Extension of Lorentz Spacetime Symmetry to Four Sectors of Flat and Curved Spacetime

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Abstract

This paper proposes an extension of Lorentz spacetime symmetry from the usual connected domain of an ordinary observer to four orientation sectors defined by one and the same invariant null boundary. The motivation is that the standard Lorentz group describes all finite boosts available inside a chosen causal sector, but it does not by itself supply a real clock–ruler description of the neighbouring non-null domains separated from that sector by lightlike directions. The proposed extension completes this local orientation structure without accelerating ordinary matter through the speed of light and without introducing additional spacetimes. Every observer still has one real time axis and three real spatial axes. The distinction between sectors is which non-null direction supplies the observer's clock and which three directions form its rest space.

The local speed of light remains exactly equal to c in every sector. This is not imposed by assigning a different limiting velocity to each description; it follows because all sector frames share the same null set and their admissible transformations preserve that set. Ordinary Lorentz boosts then describe motion inside any fixed sector, while the sector extension describes the four possible causal orientations of the complete local frame. A side sector is therefore not a superluminal state of ordinary matter. It is a Lorentzian frame in which one direction regarded as spatial by an ordinary observer supplies the single local time axis, whereas the inherited ordinary time direction belongs to the side observer's spatial rest space. Proper time remains real and future-directed, and a change of sector changes the clock–ruler decomposition rather than reversing or discontinuously replacing the physical worldline.

The same principle is generalised to curved spacetime by applying the sector structure to local orthonormal coframes and to the proper geometric scales on which a general-relativistic solution depends. When the scale used by one chart reaches its null boundary, the neighbouring sector evaluates the same gravitational field with the reciprocal active scale and with the corresponding sector metric. The purpose is to replace an extrapolation toward a vanishing, singular scale by a geometrically regular continuation whose curvature and causal distance can be tested directly. The resulting picture is one spacetime equipped with a four-sector Lorentz atlas: local causality and the invariant light speed are preserved, while black-hole and cosmological limits can be re-examined as boundaries of sector descriptions rather than being assumed in advance to be physical singular endpoints.

Keywords: Lorentz symmetry; particle sectors; invariant light speed; sector atlas; reciprocal normalisation; curvature; one-field spacetime; geodesic completeness; black holes; closed timelike curves; CPT symmetry

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1 Physical starting point: one light cone and one spacetime

Special relativity begins with the empirical statement that every inertial observer measures the same vacuum light speed c [1]. The statement is not only about a number obtained by dividing distance by time. It says that all observers agree on the null boundary separating timelike and spacelike directions. Without that common boundary they would disagree on the causal type of the same displacement, and there would be no single causal spacetime shared by them.

The present construction takes this geometrical content as its only physical starting point. It does not begin with four universes or four independently chosen metrics. It begins with one four-dimensional manifold, one local null set and one physical history. The four sectors arise only when the complete orientation of a clock–ruler plane is followed around that common null set.

Physical principle 1.1 (Observer-independent local light speed)**.** *Every local observer measures the same null speed c, independently of the observer's worldline and of which admissible sector orientation supplies the local clock.*

1.1 The light cone precedes a numerical velocity

A light cone is a property of the Lorentzian geometry before any observer assigns coordinate slopes to it. An observer chooses a unit timelike clock axis and three orthogonal ruler axes; only after this choice is the cone read numerically as propagation at c. A different observer uses tilted axes, but an ordinary Lorentz transformation preserves the same null directions. In a spacetime diagram the clock and the corresponding ruler tilt together. The cone is not deformed by the observer's motion.

This order prevents a common misinterpretation. A tangent may have an infinite or apparently superluminal slope when divided by the time coordinate of a sector whose clock it does not use. That quotient is not automatically a locally measured speed. Physical velocity is defined only after the tangent has been resolved in its active real clock–ruler frame.

An ordinary Lorentz boost explores one connected orientation domain. As its clock approaches a null ray, the normalisation of a unit timelike vector diverges because a null vector cannot carry a massive observer's proper time. No finite boost crosses that boundary. The four-sector extension therefore does not permit ordinary matter to pass through c. It asks instead how the neighbouring orientation domain is described when the old clock itself has reached the edge of its valid chart.

1.2 Why one Lorentz plane has four sectors

An ordinary observer has one time axis and three spatial axes. A side observer has exactly the same local structure: one time axis and three spatial axes. There are not several times attached to one side particle. The difference is only the orientation of its frame relative to ours. Its single time axis points along one direction that our observer calls spatial. If that direction is denoted by n^a, then n^a is the side observer's clock direction, while our old clock direction becomes one of the side observer's spatial directions. The two directions orthogonal to this clock–ruler plane remain spatial.

The direction of the side clock must not be confused with the direction of the particle's spatial velocity. A side particle is not forced to travel only along n^a. Just as an ordinary particle can move in any direction of our three-dimensional rest space while its time axis remains our time axis, a side particle can move in any direction of its own three-dimensional rest space while its single time axis remains the side-clock axis. Its complete material trajectory lies inside one side Lorentz cone. That cone looks rotated by a quarter turn in our inherited diagram, but for the side observer it is an ordinary future cone with the usual local light boundary and ordinary subluminal motion inside it.

Different possible side observers may have their one clock axis oriented along different directions of our three-space. The unit possibilities form the sphere S^2, or equivalently all nonzero directions of the old $\mathbb{R}^3$ before normalisation. Thus S^2 describes the possible orientations of one time axis across the family of side frames. It does not describe many simultaneous time axes of one particle. For a particular particle the realised direction follows from its actual tangent and Lorentz transformation.

Choose the ordinary clock direction and one realised side-clock direction n^a. Their two null lines divide the corresponding oriented Lorentz plane into four connected non-null domains. Their centres are the future clock $+t$, the first side clock $+n^a$, the opposite clock $-t$, and the reflected side clock $-n^a$:

$$+t \longrightarrow +n^a \longrightarrow -t \longrightarrow -n^a \longrightarrow +t.$$

Each domain contains its own ordinary Lorentz boosts. Such boosts change the spatial motion inside one cone but do not create a new sector and do not carry a massive observer across the null boundary. The fourfold cycle therefore completes the orientation atlas; it is not by itself a journey completed by one worldline.

1.3 Geometry orientation and particle orientation are different data

The same fourfold structure has two uses which must not be conflated. The geometry has an active sector n_g, telling us which orientation and which regular chart currently represent the common field. A particle X has a relative sector s_X, telling us which extended Lorentz transformation acts on the local geometric coframe. Its ordinary Lorentz tensor ${\Lambda_{(X)}}^a{}_b$ then describes motion inside that fixed particle sector, including the boost direction and the orientation of its spatial triad. No additional unit vector on the side-direction sphere is required as independent particle data.

These operations occur in a definite order. The geometry first supplies its sector coframe. The particle then applies its own sector phase and ordinary Lorentz motion to the components read from that coframe. The sum $N_X = n_g + s_X \pmod 4$ appears only after those two independent operations have been composed. A geometry-chart transition changes n_g and hence the total phase N_X, but it changes neither s_X nor the particle's local boost. Conversely, ordinary acceleration changes $\Lambda_{(X)}$ but not the two sector labels.

This separation allows ordinary, side-oriented and oppositely oriented particles to coexist in one flat or curved geometry. They are not copies of one particle in different universes. They are distinct local sector readings of particle motion in one spacetime, all bounded by the same null set.

1.4 Why curvature and geodesic distance must be tested

In flat spacetime the four sectors concern orientation alone. Curvature adds a second problem: the proper separations between neighbouring worldlines can focus, stretch or rotate. A sector continuation must therefore identify which physical scale reaches the null chart boundary and whether the neighbouring chart remains a regular representation of the same gravitational field.

This requirement cannot be replaced by a tetrad choice. A freely falling frame can remove the connection at one event, but it cannot remove a nonzero Riemann tensor. If invariant curvature diverges, the relative acceleration of nearby geodesics becomes unbounded and no regular neighbourhood remains in which the universal cone structure can be transported. A different particle sector also cannot cure this failure: it only changes the particle's local reading of the common tensor.

The curved construction below consequently has three tests. Proper scale tensors determine the direction whose inherited chart is failing and select the reciprocal scale of the neighbouring sector. Curvature invariants are then evaluated on that active domain. Finally, timelike proper time and null affine parameter determine whether the reciprocal centre is crossed, reached asymptotically, or not reached at all. Finite curvature is not assumed from the phase i^{n_g}, and finite curvature alone does not establish geodesic completeness. These separate tests decide whether a Schwarzschild, LTB, Kerr or FLRW continuation succeeds.

2 Four-sector Lorentz kinematics in flat spacetime

The sectors must first be defined in flat spacetime. Otherwise their meaning would depend on a particular gravitational solution and regularisation would be built into the notation. In inertial coordinates the common starting point is

$$\mathrm{d}s^2_{(0)} = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -c^2\mathrm{d}t^2 + \mathrm{d}\boldsymbol{x}^2, \qquad \mathrm{d}s^2_{(0)} = 0 \text{ for light.} \tag{2.1}$$

2.1 One spacetime and one notation

Lower-case Latin indices $a, b, c, \ldots$ are abstract spacetime indices. Greek indices $\mu, \nu, \ldots$ are used only for coordinate components, and upper-case Latin indices $A, B, \ldots$ label an orthonormal Lorentz coframe with $\eta_{AB} = \mathrm{diag}(-1, 1, 1, 1)$. Capital indices $I, J = 1, 2, 3$ label its spatial components. Labels in parentheses enumerate principal spatial modes but are not tensor indices. Sector labels take values in $\mathbb{Z}_4$ and are never summed.

The symbol x denotes one selected direction in the old spatial $\mathbb{R}^3$. It can later be represented by a spherical radial ray r, but the side sector contains every possible spatial ray rather than one preferred axis.

Let

$$\vartheta^A = \xi^A{}_a\,\mathrm{d}x^a, \qquad g^{(0)}_{ab} = \eta_{AB}\xi^A{}_a\xi^B{}_b \tag{2.2}$$

be a real reference coframe and its metric. The coefficients $\xi^A{}_a(x)$ may already contain the local deformation of a curved solution; the sector construction will not multiply them by an additional physical strain.

Three pieces of information must remain distinct. The active geometry sector is $n_g(x)$. A particle X carries a sector s_X relative to that geometry. Its ordinary Lorentz transformation $\Lambda_{(X)}{}^a{}_b$ describes continuous motion inside the fixed particle sector, including the directional information of the trajectory relative to the reference coframe. A separate side-time direction is therefore not added to the particle state. Only after the geometry and particle operations are composed do we use the total label $N_X = n_g + s_X \pmod 4$ relative to the reference coframe.

All these descriptions concern one event of one manifold. There is one gravitational metric field, one null set and one tensor $R^a{}_{bcd}$. The geometry has sector representatives and particles have different tetrad-component readouts, but no particle acquires a private metric or private curvature. A geometric centre is flat only if the common Riemann tensor vanishes there; a particle being at rest is a separate kinematic statement.

2.2 The ordinary Lorentz transformation and the tilt of the axes

Before extending the set of orientations, consider what an ordinary boost does inside an already fixed sector. Let u^a be the unit clock of the reference observer,

$$g_{ab}u^a u^b = -1, \qquad h^a{}_b = \delta^a{}_b + u^a u_b, \tag{2.3}$$

and let n^a be the unit direction of the particle trajectory in the old rest space,

$$u_a n^a = 0, \qquad n_a n^a = 1. \tag{2.4}$$

The vector n^a is not an additional clock label. It is the direction already contained in the particle tangent and in the directional part of its Lorentz transformation relative to u^a.

The active time–space plane and its hyperbolic exchange tensor are

$$P^a{}_b = -u^a u_b + n^a n_b, \qquad B^a{}_b = u^a n_b - n^a u_b, \qquad B^a{}_c B^c{}_b = P^a{}_b. \tag{2.5}$$

For $\gamma_X = (1-\beta_X^2)^{-1/2}$, the ordinary Lorentz transformation is the mixed tensor

$$\boxed{\Lambda_{(X)}{}^a{}_b = \delta^a{}_b + (\gamma_X - 1)P^a{}_b + \gamma_X \beta_X B^a{}_b, \qquad |\beta_X| < 1.} \tag{2.6}$$

It acts on the clock and the selected ruler as

$$u_X^a = \Lambda_{(X)}{}^a{}_b u^b = \gamma_X(u^a + \beta_X n^a), \qquad n_X^a = \Lambda_{(X)}{}^a{}_b n^b = \gamma_X(n^a + \beta_X u^a), \tag{2.7}$$

and leaves the two-dimensional screen orthogonal to u^a and n^a unchanged. Equation (2.6) is tensorial; a 2×2 array is only its component display in the selected plane and is not used as a second notation.

The transformation preserves the metric and the null set:

$$g_{cd} \Lambda_{(X)}{}^c{}_a \Lambda_{(X)}{}^d{}_b = g_{ab}. \tag{2.8}$$

At $\beta_X = 0$ the transformed and reference axes coincide. As $|\beta_X|$ approaches unity, the clock and its paired ruler approach one of the common null directions, while γ_X diverges because a null vector cannot be normalised as the clock of a massive observer. No finite ordinary boost crosses that boundary.

A side observer should now be read in exactly the same physical way as an ordinary observer. It has one clock axis, not a sphere of simultaneous clock axes. Relative to the ordinary reference frame, that single clock points along one direction of the ordinary spatial rest space. Across the family of all possible side observers this direction may be any

$$n^a \in S^2 \subset u^\perp. \tag{2.9}$$

Before normalisation the same family is represented by the nonzero vectors of the old $\mathbb{R}^3$. For one particular particle the realised direction is not chosen as extra data: it is already fixed by the particle tangent and by the directional content of $\Lambda_{(X)}{}^a{}_b$.

The fact that this one side clock points along one of our spatial directions does not mean that the particle can move only along that direction. The side observer has its own three-dimensional rest space, orthogonal to its clock, just as we do. Its tangent can therefore be written as

$$U_X^a = c\gamma_X(u^a + v_X^a/c), \qquad u_a v_X^a = 0, \qquad g_{ab}^{(1)} v_X^a v_X^b < c^2, \tag{2.10}$$

where u^a is its one future-directed time axis and v_X^a may point in any direction of its own spatial rest space. The whole worldline remains inside the single Lorentz cone centred on u^a. In our inherited diagram that cone is quarter-turned, but for the side observer it is simply its ordinary future cone. A spatial turn changes v_X^a; it does not rotate the particle into another time orientation or another sector.

2.3 Four orientation sectors of one Lorentz plane

The ordinary boost has so far explored only the domain centred on $+t$. To see the missing orientations, keep the same two null lines and follow an oriented axis around the complete time–space plane. The null rays divide that plane into four open connected domains. Starting from $+t$, one meets a domain centred on a selected positive spatial direction, then the domain centred on $-t$, then the reflected spatial domain, and finally returns to $+t$. The complete cycle is

n	i^n	clock orientation	sector
0	1	$+t$	ordinary sector
1	i	$+n^a$	first side sector
2	-1	$-t$	PT reflection of sector 0
3	$-\mathrm{i}$	$-n^a$	reflected side sector
$4 \equiv 0$	1	$+t$	closed orientation cycle.

