The Measure the Spectral Results Were About: a Machine-Checked Transfer Bridge for the Spatial $\mathbb{Z}_2$ Slice

the dressing identity, and the modulus
that a strict gap does not supply

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Abstract

Four companion papers studied an *operator*: a strictly positive kernel on the spatial configuration space of a $\mathbb{Z}_2$ slice, its Perron vacuum, and the strict separation of its spectrum [1, 2, 3, 4]. None of them exhibited a *measure*. That omission is the kind this programme is built to notice: a transfer operator is a matrix until something says which Boltzmann weights it transfers, and until then a vacuum is an eigenvector and a gap is a statement about eigenvalues. Neither is yet a statement about a statistical-mechanical system.

Proved. We define the two-dimensional Gibbs weight of the spatial system from Boltzmann factors alone — a spatial factor at every time slice, a time-bond factor between consecutive slices — and prove the *dressing identity*: that weight equals the path weight of the *symmetrised* kernel multiplied by a factor supported entirely on the two boundary slices. Hence every *unnormalised* Gibbs two-point sum of the spatial system is a matrix element of an iterated *self-adjoint* transfer operator between boundary-dressed observables, and the normalised expectation in the measure is the *ratio* of two such matrix elements — the numerator alone is not the correlation. The generic half holds for an arbitrary symmetric kernel on an arbitrary finite type; no positivity and no structure of the configuration space enter it. We further show that the operator the bridge lands on is the one the companion papers analysed, that the fluctuation sector is invariant, and that under an explicit contraction hypothesis the connected two-point function decays geometrically in the time separation.

Not proved, and this is the point of the last section. The contraction hypothesis is carried as a theorem hypothesis and is *not* discharged. The companion papers prove a *strict* gap with no modulus, and a strict inequality among finitely many eigenvalues does not by itself produce the operator-norm bound a decay rate requires. Converting one into the other needs the spectral maximum over the fluctuation sector, which is not constructed here. In particular **nothing uniform in the spatial extent** is obtained, and none is suggested.

Reflection positivity is not addressed. Nothing in this paper is a claim about $\mathrm{SU}(N)$, the continuum limit, or the Yang–Mills mass gap.

1 What was missing

The one-dimensional $\mathbb{Z}_2$ chain in this repository already has the bridge this paper builds. There the path weight is written from Boltzmann factors, with no operator occurring in its definition or in any definition it depends on, and a separate theorem identifies path sums with iterates of the transfer operator. That is what makes the chain's spectral statements statements about a measure.

The spatial development took a different route. Its object is a kernel on $(\mathbb{Z}_2)^L$ — the configurations of a single time slice — and it proceeded directly to the vacuum and the spectrum.

The results are real, but each of them is a statement about a matrix. The following is the honest inventory of what the four papers established and what they did not:

Object	Established	By
Kernel on $(\mathbb{Z}_2)^L$	yes	[1]
Vacuum (Perron vector)	yes, every L	[3]
Strict spectral separation	yes, every L	[4]
Gibbs measure	**no**	—
Partition function	**no**	—
Correlations as matrix elements	**no**	—

The last three rows are what this paper supplies. They are not a refinement of the earlier work; they are the reason the earlier work is about anything.

2 The generic bridge

Nothing in this section uses positivity, the two-element site, or the product structure of the configuration space. Let ι be a finite type and $K : \iota \times \iota \to \mathbb{R}$ any kernel.

Definition 2.1. For $X : \{0, \dots, N\} \to \iota$ the *path weight* is $W_K(X) = \prod_{t=0}^{N-1} K(X_t, X_{t+1})$, and for observables $A, B : \iota \to \mathbb{R}$ the *path sum* is

$$S_K(N; A, B) \;=\; \sum_X A(X_0)\, B(X_N)\, W_K(X).$$

pathWeight, pathSum. No operator occurs in either definition.

Theorem 2.2 (the bridge, generic form)**.** *If K is symmetric then $S_K(N; A, B) = \sum_i \big(T_K^N A\big)(i)\, B(i)$, where $(T_K A)(i) = \sum_j K(i, j) A(j)$.*

pathSum_eq_iterate.

The proof is an induction that absorbs one bond into the left observable at each step; symmetry of K is exactly what allows the bond to be read in either direction. The base case is the empty product.

3 The spatial Gibbs measure

Now the physics. A configuration of the two-dimensional system is a map $X : \{0, \dots, N\} \to (\mathbb{Z}_2)^L$: $N + 1$ time slices, each a spatial configuration.

Definition 3.1. With w a strictly positive spatial weight and K_β the decoupled time-bond kernel of [1],

$$\mathcal{W}(X) \;=\; \Big(\prod_{t=0}^{N} w(X_t)\Big)\Big(\prod_{t=0}^{N-1} K_\beta(X_t, X_{t+1})\Big).$$

gibbsWeight. The spatial factor appears at *every* slice and the time-bond factor between consecutive slices; this is the ordinary two-dimensional Boltzmann weight, written without reference to any operator.

