 0$; hence A is enclosed over any z-interval by its two endpoint values. *(b)* The kernel–score product collapses algebraically, $K\,\partial_t \log K = 32\beta^3 \cos(t/4)(\cos(t/4))'(1 - P - Q)\,H_B(z)$ (the factor $I_1(z)/z$ cancels between K and the denominator of the regular score), so the covariance integrands involve only $B(z) = e^{-z} H_B(z)$. *(c)* B is enclosed *unconditionally* by $[e^{-z_{\rm hi}} H_B(z_{\rm lo}),\, e^{-z_{\rm lo}} H_B(z_{\rm hi})]$, since e^{-z} decreases and H_B (a positive-coefficient entire series) increases; after centering, the B-terms are subdominant and this crude-but-safe bound suffices. Together with the exact linear-exponential cell integrals, every source of exponential interval growth is either integrated analytically or carried by a monotone one-dimensional profile — the certificate is structure, not brute force.

3 The spectral framework

Convention 36 (normalizations)*.* On $(0,\pi)$ define the kernel, entry law, and smoothing operator

$$q_1(u,v) = 2\,e^{\beta\cos u\cos v}\sinh(\beta\sin u\sin v) \;=\; 4\sum_{m\ge1} I_m \sin(mu)\sin(mv), \tag{4}$$

$$s_1(u) = \tfrac{\beta}{2}\sin u\, e^{\beta\cos u} \;=\; \sum_{m\ge1} m\, I_m \sin(mu), \tag{5}$$

$$(Qf)(v) = \frac{1}{2\pi}\int_0^\pi q_1(v,u) f(u)\,du, \tag{6}$$

so that $Q\sin(m\cdot) = I_m \sin(m\cdot)$. Both closed forms are classical consequences of the generating function $e^{\beta\cos\theta} = I_0 + 2\sum_{m\ge1} I_m \cos(m\theta)$; (5) follows by differentiation. All ten verification artifacts in the repository use exactly these normalizations.

Since $q_1 > 0$ on $(0,\pi)^2$ and $s_1 > 0$ on $(0,\pi)$, the walk with step kernel q_1 killed on $\{0,\pi\}$ is well defined; $H(\psi) = \beta\sin(\psi/2) I_1(2\beta\cos(\psi/2))$ and $K_2(\varphi) = I_0(2\beta\cos(\varphi/2))$ appear below. Note $F_B = Q^3 s_1$ and, by the sine orthogonality computation of Lemma 43, F_A is the companion cosine-weighted integral.

4 Theorem A: positivity of F_B

Theorem 37 (exact)**.** *$F_B(t) > 0$ for all $t \in (0,\pi)$ and all $\beta > 0$.*

First proof (kernel positivity). $F_B = Q^3 s_1$ by (6) and $b_m = mI_m^4$. Each application of Q integrates a positive function against the strictly positive kernel (4); $s_1 > 0$ on $(0,\pi)$. Hence $F_B > 0$ pointwise. $\square$

Second proof (symmetric-decreasing convolution). With $K_2(\varphi) = I_0(2\beta\cos(\varphi/2))$ and $P_4 = K_2 * K_2$ (circle convolution; Section 7), $F_B = -\frac{1}{2}P_4'$. The function K_2 is even and *strictly decreasing in* $|\varphi|$ *on* $[0,\pi]$ (I_0 strictly increasing, $\cos(\varphi/2)$ strictly decreasing). By the layer-cake representation, $K_2 = \int_0^\infty \mathbf{1}_{A_\lambda}\, d\lambda$ with A_λ centered arcs whose half-lengths $a(\lambda)$ fill $(0,\pi)$ continuously (strict decrease). For centered arcs, $(\mathbf{1}_A * \mathbf{1}_B)(t) = |A \cap (t+B)|$ is even and nonincreasing in $|t|$ on $[0,\pi]$, and *strictly* decreasing on $||a-b|, \min(a+b, 2\pi-a-b)|$. Hence $P_4(t) = \iint |A_\lambda \cap (t+A_\mu)|\, d\lambda\, d\mu$ is even, nonincreasing on $[0,\pi]$, and strictly decreasing on $(0,\pi)$ because for every $t \in (0,\pi)$ a positive measure of pairs (λ,μ) has $|a(\lambda)-a(\mu)| < t < \min(a+b, 2\pi-a-b)$ (the half-lengths fill $(0,\pi)$). At the derivative level: each overlap $t \mapsto |A_\lambda \cap (t+A_\mu)|$ is Lipschitz with a.e. derivative -1 in the strict regime and 0 otherwise (on $[0,\pi]$); since P_4 is analytic, dominated convergence gives $P_4'(t) = \iint \partial_t |A_\lambda \cap (t+A_\mu)|\, d\lambda\, d\mu \le -\mu\{(\lambda,\mu) : \text{strict at } t\} < 0$ pointwise on $(0,\pi)$. Thus $F_B = -\frac{1}{2}P_4' > 0$ on $(0,\pi)$. $\square$

5 Theorem B: ratio monotonicity for $0 < \beta \le 3$

Lemma 38 (exact; Wronskian identity)**.** *For real arrays with* $\sum m(|a_m| + |b_m|) < \infty$*,*

$$W := 2(F_A' F_B - F_A F_B') = \sum_{1 \le m < n} c_{mn}\, T_{mn}(t), \qquad c_{mn} = a_m b_n - a_n b_m, \tag{7}$$

where $T_{mn}(t) = (m-n)\sin((m+n)t) + (m+n)\sin((n-m)t)$*.*

Proof. Product-to-sum: $T_{mn} = 2[m\cos(mt)\sin(nt) - n\sin(mt)\cos(nt)] =: 2G(m,n)$, antisymmetric with $G(n,n) = 0$; since c_{mn} is antisymmetric, $\sum_{m<n} c_{mn} 2G = 2\sum_{m,n} a_m b_n G(m,n) = 2[F_A' F_B - F_A F_B']$ (first recorded in [7]). $\square$

The coefficient input is the weighted Turán-type lemma (Appendix A, self-contained): $\varphi_m = a_m/b_m$ is strictly increasing in $m \ge 1$ for every $\beta > 0$, hence *every minor is negative*: $c_{mn} < 0$ for $m < n$ (exact).

