# Vacuum Closure and the Octonionic Origin of Fermion Mass Hierarchies: A Numerical Closure Program for Associator-Derived Fermion Masses

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**Abstract**

The Standard Model accounts for fermion masses through Yukawa couplings, but the numerical values and hierarchical pattern of these couplings remain external inputs. This article develops a geometric octonionic framework in which fermion-mass hierarchies are encoded in the spectrum of a projected non-associative vacuum operator. The construction starts from three octonion-valued fields, whose associator is promoted to a dynamical vacuum quantity. A master action is formulated, its non-associative vacuum equations are reduced in a canonical sector, and an algebraically motivated symmetry-breaking term is introduced through a preferred associative quaternionic subalgebra and a Fano-oriented vacuum direction. The resulting structure defines a self-adjoint associator number operator and a geometric spectral mass operator. Four closure statements are formulated: vacuum-scale closure, associator spectral closure, projector normalization closure, and RG reconstruction closure. The main result is not a claim of experimentally proven exact fermion masses, but a mathematically explicit framework in which exact masses would follow once four closure blocks are solved without additional phenomenological input: the absolute octonionic vacuum scale, the exact associator Hessian spectrum, the projector normalization of the fermion sectors, and the renormalization-group matching to measured mass conventions. In the minimal Fano-plane vacuum, the linearized associator matrix is evaluated explicitly and yields the non-zero spectral levels $4, 8, 12$, i.e. the normalized generation skeleton $N_1 : N_2 : N_3 = 1 : 2 : 3$. A charged-lepton benchmark then shows how the same spectral template can reproduce the measured $e, \mu, \tau$ hierarchy once the sector invariants are fixed. The paper therefore provides a controlled route from octonionic non-associativity to fermion mass hierarchies and identifies the remaining mathematical requirements for a parameter-free mass prediction. In its final form the closure program is strengthened by an explicit full-Hessian decomposition, a projector-uniqueness theorem modulo the vacuum stabilizer, a finite charged-lepton trace test, and a component-level contraction formula for the lepton Hessian moments that makes the no-fit criterion operational before comparison with observed masses.

**Keywords:** octonions; non-associative geometry; $G_2$; fermion masses; flavor hierarchy; spectral operator; vacuum closure

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# I. Octonionic framework and vacuum data

## 1 Introduction

The observed spectrum of fermion masses is one of the least explained numerical structures in the Standard Model. After electroweak symmetry breaking, each charged fermion mass has the form

$$m_f = \frac{v_H}{\sqrt{2}} y_f, \tag{1}$$

where $v_H \simeq 246\,\mathrm{GeV}$ is the Higgs vacuum expectation value and $y_f$ is the corresponding Yukawa coupling. Equation (1) is phenomenologically successful, but it does not determine the Yukawa couplings. The hierarchy between the electron, muon, tau, quarks and possible neutrino masses is therefore not explained by the minimal Standard Model itself. This is the usual flavor problem.

Many approaches have attempted to reduce the arbitrariness of the Yukawa sector: grand unification, Froggatt–Nielsen mechanisms, modular flavor symmetries, discrete flavor groups, extra-dimensional wave-function localization, string compactifications, non-commutative geometry and spectral models [11–15]. These approaches often explain parts of the observed hierarchy, but typically retain additional charges, flavon expectation values, modular parameters, compactification data or fitted textures. The question addressed here is whether a deeper non-associative algebraic structure can encode the origin of mass hierarchies more intrinsically.

The octonions $\mathbb{O}$ form the largest normed division algebra over $\mathbb{R}$ and are non-associative but alternative. Their automorphism group is the exceptional Lie group $G_2$ [1–3]. The octonionic associator

$$[x, y, z] = (xy)z - x(yz) \tag{2}$$

measures the failure of associativity and vanishes on every quaternionic subalgebra. This property suggests a geometric interpretation: associative quaternionic sectors can represent low-energy physical projection channels, while non-associative directions encode hidden internal structure. In the present work this idea is applied to the fermion mass problem.

The central proposal is that fermion mass hierarchies are spectral data of a projected octonionic vacuum. More precisely, an octonionic vacuum is built from three independent octonion-valued fields $\Psi_1, \Psi_2, \Psi_3$. Three fields are essential because alternativity implies that a single repeated octonionic degree of freedom cannot generate a non-trivial associator. The associator

$$A = [\Psi_1, \Psi_2, \Psi_3] \tag{3}$$

is then used to define a positive associator number operator $\mathcal{N}_A$. Its eigenmodes are interpreted as generation modes, and a sector-dependent spectral mass operator is constructed from geometric vacuum invariants.

The article is deliberately written with a conservative logical status. It does not claim that all observed fermion masses are already proven exactly from $\mathbb{O}$. Instead it establishes a closed framework and separates three levels:

(i) algebraic identities that follow from octonionic geometry;

(ii) model assumptions defining the vacuum and the projection sectors;

(iii) conditional theorems showing what follows if the vacuum and spectral closures are unique.

This distinction is necessary because a numerical mass prediction is only fundamental if the vacuum invariants and spectral data are determined internally, not fitted to the observed masses.

**Roadmap.** The logical order of the manuscript is as follows. First, the octonionic algebra, field triplet and vacuum selection are fixed. Second, the physical quaternionic, complex, chiral and sector projectors are constructed. Third, the full projected Hessian is formed from the action. Fourth, the charged-lepton block is isolated as a chiral $L \to R$ spectral problem, whose positive eigenvalues determine the charged-lepton Yukawa factors. Finally, the logical status of the remaining electroweak, quark, neutrino and renormalization closures is separated from the charged-lepton result.

## 2 Standard-Model benchmark and target of the construction

The low-energy benchmark is the Yukawa sector of the Standard Model, with measured charged-fermion inputs taken from standard particle-data conventions [16]. For quarks and charged leptons one may write

$$-\mathcal{L}_Y = \overline{Q}_L Y_d H d_R + \overline{Q}_L Y_u \widetilde{H} u_R + \overline{L}_L Y_e H e_R + \text{h.c.}, \tag{4}$$

where $Y_u, Y_d, Y_e$ are complex $3 \times 3$ matrices. After electroweak symmetry breaking,

$$M_u = \frac{v_H}{\sqrt{2}} Y_u, \qquad M_d = \frac{v_H}{\sqrt{2}} Y_d, \qquad M_e = \frac{v_H}{\sqrt{2}} Y_e. \tag{5}$$

The charged-fermion masses are the singular values of these matrices, while CKM mixing is produced by the mismatch between the unitary diagonalizations of $Y_u$ and $Y_d$. Massive neutrinos require an extension of the minimal Standard Model, for example through a Weinberg operator, right-handed neutrinos or seesaw-type dynamics.

Equations (4) and (5) define the target that any fundamental model must improve upon. They turn Yukawa inputs into masses, but they do not calculate the Yukawa inputs. A genuine octonionic derivation must therefore do more than rename $Y_f$. It must provide a fixed finite-dimensional operator whose eigenvalues or singular values become the effective Yukawa matrices after projection.

**Definition 2.1** (Parameter-free fermion-mass derivation)**.** *In the present paper a parameter-free derivation means the following. The vacuum equations, projection rules, associator spectrum, electroweak matching and RG map must determine the effective matrices $Y_f^{\mathbb{O}}$ without inserting observed masses, Yukawa entries or fitted flavor textures. Once these data are fixed, all charged-fermion masses and mixing matrices must be mathematical consequences of the octonionic construction.*

This benchmark is stricter than reproducing qualitative hierarchies. It is also stricter than fitting masses with a new set of geometric-looking parameters. The central purpose of the manuscript is to formulate an explicit route to this benchmark and to state where the remaining closure proofs are required.

## 3 Octonionic algebra and associator geometry

### 3.1 Basic conventions

Let $\mathbb{O}$ denote the real algebra of octonions. A general element is written as

$$x = x_0 + \sum_{a=1}^{7} x_a e_a, \tag{6}$$

where $x_0, x_a \in \mathbb{R}$, $e_a^2 = -1$, and multiplication is determined by the oriented Fano plane. The conjugate and norm are

$$\bar{x} = x_0 - \sum_{a=1}^{7} x_a e_a, \qquad \|x\|^2 = x\bar{x} = \bar{x}x. \tag{7}$$

The real inner product is

$$\langle x, y \rangle = \operatorname{Re}(x\bar{y}). \tag{8}$$

The imaginary part $\operatorname{Im}\mathbb{O}$ is seven-dimensional and carries a natural $G_2$ structure familiar from exceptional holonomy and calibration theory [3–5]. The structure constants $f_{abc}$ are defined by

$$e_a e_b = -\delta_{ab} + f_{abc} e_c, \tag{9}$$

where $f_{abc}$ is totally antisymmetric.

**Definition 3.1** (Associator)**.** *For $x, y, z \in \mathbb{O}$, the associator is*

$$[x, y, z] = (xy)z - x(yz). \tag{10}$$

The octonions are alternative, meaning that any two elements generate an associative subalgebra. Equivalently,

$$[x,x,y]=[x,y,x]=[y,x,x]=0. \tag{11}$$

Thus a non-zero associator is intrinsically trilinear and requires three independent directions.

### 3.2 Quaternionic subalgebras and physical projection

Every oriented line of the Fano plane generates an associative quaternionic subalgebra $\mathbb{H}\subset\mathbb{O}$ [1, 2, 9]. We denote by

$$\mathbb{H}_{\text{phys}}=\operatorname{span}\{1,e_1,e_2,e_3\} \tag{12}$$

a preferred physical quaternionic subalgebra. This choice is not invariant under all of $G_2$; it represents the selection of a physical low-energy projection sector. The orthogonal decomposition is

$$\mathbb{O}=\mathbb{H}_{\text{phys}}\oplus\mathbb{H}_{\text{phys}}^{\perp}. \tag{13}$$

Let $P_H$ and $P_\perp$ denote the associated orthogonal projectors.

The stabilizer of an imaginary unit in $G_2$ is isomorphic to $SU(3)$, and $G_2$ naturally contains substructures related to complex and quaternionic geometry [1, 3, 9]. In the present framework, the choice of $\mathbb{H}_{\text{phys}}$ is interpreted as the emergence of an associative low-energy sector. Non-associative directions in $\mathbb{H}_{\text{phys}}^{\perp}$ are not discarded; instead they encode internal spectral data.

### 3.3 Canonical non-associative block

Consider the oriented triple $(e_1,e_2,e_4)$. It does not lie in a single quaternionic subalgebra. With a standard Fano convention one obtains

$$[e_1,e_2,e_4]=2e_7. \tag{14}$$

This elementary identity is the minimal non-associative block used below. Its norm is

$$\|[e_1,e_2,e_4]\|^2=4. \tag{15}$$

For a vacuum scale $v_{\mathbb{O}}$ this becomes

$$\|[v_{\mathbb{O}}e_1,v_{\mathbb{O}}e_2,v_{\mathbb{O}}e_4]\|^2=4v_{\mathbb{O}}^6. \tag{16}$$

## 4 Master action and field content

### 4.1 Octonionic field triplet

The fundamental field content for the mass-sector construction is a triplet

$$\Psi_I(x)\in\mathbb{O},\qquad I=1,2,3. \tag{17}$$

The triplet is minimal in the following sense: with fewer than three independent octonionic directions, alternativity prevents a genuinely non-associative invariant. The associator field is

$$A(x)=[\Psi_1(x),\Psi_2(x),\Psi_3(x)]. \tag{18}$$

We assign mass dimension

$$[\Psi_I]=1,\qquad [A]=3,\qquad [\|A\|^2]=6. \tag{19}$$

Thus a term $\lambda_A\|A\|^2$ in a four-dimensional effective Lagrangian requires $[\lambda_A]=-2$.

### 4.2 Effective master action

The effective action is written as

$$S=S_{\text{grav}}+S_\Psi+S_A+S_{\text{br}}+S_{\text{proj}}. \tag{20}$$

The gravitational part is standard:

$$S_{\text{grav}} = \int d^4x\sqrt{-g}\,\frac{R}{2\kappa}. \tag{21}$$

The octonionic sector is

$$S_\Psi = \int d^4x\sqrt{-g}\left[\sum_{I=1}^{3}\langle D_\mu\Psi_I, D^\mu\Psi_I\rangle - V_0(\Psi_1,\Psi_2,\Psi_3)\right], \tag{22}$$

with a local potential

$$V_0 = -\mu^2 R_\Psi + \lambda R_\Psi^2 + \lambda_2\sum_{I<J}\langle\Psi_I,\Psi_J\rangle^2 + \lambda_A\,\|A\|^2\,, \tag{23}$$

where

$$R_\Psi = \sum_{I=1}^{3}\|\Psi_I\|^2\,. \tag{24}$$

The $\lambda_2$ term energetically favors orthogonal vacuum directions for $\lambda_2 > 0$. The associator term controls the cost or preference of non-associative vacuum alignment, depending on the full sign convention. For stability at large field norm, the total sextic and quartic contributions must be bounded from below.

**Remark 4.1.** *In the mass-hierarchy application, $V_0$ is an effective potential. It is not assumed to be the final ultraviolet action. The required condition is that it define a well-posed vacuum problem whose invariants can be calculated.*

# 5 Vacuum closure

## 5.1 Canonical reduction

Take the orthogonal non-associative ansatz

$$\Psi_1 = v_{\mathbb{O}}e_1, \qquad \Psi_2 = v_{\mathbb{O}}e_2, \qquad \Psi_3 = v_{\mathbb{O}}e_4. \tag{25}$$

Then

$$R_\Psi = 3v_{\mathbb{O}}^2, \qquad \langle\Psi_I,\Psi_J\rangle = 0 \quad (I \neq J), \tag{26}$$

and, by (14),

$$A_0 = 2v_{\mathbb{O}}^3 e_7, \qquad \|A_0\|^2 = 4v_{\mathbb{O}}^6. \tag{27}$$

The reduced potential is therefore

$$V_{\text{red}}(v_{\mathbb{O}}) = -3\mu^2 v_{\mathbb{O}}^2 + 9\lambda v_{\mathbb{O}}^4 + 4\lambda_A v_{\mathbb{O}}^6. \tag{28}$$

The stationary equation $dV_{\text{red}}/dv_{\mathbb{O}} = 0$ gives

$$-6\mu^2 v_{\mathbb{O}} + 36\lambda v_{\mathbb{O}}^3 + 24\lambda_A v_{\mathbb{O}}^5 = 0. \tag{29}$$

For $v_{\mathbb{O}} \neq 0$ and $x = v_{\mathbb{O}}^2$,

$$4\lambda_A x^2 + 6\lambda x - \mu^2 = 0. \tag{30}$$

If $\lambda_A > 0$, the positive solution is

$$v_{\mathbb{O}}^2 = \frac{-6\lambda + \sqrt{36\lambda^2 + 16\lambda_A\mu^2}}{8\lambda_A}. \tag{31}$$

**Theorem 5.1** (Vacuum closure in the canonical radial sector)**.** *Assume $\mu^2 > 0$, $\lambda_A > 0$ and that the quartic and sextic contributions make the effective potential bounded from below. In the canonical* *non-associative sector* (25)*, the non-zero stationary vacuum scale is uniquely determined by* (31)*. The associated normalized associator curvature is*

$$K_A = \frac{\|A_0\|^2}{v_{\mathbb{O}}^6} = 4. \tag{32}$$

*Proof.* Substitution of the canonical ansatz into the potential gives (28). Its non-zero stationary points satisfy the quadratic equation (30) in $x = v_{\mathbb{O}}^2$. For $\lambda_A > 0$ the root with the plus sign in (31) is non-negative for $\mu^2 > 0$, whereas the other root is non-positive or physically irrelevant. The associator identity (14) gives $\|A_0\|^2 = 4v_{\mathbb{O}}^6$, hence $K_A = 4$. □

This theorem closes only the radial vacuum sector. It does not yet distinguish charged lepton, up-quark and down-quark sectors. For realistic fermion masses an additional algebraic breaking mechanism is required.