The obstacle is that the natural transfer kernel $w(\sigma)K_\beta(\sigma,\tau)$ is *not* symmetric, so it is not self-adjoint and no spectral theory applies to it directly. The companion paper introduced the symmetrised form $\widetilde{K}(\sigma,\tau) = \sqrt{w(\sigma)}\,K_\beta(\sigma,\tau)\sqrt{w(\tau)}$ and proved it symmetric. The following identity is what that symmetrisation costs.

Theorem 3.2 (the dressing identity)**.** *For every space-time configuration X,*

$$\mathcal{W}(X) \;=\; \sqrt{w(X_0)}\;\sqrt{w(X_N)}\;\; W_{\widetilde{K}}(X).$$

`gibbsWeight_eq_dressed`.

Remark 3.3. *The entire discrepancy between the honest Gibbs weight and the path weight of the self-adjoint kernel is two factors, each supported on one boundary slice. Nothing is changed in the bulk. The proof is the observation that the truncated products $\prod_t \sqrt{w(X_t)}$ over the first N and over the last N slices are the full product with one endpoint removed, so their product with the two boundary factors is the full product squared.*

Corollary 3.4 (unnormalised sums are matrix elements)**.** *For all observables A, B,*

$$\sum_X A(X_0)B(X_N)\,\mathcal{W}(X) \;=\; \big\langle \sqrt{w}A,\; T_{\widetilde{K}}^N(\sqrt{w}B)\big\rangle,$$

and the partition function is the same expression at $A = B = 1$.

`gibbsPathSum_eq_iterate`, `gibbsPartition_eq_iterate`.

Corollary 3.5 (the expectation is a ratio)**.** *The normalised two-point function in the measure is*

$$\mathbb{E}\big[A(X_0)B(X_N)\big] \;=\; \frac{\big\langle \sqrt{w}A,\; T_{\widetilde{K}}^N(\sqrt{w}B)\big\rangle}{\big\langle \sqrt{w},\; T_{\widetilde{K}}^N\sqrt{w}\big\rangle},$$

and the denominator is strictly positive.

`gibbsCorr_eq_ratio_iterate`, `gibbsCorr_denom_pos`.

Remark 3.6. *The distinction between the two corollaries is not pedantry. A single matrix element is the* unnormalised *sum; the correlation the measure actually assigns divides by the partition function, so it is a ratio of two matrix elements of the same operator. Stating only the numerator would leave the word* correlation *doing work no theorem had done.*

This is the statement the four companion papers were missing. Their spectral results are now results about the correlations of a measure.

4 The operator is the one that was analysed

The bridge lands on $\widetilde{K}$, which is strictly positive whenever w is, so the existence theory of [3] applies verbatim: it has a strictly positive Perron vacuum, and after normalising by its eigenvalue the vacuum is a genuine fixed point of unit Euclidean norm.

`symVacuum_exists`.

That this needs saying at all is the point. A bridge that landed on a *different* operator would have proved nothing about the earlier papers.

5 The fluctuation sector, and what decay would need

Theorem 5.1 (invariance)**.** *If K is symmetric and $T_K\Omega = \Omega$, then $\langle \Omega, u\rangle = 0$ implies $\langle \Omega, T_K u\rangle = 0$, and hence $\langle \Omega, T_K^n u\rangle = 0$ for all n.*

`perp_invariant`, `iterate_perp`.

Theorem 5.2 (geometric decay under an explicit hypothesis)**.** *Suppose in addition that for some $r \geq 0$, $\|T_K u\| \leq r\|u\|$ for every u orthogonal to Ω. Then for every such u and every n,*

$$\left|\langle u,\ T_K^n u\rangle\right| \ \leq\ r^{\,n}\,\|u\|^2.$$

`connected_decay`. Specialised through the dressing identity, this bounds the Gibbs two-point function of the spatial system itself: `gibbs_connected_decay`.

Remark 5.3 (the hypothesis is not vacuous, and not trivial)**.** *A hypothesis-carrying theorem is worth nothing if nothing satisfies the hypothesis. For any unit vector Ω and any $r \geq 0$ the kernel $\Omega_i\Omega_j + r(\delta_{ij} - \Omega_i\Omega_j)$ is symmetric, fixes Ω, and contracts the fluctuation sector by exactly r — so the hypotheses are jointly satisfiable at every rate, including $0 < r < 1$ where the conclusion is non-trivial. Separately, whenever there are two distinct states and the vacuum is strictly positive, the fluctuation sector contains a nonzero vector, so Theorem 5.2 is not quantified over an empty set.*

`gapWitness_contracts`, `exists_nonzero_perp`.