Lemma 39 (exact; the pair identities and bounds)**.** *For* $t \in (0,\pi)$*, with* $p = n-m$*,* $q = n+m$*:*

(a) $T_{12} = 4\sin^3 t$*;* $T_{13} = 16\sin^3 t\cos t$*;* $T_{24} = 8\sin^3(2t)$*, so* $|T_{13}| \le 16\sin^3 t$ *and* $|T_{24}| \le 64\sin^3 t$*.*

(b) $T_{m,m+1} \ge 0$ *for every* $m \ge 1$*.*

(c) $|T_{mn}| \le \dfrac{\pi^3}{48}\, pq(q^2+p^2)\, \sin^3 t$ *for all* $m < n$*.*

Proof. (a) $3\sin u - \sin 3u = 4\sin^3 u$ gives T_{12} (at $u = t$) and T_{24} (at $u = 2t$); $T_{13} = 4\sin 2t - 2\sin 4t = 4\sin 2t(1-\cos 2t) = 16\sin^3 t\cos t$. (b) $T_{m,m+1} = (2m+1)\sin t - \sin((2m+1)t) \ge 0$ since $|\sin(qt)| \le q\sin t$ on $(0,\pi)$ (induction on q). (c) T_{mn} and its first two derivatives vanish at $t = 0$ and $t = \pi$ (the parities of $m \pm n$ agree), and $|T_{mn}'''| \le pq^3 + qp^3 = pq(q^2+p^2)$; hence $|T_{mn}(t)| \le \frac{pq(q^2+p^2)}{6}\min(t, \pi-t)^3$, and $\sin t \ge \frac{2}{\pi}\min(t,\pi-t)$ gives the claim. $\square$

Theorem 40 (exact modulo the certified computation `certify_thmB.py`). *For $0 < \beta \le 3$: $W < 0$ on $(0, \pi)$, hence $E' < 0$ on $(0, \pi)$.*

Proof. Drop the adjacent pairs $m \ge 2$ (each contributes $c_{m,m+1}T_{m,m+1} \le 0$ by Lemma 39(b) and $c < 0$); keep T_{12}, T_{13}, T_{24} exactly and majorize every other pair by Lemma 39(c). Pointwise in t,

$$\frac{W(t)}{\sin^3 t} \;\le\; \mathrm{Crit}(\beta) := -4|c_{12}| + 16|c_{13}| + 64|c_{24}| + \frac{\pi^3}{48} \sum_{\substack{p \ge 2 \\ (m,n) \ne (1,3),(2,4)}} pq(q^2 + p^2)\,|c_{mn}|.$$

The certificate `certify_thmB.py` (Section 10) encloses Crit in interval arithmetic over adaptive β-boxes covering $[1/20, 3]$ (I_m increasing in β gives rigorous boxes; cutoff $M = 60$ with the tail itself computed in intervals and closed by a derived, per-box-asserted geometric ratio) and certifies $\mathrm{Crit} < 0$ on all of $[1/20, 3]$; the script then re-certifies *every* box at precision $+70$ before reporting success (the box count is invocation- and implementation-dependent; the independent Arb implementation `certify_thmB_arb.py` certifies the same coverage with 86 boxes and a full stability pass, transcript archived). For $0 < \beta \le 1/20$, Lemma 41 applies. □

Lemma 41 (exact; small β). *For $0 < \beta \le 1/20$:* $\mathrm{Crit}(\beta) < 0$.

Proof. Write $\kappa = \beta/2 \le 1/40$, $L_m = \kappa^m/m!$, $U_m = e^* L_m$ with $e^* = e^{\beta^2/4} \le e^{1/1600} < 1.001$; then $L_m \le I_m \le U_m$ (positive series). *Leading minor:* $a_2 b_1 \ge (I_1^2 I_2^2)(I_1^4) \ge L_1^6 L_2^2 = \kappa^{10}/4$ and $a_1 b_2 = (2I_1^2 I_2^2)(2I_2^4) \le 4U_1^2 U_2^6 = (e^*)^8 \kappa^{14}/16$, so

$$|c_{12}| \;\ge\; \frac{\kappa^{10}}{4}\Big(1 - \frac{(e^*)^8 \kappa^4}{4}\Big) \;\ge\; \frac{\kappa^{10}}{4}\,(1 - 10^{-6}).$$

The three visible positive terms: $a_3 \le I_3^2 \cdot (2I_2^2 + 4I_4^2) \le (e^*)^4 \kappa^{10}/24 \cdot (1 + 10^{-4})$ and $b_1 \le (e^*)^4 \kappa^4$ give $16|c_{13}| \le 16(a_3 b_1 + a_1 b_3) \le 0.70\,\kappa^{14}$; $64|c_{24}| \le 64(a_4 b_2 + a_2 b_4) \le 10^{-3}\kappa^{14}$ (both products have κ-degree ≥ 22); the largest remaining weighted pair is $(1,4)$: $\frac{\pi^3}{48} \cdot 510 \cdot (a_4 b_1 + a_1 b_4) \le 0.2\,\kappa^{18} \le 10^{-6}\kappa^{14}$. *All other pairs* (those with $s := m + n \ge 6$, $p \ge 2$): using $a_n b_m + a_m b_n \le 2mn(e^*)^8 \kappa^{4s-2}[((m-1)!m!)^{-2} n!^{-4} + ((n-1)!n!)^{-2} m!^{-4}] \le s^2 (e^*)^8 \kappa^{4s-2}$ (the factorial bracket is ≤ 2 and $2mn \le s^2/2$), $pq(q^2 + p^2) \le 2s^4$, and at most s pairs per level,

$$\frac{\pi^3}{48}\sum_{s \ge 6}(\text{level } s) \le \frac{\pi^3}{48}(e^*)^8 \sum_{s \ge 6} 2s^7 \kappa^{4s-2} \le 0.66\,\kappa^{22} \cdot 6^7 \sum_{j \ge 0} [(7/6)^7 \kappa^4]^j \le 3.7 \times 10^5\,\kappa^{22} \le 10^{-7}\kappa^{14},$$

where the level ratio $s \mapsto s + 1$ is at most $(7/6)^7 \kappa^4 < 10^{-5} < \frac{1}{2}$ (so the geometric closure is valid), and $\kappa^8 \le 40^{-8}$ absorbs the constant. Collecting:

$$\mathrm{Crit} \le -\kappa^{10}(1 - 10^{-6}) + 0.71\,\kappa^{14} = -\kappa^{10}\big(1 - 10^{-6} - 0.71\,\kappa^4\big) < 0,$$

since $\kappa^4 \le 40^{-4} < 4 \cdot 10^{-7}$. (All numbered constants above are crude upper counts; each step loses factors, none gains.) □

Remark 42 (verified; the sharp family and saturation). The sharper per-pair bound $|T_{mn}| \le \frac{pq(q^2 - p^2)}{6} \sin^3 t$ and the adjacent floor $T_{m,m+1} \ge 2m \sin^3 t$ (note the sign: the floor is a *lower* bound on a positive quantity) are verified far beyond the range used ($q \le 151$; 13 pairs to $(10, 30)$, max ratio 1.0) and would extend Theorem 40 to $\beta \approx 3.5$; their per-pair certification is a scheduled upgrade. Beyond that the decomposition is *saturated*: with the exact far block (cancellation included) the criterion fails at $\beta = 6$ by $+2.2\times 10^{14}$ while the true $W/\sin^3 t$ remains negative (-6.9×10^{12}) — a verified adversarial finding about the method, and the reason the large-β closure uses different machinery (Sections 7–9).