# II. Projection, associator spectrum and mass operators

## 6 Algebraic symmetry breaking

### 6.1 Quaternionic selection

The first breaking ingredient is the selection of $\mathbb{H}_{\text{phys}} \subset \mathbb{O}$. This is represented by

$$V_H = \epsilon_H \sum_{I=1}^{3} \left( \|P_H \Psi_I\|^2 - \eta_H \|P_\perp \Psi_I\|^2 \right)^2. \tag{33}$$

This term is algebraically motivated because it distinguishes associative and non-associative projection channels. It breaks the full $G_2$ covariance down to the subgroup preserving the chosen physical projection data.

### 6.2 Fano-oriented breaking and up/down splitting

To split two non-associative directions inside $\mathbb{H}_{\text{phys}}^{\perp}$, use a Fano-oriented vacuum vector

$$\Omega_{\text{Fano}} = \cos\varphi\, e_7 + \sin\varphi\, e_4. \tag{34}$$

Add

$$V_{\text{Fano}} = \zeta \left|\langle A, \Omega_{\text{Fano}}\rangle\right|^2. \tag{35}$$

Now take the angular vacuum ansatz

$$\Psi_1 = v_{\mathbb{O}} e_1, \qquad \Psi_2 = v_{\mathbb{O}} e_2, \qquad \Psi_3 = v_{\mathbb{O}}(\cos\theta\, e_4 + \sin\theta\, e_7). \tag{36}$$

Using

$$[e_1, e_2, e_4] = 2e_7, \qquad [e_1, e_2, e_7] = -2e_4, \tag{37}$$

we get

$$A_0 = 2v_{\mathbb{O}}^3(\cos\theta\, e_7 - \sin\theta\, e_4). \tag{38}$$

Consequently,

$$V_{\text{Fano}} = 4\zeta v_{\mathbb{O}}^6 \cos^2(\theta + \varphi). \tag{39}$$

The stationary angular equation is

$$\frac{\partial V_{\text{Fano}}}{\partial\theta} = 0 \quad \Longleftrightarrow \quad \sin(2(\theta+\varphi)) = 0. \tag{40}$$

Therefore

$$\theta = \frac{k\pi}{2} - \varphi, \qquad k \in \mathbb{Z}. \tag{41}$$

The special choice $\varphi = \pi/6$ and $k = 1$ yields $\theta = \pi/3$. This gives

$$K_u = 4\cos^2\theta = 1, \qquad K_d = 4\sin^2\theta = 3. \tag{42}$$

**Remark 6.1.** *The value $\varphi = \pi/6$ is geometrically suggestive because it leads to a $1:3$ splitting related to singlet/triplet counting. However, unless $\varphi$ is derived from a prior Fano-plane orientation principle, it remains a model axiom. This distinction is crucial for evaluating the logical status of exact mass predictions.*

### 6.3 Chiral projectors

Let $\Gamma_{\mathbb{O}}$ be an involutive grading operator on the projected spinorial sector, with $\Gamma_{\mathbb{O}}^2 = 1$. Define

$$P_L = \frac{1}{2}(1 + \Gamma_{\mathbb{O}}), \qquad P_R = \frac{1}{2}(1 - \Gamma_{\mathbb{O}}). \tag{43}$$

For each sector $s \in \{\ell, u, d\}$ define the chiral vacuum invariant

$$\chi_s = \frac{\|P_L P_s A_0\|^2 - \|P_R P_s A_0\|^2}{\|P_s A_0\|^2}. \tag{44}$$

A minimal geometric closure assigns

$$\chi_\ell = 1, \qquad \chi_u = \frac{1}{3}, \qquad \chi_d = -\frac{1}{3}. \tag{45}$$

The lepton sector is maximally chiral in this minimal projection, while the quark sectors are color averaged and Fano oriented in opposite directions.

## 7 Explicit sector projectors

A central requirement for any mass calculation is that the symbols $P_\ell$, $P_u$ and $P_d$ are not merely labels. They must be represented as algebraic maps on the octonionic state space. We give here a minimal projector construction sufficient for the spectral framework.

Let

$$\Pi_0(x) = \frac{1}{2}(x + e_3 x e_3), \qquad \Pi_1(x) = \frac{1}{2}(x - e_3 x e_3), \tag{46}$$

where $e_3 \in \mathbb{H}_{\text{phys}}$ is the distinguished complex direction inside the physical quaternionic subalgebra. These two maps separate components commuting and anticommuting with the chosen complex structure. In an effective low-energy interpretation, this is the first step in the transition

$$\mathbb{O} \longrightarrow \mathbb{H}_{\text{phys}} \longrightarrow \mathbb{C}_{\text{phys}}. \tag{47}$$

Motivated by earlier octonionic and division-algebraic realizations of Standard-Model quantum numbers [6–8], the lepton projector is then modeled as the singlet component under the color-like stabilizer of the complex line:

$$P_\ell = P_{\text{sing}}\Pi_0, \tag{48}$$

whereas the quark projector is modeled as the triplet complement

$$P_q = P_{\text{trip}}\Pi_1. \tag{49}$$

The quark triplet is subsequently decomposed into two Fano-oriented branches,

$$P_u = P_q P_+, \qquad P_d = P_q P_-, \tag{50}$$

with

$$P_\pm A_0 = \frac{1}{2}(A_0 \pm \mathcal{J}_{\text{Fano}} A_0), \tag{51}$$

where $\mathcal{J}_{\text{Fano}}$ is the involution induced by the angular splitting $(e_4, e_7)$. At the level of the canonical angular vacuum, this construction reproduces

$$\|P_u A_0\|^2 = 4v_{\mathbb{O}}^6 \cos^2\theta, \qquad \|P_d A_0\|^2 = 4v_{\mathbb{O}}^6 \sin^2\theta. \tag{52}$$

Therefore the sector curvature invariants are not independent parameters but projections of the same vacuum associator:

$$K_s = \frac{\|P_s A_0\|^2}{v_{\mathbb{O}}^6}. \tag{53}$$

**Lemma 7.1** (Orthogonal sector decomposition)**.** *Assume that the singlet/triplet splitting is orthogonal with respect to $\langle\cdot,\cdot\rangle$ and that $\mathcal{J}_{\text{Fano}}$ is an orthogonal involution. Then*

$$P_\ell P_q = 0, \qquad P_u P_d = 0, \qquad P_\ell + P_u + P_d \le 1 \tag{54}$$

*on the finite projected internal space.*

*Proof.* The maps $P_{\text{sing}}$ and $P_{\text{trip}}$ project onto inequivalent orthogonal components of the stabilizer representation, so their product vanishes. Since $\mathcal{J}_{\text{Fano}}^2 = 1$, the maps $(1 \pm \mathcal{J}_{\text{Fano}})/2$ are complementary projectors. Orthogonality of the involution implies orthogonality of their images. The final relation allows for possible neutrino or sterile sectors not included in the charged-fermion decomposition. □

This explicit projector description clarifies what must be done in a future numerical implementation: one must choose a concrete finite representation of the projected octonionic module, construct the matrices corresponding to $P_\ell$, $P_u$, $P_d$ and $\mathcal{N}_A$, and diagonalize the resulting operator.

# 8 Projection entropies and the octonionic coupling

## 8.1 Non-associative projection coupling

The universal octonionic projection coupling is defined by the inverse dimension of the effective non-associative channel:

$$\alpha_{\mathbb{O}} = \frac{1}{D_{\text{NA}}}. \tag{55}$$

From (13), $\mathbb{H}_{\text{phys}}^{\perp}$ has dimension four. Removing one radial vacuum modulus leaves three angular non-associative directions, so

$$D_{\text{NA}} = 3, \qquad \alpha_{\mathbb{O}} = \frac{1}{3}. \tag{56}$$

## 8.2 Geometric projection probabilities

A compact way to encode sector suppressions is to define

$$S_s^{\text{eff}} = -\ln p_s, \tag{57}$$

where $p_s$ is the projection probability into sector $s$. The minimal closure used here is

$$p_\ell = \frac{1}{D_{\text{NA}} D_{\text{ch}}}, \qquad p_u = \frac{1}{D_{\text{NA}} D_{\text{color}}}, \qquad p_d = \frac{1}{D_{\text{NA}} D_{\text{color}} D_{\text{Fano}}}. \tag{58}$$

With

$$D_{\text{NA}} = 3, \quad D_{\text{ch}} = 2, \quad D_{\text{color}} = 3, \quad D_{\text{Fano}} = 2, \tag{59}$$

one obtains

$$p_\ell = \frac{1}{6}, \qquad p_u = \frac{1}{9}, \qquad p_d = \frac{1}{18}, \tag{60}$$

and therefore

$$S_\ell^{\text{eff}} = \ln 6, \qquad S_u^{\text{eff}} = \ln 9, \qquad S_d^{\text{eff}} = \ln 18. \tag{61}$$

The exponential suppression coefficient is

$$\Gamma_s = \frac{S_s^{\text{eff}}}{\alpha_{\mathbb{O}}} = 3S_s^{\text{eff}}. \tag{62}$$

**Remark 8.1.** *Equations (61) and (62) are not empirical fits to the observed fermion masses. They are a geometric closure rule. Their physical validity depends on whether the projection probabilities can be derived as the unique measures induced by the octonionic vacuum and its sector projectors.*

# 9 Associator spectral theory

## 9.1 Associator number operator

Let $\mathcal{H}_{\text{phys}}$ be the projected Hilbert space carrying the low-energy fermionic degrees of freedom. The linearized associator variation around the vacuum defines an operator

$$\mathcal{A} : \mathcal{H}_{\text{phys}} \to \mathcal{H}_{\text{NA}}, \tag{63}$$

where $\mathcal{H}_{\text{NA}}$ denotes the non-associative fluctuation sector. The positive associator number operator is

$$\mathcal{N}_A = \frac{1}{v_{\mathbb{O}}^6} \mathcal{A}^\dagger \mathcal{A}. \tag{64}$$

By construction, $\mathcal{N}_A$ is positive if $\mathcal{A}^\dagger$ is the adjoint with respect to the projected inner products.

**Assumption 9.1** (Spectral domain)**.** *The domain of $\mathcal{N}_A$ is chosen such that $\mathcal{A}$ is densely defined and $\mathcal{A}^\dagger\mathcal{A}$ is self-adjoint on $\mathcal{H}_{\text{phys}}$.*

**Theorem 9.2** (Associator spectral closure)**.** *If $\mathcal{N}_A$ is self-adjoint and has a discrete three-dimensional low-energy invariant subspace $\mathcal{G} \subset \mathcal{H}_{\text{phys}}$, then there exists an orthonormal generation basis $\{\chi_1, \chi_2, \chi_3\}$ satisfying*

$$\mathcal{N}_A \chi_n = N_n \chi_n, \qquad n = 1, 2, 3. \tag{65}$$

*If the spectrum is non-degenerate and ordered, $N_1 < N_2 < N_3$, the three eigenvectors define a canonical generation hierarchy.*

*Proof.* The statement follows from the spectral theorem for self-adjoint operators restricted to the finite-dimensional invariant subspace $\mathcal{G}$. Positivity of $\mathcal{A}^\dagger\mathcal{A}$ implies $N_n \geq 0$. Non-degeneracy gives an ordered basis unique up to phases. □

The minimal algebraic closure uses the normalized eigenvalues

$$N_1 = 0, \qquad N_2 = 1, \qquad N_3 = 2. \tag{66}$$

This assignment is the simplest three-level spectrum. A stronger future result would derive (66) directly from an explicit octonionic matrix representation of $\mathcal{N}_A$.

# 10 Geometric fermion mass operator

## 10.1 Sector-dependent operator

For each sector $s \in \{\ell, u, d\}$ define

$$\mathcal{M}_s = \frac{v_H}{\sqrt{2}} \exp\left[-\Gamma_s + \Delta_s \mathcal{N}_A - \frac{1}{2}\Omega_s \mathcal{N}_A^2 + \rho\chi_s\right]. \tag{67}$$

Here $v_H$ is the electroweak scale. The coefficients are geometric invariants:

$$\Gamma_s = 3S_s^{\text{eff}}, \tag{68}$$

$$\Delta_s = \ln\left(1 + \frac{C_s}{I_s}\right), \tag{69}$$

$$\Omega_s = \frac{K_s}{D_s}, \tag{70}$$

$$\rho = \ln 3. \tag{71}$$

The quantities $I_s$ and $D_s$ are the idempotent rank and projected dimension of the sector. For the quark sectors we use

$$I_u = I_d = 3, \qquad D_u = D_d = 3, \qquad C_u = C_d = 2. \tag{72}$$

Thus

$$\Delta_u = \Delta_d = \ln\left(\frac{5}{3}\right). \tag{73}$$

Using (42),

$$\Omega_u = \frac{1}{3}, \qquad \Omega_d = 1. \tag{74}$$

For the charged lepton sector, the minimal closure leaves $C_\ell, I_\ell, K_\ell, D_\ell$ to be determined by the precise lepton projector. A simple benchmark is $I_\ell = 1$, $D_\ell = 1$ with sector-specific $C_\ell$ and $K_\ell$ determined by the chiral vacuum projection.

### 10.2 Mass eigenvalues

Given a generation eigenstate $\chi_n$, define

$$m_{s,n} = \langle \chi_n, \mathcal{M}_s \chi_n \rangle. \tag{75}$$

Since $\chi_n$ diagonalizes $\mathcal{N}_A$, this gives

$$m_{s,n} = \frac{v_H}{\sqrt{2}} \exp\left[-\Gamma_s + \Delta_s N_n - \frac{1}{2}\Omega_s N_n^2 + \rho\chi_s\right]. \tag{76}$$

This is the central mass formula of the framework.

For up-type quarks:

$$m_{u,n} = \frac{v_H}{\sqrt{2}} \exp\left[-3\ln 9 + \ln\left(\frac{5}{3}\right) N_n - \frac{1}{6}N_n^2 + \frac{1}{3}\ln 3\right]. \tag{77}$$

For down-type quarks:

$$m_{d,n} = \frac{v_H}{\sqrt{2}} \exp\left[-3\ln 18 + \ln\left(\frac{5}{3}\right) N_n - \frac{1}{2}N_n^2 - \frac{1}{3}\ln 3\right]. \tag{78}$$

For charged leptons:

$$m_{\ell,n} = \frac{v_H}{\sqrt{2}} \exp\left[-3\ln 6 + \Delta_\ell N_n - \frac{1}{2}\Omega_\ell N_n^2 + \ln 3\right]. \tag{79}$$

The lepton parameters $\Delta_\ell$ and $\Omega_\ell$ must be fixed by the explicit lepton projector. This is where a future exact mass calculation must become more detailed.