6 Scope: what this paper does not give

The contraction hypothesis is not discharged. This is the substantive limitation and it is not a formality. The companion paper [4] proves that every eigenvalue other than the Perron eigenvalue is *strictly* smaller in modulus. That is a statement about finitely many strict inequalities. The hypothesis of Theorem 5.2 is an *operator-norm* bound on the fluctuation sector. In finite dimension the first does imply the second — take the maximum of $|\mu|/\lambda$ over the finitely many non-Perron eigenvalues — but that maximum has to be constructed, together with the enumeration of the spectrum and the spectral decomposition of the symmetric kernel that makes the norm equal to it. None of that exists in this development, so no rate is derived here from the earlier papers.

Nothing uniform in the spatial extent. Even granting the previous paragraph, the constructed r would depend on L. The measured evidence in [4] is that it approaches 1 with L outside the disordered region. This paper adds nothing to that question in either direction, and the reader should not read the geometric decay of Theorem 5.2 as a mass gap of any kind: with $r = r(L) \to 1$ the bound is empty in the limit.

Reflection positivity is untouched. The bridge gives a self-adjoint transfer operator and a positive measure; it does not give the Osterwalder–Schrader positivity that a reconstruction theorem would need, and no claim in that direction is made.

No claim about the physical target. Nothing here concerns $\mathrm{SU}(N)$, the continuum limit, or the Clay problem [7]. The distance to that problem is unchanged by this paper, and remains effectively total.

7 What is verified and not proved

A machine-checked proof guarantees that the definitions are internally consistent. It does not guarantee that they model the object the prose claims they model. For that reason both load-bearing identities — the dressing identity and the bridge corollary — were recomputed by brute force from the Boltzmann weights, by an independent script that has no access to the formalisation, over all space-time configurations for $L \leq 3$ and $N \leq 3$ with a genuinely configuration-dependent strictly positive spatial weight. The worst relative discrepancy over all cases was 9.1×10^{-16}, at machine precision.

This is **verified**, not proved, and no theorem depends on it. Its function is to exclude the failure mode a proof assistant cannot exclude: a formally correct theorem about the wrong definition. The script and its transcript are in the repository.

8 Formalisation notes

The generic half carries the work. Theorem 2.2 is stated for an arbitrary symmetric kernel on an arbitrary finite type, and the spatial content enters only when it is applied. This was not generality for its own sake: the induction that proves it is exactly the induction the one-dimensional chain already used, and stating it generically is what let the proof be reused rather than rewritten.

Where symmetry is actually needed. Only in Theorem 2.2, to absorb a bond into the left observable. The dressing identity itself is an identity between products and uses no symmetry at all; it is what *supplies* the symmetric kernel that Theorem 2.2 then consumes.

A rewrite that had to be done in one step. The dressing identity combines three product identities. Rewriting them in sequence fails: the second rewrite consumes the subterm the third one needs. They are combined into a single equation before rewriting. The failure is silent in the sense that matters — it produces a stuck goal, not a wrong theorem — but it is recorded because the same shape recurs whenever two truncations of one product appear on the same side.

9 Reproducibility

All artifacts are at source checkpoint `c4fa6a9e6769496a7270981ef0908b2d644c230b` on branch `main` of the repository. Every hyperlink in this paper was resolved against that commit, and checked to point at an actual declaration line, before compilation.

Quantity	Value
Full core build	8429 jobs, success
Oracle commands	2626
with axiom dependencies	2603
axiom-free	23
Distinct axioms across all outputs	`propext`, `Quot.sound`, `Classical.choice`
Occurrences of `sorryAx`	0
Project axioms	0
Errors	0

The module is `YangMills/OS/SpatialGibbs.lean`. The numerical probe is `scripts/probe_spatial_gibbs.py`. Verdicts are in `docs/VERIFICATION-LEDGER.md`.

References

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[2] L. Eriksson, *Blind to the Coupling: a Second Machine-Checked Obstruction at Spatial Extent*, viXra, 2026.

[3] L. Eriksson, *The Vacuum Was Never Absent: A Machine-Checked Perron Theorem for Strictly Positive Kernels, and the Coupled-Slice Vacuum at Every Spatial Extent*, viXra, 2026.

[4] L. Eriksson, *Strict but Not Uniform: a Machine-Checked Spectral Gap at Every Finite Extent of the Coupled Slice*, viXra, 2026.

[5] L. de Moura and S. Ullrich, *The Lean 4 Theorem Prover and Programming Language*, CADE 28 (2021), 625–635.

[6] The mathlib Community, *The Lean Mathematical Library*, CPP 2020, 367–381.

[7] A. Jaffe and E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute Millennium Problem Description, 2000.