Blind to the Coupling: a Second Machine-Checked Obstruction at Spatial Extent

and the strictly positive eigenvector the elementary route failed to produce, in closed form at two sites

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Abstract

A companion paper proved that when a spatial coupling is switched on in a $\mathbb{Z}_2$ lattice gauge slice, the transfer kernel loses constant row sums, so the uniform vector is no longer fixed and the elementary route to the vacuum stops. The standard replacement, when row sums fail, is the Hilbert projective metric: a strictly positive kernel contracts it, and the contraction factor bounds the subdominant spectral ratio. This paper asks what that replacement gives here, and answers in Lean 4 with `mathlib`.

It gives no coupling-sensitive and no volume-uniform information, for two independent reasons, and both are proved. First, *blindness*: the projective cross-ratio is invariant under multiplication by any nowhere-zero function of the source configuration alone. The coupled kernel is exactly such a product, so the metric assigns the interacting and the non-interacting kernels the same diameter at every spatial extent — the route cannot see the coupling at all. Second, *volume degeneration*: two constant configurations realise the cross-ratio $e^{4\beta L}$, so every admissible projective diameter is at least $4\beta L$ and every contraction factor obtainable this way is at least $\tanh(\beta L)$, which lies within $2e^{-2\beta L}$ of the trivial bound 1. At the one place where the truth is known — the decoupled kernel, whose subdominant ratio the companion paper computes to be exactly $\tanh \beta$ at every L — this route already returns $\tanh(\beta L)$ instead. The degeneration is the method's, not the model's.

We then hand over the object the elementary route stopped producing, at the smallest interacting size. In the character basis the coupled two-site kernel splits into two 2×2 blocks, and we exhibit a *strictly positive* eigenvector in closed form, together with a second exact eigenpair. The identity $A - B = 4$ between the two *decoupled* even-sector eigenvalues drives every estimate. The blindness is proved two-sided, so it covers the symmetrised convention $w^{1/2}Kw^{1/2}$ as well, and the positive eigenvector is proved to dominate every eigenvalue, real or complex — so its eigenvalue is the spectral radius, which is the Perron statement this development needs and proves without a Perron–Frobenius theorem in the library.

No volume-uniform statement about an interacting system is proved here, and none is claimed; the general-L behaviour is recorded separately as measured and unproved. Nothing in this paper is a claim about $\mathrm{SU}(N)$, the continuum limit, or the Yang–Mills mass gap.

1 Introduction

1.1 What the predecessor left

The companion paper [1] proved, for a $\mathbb{Z}_2$ slice carrying L spins, that the decoupled transfer kernel has constant row sums at every L — so the uniform vacuum survives — and that switching on a coupling inside the slice destroys that constancy, because the spatial weight depends on the *source* only and therefore factors out of the sum over the target. With constant row sums goes $T\Omega = \Omega$, and with it the starting point of every later step of the reconstruction.

That paper ended by naming two honest continuations: a Perron–Frobenius route to the vacuum at spatial extent, or a statement that the elementary methods end there. It declined to say which. This paper answers, and the answer has both signs: the natural elementary replacement is proved useless here, and the object it failed to produce is written down at the smallest interacting size — as a strictly positive eigenvector in closed form, which is what is proved about it.

1.2 The natural next attempt, and why it is the right one to test

When constant row sums fail, the textbook instrument is the *Hilbert projective metric*. A strictly positive kernel contracts it, the contraction factor is $\tanh(\Delta/4)$ where Δ is the projective diameter [2, 3], and that factor bounds the subdominant spectral ratio without any need to know the leading eigenvector. It is exactly the tool one reaches for when the eigenvector is unavailable, which is the situation [1] created. So it is the right next thing to test, and testing it is what this paper does.

1.3 What is proved

Write k_β for the Boltzmann bond weight, $Z_\beta = e^\beta + e^{-\beta}$, $K(\sigma,\tau) = \prod_j k_\beta(\sigma_j,\tau_j)$ for the decoupled kernel, and w for a weight depending on the source configuration only. Define the projective cross-ratio

$$R_A(\sigma,\sigma';\tau,\tau') \;=\; \frac{A(\sigma,\tau)\,A(\sigma',\tau')}{A(\sigma,\tau')\,A(\sigma',\tau)}.$$

Blindness.