(2.11)

Thus the ordinary flat reference has its future cone centred on $+t$. The first side reference is the family of equally admissible flat cones centred on directions that were spatial in the ordinary projection. For a particular side trajectory, one direction $+n^a$ is determined by its tangent and by the directional content of $\Lambda_{(X)}$ relative to the reference coframe; it is not selected as an additional independent clock variable. The opposite sector is the PT reflection of the ordinary sector, and the second side sector is the reflected family $-n^a$. These are four views of one flat Lorentz geometry, not four flat spacetimes.

The entries in the table are the centres of the four domains, not the only directions allowed inside them. Every sector has its own finite $|\beta_X| < 1$ Lorentz boosts, and none of those boosts creates another sector. In the first side class a complete material trajectory lies inside one cone whose clock centre is $+n^a$ relative to the ordinary observer. The cone appears quarter-turned in the inherited diagram, but it is an ordinary Lorentz cone for the side observer and has the same null boundary. Across all possible trajectories its centre may point along any unit direction of S^2, or equivalently along any nonzero direction of the old $\mathbb{R}^3$ before normalisation. One trajectory realises one such direction through its Lorentz and tangent data; it does not carry the whole sphere as simultaneous time axes.

The discrete label records the connected orientation component, whereas ${\Lambda_{(X)}}^a{}_b$, including its boost direction and spatial rotation, records continuous particle motion inside that component. The geometric ratio q_i introduced in Section 3.4 instead tracks the orientation of an active proper scale; it is not a particle velocity.

The table describes orientations, not four kinds of matter and not four disconnected regions. At a fixed event the null cone is one geometric set. What changes from one entry to the next is the direction chosen to bisect that cone and to carry the observer's clock. In sector 0 this direction is the usual $+t$ axis. At the neighbouring null boundary that axis can no longer be normalised as timelike; sector 1 then reads one of the old spatial directions as its real clock direction. Repeating the same construction produces the reflected $-t$ and $-n^a$ orientations and finally returns to the starting frame. This is why the cycle completes the Lorentz description of one spacetime rather than adding three spacetimes.

The null rays are boundaries of the four descriptions, not additional sectors. At such a ray the old unit clock is undefined because its norm is zero. The neighbouring sector begins by assigning a new timelike centre on the other side of that boundary. This change concerns the frame used to read the geometry. It does not say that a massive tangent has crossed the null ray or changed from timelike to spacelike.

Only one old spatial direction supplies the clock of a given side observer at one event. If it is denoted by the unit vector n^a, the old clock becomes one direction in the new rest space, n^a becomes the new clock, and the two directions orthogonal to their plane remain spatial. The two-sphere parametrises all possible directions relative to the reference observer, but the direction of a particular particle is derived from its actual tangent U^a_X and $\Lambda_{(X)}$ rather than appended as a new label. For inertial motion in flat spacetime it is constant because the tangent is constant; if the trajectory is physically accelerated, it changes continuously only with that trajectory while s_X remains fixed. In spherical language (θ, φ) label this family of possible directions; they are not two additional times on one worldline.

2.4 Geometry sector, particle transformation, and total orientation

The sector factors enter in two successive operations. First the geometry chooses its active coframe:

$$\vartheta^A_{(g)} = \mathrm{i}^{n_g} \vartheta^A = \mathrm{i}^{n_g} \xi^A{}_a \, \mathrm{d}x^a, \qquad g^{(n_g)}_{ab} = \eta_{AB} \vartheta^A_{(g)a} \vartheta^B_{(g)b} = \mathrm{i}^{2n_g} g^{(0)}_{ab}. \tag{2.12}$$

The phase i^{n_g} belongs to the geometry coframe, while its square appears in the sector representation of the metric. Odd sectors reverse the sign convention of the quadratic form and even sectors preserve it. Because multiplication by a nonzero constant leaves the null set unchanged, all four representations describe the same local light boundary. The full orientation still resides in the coframe: the metric alone cannot distinguish sectors 0 and 2, or sectors 1 and 3.

Only after the geometry has supplied $\vartheta^A_{(g)}$ does particle X apply its own extended Lorentz transformation

$$\mathcal{G}_{(X)}{}^A{}_B = \mathrm{i}^{s_X} \Lambda_{(X)}{}^A{}_B, \qquad \mathrm{d}X^A_{(X)} = \mathcal{G}_{(X)}{}^A{}_B \vartheta^B_{(g)} = \mathrm{i}^{n_g + s_X} \Lambda_{(X)}{}^A{}_B \xi^B{}_a \, \mathrm{d}x^a. \tag{2.13}$$

This is the central composition law. The factor i^{s_X} belongs to the particle transformation, not to a second geometric coframe. The ordinary Lorentz tensor $\Lambda_{(X)}{}^a{}_b$ describes motion inside that particle sector. In three spatial dimensions its boost and rotation data also determine which member of the side-direction family is realised relative to the reference coframe; no independent n^a_X is added to Eq. (2.13). The total label $N_X = n_g + s_X \pmod 4$ records the final phase relative to the reference coframe, but it does not replace the two independent sources of that phase.

The four relative classes are

s_X	relative orientation	name used here
0	ordinary	ordinary particle
1	first side	side particle I
2	opposite	anti-sector particle
3	reflected side	side particle II.

(2.14)

The word *tachyon* will be used as shorthand for the two side classes $s_X = 1, 3$. It does not mean ordinary matter accelerated through c. Likewise, $s_X = 2$ denotes the opposite transformation orientation. Identifying this class with observed antimatter is an additional physical interpretation, not a result of standard quantum field theory.

When the geometry advances by k charts, both n_g and the total readout N_X advance by k, while s_X and $\Lambda_{(X)}$ remain unchanged. By contrast, ordinary acceleration changes $\Lambda_{(X)}$ while leaving s_X fixed. Ordinary matter, a side particle and an anti-sector particle therefore retain their class when the geometry changes sector.

Equation (2.13) may be used to form the local quadratic readout $\eta_{AB} \mathrm{d}X^A_{(X)} \mathrm{d}X^B_{(X)}$. It must not be promoted to a new dynamic tensor $g^{(X)}_{ab}$. The only gravitational metric is the geometry field represented in Eq. (2.12); particles differ by the transformations through which they resolve that common field.

2.5 Particle rest and geometric flatness

Particle rest is defined from the readout in Eq. (2.13), not from the value of a coordinate slope. If U^a_X is its tangent, its local components are obtained by applying $\mathcal{G}_{(X)}{}^A{}_B \vartheta^B_{(g)a}$ to that tangent. The particle is at rest relative to the local geometry inside its own sector when its three spatial components vanish. In the adapted representative this is equivalently

$$U^I_{(X)} = 0 \qquad \Longleftrightarrow \qquad \Lambda_{(X)}{}^a{}_b = \delta^a{}_b. \tag{2.15}$$

For $s_X = 0$ this is ordinary rest relative to the geometric coframe. For $s_X = 1$ or 3 the clock axis is side-oriented, so an inherited ordinary chart can assign it an infinite or superluminal-looking slope even though $\Lambda_{(X)}$ is the identity. For $s_X = 2$ the particle transformation has the opposite orientation. There is no absolute rest common to all four readouts.

The sector coframe changes the decomposition of the tangent, not the tangent itself. Relative to the future clock $u^a_{(X)}$ of the particle's sector, write only

$$U^a_X = \gamma_X\big(c\, u^a_{(X)} + v^a_X\big), \qquad u_{(X)a} v^a_X = 0, \qquad \gamma_X > 0. \tag{2.16}$$

An ordinary spatial rotation acts on v^a_X inside $u^\perp_{(X)}$ and leaves $u^a_{(X)}$ fixed. It may reverse radial, angular, or inherited-$X = ct$ motion, but it cannot reverse the positive temporal component of U^a_X. Thus a side particle may turn around in its own space without turning around in proper time.

Geometric sector rest is a different statement. Let u^a_g be the clock of the active geometry chart and $h_g{}^a{}_b = \delta^a{}_b + u^a_g u_{gb}$ its rest projector. A reference congruence is aligned with that chart when $U^a = c\, u^a_g$. It defines a flat sector centre only when the alignment extends through a neighbourhood without tidal deviation:

$$\begin{aligned} &\text{geometric sector rest:} && U^a = c\, u^a_g \quad \Longleftrightarrow \quad h_g{}^a{}_b U^b = 0, \\ &\text{flat geometric centre:} && \nabla_b u^a_g = 0 \text{ throughout a neighbourhood}, \qquad R^a{}_{bcd} = 0. \end{aligned} \tag{2.17}$$

The first line is pointwise and can hold in curved spacetime. The second is a property of the one geometry. Hence a tachyon can be at particle-sector rest in a curved region, and an ordinary particle can move through a flat geometric centre. Neither fact changes the Riemann tensor. Regularisation below relies on the second line, not on setting $\Lambda_{(X)}{}^a{}_b = \delta^a{}_b$ for one particle.

The particle phase must be removed when its components are compared with the geometry norm. From Eq. (2.13),

$$\mathrm{i}^{-2s_X} \eta_{AB} U^A_{(X)} U^B_{(X)} = -c^2, \qquad \eta_{AB} k^A_{(X)} k^B_{(X)} = 0. \tag{2.18}$$

The first expression is the geometry norm written through the particle transformation; it is not a particle metric. The null equation needs no compensating factor because zero is unchanged. The numerical value c and the null boundary are identical in all four orientations. A sector-0 observer may assign a side particle a formal coordinate slope greater than c, but that quotient is not the velocity $\beta_X c$ appearing in its own $\Lambda_{(X)}$.

2.6 Cone orientation and the uncrossed null boundary

The four sector domains share their null boundary. For a particle with fixed sector class, $\Lambda_{(X)}$ tilts its clock and ruler axes inside that domain and Eq. (2.18) places light on its boundary.

The boost can approach the boundary but cannot cross it. Hence ordinary matter cannot become a side particle by acceleration, a side particle cannot be slowed into ordinary matter, and an anti-sector particle cannot be converted into ordinary matter by changing observer.

This statement is local and independent of n_g. In flat geometry with $n_g = 0$, particles with $s_X = 0, 1, 2, 3$ may coexist at the same event and use the same null set. A side particle is already centred on $+n^a$ or $-n^a$ before any gravitational sector transition occurs, with the actual direction fixed by its tangent and $\Lambda_{(X)}$ relative to the reference observer. Its apparent ordinary-sector quantity $\mathrm{d}x/\mathrm{d}t$ can be infinite or formally exceed c, but this is a projection on a clock it does not use. Its measured velocity is the finite $\beta_X c$ obtained from its own sector-transformed components.

A geometry-sector transition is different. When the active field chart moves from n_g to $n_g + 1$, the geometry coframe acquires the next phase before each unchanged particle transformation acts on it. Every N_X shifts together, while s_X and $\Lambda_{(X)}$ are preserved. Thus an ordinary particle inside side-oriented black-hole geometry has a side-looking total phase $N_X = 1$, but it is still ordinary relative to the local field because $s_X = 0$.

The global speed limit is therefore the conjunction of two facts. Every ordinary Lorentz motion remains inside its own sector domain, and all four domains possess the same null set. No physical process described here moves a particle through that set. What can change is the active geometry chart or the observer used to project the tangent; neither is a measurement of a particle outrunning light in its own sector.

This also fixes the meaning of the inherited coordinate $X = ct$. The clock–ruler exchange does not erase the real differential $\mathrm{d}X = c\,\mathrm{d}t$ and does not redirect the physical tangent. It changes the role assigned to that component by the active coframe. In a side sector X belongs to the spatial decomposition, so $\mathrm{d}t < 0$ can describe motion toward decreasing X while the particle's own proper time still increases. Neither the sign of $\mathrm{d}t$, a spatial turn in X, nor a relabelling of the axes reverses the future orientation of U_X^a.

In flat spacetime this already excludes a sector-generated closed timelike curve. The geometry sector n_g is constant, the particle label s_X is conserved, and every finite $\Lambda_{(X)}$ remains within one connected null domain. A particle may belong to any of the four classes from the outset, but no classical boost turns its worldline into the next class and no single particle completes the orientation cycle. The algebraic sequence of sector phases is therefore not a timelike loop in Minkowski spacetime.

2.7 The sector continuation and the role of i^n

The extended Lorentz transformation is the mixed tensor

$$\mathcal{G}_{(n)}{}^a{}_b[\Lambda] = \mathrm{i}^n \Lambda^a{}_b, \qquad n = 0, 1, 2, 3. \tag{2.19}$$

The phase labels the orientation component and $\Lambda^a{}_b$ is the ordinary Lorentz transformation acting inside it. Composition and inversion are written without suppressing tensor indices:

$$\begin{aligned} \mathcal{G}_{(m)}{}^a{}_c[\Lambda_2]\mathcal{G}_{(n)}{}^c{}_b[\Lambda_1] &= \mathcal{G}_{(m+n)}{}^a{}_b[\Lambda_2 \circ \Lambda_1] \\ &= \mathrm{i}^{m+n}(\Lambda_2)^a{}_c(\Lambda_1)^c{}_b, \end{aligned} \tag{2.20}$$

$$\left(\mathcal{G}_{(n)}^{-1}\right)^a{}_b[\Lambda] = \mathcal{G}_{(-n)}{}^a{}_b[\Lambda^{-1}]. \tag{2.21}$$

Sector addition is understood modulo four. The metric contraction is

$$g_{cd}\mathcal{G}_{(n)}{}^c{}_a\mathcal{G}_{(n)}{}^d{}_b = \mathrm{i}^{2n} g_{ab}. \tag{2.22}$$

Thus the quadratic sign alternates, but the null equation is unchanged. No transpose notation or unindexed matrix product is required.

For a particle, Eq. (2.19) is used with $n = s_X$ and $\Lambda = \Lambda_{(X)}$. For the geometry, the sector factor appears separately in its coframe as in Eq. (2.12). These must not be collapsed into one unexplained i^n. Only their ordered composition in Eq. (2.13) has the total phase i^{N_X}.

Once a sector is fixed, the derivation in Section 2.2 applies without modification: $g_{cd}\Lambda^c{}_a\Lambda^d{}_b = g_{ab}$ preserves the null rays and the measured relative speed remains below c. The four sectors therefore do not carry four different Lorentz laws. They are four orientation components on which the same Lorentz law acts.

The continuous orbit and the discrete operator label must also be distinguished. An oriented axis can approach a null boundary in one chart and recede from it in the next, so its geometric trajectory is continuous. The algebraic factors $1, i, -1, -i$ only label the four normalisation branches used along that trajectory; the operator i^n is not itself a continuous rotation parameter. Section 3.4 gives the single angle and the projective coordinates that cover this orbit without identifying them with a particle speed. This continuity belongs to the space of oriented frames. It does not assert that one timelike or null worldline completes the orbit at finite physical parameter.

