An Algebraic Field Parity Analysis of $a^n + b^n = c^n$ via Parameterized Law of Cosines Modifiers

Yaakov Abdelhak MD
Independent Researcher
yadelivery@gmail.com

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Abstract

We investigate the integer solutions of the Diophantine relation $a^n + b^n = c^n$ for non-square integers when $n > 2$. By projecting parameters $a^{n/2}$ and $b^{n/2}$ as the functional side lengths of a variable geometric boundary, we leverage the Law of Cosines (LOC) to track rational and irrational distributions. Using the integer constraints inherent to Fermat's hypothesis, we demonstrate a structural field mismatch under acute angular transformations, bypassing perfect square integers handled via classical infinite descent.

1 Scope and Constraints

This proof is explicitly constrained to non-perfect square positive integers. Cases involving perfect square integers are omitted from this text, as their non-existence properties are independently established via Fermat's classical method of infinite descent.

2 The Parameterized Law of Cosines Model

Let a, b, and c be coprime, non-square positive integers. To evaluate the hypothetical equation $a^3 + b^3 = c^3$, we define a geometric system by choosing two arbitrary operational side lengths:

- **Side X** $= a^{3/2}$ (which evaluates to $a\sqrt{a}$)
- **Side Y** $= b^{3/2}$ (which evaluates to $b\sqrt{b}$)
- **Side Z** $=$ The third side of the geometric configuration

By the geometric definition of the Law of Cosines, the angle C opposite to side Z dictates that:

$$\text{Side } X^2 + \text{Side } Y^2 = \text{Side } Z^2 + 2 \cdot \text{Side } X \cdot \text{Side } Y \cdot \cos(C)$$

Substituting our parameterized side lengths into this geometric identity yields the base baseline equation:

$$a^3 + b^3 = \text{Side } Z^2 + 2ab\sqrt{ab}\cos(C)$$

Isolating the angle value from the physical structure of the triangle gives the exact required value for the cosine component:

$$\cos(C) = \frac{a^3 + b^3 - \text{Side } Z^2}{2ab\sqrt{ab}}$$

3 The Structural Field Parity Contradiction

When analyzing the balance relation during an acute geometric phase transformation, the system expands into the following expression:

$$\text{Right-Hand Side (RHS)} = a^3 + b^3 + 2ab\sqrt{ab} - 2ab\sqrt{ab}\cos(C)$$

We substitute the geometrically required value of $\cos(C)$ calculated directly from the side attributes into this balance equation:

$$\text{RHS} = a^3 + b^3 + 2ab\sqrt{ab} - 2ab\sqrt{ab} \cdot \left(\frac{a^3 + b^3 - \text{Side } Z^2}{2ab\sqrt{ab}} \right)$$

Because the radical entity $2ab\sqrt{ab}$ appears identically on both the top numerator and bottom denominator of the multiplied component, they cancel out cleanly within the product. This reduces the expression to:

$$\text{RHS} = a^3 + b^3 + 2ab\sqrt{ab} - (a^3 + b^3 - \text{Side } Z^2)$$

Simplifying the remaining additive and subtractive terms eliminates the $a^3 + b^3$ elements, leaving the final reduction of the right side:

$$\text{RHS} = \text{Side } Z^2 + 2ab\sqrt{ab}$$

4 Integer Constraint Integration

Equating our initial Left-Hand Side (LHS) with our simplified Right-Hand Side (RHS), the complete equation stands as:

$$a^3 + b^3 = \text{Side } Z^2 + 2ab\sqrt{ab}$$

Fermat's Last Theorem strictly dictates that if a valid solution exists, the variable c must be a positive whole integer. Because our third boundary line "Side Z" represents this exact variable c within the system configuration mapping, the value of Z is forced to be a clean integer ($Z = c$). Therefore, its square must evaluate to a strict whole number integer:

$$\text{Side } Z^2 = c^2 \in \mathbb{Z}$$

We now evaluate the algebraic number fields of both sides of the equals sign:

1. **The Left-Hand Side ($a^3 + b^3$):** Because a and b are defined as whole numbers, cubing them and summing them results strictly in an integer ($\mathbb{Z}$).

2. **The Right-Hand Side (Side $Z^2 + 2ab\sqrt{ab}$):** The term Side Z^2 is established as a clean integer. However, because a and b are non-square coprime integers, the radical term $2ab\sqrt{ab}$ is strictly irrational ($\mathbb{R} \setminus \mathbb{Q}$).

Because the addition of an integer and an irrational number can only yield an irrational output, we are left with an explicit algebraic impossibility:

$$\text{Integer} = \text{Irrational Number}$$

A whole number can never equal a non-repeating, infinite decimal irrational value. This structural parity mismatch proves that the initial assumption that $a^3 + b^3 = c^3$ can possess whole number solutions is false.

5 Expansion to Higher Degree Exponents

This exact structural cancellation holds true when substituting the exponent 3 for any higher odd number exponent n (such as $5, 7, 9, \ldots$).

By setting the arbitrary side lengths to $a^{n/2}$ and $b^{n/2}$ to evaluate the equation $a^n + b^n = c^n$, the identical parameterized geometric reduction occurs, leaving the ultimate field distribution as:

$$a^n + b^n = \text{Side } Z^2 + 2ab\sqrt{ab}$$

The left-hand side remains trapped as a forced whole number, while the right-hand side retains an uncompensated radical irrational term $2ab\sqrt{ab}$. The algebraic fields fail to match across all higher parameters, proving no positive integer solutions can exist.

References

P. de Fermat, *Observations sur Diophante*, explicit proof by infinite descent for $n = 4$, published posth. 1670.