- $R_{wK} = R_K$ for *every* nowhere-zero source-only weight w, at every spatial extent. The weight occurs once above and once below the bar, at the same two configurations, and cancels identically;
- and the same computation gives it *two-sided*: $R_{uAv} = R_A$ for nowhere-zero u on the source and v on the target. So the symmetrised convention $w^{1/2}Kw^{1/2}$ is invisible too, which settles by theorem what the companion paper could only record as prose.

Volume degeneration.

- at the all-0 and all-1 configurations, $R_K = e^{4\beta L}$ exactly;
- hence every D with $\log R_K \le D$ everywhere satisfies $4\beta L \le D$, and every contraction factor $\tanh(D/4)$ obtainable this way satisfies $\tanh(\beta L) \le \tanh(D/4)$;
- and $1 - \tanh(\beta L) \le 2e^{-2\beta L}$: that lower bound is itself exponentially close, in the volume, to the trivial bound 1;
- while the truth for the decoupled kernel is $\tanh\beta$ at every L [1], strictly better for $L \ge 2$, $\beta > 0$.

The two-site positive eigenvector. With $A = (e^\beta + e^{-\beta})^2$ and $B = (e^\beta - e^{-\beta})^2$:

- the coupled kernel preserves the two-dimensional span of the constant and the top character, acting there by $\left(\begin{smallmatrix} A\cosh\gamma & B\sinh\gamma \\ A\sinh\gamma & B\cosh\gamma \end{smallmatrix}\right)$;

- the vector $\alpha \cdot 1 + \delta \cdot s_0 s_1$ with $\alpha = B\sinh\gamma$ and $\delta = \lambda_+ - A\cosh\gamma$ is an eigenvector with eigenvalue $\lambda_+ = \frac{1}{2}\big((A+B)\cosh\gamma + \sqrt{(A+B)^2\cosh^2\gamma - 4AB}\big)$, and it is *strictly positive* at every configuration for $\beta > 0$, $\gamma > 0$;

- $s_0 + s_1$ is a second eigenvector, with eigenvalue $\mu_+ = (e^{\beta} - e^{-\beta})Z_\beta e^{\gamma}$, and it does not vanish;

- $\mu_+ < \lambda_+$ for $\beta > 0$, $\gamma > 0$, so the two exhibited eigenpairs are ordered. The identity $A + B - 2(e^{\beta} - e^{-\beta})Z_\beta = 4e^{-2\beta}$, companion of $A - B = 4$, reduces that to one line;

- and *every* eigenvalue is dominated, real or complex: $|\mu| \le \lambda_+$. So λ_+ is the spectral radius. This is the one consequence of Perron–Frobenius the paper needs, and it is proved here rather than cited, because the pinned library has no Perron–Frobenius theorem at all.

1.4 What is deliberately not claimed

1. **No volume-uniform statement about an interacting system.** None is proved here and none is claimed. Section 6 records what is *measured* about general L and says explicitly that it is not proved.

2. **No claim that contraction arguments cannot work.** What is proved is that this one — the Hilbert metric applied to this kernel — is blind and degenerate. A different metric, a power of the kernel, or a Dobrushin-type condition are untouched by these theorems.

3. **The Birkhoff contraction theorem itself is not mechanised.** What is machine-checked is the algebra that forces the blindness and the degeneration: the invariance of R_A, the exact witness, the resulting bounds. That $\tanh(\Delta/4)$ is a contraction factor for the projective metric is classical [2, 3] and enters only as the reason those bounds are the ones a reader would want; no theorem below takes it as a hypothesis. A reader who wants the whole chain inside the kernel should read the title as naming the obstruction, not the classical interpretation of it.

4. **The spectral-radius claim covers all eigenvalues, but not uniqueness or simplicity.** Theorems 5.9 and 5.10 prove $|\mu| \le \lambda_+$ for every eigenvalue, real or complex, with any nonzero eigenvector; that is what makes λ_+ the spectral radius. Uniqueness of the positive eigenvector up to scale, simplicity of λ_+, and completeness of the spectrum are *not* covered — those are the rest of Perron–Frobenius, and the library has none of it (Remark 5.12).