2.8 The interval cycle and its physical meaning

For a generic orientation label $m \in \mathbb{Z}_4$, successive multiplication by the sector phase gives $\mathrm{d}x^\mu$, $\mathrm{i}\mathrm{d}x^\mu$, $-\mathrm{d}x^\mu$, $-\mathrm{i}\mathrm{d}x^\mu$, and again $\mathrm{d}x^\mu$. The corresponding interval is

$$\mathrm{d}s^2_{(m)} = (-1)^m \mathrm{d}s^2_{(0)}. \tag{2.23}$$

For the geometry coframe one sets $m = n_g$. Applying a particle transformation adds the independent phase s_X, so its final local quadratic readout has $m = N_X$. This does not define a particle-dependent metric. The four steps have a direct reading. Sector 0 uses the original differential and the clock $+t$. The first factor i selects the side normalisation and makes the old positive spatial direction the active clock. The second turn gives $-\mathrm{d}x^\mu$ and reverses both inherited axes, producing the $-t$ orientation. The third gives the reflected side clock $-n^a$; the fourth returns the orientation and the differential to their starting values. The phase is therefore a compact account of the complete axis cycle already listed in Eq. (2.11).

In one selected plane, $\mathrm{d}s^2_{(0)} = -c^2\mathrm{d}t^2 + \mathrm{d}x^2$, while the inherited sector-1 components give $\mathrm{d}s^2_{(1)} = -\mathrm{d}x^2 + c^2\mathrm{d}t^2$. The sign change is the algebraic record that the old spatial direction occupies the active time slot and the old time direction belongs to the active spatial decomposition. The same exchange with reversed orientation occurs in sector 3.

Equation (2.23) does not prescribe an independent metric field. In the real reading of every active sector the physical line element has one clock direction and three ruler directions. The factor $(-1)^m$ records the sign convention seen in inherited components. The manifold, common null set and scalar curvature invariants still belong to the one geometry.

2.9 What flat spacetime means in every sector

A cone does not make a region curved merely because it appears tilted or sideways in an inherited coordinate diagram. Flatness means that one active clock–ruler tetrad can be extended throughout a neighbourhood without path-dependent rotation and that the corresponding reference geodesics have no tidal acceleration. Equivalently, the second line of Eq. (2.17) holds there. A constant Lorentz tilt applied to every cone changes the observer, but it does not change this property and therefore does not create curvature.

For geometry sector $n_g = 0$, the flat reference congruence follows the $+t$ axes of a family of parallel cones. Geometry sector $n_g = 1$ uses the same flat construction after the time–space roles have been exchanged, so the old spatial direction identified by the active plane supplies its reference clock. Curvature vanishes in both descriptions because the coframes remain parallel throughout the neighbourhood.

Particle sectors are superposed on this geometric choice. Even when $n_g = 0$ everywhere, a particle with $s_X = 1$ can have a side-oriented transformation and be at rest in its own side frame. Relative to the reference coframe, the realised side direction is contained in the rotation and boost-direction data of $\Lambda_{(X)}$; $\Lambda_{(X)}{}^a{}_b = \delta^a{}_b$ denotes the canonical orientation of that frame rather than a missing direction. A sector-0 chart may draw its clock horizontally and assign it an infinite slope, but this is neither a divergent physical speed nor a change of the background geometry. Particles with all four values of s_X can therefore coexist in one flat region.

Geometry sectors 2 and 3 repeat the two flat references with opposite orientation. The reversal changes which direction is called future, not the vanishing of curvature. Thus all four geometric sector centres and all four constant particle-sector transformations are compatible with one Minkowski field and the same null boundary c.

The sign cycle in Eq. (2.23) is consistent with this statement. Multiplying an interval by -1 leaves its zero set unchanged, so the null rays that separate the sectors are common. Each active geometry coframe places its selected clock in the $A = 0$ slot. The sign change records the exchange of inherited temporal and spatial roles; it does not introduce a second metric field or a second spacetime.

This also clarifies why particle rest is not a curvature criterion. $\Lambda_{(X)}{}^a{}_b = \delta^a{}_b$ concerns one particle transformation, whereas a flat geometric centre requires a whole neighbouring congruence with no relative turning and no geodesic deviation. The latter is a property of an extended region and cannot be inferred from the sector or velocity of one body.

2.10 Null boundaries and continuity

Neighbouring orientation charts meet only at a null direction. Ordinary Lorentz motion remains inside the particle's chosen sector; only light lies on the common boundary. The geometry can reach that boundary and require a neighbouring chart, but this shifts n_g and N_X together rather than changing the relative label s_X.

The geometry orientation itself is nevertheless continuous. Starting at one sector centre, the clock–ruler plane tilts toward a common null ray. The old unit clock ceases to exist exactly at that ray; the neighbouring chart begins at the same oriented limit and follows the plane away from it toward its own centre. The discrete value n_g therefore records which finite chart is being used, not an impulse delivered to an object and not a discontinuity of the oriented-frame atlas.

No single projective coordinate covers the complete cycle. The explicit relation between the continuous angle, the scale ratio q_i, and the reciprocal rule is derived in Section 3.4. The important distinction is that q_i locates the field orientation, whereas β_X is the within-sector velocity contained in $\Lambda_{(X)}$. A side particle can exist without traversing the geometry orbit. Ordinary matter may also enter a neighbouring geometry chart without becoming a side particle, but whether it reaches that chart's centre is decided by its proper-time equation rather than by the orientation diagram.

2.11 PT and CPT closure in the same spacetime

After two geometry turns $\mathrm{i}^2 = -1$, so $\mathrm{d}x'^{\mu} = -\mathrm{d}x^{\mu}$ and the inherited coordinates are read as $(t, \boldsymbol{x}) \mapsto (-t, -\boldsymbol{x})$. Sector 2 is therefore the PT reflection of sector 0, while sector 3 is the corresponding reflection of the side family. A particle with $s_X = 2$ applies the analogous opposite Lorentz-sector transformation even when $n_g = 0$. Charge conjugation, if imposed on the matter fields, acts at the same events, so the construction is compatible with a closed CPT assignment in one spacetime. The sector phase by itself supplies the PT orientation and does not prove that physical antiparticles are identical to the $s_X = 2$ class.

2.12 From a uniform cone field to curvature

Up to this point every statement concerns one flat geometry. The cone axis may be boosted, exchanged with an inherited spatial direction, or reflected, yet the same choice can be repeated uniformly at neighbouring events. All these operations change the projection of the cone field without changing the fact that its axes can be made parallel throughout the region. Sector orientation alone therefore cannot be a source of gravity.

The passage to curved spacetime begins only when the local construction can no longer be extended uniformly. Each individual event still admits the four orientations and the same null boundary, but the active axes selected at neighbouring events need not remain parallel under transport. If their variation can be removed by one coordinate change or one Lorentz transformation over the whole neighbourhood, the geometry is still flat. If transport along different paths gives different relative orientations and neighbouring geodesics acquire tidal acceleration, the cone field is curved.

This is the precise bridge to general relativity. The sector construction supplies the possible local clock axes and the flat rest configuration against which their variation is compared. The connection describes how the axes change from event to event, and the Riemann tensor decides whether that change is merely a frame choice or genuine curvature. The next section therefore does not add gravity to four separate spaces. It projects one curved metric and one curvature tensor onto the four local orientations already derived in flat spacetime. Particles with different s_X perform different tetrad projections of that same tensor; none of them creates a separate curvature field.

The same bridge explains why regularity is required rather than merely hoped for. If the active curvature diverged, neighbouring timelike geodesics could acquire unbounded relative acceleration and no regular sector-rest neighbourhood would remain in which the universal null limit could be applied to arbitrary objects. A sector continuation is therefore physically complete only when its active metric and invariant curvature stay finite. The causal construction identifies the necessary chart; the tensor calculation in the next sections decides whether that chart actually satisfies the requirement.

3 Curved spacetime: active scales and sector charts

The flat orbit determines which local clock orientations must remain available. Curvature adds a new question: which proper spatial mode is being focused toward a null chart boundary? A coordinate matrix cannot answer this invariantly. The construction must live on the observer's spatial rest bundle and must then be embedded back into the full tangent bundle.

This section concerns the sector n_g of the geometry. A particle's s_X does not choose the metric domain and does not alter the tensors used to regularise it. Once the active geometric coframe has been found, particle readouts are obtained from Eq. (2.13); all of them resolve the same curvature.

The order of the construction is fixed. A congruence determines its proper deformation scales; the solution determines the corresponding critical scales; a tensorial eigenproblem selects the active direction; and that direction is paired with the local clock. Only after this geometric identification is the reciprocal chart of the same GR solution selected. No operator introduced below is an extra metric deformation.

3.1 The rest bundle and the tensor of proper scales

Let u^a be the unit clock field of the active geometry sector. Its local rest space and projector are

$$u^a u_a = -1, \qquad h^a{}_b = \delta^a{}_b + u^a u_b, \qquad h_{ab} = g_{ab} + u_a u_b. \tag{3.1}$$

All scale tensors below are endomorphisms of $u^\perp$: contraction with u^a on either index vanishes. This makes their eigenvalues proper spatial scales rather than coordinate coefficients.

Let ξ_0^a and ξ^a be spatial separation vectors between neighbouring worldlines of a reference congruence. A Jacobi map $\mathcal{D}^a{}_b$ relates them, and its polar part defines the positive scale tensor

$$\xi^a = \mathcal{D}^a{}_b \xi_0^b, \qquad C^a{}_b = (\mathcal{D}^\dagger)^a{}_c \mathcal{D}^c{}_b, \qquad L^a{}_b = (C^{1/2})^a{}_b. \tag{3.2}$$

The adjoint is taken with respect to h_{ab}. Consequently $L^a{}_b$ is self-adjoint and positive on $u^\perp$. Its orthonormal eigenvectors $n^a_{(i)}$ and eigenvalues $L_i > 0$ satisfy $u_a n^a_{(i)} = 0$ and $h_{ab} n^a_{(i)} n^b_{(j)} = \delta_{ij}$. Rigid rotation carried by $\mathcal{D}^a{}_b$ is separated from the physical linear scales.

3.2 Critical scales and the active tensor mode

The deformation tensor alone does not say where a sector chart ends. The given GR solution must also provide a positive spatial tensor $G^a{}_b$ with the same dimensions as $L^a{}_b$. It contains the critical scale in each principal direction. The active modes are defined by the covariant generalised eigenproblem

$$G^a{}_b n^b_{(i)} = q_i L^a{}_b n^b_{(i)}. \tag{3.3}$$

When the two tensors share eigenvectors, $q_i = G_i/L_i$. In the principal basis they simply have the tensor decompositions $L^a{}_b = \sum_i L_i n^a_{(i)} n_{(i)b}$ and $G^a{}_b = \sum_i G_i n^a_{(i)} n_{(i)b}$. Their vanishing contraction with u^a states that they measure proper spatial scales on $u^\perp$; their eigenvectors identify which ruler direction becomes critical. The next subsection pairs that direction with the clock.

The number q_i is a geometric scale ratio, not a material velocity. The active chart contains $q_i < 1$; $q_i = 1$ is its null boundary; and a formal value $q_i > 1$ must be represented from the neighbouring sector centre. Schwarzschild reduces this construction to one areal scale, LTB separates longitudinal and transverse modes, FLRW makes all three modes degenerate, and Kerr selects a position-dependent principal plane.

3.3 Covariant embedding in the active time–space plane

Let n^a be a unit eigenvector of the mode that reaches $q = 1$. The projector onto the active plane $\mathrm{span}\{u, n\}$ is

$$P^a{}_b = -u^a u_b + n^a n_b. \tag{3.4}$$

Its complement $\delta^a{}_b - P^a{}_b$ is the transverse two-dimensional screen. The projector makes explicit which old ruler is paired with the clock and which two directions remain spectators.

The covariant quarter-turn generator on that plane is

$$J^a{}_b = u^a n_b + n^a u_b, \qquad J^a{}_c u^c = -n^a, \quad J^a{}_c n^c = u^a, \quad J^a{}_c J^c{}_b = -P^a{}_b. \tag{3.5}$$

It vanishes on the transverse screen. Its exponential therefore has the closed tensorial form

$$\mathcal{R}(\alpha)^a{}_b = [\exp(\alpha J)]^a{}_b = \delta^a{}_b - P^a{}_b + \cos\alpha\, P^a{}_b + \sin\alpha\, J^a{}_b. \tag{3.6}$$

At $\alpha = \pi/2$, u^a is carried to $-n^a$ and n^a to u^a; the local clock and active ruler exchange roles while the transverse screen is unchanged.

Equations (3.4)–(3.6) are already four-dimensional tensor statements; no second matrix notation is needed. The active geometric mode identifies one n^a in the old rest space as the new clock direction. This geometric eigenvector must not be confused with an additional particle label. The remaining two-dimensional screen stays spatial, so changing its angular coordinates cannot reverse the active time direction.

The tensor $\mathcal{R}(\alpha)^a{}_b$ records orientation and the selected plane. It is not an ordinary Lorentz boost and is not multiplied into a completed GR coframe to create a new metric. Such an operation would alter the Einstein tensor and manufacture an effective source even in a vacuum solution. Ordinary particle motion inside each active sector remains described by $\Lambda_{(X)}$ from Section 2.2.

3.4 One continuous angle and four finite projective charts

For one principal mode, a single angle α_i follows the orientation of the selected clock–ruler plane. In the chart whose centre is $\alpha_c = n_g\pi/2$, use the finite scale ratio

$$q_i = |\tan(\alpha_i - \alpha_c)|\,, \qquad 0 \le q_i \le 1. \tag{3.7}$$

At the chart centre $\alpha_i = \alpha_c$ the ratio is zero; at either null edge it equals one. The neighbouring chart represents the same null orientation and then follows the plane toward its own zero. If the old ratio is formally continued beyond the edge, the neighbouring finite variable is $\widetilde{q}_i = q_i^{-1}$, which is the scalar content of Eq. (3.8).

The angle is neither a rapidity nor a particle velocity. It describes how the geometry coframe is oriented, while β_X remains the velocity inside the fixed particle sector. The geometry may therefore move from $+t$ toward $+n^a$ without rotating the physical tangent U^a_X. In the Schwarzschild, Kerr, and regular FLRW examples the exact side orientation is only an asymptotic limit because its reciprocal centre lies at infinite physical parameter.

3.5 Reciprocal normalisation of the active solution

At $q_i = 1$ two adjacent charts meet at the same critical scale. Beyond that boundary the same mode must be measured from the rest centre of the new sector. The tensorial reciprocal rule fixes the boundary and exchanges a large old ratio with a small new one:

$$\widetilde{L}^a{}_b = G^a{}_c (L^{-1})^c{}_d G^d{}_b. \tag{3.8}$$

For a principal scalar mode it reads $\widetilde{L}_i = G_i^2/L_i$. Applying the rule twice returns the original scale algebraically. This involution constructs the reflected chart data but does not continue a physical worldline through the reciprocal centre. Moreover, Eq. (3.3) shows that the same eigenvector has the reciprocal chart ratio $\widetilde{q}_i = q_i^{-1}$. Thus the regime described by an arbitrarily large old ratio becomes the zero-ratio rest centre of the neighbouring chart.

The rule acts on the variables with which a known GR solution is evaluated, not as a conformal or matrix correction to its finished metric. Schematically,

$$g_{ab}^{(n_g)} = \mathrm{i}^{2n_g}\, \mathcal{I}_{n_g}^{*} g_{ab}^{\mathrm{GR}}[L^a{}_b]. \tag{3.9}$$

The pullback means that the same functional GR solution is written in the active proper scales. The constant factor $\mathrm{i}^{2n_g} = (-1)^{n_g}$ records the sector sign of its tetrad representation; it leaves the Levi–Civita connection and null set unchanged and does not create another metric field on another spacetime. At the critical surface $L_i = G_i$ the unsigned proper scale agrees in both charts. For a single scale $\widetilde{X} = X_h^2/X$, the Jacobian equals -1 at $X = X_h$; the radial coordinate orientation therefore reverses while invariant curvature remains continuous.

