A Machine-Checked Reflection-Positivity Framework for $\mathbb{Z}_N$ Lattice Gauge Theory, with the $\mathbb{Z}_2$ Wilson Instance

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Abstract

We machine-check, in Lean 4 with no `sorry` and no project axioms, the Osterwalder–Seiler reflection positivity of a lattice gauge theory with finite abelian gauge group. The development is organised so that the three ingredients are separated and each is proved on its own: an analytic step, a geometric step, and the single place where a property of the Boltzmann factor is actually used.

The analytic step is that a crossing kernel of the form $K(x,y) = \sum_i c_i \, \varphi_i(x)\overline{\varphi_i(y)}$ with $c_i \geq 0$ is positive semidefinite, that this class is closed under products, and that it is closed under conjugation by a positive diagonal. Formulating the hypothesis as a non-negative combination of characters rather than as non-negativity of Fourier coefficients removes any need for Bochner's theorem on a finite abelian group and for the Schur product theorem: the development uses no spectral and no matrix-positivity API.

The geometric step is a splitting of the configuration space across the reflection plane under which the reflection is the swap and the Gibbs weight factors as $w(x)\,w(y)\,K(x,y)$. We prove that the Osterwalder–Seiler pairing of an observable of one half against its reflection is then *exactly* the quadratic form of $w(x)K(x,y)w(y)$, so that reflection positivity follows from the analytic step.

The physical step is the instance. For $\mathbb{Z}_2$ the Wilson factor $e^{\beta s}$, $s = \pm 1$, expands in the two characters with coefficients $(e^{\beta} \pm e^{-\beta})/2$, both non-negative exactly when $\beta \geq 0$; so the $\mathbb{Z}_2$ Wilson crossing kernel is positive semidefinite at non-negative coupling. For $\mathbb{Z}_N$ with $N > 2$ the coefficients are discrete Bessel-type sums and their non-negativity is *not* established here.

We are explicit about what is absent: no Gelfand–Naimark–Segal quotient, no transfer operator, no identification of a Euclidean correlator with a matrix element, and therefore no mass gap. Nothing here is a claim about $\mathrm{SU}(N)$, the continuum limit, or the Clay problem.

1 What is proved, and what is not

Reflection positivity is the hypothesis from which Osterwalder and Seiler [1] construct the physical Hilbert space of a lattice gauge theory and its transfer operator. It is the first of four steps between a Euclidean lattice measure and a statement about a spectrum, the others being the Gelfand–Naimark–Segal quotient, the transfer operator itself, and the identification of Euclidean correlators with its matrix elements.

This paper mechanizes the first step for a finite abelian gauge group, and *only* the first step. In particular the following are not established here, nor anywhere in the Lean development accompanying this paper: the GNS quotient; the existence of a transfer operator for any gauge theory; the identification $\mathbb{E}[A \cdot \theta_n B] = \langle A\Omega,\, T^n B\Omega\rangle$; and any statement whatever about a mass gap. A companion development [3] proves the operator-theoretic criterion that would consume such a transfer operator once it exists; the two are not yet joined, and we do not claim they are.

2 The gauge system

Let $N \geq 1$ and let Edge, Plaq be finite types. A *gauge system* is an incidence inc : Plaq $\to$ Edge $\to \mathbb{Z}_N$, a strictly positive single-plaquette Boltzmann factor weight : $\mathbb{Z}_N \to \mathbb{R}$, and an involution θ of the edge set. A configuration is U : Edge $\to \mathbb{Z}_N$; the holonomy of a plaquette is the incidence-weighted sum

$$\mathrm{hol}(U,p) \;=\; \sum_e \mathrm{inc}(p,e)\, U(e),$$

which for an abelian group is the whole content of "the product around the boundary"; and the Gibbs weight is $\mathrm{boltz}(U) = \prod_p \mathrm{weight}(\mathrm{hol}(U,p))$. Every expectation is the finite sum

$$\mathbb{E}[F] \;=\; \Big(\sum_U \mathrm{boltz}(U)\, F(U)\Big)\Big/ Z, \qquad Z = \sum_U \mathrm{boltz}(U) > 0,$$

so no measure theory occurs anywhere in the development.

Carrying the lattice geometry as data rather than fixing a box means the analytic content is proved once, for every lattice and every reflection at once. The price is that reflection invariance of the weight is a hypothesis rather than a consequence; it is a property of the geometry, and each concrete system discharges it.

3 The analytic step

Definition 3.1. For a kernel $K : X \times X \to \mathbb{C}$ on a finite type write

$$Q_K(F) \;=\; \sum_x \sum_y \overline{F(x)}\, K(x,y)\, F(y),$$

and call K *positive semidefinite* when $Q_K(F)$ is a non-negative real for every F. (Stated in that form, reality is part of the conclusion rather than a side condition.)

Definition 3.2 (non-negative character combination)**.** K is a non-negative combination for the family φ when $c_i \geq 0$ for all i and $K(x,y) = \sum_i c_i\, \varphi_i(x)\overline{\varphi_i(y)}$.

