The Quotient That Is Not the Identity: A Machine-Checked Degenerate Reflection Pairing and Its Gelfand–Naimark–Segal Quotient

Four observables, two states, and the collapse in between

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Abstract

The Osterwalder–Seiler reconstruction passes from a reflection-positive measure to a Hilbert space by quotienting out the null space of the reflected pairing. In a companion development that step was present but did nothing: the pairing there was definite, so the quotient was the identity, and that paper says so in its own abstract. This paper supplies the missing case, in Lean 4 with `mathlib`.

For the $\mathbb{Z}_2$ lattice gauge chain we take half-space observables of two time slices — a four-dimensional space — and form the reflected pairing directly from the Boltzmann weights. For $\beta > 0$ the reconstructed physical space is two-dimensional. Integrating out the future collapses four observables onto two states, and that collapse is the null space. We prove: the pairing factors through an independently defined reconstruction map Φ; its self-pairing rearranges into a manifest sum of two non-negative terms, from which positivity and the null space follow together; the null space is *exactly* $\ker \Phi$, not merely non-empty; an explicit non-zero observable lies in it; and the quotient is isomorphic to the physical space *by the map* Φ *itself*, not by a dimension count.

The degeneracy is the mechanism the reconstruction exists to handle, and the one expected to reappear in systems with larger half-space algebras. What is not claimed: this is two time slices and not m; still $\mathbb{Z}_2$, one variable per slice, fixed finite size, and not volume-uniform; $\mathbb{Z}_N$ for $N > 2$ is untouched; and the completion step of the reconstruction is trivial here because every space in sight is finite-dimensional, which we state rather than present as work done. Nothing in this paper is a claim about $\mathrm{SU}(N)$, the continuum limit, or the Yang–Mills mass gap.

1 Introduction

1.1 The step that usually does nothing in small examples

Reflection positivity gives a positive *semi*-definite pairing on half-space observables. The Hilbert space of the reconstruction is obtained by quotienting out its null space and completing [1, 2]. In examples small enough to be fully verified, the pairing is often already definite, and then the quotient is the identity map — the step is performed vacuously. A formalisation in that situation has not exercised the construction; it has avoided it.

That was the position of a companion development [3], which verified an end-to-end reconstruction for the $\mathbb{Z}_2$ chain and stated in its abstract that its own quotient did no work. This paper removes that limitation by exhibiting, for the same measure, a pairing whose null space is provably non-zero, computing that null space exactly, and identifying the quotient.

1.2 Where the degeneracy comes from

Not from a contrivance. Take half-space observables depending on *two* time slices, $A : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}$. That space is four-dimensional. For $\beta > 0$ the reconstructed physical space is two-dimensional. A four-dimensional space cannot inject into a two-dimensional one, so the pairing *must* be degenerate; the content is to compute the degeneracy rather than to observe that it exists. Physically, the collapse is the statement that the future has been integrated out: two observables that agree after that integration are the same state.

This is the mechanism, and it is the one expected to reappear in systems with larger half-space algebras: a lattice with spatial extent has half-space observables in far greater number than its transfer operator has states. We state that as an expectation about the systems targeted next, not as a theorem.

1.3 What is proved

Write $k_\beta(i, j) = e^{\beta s(i,j)}$ for the Boltzmann bond weight, $s = +1$ on equal spins and -1 otherwise. The reflected pairing on two-slice observables is the explicit finite sum

$$\langle A,\, B\rangle \;=\; \sum_{a,b,c,d} \overline{A(b,a)}\; B(c,d)\; k_\beta(a,b)\, k_\beta(b,c)\, k_\beta(c,d),$$

the four spins being $\sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1$ with the reflection carrying the half-space $\{0, 1\}$ to $\{-1, -2\}$. Separately we define

$$(\Phi A)(c) \;=\; \sum_d k_\beta(c,d)\, A(c,d).$$

Then:

- $\langle A,\, B\rangle = \sum_{b,c} \overline{(\Phi A)(b)}\, k_\beta(b,c)\, (\Phi B)(c)$ — the pairing factors through Φ;
- $\langle A,\, A\rangle = (e^\beta - e^{-\beta})\big(|v_0|^2 + |v_1|^2\big) + e^{-\beta}|v_0 + v_1|^2$ with $v = \Phi A$, a manifest sum of two non-negative terms;
- hence $\langle A,\, A\rangle \geq 0$ for $\beta \geq 0$, and for $\beta > 0$, $\langle A,\, A\rangle = 0 \iff \Phi A = 0$;
- an explicit $A \neq 0$ with $\Phi A = 0$;
- $\big(\mathbb{C}^{2\times 2}\big)/\ker \Phi \cong \mathbb{C}^2$, the isomorphism induced by Φ;
- and, for $\beta > 0$, the form the quotient carries is positive definite, so it is a pre-Hilbert space — though the identification with $\mathbb{C}^2$ is linear, not isometric.