This sign reversal is the differential expression of the clock–ruler exchange. Along a smooth worldline, $\mathrm{d}\widetilde{X}/\mathrm{d}\tau = -(X_h^2/X^2)\mathrm{d}X/\mathrm{d}\tau$. The sign of one coordinate component changes at the transition, but the tangent itself is carried by the Jacobian of the chart map and remains timelike. There is no reflection of the body and no physical crossing of its null cone.

3.6 One geometry coframe and the particle readout

Let $\mathcal{I}_{n_g} : U_{n_g} \to M$ be the active geometry chart and $\vartheta^A_{\mathrm{GR}}[L]$ a coframe of the same GR solution evaluated in its active scale tensor. The geometric sector coframe is

$$\vartheta^A_{(g)} = \mathrm{i}^{n_g}\, \mathcal{I}_{n_g}^{*} \vartheta^A_{\mathrm{GR}}[L]. \tag{3.10}$$

The pullback selects a regular chart of the field, i^{n_g} selects its orientation component, and the coframe of the GR solution already contains the local geometric deformation. None of these operations creates another geometry.

The particle operation is applied only afterward. In Eq. (2.13), the reference coframe is then replaced by the pulled-back GR coframe in Eq. (3.10); the final phase is still i^{N_X}, with $N_X = n_g + s_X \pmod 4$. No additional formula is needed because this is exactly the ordered composition already defined in flat spacetime. The result is a transformed differential, not a second coframe field assigned to the manifold. Changing n_g first changes the geometry coframe; changing $\Lambda_{(X)}$ changes one particle's motion; and the conserved s_X fixes which extended Lorentz transformation that particle uses.

The metric still belongs only to the geometry and is the tensor already defined in Eq. (3.9). Its sign alternation is the quadratic trace of the sector orientation. It distinguishes even from odd sectors but not the two opposite orientations within either parity class; the missing information remains in $\vartheta^A_{(g)}$. No analogous $g_{ab}^{(X)}$ is introduced.

Before the orientation sign is attached, every physical or geometric field has one value on M and only chart pullbacks

$$\mathcal{F}_{(n_g)} = \mathcal{I}_{n_g}^{*}\mathcal{F}, \qquad \mathcal{F} \in \{g_{ab}, T_{ab}, R^a{}_{bcd}, \ldots\}. \tag{3.11}$$

The metric readout $g^{(n_g)}$ is obtained from its pullback by the single orientation factor displayed in Eq. (3.9); it is not a second freely specifiable tensor. The global matter configuration may contain several values of s_X, but there is no signal between separate sector worlds because no such worlds exist. Every particle transformation acts on fields at the same events.

3.7 Curvature and unchanged field equations

Let $\mathcal{I}$ denote an adjacent reciprocal chart map and let $H^A{}_B$ be the corresponding coframe transition. The Cartan connection has the usual inhomogeneous change-of-frame term, whereas its curvature two-form transforms tensorially:

$$\Omega_{(n_g+1)}{}^A{}_B = H^A{}_C\,\mathcal{I}^*\Omega_{(n_g)}{}^C{}_D(H^{-1})^D{}_B. \tag{3.12}$$

This is ordinary tensorial covariance, not a new sector curvature law [2, 3].

Scalar contractions are therefore pullbacks of one curvature field. In particular, $K_{(n_g)} = \mathcal{I}^*_{n_g}(R_{abcd}R^{abcd})$. Adjacent sectors give the same value at the same boundary event. A normal derivative may reverse sign because the reciprocal Jacobian reverses the normal orientation; this is not a jump in the curvature tensor.

Each particle resolves this same $R^a{}_{bcd}$ through its transformation $\mathcal{G}_{(X)}$; only its tetrad components change. The field equation and the causal propagation laws remain those of the one geometry:

$$G_{ab} = \frac{8\pi G_N}{c^4}T_{ab}, \qquad U^b_X\nabla_b U^a_X = 0, \qquad g_{ab}k^ak^b = 0. \tag{3.13}$$

There is no sector-dependent Einstein tensor or independently adjustable stress tensor. Different s_X values change the temporal–spatial split measured by a particle, not the scalar invariants. In particular, inserting $\mathcal{R}(\alpha)^a{}_b$ into a completed coframe while keeping η_{AB} fixed would create a different metric. The orientation tensor selects the active plane; the pullback and reciprocal scale select the regular representative of the original solution.

The flat-spacetime kinematic meaning has already been fixed in Sections 2.2–2.6: a sector change re-resolves one tangent rather than replacing it. In curved spacetime the only additional statement is the tensorial chart law. On an overlap,

$$U^{a'}_X = \frac{\partial x^{a'}}{\partial x^b}U^b_X, \qquad k^{a'} = \frac{\partial x^{a'}}{\partial x^b}k^b, \tag{3.14}$$

while s_X and the physical future orientation remain unchanged. The active coframe decides which component is temporal. It does not redirect the worldline, and an inherited differential such as $\mathrm{d}X = c\,\mathrm{d}t$ may become spatial without becoming unreal or acquiring the power to reverse proper time.

3.8 Reciprocal centres as asymptotic boundaries

Two geometrically different limits must now be kept apart. The first is the common null boundary of adjacent finite charts. A horizon-regular frame can carry a causal curve through this boundary in finite proper or affine parameter. The second is the reciprocal centre approached after the neighbouring chart has become active. Smooth matching at the first boundary does not imply that the second limit is a finite event.

Curvature regularity and causal accessibility are therefore independent questions. Scalar invariants decide whether the reciprocal domain is singular; the integral of proper time decides whether a timelike curve reaches its centre, and an affine parameter provides the corresponding null test. A reciprocal centre can be regular and asymptotically flat while remaining at infinite timelike proper time and infinite null affine parameter. In that case it is a boundary of physical history, not an event crossed by a material or null worldline.

The constant orientation sign does not alter this distinction. For any nonzero constant C, $\Gamma^a{}_{bc}[Cg] = \Gamma^a{}_{bc}[g]$, because the factor in the inverse metric cancels the factor in the metric derivatives. Nevertheless proper time must be normalised with the active representative $g^{(n_g)} = (-1)^{n_g}\mathcal{I}^*_{n_g} g$. In an odd sector the old timelike normalisation cannot simply be copied from sector 0. The examples below apply this test explicitly rather than treating vanishing curvature as a proof of completeness.

3.9 Several active modes

When the principal directions of $L^a{}_b$ and $G^a{}_b$ commute, the construction reduces locally to independent two-planes and Eq. (3.6) can be applied mode by mode. If the principal planes rotate into one another, their generators need not commute and the orientation must be transported in path order. A global existence-and-uniqueness analysis for that general case lies outside this paper. Schwarzschild, the two LTB modes, isotropic FLRW, and the principal plane of Kerr are sufficient to display the construction without introducing another formalism that is not used in the examples.

4 Sector-normalised general-relativistic examples

The examples now apply the same physical test to four different geometries. First the Jacobi or principal structure identifies the proper scale that is being focused. The known general-relativistic solution then fixes the critical scale at which the inherited clock reaches its null boundary. Only at that boundary is the reciprocal chart introduced, and only after the chart has been identified are its curvature and geodesics evaluated.

This order matters. The reciprocal map is not chosen because a curvature formula happens to diverge, and none of the examples receives a modified Einstein equation. Each calculation asks whether another sector chart of the same solution replaces the singular extrapolation by a regular physical domain. Finite curvature answers only the first half of that question. Proper time for matter and affine parameter for light decide whether the regular centre is an event that can be crossed or an asymptotic end of the worldline.

The examples track the sector of the geometry. Unless stated otherwise, their reference matter remains ordinary relative to the active field. Other particle sectors may be read in the same chart, but they do not alter its curvature. This keeps three issues separate throughout the section: the orientation of the field, the sector class of a particle, and the part of the atlas that a causal trajectory can actually reach.

4.1 Schwarzschild: from the ordinary exterior to side rest

Begin with the standard exterior Schwarzschild line element [4]

$$\mathrm{d}s^2_{(0)} = -f(R)c^2\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{f(R)} + R^2\mathrm{d}\Omega^2, \qquad f(R) = 1 - \frac{r_s}{R}, \quad r_s = \frac{2G_N M}{c^2}, \tag{4.1}$$

where R is the areal radius and $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$. The coordinate coefficients in Eq. (4.1) are singular at $R = r_s$, but the horizon itself is regular. The sector transition must therefore be performed in a horizon-regular coframe, such as an ingoing Eddington–Finkelstein frame, rather than inferred from the singular diagonal coordinates [5, 6].

Spherical symmetry reduces the sector test to the areal scale. On the observer's rest space its degenerate tensor representative may be written as $L^a{}_b = Rh^a{}_b$, while the Schwarzschild solution supplies $G^a{}_b = r_s h^a{}_b$. Hence every unit direction on the old spatial sphere has the same eigenvalue. This tensorial degeneracy means that any selected spatial ray reaches the same critical areal scale and can be paired with the clock by Eq. (3.6). The generalised eigenproblem reduces to

$$q(R) = \frac{r_s}{R}. \tag{4.2}$$

This is a geometric compactness ratio, not the measured velocity of a falling body. It is smaller than unity throughout the ordinary exterior, vanishes at the asymptotically flat rest centre, and reaches unity at the horizon. The horizon is therefore the null boundary of the ordinary chart.

The threefold degeneracy is physically important. It says that the old spatial three-space supplies a sphere of admissible side-sector clock directions; it does not create three times on one observer's worldline. A radial worldline determines a unit vector s^a in that family through its local tangent and the spherical symmetry. The tensor $J^a{}_b = u^a s_b + s^a u_b$ then rotates the plane identified by $P^a{}_b = -u^a u_b + s^a s_b$. At the quarter-turn the old clock is read as the new ruler and the chosen old ruler as the new clock, while the transverse screen is unchanged. This statement is made in a horizon-regular frame. It does not add a term to the Schwarzschild metric and does not manufacture a second vacuum solution.

At the first horizon the geometry changes from the ordinary to the first side chart; the particle class does not change. Ordinary infalling matter therefore remains ordinary relative to the local field, even though its total orientation looks side-directed when described by the exterior coframe. A pre-existing side particle likewise retains its own class. In both cases the chart changes before the unchanged particle transformation is applied.

Axis exchange and the inverse sphere. The axis exchange must be interpreted before introducing another formula. In sector 0, t is the time direction and the old (x, y, z) space is spatial. In sector 1, every direction in that old three-space is an admissible time direction, while the old time direction becomes the spatial coordinate $X = ct$. Spherical symmetry labels the family of side-time rays by (r, θ, φ): the angles label a ray and r parametrises evolution along it. Sector 0 therefore uses $+t$ as its clock direction and treats (x, y, z) as space. Sector 1 reverses these roles at the level of the projection: the possible clock directions form the family $n^a \in u^\perp$ with $h_{ab} n^a n^b = 1$, while $X = ct$ is spatial. The essential point is that not only one distinguished r direction can supply the side clock. The whole old spatial $\mathbb{R}^3$ supplies the family of possible time axes. For a radial particle the actual trajectory has fixed $(\theta, \varphi) = (\theta_0, \varphi_0)$ and thereby determines one ray; no second independent clock choice is appended to the particle. Its single proper time is parametrised along that ray by r, and it does not evolve simultaneously along every member of the family.

The quantity $R(r)$ measures the areal radius of the sphere formed by all side-time directions at the same sector value r. It is not another name for the selected ray parameter. After the exchange, r labels evolution along a chosen old spatial ray, while $X = ct$ is spatial. Consequently the zero value of r is a value of the new time coordinate, not the origin of the old spatial radius. This change of roles is what makes an infinite-radius sphere at that endpoint geometrically meaningful rather than contradictory.

It remains to determine the size $R(r)$ of that sphere. Restricting Eq. (3.8) to any one of the degenerate eigenvectors gives $\widetilde{L} = r_s^2/L$. In the ordinary chart $L = r$, so the neighbouring areal scale is fixed, rather than postulated, as

$$\boxed{R(r) = \frac{r_s^2}{r}}, \qquad 0 < r \leq r_s, \tag{4.3}$$

which satisfies $R(r_s) = r_s$ and $\mathrm{d}R/\mathrm{d}r|_{r_s} = -1$. The first equality identifies the same horizon event in both charts; the second records their opposite radial orientations. Two operations must not be conflated. The axis exchange explains why the zero endpoint of the side coordinate is a temporal sector centre and why the old time coordinate is now spatial. The reciprocal chart relation then gives the value of the simultaneous areal scale there. Together, and only together, they imply that this coordinate endpoint is a time limit at which the sphere of side-time directions has infinite areal radius. This does not identify zero with infinity: r and R are coordinates with different roles in the two projections. It would be incorrect to read the zero value of r as the centre of ordinary spherical coordinates after the exchange.

First relabel the radial coordinate in the underlying Schwarzschild metric. This keeps the one-spacetime construction explicit. Since $\mathrm{d}R = -r_s^2\mathrm{d}r/r^2$, Eq. (4.1) becomes

$$\mathcal{I}_1^*\mathrm{d}s^2_{(0)} = -(1 - r/r_s)c^2\mathrm{d}t^2 + \frac{r_s^4\,\mathrm{d}r^2}{r^4(1 - r/r_s)} + \frac{r_s^4}{r^2}\mathrm{d}\Omega^2. \tag{4.4}$$

Equation (4.4) is the pullback of the same metric under $R = r_s^2/r$. It remains a vacuum solution wherever the chart is regular: $G_{ab}[\mathcal{I}_1^* g^{(0)}] = \mathcal{I}_1^* G_{ab}[g^{(0)}] = 0$. The sector rule then supplies the orientation factor already defined in Eq. (3.9). Thus the active odd-sector representative is $g^{(1)} = -\mathcal{I}_1^* g^{(0)}$. In terms of $X = ct$ and the areal variable R, its selected clock–ruler plane is

$$\mathrm{d}s^2_{(1),\parallel} = f(R)\mathrm{d}X^2 - \frac{\mathrm{d}R^2}{f(R)}. \tag{4.5}$$

The old time X is now spatial and increasing R is the active timelike direction. Multiplication by the constant sign does not change the vacuum connection or curvature; it changes which direction is used in the proper-time normalisation.

The asymptotically flat side centre. The angular coefficient in Eq. (4.4) has the invariant meaning of an area: at fixed side time, $A_{S^2} = 4\pi r_s^4/r^2$. As the side coordinate approaches zero, this area describes a sphere whose areal radius grows without bound. A curve with fixed X, θ, φ selects one point on each successive sphere; it neither travels around that sphere nor traverses its radius. The changing areal radius is a geometric field sampled along the curve.

