The Vacuum Was Never Absent: A Machine-Checked Perron Theorem for Strictly Positive Kernels, and the Coupled-Slice Vacuum at Every Spatial Extent

existence by compactness, without invoking a fixed-point theorem in the pinned `mathlib` revision

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Abstract

Two companion papers established, for a $\mathbb{Z}_2$ lattice gauge slice with a spatial coupling, that the elementary route to the vacuum stops [1] and that the natural replacement — the Hilbert projective metric — is blind to the coupling and degenerates in the volume [2]. Both had to work around the same absence: the pinned `mathlib` carries no Perron–Frobenius theorem. The first paper could therefore only say the vacuum had become unavailable; the second had to build its domination bound from scratch, and could exhibit the vacuum in closed form only at two sites.

This paper discharges that dependency. For a strictly positive kernel on a finite nonempty type we prove, in Lean 4 with `mathlib`: a strictly positive eigenvector *exists*; its eigenvalue is strictly positive; any two strictly positive eigenvectors are proportional and share their eigenvalue; and every real eigenvector for that eigenvalue is a scalar multiple of it. Together with the domination theorem of [2] this gives the Perron statement the lane needs: the eigenvalue is the spectral radius.

The existence proof does not use a fixed-point theorem, because the pinned `mathlib` revision contains none. It maximises r over the compact set of pairs (r, x) with x in the simplex and $rx \leq Ax$; maximality forces equality, since a strict inequality anywhere would let one further application of A produce an admissible pair with a larger r. The bound that keeps the set compact is obtained by summing the constraint: $r = r\sum x \leq \sum(Ax)$.

The application is the point. At *every* spatial extent, and for *every* strictly positive weight on the source configuration — the class that contains the coupled kernel of [1] — the vacuum exists, is unique up to scale, and carries the spectral radius. The obstruction of the first paper was never an absence; it was an unavailability, and it was an unavailability of one route rather than of the object.

No spectral gap is proved here, uniform in the volume or otherwise, and none is claimed. Nothing in this paper is a claim about $\mathrm{SU}(N)$, the continuum limit, or the Yang–Mills mass gap.

1 Introduction

1.1 A dependency that bit twice

[1] proved that switching on a coupling inside a slice destroys the constancy of the row sums, so the uniform vector stops being fixed and the elementary construction of the vacuum halts. It

phrased the conclusion carefully: the vacuum becomes a Perron vector that row-sum normalisation no longer supplies. It did not claim the vacuum was gone — but it could not produce it either.

[2] then showed the natural replacement fails for two independent reasons, and wrote the vacuum down at $L = 2$ in closed form. To identify its eigenvalue as the spectral radius it had to prove a domination bound from first principles, for the same reason: no Perron–Frobenius in the library.

A dependency that forces a detour twice is not bad luck. This paper removes it.

1.2 What is proved

Throughout, ι is a finite nonempty type and $A : \iota \times \iota \to \mathbb{R}$ is *strictly positive*, $A_{ij} > 0$. Vectors are functions $\iota \to \mathbb{R}$ and $(Ax)_i = \sum_j A_{ij} x_j$.

- **Existence.** There are v and λ with $v_i > 0$ for all i, $\lambda > 0$, and $Av = \lambda v$.
- **Uniqueness up to scale.** If v, w are strictly positive eigenvectors with eigenvalues λ, μ, then $\lambda = \mu$ and $w = cv$ for some $c > 0$.
- **Geometric simplicity.** Every real eigenvector for λ is a scalar multiple of v.
- **Spectral radius.** Every eigenvalue, real or complex, is dominated in modulus by λ [2].

The application. For every spatial extent L and every strictly positive w on configurations, the kernel $K_w(\sigma, \tau) = w(\sigma)K(\sigma, \tau)$ — with K the decoupled kernel of [1] — has all four properties. The coupled kernel of that paper is the case of a single spatial bond, and a bond around the slice is another.

1.3 What is deliberately not claimed

1. **No spectral gap.** Nothing here bounds the subdominant eigenvalue, at any L, in any region of parameters. That is a different problem and this paper does not touch it. In particular nothing here contradicts, or improves on, the obstructions of [2].
2. **Algebraic simplicity is not proved.** The eigenspace is shown to be one-dimensional; that the eigenvalue is a simple root of the characteristic polynomial is not formalised.
3. **Strict positivity is assumed, not irreducibility.** The Perron–Frobenius theory of merely irreducible nonnegative kernels, and the periodicity theory, are outside this development.
4. **Still** $\mathbb{Z}_2$, one variable per site, in the application.
5. **Nothing about** $\mathrm{SU}(N)$**, the continuum limit, or the Clay problem** [6]. No reduction is claimed and none is known to the author.

2 One comparison, used three times

Lemma 2.1 (Positivity of the image)**.** *If $z \geq 0$ and $z_{j_0} > 0$ for some j_0, then $(Az)_i > 0$ for every i.*

`apply_pos_of_nonneg`.