### 10.3 Relation to the linear projected operator form

The exponential mass operator (67) refines the primitive projected-operator statement

$$m_{f,n} = v_{\mathbb{O}} \left|\langle \chi_{f,n}, (\alpha P_{\text{assoc}} + \beta P_{G_2} + \gamma P_{\text{idemp}} + \delta P_{\text{gen}})\, \chi_{f,n}\rangle\right|. \tag{80}$$

Here $P_{\text{assoc}}$ denotes the non-associative projection channel, $P_{G_2}$ the $G_2$-breaking channel, $P_{\text{idemp}}$ the lepton-quark idempotent channel, and $P_{\text{gen}}$ the generation projection. In a minimal factorized limit this reduces to

$$m_{f,n} = v_{\mathbb{O}} |y_f \lambda_n|. \tag{81}$$

Equation (81) displays the basic physical interpretation: masses are eigenvalues of projected associator dynamics. However, it is too coarse for realistic phenomenology, since a universal generation spectrum with only three sector constants would force overly similar generation ratios across charged leptons, up quarks and down quarks. The exponential operator (67) is therefore the invariant completion of this linear form. If its coefficients are calculated from the vacuum, the construction is a derivation; if they are adjusted to experiment, it is only a reparametrization.

**Proposition 10.1** (Hierarchy generation)**.** *Assume $\Omega_s > 0$ and $\Delta_s$ is finite. Then (76) generates an exponential hierarchy across generation eigenvalues $N_n$.*

*Proof.* For two generations $n > m$,

$$\ln \frac{m_{s,n}}{m_{s,m}} = \Delta_s (N_n - N_m) - \frac{1}{2} \Omega_s (N_n^2 - N_m^2). \tag{82}$$

The ratio is therefore exponential in the spectral separation of $\mathcal{N}_A$. Positive $\Omega_s$ supplies nonlinear saturation at large $N$. □

## 11 Pre-closure mass parametrization

The previous sections define a geometric spectral mass operator. To convert it into an exact prediction, three closures are needed:

(a) **Vacuum closure:** the vacuum solution and all angles, projection probabilities and chiral invariants are uniquely fixed by the octonionic action.

(b) **Spectral closure:** the eigenvalues $N_n$ of $\mathcal{N}_A$ are uniquely determined in the physical generation sector.

(c) **RG closure:** the relation between the high-scale spectral masses and the experimentally quoted running masses is fully specified.

**Theorem 11.1** (Conditional fermion mass map)**.** *Suppose the following conditions hold:*

*(i) the octonionic vacuum equations derived from the full action have a unique physical vacuum class;*

*(ii) this vacuum uniquely determines $S_s^{\text{eff}}$, $C_s$, $I_s$, $K_s$, $D_s$ and $\chi_s$ for all fermion sectors;*

*(iii) the self-adjoint associator number operator has a uniquely determined three-generation spectrum $\{N_1, N_2, N_3\}$;*

*(iv) the renormalization-group map $\mathcal{R}_{\mu_0 \to \mu}$ from the octonionic matching scale $\mu_0$ to the experimental scale $\mu$ is specified.*

*Then the fermion masses at scale $\mu$ are determined without independent Yukawa parameters by*

$$m_{s,n}(\mu) = \mathcal{R}_{\mu_0 \to \mu} \left[ \frac{v_H(\mu_0)}{\sqrt{2}} \exp\left( -\Gamma_s + \Delta_s N_n - \frac{1}{2} \Omega_s N_n^2 + \rho \chi_s \right) \right]. \tag{83}$$

*Proof.* By assumption (i) and (ii), all sector coefficients in the exponent are uniquely determined by the octonionic vacuum. By assumption (iii), the generation eigenvalues are uniquely fixed. Thus the matching-scale masses are fixed by the displayed spectral formula. Assumption (iv) then maps these masses to the experimental scale. No additional Yukawa parameters enter. □

**Corollary 11.2.** *If any of the four conditions in the theorem is not established, the construction remains a geometric mass framework rather than a parameter-free exact prediction.*

This corollary is important for the scientific interpretation of the model. The present article establishes the framework and several conditional statements. It does not assert that the four required closure conditions have already been proven in full generality.

**Theorem 11.3** (Vacuum-closure criterion for parameter-free masses)**.** *The octonionic model yields exact parameter-free fermion masses if and only if the following data are determined by the octonionic vacuum equations and projection consistency conditions without phenomenological mass input:*

*(i) the vacuum triplet $(\Phi_1, \Phi_2, \Phi_3)$ up to physical $G_2$ equivalence;*

*(ii) the physical quaternionic subalgebra $\mathbb{H}_{\mathrm{phys}}$ and complex line $\mathbb{C}_{\mathrm{phys}}$;*

*(iii) the lepton, quark, color, weak and chiral projectors;*

*(iv) the full projected mass operator, either in the linear form (80) or in the invariant exponential form (67);*

*(v) the absolute matching scale or its relation to $v_H$;*

*(vi) the RG map from the octonionic matching scale to the experimental scale.*

*Under these conditions the masses are fixed eigenvalues or singular values of a finite operator. If one listed datum remains free, the result is not a parameter-free prediction.*

*Proof.* If all listed data are fixed, each sector operator is a fixed self-adjoint or polar-decomposable finite operator on the projected internal space. Its spectral values are then mathematical consequences. Conversely, any unfixed vacuum scale, projector angle, sector invariant or RG matching prescription leaves at least one ambiguity in the mass matrix. That ambiguity is physically equivalent to a remaining Yukawa input. □

## 12 Matrix realization of the closure problem

A practical route toward exact calculation is to construct a finite matrix representation of the internal projected module. Let $\{u_a\}_{a=1}^d$ be an orthonormal basis of a finite projected internal space. Define the matrices

$$(\mathcal{A})_{ab} = \langle u_a, \delta A[u_b] \rangle \,, \tag{84}$$

where $\delta A$ denotes the linearized associator variation around the vacuum. Then

$$(\mathcal{N}_A)_{ab} = \frac{1}{v_{\mathbb{O}}^6} (\mathcal{A}^\dagger \mathcal{A})_{ab}. \tag{85}$$

The exact spectral closure problem is the finite-dimensional eigenvalue problem

$$\det(\mathcal{N}_A - N\mathbf{1}) = 0. \tag{86}$$

If the characteristic polynomial factorizes as

$$\det(\mathcal{N}_A - N\mathbf{1}) = N(N-1)(N-2)Q(N), \tag{87}$$

and the remaining roots in $Q(N)$ are heavy or projected out, then the minimal spectrum (66) is not an assumption but a derived result.

This gives a concrete computational target. The proposed theory becomes predictive only when the matrices are constructed from the octonion multiplication table, the vacuum solution and the projectors without inserting observed fermion masses.

# III. Charged-lepton Hessian closure and numerical masses

## 13 Associator spectrum and generation operator

The previous sections identify the structural origin of fermion masses. To turn this structure into exact numerical values one must close four data channels: the generation eigenvalues, the effective Yukawa factors, the absolute vacuum scale and the renormalization-group map to experimental mass conventions. This section formulates the required closure steps as a concrete finite-dimensional calculation. It is included to make clear that the remaining problem is not the form of the mass formula, but the derivation of all its input data from the octonionic vacuum.

**Definition 13.1** (Four-block numerical closure)**.** *A numerical derivation of fermion masses from the associator ansatz is called closed if the following four objects are determined internally:*

$$v_{\mathbb{O}} \text{ from the vacuum equations and the projected norm,} \tag{88}$$

$$(N_1, N_2, N_3) \text{ from the physical spectrum of the associator Hessian,} \tag{89}$$

$$(P_H, P_{\mathbb{C}}, P_L, P_R, P_{\rm gen}, P_s) \text{ from a projector variational problem and vacuum stabilizers,} \tag{90}$$

$$\mathcal{R}_{\rm RG}(\mu_{\rm exp}, \mu_0) \text{ from a specified matching scale and running convention.} \tag{91}$$

*No observed fermion mass, Yukawa entry or fitted flavor texture may enter these four steps. Experimental masses enter only after the final RG evolution as comparison data.*

In this notation the exact target formula is

$$\boxed{m_{s,n}^{\rm phys}(\mu_{\rm exp}) = \mathcal{R}_{s,n}(\mu_{\rm exp}, \mu_0)\frac{F_{\rm EW} v_{\mathbb{O}}}{\sqrt{2}} y_{s,n}^{\mathbb{O}}(\mu_0).} \tag{92}$$

Here $F_{\rm EW} = \left\| P_H \Psi^{(0)} \right\| / v_{\mathbb{O}}$ is the electroweak projection factor. The exact Yukawa replacement may be written at the operator level as

$$y_{s,n}^{\mathbb{O}} = \text{eig}_n \left( P_{s,L} \mathcal{D}_A P_{s,R} \right), \tag{93}$$

or, in the reduced scalar spectral model used below, as

$$y_{s,n}^{\mathbb{O}} = \exp\left[ -\Gamma_s + \Delta_s N_n - \frac{1}{2}\Omega_s N_n^2 + \rho \chi_s \right]. \tag{94}$$

The exact calculation therefore has two logically separate parts. First, one calculates the internal octonionic data

$$(v_{\mathbb{O}}, F_{\rm EW}, N_n, \Gamma_s, \Delta_s, \Omega_s, \chi_s, \mu_0) \tag{95}$$

from the vacuum and its projectors. Second, one applies the chosen Standard-Model or beyond-Standard-Model RG map. The first part is new to the associator framework; the second part is a conventional matching calculation once the high-scale Yukawa data are known.

**Proposition 13.2** (No-fit mass criterion)**.** *The formula* (92) *is a parameter-free fermion-mass prediction if and only if all entries in*

$$\{v_{\mathbb{O}}, F_{\rm EW}, \mu_0, P_H, P_{\mathbb{C}}, P_L, P_R, P_{\rm gen}, P_s, N_n, \Gamma_s, \Delta_s, \Omega_s, \chi_s\} \tag{96}$$

*are obtained from the octonion multiplication table, the master action and the selected vacuum solution without using the measured values of*

$$m_e, m_\mu, m_\tau, m_u, m_d, m_s, m_c, m_b, m_t. \tag{97}$$

*Proof.* If all listed quantities are determined by the vacuum problem, then the right-hand side of (92) is a fixed number for every fermion and every specified mass convention. Conversely, an unfixed entry can be varied continuously or discretely without changing the octonion multiplication table. Such a variation changes at least one mass prediction and is therefore equivalent to a remaining Yukawa or matching input. □

### 13.1 Associator spectrum and the values of $N_1, N_2, N_3$

The generation numbers must be obtained as eigenvalues of a fully specified associator number operator. Let

$$\mathbb{O} = \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_7\}, \qquad e_i e_j = -\delta_{ij} + f_{ijk} e_k, \tag{98}$$

with a fixed Fano-plane orientation. Let the octonionic vacuum be

$$\Psi_I = \Psi_I^{(0)} + \delta\Psi_I, \qquad I = 1, 2, 3, \tag{99}$$

and define

$$A(\Psi_1, \Psi_2, \Psi_3) = [\Psi_1, \Psi_2, \Psi_3]. \tag{100}$$

The associator energy is

$$V_A = \lambda_A \left\| A(\Psi_1, \Psi_2, \Psi_3) \right\|^2. \tag{101}$$

Linearizing around the vacuum gives a map

$$\delta A : \delta\Psi \mapsto [\delta\Psi_1, \Psi_2^{(0)}, \Psi_3^{(0)}] + [\Psi_1^{(0)}, \delta\Psi_2, \Psi_3^{(0)}] + [\Psi_1^{(0)}, \Psi_2^{(0)}, \delta\Psi_3]. \tag{102}$$

For an orthonormal basis $\{u_a\}_{a=1}^{d}$ of the finite projected internal module one obtains

$$\mathcal{A}_{ab} = \langle u_a, \delta A[u_b] \rangle, \qquad \mathcal{N}_A = \frac{1}{v_{\mathbb{O}}^6} \mathcal{A}^\dagger \mathcal{A}. \tag{103}$$

Equivalently, in coordinates $\phi^a$ on the internal fluctuation space,

$$(\mathcal{N}_A)_{ab} = \frac{1}{v_{\mathbb{O}}^6} \frac{\partial^2 \left\| A \right\|^2}{\partial\phi^a \partial\phi^b} \bigg|_{\Psi = \Psi^{(0)}}. \tag{104}$$

The exact generation values are then not adjustable constants but roots of the characteristic equation

$$\det\left(P_{\mathrm{gen}} \mathcal{N}_A P_{\mathrm{gen}} - N\mathbf{1}\right) = 0. \tag{105}$$

The desired closure theorem has the form

$$\boxed{\mathrm{Spec}\left(P_{\mathrm{gen}} \mathcal{N}_A P_{\mathrm{gen}}\right)_{\mathrm{light}} = \{N_1, N_2, N_3\}.} \tag{106}$$

Thus the currently open data $N_1, N_2, N_3$ are obtained by solving a finite spectral problem once the vacuum $\Psi_I^{(0)}$, the Fano orientation and the generation projector are fixed internally.

**Proposition 13.3** (Generation closure criterion)**.** *If the vacuum equations determine $\Psi_I^{(0)}$ uniquely up to the residual physical symmetry group, and if $P_{\mathrm{gen}}$ is fixed by the stabilizer of this vacuum, then the three light eigenvalues of $P_{\mathrm{gen}} \mathcal{N}_A P_{\mathrm{gen}}$ are parameter-free octonionic invariants.*

*Proof.* Under the stated assumptions all entries of $\mathcal{A}$, hence of $\mathcal{N}_A$, are fixed by the multiplication constants $f_{ijk}$ and by $\Psi_I^{(0)}$. Since $P_{\mathrm{gen}}$ is also fixed, the characteristic polynomial is fixed. Its light roots are therefore invariants of the closed vacuum problem and cannot be tuned independently. $\square$

### 13.2 Minimal explicit computation of the associator spectrum

The most immediate calculable part of the closure program is the spectrum of the linearized associator. This subsection carries out that calculation in a minimal Fano-plane vacuum. The purpose is not yet to claim the final physical fermion spectrum, but to demonstrate that the three generation levels can be obtained from an explicit finite-dimensional octonionic matrix rather than postulated.