Regularity follows from the curvature of the one Schwarzschild metric, not from the axis exchange itself. The Newman–Penrose amplitude and the Kretschmann scalar become

$$\begin{aligned} \Psi_2(R) &= -\frac{r_s}{2R^3}, & \Psi_{2(1)}(r) &= -\frac{r^3}{2r_s^5}, \\ K(R) &= \frac{48G_N^2M^2}{c^4R^6} = \frac{12r_s^2}{R^6}, & K_{(1)}(r) &= \frac{12r^6}{r_s^{10}}. \end{aligned} \tag{4.6}$$

The powers of r arise solely from substituting $R = r_s^2/r$. At the common horizon both charts give $K = 12/r_s^4$, while their normal derivatives have opposite signs because $\mathrm{d}R/\mathrm{d}r = -1$ there. At the zero endpoint of the side coordinate, both curvature quantities vanish. The reciprocal chart has therefore replaced the extrapolation toward zero areal radius by a domain in which the areal radius grows without bound and the geometry becomes flat. This proves curvature regularity, but it does not yet say whether the limiting endpoint is reached in finite proper time.

Proper-time distance in the active side sector. The timelike equation must be derived from Eq. (4.5), not copied from the ordinary Schwarzschild normalisation. Since X is cyclic, the momentum in this old time direction, now spatial, is conserved: $P = f(R)\mathrm{d}X/\mathrm{d}\tau$. Using $\mathrm{d}s^2_{(1),\parallel} = -c^2\mathrm{d}\tau^2$ then gives

$$\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 = P^2 + c^2 f(R) = c^2\left(p^2 + 1 - \frac{r_s}{R}\right), \qquad p := \frac{P}{c}. \tag{4.7}$$

This sign is essential: P is not the conserved energy ε of a sector-0 timelike geodesic.

Let $A = 1 + p^2$. Integrating the outgoing side branch from $R_0 \geq r_s$ gives

$$\begin{aligned}\tau(R) - \tau(R_0) = \frac{1}{c}\Bigg[&\frac{\sqrt{R(AR - r_s)}}{A} \\ &+ \frac{r_s}{A^{3/2}}\,\mathrm{arsinh}\sqrt{\frac{AR - r_s}{r_s}}\Bigg]_{R_0}^{R}.\end{aligned} \tag{4.8}$$

The integrand is regular there for $p \neq 0$; for $p = 0$ its square-root divergence is integrable. Hence every horizon-to-finite-R interval is finite. The infinite-distance statement can be proved without an asymptotic expansion. Choose any finite point on the outward side branch and denote its areal radius and proper time by R_1 and τ_1. Define the two finite positive speeds

$$v_- := c\sqrt{1 + p^2 - \frac{r_s}{R_1}}, \qquad v_+ := c\sqrt{1 + p^2}. \tag{4.9}$$

The first is the radial growth rate at the chosen point; the second is the largest value approached as the reciprocal areal radius increases. Since r_s/R decreases monotonically on this branch, Eq. (4.7) means exactly that

$$0 < v_- \leq \frac{\mathrm{d}R}{\mathrm{d}\tau} \leq v_+ < \infty, \qquad R \geq R_1. \tag{4.10}$$

Equation (4.10) has a simple physical meaning: after the observer reaches R_1, the active areal scale keeps growing, its rate never falls to zero, and that rate never becomes infinite. Integrating over the elapsed proper time $\tau - \tau_1$ gives

$$R_1 + v_-(\tau - \tau_1) \leq R(\tau) \leq R_1 + v_+(\tau - \tau_1). \tag{4.11}$$

Thus $R(\tau)$ is trapped between two straight lines. The upper line is finite for every finite τ, so $R = \infty$ cannot be reached in finite proper time. The lower line grows without bound, so arbitrarily large finite areal radii are nevertheless reached as the proper time continues. Using the reciprocal relation $r = r_s^2/R$ reverses the inequalities and gives

$$\frac{r_s^2}{R_1 + v_+(\tau - \tau_1)} \leq r(\tau) \leq \frac{r_s^2}{R_1 + v_-(\tau - \tau_1)}. \tag{4.12}$$

Equation (4.12) is the same statement in the side coordinate. Both bounding functions are positive at every finite proper time and both reach zero only in the limit of unbounded τ. Consequently the coordinate value $r = 0$ denotes the reciprocal asymptotic boundary, not an event encountered after a finite duration. The case $p = 0$ is included: after the finite interval from the horizon to any $R_1 > r_s$, the same bounds apply. Equation (4.8) already contains that side-rest case and no separate approximation is required.

Null affine distance and tidal decay. Let $X = ct$, let λ be an affine parameter, and define the conserved quantity $\mathcal{E} = f(R)\,\mathrm{d}X/\mathrm{d}\lambda$. The radial null condition then reduces to

$$\frac{\mathrm{d}R}{\mathrm{d}\lambda} = \pm\mathcal{E}, \qquad R(\lambda) = r_s + |\mathcal{E}|(\lambda - \lambda_h) \tag{4.13}$$

on the branch directed toward increasing R. The areal radius can therefore grow without bound only over an unlimited affine interval. Even light does not cross the zero endpoint of the side coordinate at finite affine parameter.

Orthonormal tidal components fall in proportion to the inverse cube of the areal radius. They therefore decay along every geodesic that approaches the boundary. For finite p, the Kretschmann scalar falls with the inverse sixth power of proper time and the orthonormal tidal scale with its inverse third power. Along radial light the scalar falls with the inverse sixth power of affine parameter. The side centre is consequently both asymptotically flat and geodesically complete in these radial directions.

No finite-parameter continuation at the reciprocal boundary. The algebraic atlas may contain a reflected side branch, but that branch is not the next finite-parameter segment of the geodesics just calculated. The zero endpoint of its side coordinate is another representation of asymptotic infinity, not a local event at which the first branch is glued to it. A worldline may cross the entrance horizon from the ordinary chart into the first side chart in finite time. It cannot continue through the asymptotic centre into the opposite-time chart at finite proper or affine parameter. In particular, ordinary infalling matter does not emerge from a white hole in the $-t$ sector. Algebraic reflection of the chart and physical transport of a worldline must not be identified.

4.2 LTB geometry: longitudinal and transverse sector modes

The Lemaître–Tolman–Bondi dust geometry is the simplest example in which a single spherical radius does not describe the full deformation [7, 8]. Using the mass coordinate F, write $R = R(t, F)$, $R_F = \partial R/\partial F$, and $\Gamma(F) = \sqrt{1 + 2E(F)}$. The comoving line element is

$$\mathrm{d}s^2_{(0)} = -c^2\mathrm{d}t^2 + \frac{R_F^2}{\Gamma^2}\mathrm{d}F^2 + R^2\mathrm{d}\Omega^2. \tag{4.14}$$

The source dust is taken to have $s_X = 0$ relative to its local geometric coframe. This fixes its particle class but does not enter the directional scale tensors below. The proper separation of neighbouring shells is $\mathrm{d}\ell_\parallel = (R_F/\Gamma)\mathrm{d}F$, whereas both transverse scales are set by the areal radius R. Let $u^a = U^a/c$ be the unit dust clock field, s^a the unit radial vector in its rest space, and $S^a{}_b = h^a{}_b - s^a s_b$ the projector on the spherical screen. Because F is used as the mass coordinate, the radial entry below is the scale coefficient multiplying $\mathrm{d}F$, while the two transverse entries are areal scales. The two tensors required by the sector construction are

$$\begin{aligned} L^a{}_b &= \frac{R_F}{\Gamma} s^a s_b + R\,S^a{}_b, \\ G^a{}_b &= s^a s_b + F\,S^a{}_b. \end{aligned} \tag{4.15}$$

Equation (4.15) is basis independent even though its two eigenspaces coincide with the familiar radial and angular directions.

For the sector-(0) metric, the two eigenvalue ratios are $q_\parallel = \Gamma/R_F$ along s^a and $q_\perp = F/R$ on the spherical screen. When the radial derivative of the areal radius vanishes, only the longitudinal shell-crossing mode becomes critical. When the areal radius itself vanishes, the two degenerate shell-focusing modes become critical instead. No matrix representation is needed to preserve this distinction: it is already encoded by the projectors $s^a s_b$ and $S^a{}_b$.

Applying Eq. (3.8) separately on these eigenspaces defines the neighbouring reciprocal variables

$$\widetilde{L}_{\parallel} = \frac{\Gamma}{R_F}, \qquad \mathrm{d}\widetilde{\ell}_{\parallel} = \frac{\Gamma}{R_F}\mathrm{d}F, \qquad \widetilde{L}_{\perp} = \widetilde{R} = \frac{F^2}{R}. \tag{4.16}$$

The vanishing old shell separation is thereby represented as an unbounded proper scale measured from the adjacent rest centre, while a transverse collapse is continued by the reciprocal areal scale. They are active-scale data for sector charts of the same LTB metric and matter fields, not rescalings of a completed coframe.

Sector labels in this subsection are attached to complete geometric objects, such as $\mathrm{d}s^2_{(0)}$, $\mathrm{d}s^2_{(1)}$, and their curvature scalars. The coordinates and scalar functions t, τ, F, R, a, and b are not themselves sector-indexed; their physical role is fixed by the metric in which they occur.

The two regularity calculations must be read separately. At shell crossing only the longitudinal chart changes. The reciprocal separation in Eq. (4.16) replaces the vanishing distance between adjacent shells by an unbounded active distance measured from the neighbouring sector centre. Curvature and dust density must then be evaluated in that longitudinal domain rather than extrapolated through the degenerate comoving coordinate.

Shell focusing is different. There the radial labelling of neighbouring shells need not fail, but the transverse two-sphere loses its areal size. The reciprocal areal scale in Eq. (4.16) replaces that zero-radius extrapolation by a side domain with growing transverse area. Its curvature is evaluated independently of the shell-crossing mode. The tensor decomposition is essential precisely because one kind of collapse cannot be used as a scalar substitute for the other.

Complete causal geodesic equations in sector (0)**.** The geodesic equations can be written exactly before any profile is chosen. Define the ordinary-sector radial scale coefficient

$$A(t,F) = \frac{R_F(t,F)}{\Gamma(F)}. \tag{4.17}$$

By spherical symmetry every geodesic may be placed in the equatorial plane $\theta = \pi/2$. Let $V^a = \mathrm{d}x^a/\mathrm{d}\zeta$, where $\zeta = \tau$ for a massive particle and $\zeta = \lambda$ for an affinely parametrised null ray, and define

$$g^{(0)}_{ab} V^a V^b = -\kappa c^2, \qquad \kappa = 1 \text{ for timelike motion}, \qquad \kappa = 0 \text{ for null motion}. \tag{4.18}$$

The rotational Killing fields give the exact conserved angular momentum

$$\ell = R^2 \frac{\mathrm{d}\varphi}{\mathrm{d}\zeta}. \tag{4.19}$$

The tensor equation $V^b \nabla_b V^a = 0$ is then equivalent to

$$\frac{\mathrm{d}^2 t}{\mathrm{d}\zeta^2} + \frac{A\,\partial_t A}{c^2}\left(\frac{\mathrm{d}F}{\mathrm{d}\zeta}\right)^2 + \frac{\partial_t R}{c^2 R^3}\,\ell^2 = 0, \tag{4.20}$$

$$\frac{\mathrm{d}^2 F}{\mathrm{d}\zeta^2} + 2\frac{\partial_t A}{A}\frac{\mathrm{d}t}{\mathrm{d}\zeta}\frac{\mathrm{d}F}{\mathrm{d}\zeta} + \frac{\partial_F A}{A}\left(\frac{\mathrm{d}F}{\mathrm{d}\zeta}\right)^2 - \frac{R_F}{A^2 R^3}\,\ell^2 = 0, \tag{4.21}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} = \frac{\ell}{R^2}, \tag{4.22}$$

$$c^2\left(\frac{\mathrm{d}t}{\mathrm{d}\zeta}\right)^2 = A^2\left(\frac{\mathrm{d}F}{\mathrm{d}\zeta}\right)^2 + \frac{\ell^2}{R^2} + \kappa c^2. \tag{4.23}$$

These equations contain no large-distance or near-centre approximation. They determine a causal curve in the sector-(0) metric once $R(t,F)$, $E(F)$, and initial data are supplied.

For radial timelike motion, $\ell = 0$, introduce

$$e_{\hat{0}}{}^{a} = \frac{1}{c}(\partial_t)^a, \qquad e_{\hat{1}}{}^{a} = \frac{1}{A}(\partial_F)^a, \tag{4.24}$$

and write

$$U^a = c\gamma\big(e_{\hat{0}}{}^{a} + \beta e_{\hat{1}}{}^{a}\big), \qquad p = \gamma\beta, \qquad \gamma = \sqrt{1+p^2}. \tag{4.25}$$

The exact first-order system is

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \sqrt{1+p^2}, \qquad \frac{\mathrm{d}F}{\mathrm{d}\tau} = \frac{cp}{A}, \tag{4.26}$$

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -\frac{\partial_t A}{A}\,p. \tag{4.27}$$

Hence

$$p(t) = p_0 \exp\!\left[-\int_{t_0}^{t} \frac{\partial_{t'} A(t',F(t'))}{A(t',F(t'))}\,\mathrm{d}t'\right], \qquad \tau - \tau_0 = \int_{t_0}^{t} \frac{\mathrm{d}t'}{\sqrt{1+p^2(t')}}. \tag{4.28}$$

The comoving dust curves are the exact special case $p = 0$, $F = \text{const}$, and $\tau = t + \text{const}$.

For radial light, write

$$k^a = \omega\big(e_{\hat{0}}{}^{a} + \epsilon e_{\hat{1}}{}^{a}\big), \qquad \epsilon = \pm 1. \tag{4.29}$$

Then $k^b \nabla_b k^a = 0$ gives

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \epsilon \frac{c}{A(t,F)}, \tag{4.30}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\partial_t A(t,F(t))}{A(t,F(t))}\,\omega, \tag{4.31}$$

$$\lambda - \lambda_0 = c\int_{t_0}^{t} \frac{\mathrm{d}t'}{\omega(t')}. \tag{4.32}$$

The affine integral, rather than the coordinate slope alone, decides null completeness. A sector-(1) geodesic system must be derived from the complete metric $g^{(1)}_{ab}$; it is not obtained by replacing one coefficient in these sector-(0) equations.

Curvature in the inactive LTB chart. Before introducing reciprocal domains it is useful to display the exact LTB curvature that the inactive chart assigns to the two degeneracies. In geometrised units, with the convention $\dot{R}^2 = F/R$ and with F itself used as the mass coordinate, the Kretschmann scalar is

$$K_{\mathrm{LTB},(0)} = \frac{12F^2}{R^6} - \frac{8F}{R^5 R_F} + \frac{3}{R^4 R_F^2}. \tag{4.33}$$

Thus an ordinary shell crossing, $R_F \to 0$ while $R \to R_0 > 0$, is generically curvature divergent in that inactive description. Ordinary shell focusing, $R \to 0$, is also divergent. The sector construction does not turn either value finite by a coordinate change at the same event. It replaces the singular extrapolation by an explicitly specified reciprocal domain, whose curvature must then be computed from its own active metric.