Theorem 3.3. *A non-negative character combination is positive semidefinite.*

Proof. Put $w_i = \sum_y \overline{\varphi_i(y)} F(y)$. Resumming, $Q_K(F) = \sum_i c_i\, \overline{w_i}\, w_i = \sum_i c_i |w_i|^2 \geq 0$. □

Proposition 3.4. *The class is closed under entrywise products and under conjugation by a diagonal.*

Proof. For products, index by pairs: $\varphi_i \psi_j$ is again a family and $c_i d_j \geq 0$. For the diagonal, replace φ_i by $D\varphi_i$; the coefficients are unchanged. □

Remark 3.5. *Proposition 3.4 is what would otherwise be the Schur product theorem, and Theorem 3.3 is what would otherwise be one direction of Bochner's theorem on a finite abelian group. Defining the class by the shape of the combination rather than by a condition on Fourier coefficients makes both elementary, and the development consequently uses no spectral and no matrix-positivity API at all.*

4 The geometric step

Definition 4.1 (reflection splitting)**.** A *splitting* of a gauge system consists of a finite type Half, an equivalence Config $\simeq$ Half $\times$ Half under which the reflection acts as the swap, a strictly positive half-space factor w, and a crossing kernel K, such that

$$\mathrm{boltz}(U) \;=\; w(x)\,w(y)\,K(x,y), \qquad (x,y) = \text{the two halves of } U.$$

Definition 4.2. For F a function of one half-space configuration, the Osterwalder–Seiler pairing is

$$\langle F\rangle_{\mathrm{OS}} \;=\; \sum_U \mathrm{boltz}(U)\,\overline{F(x_U)}\,F(y_U).$$

Theorem 4.3 (the identity that does the work)**.** *$\langle F\rangle_{\mathrm{OS}} = Q_M(F)$ with $M(x,y) = w(x)K(x,y)w(y)$.*

Proof. Reindex the sum over configurations by the splitting and factor the weight. □

Corollary 4.4 (reflection positivity)**.** *If the crossing kernel is a non-negative character combination then $\langle F\rangle_{\mathrm{OS}}$ is a non-negative real for every F.*

Proof. By Proposition 3.4 the half-space factors are absorbed into the combination, and Theorem 3.3 applies to M; conclude with Theorem 4.3. □

5 The physical step: where a Boltzmann factor is used

Sections 3 and 4 assume nothing about any particular weight. The single place a property of the Boltzmann factor is used is here.

Lemma 5.1 (weights to kernels)**.** *Let X be a finite abelian group and let χ be a family with $\chi_i(x-y) = \chi_i(x)\overline{\chi_i(y)}$. If $k(u) = \sum_i c_i\chi_i(u)$ with $c_i \geq 0$, then $(x,y) \mapsto k(x-y)$ is a non-negative character combination.*

Theorem 5.2 (the $\mathbb{Z}_2$ Wilson instance)**.** *For $\mathbb{Z}_2$ the Wilson factor is $e^{\beta s}$ with $s = \pm 1$. Its expansion in the two characters of $\mathbb{Z}_2$ is*

$$e^{\beta s} \;=\; c_0 + c_1\,\chi_1, \qquad c_0 = \frac{e^{\beta}+e^{-\beta}}{2}, \quad c_1 = \frac{e^{\beta}-e^{-\beta}}{2},$$

and $c_0 \geq 0$ always while $c_1 \geq 0$ exactly when $\beta \geq 0$. Hence for $\beta \geq 0$ the $\mathbb{Z}_2$ Wilson crossing kernel is a non-negative character combination, and by Corollary 4.4 the pairing is positive. For $\beta > 0$ the coefficient c_1 is strictly positive, so the weight is genuinely non-constant and the instance is not the degenerate $\beta = 0$ one.

The coefficients are written in that explicit form, rather than as cosh and sinh, so that the mechanized proof needs no hyperbolic-function lemma: the non-negativity of c_1 is monotonicity of exp.

Remark 5.3 (the limit of the instance)**.** *Theorem 5.2 settles $\mathbb{Z}_2$ at non-negative coupling and nothing more. For $\mathbb{Z}_N$ with $N > 2$ the character coefficients of the Wilson factor are discrete Bessel-type sums; their non-negativity is a genuine analytic fact and is* not *established here. We state it as the natural next target rather than as folklore we are entitled to.*

6 One instance carrying all four ingredients

The predicates above are of a shape that admits degenerate inhabitants — positive semidefiniteness is satisfied by the zero kernel, a splitting by a one-point system — so a single named endpoint is given in which nothing is degenerate.