We fix the standard Fano orientation

$$\mathcal{F} = \{(123), (145), (176), (246), (257), (347), (365)\}. \tag{107}$$

meaning that $e_i e_j = e_k$ for each oriented triple $(ijk) \in \mathcal{F}$, with antisymmetry and $e_i^2 = -1$. A minimal non-associative vacuum is chosen as

$$\Psi_1^{(0)} = v_{\mathbb{O}} e_1, \qquad \Psi_2^{(0)} = v_{\mathbb{O}} e_2, \qquad \Psi_3^{(0)} = v_{\mathbb{O}} e_4. \tag{108}$$

The three directions $e_1, e_2, e_4$ do not lie in a common quaternionic subalgebra. Hence their associator is non-vanishing:

$$A(e_1, e_2, e_4) = 2e_7, \qquad \|A(e_1, e_2, e_4)\|^2 = 4. \tag{109}$$

Let the imaginary fluctuations be

$$\delta\Psi_I = \sum_{a=1}^{7} \phi_I^a e_a, \qquad I = 1, 2, 3. \tag{110}$$

The linearized associator map is

$$L(\delta\Psi) = A(\delta\Psi_1, e_2, e_4) + A(e_1, \delta\Psi_2, e_4) + A(e_1, e_2, \delta\Psi_3). \tag{111}$$

In the ordered fluctuation basis

$$\mathcal{B} = \{(1, e_1), \ldots, (1, e_7), (2, e_1), \ldots, (2, e_7), (3, e_1), \ldots, (3, e_7)\}, \tag{112}$$

we define the explicit positive associator matrix

$$G_{(I,a)(J,b)} = \langle L(E_{I,a}), L(E_{J,b}) \rangle, \qquad E_{I,a} : \ \delta\Psi_I = e_a, \tag{113}$$

where the inner product is the Euclidean norm on $\Im\mathbb{O}$. Equivalently, if $L$ is represented as an $8 \times 21$ real matrix, then

$$G = L^T L. \tag{114}$$

A direct multiplication table calculation gives

$$\mathrm{Spec}(G) = \{0^{\times 14}, 4^{\times 3}, 8^{\times 3}, 12^{\times 1}\}. \tag{115}$$

Thus the non-zero spectrum organizes itself into three levels,

$$4, \qquad 8, \qquad 12, \tag{116}$$

with degeneracies $3, 3, 1$. Dividing by the smallest non-zero level gives the normalized generation spectrum

$$\boxed{N_1 : N_2 : N_3 = 1 : 2 : 3.} \tag{117}$$

This is the first explicit no-fit output of the associator spectral program.

**Proposition 13.4** (Minimal Fano-vacuum spectrum)**.** *For the Fano orientation $\mathcal{F}$ in Eq. (107) and the minimal vacuum $(e_1, e_2, e_4)$, the positive linearized associator number operator $G = L^T L$ on imaginary triplet fluctuations has non-zero eigenvalue levels*

$$4, \qquad 8, \qquad 12. \tag{118}$$

*Therefore the normalized generation levels are*

$$N_1, N_2, N_3 = (1, 2, 3). \tag{119}$$

*Proof.* The result follows by evaluating (111) on the twenty-one basis fluctuations $E_{I,a}$. The octonion products are fixed entirely by $\mathcal{F}$. In the output basis $(e_1, \ldots, e_7)$, the corresponding $7 \times 21$ matrix $L$

has the images listed explicitly in Appendix C. These entries give the diagonal row Gram matrix

$$LL^T = \mathrm{diag}(4,4,8,4,8,8,12). \tag{120}$$

Since the non-zero eigenvalues of $L^TL$ and $LL^T$ coincide, while $L^TL$ acts on a 21-dimensional fluctuation space and rank $L = 7$, the characteristic polynomial is

$$\chi_G(x) = x^{14}(x-4)^3(x-8)^3(x-12). \tag{121}$$

The three distinct non-zero levels are therefore $4, 8, 12$, and normalization by the lowest level gives $(1,2,3)$. This proves the stated spectrum without numerical diagonalization. □

**Remark 13.5** (Relation to the full Hessian)**.** *The full Hessian of $\|A\|^2$ at a non-zero-associator point also contains terms proportional to the background associator $A(e_1, e_2, e_4)$. These terms are not by themselves physical until the local vacuum potential and the stationarity equations are included. The operator $G = L^TL$ is the positive spectral block that controls the leading associator fluctuation norm. In a complete vacuum solution, the same computation must be repeated for the dynamically selected $\Psi_I^{(0)}$; the spectrum (115) is therefore an explicit minimal benchmark and a candidate generation skeleton, not yet the final measured mass spectrum.*

### 13.3 Full-Hessian closure of the associator spectrum

The previous computation proves the positive Gram spectrum of the linearized associator. One must also state exactly how this positive block sits inside the Hessian of the complete vacuum potential. Let

$$A_0 = A(\Psi_1^{(0)}, \Psi_2^{(0)}, \Psi_3^{(0)}), \qquad \delta A = L\delta\Psi + Q(\delta\Psi, \delta\Psi) + O(\delta\Psi^3), \tag{122}$$

where $L$ is the linearized associator map and $Q$ is the quadratic part of the associator variation. For

$$V_{\mathrm{full}} = V_{\mathrm{loc}} + \lambda_A \|A\|^2 + V_{\mathrm{break}} + V_{H\mathbb{O}}, \tag{123}$$

one obtains on the real fluctuation space

$$\boxed{\mathcal{H}_{\mathrm{full}} = \mathcal{H}_{\mathrm{loc}} + \mathcal{H}_{\mathrm{break}} + \mathcal{H}_{H\mathbb{O}} + 2\lambda_A L^\dagger L + 2\lambda_A \mathcal{B}_{A_0}.} \tag{124}$$

Here the background-associator bilinear form is

$$\langle u, \mathcal{B}_{A_0} v\rangle = \langle A_0, Q(u,v) + Q(v,u)\rangle\,, \tag{125}$$

and $\mathcal{H}_{\mathrm{loc}}$, $\mathcal{H}_{\mathrm{break}}$ and $\mathcal{H}_{H\mathbb{O}}$ are the Hessians of the remaining local, symmetry-breaking and Higgs-octonion terms. The physically meaningful mass operator is not $L^\dagger L$ alone but the restriction

$$\mathcal{H}_{\mathrm{phys}} = P_{\mathrm{phys}} \mathcal{H}_{\mathrm{full}} P_{\mathrm{phys}}\big|_{\mathcal{G}^\perp}, \tag{126}$$

where $\mathcal{G}$ is the tangent space to gauge, automorphism and projector-orbit zero modes.

**Theorem 13.6** (Full-Hessian generation closure)**.** *Assume that the vacuum $\Psi^{(0)}$ is a stationary point of $V_{\mathrm{full}}$, that all zero modes generated by the residual symmetry group have been quotiented out, and that $P_{\mathrm{gen}}$ is fixed by the vacuum stabilizer. Then the generation values are the three light positive eigenvalue classes of*

$$P_{\mathrm{gen}} \mathcal{H}_{\mathrm{phys}} P_{\mathrm{gen}}, \tag{127}$$

*not merely of the positive block $L^\dagger L$. The minimal values $1:2:3$ remain a valid generation skeleton precisely when the additional operator*

$$\Delta\mathcal{H} = \mathcal{H}_{\mathrm{loc}} + \mathcal{H}_{\mathrm{break}} + \mathcal{H}_{H\mathbb{O}} + 2\lambda_A \mathcal{B}_{A_0} \tag{128}$$

*is diagonal on the three light spectral classes or shifts them by a common affine rescaling.*

*Proof.* Equation (122) gives

$$\delta^2 \|A\|^2 [u, v] = 2 \langle Lu, Lv \rangle + 2 \langle A_0, Q(u, v) + Q(v, u) \rangle . \tag{129}$$

Adding the second variations of the remaining potential terms gives (124). Stationarity removes all first-order terms. Quotienting by $\mathcal{G}$ removes symmetry directions with zero physical restoring force. Therefore the spectral problem for physical masses is (127). If $\Delta\mathcal{H}$ is block-diagonal on the three light eigenspaces of $L^\dagger L$, it cannot mix the generation classes; if its restriction is affine in the level, the ratios are preserved after normalization. Otherwise the minimal $1 : 2 : 3$ skeleton is replaced by the exact spectrum of $\mathcal{H}_{\text{phys}}$, which is still a finite no-fit invariant. □

**Definition 13.7** ( spectral data)**.** *The mass program uses the ordered data*

$$\left(\Psi^{(0)}, P_H, P_{\mathbb{C}}, P_L, P_R, P_s, \mathcal{H}_{\text{phys}}, \mu_0, \mathcal{R}_f\right) \tag{130}$$

*with $\mathcal{H}_{\text{phys}}$ defined by* (126)*. A claim of exact fermion masses is admissible only after this full data vector has been computed without observed fermion masses as input.*

Table 1 summarizes the output.

| Raw level | Degeneracy | Normalized level | Interpretation |
|---|---|---|---|
| 4 | 3 | $N_1 = 1$ | first generation level |
| 8 | 3 | $N_2 = 2$ | second generation level |
| 12 | 1 | $N_3 = 3$ | third generation level |

Table 1: Minimal explicit spectrum of the linearized associator operator in the Fano-plane vacuum $(e_1, e_2, e_4)$. The table shows the first concrete no-fit generation skeleton obtained from the octonionic associator calculation.

Inserted into the exponential mass formula, this benchmark gives the immediately computable hierarchy template

$$m_{s,n}^{\min} = \frac{v_H}{\sqrt{2}} \exp\left[-\Gamma_s + \Delta_s n - \frac{1}{2}\Omega_s n^2 + \rho\chi_s\right], \qquad n = 1, 2, 3. \tag{131}$$

Consequently, the remaining non-trivial task is shifted from the generation numbers to the sector invariants $\Gamma_s, \Delta_s, \Omega_s, \chi_s$ and to the dynamical selection of the physical vacuum.

## 14 Charged-lepton Hessian closure

### 14.1 Charged-lepton benchmark from the explicit spectrum

The minimal spectrum $N = (1, 2, 3)$ can be tested immediately in the cleanest charged sector, namely the charged leptons. This is the phenomenologically safest first test because the charged-lepton masses are not obscured by strong-interaction scheme dependence. The purpose of this subsection is carefully limited: it is a calibrated closure benchmark showing that the spectral template has enough curvature to reproduce the observed charged-lepton hierarchy. It is not yet a no-fit prediction, because the lepton-sector invariants must still be derived from the lepton projector.

For $\chi_\ell = 1$ and $\rho = \ln 3$, the charged-lepton template is

$$m_{\ell,n} = \frac{v_H}{\sqrt{2}} \exp\left[-\Gamma_\ell + \Delta_\ell n - \frac{1}{2}\Omega_\ell n^2 + \rho\right], \qquad n = 1, 2, 3. \tag{132}$$

The three logarithmic equations

$$\ln\left(\frac{m_{\ell,n}}{v_H/\sqrt{2}}\right) - \rho = -\Gamma_\ell + \Delta_\ell n - \frac{1}{2}\Omega_\ell n^2 \tag{133}$$

are linear in the three unknown invariants $\Gamma_\ell, \Delta_\ell, \Omega_\ell$. Using

$$v_H = 246.21965\,\text{GeV}, \qquad (m_e, m_\mu, m_\tau) = (0.00051099895, 0.1056583755, 1.77686)\,\text{GeV}, \tag{134}$$

only as external benchmark values, one obtains

$$\boxed{\Gamma_\ell = 21.67821266, \qquad \Delta_\ell = 9.09540887, \qquad \Omega_\ell = 2.50920674.} \tag{135}$$

Inserted back into (132), these values reproduce the charged-lepton masses shown in Table 2.

| State | $N_n$ | Model benchmark [GeV] | Reference value [GeV] | Relative deviation |
|---|---|---|---|---|
| $e$ | 1 | 0.00051099895 | 0.00051099895 | 0 |
| $\mu$ | 2 | 0.1056583755 | 0.1056583755 | 0 |
| $\tau$ | 3 | 1.77686 | 1.77686 | 0 |

Table 2: Calibrated charged-lepton benchmark using the explicit associator levels $N = (1, 2, 3)$. The table is not claimed as a no-fit prediction: the measured masses are used here only to reconstruct the lepton-sector invariants that a complete octonionic projector calculation must derive independently.

This benchmark is physically useful because it converts the charged-lepton problem into a sharply defined algebraic target. A no-fit octonionic derivation of the charged-lepton masses is equivalent to proving

$$\Gamma_\ell = 21.67821266, \qquad \Delta_\ell = 9.09540887, \qquad \Omega_\ell = 2.50920674 \tag{136}$$

from the lepton idempotent projector, the Fano orientation and the physical vacuum, without using (134). Thus the numerical content of the flavor problem has been reduced to a concrete invariant-computation problem. In particular, the large ratio

$$m_e : m_\mu : m_\tau \simeq 1 : 206.77 : 3477.15 \tag{137}$$

is not expected to equal $1 : 2 : 3$. Instead, $1 : 2 : 3$ is the generation coordinate, while the exponential curvature $\Omega_\ell$ and slope $\Delta_\ell$ generate the physical hierarchy.

**Proposition 14.1** (Charged-lepton no-fit criterion). *Given the explicit associator spectrum $N = (1, 2, 3)$, the charged-lepton masses are no-fit consequences of the octonionic model if and only if the lepton projector determines $\Gamma_\ell, \Delta_\ell, \Omega_\ell$ algebraically and these values coincide with (136) after electroweak normalization and lepton-sector RG matching.*

*Proof.* For fixed $v_H$, $\rho$, $\chi_\ell$ and $N = (1, 2, 3)$, Eq. (133) is an invertible three-by-three linear system for $(\Gamma_\ell, \Delta_\ell, \Omega_\ell)$. Hence any independently derived values of these invariants determine the three charged-lepton masses uniquely. Conversely, reproducing the three masses fixes the three invariant targets in (136). Therefore no-fit prediction is equivalent to deriving these targets from the projector geometry rather than fitting them to the observed masses. □

### 14.2 No-fit charged-lepton closure test

The benchmark in Subsection 14.1 identifies the numerical target. The next step is to remove the backward use of the charged-lepton masses. This can be done by replacing the three fitted quantities $\Gamma_\ell, \Delta_\ell, \Omega_\ell$ by three trace invariants of the lepton projector. The charged-lepton sector then becomes a finite no-fit test of the octonionic vacuum.

Let $P_\ell$ be the lepton idempotent, let $P_L, P_R$ be the chiral projectors, and let $P_n$ denote the spectral projector of the physical associator operator associated with the level $N_n = n$. Define the charged-lepton generation projector by

$$P_{\ell n} = P_\ell P_n P_{\mathbb{C}} P_H, \qquad n = 1, 2, 3, \tag{138}$$

with the product understood as the symmetrized product if the individual projectors do not commute exactly before restriction to the physical module. The no-fit lepton weight is then defined by the finite-dimensional matrix element

$$\mathcal{Y}_{\ell n}^{\mathbb{O}} = \frac{1}{d_{\ell n}} \operatorname{Tr}_{\mathcal{H}_{\text{phys}}} \left[ P_{\ell n} P_L \mathcal{M}_A P_R \mathcal{M}_A^\dagger P_L P_{\ell n} \right]^{1/2}, \qquad d_{\ell n} = \operatorname{Tr} P_{\ell n}, \tag{139}$$

where $\mathcal{M}_A$ is the linearized associator mass map restricted to the physical module. In components it is the projected version of the bilinear variation of the associator potential,

$$\mathcal{M}_A = P_{\text{phys}} \frac{\delta}{\delta \Psi} \left( \frac{1}{2} \|[\Psi_1, \Psi_2, \Psi_3]\|^2 + V_{\text{break}} \right)\bigg|_{\Psi = \Psi^{(0)}} P_{\text{phys}}. \tag{140}$$

The symmetry-breaking term $V_{\text{break}}$ is not a phenomenological Yukawa term; it is the algebraic term that lifts the degeneracy between the lepton, up-quark, down-quark and neutrino idempotent sectors. In a fully closed model it must be fixed by the same vacuum equations as $P_H$ and $P_\ell$.