6 The bridge structure of E (exact)

Let $k = q_1 \circ q_1$ denote the two-step kernel, $k(\psi, t) = 4\sum_{m\geq 1} I_m^2 \sin(m\psi)\sin(mt)$, and define

$$\omega_t(\,d\psi) \;\propto\; (Qs_1)(\psi)\, k(\psi, t)\, d\psi \qquad \text{on } (0, \pi), \tag{8}$$

the law of the time-2 position (midpoint) of the four-step bridge entering from 0 with law s_1 and conditioned to end at t.

Lemma 43 (exact)**.** $\displaystyle\int_0^\pi (Qs_1)\, k(\cdot, t)\, d\psi \;=\; 2\pi F_B(t)$ *and* $\displaystyle\int_0^\pi \cos\psi\,(Qs_1)\, k(\cdot, t)\, d\psi \;=\; \pi F_A(t)$. *Hence, as an identity,*

$$E(t) \;=\; \mathbb{E}_{\omega_t}[\cos\psi] \;=\; \frac{F_A(t)}{2F_B(t)}.$$

Proof. $(Qs_1)(\psi) = \sum m I_m^2 \sin(m\psi)$. By sine orthogonality the first integral picks the diagonal, giving $2\pi \sum m I_m^4 \sin(mt)$. In the second, $\cos\psi\, \sin(m\psi)\sin(n\psi)$ integrates to $\frac{\pi}{4}(\delta_{n,m+1}+\delta_{n,m-1})$, which produces exactly the weights $(n-1)I_{n-1}^2 + (n+1)I_{n+1}^2$ of a_n. ☐

Corollary 44 (exact; $|E| < 1$)**.** *For all $t \in (0, \pi)$ and $\beta > 0$, $|E(t, \beta)| < 1$. Indeed $q_1(u, v) = 2e^{\beta\cos u\cos v}\sinh(\beta\sin u\sin v) > 0$ on $(0, \pi)^2$ and $s_1 > 0$, so $(Qs_1) > 0$ and $k = q_1 \circ q_1 > 0$ on $(0, \pi)$; hence ω_t in (8) is a genuine probability measure and $E = \mathbb{E}_{\omega_t}[\cos\psi] \in (-1, 1)$.*

Lemma 45 (exact; entry law)**.** $\partial_t k(0^+, \psi) = 2H(\psi)$, *where* $H(\psi) = \beta\sin(\psi/2)\, I_1(2\beta\cos(\psi/2))$.

Proof. Differentiate in ψ the squared Graf identity $I_0^2 + 2\sum_{m\geq 1} I_m^2 \cos(m\psi) = I_0(2\beta\cos(\psi/2))$ [4, §10.23(ii)] and compare with $\partial_t k(0^+, \psi) = 4\sum m I_m^2 \sin(m\psi)$. ☐

Proposition 46 (certified negative; no monotone full-path coupling)**.** *At $\beta = 20$ the time-3 marginal $\rho_t(u) \propto (Q^2 s_1)(u)\, q_1(u, t)$ is not stochastically monotone in t: with $a = \frac{4802}{10^4}$, $t_1 = \frac{29621}{10^4}$, $t_2 = \frac{30070}{10^4}$, the tail masses satisfy $T(t_2, a) < T(t_1, a)$, with certified enclosure*

$$\begin{aligned} T(t_2, a) - T(t_1, a) \in [&-2.52991509800817866900281 07, \\ &-2.5299150980081786690028074] \times 10^{-10}. \end{aligned}$$

Consequently no monotone coupling of full bridge paths exists at $\beta = 20$. The certificate uses exact termwise tails ($J(m, n, a) = \int_a^\pi \sin mu \sin nu\, du$ in closed form: no quadrature error), positive-series Bessel enclosures with geometric tails, explicit truncation bounds, nested passes $(M, \text{prec}) = (120, 350) \subset (140, 500)$, and two independent interval implementations (`mpmath.iv`; Arb via `python-flint`).

Proposition 47 (certified negatives; the weight–kernel matrix)**.** *At $\beta = 20$, for every entry weight $Q^{k-1}s_1$ ($k = 1, 2, 3, 4$) the marginal one q_1-step from the moving endpoint fails stochastic monotonicity, with certified witnesses sharing $(t_1, t_2) = (3, 61/20)$: $k = 1$ (two-step bridge midpoint, the minimal counterexample), $a = 9/10$, margin $\approx -1.34 \times 10^{-2}$; $k = 2$, $a = 3/4$, $\approx$*

-1.80×10^{-6}*;* $k=4$*,* $a=3/10$*,* $\approx -4.75\times10^{-13}$ *(*$k=3$ *is Proposition 46). Two smoothing steps repair all tested weights (verified); one never does. The statement is relative to the natural weight family:* k_2 *possesses its own total-positivity corner (*$\beta \gtrsim 8$*) reachable by two-atom adversarial weights.*

Lemma 48 (exact; the 3/2 mechanism)**.** $\lim_{u\downarrow 0} \dfrac{\partial_u\partial_t \log q_1(u,t)}{\beta \sin t\, u} = 1 + \dfrac{2\beta}{3}\cos t$*. For* $t > \pi/2$ *the corner sign change of* $\partial_u\partial_t \log q_1$ *occurs exactly when* $\beta|\cos t| > 3/2$ *— the same threshold as the* TP_2 *failure of* q_1*.*

Proof. $\partial_t \log q_1 = -\beta \sin t \cos u + \beta \cos t \sin u \coth(\beta \sin t \sin u)$ (direct differentiation of (4)); expand $\coth z = z^{-1} + z/3 + O(z^3)$ and differentiate in u; the stated limit is a one-line computation (symbolic residual zero). □

Remark 49 (verified; the mechanism reading)*.* The identification of this local defect as *the* engine of Propositions 46–47 — sine-type entry weights, vanishing linearly at 0 with mass on both sides of the non-monotone region, flip the covariance $\mathrm{Cov}_{\rho_t}(\mathbf{1}_{u\geq a}, \partial_t \log q_1)$, while the flat weight does not — is supported by the full weight–kernel matrix and the β-scan of the violations (onset at $\beta \gtrsim 12$, consistent with the threshold), but is a verified reading, not a theorem.

7 The single-Bessel reduction and the master formula

Lemma 50 (exact; Neumann product formulas)**.** *For* $x, y \geq 0$*, with* $w = \sqrt{x^2+y^2+2xy\cos\alpha}$*,*

$$I_0(x)I_0(y) = \frac{1}{2\pi}\int_0^{2\pi} I_0(w)\, d\alpha, \qquad I_0(x)I_1(y) = \frac{1}{2\pi}\int_0^{2\pi} I_1(w)\, \frac{y + x\cos\alpha}{w}\, d\alpha.$$

Proof. The first is the integrated Graf/Neumann addition theorem [4, §10.23(ii)]; the second is its ∂_y-derivative. □

Remark 51 (signs)*.* Both formulas extend to all real x, y, as used below (on $[-\pi,\pi]^2$ one has $c_1 = \cos(\varphi/2) \geq 0$ but c_2 may be negative): $w(x,-y,\alpha) = w(x,y,\pi-\alpha)$, and the substitution $\alpha \mapsto \pi - \alpha$ over a full period maps the first identity to itself (both sides even in y) and the second to its negative on both sides simultaneously (I_1 odd; $(-y + x\cos(\pi-\alpha)) = -(y + x\cos\alpha)$).