Theorem 6.1 (endpoint)**.** *Consider the $\mathbb{Z}_2$ gauge system with two edges exchanged by the time reflection and one plaquette whose boundary is both of them, carrying the genuine Wilson factor $e^{\beta s}$. Its plaquette holonomy is $x+y$, which in $\mathbb{Z}_2$ equals $x-y$: the plaquette* straddles the reflection plane*, so the entire Gibbs weight is the crossing kernel and there is no half-space factor at all. Then for every $\beta \geq 0$ and every observable F of one half, $\langle F\rangle_{\mathrm{OS}}$ is a non-negative real. For $\beta > 0$ the weight takes two distinct values, so the Gibbs measure is genuinely non-uniform and the reflection genuinely exchanges the two edges.*

This is reflection positivity obtained by *discharging* a gauge system, a splitting and a character expansion — not by assuming a pairing, and not with a constant weight.

Remark 6.2 (the size of the instance)**.** *It is a two-edge system. A full temporal box, with spatial extent, several time slices and plaquettes of all three kinds — wholly on one side, wholly on the other, and straddling — is* not *treated here. The splitting structure is precisely the interface such a box would have to discharge, and discharging it is geometric bookkeeping this paper does not perform. What the endpoint establishes is that the three steps compose on a system whose weight is the real Wilson factor at positive coupling, not that any physically interesting lattice has been handled.*

7 What remains

Within the Osterwalder–Seiler chain the remaining obligations are the GNS quotient of the pairing proved positive here, the transfer operator with $0 \leq T \leq 1$ fixing the vacuum, and the identification of Euclidean correlators with its matrix elements. The first of these is within reach of the library as it stands: a positive semidefinite sesquilinear form is a `PreInnerProductSpace.Core`, which requires no definiteness, and the quotient by its null space is the separation quotient of the induced seminorm, which carries an inner-product-space instance already. We record that as a measured observation about the library, not as work done.

We do not claim that list is exhaustive for any particular notion of mass gap, and we do not claim any of it here.

8 Reproducibility and verification

All statements are machine-checked in Lean 4 (toolchain `leanprover/lean4:v4.29.0-rc6`) against Mathlib pinned to commit `07642720480157414db592fa85b626dafb71355b`. The freeze is the source tree at commit `bbadf68cb9d2433e2e3f88f910e7240acb6ae21d`. Build and oracle figures are recorded in the repository's verification ledger at that commit. Every declaration reachable from the core root is subject to an axiom oracle; the transcript records that the axiom sets occurring are subsets of `[propext, Classical.choice, Quot.sound]`, with zero occurrences of `sorryAx` and no project axioms.

Statement here	Lean name	Mod.
Gauge system, weight, expectation	`GaugeData`, `boltz`, `expect`	Z
Reflection on configurations	`reflConfig`, `ReflInvariant`	Z
Q_K, positive semidefiniteness	`kernelForm`, `IsPSDKernel`	P
Character combination	`IsCharCombo`	P
Theorem 3.3	`isPSDKernel_of_charCombo`	P
Prop. 3.4	`isCharCombo_mul`, `isCharCombo_diagConj`	P
Splitting	`ReflectionSplitting`	R
OS pairing	`osPairing`, `osKernel`	R
Theorem 4.3	`osPairing_eq_kernelForm`	R
Cor. 4.4	`osPairing_nonneg`	R
Lemma 5.1	`isCharCombo_of_convolution`	W
Theorem 5.2	`z2Wilson_isCharCombo`, `z2Wilson_isPSDKernel`, `z2Coeff_one_pos`	W
Non-vacuity of the chain	`minimalSplitting_reflectionPositive`	R
Thm. 6.1	`z2Wilson_reflectionPositive`, `z2WilsonSystem_nondegenerate`, `z2WilsonSystem_refl_ne_id`	E

Modules: Z = `YangMills/OS/ZNSubstrate.lean`; P = `YangMills/OS/PSDKernel.lean`; R = `YangMills/OS/ReflectionSplitting.lean`; W = `YangMills/OS/WilsonCharCombo.lean`; E = `YangMills/OS/Z2Endpoint.lean`. Core root: `YangMillsCore.lean`; oracle driver: `oracle_check.lean`; governance and scope record: `docs/O-BRIDGE-CHARTER.md`.

Provenance

Reflection positivity for lattice gauge theories is due to Osterwalder and Seiler [1], and the character-expansion criterion for abelian models is classical; **no mathematical novelty is claimed for either**. What is offered is the mechanization, and the separation of the argument into an analytic step, a geometric step, and a single point of contact with a Boltzmann factor, which is what makes the $\mathbb{Z}_N$, $N > 2$ gap in Remark 5.3 visible as a precise open statement rather than a hand-wave.

References

[1] K. Osterwalder and E. Seiler, *Gauge field theories on a lattice*, Ann. Physics **110** (1978), 440–471.

[2] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, 2nd ed., Springer, 1987.

[3] L. Eriksson, *Clustering and the transfer-operator gap: a machine-checked dense-family criterion*, 2026.

[4] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, CADE 28 (2021), 625–635.

[5] The mathlib Community, *The Lean mathematical library*, CPP 2020, 367–381.