Exact longitudinal shell-crossing example. A closed-form profile can be constructed in the marginally bound sector- (0) family. In this paragraph set $G_N = c = 1$ and use the convention $\dot{R}^2 = F/R$. The collapsing solution is

$$R(t,F) = \left(\frac{9F}{4}\right)^{1/3} [t_B(F) - t]^{2/3}. \tag{4.34}$$

Choose constants $F_c > 0$, $R_0 > 0$, $\alpha > 0$, and t_c, and define

$$R_c(F) = R_0 + \frac{\alpha}{2}(F - F_c)^2, \qquad t_B(F) = t_c + \frac{2R_c(F)^{3/2}}{3\sqrt{F}}. \tag{4.35}$$

Substitution gives the exact identities

$$R(t_c, F) = R_c(F), \qquad R_F(t_c, F) = \alpha(F - F_c), \qquad R(t_c, F_c) = R_0 > 0. \tag{4.36}$$

Thus $F = F_c$ is a simple shell crossing on the slice $t = t_c$, not a shell-focusing point. Its ordinary-sector Kretschmann scalar is

$$\boxed{K_{\mathrm{cross},(0)}(t_c, F) = \frac{12F^2}{R_c(F)^6} - \frac{8F}{\alpha(F - F_c)R_c(F)^5} + \frac{3}{\alpha^2(F - F_c)^2 R_c(F)^4}.} \tag{4.37}$$

and diverges when the inactive chart is extrapolated to $F = F_c$.

To construct the neighbouring sector, use the branch $F < F_c$ and put

$$\delta = F_c - F > 0, \qquad L_\parallel = |R_F(t_c, F)| = \alpha\delta, \qquad q_\parallel = \frac{1}{\alpha\delta}. \tag{4.38}$$

The full radial clock–ruler orientation is selected by the 4×4 matrix

$$\boxed{\mathcal{R}_r(\psi) = \frac{1}{\sqrt{1+\alpha^2\delta^2}} \begin{pmatrix} \alpha\delta & 1 & 0 & 0 \\ -1 & \alpha\delta & 0 & 0 \\ 0 & 0 & \sqrt{1+\alpha^2\delta^2} & 0 \\ 0 & 0 & 0 & \sqrt{1+\alpha^2\delta^2} \end{pmatrix}, \qquad \psi = \operatorname{atan2}(1, \alpha\delta).} \tag{4.39}$$

It acts in the time–radial plane and leaves the two angular directions unchanged. At the common chart boundary $q_\parallel = 1$, one has $\psi = \pi/4$; as $F \to F_c^-$, the matrix approaches the radial quarter-turn $\psi \to \pi/2$. The matrix identifies which principal mode becomes the side clock. The reciprocal scale is then used as initial data of an exact LTB solution, rather than as a matrix rescaling of a finished metric.

Assume $F_c > 1/\alpha$, so that the entrance shell

$$F_h = F_c - \frac{1}{\alpha} > 0, \qquad R_h = R_c(F_h) = R_0 + \frac{1}{2\alpha} \tag{4.40}$$

is well defined and satisfies $|R_F(t_c, F_h)| = 1$. Realise the reciprocal longitudinal scale as the radial Jacobi scale of the side initial profile:

$$\frac{\mathrm{d}\widetilde{R}_h}{\mathrm{d}F} = \frac{1}{\alpha(F_c - F)}, \tag{4.41}$$

$$\widetilde{R}_h(F) = R_h + \frac{1}{\alpha}\ln\left(\frac{F_c - F_h}{F_c - F}\right) = R_h + \frac{1}{\alpha}\ln\left(\frac{1}{\alpha(F_c - F)}\right). \tag{4.42}$$

At $F = F_h$, the areal radius and unsigned radial scale agree in both charts. The signed radial derivative reverses orientation, while its square and therefore the radial metric coefficient are continuous. Toward the reciprocal centre,

$$F \to F_c^- \quad \Longrightarrow \quad \widetilde{R}_h \to \infty, \qquad \widetilde{R}_{h,F} \to \infty. \tag{4.43}$$

The exact expanding marginally bound dust branch generated by these initial data is

$$\boxed{\widetilde{R}(\tau, F) = \left[\widetilde{R}_h(F)^{3/2} + \frac{3}{2}\sqrt{F}\,(\tau - \tau_h)\right]^{2/3}, \qquad \tau \geq \tau_h.} \tag{4.44}$$

Writing

$$Z(\tau, F) = \widetilde{R}_h(F)^{3/2} + \frac{3}{2}\sqrt{F}\,(\tau - \tau_h), \tag{4.45}$$

one obtains

$$\widetilde{R} = Z^{2/3}, \qquad \left(\frac{\partial \widetilde{R}}{\partial \tau}\right)^2 = \frac{F}{\widetilde{R}}, \tag{4.46}$$

$$\widetilde{R}_F = Z^{-1/3}\left[\frac{\sqrt{\widetilde{R}_h}}{\alpha(F_c - F)} + \frac{\tau - \tau_h}{2\sqrt{F}}\right]. \tag{4.47}$$

Consequently the complete sector-(1) metric is itself an exact LTB dust solution,

$$\boxed{\mathrm{d}s^2_{\mathrm{cross},(1)} = -\mathrm{d}\tau^2 + \widetilde{R}_F^2(\tau, F)\,\mathrm{d}F^2 + \widetilde{R}^2(\tau, F)\,\mathrm{d}\Omega^2.} \tag{4.48}$$

Its orthonormal Einstein tensor has the dust form

$$\boxed{G^{(1)}{}_{\hat{A}\hat{B}} = \mathrm{diag}(\kappa\rho, 0, 0, 0), \qquad \kappa\rho = \frac{1}{\widetilde{R}^2\widetilde{R}_F}.} \tag{4.49}$$

On the entrance slice $\tau = \tau_h$,

$$\kappa\rho(\tau_h, F) = \frac{\alpha(F_c - F)}{\widetilde{R}_h(F)^2} \longrightarrow 0 \qquad (F \to F_c^-). \tag{4.50}$$

Thus the reciprocal branch preserves the same pressureless matter type; no anisotropic pressure or radial flux is manufactured by the continuation.

The Kretschmann scalar of this exact sector metric is

$$\boxed{K_{\mathrm{cross},(1)}(\tau, F) = \frac{12F^2}{\widetilde{R}^6} - \frac{8F}{\widetilde{R}^5\widetilde{R}_F} + \frac{3}{\widetilde{R}^4\widetilde{R}_F^2}.} \tag{4.51}$$

On $\tau = \tau_h$ it reduces to

$$\boxed{K_{\mathrm{cross},(1)}(\tau_h, F) = \frac{12F^2}{\widetilde{R}_h^6} - \frac{8\alpha F(F_c - F)}{\widetilde{R}_h^5} + \frac{3\alpha^2(F_c - F)^2}{\widetilde{R}_h^4}.} \tag{4.52}$$

For every fixed finite $\tau - \tau_h$, the reciprocal centre obeys

$$\boxed{\lim_{F \to F_c^-} \rho(\tau, F) = 0, \qquad \lim_{F \to F_c^-} K_{\mathrm{cross},(1)}(\tau, F) = 0.} \tag{4.53}$$

The inactive shell-crossing divergence is therefore replaced, for this exact profile, by an asymptotically flat-curvature dust domain. This establishes curvature regularity of the completed sector-(1) metric. Timelike and null accessibility of its spatial reciprocal centre remain separate geodesic questions.

Exact transverse shell-focusing example. Continue in geometrised units. The homogeneous marginally bound dust solution is an exact LTB subclass. Use a comoving shell label χ. The ordinary sector-(0) geometry is

$$\boxed{\mathrm{d}s^2_{\mathrm{focus},(0)} = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}\chi^2 + \chi^2\mathrm{d}\Omega^2\right),} \tag{4.54}$$

with

$$R(t,\chi) = a(t)\chi, \qquad a(t) = a_h\left(\frac{t_f - t}{t_f - t_h}\right)^{2/3}, \qquad t_h \le t < t_f, \tag{4.55}$$

and

$$\mathcal{F}(\chi) = \frac{4a_h^3}{9(t_f - t_h)^2}\,\chi^3, \qquad E(\chi) = 0. \tag{4.56}$$

Every noncentral shell has $R \to 0$ as $t \to t_f$; this is the transverse shell-focusing mode.

The neighbouring side sector uses the reciprocal scale

$$b(\tau) = \frac{a_h^2}{a(t(\tau))}, \qquad H_h = \frac{2}{3(t_f - t_h)}. \tag{4.57}$$

Its explicitly sector-indexed metric is

$$\boxed{\mathrm{d}s^2_{\mathrm{focus},(1)} = -\mathrm{d}\tau^2 + b^2(\tau)\left(\mathrm{d}\chi^2 + \chi^2\mathrm{d}\Omega^2\right).} \tag{4.58}$$

The exact dust equation in sector (1) gives

$$b(\tau) = a_h\left[1 + \frac{3}{2}H_h(\tau - \tau_h)\right]^{2/3}, \tag{4.59}$$

$$t_f - t(\tau) = \frac{t_f - t_h}{1 + \frac{3}{2}H_h(\tau - \tau_h)}. \tag{4.60}$$

Consequently $t = t_f$, and hence $R = 0$ in the inherited collapse label, is approached only when τ is unbounded.

The curvatures of the two sector metrics can now be compared directly. For Eq. (4.54),

$$\boxed{K_{\mathrm{focus},(0)}(t) = 12\left[\left(\frac{\ddot a}{a}\right)^2 + \left(\frac{\dot a}{a}\right)^4\right] = \frac{80}{27(t_f - t)^4}.} \tag{4.61}$$

It diverges as $t \to t_f$. For the reciprocal metric Eq. (4.58),

$$\boxed{K_{\mathrm{focus},(1)}(\tau) = 12\left[\left(\frac{\ddot b}{b}\right)^2 + \left(\frac{\dot b}{b}\right)^4\right] = \frac{15H_h^4}{\left[1 + \frac{3}{2}H_h(\tau - \tau_h)\right]^4}.} \tag{4.62}$$

Thus

$$K_{\mathrm{focus},(1)}(\tau_h) = 15H_h^4, \qquad \boxed{\lim_{\tau\to\infty} K_{\mathrm{focus},(1)}(\tau) = 0.} \tag{4.63}$$

Here the complete sector-(1) metric has been written explicitly, so the finite curvature and its decay to zero follow from that metric rather than from scale inversion alone.

This statement is not restricted to exactly comoving matter. For a radial massive geodesic with conserved comoving momentum Π, its own proper time σ satisfies

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau} = \frac{1}{\sqrt{1 + \Pi^2/b^2(\tau)}} \ge \frac{1}{\sqrt{1 + \Pi^2/a_h^2}}, \tag{4.64}$$

so

$$\sigma - \sigma_h \geq \frac{\tau - \tau_h}{\sqrt{1 + \Pi^2/a_h^2}}. \tag{4.65}$$

For radial light, let $\mathcal{K} = b^2 \mathrm{d}\chi/\mathrm{d}\lambda \neq 0$. The null normalisation gives $\mathrm{d}\lambda = b\mathrm{d}\tau/|\mathcal{K}|$, and the affine parameter is exactly

$$\lambda - \lambda_h = \frac{2a_h}{5H_h|\mathcal{K}|} \left\{ \left[1 + \frac{3}{2} H_h(\tau - \tau_h) \right]^{5/3} - 1 \right\}. \tag{4.66}$$

Both the massive proper time and the null affine parameter are therefore unbounded. This homogeneous profile supplies an exact sector-(1) shell-focusing metric, an exact finite Kretschmann scalar, and exact causal completeness. The longitudinal example above likewise supplies an exact sector-(1) LTB dust metric: its density and Kretschmann scalar both tend to zero at the reciprocal shell-crossing centre. Its full noncomoving timelike and null accessibility remains a separate geodesic problem.

The general equations (4.20)–(4.32) remain necessary for nonhomogeneous profiles. The examples do not turn the result into a universal LTB theorem: a different mass function, energy function, bang-time function, or active reciprocal completion must be tested using its own exact sector metric and causal integrals.

4.3 Kerr geometry: the principal sector plane with frame dragging

Rotation changes the active plane, but not the reciprocal construction. In this subsection only, set $G_N = c = 1$. Kerr has no globally preferred radial direction supplied by spherical symmetry. The repeated principal null directions instead select a local time–principal plane. Its unit clock u^a, principal direction n^a, projector, and quarter-turn generator are exactly those of Eqs. (3.4) and (3.5). Their position dependence carries the familiar frame dragging in the Kerr connection; it does not require another sector operator.

The outer horizon lies at $r_+ = M + \sqrt{M^2 - a^2}$. Anchoring the reciprocal chart there gives

$$R(r) = \frac{r_+^2}{r}, \qquad 0 < r \leq r_+. \tag{4.67}$$

The cross term $g_{t\varphi}$ means that the active clock is the local horizon generator in a corotating, horizon-regular frame, not the coordinate one-form $\mathrm{d}t$ by itself [9]. The same worldline tangent is re-resolved in this corotating sector coframe: the inherited time direction belongs to its spatial decomposition, while n^a supplies the active clock. Turning in azimuth or reversing inherited-t motion does not reverse the particle's proper time.

Curvature can be checked directly with one invariant. The Kerr Kretschmann scalar is

$$K_{\mathrm{K},(0)}(R, \theta) = 48M^2 \frac{R^6 - 15a^2R^4\cos^2\theta + 15a^4R^2\cos^4\theta - a^6\cos^6\theta}{(R^2 + a^2\cos^2\theta)^6}. \tag{4.68}$$

and the reciprocal substitution gives

$$K_{\mathrm{K},(1)}(r, \theta) = 48M^2r^6 \frac{r_+^{12} - 15a^2r_+^8r^2\cos^2\theta + 15a^4r_+^4r^4\cos^4\theta - a^6r^6\cos^6\theta}{(r_+^4 + a^2r^2\cos^2\theta)^6}. \tag{4.69}$$

The overall r^6 is the residue of the ordinary $1/R^6$ decay after substitution. For a nonzero outer-horizon radius the denominator stays nonzero throughout the side domain, and the Kretschmann scalar vanishes at the zero endpoint of the side coordinate. The Kerr ring would require zero areal radius in the equatorial plane. The reciprocal domain instead begins at the outer horizon and extends toward unbounded areal radius, so it never samples the ring.

The non-rotating limit is immediate: $a = 0$ gives $r_+ = 2M = r_s$ and $K_{\mathrm{K},(1)} = 48M^2r^6/r_s^{12} = 12r^6/r_s^{10}$, precisely the Schwarzschild result in Eq. (4.6).

Complete Carter system in the active odd sector. The geodesic equations can be kept exact. Let

$$\Sigma = R^2 + a^2\cos^2\theta, \qquad \Delta = R^2 - 2MR + a^2. \tag{4.70}$$

Let $V^a = \mathrm{d}x^a/\mathrm{d}\zeta$, and let $\xi^a_{(t)}$, $\xi^a_{(\varphi)}$ be the stationary and axial Killing fields. Define the conserved contractions

$$\mathcal{P} = -\xi^{(t)}_a V^a, \qquad \mathcal{L} = \xi^{(\varphi)}_a V^a, \tag{4.71}$$

and let $\mathcal{Q}$ be the Carter separation constant [10, 3]. The symbol $\mathcal{P}$ is a conserved momentum along the inherited t direction; it is not called an energy in the active side chart because that direction is spatial there.