With $K_2(\varphi) = I_0(2\beta\cos(\varphi/2)) = \sum_{m\in\mathbb{Z}} I_m^2 e^{im\varphi}$ and $P_4 = K_2 * K_2$ (circle convolution), $P_4 = \sum_m I_m^4 e^{imt}$ and $F_B = -\frac{1}{2}P_4'$. Applying Lemma 50 to both convolution factors:

Lemma 52 (exact; master representation)**.** *Let* $s = \varphi - t/2$*,* $c_1 = \cos(\varphi/2)$*,* $c_2 = \cos((t-\varphi)/2)$*,* $P = \sin^2(s/2)$*,* $Q = \sin^2(\alpha/2)$*, and*

$$R^2 = c_1^2 + c_2^2 + 2c_1c_2\cos\alpha = 4\big[\cos^2(t/4)\,(1-P-Q) + PQ\big]. \tag{9}$$

Then

$$F_B(t) = \frac{\beta\sin(t/2)}{8\pi^2}\iint_{[-\pi,\pi]^2} \frac{I_1(2\beta R)}{R}\,\big[\cos^2(s/2) - \sin^2(\alpha/2)\big]\, ds\, d\alpha, \tag{10}$$

and, under the common positive kernel $K = I_1(2\beta R)/R$*,*

$$E(t) = \frac{\langle \cos(t/2)\cos 2s + \cos\alpha\,[\cos(t/2)\cos s - \sin^2 s]\,\rangle}{\langle \cos s + \cos\alpha \,\rangle}, \tag{11}$$

where $\langle\cdot\rangle$ *denotes the* $K\,ds\,d\alpha$*-integral over* $[-\pi,\pi]^2$*. The prefactor collapse behind* (10) *is the sum-to-product symmetrization* $\frac{1}{2}[\sin(\frac{t-\varphi}{2})(c_2+c_1\cos\alpha)+(\varphi\leftrightarrow t-\varphi)] = \frac{1}{2}\sin(t/2)(\cos s+\cos\alpha)$*, and* $\cos s+\cos\alpha = 2[\cos^2(s/2)-\sin^2(\alpha/2)]$.

Proof. (i) The squared Graf identity gives $K_2(\varphi)=\sum_m I_m^2 e^{im\varphi}$; convolution multiplies Fourier coefficients, so $P_4=\sum_m I_m^4 e^{im\varphi}$ and, termwise (super-exponential decay), $F_B=-\frac{1}{2}P_4'$. (ii) Insert Lemma 50 (first formula) with $x=2\beta c_1$, $y=2\beta c_2$ into $P_4(t)=\frac{1}{2\pi}\int K_2(\varphi)K_2(t-\varphi)\,d\varphi$: a two-dimensional integral of $I_0(2\beta R)$ with R as in (9); the (P,Q) form of R^2 is product-to-sum. (iii) Differentiate under the integral (smooth integrand, compact domain): $\partial_t I_0(2\beta R) = I_1(2\beta R)\cdot 2\beta\,\partial_t R$ and $2R\,\partial_t R = 2(c_2+c_1\cos\alpha)\,\partial_t c_2$ with $\partial_t c_2 = -\frac{1}{2}\sin((t-\varphi)/2)$. (iv) Symmetrize over $\varphi\leftrightarrow t-\varphi$ (R invariant): with $\sin(x/2)\cos(x/2)=\frac{1}{2}\sin x$ and the angle-addition formula,

$$\begin{aligned}\tfrac{1}{2}\big[\sin(\tfrac{t-\varphi}{2})(c_2+c_1\cos\alpha)+\sin(\tfrac{\varphi}{2})(c_1+c_2\cos\alpha)\big] &= \tfrac{1}{4}[\sin(t-\varphi)+\sin\varphi]+\tfrac{1}{2}\sin(\tfrac{t}{2})\cos\alpha\\ &= \tfrac{1}{2}\sin(\tfrac{t}{2})(\cos s+\cos\alpha),\end{aligned}$$

using $\sin(t-\varphi)+\sin\varphi = 2\sin(t/2)\cos s$ and $\sin(\frac{t-\varphi}{2})c_1+\sin(\frac{\varphi}{2})c_2=\sin(t/2)$. Collecting constants yields (10). The same substitution in the companion representation of F_A (Lemma 43 with Lemma 50, second formula) and cancellation of the common factor $\beta\sin(t/2)$ yields (11). As an end-to-end check, (10) and (11) match the quartic series with ratio 1.000000000000 at $t\in\{0.4,1.1,2.3,2.9\}$, $\beta=4$. ☐

Lemma 53 (exact; geometry of the (P,Q) square, $0<t<\pi$)**.** *On* $[0,1]^2$*: the main saddle is* $(0,0)$ *with* $R=2\cos(t/4)$*; the secondary saddle is* $(1,1)$ *with* $R=2\sin(t/4)$ *and gap* $2(\cos(t/4)-\sin(t/4))>0$ *degenerating only at* $t=\pi$*; the zero set* $\{R=0\}$ *consists of exactly the two corners* $(1,0)$ *and* $(0,1)$*, and at both of them* both *brackets of* (10) *and* (11) *vanish, so the singular points of* K *are switched off (the extension* $I_1(z)/z\to\frac{1}{2}$ *makes* K *continuous with value* β*). At* $t=0$ *the zero set degenerates to the edges* $P=1$*,* $Q=1$*. The saddle deficit is the exact sum of two squares*

$$4\cos^2(t/4)-R^2 = 4\cos^2(t/4)\sin^2(s/2)+4c_1c_2\sin^2(\alpha/2), \qquad c_1c_2=\cos^2(t/4)-\sin^2(s/2), \quad (12)$$

with sign region $c_1c_2\ge 0 \iff |s|\le \pi-t/2$.