The logarithmic no-fit data are

$$L_n^{\mathbb{O}} = \log \mathcal{Y}_{\ell n}^{\mathbb{O}}, \qquad n = 1, 2, 3. \tag{141}$$

The three effective invariants are then not fitted, but obtained by the discrete finite-difference map

$$\Omega_\ell^{\mathbb{O}} = 2L_2^{\mathbb{O}} - L_1^{\mathbb{O}} - L_3^{\mathbb{O}}, \tag{142}$$

$$\Delta_\ell^{\mathbb{O}} = L_2^{\mathbb{O}} - L_1^{\mathbb{O}} + \frac{3}{2}\Omega_\ell^{\mathbb{O}}, \tag{143}$$

$$\Gamma_\ell^{\mathbb{O}} = \Delta_\ell^{\mathbb{O}} - \frac{1}{2}\Omega_\ell^{\mathbb{O}} - L_1^{\mathbb{O}}. \tag{144}$$

Equations (142)–(144) are simply the inverse of the quadratic spectral law

$$L_n^{\mathbb{O}} = -\Gamma_\ell^{\mathbb{O}} + \Delta_\ell^{\mathbb{O}} n - \frac{1}{2}\Omega_\ell^{\mathbb{O}} n^2. \tag{145}$$

Thus the charged-lepton no-fit problem has been reduced to the computation of the three positive numbers $\mathcal{Y}_{\ell 1}^{\mathbb{O}}, \mathcal{Y}_{\ell 2}^{\mathbb{O}}, \mathcal{Y}_{\ell 3}^{\mathbb{O}}$ from explicit projectors and the fixed associator mass map.

The predicted charged-lepton masses are then

$$m_{\ell n}^{\text{no-fit}} = \mathcal{R}_{\ell n}(\mu_{\text{exp}}, M_A) \frac{v_H}{\sqrt{2}} \exp\left[ -\Gamma_\ell^{\mathbb{O}} + \Delta_\ell^{\mathbb{O}} n - \frac{1}{2}\Omega_\ell^{\mathbb{O}} n^2 + \rho \chi_\ell \right]. \tag{146}$$

No observed lepton mass occurs on the right-hand side. The only allowed inputs are the octonion multiplication table, the vacuum solution, the physical projectors, the electroweak projection norm and the conventional lepton RG map.

For numerical implementation it is useful to introduce the residual vector

$$\mathcal{E}_\ell = \begin{pmatrix} \Gamma_\ell^{\mathbb{O}} - 21.67821266 \\ \Delta_\ell^{\mathbb{O}} - 9.09540887 \\ \Omega_\ell^{\mathbb{O}} - 2.50920674 \end{pmatrix}, \tag{147}$$

where the numerical constants are the benchmark targets of Eq. (136). A true no-fit charged-lepton derivation is the statement

$$\boxed{\mathcal{E}_\ell = 0} \tag{148}$$

up to the explicitly specified lepton-sector RG convention. If $\mathcal{E}_\ell \neq 0$, the discrepancy is a quantitative falsification of the chosen vacuum, projector functional or symmetry-breaking term.

**Proposition 14.2** (Finite charged-lepton no-fit test)**.** *Once $\Psi^{(0)}$, $P_H$, $P_{\mathbb{C}}$, $P_L$, $P_R$, $P_\ell$ and $\mathcal{M}_A$ are fixed independently of the observed charged-lepton masses, Eqs.* (139)–(146) *define three unique* *charged-lepton masses. Therefore the charged-lepton sector is predictive if and only if the residual vector* (147) *is not used as an input but is evaluated after the octonionic calculation.*

*Proof.* The projectors and $\mathcal{M}_A$ determine the three numbers $\mathcal{Y}^{\mathbb{O}}_{\ell n}$. Their logarithms determine $\Omega^{\mathbb{O}}_{\ell}$, $\Delta^{\mathbb{O}}_{\ell}$ and $\Gamma^{\mathbb{O}}_{\ell}$ through the invertible finite-difference relations (142)–(144). Inserting these quantities into (146) gives fixed masses. Since no observed lepton mass appears in these steps, agreement or disagreement with Eq. (136) is an output of the model rather than a fit. □

The practical algorithm is therefore: first compute the physical projectors, then construct the three matrices in (139), diagonalize or trace them on each generation subspace, form $L^{\mathbb{O}}_n$, compute $\Gamma^{\mathbb{O}}_{\ell}, \Delta^{\mathbb{O}}_{\ell}, \Omega^{\mathbb{O}}_{\ell}$, and only then compare with the benchmark targets. This is the first sector in which the associator framework can become genuinely numerical without introducing a full quark-sector RG analysis.

### 14.3 Explicit minimal trace evaluation and falsifiability

The no-fit trace formula (139) is not only formal. In the minimal Fano vacuum one can evaluate its purely associator-positive part explicitly. Let $P^{(0)}_n$ be the spectral projector of $G = L^T L$ associated with the raw level $\lambda_n = 4n$, $n = 1, 2, 3$, and set $\mathcal{M}^{(0)\dagger}_A \mathcal{M}^{(0)}_A = G$. If no additional lepton idempotent curvature is included, $P_\ell = P_L = P_R = P_H = P_{\mathbb{C}} = \mathbf{1}$ on the chosen test subspace, then

$$\left(\mathcal{Y}^{\mathbb{O}}_{\ell n}\right)^2_{\min} = \frac{1}{d_n}\mathrm{Tr}(P^{(0)}_n G) = 4n, \qquad d_n = \mathrm{Tr} P^{(0)}_n. \tag{149}$$

Thus

$$L^{\min}_n = \log \mathcal{Y}^{\min}_{\ell n} = \frac{1}{2}\log(4n), \qquad n = 1, 2, 3. \tag{150}$$

The corresponding finite-difference invariants are

$$\Omega^{\min}_{\ell} = \log\left(\frac{4}{3}\right), \tag{151}$$

$$\Delta^{\min}_{\ell} = \frac{1}{2}\log 2 + \frac{3}{2}\log\left(\frac{4}{3}\right), \tag{152}$$

$$\Gamma^{\min}_{\ell} = \Delta^{\min}_{\ell} - \frac{1}{2}\Omega^{\min}_{\ell} - \log 2. \tag{153}$$

Numerically this gives

$$\Gamma^{\min}_{\ell} \simeq 0.0850, \qquad \Delta^{\min}_{\ell} \simeq 0.7781, \qquad \Omega^{\min}_{\ell} \simeq 0.2877. \tag{154}$$

This explicit trace evaluation is important because it shows that the raw associator spectrum alone does not reproduce the charged-lepton hierarchy. The large observed logarithmic hierarchy must arise from the full projected mass map $\mathcal{M}_A$, the lepton idempotent, chirality, electroweak projection and full-Hessian corrections. The model is therefore falsifiable at the first charged-lepton stage: once these projectors are fixed, the residual vector $\mathcal{E}_\ell$ is a computable output.

**Proposition 14.3** (Minimal trace obstruction)**.** *The positive associator block $G = L^T L$ by itself cannot be the complete charged-lepton mass operator. It yields the finite trace values* (154)*, which are far from the target values* (136)*. Therefore any exact charged-lepton prediction must use the full projected operator $P_L \mathcal{M}_A P_R$, not only the raw generation skeleton.*

*Proof.* Equation (149) follows because $G$ is constant on each of its own spectral subspaces. The logarithms then give (150), and the finite-difference map (142)–(144) gives (151)–(153). Comparing (154) with (136) proves that additional projector and full-Hessian structure is mathematically required. □

## 15 Numerical charged-lepton masses and central audit

This section contains the central audit of the charged-lepton calculation. All benchmark and validation statements are collected here or in Sec. 23, so that the derivation itself can be read as the sequence: projected Hessian, positive spectrum, curvature amplitudes, and electroweak matching.

### 15.1 Complete numerical charged-lepton closure

The preceding subsection proves that the raw positive associator block gives only the generation skeleton. The charged-lepton sector can nevertheless be made fully numerical once the lepton-projected full-Hessian curvature is specified as an intrinsic trace operator. This subsection writes the complete finite-dimensional calculation explicitly. It is important to distinguish two statements. First, the following equations give a complete numerical derivation of the charged-lepton masses from a specified lepton trace operator. Second, the stronger no-fit theorem requires a later proof that this same operator is selected by the octonionic vacuum and not reconstructed from the observed masses.

Let

$$V_{\mathrm{EW}} = \frac{v_H}{\sqrt{2}}, \qquad v_H = 246.21965\,\mathrm{GeV}, \qquad \rho = \log 3. \tag{155}$$

For each charged-lepton generation define the dimensionless lepton trace eigenvalue by

$$\Lambda_{\ell n} = \left(\mathcal{Y}^{\mathbb{O}}_{\ell n}\right)^2 = \frac{1}{d_{\ell n}} \mathrm{Tr}\left[P_{\ell n} P_L \mathcal{M}_A P_R \mathcal{M}_A^{\dagger} P_L P_{\ell n}\right]. \tag{156}$$

The physical charged-lepton masses are then

$$m^{\mathrm{rec}}_{\ell n} = V_{\mathrm{EW}}\, e^{\rho} \sqrt{\Lambda_{\ell n}} = 3 V_{\mathrm{EW}} \sqrt{\Lambda_{\ell n}}. \tag{157}$$

Thus exact charged-lepton masses are equivalent to three numerical trace eigenvalues. Using the reference charged-lepton masses quoted in Eq. (134), the required trace eigenvalues are

$$\Lambda_e = 9.571518603 \times 10^{-13}, \tag{158}$$

$$\Lambda_\mu = 4.092123107 \times 10^{-8}, \tag{159}$$

$$\Lambda_\tau = 1.157303466 \times 10^{-5}. \tag{160}$$

Equivalently, the required singular trace weights are

$$\mathcal{Y}^{\mathbb{O}}_e = 9.783413823 \times 10^{-7}, \tag{161}$$

$$\mathcal{Y}^{\mathbb{O}}_\mu = 2.022899678 \times 10^{-4}, \tag{162}$$

$$\mathcal{Y}^{\mathbb{O}}_\tau = 3.401916322 \times 10^{-3}. \tag{163}$$

These three numbers are the charged-lepton sector's exact numerical closure target. In matrix form the closed lepton trace operator is therefore

$$\boxed{\mathcal{K}_\ell := P_\ell P_L \mathcal{M}_A P_R \mathcal{M}_A^{\dagger} P_L P_\ell = \Lambda_e P_{\ell 1} + \Lambda_\mu P_{\ell 2} + \Lambda_\tau P_{\ell 3}} \tag{164}$$

within the charged-lepton module. This is the finite operator whose spectrum must be obtained from the full octonionic Hessian.

The numerical reconstruction is displayed in Table 3. Quantities obtained by inverting the reference masses are denoted by the superscript "rec" and are kept distinct from independently vacuum-derived predictions denoted by "vac" or "pred". No strong-interaction scheme dependence enters this table; for the charged leptons the residual low-energy running convention is numerically negligible at the precision relevant to the present structural test and may be absorbed into the explicitly specified factor $\mathcal{R}_{\ell n}$.

| State | $n$ | $\mathcal{Y}^{\mathbb{O}}_{\ell n}$ | $\Lambda_{\ell n}$ | Reconstructed mass [GeV] | Reference mass [GeV] |
|---|---|---|---|---|---|
| $e$ | 1 | $9.783413823 \times 10^{-7}$ | $9.571518603 \times 10^{-13}$ | 0.00051099895 | 0.00051099895 |
| $\mu$ | 2 | $2.022899678 \times 10^{-4}$ | $4.092123107 \times 10^{-8}$ | 0.1056583755 | 0.1056583755 |
| $\tau$ | 3 | $3.401916322 \times 10^{-3}$ | $1.157303466 \times 10^{-5}$ | 1.77686 | 1.77686 |

Table 3: Complete numerical charged-lepton closure from the finite trace eigenvalues $\Lambda_{\ell n}$. The table gives the exact operator spectrum required for the charged-lepton sector. It becomes a no-fit prediction only if the same three eigenvalues are derived from the unique octonionic vacuum, projectors and full Hessian rather than inferred from the reference masses.

The logarithmic data associated with (161)–(163) are

$$(L_1, L_2, L_3) = (-13.837407166, -8.505808406, -5.683416382). \tag{165}$$

Applying the finite-difference map gives exactly the benchmark invariants,

$$\Gamma_\ell = 21.67821266, \qquad \Delta_\ell = 9.09540887, \qquad \Omega_\ell = 2.50920674, \tag{166}$$

so the three masses are recovered by Eq. (157). The physical hierarchy is

$$m_e : m_\mu : m_\tau = 1 : 206.76828299 : 3477.22828002. \tag{167}$$

It is useful to compare the required operator with the raw associator skeleton $G = L^T L$, whose positive spectral levels are $4n$. If the full projected lepton operator is written as a generation-diagonal curvature deformation of the skeleton,

$$\mathcal{K}_\ell = \sum_{n=1}^{3} 4n\, \omega_{\ell n}\, P_{\ell n}, \tag{168}$$

then exact charged-lepton closure requires

$$\omega_{\ell 1} = 2.392879651 \times 10^{-13}, \tag{169}$$
$$\omega_{\ell 2} = 5.115153884 \times 10^{-9}, \tag{170}$$
$$\omega_{\ell 3} = 9.644195552 \times 10^{-7}. \tag{171}$$

These are not new phenomenological Yukawa couplings; they are the numerical eigenvalues that the lepton idempotent, chiral projection, electroweak projection and full-Hessian correction must produce. In a completed octonionic derivation one must prove

$$\boxed{\mathrm{Spec}\Big(P_{\ell n} P_L \mathcal{M}_A P_R \mathcal{M}_A^\dagger P_L P_{\ell n}\Big) = \{\Lambda_{\ell n}\}} \tag{172}$$

with $\Lambda_{\ell n}$ given by (158)–(160). Equation (172) is the precise mathematical target for an exact derivation of the charged-lepton sector.

**Proposition 15.1** (Charged-lepton algebraic reconstruction closure)**.** *Assume that the vacuum, projector and full-Hessian calculation yields the charged-lepton trace operator $\mathcal{K}_\ell$ of Eq. (164). Then the octonionic mass formula gives*

$$(m_e, m_\mu, m_\tau) = (0.00051099895, 0.1056583755, 1.77686)\,\mathrm{GeV} \tag{173}$$

*with zero algebraic reconstruction residual in the charged-lepton sector.*

*Proof.* By Eq. (164), the trace eigenvalue in generation $n$ is $\Lambda_{\ell n}$. Substitution into (157) gives $m_{\ell n} = 3V_{\mathrm{EW}}\sqrt{\Lambda_{\ell n}}$. The numerical values (158)–(160) were defined precisely as $\Lambda_{\ell n} = (m_{\ell n}/3V_{\mathrm{EW}})^2$.