5. **No completeness of the two-site spectrum.** Two eigenpairs are exhibited. That they and their partners exhaust the four-dimensional space is not formalised.

6. **Still** $\mathbb{Z}_2$, one variable per site, and the closed form is at $L = 2$.

7. **Nothing about** $\mathrm{SU}(N)$**, the continuum limit, or the Clay problem** [7]. No reduction is claimed and none is known to the author.

2 The projective cross-ratio

Definition 2.1. For a kernel $A : X \times X \to \mathbb{R}$ on a set X,

$$R_A(\sigma, \sigma'; \tau, \tau') \;=\; \frac{A(\sigma,\tau)\,A(\sigma',\tau')}{A(\sigma,\tau')\,A(\sigma',\tau)}.$$

`crossRatio`. The Hilbert projective diameter of a strictly positive kernel is $\sup \log R_A$, and the Birkhoff contraction factor is $\tanh(\Delta/4)$ [2, 3]. Everything below is phrased so that only R_A itself is formalised: the two classical facts just quoted are used as *context*, never as hypotheses of a theorem.

3 Blindness: no source-only weight is visible

Theorem 3.1 (The metric cannot see a source-only weight)**.** *Let w be nowhere zero and depend on the source only, and let $(wK)(\sigma,\tau) = w(\sigma)K(\sigma,\tau)$. Then $R_{wK} = R_K$ at every four configurations, at every spatial extent.*

Proof. $w(\sigma)$ and $w(\sigma')$ each occur once in the numerator and once in the denominator, so they cancel. □

`crossRatio_sourceWeighted`, and for the coupled kernel of [1] as the special case `crossRatio_coupledKernel`.

Remark 3.2. *The hypothesis is not a technicality chosen to make the proof work: it is* exactly *the structural feature that [1] used to break the row sums. The same one-line fact — the weight depends on the source only — destroys the first route and makes the second route unable to detect that anything happened. One property, two opposite-looking failures.*

The obvious objection is that the source-only convention is doing the work. It is not, and the same computation shows why.

Theorem 3.3 (The blindness is two-sided)**.** *Let u, v be nowhere zero. Then $R_{u\,A\,v} = R_A$, where $(uAv)(\sigma,\tau) = u(\sigma)A(\sigma,\tau)v(\tau)$.*

`crossRatio_twoSided`.

Corollary 3.4 (The symmetrised kernel is invisible as well)**.** $R_{\widetilde{K}_\gamma} = R_K$ *for* $\widetilde{K}_\gamma(\sigma,\tau) = w_\gamma(\sigma)^{1/2}K(\sigma,\tau)w_\gamma(\tau)^{1/2}$.

`crossRatio_symCoupledKernel`, with the kernel itself at `symCoupledKernel`. The companion paper could record the corresponding statement only as prose; here it is a theorem, and it removes the last reading on which the blindness could be an artifact of the convention.

4 Volume degeneration

Theorem 4.1 (The witness)**.** *R_K at the all-0 and all-1 configurations equals $e^{4\beta L}$.*

Proof. The four kernel values are $e^{\beta L}$, $e^{\beta L}$, $e^{-\beta L}$, $e^{-\beta L}$. □

`crossRatio_const`.

Theorem 4.2 (The wall)**.** *If D satisfies* $\log R_K \le D$ *at every four configurations, then* $4\beta L \le D$*, and consequently* $\tanh(\beta L) \le \tanh(D/4)$*.*

projDiameter_ge, birkhoff_bound_ge.

Theorem 4.3 (The degeneration is exponential in the volume)**.** *For* $\beta \ge 0$*,* $1 - \tanh(\beta L) \le 2e^{-2\beta L}$*.*

birkhoff_bound_near_one.

Theorem 4.4 (And the bound is not tight)**.** *For* $\beta > 0$ *and* $L \ge 2$*,* $\tanh\beta < \tanh(\beta L)$*.*

birkhoff_bound_not_tight.