Proof. (9) and (12) are product-to-sum computations (symbolic zeros). $R^2\ge(|c_1|-|c_2|)^2$ with equality forcing $|c_1|=|c_2|$ (i.e. $s=0$ or $s=\pm\pi$) and $\cos\alpha=\mp\mathrm{sign}(c_1c_2)$, which lands exactly on $(P,Q)\in\{(0,1),(1,0)\}$. Bracket vanishing at both points is direct substitution. The Jacobian of $(s,\alpha)\mapsto(P,Q)$ carries the arcsine weight $ds\,d\alpha = dP\,dQ/\sqrt{P(1-P)Q(1-Q)}$ (per quadrant), singular at all four corners; near the main saddle one therefore works in (s,α), and in (P,Q) only for the global geometry. ☐

Lemma 54 (exact; saddle cancellations)**.** *Write* N,D *for numerator and denominator brackets of* (11)*,* $C=\cos(t/2)$*,* $c_0=\cos(t/4)$*. At the saddle* $(s,\alpha)=(0,0)$*:*

$$\frac{N}{D}\Big|_{(0,0)} = C, \qquad N_{\alpha\alpha}-C\,D_{\alpha\alpha}=0, \qquad N_{ss}-C\,D_{ss}=-4C-2,$$

and for $N_t=-\frac{1}{2}\sin(t/2)[\cos 2s+\cos\alpha\cos s]$*:*

$$N_{t,\alpha\alpha}-m\,D_{\alpha\alpha}=0 \quad (m=-\tfrac{1}{2}\sin(t/2)), \qquad N_{t,ss}-m\,D_{ss}=2\sin(t/2).$$

Also exact: $2C+1=4c_0^2-1$ *and* $C\,c(t)=2c_0-1/(2c_0)$ *for* $c(t)=(4c_0^2-1)/(2c_0C)$.

Proposition 55 (verified second-order laws)**.** *The frozen-kernel expansion built on Lemma 54 (with $\langle s^2\rangle = 1/(\beta c_0) + O(\beta^{-2})$ from (12)) predicts*

$$E = C\Big(1 - \frac{c(t)}{\beta}\Big) + O(\beta^{-2}), \qquad E' = -\tfrac{1}{2}\sin(t/2) + \frac{\sin(t/4)(4c_0^2+1)}{8\beta c_0^2} + O(\beta^{-2}). \tag{13}$$

The coefficient identities are exact (Lemma 54); the laws with remainder *are verified, not yet certified: measured remainders are $\approx 0.15/\beta^2$ with true convergence at fixed t (e.g. β^2-scaled residues* 0.1776/0.1767/0.1762 *at* $t = 1$, $\beta = 30/60/120$*), and* $\beta\,(E' + \frac{1}{2}\sin(t/2)) \to \sin(t/4)(4c_0^2 + 1)/(8c_0^2)$ *at $t = 1, 2$. The remainder control is exactly what the closure map owes. Note the $1/\beta$ correction to E' is* positive*: it opposes the target inequality and must be beaten, not used. The α-cancellations imply every kernel correction multiplies a vanishing fluctuation and enters at $O(\beta^{-2})$ only.*

Remark 56 (exact decomposition of E')*.* With the normalized positive measure $d\mu \propto K\,ds\,d\alpha$, $E' = \langle N_t\rangle/\langle D\rangle + \langle (N - ED)\,\partial_t \log K\rangle/\langle D\rangle$, $D_t = 0$, and the regular form $\partial_t \log K = 4cc'(1-P-Q)\,[zI_0(z)/I_1(z) - 2]/R^2$, $z = 2\beta R$, extends continuously by $4\beta^2 cc'(1-P-Q)$ at $R = 0$. Verified split at $(\beta,t) = (5, 1.3)$: $-0.238212 - 0.024552 = -0.262764$ = the finite difference exactly. Bounds for E' are taken from this exact quotient — never by differentiating an estimate of E.

8 The π-endpoint package

Lemma 57 (exact; telescoping)**.** *With a_m as in (1):*

$$A_\infty := \sum_{m\ge 1} (-1)^{m+1} m\, a_m = 0,$$
$$\sum_{m\ge 1} (-1)^{m+1} m^3 a_m = -\sum_{m\ge 1} (-1)^{m+1} m(m+1)(2m+1)\, I_m^2 I_{m+1}^2.$$

Hence $F_A(\pi - \varepsilon) = c_3\,\varepsilon^3 + O(\varepsilon^5)$ with

$$c_3 = \frac{1}{6}\sum_{m\ge 1} (-1)^{m+1} m(m+1)(2m+1)\, I_m^2 I_{m+1}^2. \tag{14}$$

Proof. Split $m^r a_m = m^r(m-1)I_{m-1}^2 I_m^2 + m^r(m+1) I_m^2 I_{m+1}^2$ and shift $m \to m+1$ in the first sum: for $r = 1$ the two weights cancel ($m(m+1) - m(m+1) = 0$); for $r = 3$ they combine to $m(m+1)[m^2 - (m+1)^2] = -m(m+1)(2m+1)$. Absolute convergence justifies the reindexing; $\sin(m(\pi - \varepsilon)) = (-1)^{m+1}\sin(m\varepsilon)$ expands the sine. □

Lemma 58 (exact integral form)**.** *Let* $g(u) = \frac{1}{2} I_1(2\beta\cos u) = \sum_{m\ge 0} I_m I_{m+1}\cos((2m+1)u)$ (addition theorem)*. Then*

$$c_3 = \frac{S_3 - S_1}{24}, \qquad S_1 = \frac{2}{\pi}\int_0^\pi g(\tfrac{\pi}{2} - u)\, g'(u)\, du, \quad S_3 = \frac{2}{\pi}\int_0^\pi g'(\tfrac{\pi}{2} - u)\,(-g''(u))\, du.$$

Moreover $g(-u) = g(u)$ and $g(\pi - u) = -g(u)$ (parity of I_1), hence $g'(\pi - u) = g'(u)$, $g''(\pi - u) = -g''(u)$, and both *integrands are invariant under $u \mapsto \pi - u$: the two halves $[0, \pi/2]$ and $[\pi/2, \pi]$ contribute exactly equally, and one may integrate a half range and double. In particular no cancellation between the two phase maxima $u = \pi/4$ and $u = 3\pi/4$ is possible — exactly, not asymptotically.*

Proof. The expansion of g is the Graf addition theorem with unit shift. Quarter turn: $\cos((2m+1)(\frac{\pi}{2}-u)) = (-1)^m \sin((2m+1)u)$, so $g(\frac{\pi}{2}-u) = \sum_m (-1)^m I_m I_{m+1} \sin((2m+1)u)$ while $g'(u) = -\sum_k (2k+1) I_k I_{k+1} \sin((2k+1)u)$ and $g''(u) = -\sum_k (2k+1)^2 I_k I_{k+1} \cos((2k+1)u)$, whence $g'(\frac{\pi}{2}-u) = -\sum_k (-1)^k (2k+1) I_k I_{k+1} \cos((2k+1)u)$. Orthogonality of $\{\sin((2m+1)u)\}$ and $\{\cos((2m+1)u)\}$ on $[0,\pi]$ (norm $\pi/2$) gives $S_1 = -\sum_m (-1)^m (2m+1)(I_m I_{m+1})^2$ and $S_3 = -\sum_m (-1)^m (2m+1)^3 (I_m I_{m+1})^2$, so

$$\begin{aligned}\frac{S_3 - S_1}{24} &= \sum_{m\geq 0} (-1)^{m+1} \frac{(2m+1)^3 - (2m+1)}{24} (I_m I_{m+1})^2 \\ &= \frac{1}{6} \sum_{m\geq 1} (-1)^{m+1} m(m+1)(2m+1) I_m^2 I_{m+1}^2 = c_3,\end{aligned}$$

using $(2m+1)^3 - (2m+1) = 4m(m+1)(2m+1)$ and the vanishing of the $m=0$ term. The parity claims follow from $I_1(-z) = -I_1(z)$ applied to $g(\pi - u) = \frac{1}{2} I_1(-2\beta \cos u)$, and by differentiating the two symmetry relations. □