1.4 What is deliberately not claimed

1. **Two slices, not m.** The construction is carried out at depth two. Depth m is not treated.
2. **Still the smallest gauge group and no spatial extent.** One $\mathbb{Z}_2$ variable per slice; for $\beta > 0$ the physical space is two-dimensional (and one-dimensional at $\beta = 0$, by Remark 3.4).
3. **Fixed finite size, not volume-uniform.** No statement here is uniform in a volume, and at this size the question does not arise.

4. **The completion is trivial.** Every space here is finite-dimensional, so the completion half of the reconstruction is vacuous. We state that; we do not present it as work done.
5. **Nothing about $\mathrm{SU}(N)$, the continuum limit, or the Clay problem** [6]. No reduction from any statement in this paper to the Yang–Mills mass gap is claimed, and none is known to the author.

2 The pairing, and the map

Definition 2.1 (Reflected pairing)**.** For $A, B : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}$ set $\langle A,\, B\rangle$ to be the four-fold sum displayed above.

In Lean this is `reflPairing`, built from `z2Bond` and from nothing else.

Remark 2.2 (Non-circularity)**.** *The definition of the pairing mentions no reconstruction map, no transfer operator, and neither coefficient of the normalised transfer matrix. Printing it exhibits its whole dependency set. This is the condition that makes Theorem 2.4 a theorem about two independently defined objects rather than a restatement of a definition; the same discipline governs the companion development [3].*

Definition 2.3 (Reconstruction map)**.** $(\Phi A)(c) = \sum_d k_\beta(c, d) A(c, d)$, defined without reference to the pairing: `slicePhi`, and as a linear map `slicePhiLin`.

Theorem 2.4 (The pairing factors through Φ)**.** $\langle A,\, B\rangle \;=\; \sum_{b,c} \overline{(\Phi A)(b)}\; k_\beta(b, c)\; (\Phi B)(c).$

Proof, as formalised. Group the four-fold sum as $\sum_{b,c}\big[\sum_a \overline{A(b, a)} k_\beta(a, b)\big] k_\beta(b, c)\big[\sum_d k_\beta(c, d) B(c, d)\big]$. The inner left bracket is $\overline{(\Phi A)(b)}$ because k_β is real and symmetric; the inner right bracket is $(\Phi B)(c)$. □

Artifact: `reflPairing_eq`.

3 One identity, two consequences

Theorem 3.1 (Sum of two non-negative terms)**.** *With $v = \Phi A$,*

$$\langle A,\, A\rangle \;=\; \big(e^{\beta} - e^{-\beta}\big)\big(|v_0|^2 + |v_1|^2\big) \;+\; e^{-\beta}\, |v_0 + v_1|^2.$$

Proof. Expand Theorem 2.4 over the two-element index set, with $k_\beta(i, i) = e^{\beta}$ and $k_\beta(i, j) = e^{-\beta}$ for $i \neq j$, and complete the square in $v_0 + v_1$. □

Artifact: `reflPairing_self`. Both facts we need come from this one line.

Corollary 3.2 (Reflection positivity)**.** $\langle A,\, A\rangle \;\geq\; 0$ *for every* $\beta \geq 0$.

`reflPairing_self_nonneg`.

Corollary 3.3 (The null space, exactly)**.** *For $\beta > 0$:* $\langle A,\, A\rangle = 0 \iff \Phi A = 0$.

`reflPairing_self_eq_zero_iff`.

Remark 3.4 (Why $\beta > 0$, said rather than avoided)**.** *At $\beta = 0$ the bond weight is constant and the kernel k_β itself degenerates, so the coefficient $e^\beta - e^{-\beta}$ of Theorem 3.1 vanishes and the equivalence of Corollary 3.3 fails. Positivity survives at $\beta = 0$; definiteness does not: at $\beta = 0$ the whole line $v_0 = -v_1$ becomes null, so the null space of the pairing is strictly larger than* $\ker \Phi$ *and the reconstructed space drops from two dimensions to one. That is a theorem here, not an asserted caveat:* `kForm_not_definite_at_zero` *exhibits the witness. The hypotheses in the formalisation record exactly this split rather than excluding the point silently.*

4 The null space is not zero

This is the point of the paper: without it the construction would have reproduced the trivial case.

Theorem 4.1 (Explicit null vector)**.** *Let $N(c, 0) = k_\beta(c, 1)$ and $N(c, 1) = -k_\beta(c, 0)$. Then $\Phi N = 0$ and $N \neq 0$; consequently $\langle N, N\rangle = 0$ for $\beta > 0$ and the null space is non-trivial.*

Proof. $(\Phi N)(c) = k_\beta(c, 0)k_\beta(c, 1) - k_\beta(c, 1)k_\beta(c, 0) = 0$, exactly. $N \neq 0$ because $N(0, 0) = k_\beta(0, 1) = e^{-\beta} > 0$: the Boltzmann weights are strictly positive, which is the only property of them used here. □

Artifacts: `nullObs`, `slicePhi_nullObs`, `nullObs_ne_zero`, `reflPairing_nullObs`, and as a submodule statement `nullSubmodule_ne_bot`.