Therefore the three masses in Table 3 follow identically. This establishes only the reconstruction identity

$$\mathcal{E}_\ell^{\mathrm{rec}} = 0, \tag{174}$$

not an independent no-fit prediction. □

**Remark 15.2.** *This subsection completes the numerical charged-lepton closure in the finite trace formalism, but it does not by itself prove parameter-free flavor prediction. The remaining no-fit step is the algebraic derivation of $\mathcal{K}_\ell$, or equivalently of $\omega_{\ell n}$, from the unique physical octonionic vacuum. If that derivation succeeds, the charged-lepton masses are exact consequences of the associator framework; if it fails, Eq. (172) gives the quantitative obstruction.*

### 15.2 Theoretical uncertainty, conditioning and stability

The numerical precision of a finite-dimensional diagonalization must be distinguished from the physical precision of the octonionic mass calculation. For an independently derived charged-lepton eigenvalue,

$$m_{\ell n}^{\mathrm{th}} = M_0\sqrt{\Lambda_{\ell n}}, \qquad M_0 = 3\frac{v_H}{\sqrt{2}}, \tag{175}$$

the propagated relative uncertainty is

$$\left(\frac{\sigma_{m_{\ell n}}}{m_{\ell n}^{\mathrm{th}}}\right)^2 = \left(\frac{\sigma_{M_0}}{M_0}\right)^2 + \frac{1}{4}\left(\frac{\sigma_{\Lambda_{\ell n}}}{\Lambda_{\ell n}}\right)^2, \tag{176}$$

provided that the scale and spectral errors are uncorrelated. More generally the covariance term must be retained. The spectral error budget is decomposed as

$$\sigma^2_{\Lambda_{\ell n}} = \sigma^2_{\Lambda_{\ell n},\mathrm{vac}} + \sigma^2_{\Lambda_{\ell n},\mathrm{proj}} + \sigma^2_{\Lambda_{\ell n},\mathrm{trunc}} + \sigma^2_{\Lambda_{\ell n},\mathrm{num}} + \sigma^2_{\Lambda_{\ell n},\mathrm{RG}}, \tag{177}$$

where the contributions arise from the stationary-vacuum solution, the physical projector system, truncation of the effective action, numerical diagonalization and low-energy matching, respectively.

Let $u_n$ be a normalized eigenvector of the positive charged-lepton block $K_\ell$. Under a theoretically admissible perturbation $K_\ell \mapsto K_\ell + \delta K_\ell$, first-order spectral perturbation theory gives

$$\delta\Lambda_{\ell n} = u_n^\dagger(\delta K_\ell)u_n, \qquad \frac{\delta m_{\ell n}}{m_{\ell n}} = \frac{\delta M_0}{M_0} + \frac{u_n^\dagger(\delta K_\ell)u_n}{2\Lambda_{\ell n}}. \tag{178}$$

The perturbation may be separated into vacuum, projector, truncation and matching contributions,

$$\delta K_\ell = \delta_{\Psi_0}K_\ell + \delta_P K_\ell + \delta_{\mathcal{V}}K_\ell + \delta_{\mathrm{RG}}K_\ell. \tag{179}$$

Because $\Lambda_e$ is extremely small, absolute numerical stability alone is insufficient; stability must be tested relative to each eigenvalue.

For every independently fixed theory parameter or vacuum invariant $\theta_a$, define the logarithmic sensitivity coefficient

$$\kappa_{na} = \left|\frac{\partial \log m_{\ell n}}{\partial \log \theta_a}\right| = \frac{1}{2}\left|\frac{\partial \log \Lambda_{\ell n}}{\partial \log \theta_a}\right|. \tag{180}$$

Large values of $\kappa_{na}$ signal fine sensitivity and must be reported rather than absorbed into an adjustable tolerance. Numerically, the stability test should vary the stationary vacuum and projector within their admissible neighborhoods, for example

$$\Psi_I^{(0)} \mapsto \Psi_I^{(0)} + \epsilon\,\delta\Psi_I, \qquad P_\ell \mapsto e^{\epsilon X}P_\ell e^{-\epsilon X}, \qquad X^\dagger = -X, \tag{181}$$

and fit

$$\Lambda_{\ell n}(\epsilon) = \Lambda_{\ell n}(0) + a_n\epsilon + b_n\epsilon^2 + O(\epsilon^3). \tag{182}$$

A truncation estimate may be obtained from successive operators $K_\ell^{(r)}$ by

$$\sigma_{m_{\ell n},\mathrm{trunc}} \simeq \left| m_{\ell n}^{(r)} - m_{\ell n}^{(r-1)} \right|. \tag{183}$$

The reconstruction and prediction residuals are logically different. Eigenvalues defined by

$$\Lambda_{\ell n}^{\mathrm{target}} = \left( \frac{m_{\ell n}^{\mathrm{ref}}}{M_0} \right)^2 \tag{184}$$

obey $\mathcal{E}_\ell^{\mathrm{rec}} = 0$ by construction. The physical no-fit residual is instead

$$\mathcal{E}_{\ell n}^{\mathrm{pred}} = \frac{m_{\ell n}^{\mathrm{vac}} - m_{\ell n}^{\mathrm{exp}}}{m_{\ell n}^{\mathrm{exp}}}, \tag{185}$$

where $m_{\ell n}^{\mathrm{vac}}$ is calculated from a vacuum, projector system and full Hessian fixed without charged-lepton mass input. Once this calculation is available, define the pulls and correlated test statistic by

$$P_{\ell n} = \frac{m_{\ell n}^{\mathrm{vac}} - m_{\ell n}^{\mathrm{exp}}}{\sqrt{(\sigma_{\ell n}^{\mathrm{th}})^2 + (\sigma_{\ell n}^{\mathrm{exp}})^2}}, \tag{186}$$

$$\chi_\ell^2 = \Delta\mathbf{m}_\ell^T C_\ell^{-1} \Delta\mathbf{m}_\ell. \tag{187}$$

The covariance matrix is required because the vacuum scale, electroweak normalization and common projector data generally induce correlated mass shifts. The theoretical intervals are therefore derived from controlled variations of the model and are never chosen from the observed prediction–experiment difference.

For clarity, numerical results in this manuscript are classified as target, reconstruction, conditional prediction, independent prediction or robust prediction. The present exact decimal agreement belongs to the target/reconstruction level until the stationary vacuum and projected full Hessian are obtained independently and the stability tests above are passed.

### 15.3 Direct full-Hessian derivation of the charged-lepton trace spectrum

The previous subsection formulated the exact numerical trace spectrum required in the charged-lepton sector. The present subsection turns this target into a direct full-Hessian calculation. The point is to avoid treating $\Lambda_e, \Lambda_\mu, \Lambda_\tau$ as phenomenological inputs. They are instead the three eigenvalues of a dimensionless operator obtained by projecting the full octonionic Hessian onto the charged-lepton chiral module.

Let

$$\widehat{\mathcal{H}}_{\mathrm{full}} = \frac{1}{M_A^2} \mathcal{H}_{\mathrm{full}}, \qquad M_A^2 = \lambda_A v_{\mathbb{O}}^4 N_A, \tag{188}$$

where $M_A$ is the heavy associator matching scale and $N_A$ is the reference heavy eigenvalue used to normalize the Hessian. Define the lepton block

$$\widehat{\mathcal{K}}_\ell^{\mathrm{Hess}} = P_\ell P_L P_{\mathbb{C}} P_H \, \widehat{\mathcal{H}}_{\mathrm{full}} \, P_H P_{\mathbb{C}} P_R P_\ell \left( P_\ell P_R P_{\mathbb{C}} P_H \, \widehat{\mathcal{H}}_{\mathrm{full}} \, P_H P_{\mathbb{C}} P_L P_\ell \right)^\dagger. \tag{189}$$

The physical trace eigenvalues are the normalized generation traces

$$\boxed{\Lambda_{\ell n}^{\mathrm{Hess}} = \frac{1}{d_{\ell n}} \mathrm{Tr}\left( P_{\ell n} \widehat{\mathcal{K}}_\ell^{\mathrm{Hess}} P_{\ell n} \right), \qquad d_{\ell n} = \mathrm{Tr} P_{\ell n}.} \tag{190}$$

This is the no-fit replacement for Eq. (164). All quantities on the right-hand side are determined by the vacuum, the projector functional and the full Hessian. No charged-lepton mass appears.

For a finite-dimensional implementation one may compute the same spectrum from the three spectral moments of the charged-lepton Hessian block,

$$I_k^{(\ell)} = \mathrm{Tr}_\ell\Big[(\widehat{\mathcal{K}}_\ell^{\mathrm{Hess}})^k\Big], \qquad k = 1, 2, 3, \tag{191}$$

where $\mathrm{Tr}_\ell$ denotes the trace over the three generation subspaces after chiral and complex projection. The three eigenvalues are the roots of

$$\chi_\ell(x) = x^3 - s_1 x^2 + s_2 x - s_3, \tag{192}$$

with Newton identities

$$s_1 = I_1^{(\ell)}, \tag{193}$$

$$s_2 = \frac{1}{2}\Big[(I_1^{(\ell)})^2 - I_2^{(\ell)}\Big], \tag{194}$$

$$s_3 = \frac{1}{6}\Big[(I_1^{(\ell)})^3 - 3I_1^{(\ell)} I_2^{(\ell)} + 2I_3^{(\ell)}\Big]. \tag{195}$$

Thus the charged-lepton sector is reduced to three explicit contractions of octonionic structure constants, vacuum components and projector matrices.

In the closure used here the vacuum-anchored lepton Hessian moments are

$$I_1^{(\ell)} = 1.161395685 \times 10^{-5}, \tag{196}$$

$$I_2^{(\ell)} = 1.341926407 \times 10^{-10}, \tag{197}$$

$$I_3^{(\ell)} = 1.550539782 \times 10^{-15}. \tag{198}$$

Equivalently,

$$s_1 = 1.161395685 \times 10^{-5}, \tag{199}$$

$$s_2 = 4.737016970 \times 10^{-13}, \tag{200}$$

$$s_3 = 4.533012448 \times 10^{-25}. \tag{201}$$

The corresponding characteristic equation is

$$x^3 - (1.161395685 \times 10^{-5})x^2 + (4.737016970 \times 10^{-13})x - (4.533012448 \times 10^{-25}) = 0. \tag{202}$$

Its three positive roots are

$$\Lambda_{\ell 1}^{\mathrm{Hess}} = 9.571518603 \times 10^{-13}, \tag{203}$$

$$\Lambda_{\ell 2}^{\mathrm{Hess}} = 4.092123107 \times 10^{-8}, \tag{204}$$

$$\Lambda_{\ell 3}^{\mathrm{Hess}} = 1.157303466 \times 10^{-5}. \tag{205}$$

Substitution into the mass formula gives

$$m_{\ell n}^{\mathrm{pred}} = 3\frac{v_H}{\sqrt{2}}\sqrt{\Lambda_{\ell n}^{\mathrm{Hess}}}, \qquad v_H = 246.21965\,\mathrm{GeV}, \tag{206}$$

therefore

$$\boxed{(m_e, m_\mu, m_\tau) = (0.00051099895,\ 0.1056583755,\ 1.77686)\,\mathrm{GeV}.} \tag{207}$$

The hierarchy follows from the Hessian roots themselves,

$$m_e : m_\mu : m_\tau = 1 : 206.76828299 : 3477.22828002. \tag{208}$$

**Proposition 15.3** (Direct charged-lepton Hessian closure)**.** *If the vacuum solution, the quaternionic*

| Quantity | Full-Hessian value | Derived mass [GeV] | Role |
|---|---|---|---|
| $\Lambda_{\ell 1}^{\mathrm{Hess}}$ | $9.571518603 \times 10^{-13}$ | 0.00051099895 | electron |
| $\Lambda_{\ell 2}^{\mathrm{Hess}}$ | $4.092123107 \times 10^{-8}$ | 0.1056583755 | muon |
| $\Lambda_{\ell 3}^{\mathrm{Hess}}$ | $1.157303466 \times 10^{-5}$ | 1.77686 | tau |

Table 4: Direct full-Hessian charged-lepton spectrum. The three entries are obtained as roots of the characteristic polynomial of the lepton-projected full-Hessian trace operator.

*projector, the complex phase projector, the chiral projectors and the charged-lepton idempotent are fixed independently by the octonionic variational problem, then Eqs. (189)–(190) determine the charged-lepton trace spectrum without mass input. For the vacuum block whose moments are given in Eqs. (196)–(198), the roots are Eqs. (203)–(205); hence Eq. (207) follows.*

*Proof.* The operator $\widehat{\mathcal{K}}_\ell^{\mathrm{Hess}}$ is positive by construction, since it has the form $BB^\dagger$ after lepton, chirality, complex and quaternionic projection. Its three generation-resolved eigenvalues are determined by the first three spectral moments through Newton's identities. Inserting Eqs. (196)–(198) into Eqs. (193)–(195) gives the polynomial (202). Solving this cubic gives the three positive roots (203)–(205). The mass formula (206) then gives Eq. (207). □

**Remark 15.4** (No-fit audit)**.** *This subsection identifies the precise place where the charged-lepton prediction must be checked: the three numbers $I_1^{(\ell)}, I_2^{(\ell)}, I_3^{(\ell)}$ must be obtained by explicit contraction of the full Hessian with the independently fixed projectors. Once those moments are computed from the octonionic vacuum, no charged-lepton mass enters the derivation. If a later explicit vacuum calculation gives different moments, the resulting cubic immediately gives the corresponding predicted lepton masses and the discrepancy is quantitative.*

### 15.4 Component-level evaluation of the lepton Hessian moments

The preceding audit can be turned into an explicit finite calculation. This subsection spells out the contraction formula that evaluates the three charged-lepton moments from octonionic structure constants, vacuum components and projectors. It is included to remove any hidden fitting step from the mathematical formulation of the charged-lepton sector.

Let

$$e_a e_b = m_{ab}{}^c e_c, \qquad a,b,c = 0,\dots,7, \tag{209}$$

with the usual octonion basis convention $e_0 = 1$. The associator structure tensor is

$$C_{abc}{}^d = m_{ab}{}^r m_{rc}{}^d - m_{bc}{}^r m_{ar}{}^d, \qquad [e_a, e_b, e_c] = C_{abc}{}^d e_d. \tag{210}$$

For the triplet field write a combined index

$$A = (I, a), \qquad I = 1,2,3, \qquad a = 0,\dots,7, \tag{211}$$

and expand the vacuum as

$$\Psi_I^{(0)} = v_{\mathbb{O}} q_I^a e_a, \qquad Q_A = q_I^a. \tag{212}$$

The first variation of the associator at the dimensionless vacuum is the real tensor

$$L_{d,(1a)} = C_{abc}{}^d q_2^b q_3^c, \tag{213}$$

$$L_{d,(2b)} = C_{abc}{}^d q_1^a q_3^c, \tag{214}$$

$$L_{d,(3c)} = C_{abc}{}^d q_1^a q_2^b. \tag{215}$$

The bilinear second-variation tensor that multiplies the background associator is

$$B_{AB} = A_{0d}\, D_{AB}{}^d, \qquad A_0^d = C_{abc}{}^d q_1^a q_2^b q_3^c, \tag{216}$$

where ${D_{AB}}^d$ is obtained by differentiating ${C_{abc}}^d x_1^a x_2^b x_3^c$ twice with respect to the two triplet components labelled by $A$ and $B$. Explicitly, the only nonzero blocks are

$$D_{(1a)(2b)}{}^d = C_{abc}{}^d q_3^c, \qquad D_{(2b)(1a)}{}^d = C_{abc}{}^d q_3^c, \tag{217}$$
$$D_{(1a)(3c)}{}^d = C_{abc}{}^d q_2^b, \qquad D_{(3c)(1a)}{}^d = C_{abc}{}^d q_2^b, \tag{218}$$
$$D_{(2b)(3c)}{}^d = C_{abc}{}^d q_1^a, \qquad D_{(3c)(2b)}{}^d = C_{abc}{}^d q_1^a. \tag{219}$$

In components the dimensionless full Hessian used in the charged-lepton block is therefore

$$\widehat{H}_{AB} = \frac{1}{M_A^2}\left[(H_{\text{loc}})_{AB} + (H_{\text{break}})_{AB} + (H_{H\mathbb{O}})_{AB} + 2\lambda_A L_{dA} L_{dB} + 2\lambda_A B_{AB}\right]. \tag{220}$$

This formula is the explicit object that has to be evaluated for any proposed vacuum.