Lemma 59 (exact; shifted right-edge correlation)**.** *Put* $h_n = I_n(\beta) I_{n+1}(\beta)$ *and*

$$S(x) = \frac{2}{\pi} \int_0^\pi g(\tfrac{\pi}{2} - u) g(u - x)\, du, \qquad g(u) = \tfrac{1}{2} I_1(2\beta \cos u).$$

Then

$$S(x) = \sum_{n\geq 0} (-1)^n h_n^2 \sin((2n+1)x)$$

and, for $d \in [0,\pi]$,

$$F_A(\pi - d) = \sin(\tfrac{d}{2}) S'(\tfrac{d}{2}) - \cos(\tfrac{d}{2}) S(\tfrac{d}{2}). \tag{15}$$

If $p_m = I_m(\beta)^2$ *and* $R(d) = \sum_{m\geq 1} (-1)^m p_m^2 \cos(md)$, *then also* $F_B(\pi - d) = R'(d)$.

Proof. Odd-frequency orthogonality, together with $\cos((2n+1)(\pi/2-u)) = (-1)^n \sin((2n+1)u)$, diagonalizes the integral defining S. Next $a_m = (m-1)h_{m-1}^2 + (m+1)h_m^2$ gives, without a boundary term,

$$F_A(\pi - d) = \sum_{n\geq 0} (-1)^n h_n^2 \{n \sin((n+1)d) - (n+1)\sin(nd)\}.$$

For $k = 2n+1$ and $x = d/2$, the expression in braces is $k \sin x \cos(kx) - \cos x \sin(kx)$, which proves (15). The formula for F_B follows by direct differentiation. Super-exponential Bessel decay justifies every exchange. □

Lemma 60 (exact; cancellation-free right-edge target)**.** *Put* $d = \pi - t$, $x = d/2$, $y = x^2$, *and write* $S(x) = xU(y)$ *and* $R(2x) = V(y)$. *Define the entire elementary functions*

$$s(y) = \frac{\sin x}{x}, \qquad j(y) = \frac{\sin x - x\cos x}{x^3},$$

with their removable values at $y = 0$, *and set*

$$a = jU + 2sU_y, \qquad b = V_y.$$

Then

$$F_A(\pi - 2x) = x^3 a(y), \qquad F_B(\pi - 2x) = x b(y), \qquad E(\pi - 2x, \beta) = \frac{x^2 a(y)}{2b(y)}.$$

Consequently, for $\lambda = 2\beta x$,

$$-\frac{E_t}{\lambda} = \frac{\delta}{4}\frac{P(y)}{b(y)^2}, \qquad P = ab + y(a_y b - a b_y), \qquad \delta = \beta^{-1}. \tag{16}$$

In particular the forced cubic and linear endpoint zeros have been removed before interval evaluation.

Proof. Equation (15) gives $F_A = \sin x\, S' - \cos x\, S = x^3(jU + 2sU_y)$, while $F_B = R'(2x) = \frac{1}{2}\frac{d}{dx}V(x^2) = xV_y$. Differentiate the resulting quotient with respect to x, use $E_t = -E_x/2$, and collect the quotient numerator. This gives (16) identically. $\square$

Lemma 61 (exact; five-family saddle normalization)**.** *Let $\rho = e^{-2\sqrt{2}\beta}$ and define*

$$\widehat{U}_0 = \rho\beta^{1/2}U, \quad \widehat{U}_1 = \rho\beta^{-3/2}U_y, \quad \widehat{U}_2 = \rho\beta^{-7/2}U_{yy},$$

$$\widehat{B}_0 = \rho\beta^{-1/2}b, \qquad \widehat{B}_1 = \rho\beta^{-5/2}b_y.$$

If

$$\begin{aligned}
\widehat{A}_0 &= j\delta^2\widehat{U}_0 + 2s\widehat{U}_1, \\
\widehat{A}_1 &= j_y\delta^4\widehat{U}_0 + (j + 2s_y)\delta^2\widehat{U}_1 + 2s\widehat{U}_2, \\
\widehat{P} &= \widehat{A}_0\widehat{B}_0 + \frac{\lambda^2}{4}(\widehat{A}_1\widehat{B}_0 - \widehat{A}_0\widehat{B}_1),
\end{aligned}$$

then the target has the scale-free form

$$-\frac{E_t}{\lambda} = \frac{\widehat{P}}{4\widehat{B}_0^2}. \tag{17}$$

The only non-elementary inputs in this formula are the five displayed families.

Proof. Since $y = \lambda^2\delta^2/4$, substitution of the five definitions into a, a_y, b, b_y gives $\widehat{A}_0 = \rho\beta^{-3/2}a$, $\widehat{A}_1 = \rho\beta^{-7/2}a_y$, and $\widehat{P} = \rho^2\beta^{-2}P$. Also $b = \rho^{-1}\beta^{1/2}\widehat{B}_0$. All exponential and polynomial factors therefore cancel in (16), proving (17). $\square$

Remark 62 (zero-face algebra and remaining analytic passage)*.* The extracted main and mirror coefficients assemble to the unique candidate

$$Q_0(\lambda) = \frac{\lambda}{2}\coth\left(\frac{\lambda}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}, \qquad H_0(\lambda) = \frac{Q_0'(\lambda)}{\lambda} = \frac{\sinh z\cosh z - z}{2\lambda\sinh^2 z}, \quad z = \frac{\lambda}{\sqrt{2}}.$$

Exact coefficient comparison shows that H_0 is decreasing and $H_0 > 1/5$ on $[0, 3/2]$: after a positive factor, its derivative has coefficients $8n^2 - 6n - (9^n - 1)/4$, zero for $n = 1, 2$ and negative for $n \geq 3$. This closes the algebra and sign of the candidate face. Promotion to the actual $\delta = 0$ face, and hence the moving-edge row, still requires the uniform two-saddle enclosure of the five families in Lemma 61; no limiting interchange is asserted here.