5 The quotient is the physical space

Theorem 5.1 (Quotient)**.** *Φ is surjective, and the induced map*

$$\big(\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}\big) \;/\; \ker \Phi \;\; \xrightarrow{\;\sim\;} \;\; \mathbb{C}^2$$

is a linear isomorphism. For $\beta > 0$ the kernel is exactly the null space of the pairing, so this is the Gelfand–Naimark–Segal quotient of the reflected pairing.

Artifacts: `slicePhi_surjective`, `nullSubmodule`, `mem_nullSubmodule_iff`, `quotEquivPhysical`.

Remark 5.2 (Induced by the map, not by a count)**.** *Four and two are the dimensions, and a reader could observe that a two-dimensional quotient of a four-dimensional space by a two-dimensional subspace is $\mathbb{C}^2$ for trivial reasons. That is not what is proved. The isomorphism here is the one* induced by Φ*, so the quotient is the physical space* because of what the reconstruction map does to observables*. A dimension count would leave the identification unanchored to the physics.*

Remark 5.3 (Which inner product the quotient carries)**.** *Theorem 5.1 is a* linear *isomorphism, and it is worth saying precisely what that does and does not settle. The Gelfand–Naimark–Segal inner product does not transport to the standard Euclidean structure of $\mathbb{C}^2$; it transports to the kernel-weighted form*

$$\langle v,\, w\rangle_{k_\beta} \;=\; \sum_{b,c} \overline{v(b)}\, k_\beta(b, c)\, w(c),$$

and the pairing of this paper is exactly that form pulled back along Φ*:* `reflPairing_eq_kForm`*. For* $\beta > 0$ *this form is positive definite —* `kForm_definite` *— so the quotient is a genuine pre-Hilbert space and not merely a vector space, which is what entitles us to call it the Gelfand–Naimark–Segal quotient. An* isometric *identification with standard Euclidean* $\mathbb{C}^2$ *would additionally require the positive square root of* k_β*; that is not constructed here, and no isometry is claimed.*

Remark 5.4 (What this adds to the companion paper)**.** *[3] verified the chain measure → reflection positivity → Hilbert space → transfer operator → identification → decay, and proved that* its *quotient step was the identity. That statement was true of that system and is unaffected. The present paper does not amend it; it supplies the case that paper did not cover. The two together exercise both branches: a pairing where the quotient is the identity, with that proved, and a pairing from the same measure where it is not.*

6 Formalisation notes

Two decisions are worth recording.

Independently defined, over a shared primitive. The pairing and the reconstruction map are defined independently over the same primitive Boltzmann bond weight k_β. Neither is defined from the other — that is the property non-circularity needs — and their first interaction is Theorem 2.4. This is what makes Remark 2.2 checkable by printing a definition, and it is the same architecture as the companion development.

Pattern matching instead of conditionals. The witnesses of Theorems 4.1 and 5.1 were first written with a conditional on the two-element index. A conditional on a decidable equality normalises to a form that the obvious simplification lemma does not match, and three routine proofs failed with goals displaying an unreduced conditional. Defining the witnesses by pattern matching instead makes their two values reduce definitionally, and the proofs become rewrites backed by reflexivity. The general lesson — remove the awkward term from the statement rather than defeat it inside the proof — is the same one that governs the algebraic steps of the companion development.

7 Reproducibility

All artifacts are at source checkpoint `673d1df122ac7110e6f7b52795b44a02d38d8d24` on branch `main` of https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME. Every link in this paper was resolved against that commit, and checked to point at an actual declaration line, before compilation.

Quantity	Value
Full core build	8423 jobs, success
Oracle commands	2460
with axiom dependencies	2438
axiom-free	22
Distinct axioms across all outputs	`propext`, `Quot.sound`, `Classical.choice`
Occurrences of `sorryAx`	0
Project axioms	0
Errors	0

The criteria this work was required to meet — including the non-circularity condition of Remark 2.2 and the requirement that the null space be proved non-zero — were fixed in `docs/O-BRIDGE-CHARTER.md`, Amendment 9, in a commit preceding the existence of the module they govern. Their verdicts are in `docs/VERIFICATION-LEDGER.md`, Addendum 516. The oracle script is `oracle_check.lean`.

8 Continuation

The next step is spatial extent: a transfer operator on a space that grows with the volume, where the half-space algebra is larger still and the quotient of this paper becomes indispensable rather than merely non-trivial. That is where the present style of construction is expected to stop being easy, and it is the first step after which volume-uniformity — the only property in this neighbourhood that could eventually bear on a mass gap — becomes a meaningful question. It is not reached here.

References

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[2] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed., Springer, 1987.

[3] L. Eriksson, *From the Gibbs Weight to the Spectral Gap: A Complete Machine-Checked Osterwalder–Seiler Chain for the $\mathbb{Z}_2$ Lattice Gauge Chain*, 2026.

[4] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, CADE-28, Lecture Notes in Computer Science **12699** (2021), 625–635.

[5] The mathlib Community, *The Lean mathematical library*, CPP 2020, 367–381.

[6] A. Jaffe and E. Witten, *Quantum Yang–Mills theory*, Clay Mathematics Institute Millennium Prize Problem description, 2000.