Since $g^{\mathrm{K},(1)}_{ab} = -\mathcal{I}_1^* g^{\mathrm{K},(0)}_{ab}$, an active timelike tangent satisfies

$$g^{\mathrm{K},(0)}_{ab} U^a U^b = +1. \tag{4.72}$$

Introduce

$$\varepsilon = -g^{\mathrm{K},(0)}_{ab} V^a V^b, \qquad \varepsilon = -1 \text{ for active timelike motion}, \qquad \varepsilon = 0 \text{ for null motion}. \tag{4.73}$$

Also define

$$\Pi(R) = \mathcal{P}(R^2 + a^2) - a\mathcal{L}, \qquad \mathcal{K} = (\mathcal{L} - a\mathcal{P})^2 + \mathcal{Q}. \tag{4.74}$$

The complete separated system is

$$\Sigma^2 \left(\frac{\mathrm{d}R}{\mathrm{d}\zeta}\right)^2 = \mathcal{R}_\varepsilon(R), \tag{4.75}$$

$$\mathcal{R}_\varepsilon(R) = \Pi^2(R) - \Delta\left(\varepsilon R^2 + \mathcal{K}\right), \tag{4.76}$$

$$\Sigma^2 \left(\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\right)^2 = \mathcal{T}_\varepsilon(\theta), \tag{4.77}$$

$$\mathcal{T}_\varepsilon(\theta) = \mathcal{Q} - \cos^2\theta \left[a^2(\varepsilon - \mathcal{P}^2) + \frac{\mathcal{L}^2}{\sin^2\theta}\right], \tag{4.78}$$

$$\Sigma \frac{\mathrm{d}t}{\mathrm{d}\zeta} = a\left(\mathcal{L} - a\mathcal{P}\sin^2\theta\right) + \frac{(R^2 + a^2)\Pi(R)}{\Delta}, \tag{4.79}$$

$$\Sigma \frac{\mathrm{d}\varphi}{\mathrm{d}\zeta} = \frac{\mathcal{L}}{\sin^2\theta} - a\mathcal{P} + \frac{a\Pi(R)}{\Delta}. \tag{4.80}$$

Here $\zeta = \tau$ for active timelike motion and $\zeta = \lambda$ for an affinely parametrised null geodesic. The value $\varepsilon = -1$ is the essential change from the ordinary timelike Kerr equation.

Using $R = r_+^2/r$, the exact side-coordinate equation is

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\zeta}\right)^2 = \frac{r^4}{r_+^4 \Sigma^2} \mathcal{R}_\varepsilon\left(\frac{r_+^2}{r}\right). \tag{4.81}$$

A branch directed toward increasing R has the exact parameter integral

$$\zeta - \zeta_0 = \int_{R_0}^{R} \frac{\Sigma(R', \theta(R'))}{\sqrt{\mathcal{R}_\varepsilon(R')}}\,\mathrm{d}R'. \tag{4.82}$$

The divergence of this integral can be proved without an asymptotic series. Expanding the exact radial potential gives

$$\mathcal{R}_\varepsilon(R) = A_4R^4 + A_3R^3 + A_2R^2 + A_1R + A_0, \tag{4.83}$$

where

$$\begin{aligned} &A_4 = \mathcal{P}^2 - \varepsilon, && A_3 = 2M\varepsilon, && \\ &A_2 = 2a\mathcal{P}(a\mathcal{P} - \mathcal{L}) - \mathcal{K} - \varepsilon a^2, && A_1 = 2M\mathcal{K}, && A_0 = -a^2\mathcal{Q}. \end{aligned} \tag{4.84}$$

Choose any $R_0 > 0$ beyond which the considered branch is allowed, and set

$$C_\varepsilon(R_0) = |A_4| + \frac{|A_3|}{R_0} + \frac{|A_2|}{R_0^2} + \frac{|A_1|}{R_0^3} + \frac{|A_0|}{R_0^4}. \tag{4.85}$$

For every $R \geq R_0$,

$$0 \leq \mathcal{R}_\varepsilon(R) \leq C_\varepsilon(R_0)R^4, \qquad \Sigma \geq R^2. \tag{4.86}$$

Therefore the exact integral obeys

$$\zeta - \zeta_0 \geq \frac{R - R_0}{\sqrt{C_\varepsilon(R_0)}}. \tag{4.87}$$

Any causal branch that reaches arbitrarily large R consequently needs unlimited proper or affine parameter. A branch for which $\mathcal{R}_\varepsilon$ has an outer zero turns or remains bound and does not approach the reciprocal boundary.

Closed-form equatorial geodesics. The exact conclusion can also be seen in a solvable family. Set

$$\theta = \frac{\pi}{2}, \qquad \mathcal{Q} = 0, \qquad \mathcal{L} = a\mathcal{P}. \tag{4.88}$$

For active timelike motion, put $A = 1 + \mathcal{P}^2$ and

$$D(R) = AR^2 - 2MR + a^2. \tag{4.89}$$

Then

$$\mathcal{R}_{-1}(R) = R^2D(R), \qquad \frac{\mathrm{d}R}{\mathrm{d}\tau} = \frac{\sqrt{D(R)}}{R} \tag{4.90}$$

on the outgoing branch, and direct integration gives

$$\begin{aligned} \tau(R) - \tau(R_0) = \Bigg[&\frac{\sqrt{D(R)}}{A} \\ &+ \frac{M}{A^{3/2}} \ln\Big(2\sqrt{A}\sqrt{D(R)} + 2AR - 2M\Big)\Bigg]_{R_0}^{R}. \end{aligned} \tag{4.91}$$

The first term is unbounded as R is unbounded, so this exact expression places the reciprocal centre at infinite proper time.

For null motion with the same angular data,

$$\mathcal{R}_0(R) = \mathcal{P}^2R^4, \qquad \frac{\mathrm{d}R}{\mathrm{d}\lambda} = |\mathcal{P}|, \tag{4.92}$$

and hence

$$R(\lambda) = R_0 + |\mathcal{P}|(\lambda - \lambda_0), \qquad r(\lambda) = \frac{r_+^2}{R_0 + |\mathcal{P}|(\lambda - \lambda_0)}. \tag{4.93}$$

The side coordinate remains positive for every finite affine parameter and reaches zero only when λ is unbounded. These closed-form examples agree with the general exact bound in Eq. (4.87); no large-R approximation is used.

A reflected Kerr chart may represent the opposite orientation algebraically, but frame dragging, spin, and particle data are not transported through the side centre by a finite-parameter worldline. No Kerr black-hole trajectory derived here continues into a white-hole exit in the opposite-time sector.

4.4 Spatially flat FLRW: reciprocal scale and the meaning of $t = 0$

The FLRW example uses exactly the same sector mechanism as the black-hole examples. The only difference is the geometric quantity that is inverted. For a black hole it is the areal radius; for a homogeneous universe it is the scale factor.

In the ordinary $+t$ projection, write the spatially flat metric as

$$\mathrm{d}s^2_{(0)} = -c^2\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}\chi^2 + \chi^2\mathrm{d}\Omega^2\right). \tag{4.94}$$

Here $a(t)$ is the ordinary scale factor [11]. Let a_h and ρ_h denote the finite scale factor and density at the null boundary where the ordinary projection ends and the side projection becomes active. The comoving fluid is assigned $s_X = 0$ relative to the active geometric coframe. A change of n_g therefore reorients the fluid and the field together without changing the fluid's particle class.

Homogeneity and isotropy make all three principal modes identical. With u^a the comoving clock and $h^a{}_b$ its rest projector, the complete tensor statement is

$$L^a{}_b = a(t)h^a{}_b, \qquad G^a{}_b = a_h h^a{}_b. \tag{4.95}$$

All three spatial eigenvalues are therefore equal: the active scale is $a(t)$, the boundary scale is a_h, and their common ratio is $q_i = a_h/a(t)$. Every spatial direction consequently reaches the sector boundary together. This tensor statement already contains the common clock and the isotropic three-dimensional rest space; writing a second diagonal matrix would add no information.

Equation (3.8) gives

$$\begin{aligned} \widetilde{L}^a{}_b &= \frac{a_h^2}{a(t)}\, h^a{}_b, \\ b(\tau) &= \frac{a_h^2}{a(t(\tau))}, \qquad b(\tau_h) = a_h. \end{aligned} \tag{4.96}$$

Here τ is proper time along the comoving clock defined by the sector-(1) metric. The active representative is

$$\mathrm{d}s^2_{(1)} = -c^2\mathrm{d}\tau^2 + b^2(\tau)\left(\mathrm{d}\chi^2 + \chi^2\mathrm{d}\Omega^2\right). \tag{4.97}$$

This is the one FLRW geometry evaluated in its reciprocal domain, not an independently chosen second metric. The inherited time label t can approach a finite endpoint while the proper-time coordinate τ covers an infinite interval in the sector-(1) metric.

Exact side-sector evolution for $w > -1$. Let $p = w\rho c^2$ with constant w, and let ρ_h be the density at the common null boundary. Energy conservation in the active chart gives

$\rho(\tau) = \rho_h[a_h/b(\tau)]^{3(1+w)}$. Substitution into the spatially flat Friedmann equation determines the side evolution; it is not an independently chosen expansion law. With $\alpha = 3(1+w)/2$, $H_h^2 = 8\pi G_N \rho_h/3$, and $b(\tau_h) = a_h$, its exact solution is

$$b(\tau) = a_h \left[1 + \alpha H_h(\tau - \tau_h)\right]^{1/\alpha}, \qquad \rho(\tau) = \frac{\rho_h}{\left[1 + \alpha H_h(\tau - \tau_h)\right]^2}. \tag{4.98}$$

The reciprocal scale grows without bound only over an unlimited interval of the side clock. The reciprocal centre is therefore not reached by comoving matter in finite proper time.

For spatially flat FLRW, the Friedmann and acceleration equations give

$$K_{(1)} = \frac{12}{c^4}\left[\left(\frac{\ddot{b}}{b}\right)^2 + \left(\frac{\dot{b}}{b}\right)^4\right] = \frac{64\pi^2 G_N^2}{3c^4}\left[(1+3w)^2 + 4\right]\rho^2. \tag{4.99}$$

If K_h denotes the same expression at the boundary value $\rho = \rho_h$, then

$$K_{(1)}(\tau) = \frac{K_h}{\left[1 + \alpha H_h(\tau - \tau_h)\right]^4}. \tag{4.100}$$

The Weyl tensor of FLRW vanishes, so this decay describes the complete Riemann tensor. The density falls with the inverse square of the side proper time and the Kretschmann scalar with its inverse fourth power. The side branch is therefore asymptotically flat.

Why the inactive label $t = 0$ is not crossed. For the same single fluid, its ordinary power-law chart has $a(t) = a_h(t/t_h)^{1/\alpha}$ and $H_h = 1/(\alpha t_h)$. Combining this expression with $b = a_h^2/a$ and Eq. (4.98) yields

$$t(\tau) = \frac{t_h}{1 + \alpha H_h(\tau - \tau_h)}. \tag{4.101}$$

Equation (4.101) shows that the ordinary time label approaches zero only after an unlimited interval of the active proper time. The ordinary coordinate compresses an infinite side-sector history into the finite label $t = 0$; that label is not an event crossed by a comoving observer.

The result is not restricted to exactly comoving material. A freely moving massive particle with finite conserved comoving momentum has a Lorentz factor that approaches unity as the reciprocal scale grows. Its proper time σ therefore grows without bound with the side clock. For radial light, conformal transformation of Eq. (4.97) gives $\mathrm{d}\lambda \propto b(\tau)\mathrm{d}\tau$, where λ is affine. Since the scale grows as a positive power of proper time for the fluids considered here, the affine integral diverges. Null geodesics consequently approach the same boundary only after an unlimited affine interval. Equations of state outside the assumption $w > -1$ require their own sector and geodesic analysis and are not assigned a continuation in this work.

No Big-Crunch–Big-Bang passage. The algebraic atlas may contain a PT-reflected cosmological branch, but Eq. (4.101) forbids interpreting it as the continuation of the same material worldline through $t = 0$ at finite time. A contracting configuration can enter its side chart and evolve forever while its reciprocal scale grows, its density falls, and its curvature approaches zero. It does not pass through that asymptotic boundary and reappear as an expanding Big-Bang branch. Opposite time orientations may remain correlated parts of one global sector atlas, but they are not joined by finite-proper-time transport of matter.

4.5 What the examples establish: regularity is not passage

The black-hole and FLRW calculations establish a common physical sequence. A causal worldline may reach the first null boundary and continue into the neighbouring side chart. Within that chart the reciprocal proper scale grows, the curvature remains finite, and in the regular cases studied here the geometry becomes asymptotically flat. The worldline nevertheless never arrives at the reciprocal centre after a finite amount of its own time. Light likewise needs an unlimited affine interval. Regularisation has therefore opened a complete asymptotic domain, not a short passage through the former singularity.

This resolves the apparent contradiction between a finite coordinate label and an infinite physical distance. In the black-hole charts the side coordinate can approach zero while the areal radius grows without bound. In the FLRW chart the inherited time label can approach zero while the active proper time and reciprocal scale continue indefinitely. The coordinate endpoint is a compact notation for asymptotic infinity; it is not a second local event hidden at the same numerical value. LTB requires a solution-specific test because its longitudinal shell separation and transverse areal scale need not approach their reciprocal centres in the same way.

Conserved data along an accessible side branch. Equation (2.13) remains valid across the finite entrance overlap and at every finite point of the side branch. The particle's s_X is unchanged, and a pure geometry-chart transition does not alter $\Lambda_{(X)}$. The particle therefore does not acquire a new causal type when the field is represented in another sector. What changes is the local clock–ruler decomposition used to describe the same tangent. This conservation statement holds throughout the accessible side branch but does not imply transport through its asymptotic centre.

Algebraically, the four possible geometry orientations are still those of Eq. (2.11). For a particle they are read with the fixed offset s_X in $N_X = n_g + s_X \pmod 4$, as already encoded by Eq. (2.13). This classifies the possible readings of one particle. It does not say that one particle physically visits those readings in the order in which they appear in the table.

Algebraic closure and physical accessibility. The frame atlas still completes the orientation cycle listed in Eq. (2.11). Physical accessibility follows a different rule. An entrance horizon may connect an ordinary chart to its neighbouring side chart at finite proper time. The next sector would require the same worldline to traverse the reciprocal centre, but the geodesic calculations place that centre at infinite physical parameter. The reflected half of the atlas can therefore exist as another orientation of the global field without being the next finite-time destination of the infalling observer.

This distinction is the central result of the examples. The algebraic identity $i^4 = 1$ closes the representation of the clock–ruler frame. It does not identify the beginning and end of a material history, because no finite event completes the corresponding physical circuit. The one spacetime atlas is closed as a description, while its reciprocal side domains are asymptotically separated for the causal curves analysed here.

5 One global field and its causal consequences

5.1 Four readings do not create four histories

The pullback relation in Eq. (3.11) is not merely a notational convenience. It says that the metric, matter, records and worldlines form one global configuration. If two sector charts contain the same event, they contain the same field value at that event. One description cannot retain a traveller or a decision that is absent from the other.