Proposition 63 (exact; half-line sign of the cubic coefficient)**.** *For every $\beta \geq 125$, the coefficient c_3 in* (14) *is strictly positive.*

Proof. Write $f(x) = I_1(x)/2$, $y = 2\beta \sin u$, $z = 2\beta \cos u$. Exact parity in Lemma 58 gives

$$c_3 = \frac{1}{6\pi} \int_0^{\pi/2} J(u, \beta)\ du, \quad J = -z^2 f'(y) f'(z) + zy^2 f'(y) f''(z) + y f(y) f'(z).$$

Set $\ell(x) = f'(x)/f(x)$. The modified-Bessel equation and $y^2 + z^2 = 4\beta^2$ yield the regular identity

$$\frac{J}{f(y)f(z)\beta^3} = 8 \sin^2 u \cos u\, \ell(y) + \beta^{-2} \left(\frac{2\sin^2 u}{\cos u} \ell(y) + 2 \sin u\, \ell(z) \right) - 4\beta^{-1} \ell(y)\ell(z).$$

On $[\pi/8, 3\pi/8]$ one has $y, z > 375/4$. The Amos lower bound at order one and strict order monotonicity give $97/100 < \ell(y), \ell(z) < 51/50$. Since sine and cosine exceed $3/8$, the last display is larger than

$$\frac{27}{64}\frac{97}{100} - \frac{4}{125}\left(\frac{51}{50}\right)^2 > \frac{3}{8}.$$

On $I = [3\pi/16, 5\pi/16]$, the lower companion of Lemma 15, together with $\sqrt{2} > 707/500$ and $\cos(\pi/16) > 49/50$, therefore gives

$$\int_I J\ du > \frac{3}{1024}\left(\frac{622}{625}\right)^2 \beta^2 e^{277\beta/100}.$$

For the exterior $X = [0, \pi/8] \cup [3\pi/8, \pi/2]$, the integral representation gives $I_n(x) \leq e^x$, hence $f, f', f'' \leq e^x/2$. Since $\sin u + \cos u < 131/100$ there,

$$\int_X |J|\ du < \frac{7\pi}{8} \beta^3 e^{131\beta/50}.$$

The intervening part is nonnegative and may be dropped. Using $\pi < 22/7$, the exterior-to-inner ratio is less than $948\beta e^{-3\beta/20}$. It decreases for $\beta \geq 125$, and at the endpoint it is less than one because $e^{75/4} > e^{18} > 18^7/7! > 948 \cdot 125$. Thus the full integral is positive. The rational constant audit is executable in `verify_surface_right_edge_c3_halfline.py`. □

Proposition 64 (verified; prefactor law)**.** *$c_3 > 0$ for $\beta \in \{1, 5, 20, 40, 80, 120\}$ at cancellation-adjusted precision* (the alternating sum cancels at scale $e^{-(4-2\sqrt{2})\beta}$; evaluations use dps $\geq 0.6\beta + 50$)*, and*

$$c_3 \ \sim\ A\, \beta^{3/2} e^{2\sqrt{2}\beta}, \qquad A = \frac{1}{24 \cdot 2^{1/4} \pi^{3/2}} = 0.0062922\ldots,$$

with measured convergence

β	40	60	90	120
ratio	0.0058646	0.0060049	0.0060996	0.0061475

and $O(1/\beta)$ gaps. The sign on the entire half-line $\beta \geq 125$ is now exact by Proposition 63; the displayed sharp prefactor and its measured convergence are not used to promote that sign. The uniform moving-edge theorem is supplied independently by the certified five-family quotient, not inferred from another pointwise or asymptotic sign check for c_3.

9 Terminal closure

Proposition 65 (verified truth map; all statements on the stated finite grids)**.** *Let* $g(t,\beta) = E' + \frac{1}{4}\sin(t/2)$*. At* $\beta = 15$*,* $g < 0$ *at every point of a* 13*-point grid spanning* $[0.05, \pi - 0.09]$ *(independently reproduced on a much finer mesh at* 110 *digits), with thinnest margin* -0.00573 *at* $t = 0.05$ *(on the seal window* $[0.05, \pi - 0.10]$*) and* -0.00216 *at* $\pi - 0.09$*; the crossing sits at* $\pi - t^* = 0.0890502225$*. At* $\beta = 12$ *the truth itself fails at* $\pi - 0.10$ *(*$g = +0.025$*); crossings scale as the mirror window:* $\pi - t^* = 0.1155, 0.0891, 0.0644$ *at* $\beta = 12, 15, 20$*, matching* $\sqrt{2}/\beta$*. A sweep* $\beta = 15 \ldots 60$ *keeps the maximum at* $t = 0.05$*, drifting to* -0.0061*: strong uniformity evidence, not proof.*

Completion of Theorem 2. Theorem 37 proves $F_B > 0$. Lemma 41 and Theorem 40 prove $E' < 0$ for $0 < \beta \leq 3$. The compact Arb and bivariate-Taylor unions, together with their left- and right-edge splices, cover $3 \leq \beta \leq 20$. The finite-role Wronskian audit covers $20 \leq \beta \leq 1000/9$ for all $0 < t < \pi$. Finally Proposition 29 covers $\beta \geq 1000/9$. The endpoints $\beta = 3, 20, 1000/9$ belong to adjacent closed certificates, so their union covers every $\beta > 0$ with no seam gap. This proves part (ii). □

The truth map in Proposition 65 and the second-order law (13) are retained as non-load-bearing diagnostics of the geometry. The theorem uses the certified partition above rather than an asymptotic extrapolation.

10 Verification, certificates, and reproducibility

All numerical artifacts live in the repository

https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME

(directories `scripts/`, `docs/`); the immutable commit hash for the version audited with this manuscript is recorded in the repository's hash registry (`docs/SURFACE-CLOSURE-NOTES.md`) and in the arXiv comment field on submission. Each certificate embeds its pinned statement and self-verifies on every execution.

Certificates (interval arithmetic, outward rounding, nested or stability passes). `certify_time3_negative.py` and its Arb twin (Proposition 46); `certify_bridge_matrix.py` and twin (Proposition 47); `certify_thmB.py` (Theorem 40: adaptive β-boxes covering $[1/20, 3]$; tails computed in intervals and closed at the level of *complete shells*: each pair maps into the next shell with term ratio r derived from the termwise-exact $I_{n+1}(x) \leq x I_n(x)/(2(n+1))$, and the one new pair per shell is bounded by a double index shift, giving shell ratio $2r$, asserted $< \frac{1}{2}$ per box before the geometric closure; a built-in full stability pass at precision +70; the Arb twin `certify_thmB_arb.py` independently certifies the full coverage — 86 boxes, both passes, transcript archived; the `mpmath.iv` canonical transcript independently certifies the same interval with 53 boxes and both passes (its source hash and runtime versions are recorded in the transcript). Bessel enclosures use the positive series with an exact-integer stopping rule and geometric tails; floats appear only in loop-termination heuristics. Nested enclosures prevent silent numerical or truncation degradation; statement auditing and the independent Arb implementation guard against common-mode errors.

Verified computations. The master-formula and reduction identities (ratios 1.0 to 12 digits); the truth map of Proposition 65 (dps 80, confirmed independently at 110 digits); the c_3 ladder (Proposition 64); the term-split coefficients of (13) at $\beta = 60, 120, 240$.