Remark 4.5. *Theorem 4.4 is what makes Theorem 4.3 a statement about the* method*. For the decoupled kernel the companion paper computes the subdominant ratio exactly — it is* $\tanh\beta$*, independent of* L *— so in the one case where the answer is known, the route already returns a number that degenerates while the truth does not. Nothing about the model forces the loss; the instrument loses it.*

5 The positive eigenvector, in closed form at two sites

The obstruction of [1] says the vacuum stops being handed to us. It does not say the vacuum is unavailable. At $L = 2$ the object it stopped handing over can be written down, and this section does so — as what is actually proved about it, an eigen-equation together with strict positivity; Remark 5.12 says precisely what would be needed to call it the vacuum and why that step is absent.

Definition 5.1. $s_i(\sigma)$ is the sign of site i, and $s_0 s_1$ is the top character. The four characters $1, s_0, s_1, s_0 s_1$ span the functions on a two-site slice.

Lemma 5.2 (The weight is affine in the top character)**.** $w_\gamma(\sigma) = \cosh\gamma + \sinh\gamma\,(s_0 s_1)(\sigma)$*.*

spatialWeight_eq. The exponent takes only the values $\pm\gamma$, which is the whole content.

Lemma 5.3 (The decoupled kernel is diagonal in this basis)**.** $\sum_\tau K(\sigma,\tau) = Z_\beta^L$, $\sum_\tau K(\sigma,\tau) s_i(\tau) = (e^\beta - e^{-\beta}) Z_\beta^{L-1} s_i(\sigma)$ [1], *and* $\sum_\tau K(\sigma,\tau)(s_0 \cdots s_{L-1})(\tau) = (e^\beta - e^{-\beta})^L (s_0 \cdots s_{L-1})(\sigma)$*.*

spatialKernel_allSign for the third; the first two are theorems of the companion development.

Theorem 5.4 (The even sector closes)**.** *With* $A = Z_\beta^2$ *and* $B = (e^\beta - e^{-\beta})^2$*, the coupled kernel maps* $\alpha \cdot 1 + \delta \cdot s_0 s_1$ *to* $(\cosh\gamma\,\alpha A + \sinh\gamma\,\delta B) \cdot 1 + (\cosh\gamma\,\delta B + \sinh\gamma\,\alpha A) \cdot s_0 s_1$*.*

coupled_even_action.

Lemma 5.5 (The identity that drives every estimate)**.** $A - B = 4$*.*

evenA_sub_evenB. Indeed $(a+b)^2 - (a-b)^2 = 4ab$ and $e^\beta e^{-\beta} = 1$.

Theorem 5.6 (A strictly positive eigenvector, in closed form)**.** *Let* $\lambda_+ = \frac{1}{2}\big((A+B)\cosh\gamma + \sqrt{(A+B)^2\cosh^2\gamma - 4AB}\big)$*. Then* $\alpha = B\sinh\gamma$*,* $\delta = \lambda_+ - A\cosh\gamma$ *gives an eigenvector of the coupled kernel with eigenvalue* λ_+*, and for* $\beta > 0$ *and* $\gamma > 0$ *that eigenvector is strictly positive at every configuration.*

`coupled_perron_eigen`, `perronVec_pos`; the discriminant is nonnegative by `evenDisc_nonneg`, using $\cosh\gamma \geq 1$.

Proof of positivity. The vector takes the two values $\alpha \pm \delta$, so positivity is $|\,2\delta\,| < 2\alpha$, i.e. $4\cosh\gamma - 2B\sinh\gamma < \sqrt{\Delta} < 4\cosh\gamma + 2B\sinh\gamma$ after using $A - B = 4$. Squaring, the two gaps are $16B\sinh\gamma(\sinh\gamma + \cosh\gamma)$ and $16B\sinh\gamma(\cosh\gamma - \sinh\gamma)$, both positive. □

Theorem 5.7 (A second exact eigenpair)**.** $s_0 + s_1$ *is an eigenvector with eigenvalue* $(e^\beta - e^{-\beta})Z_\beta e^\gamma$*, and it takes the value* 2 *at the constant configuration, so it does not vanish identically.*

`coupled_odd_eigen`, `oddVec_cfgFlat`.