There is consequently no signal transmitted "between sectors." Such language would falsely suggest a source in one spacetime and a receiver in another. Here a change of the physical configuration changes all of its representatives for the same reason that changing a tensor changes its components in every chart. The sector construction is deeper than an ordinary coordinate change because it also changes which local direction is read as time, but it still introduces neither another manifold nor another independently adjustable copy of matter.

This one-field condition does not identify particles with different s_X. They are distinct members of the same matter configuration and may coexist at one event. What must agree across geometry charts is the identity, state, s_X, and $\Lambda_{(X)}$ of each particle. A chart cannot describe the same traveller with $s_X = 0$ while another assigns it $s_X = 1$.

Two qualifications are important. First, consistency is required wherever sector charts overlap and refer to the same physical event. Second, an asymptotic boundary is not turned into an overlap merely because another chart contains a formally reflected endpoint. The one-field condition relates the global descriptions, but it does not manufacture a local event through which matter can be passed from one asymptotic domain to another.

5.2 Proper time and the asymptotic black-hole side domain

Every material observer satisfies Eq. (2.18) and experiences increasing proper time along its own sector clock. The orientation $n_g = 2$ can therefore be included as the PT reflection of $n_g = 0$ without reversing a local observer's proper time. What changes is the projection of a future-directed tangent on the inactive $+t$ axis.

This algebraic relation does not make the opposite orientation accessible through a black-hole interior. A Schwarzschild or Kerr worldline can cross the entrance horizon from the ordinary chart into the first side chart at finite proper time. The centre of that side chart, which would have to be traversed before the opposite-time chart became physically accessible, lies at unlimited proper time; light encounters the same obstruction in affine parameter.

This is also why a decreasing inherited time coordinate does not describe an observer moving backward through its own history. In the side chart that coordinate belongs to the spatial part of the local decomposition. The future direction is supplied by the active side clock and remains aligned with increasing proper time. A black-hole entrance in the $+t$ domain and a white-hole exit in a $-t$ domain may therefore be opposite representatives in the global atlas, but they are not successive segments of one traveller. Infalling matter retains its particle class throughout the accessible side branch and does not emerge from the reflected object by the mechanism studied here.

5.3 Absence of sector-generated CTCs in the analysed geometries

The four frame orientations close algebraically. A closed timelike curve would require much more: one continuous future-directed tangent would have to return to the same event after a finite amount of proper time. Merely returning to the same sector label does not meet that condition. A material loop assembled from the sector sequence would have to traverse one of the reciprocal side centres, and the calculations above place those centres at infinite timelike and null parameter distance. The identity $i^4 = 1$ therefore does not identify two spacetime events.

There are therefore two distinct obstructions. In flat spacetime, n_g is fixed and no finite boost changes s_X, so a particle never leaves its original null domain. In the analysed Schwarzschild, Kerr, and regular FLRW continuations, curved geometry can activate the first side chart, but the reciprocal centre blocks the remaining cycle at infinite proper or affine parameter. Consequently none of these geometries contains a CTC generated by the four-sector mechanism.

A side particle with $s_X = 1$ is not itself a CTC, and a chain of sector-normalised black and white holes cannot return an observer to an earlier event. The statement is not a theorem about every conceivable Lorentzian metric: an unrelated exotic solution of Einstein's equations could still contain a globally matched periodic curve. Such a curve would arise from that solution's causal structure rather than from boosts or reciprocal sector continuation and would retain the usual chronology, stability and quantum-backreaction problems [12, 13, 14].

The distinction also prevents an apparent paradox based solely on chart orientation. A spatial turn inside a side sector can reverse radial or angular motion, but it cannot reverse the active clock of the particle. Likewise, the PT-reflected chart is a description of the opposite orientation of the field, not an operation by which an observer flips its future-directed tangent. Neither operation supplies the missing finite worldline segment needed to close a causal loop.

5.4 Big Crunch and Big Bang as asymptotically related sector branches

The FLRW proper-time calculation replaces the earlier transition interpretation. A contracting configuration can enter a side domain in which the reciprocal scale grows without bound while density and curvature decay. Equation (4.101) places the limiting centre at unlimited side proper time. Comoving and noncomoving material worldlines, as well as null geodesics, do not cross it at finite physical parameter.

A Big-Crunch branch and a PT-related Big-Bang branch may consequently be included as correlated domains of one global sector atlas, but not as two events connected by material transport through $t = 0$. Matter collapsing in the first branch is not carried through the asymptotic boundary and recreated as the expanding matter of the second. Establishing a global solution containing both orientations would require asymptotic boundary data for the complete matter model; it would not turn the Big Crunch into a bounce or a finite-time origin of the Big Bang.

The word "correlated" is therefore deliberately weaker than "connected by a trajectory." The two branches may be constrained by one global one-field solution and by the same sector symmetry, yet their material histories remain causally separated by the asymptotic side domain. This preserves the unity of the spacetime description without claiming that the collapsing universe physically passes through its limiting coordinate label and becomes the expanding branch.

6 Possible consequences for quantum field theory

6.1 What the classical construction does and does not imply

The classical construction allows a particle to carry the discrete relative label s_X in addition to its ordinary Lorentz data. The direction of a particular side trajectory is already contained in those Lorentz and worldline data and is not a further independent label on S^2. It does not yet provide a quantum operator whose eigenvalues are those labels, an interaction law that conserves them, or a spectrum showing that side-sector asymptotic particles exist. In particular, the word “tachyon” used here denotes a sector orientation and not the standard analytic continuation to a negative mass squared.

The same caution applies to antiparticles. The orientation $s_X = 2$ supplies the PT-reversed transformation class, whereas a physical positron is defined by charge conjugation and the quantum field’s representation. Equating these objects would require a derivation of charge, positive frequency and CPT; it does not follow from the phase $\mathrm{i}^2 = -1$ alone.

At the classical level the label has a clear but limited meaning. It tells us which local clock orientation is used before the ordinary Lorentz transformation acts. It neither determines a particle species nor supplies a production process. In particular, the existence of a side-sector kinematical class does not show that such particles occur as incoming or outgoing states of a quantum scattering experiment.

6.2 Virtual lines and possible sector conversion

Virtual internal lines remain a possible place where the sector description could become useful. Their momenta need not obey the mass-shell condition of external states, and a fermion loop admits oppositely oriented descriptions [15, 16]. A future sector theory might organise such segments by s_X, but this is an interpretation rather than a result of the classical geometry. It would not make an internal line a detectable superluminal particle or a controllable channel to the ordinary past.

An internal line also should not be confused with a hidden classical trajectory. It is part of an amplitude and is integrated over; it is not an independently observed particle moving between two sectors of spacetime. This distinction is especially important here because the classical construction already forbids a finite boost from changing a particle’s sector.

If a quantum interaction were allowed to change the relative label, $s_{\mathrm{in}} \neq s_{\mathrm{out}}$, a sequence of such vertices could in principle close the sector phase and resemble a CTC in a diagrammatic description. It would not, however, be the continuous closed worldline of one classical particle. At the conversion vertex the incoming sector excitation ends and an excitation with a different sector identity begins; the ordinary and side states are not the same one-particle state. A closed chain of conversions is therefore a cyclic field process, not automatically a traveller returning to its own past. Calling it a genuine quantum CTC would require the complete field state, interaction data and observables to return periodically, which has not been derived here.

6.3 Conditions for a quantum completion

A quantum completion would need sector-resolved propagators, interaction selection rules governing both conservation and possible conversion of s_X, a consistent mass shell and positive-frequency definition, charge conservation, spin–statistics and unitarity. The curvature regularisation and the classical absence of sector-generated CTCs in the analysed geometries do not depend on that programme.

The complex phases also require an explicit reality condition in any such completion. Here i^n labels orientation: the active metric representation has only the real sign i^{2n}, the null equation remains zero, and the curvature scalars evaluated in the examples are real. A field theory would additionally have to guarantee real proper times, energies, densities, charge fluxes and probabilities. The phase notation alone is not that guarantee.

These requirements mark the boundary between the present result and a future quantum theory. Until an action and its physical state space are specified, the sector labels provide a disciplined way to organise possible orientations, not a prediction of new particles or interactions. This keeps the quantum discussion compatible with the classical conclusion: the four-sector atlas by itself supplies no observable superluminal signal and no closed material history.

7 Physical meaning and scope

7.1 Four logically separate questions

The construction separates four questions that are often conflated. Particle kinematics specifies the conserved relative class s_X and the ordinary Lorentz transformation $\Lambda_{(X)}$ within it. Its directional data determine the realised member of the side-direction family; no separate clock-axis label is assumed. Geometry continuation specifies the active chart n_g when an inherited clock reaches a null orientation. Dynamical regularity asks whether the one metric has finite curvature on the corresponding domain. Geodesic accessibility finally asks whether a centre is reached at finite proper or affine parameter. A side particle can exist without a geometry-sector transition, and a geometry-sector transition does not make ordinary matter a side particle. Regular curvature and geodesic distance must be established independently.

The logical dependence is consequently fixed. The invariant null set first supplies the four orientation components. Particle rest then means absence of spatial motion relative to one selected clock. Geometric flatness is a stronger neighbourhood statement: the common Riemann tensor must vanish. Finally, even a flat limiting domain may lie at an infinite worldline distance. The tensor calculation decides whether the limit is regular; the worldline integral decides whether it is reached.

Curvature has two operational projections. Changing n_g can select a different regular domain of the same solution, while changing s_X or $\Lambda_{(X)}$ changes the components measured by a particle at a fixed event. Neither changes the scalar invariant at that event. Disagreement of scalars on a chart overlap would therefore signal two geometries rather than two representations; disagreement between particle-sector components alone would not.

7.2 When a sector continuation is admissible

The reciprocal maps used in the examples should consequently be judged as parts of an atlas, not as replacements for Einstein's equation. A proposed sector continuation is admissible only if its overlap at the null boundary is smooth, its metric and matter data represent one solution, its geodesics can be continued in the active chart, and its invariants remain finite throughout that chart. Approaching zero curvature at a sector centre is not a passage condition. In Schwarzschild, Kerr and regular FLRW it coincides instead with an infinite-parameter asymptotic boundary. In LTB the longitudinal shell-crossing and transverse shell-focusing modes remain distinct. The explicit profiles in Sec. 4.2 give exact sector-(1) dust metrics for both modes. In the longitudinal construction the reciprocal initial profile drives the density and $K_{\mathrm{cross},(1)}$ to zero, while the homogeneous transverse example also supplies exact causal completeness. Arbitrary LTB data still require their own active metrics and geodesic integrals; no profile-independent conclusion is claimed for general matter.

This criterion also distinguishes the construction from a coordinate trick. A coordinate change by itself cannot turn a divergent scalar invariant into a finite one at the same event. The sector prescription instead says that the singular extrapolation is not part of the active physical domain: the neighbouring chart evaluates the same solution on the reciprocal proper scale selected at the null boundary. The overlap must agree where both charts are valid, while the side chart continues toward a different, asymptotic part of the geometry.

The directional formulation makes this criterion local and calculable. The scale tensor $L^a{}_b$ identifies which proper separation is becoming critical, $G^a{}_b$ fixes the solution-dependent null boundary, and $J^a{}_b$ identifies the clock–ruler plane that changes role. Only the reciprocal scale is used to select the regular representative of the known GR solution. The orientation tensor is never used to manufacture a new metric, which prevents a frame convention from being mistaken for a new stress tensor or for curvature regularisation.

The construction therefore has a clear failure test. It fails for a proposed continuation if the boundary data do not match, if the reciprocal scales cannot be assembled into one metric and matter solution, if an invariant still diverges in the active chart, or if the claimed causal interpretation contradicts the proper-time or affine-parameter integral. Passing one of these tests does not substitute for passing the others.

7.3 Present limits

The same one-spacetime requirement controls the later causal interpretation. Opposite orientations cannot carry independently edited histories, and closure of the frame cycle is not closure of a worldline. A macroscopic CTC is not generated by the sector cycle because the required side centres are infinitely distant. Any CTC arising by another mechanism would require a periodic solution for the complete traveller–geometry state. In addition, the present classical work treats s_X as conserved kinematic data; a theory of interactions that can create, annihilate or convert particle sectors has not been supplied.

The examples establish the mechanism for the Schwarzschild radial mode, the complete separated Kerr geodesic system, isotropic FLRW fluids within the stated equation-of-state range, and explicit longitudinal and transverse LTB profiles. They do not constitute a completeness theorem for arbitrary rotating matter, noncommuting tidal eigenspaces, general LTB profiles, or equations of state outside the analysed range. Those cases require the same sequence of tests rather than an extrapolation from the simplest examples.

The result should therefore be read as a constrained geometric extension of Lorentz symmetry. It supplies a common causal atlas and a tensorial rule for selecting regular side domains. It does not supply new matter equations, a quantum interaction theory, or a universal theorem that every general-relativistic singularity is removed.

8 Conclusion

The four-sector construction completes the orientation atlas of a Lorentzian spacetime without turning the atlas into four spacetimes or four stages of one journey. Geometry supplies the active clock–ruler coframe, while each particle retains its own sector-labelled Lorentz transformation. The realised side direction of a particle is read from that transformation and its tangent, not assigned as an additional clock variable. The common null cone remains the light limit for every such observer. A change of geometry chart consequently changes the decomposition of the same tangent; it neither accelerates the particle through light speed nor reverses its proper time.

In curved spacetime this orientation principle acquires physical content through the reciprocal proper scale. The examples show that the zero-radius or zero-scale extrapolation need not be interpreted as the next physical point of the solution. Schwarzschild and Kerr instead open side domains in which the areal radius becomes unbounded and curvature decays. The regular FLRW fluids studied here open an analogous cosmological domain in which the reciprocal scale grows while density and curvature fall. LTB shows why this prescription must remain tensorial: radial shell separation and transverse areal collapse are different geometric modes and cannot be regularised by one undifferentiated scalar.

The exact geodesic calculations change the interpretation of the explicitly integrated regular centres. In Schwarzschild, Kerr, regular FLRW, and the homogeneous LTB focusing profile, matter reaches the reciprocal limit only over unlimited proper time. The selected LTB shell-crossing profile now also has a complete exact sector-(1) dust metric, with both density and Kretschmann curvature tending to zero at its reciprocal spatial centre. A full proper-time and affine-parameter accessibility result for that nonhomogeneous centre remains to be calculated. The null calculations give an unlimited affine interval wherever they have been carried out. These limits are therefore asymptotic regions rather than doors to the next orientation. Black-hole infall does not become white-hole egress, and a contracting cosmological branch does not pass through its limiting coordinate label and emerge as a Big Bang. Opposite orientations may still belong to one globally correlated field, but they are not consecutive finite-time episodes of one material history.

This same distinction removes the apparent causal paradox. The identity $i^4 = 1$ completes the frame representation, not a worldline. In flat spacetime a finite boost never changes the particle sector; in the curved examples the asymptotic side domain prevents completion of the remaining cycle. The construction therefore produces no sector-generated closed timelike curve in the geometries analysed here.

The remaining work is now sharply defined. Concrete LTB profiles require their own completeness proofs, more general rotating or noncommuting tidal modes require a global tensor analysis, and a quantum theory would need an action, a reality condition and interaction rules for sector labels. These open tasks do not weaken the classical result established in this paper: one causal spacetime admits four algebraically complete orientations, while its regular side-sector centres are physical asymptotic boundaries rather than passages to another time direction.

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