Precision policies (each traceable to a caught artifact): dps $\geq 2.2\beta + 40$ for any evaluation with $t \geq 2$, $\beta > 30$ (mirror cancellation); dps $\geq 0.6\beta + 50$ for c_3-type alternating sums; series cutoffs scale with the maximal argument, and scaled-asymptotic evaluation of $e^{-z}I_n(z)$ is the default for $z > 35$; every measured coefficient carries a β-scaling sanity check; whatever can be computed from a positive integral representation is.

Audit trail. Fifteen numerical or formulation artifacts ("ghosts") were caught during fabrication and writing by the three-way review protocol; Appendix B tabulates them. We consider the table part of the result: it is the reproducibility record that makes the certificates credible.

Machine-checked lemmas. The unit-step and Turán-type coefficient inequalities behind Theorem 40 are formalized in Lean 4/Mathlib (`PhiMonotone.lean`, `FHBesselAmos.lean`, `WronskianIdentity.lean`; axiom oracle `[propext, Classical.choice, Quot.sound]`, no `sorry`). Mathlib has no Bessel functions: what is machine-checked is the ordered-field implication, with the analytic inputs (three-term recurrence [4, 10.29.1]; the Amos-type bound of [1, 2, 3], cited verbatim, shift $\nu \to \nu+1$ as in Appendix A) carried as named hypotheses. Correct claim everywhere: *machine-checked modulo classical Bessel inputs carried as named hypotheses.*

Acknowledgements and provenance

The theorem, the bridge reading, the reduction, and all certificates were produced inside an audit-first programme in which three independent reviewing voices challenge the load-bearing chain before release; scores, objections, and the complete ghost ledger are archived with the repository history. No terminal numerical conclusion rests on a single run: its production archive has an independent replay, and every identity marked exact has a symbolic-zero transcript.

A The weighted Turán-type lemma, self-contained

Theorem 66 (exact)**.** *For every $x > 0$ and every real $m \geq 1$, with $\varphi_m = a_m/b_m$ built from (1) at argument x: $\varphi_m < \varphi_{m+1}$. Equivalently, with $\rho_\mu = I_{\mu+1}/I_\mu$,*

$$\frac{1}{m}\Big(\frac{m-1}{\rho_{m-1}^2} + (m+1)\rho_m^2\Big) < \frac{1}{m+1}\Big(\frac{m}{\rho_m^2} + (m+2)\rho_{m+1}^2\Big).$$

Proof. Two classical inputs, valid for all real orders: the three-term recurrence $\rho_{m-1}^{-1} = \rho_m + 2m/x$, $\rho_{m+1} = \rho_m^{-1} - 2(m+1)/x$ [4, 10.29.1], and the Amos-type bound

$$\rho_m(x) < U_m(x) := \frac{x}{a + \sqrt{a^2 + x^2}}, \qquad a = m + \tfrac{1}{2},$$

proved best of its form for all real orders ≥ 0 by Segura [2]; we use [3, Thm. 2], which bounds $I_\nu(x)/I_{\nu-1}(x)$ for $\nu \geq \frac{1}{2}$ by $x/(\nu - \frac{1}{2} + \sqrt{(\nu+\frac{1}{2})^2 + x^2}\,)$; setting $\nu = m+1$ bounds $\rho_m = I_{m+1}/I_m$ by $x/(m + \frac{1}{2} + \sqrt{(m+\frac{3}{2})^2 + x^2}\,) \leq x/(a + \sqrt{a^2+x^2}\,)$ with $a = m + \frac{1}{2}$, which is the displayed bound (the last step uses $\sqrt{(m+\frac{3}{2})^2 + x^2} \geq \sqrt{a^2+x^2}$). Substitute the recurrences, write $u = \rho_m \in (0,1)$, $c = 1/x$, $S = u + 1/u$, $P = 1/u - u$; clearing the positive denominators, the claim becomes $\Delta_m > 0$ where the *exact factorization*

$$\frac{\Delta_m}{2m(m+1)} = (S - 3c)\big(P - (2m+1)c\big) + (2m+1)c^2$$

holds as an algebraic identity (expand and collect; the coefficients of $u^{\pm 2}$ are $\pm 2m(m+1)$, the linear-in-c terms are $-4m(m+1)c[(m-1)u+(m+2)/u]$, and $(m-1)u+(m+2)/u = \frac{1}{2}[(2m+1)S+3P]$, $1/u^2-u^2 = PS$). The second factor is positive because U_m is the positive root of $1/w - w = (2m+1)/x$ (the calibration $1/U_m - U_m = 2a/x$) and $w \mapsto 1/w - w$ is strictly decreasing, so $\rho_m < U_m$ gives $P > (2m+1)c$. The first factor is positive because $\sqrt{a^2+x^2} > a$ gives $U_m < x/(2a)$, so $u < x/(2m+1) \le x/3$ and $S > 1/u > (2m+1)/x \ge 3c$. The third term is positive. Strictness is strict throughout. (This is the proof of [6], reproduced so that Theorem 40 is self-contained; the ordered-field implication is machine-checked in `PhiMonotone.lean`, modulo the two classical inputs carried as named hypotheses.) □

B Audit table (the ghost ledger)

Each row: what was wrong, how it was caught, and the rule it produced. All entries are documented with transcripts in `docs/SURFACE-CLOSURE-NOTES.md`.

#	artifact	catch / rule produced
1	random TP_2 scan missed adversarial corners	targeted corner search; adversarial grids mandatory
2	verification cell with unbound parameter	explicit binding rule for inline prints
3	near-π scan at low precision, false violation	precision rule dps $\ge 2.2\beta + 40$ for $t \ge 2$, $\beta > 30$
4	stale-PDF claim misattributed between voices	provenance ledger: attribute to exact source
5	float64 CDF increments ± 11 (impossible)	positive-representation preference
6	t=2, β=240 residue 0.363 vs 0.317	boundary $t \ge 2$ included in the dps rule
7	I_1 series truncated at 60 terms, $z \le 480$	cutoffs scale with argument; scaled asymptotics for $z > 35$; β-scaling checks
8	c_3 sign flip at $\beta = 120$ (cancellation)	alternating-sum rule dps $\ge 0.6\beta + 50$
9	$R = 0$ corner misplaced at $(1, 1)$	complete critical-point census required in prose too
10	one-saddle prefactor off by exactly 2	census rule; the factor 2 later proved an exact parity
11	I_1 lower bound dropped the negative tail	sign audits are a separate mandatory pass
12	fixed quadratic window at π was provably false	interrogate obligations before attempting them
13	Thm. A second proof: rearrangement misquoted	layer-cake arc proof substituted (this revision)
14	Lemma sign error: floor stated as upper bound	corrected; per-pair statements now carry signs explicitly
15	"exact" labels on verified inequalities	tricotomy re-audit of every label (this revision)

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