Lemma 2.2 (The vanishing comparison)**.** *If $z \geq 0$, $Az = \lambda z$ and $z_{i_0} = 0$ for some i_0, then $z = 0$.*

Proof. Otherwise z is positive somewhere, so $(Az)_{i_0} > 0$ by Lemma 2.1; but $(Az)_{i_0} = \lambda z_{i_0} = 0$. □

`eq_zero_of_nonneg_of_eigen_of_zero`. This single lemma carries both uniqueness statements: in each case one subtracts the largest multiple of one eigenvector that still fits under the other, which produces a nonnegative eigenvector with a zero entry.

Lemma 2.3 (Positivity of the eigenvalue)**.** *If $v > 0$ and $Av = \lambda v$ then $\lambda > 0$.*

`eigenvalue_pos`.

3 Uniqueness and geometric simplicity

Theorem 3.1 (Geometric simplicity)**.** *If $v > 0$ with $Av = \lambda v$, then every w with $Aw = \lambda w$ satisfies $w = cv$ for some $c \in \mathbb{R}$.*

Proof. Let $c = \min_i w_i/v_i$, attained at i_0. Then $z = w - cv \geq 0$ by the choice of c, $z_{i_0} = 0$, and $Az = \lambda z$ by linearity. Lemma 2.2 gives $z = 0$. □

`eigenvector_eq_smul_of_pos`.

Theorem 3.2 (Uniqueness up to scale)**.** *Two strictly positive eigenvectors have the same eigenvalue, and are positive multiples of one another.*

Proof. Both eigenvalues are positive by Lemma 2.3, and each dominates the other in modulus by the domination theorem of [2], so they are equal; then Theorem 3.1 applies, and the constant is positive because both vectors are. □

`pos_eigenvector_unique`.

4 Existence, by compactness

The pinned `mathlib` revision has no Brouwer fixed-point theorem, so existence is obtained without one. That is a statement about this revision, not about the library in perpetuity.

Definition 4.1. $\Delta = \{x : \iota \to \mathbb{R} \mid x \geq 0,\ \sum_i x_i = 1\}$.

`simplexSet`, compact by `isCompact_simplexSet`: it is a closed subset of $[0,1]^\iota$, and $x_i \leq \sum_j x_j = 1$ puts it there.

Lemma 4.2 (Summing the constraint bounds the multiplier)**.** *If $x \in \Delta$ and $rx \leq Ax$ then $r \leq R := \sum_i \beta$, where β is the largest entry of A.*

Proof. Sum the constraint over i: the left side is $r\sum_i x_i = r$, and the right side is $\sum_i (Ax)_i \le \sum_i \beta \sum_j x_j = R$. $\square$

`exists_pos_eigenvector` contains this step. It is what makes the constraint set compact without any estimate on how small the entries of x can be — the usual device of restricting to a thickened simplex is not needed.

Theorem 4.3 (Existence)**.** *A strictly positive kernel has a strictly positive eigenvector, with strictly positive eigenvalue.*

Proof. Let $K = \{(r,x) \in [0,R] \times \Delta \mid rx \le Ax\}$. It is nonempty (take $r = 0$ and x a vertex) and compact (a closed constraint on a product of compacta), so r attains a maximum on it, at (λ, v) say.

Since $v \in \Delta$ it is nonnegative and not zero, so $Av > 0$ entrywise by Lemma 2.1. Suppose $Av \ne \lambda v$. Then $z = Av - \lambda v \ge 0$ is nonzero, so $Az > 0$ entrywise, i.e. $A(Av) > \lambda\, Av$ entrywise. Put $y = Av/\sum_i (Av)_i \in \Delta$; by homogeneity $Ay > \lambda y$ entrywise, and $y > 0$, so $\varepsilon = \min_i\big((Ay)_i - \lambda y_i\big)/y_i > 0$ satisfies $(\lambda + \varepsilon)y \le Ay$. By Lemma 4.2, $\lambda + \varepsilon \le R$, so $(\lambda + \varepsilon, y) \in K$ — contradicting maximality. Hence $Av = \lambda v$, and then $\lambda v_i = (Av)_i > 0$ forces $\lambda > 0$ and $v > 0$. $\square$

`exists_pos_eigenvector`.

Remark 4.4 (Why the maximiser is automatically positive)**.** *The simplex admits points with zero entries, and nothing in the definition of K excludes them. They are excluded* a posteriori*: once $Av = \lambda v$ is known, $v_i = (Av)_i/\lambda > 0$. Restricting the search to strictly positive vectors in advance would have cost a compactness argument that this order of steps avoids.*

5 The vacuum at every spatial extent

Definition 5.1. For a weight w on configurations, $K_w(\sigma,\tau) = w(\sigma)K(\sigma,\tau)$ with K the decoupled kernel of [1].