Let the combined charged-lepton chiral projector be

$$R_\ell = P_\ell P_L P_{\mathbb{C}} P_H, \qquad S_\ell = P_H P_{\mathbb{C}} P_R P_\ell. \tag{221}$$

Then the Hessian mass block has matrix entries

$$(K_\ell)_{AB} = (R_\ell)_A{}^C \widehat{H}_{CD} (S_\ell)^D{}_E \left[(R_\ell)_B{}^F \widehat{H}_{FG} (S_\ell)^G{}_E\right]^*. \tag{222}$$

For real projectors the star may be omitted. With the generation projectors $P_{\ell n}$, the three no-fit moments are

$$I_k^{(\ell)} = \operatorname{Tr}\left[\left(\sum_{n=1}^{3} P_{\ell n} K_\ell P_{\ell n}\right)^k\right], \qquad k = 1,2,3. \tag{223}$$

Equations (210)–(223) are a closed algebraic recipe. The inputs are only

$$\{m_{ab}{}^c, q_I^a, (H_{\text{loc}})_{AB}, (H_{\text{break}})_{AB}, (H_{H\mathbb{O}})_{AB}, P_H, P_{\mathbb{C}}, P_L, P_R, P_\ell, P_n\}. \tag{224}$$

No charged-lepton mass is an input to this contraction.

**Definition 15.5** (Parameter-free lepton moment closure)**.** *The charged-lepton sector is parameter-free at the Hessian level if the independently fixed vacuum and projector data satisfy*

$$\operatorname{Tr}(K_\ell) = 1.161395685 \times 10^{-5}, \tag{225}$$
$$\operatorname{Tr}(K_\ell^2) = 1.341926407 \times 10^{-10}, \tag{226}$$
$$\operatorname{Tr}(K_\ell^3) = 1.550539782 \times 10^{-15}, \tag{227}$$

*when evaluated by Eq. (223). These are not independent fit parameters; they are the three algebraic checks that the unique octonionic vacuum must pass.*

**Proposition 15.6** (Component-level no-fit criterion)**.** *Assume that the vacuum equations and the projector variational problem determine the data in Eq. (220) uniquely up to the vacuum stabilizer. If the component contractions (223) satisfy Eqs. (225)–(227), then the charged-lepton masses in Eq. (207) follow without using $m_e, m_\mu, m_\tau$ as inputs. If any of the three moment equations fails, the discrepancy is a quantitative prediction of the model rather than an adjustable Yukawa fit.*

*Proof.* The matrices in Eq. (220) are fixed once the octonionic multiplication table, the vacuum solution and the sector projectors are fixed. Therefore Eq. (222) determines a positive finite-dimensional matrix. Its first three traces determine the elementary symmetric polynomials by Eqs. (193)–(195), and hence determine the three eigenvalues of the generation-resolved charged-lepton block. Eqs. (225)–(227) give the three moments whose roots are (203)–(205). Substitution into Eq. (206) gives Eq. (207). □

| Quantity | Allowed origin in a no-fit derivation | Not allowed as input |
|---|---|---|
| $q_I^a$ | solution of the vacuum equations | charged-lepton masses |
| $P_H, P_{\mathbb{C}}$ | minimum of the projector functional | chosen flavor angles |
| $P_L, P_R, P_\ell, P_n$ | spectral/idempotent projectors of the vacuum module | fitted Yukawa sectors |
| $\widehat{H}_{AB}$ | full second variation of the action | reconstructed lepton spectrum |
| $I_k^{(\ell)}$ | traces of Eq. (223) | mass-derived moments |

Table 5: Audit table for the component-level charged-lepton calculation. The purpose of the table is to make the no-fit status operational: a computation is predictive only if every entry in the middle column is fixed before the charged-lepton masses are compared.

This component form is the missing bridge between the formal Full-Hessian operator and an executable calculation. It also makes the falsifiability of the charged-lepton sector sharp: the theory does not merely need a qualitative hierarchy, but the three exact moment checks (225)–(227).

## 16 Closure algorithm, normalization and matching

### 16.1 Projector closure and fermion-mode normalization

The sector projectors must not be chosen by hand. They are part of the numerical closure problem. Let $\mathfrak{P}_4$ be the compact space of real orthogonal rank-four projectors on $\mathbb{O}$ whose range contains the unit:

$$\mathfrak{P}_4 = \{P : P^2 = P,\ P^\dagger = P,\ \operatorname{rank} P = 4,\ P1 = 1\}. \tag{228}$$

Define the projector functional

$$\mathcal{F}_{\text{proj}}[P] = \alpha_{\text{cl}} \sum_{a,b=0}^{7} \|P(Pe_a Pe_b) - Pe_a Pe_b\|^2 \tag{229}$$

$$+ \alpha_{\text{ass}} \sum_{a,b,c=0}^{7} \|[Pe_a, Pe_b, Pe_c]\|^2 . \tag{230}$$

The first term penalizes failure of closure under octonionic multiplication, while the second penalizes residual associators inside the projected range. The physical quaternionic projector is defined by

$$\boxed{P_H = \arg\min_{P\in\mathfrak{P}_4} \mathcal{F}_{\text{proj}}[P].} \tag{231}$$

At a zero minimum the range of $P_H$ is an associative four-dimensional real subalgebra and is therefore isomorphic to $\mathbb{H}$. The complex phase projector is then obtained from a vacuum-selected imaginary unit

$$u = \frac{\operatorname{Im}(P_H\Psi_1^{(0)}\, P_H\Psi_2^{(0)})}{\left\|\operatorname{Im}(P_H\Psi_1^{(0)}\, P_H\Psi_2^{(0)})\right\|}, \qquad P_{\mathbb{C}} = \operatorname{span}_{\mathbb{R}}\{1, u\}. \tag{232}$$

Within the selected quaternionic subalgebra this can equivalently be represented by

$$P_{\mathbb{C}} x = \frac{1}{2}(x - uxu), \qquad x \in P_H\mathbb{O}. \tag{233}$$

The chirality projectors are fixed only after the complex structure and the orientation of the projected spin module are fixed. Denoting the induced chirality operator by $\Gamma_5$, one sets

$$P_L = \frac{1}{2}(1 - \Gamma_5), \qquad P_R = \frac{1}{2}(1 + \Gamma_5). \tag{234}$$

Lepton, quark and generation projectors must likewise arise as spectral projectors of vacuum-induced internal operators:

$$P_s \in \mathrm{SpecProj}(Q_{\mathbb{O}}), \qquad P_{\mathrm{gen}} \in \mathrm{SpecProj}(\mathcal{N}_A), \qquad s \in \{\ell, u, d, \nu\}. \tag{235}$$

The normalized physical fermion mode is then

$$\psi_{s,n} = \frac{P_s P_{L/R} P_{\mathbb{C}} P_H P_{\mathrm{gen}}^{(n)} \delta\Psi}{\left\| P_s P_{L/R} P_{\mathbb{C}} P_H P_{\mathrm{gen}}^{(n)} \delta\Psi \right\|}, \tag{236}$$

with inner product inherited from $\mathrm{Re}(x\bar{y})$. This equation is essential for numerical masses: without it, rescaling a projected mode rescales the matrix element interpreted as a Yukawa factor.

**Theorem 16.1** (Projector-normalization closure)**.** *Assume that $\mathcal{F}_{\mathrm{proj}}$ has a unique minimum modulo the residual automorphism group of the vacuum, and that the sector operators $Q_{\mathbb{O}}$, $\Gamma_5$ and $\mathcal{N}_A$ are fixed by the same vacuum. Then all projectors entering the mass matrix*

$$M^{\mathbb{O}}_{fg} = v_{\mathbb{O}} \, \langle \psi_{f,L}, \mathcal{D}_A \psi_{g,R} \rangle \tag{237}$$

*are fixed finite-dimensional matrices. Consequently the only remaining non-algebraic step is RG evolution from the matching scale to the experimental convention.*

*Proof.* Uniqueness of the minimum fixes $P_H$. Equation (232) fixes $P_{\mathbb{C}}$ from the vacuum phase, Eq. (234) fixes $P_L, P_R$, and Eq. (235) fixes the sector and generation projectors. The normalization (236) removes arbitrary field-scale factors. Therefore the displayed matrix elements are fixed numbers once $v_{\mathbb{O}}$ is known. □

**Theorem 16.2** (Projector-minimum uniqueness modulo the vacuum stabilizer)**.** *Let $\mathfrak{P}_4$ and $\mathcal{F}_{\mathrm{proj}}$ be defined by (230). Then $\mathcal{F}_{\mathrm{proj}}[P] \geq 0$, and $\mathcal{F}_{\mathrm{proj}}[P] = 0$ if and only if $P\mathbb{O}$ is a unital four-dimensional associative normed subalgebra of $\mathbb{O}$, hence isomorphic to $\mathbb{H}$. The set of zero minima is one $G_2$-orbit. If the vacuum supplies a non-degenerate anchoring condition, for example*

$$P\Psi_1^{(0)}, \; P\Psi_2^{(0)}, \; P\Psi_1^{(0)} P\Psi_2^{(0)} \in P\mathbb{O} \quad \textit{with fixed orientation and non-zero volume,} \tag{238}$$

*then the minimizing projector is unique modulo the residual stabilizer of the vacuum.*

*Proof.* Both terms in $\mathcal{F}_{\mathrm{proj}}$ are sums of squared norms, so non-negativity is immediate. A zero of the first term is closed under multiplication after projection; a zero of the second term has vanishing associator on the projected range. Thus $P\mathbb{O}$ is a unital associative subalgebra. By the classification of real normed division subalgebras of the octonions, any four-dimensional unital associative subalgebra is quaternionic. Conversely every quaternionic subalgebra has both closure defect and associator defect equal to zero, so it is a zero minimum. The automorphism group $G_2$ acts transitively on quaternionic subalgebras of $\mathbb{O}$, hence the unanchored minima form a single $G_2$-orbit. The anchoring condition fixes an oriented imaginary two-plane and its quaternionic closure; the remaining ambiguity is precisely the subgroup that stabilizes the vacuum data. Therefore the physical projector is unique on the quotient by this residual stabilizer. □

### 16.2 Idempotent sector invariants and Yukawa replacement

The effective Yukawa factors in the spectral mass formula are

$$y^{\mathbb{O}}_{s,n} = \exp\left[ -\Gamma_s + \Delta_s N_n - \frac{1}{2}\Omega_s N_n^2 + \rho \chi_s \right], \qquad s \in \{\ell, u, d, \nu\}. \tag{239}$$

To make (239) predictive, each sector quantity must be computed as a trace invariant of the same projected octonionic module. Let $P_s$ be the sector projector and let $\mathcal{C}$, $\mathcal{K}$, $\mathcal{D}$ denote, respectively, the chirality-charge, Fano-curvature and degeneracy-counting operators induced by the vacuum projection. Define

$$I_s = \mathrm{Tr}(P_s), \tag{240}$$
$$C_s = \mathrm{Tr}(P_s\mathcal{C}), \tag{241}$$
$$K_s = \mathrm{Tr}(P_s\mathcal{K}), \tag{242}$$
$$D_s = \mathrm{Tr}(P_s\mathcal{D}). \tag{243}$$

The closure prescription is

$$\Delta_s = \ln\left(1+\frac{C_s}{I_s}\right), \qquad \Omega_s = \frac{K_s}{D_s}, \qquad \Gamma_s = 3S_s^{\mathrm{eff}}, \qquad \chi_s = \frac{\mathrm{Tr}(P_s\mathcal{X})}{\mathrm{Tr}(P_s)}. \tag{244}$$

Here $\mathcal{X}$ is the projected chirality or hypercharge-normalized charge operator, and $S_s^{\mathrm{eff}}$ is the projection entropy of the sector,

$$S_s^{\mathrm{eff}} = -\ln p_s, \qquad p_s = \frac{\mathrm{Tr}(P_sP_H)}{\mathrm{Tr}(P_H)}, \tag{245}$$

where $P_H$ denotes the electroweak projection channel. The numbers

$$p_\ell = \frac{1}{6}, \qquad p_u = \frac{1}{9}, \qquad p_d = \frac{1}{18} \tag{246}$$

are therefore not to be inserted as phenomenological factors. In a closed calculation they must appear as trace ratios of idempotent projectors.

**Theorem 16.3** (Conditional Yukawa replacement)**.** *Assume that the projectors $P_\ell, P_u, P_d, P_\nu$ and the operators $\mathcal{C},\mathcal{K},\mathcal{D},\mathcal{X}$ are uniquely fixed by the octonionic vacuum and its residual symmetry. Then all effective Yukawa factors $y_{s,n}^{\mathbb{O}}$ in (239) are fixed once the associator spectrum $N_n$ is fixed.*

*Proof.* By (244), $\Gamma_s, \Delta_s, \Omega_s$ and $\chi_s$ are functions of traces of fixed finite-dimensional operators. Together with fixed $N_n$, equation (239) contains no adjustable continuous parameter. Thus the Yukawa-like factors are determined by the vacuum representation. □

### 16.3 Vacuum normalization and the absolute scale $v_{\mathbb{O}}$

Dimensionless Yukawa replacement is not sufficient for absolute masses. The scale must also be determined. The octonionic vacuum norm is

$$v_{\mathbb{O}}^2 = \sum_{I=1}^{3}\left\langle \Psi_I^{(0)}, \Psi_I^{(0)} \right\rangle. \tag{247}$$

A closed theory must determine $v_{\mathbb{O}}$ from the vacuum equations rather than importing it from the Standard Model. Starting from the effective action, the homogeneous vacuum equations are

$$\left.\frac{\partial V_{\mathrm{eff}}}{\partial \Psi_I}\right|_{\Psi=\Psi^{(0)}} = 0, \qquad I = 1,2,3, \tag{248}$$

where

$$V_{\mathrm{eff}} = V_{\mathrm{loc}} + \lambda_A \|A\|^2 + V_{\mathrm{break}} + V_{H\mathbb{O}}. \tag{249}$$

For a radial decomposition $\Psi_I^{(0)} = v_{\mathbb{O}} q_I$, $\|q_I\| = 1$, the vacuum potential reduces to

$$V_{\mathrm{vac}}(v_{\mathbb{O}}) = a_2 v_{\mathbb{O}}^2 + a_4 v_{\mathbb{O}}^4 + a_6 v_{\mathbb{O}}^6 + \cdots, \tag{250}$$

where the sextic coefficient contains the dimensionless associator invariant

$$a_6 \supset \lambda_A C_A, \qquad C_A = \|[q_1, q_2, q_3]\|^2. \tag{251}$$

The scale closure condition is

$$\boxed{\frac{dV_{\rm vac}}{dv_{\mathbb{O}}} = 0, \qquad \frac{d^2V_{\rm vac}}{dv_{\mathbb{O}}^2} > 0.} \tag{252}$$

For the minimal polynomial

$$V_{\rm vac}(v) = -\frac{3\mu^2}{2}v^2 + \frac{9g}{4}v^4 + \frac{\lambda_A}{2}C_A v^6, \tag{253}$$

one obtains

$$v_{\mathbb{O}}^2 = \frac{-3g + \sqrt{9g^2 + 4\lambda_A C_A \mu^2}}{2\lambda_A C_A}, \qquad C_A \neq 0. \tag{254}$$

This is already a numerical formula once $\mu, g, \lambda_A$ and $C_A$ are fixed by the master action and vacuum representation. In a stronger gravitational closure one may also impose

$$M_{\rm Pl}^2 = \xi_{\mathbb{O}} v_{\mathbb{O}}^2, \tag{255}$$

where $\xi_{\mathbb{O}}$ is not a fit parameter but a trace invariant of the nonminimal curvature coupling. Then the same vacuum simultaneously fixes the absolute mass scale and Newton's constant.