Theorem 5.8 (The two exhibited eigenpairs are ordered)**.** *For* $\beta > 0$ *and* $\gamma > 0$*,* $\mu_+ < \lambda_+$*.*

`oddEigen_lt_evenTop`. The proof squares once and uses the identity $A + B - 2(e^\beta - e^{-\beta})Z_\beta = 4e^{-2\beta}$ (`evenSum_sub_two_odd`) together with $AB = \big((e^\beta - e^{-\beta})Z_\beta\big)^2$ (`evenA_mul_evenB`); the resulting gap is $4\mu_0\cosh\gamma \cdot 4e^{-2\beta}(\cosh\gamma + \sinh\gamma) > 0$. So the ratio μ_+/λ_+ is a genuine number strictly below 1 — between the two eigenpairs this paper exhibits. Whether it is *the* subdominant ratio depends on completeness of the spectrum, which is not formalised.

Theorem 5.9 (The positive eigenvector dominates)**.** *Let* A *be entrywise nonnegative with a strictly positive eigenvector* v*,* $Av = \lambda v$*. Then every real eigenvalue* μ *of* A*, with any nonzero real eigenvector, satisfies* $|\mu| \leq \lambda$*. In particular, for the coupled two-site kernel with* $\beta, \gamma > 0$*, every real eigenvalue is dominated by* λ_+*.*

`abs_eigenvalue_le_of_pos_eigenvector` and, for the kernel of this paper, `evenTop_dominates`.

Proof. Scale the competing eigenvector until it touches the positive one: let t be the maximum of $|w_i|/v_i$, attained at i_0. Then $|\mu|\,|w_{i_0}| = |\sum_j A_{i_0 j} w_j| \leq \sum_j A_{i_0 j}|w_j| \leq t\sum_j A_{i_0 j} v_j = t\lambda v_{i_0} = \lambda|w_{i_0}|$, and $|w_{i_0}| > 0$. □

Theorem 5.10 (The same bound over $\mathbb{C}$, hence the spectral radius)**.** *The statement of Theorem 5.9 holds verbatim for complex eigenpairs: if* $w : \iota \to \mathbb{C}$ *is not identically zero and* $Aw = \mu w$ *with* $\mu \in \mathbb{C}$*, then* $|\mu| \leq \lambda$*. Consequently* λ_+ *is the* spectral radius *of the coupled two-site kernel — it dominates every eigenvalue, and it is itself one, with a strictly positive eigenvector.*

`norm_eigenvalue_le_of_pos_eigenvector` and `evenTop_dominates_complex`. The proof is the real one with the complex modulus in place of the absolute value; the triangle inequality is all that the argument ever used, so nothing else changes.

Remark 5.11 (What this does and does not settle)**.** *Theorems 5.9 and 5.10 are the consequence of Perron–Frobenius that this paper needs, and they are proved here rather than cited, because the pinned library has no Perron–Frobenius (Remark 5.12). With Theorem 5.6 they say:* λ_+ *is the spectral radius of the coupled two-site kernel, and the strictly positive eigenvector carries* *it. What is still* not *settled is uniqueness of that eigenvector up to scale, simplicity of λ_+, and completeness of the spectrum — so 'the' vacuum remains, strictly, 'a' strictly positive eigenvector carrying the spectral radius.*

Remark 5.12 (What Perron–Frobenius would add, and that we do not use it)**.** *For a strictly positive matrix, Perron–Frobenius says a strictly positive eigenvector is unique up to scale and its eigenvalue is the spectral radius — so Theorem 5.6 would identify λ_+ as the top of the spectrum and the vector as* the *vacuum. We do not invoke that, and the reason is worth recording because it is a fact about the tools rather than about the mathematics. The* `mathlib` *revision pinned by this development carries* no *Perron–Frobenius theorem. What it carries is the layer underneath:* `Mathlib.LinearAlgebra.Matrix.Irreducible`*, a quiver-theoretic treatment of irreducibility and primitivity for entrywise-nonnegative matrices. The statement that an irreducible nonnegative matrix has a positive eigenvector whose eigenvalue is the spectral radius is not there, and neither is the Hilbert projective metric nor the Birkhoff contraction coefficient — which is why Definition 2.1 builds the cross-ratio from nothing. So the one consequence this paper needs had to be constructed rather than cited, and Theorems 5.9 and 5.10 construct it. What is machine-checked is exactly what is stated: the eigen-equation, strict positivity of the eigenvector, and domination in modulus of every eigenvalue, real or complex — hence the identification of λ_+ as the spectral radius. What remains outside the development is uniqueness up to scale, simplicity of λ_+, and spectral completeness.*

6 What is measured and not proved

A reader will ask what happens for $L > 2$. We have looked, numerically, and we report the result with its label attached and no theorem behind it.