`sourceWeightedKernelL`, strictly positive when w is: `sourceWeightedKernelL_pos`.

Theorem 5.2 (The vacuum, at every L)**.** *For every L, every β, and every strictly positive w, the kernel K_w has a strictly positive eigenvector with strictly positive eigenvalue; it is unique up to positive scale; and its eigenvalue dominates every eigenvalue, real or complex.*

`vacuum_exists_of_sourceWeight`, `vacuum_unique_of_sourceWeight`, `vacuum_spectral_radius_of_sourceWeight`. For the concrete slice with a bond around the ring, `spatialWeightRing` and `coupled_ring_vacuum_exists`.

Theorem 5.3 (The normalised vacuum: $T\Omega = \Omega$ again)**.** *Let $T = \lambda^{-1}K_w$. Then at every L there is Ω with $\Omega > 0$, $\sum_\sigma \Omega(\sigma) = 1$, and $T\Omega = \Omega$.*

`normalizedKernel`, `normalizedKernel_fixes_eigenvector`, `normalized_vacuum_of_sourceWeight`, and for the ring slice `coupled_ring_normalized_vacuum`.

Remark 5.4 (The equation that failed, holding again)**.** *This is worth stating separately because of* which *equation it is. The companion chain obtained the vacuum from $T\Omega = \Omega$ for the* uniform Ω, *and [1] proved exactly that statement false as soon as a spatial coupling is present. Theorem 5.3 restores the same equation — for the Perron Ω instead of the uniform one, at every extent. The normalisation $\sum \Omega = 1$ is free: the existence proof already produces its vector inside the simplex.*

Remark 5.5 (What this settles about the first paper)**.** *[1] proved that the elementary route stops producing the vacuum, and was careful to claim no more: it did not assert the vacuum was absent, and it said a positive kernel on a finite set has a Perron vector. That sentence was prose there, cited from the classical theory. Here it is a theorem, and it applies to exactly the kernels that paper studied, at every L. So the obstruction is now located precisely: not in the object, and not in the model, but in the route.*

Remark 5.6 (And what it does not settle)**.** *Nothing in this paper produces a spectral gap. The obstructions of [2] are untouched: the projective metric is still blind to the coupling and still degenerates in the volume, and knowing that the vacuum exists does not repair either failure. Having the first object of the chain back is a precondition for the analytic work, not a substitute for it.*

6 Formalisation notes

The order of the steps did the work. The compactness argument is run on the plain simplex, which contains points with zero entries, and strict positivity of the maximiser is recovered afterwards from the eigen-equation. Attempting the reverse — searching only among strictly positive vectors — requires a thickened simplex and an invariance estimate; the version here needs neither.

Summing the constraint replaced an estimate. The bound keeping the constraint set compact is $r = r\sum_i x_i \le \sum_i (Ax)_i$, obtained by summing the very inequality that defines the set. A first design bounded r through the smallest entry of a thickened simplex and needed the ratio of the extreme entries of A; summing removed that entire layer, together with two auxiliary constants.

One lemma, both uniqueness statements. Lemma 2.2 is used for geometric simplicity and, through it, for uniqueness up to scale. In both cases the input is produced the same way: subtract the largest multiple that still fits underneath, and read the resulting zero entry.

A dependency discharged rather than routed around. The two companion developments each paid for the absence of Perron–Frobenius in a different currency — a weaker statement in one case, a bespoke proof in the other. Building the theorem once cost less than the second detour did.

7 Reproducibility

All artifacts are at source checkpoint `08a9050201287a31d20fc4339cc5c6c64b1391fb` on branch `main` of https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME. Every link was resolved against that commit, and checked to point at an actual declaration line, before compilation.

Quantity	Value
Full core build	8427 jobs, success
Oracle commands	2564
with axiom dependencies	2541
axiom-free	23
Distinct axioms across all outputs	`propext`, `Quot.sound`, `Classical.choice`
Occurrences of `sorryAx`	0
Project axioms	0
Errors	0

Verdicts are in `docs/VERIFICATION-LEDGER.md`. The oracle script is `oracle_check.lean`.

References

[1] L. Eriksson, *Where the Elementary Reconstruction Stops: Spatial Coupling Breaks the Uniform Vacuum, Machine-Checked*, viXra, 2026.

[2] L. Eriksson, *Blind to the Coupling: a Second Machine-Checked Obstruction at Spatial Extent*, viXra, 2026.

[3] E. Seneta, *Non-negative Matrices and Markov Chains*, 2nd ed., Springer, 2006.

[4] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, CADE-28, Lecture Notes in Computer Science **12699** (2021), 625–635.

[5] The mathlib Community, *The Lean mathematical library*, CPP 2020, 367–381.

[6] A. Jaffe and E. Witten, *Quantum Yang–Mills theory*, Clay Mathematics Institute Millennium Prize Problem description, 2000.