Direct dense *numerical* diagonalisation of the $2^L \times 2^L$ coupled kernel over $\beta \in \{0.3, 0.8\}$, $\gamma \in \{0, 0.4, 1.2\}$, $L \in \{2, 3, 4\}$ gives a subdominant ratio that rises towards 1 with L over much of that grid — for instance $0.908, 0.977, 0.994$ at $\beta = 0.8$, $\gamma = 1.2$. The rows where it stays bounded away from 1 are those with $\sinh 2\beta \sinh 2\gamma < 1$. Both statements are **verified** in the numerical sense and **not proved**; the identification of that threshold with the Kramers–Wannier disordered line of the two-dimensional Ising transfer matrix is standard literature and is not established here. The probe and its transcript are in the repository, and their consequence for the programme is recorded there too: a volume-uniform statement about the coupled system, asserted without an explicit smallness hypothesis, is aiming at something the numbers say is false.

Nothing in this section is used by any theorem of this paper.

7 Formalisation notes

One structural fact, two failures. That the spatial weight depends on the source only is what broke the row sums in [1] and what makes the projective metric blind here. The same hypothesis appears in both proofs, with opposite-looking consequences.

Characters instead of enumeration. The two-site computation never enumerates the four configurations. Writing every vector in the character basis turns each sum into an instance of the companion module's factorisation lemma, so the proofs are the same length at $L = 2$ as they would be at any L where the sector closed.

One identity carries the estimates. $A - B = 4$ (Lemma 5.5) reduces both positivity inequalities to $16B \sinh\gamma(\cosh\gamma \pm \sinh\gamma) > 0$. Without it the same inequalities are opaque to automation.

The general statement was cheaper than the special one. Theorem 3.1 was proved first for a source-only weight, because that is the case the companion paper needs. Theorem 3.3 — two factors instead of one — is the same proof with one more pair of terms to cancel, and it subsumes the special case, covers the symmetrised convention, and removes an entire class of referee objection. The lesson is narrow but real: when a cancellation argument is driven by *where* a factor sits rather than *what* it is, generalising costs nothing and buys the objection.

Cofactors computed, not guessed. The coefficients of the `linear_combination` steps and the two square-root estimates were obtained by polynomial division in a computer algebra system before being written into Lean. The division also returned something useful: the ordering identity of Theorem 5.8 needs only $\cosh^2 - \sinh^2 = 1$ and *not* $e^{\beta}e^{-\beta} = 1$, which removed a hypothesis from the estimate. Guessing those coefficients by trial would have cost more compiler round-trips than the algebra cost.

Self-contained hyperbolic facts, and a naming tax. `cosh`/`sinh` lemmas were re-derived from `Real.cosh_eq` and `Real.sinh_eq` rather than searched for by name, after the search cost more than the derivation. Where a library lemma was unavoidable the name still had to be read out of the pinned source: the division-comparison lemmas this development needs carry a $_0$ suffix in this revision, and one `simp` normal form silently undoes $e^{\gamma} = \cosh\gamma + \sinh\gamma$, so that rewrite has to be carried as a term in the linear combination instead.

8 Reproducibility

All artifacts are at source checkpoint `a70426f4c8a0ed733b4eee94fb01b158bc81fd08` on branch `main` of `https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME`. Every link was resolved against that commit, and checked to point at an actual declaration line, before compilation.

Quantity	Value
Full core build	8426 jobs, success
Oracle commands	2542
with axiom dependencies	2519
axiom-free	23
Distinct axioms across all outputs	`propext`, `Quot.sound`, `Classical.choice`
Occurrences of `sorryAx`	0
Project axioms	0
Errors	0

The design probe of Section 6 is `scripts/probe_spatial_birkhoff.py` with transcript `docs/O-LANE-PROBE-TRANSCRIPT-20260728.txt`; the continuation ladder it fixed is `docs/O-LANE-CONTINUATION-20260728.md`. Verdicts are in `docs/VERIFICATION-LEDGER.md`. The oracle script is `oracle_check.lean`.

References

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