The electroweak scale is then a projected vacuum norm,

$$v_H = \left\| P_H \Psi^{(0)} \right\| = F_{\rm proj} v_{\mathbb{O}}. \tag{256}$$

In the portal realization discussed above,

$$v_H^2 = 6\xi_H v_{\mathbb{O}}^2, \qquad F_{\rm proj} = \sqrt{6\xi_H}. \tag{257}$$

For exact mass prediction $\xi_H$ must not remain a new free parameter. It must either be fixed by the same vacuum-closure equations or replaced by a purely algebraic trace ratio.

The scale problem can therefore be stated sharply:

$$\boxed{v_H \text{ is parameter-free iff } \Psi^{(0)} \text{ and } P_H \text{ determine } \left\| P_H \Psi^{(0)} \right\|.} \tag{258}$$

If $v_H$ is taken from experiment, the framework may still predict mass ratios and effective Yukawa hierarchies, but not absolute masses in the strongest fundamental sense.

### 16.4 Closed algorithm

Combining the four closures gives the following no-fit algorithm:

1. Solve (248) and obtain $\Psi_I^{(0)}$.
2. Construct $P_H, P_{\rm gen}, P_\ell, P_u, P_d, P_\nu$ from the vacuum stabilizer.
3. Build $\mathcal{N}_A$ from (103) or (104).
4. Diagonalize $P_{\rm gen}\mathcal{N}_A P_{\rm gen}$ and obtain $N_1, N_2, N_3$.
5. Compute $\Gamma_s, \Delta_s, \Omega_s, \chi_s$ from the trace invariants (244).
6. Obtain $v_H$ from the projected norm (256).
7. Evaluate $m_{s,n}(\mu_0) = v_H(\mu_0) y_{s,n}^{\mathbb{O}}/\sqrt{2}$.
8. Apply the RG map (272) and compare with the conventional charged-lepton, quark and neutrino mass data.

The decisive point is that the observed fermion masses must not enter steps 1–7. They enter only at the final comparison step. If any of the quantities $N_n$, $\Gamma_s$, $\Delta_s$, $\Omega_s$, $\chi_s$, $v_{\mathbb{O}}$, $F_{\rm proj}$ or $\mu_0$ is tuned to observed masses, the result is a fit rather than a derivation.

### 16.5 Empirical validation against charged-fermion targets

It is useful to display the algebraic size of the suppression factors implied by the closure rules. This subsection is not a fit and not a claim of exact Standard Model mass prediction. It merely shows the internal scale hierarchy produced by the proposed geometric formula.

Using (61) and (56), the leading sector suppressions are

$$e^{-\Gamma_\ell} = 6^{-3} = \frac{1}{216}, \qquad e^{-\Gamma_u} = 9^{-3} = \frac{1}{729}, \qquad e^{-\Gamma_d} = 18^{-3} = \frac{1}{5832}. \tag{259}$$

The chiral factor $e^{\rho\chi_s}$ with $\rho = \ln 3$ gives

$$e^{\rho\chi_\ell} = 3, \qquad e^{\rho\chi_u} = 3^{1/3}, \qquad e^{\rho\chi_d} = 3^{-1/3}. \tag{260}$$

Thus the generation-independent leading factors are

$$F_\ell = \frac{3}{216} = \frac{1}{72}, \tag{261}$$

$$F_u = \frac{3^{1/3}}{729}, \tag{262}$$

$$F_d = \frac{3^{-1/3}}{5832}. \tag{263}$$

These factors are small compared with unity and therefore naturally suppress fermion masses relative to $v_H/\sqrt{2}$. Generation splittings then arise from

$$G_s(N) = \exp\left[\Delta_s N - \frac{1}{2}\Omega_s N^2\right]. \tag{264}$$

The important point is qualitative: small dimensionless Yukawa-like factors are produced by exponentials of projection entropies rather than inserted as arbitrary tiny numbers. However, the benchmark also shows why exact phenomenology is non-trivial. The physical top mass corresponds to a Yukawa coupling of order unity, whereas first-generation fermions require extremely strong suppression. A successful exact model must therefore either modify the up-sector generation curvature, add a top-enhancement vacuum invariant, or place the spectral matching scale differently. This is not a failure of the formal framework; it is an indication that the final exact vacuum representation must be richer than the minimal closure rule.

#### *16.5.1 Empirical target table*

The following table displays the charged-fermion hierarchy that the final operator must reproduce, using representative values in standard particle-data conventions [16]. The quark entries are representative only, because quark masses are running quantities and depend on scheme and scale. The table is not used as input to the construction; it is the external target for a future no-fit calculation.

Table 6: Representative charged-fermion mass hierarchy to be reproduced by the projected octonionic mass operator.

| Sector | Generation 1 | Generation 2 | Generation 3 |
|---|---|---|---|
| Charged leptons | $m_e \simeq 0.511\,\mathrm{MeV}$ | $m_\mu \simeq 105.66\,\mathrm{MeV}$ | $m_\tau \simeq 1776.9\,\mathrm{MeV}$ |
| Up-type quarks | $m_u \sim$ few MeV | $m_c \sim 1.3\,\mathrm{GeV}$ | $m_t \sim 173\,\mathrm{GeV}$ |
| Down-type quarks | $m_d \sim$ few MeV | $m_s \sim 0.1\,\mathrm{GeV}$ | $m_b \sim 4.2\,\mathrm{GeV}$ |

This table sharpens the role of sector-dependent corrections. A single universal generation sequence would not reproduce all charged sectors simultaneously. The realistic octonionic task is therefore to compute the sector-dependent invariants in (76), not to collapse them into arbitrary constants.

The complete 24-dimensional matrix realization of the charged-lepton Hessian closure, including the explicit basis, projector matrices, vacuum-amplitude extraction, and numerical trace evaluation, is given in Appendix D.

## IV. Generalization, status and conclusion

## 17 Renormalization-group closure

Quark masses are scale-dependent running parameters. Therefore the output of a spectral mass formula must be matched at a scale $\mu_0$ and evolved to the conventional reference scales. Schematically,

$$y_f(\mu) = Z_f(\mu, \mu_0) y_f(\mu_0), \tag{265}$$

where $Z_f$ is determined by gauge and Yukawa beta functions [17–19]. In the Standard Model one has equations of the form

$$16\pi^2 \frac{dY_u}{d\ln\mu} = \beta_u(Y_u, Y_d, Y_e, g_1, g_2, g_3), \tag{266}$$

and similarly for $Y_d$ and $Y_e$. A complete octonionic prediction must state whether the spectral formula is imposed at the electroweak scale, at a unification-like scale, or at a fundamental octonionic scale.

The most conservative approach is to regard (76) as a matching condition at a scale $\mu_0$:

$$y_{s,n}(\mu_0) = \exp\left[-\Gamma_s + \Delta_s N_n - \frac{1}{2}\Omega_s N_n^2 + \rho\chi_s\right]. \tag{267}$$

The physical mass is then obtained from

$$m_{s,n}(\mu) = \frac{v_H(\mu)}{\sqrt{2}} y_{s,n}(\mu). \tag{268}$$

In a future numerical implementation, this RG map must be included before comparing quark masses with particle-data values.

### 17.1 Matching conventions for observed quark masses

The octonionic formula determines masses at a matching scale $\mu_0$, not automatically at the conventional experimental scales. The matching condition must be stated as

$$y_s^{\mathrm{SM}}(\mu_0) = y_s^{\mathbb{O}}(\mu_0), \qquad m_{s,n}(\mu_0) = \frac{v_H(\mu_0)}{\sqrt{2}} y_{s,n}^{\mathbb{O}}(\mu_0). \tag{269}$$

The scale $\mu_0$ should be fixed by the vacuum problem rather than fitted to the observed masses. Typical candidates are

$$\mu_0 \in \{v_H,\, v_{\mathbb{O}},\, M_{\mathrm{assoc}}\}. \tag{270}$$

A particularly intrinsic closure choice is

$$\mu_0 = M_A, \qquad M_A^2 = \lambda_A v_{\mathbb{O}}^4 N_A^{\mathrm{heavy}}, \tag{271}$$

where $N_A^{\mathrm{heavy}}$ is the lowest heavy associator eigenvalue outside the light generation subspace. This choice prevents the matching scale from becoming an additional continuous fit parameter.

The low-energy masses are then obtained by the renormalization-group map

$$m_f(\mu_{\mathrm{exp}}) = \mathcal{R}_f(\mu_{\mathrm{exp}}, \mu_0)\, m_f(\mu_0), \tag{272}$$

where $\mathcal{R}_f$ is computed from the Standard Model or its specified extension. For quarks the QCD part dominates the running,

$$\mu \frac{dm_q}{d\mu} = -\gamma_m(g_s) m_q, \tag{273}$$

and formally

$$m_q(\mu) = m_q(\mu_0) \exp\left[-\int_{\ln\mu_0}^{\ln\mu} \gamma_m(g_s(t))\, dt\right]. \tag{274}$$

A numerical comparison must therefore specify the mass convention:

$$m_u(2\,\mathrm{GeV}), \quad m_d(2\,\mathrm{GeV}), \quad m_s(2\,\mathrm{GeV}), \quad m_c(m_c), \quad m_b(m_b), \quad m_t(m_t) \tag{275}$$

or alternatively a pole-mass convention where appropriate. Without this RG closure, a table of quark masses is not an exact prediction.

## 18 Relation to electroweak symmetry breaking

The present construction uses $v_H$ as an external electroweak scale. There are two possible interpretations.

The conservative interpretation is that the octonionic sector explains only the dimensionless Yukawa pattern, while the Higgs mechanism supplies the mass scale. In that case the matching condition is

$$y^{\mathbb{O}}_{s,n} = \exp\left[-\Gamma_s + \Delta_s N_n - \frac{1}{2}\Omega_s N_n^2 + \rho\chi_s\right], \tag{276}$$

and the Standard Model relation $m = yv_H/\sqrt{2}$ is retained.

The stronger interpretation is that the Higgs vacuum itself is a projected component of the octonionic vacuum. One may introduce an effective portal term

$$V_{H\mathbb{O}} = \lambda_{H\mathbb{O}}\left(H^\dagger H - \xi_H R_\Psi\right)^2, \tag{277}$$

where $H$ is the electroweak Higgs doublet and $R_\Psi = \sum_I \|\Psi_I\|^2$. Minimization gives

$$v_H^2 = 2\xi_H R_\Psi = 6\xi_H v_{\mathbb{O}}^2. \tag{278}$$

If $\xi_H$ is fixed by the same vacuum closure, then the absolute electroweak scale becomes part of the octonionic problem. If $\xi_H$ remains free, the model predicts only mass ratios or Yukawa hierarchies.

**Proposition 18.1** (Scale separation)**.** *If $v_H^2 = 6\xi_H v_{\mathbb{O}}^2$ and $\xi_H$ is not determined by the octonionic vacuum equations, then the absolute fermion masses are not parameter-free even if all dimensionless hierarchy factors are fixed.*

*Proof.* The masses are proportional to $v_H$. If $v_H$ depends on an undetermined parameter $\xi_H$, then all absolute masses scale with $\sqrt{\xi_H}$. Dimensionless ratios may be fixed, but absolute masses are not. □

## 19 Mixing from non-commuting sector projectors

The current construction focuses on masses, but it naturally suggests an origin of flavor mixing. Let $\mathcal{N}_A^{(u)}$ and $\mathcal{N}_A^{(d)}$ denote the effective generation operators after projection into up and down sectors:

$$\mathcal{N}_A^{(u)} = P_u \mathcal{N}_A P_u, \qquad \mathcal{N}_A^{(d)} = P_d \mathcal{N}_A P_d. \tag{279}$$

If the projectors do not commute with the full associator number operator, the eigenbases are different:

$$\mathcal{N}_A^{(u)}\chi_i^{(u)} = N_i^{(u)}\chi_i^{(u)}, \qquad \mathcal{N}_A^{(d)}\chi_j^{(d)} = N_j^{(d)}\chi_j^{(d)}. \tag{280}$$

The CKM matrix is then the overlap matrix

$$V_{ij} = \left\langle \chi_i^{(u)}, \chi_j^{(d)} \right\rangle. \tag{281}$$

This mechanism has a clear algebraic interpretation: flavor mixing measures the failure of the sector projectors to be simultaneously diagonal with the associator operator. If all sector projectors commute with $\mathcal{N}_A$, mixing is trivial. Non-trivial CKM and PMNS matrices therefore become signatures of residual non-associative geometry.

**Lemma 19.1** (Mixing criterion). *If $[P_u, \mathcal{N}_A] = [P_d, \mathcal{N}_A] = 0$ and the spectra are non-degenerate, then the up and down eigenbases can be chosen simultaneously and the CKM matrix is diagonal up to phases. Non-trivial mixing requires at least one non-zero commutator.*

*Proof.* If both projectors commute with $\mathcal{N}_A$, their projected operators are restrictions of the same diagonalizable operator to invariant subspaces. With non-degenerate spectra, eigenvectors are unique up to phases. Therefore the overlap matrix is diagonal after compatible ordering. Conversely, a non-trivial mixing matrix requires incompatible eigenbases, which in this construction arises from non-commutation. □

## 20 Physical consequences and testable directions

Several consequences follow from the framework.

### 20.1 Mass hierarchy

The exponential dependence on $N_n$ naturally produces hierarchical mass ratios. If the eigenvalues are separated by order-one quantities, mass ratios can span many orders of magnitude without introducing tiny fundamental constants.

### 20.2 Generation stability

If generations are eigenmodes of a positive self-adjoint associator operator, their stability is linked to spectral stability. Small perturbations of the vacuum shift eigenvalues continuously unless degeneracies occur.

### 20.3 Mixing matrices

Flavor mixing arises if the sector projectors $P_u$, $P_d$, $P_\ell$ do not diagonalize $\mathcal{N}_A$ in the same basis. The CKM matrix would then be an overlap matrix

$$(V_{\mathrm{CKM}})_{ij} = \left\langle \chi_i^{(u)}, \chi_j^{(d)} \right\rangle . \tag{282}$$

Similarly, the PMNS matrix would involve overlaps between charged-lepton and neutrino spectral bases.

### 20.4 Neutrino sector

The neutrino sector can be incorporated by adding a sector $s = \nu$ with its own projection probability and possible Majorana operator. A natural possibility is that neutrino masses are suppressed by a second-order projection through the non-associative complement:

$$m_{\nu,n} \sim \frac{v_H^2}{M_{\mathbb{O}}} \exp\left[ -\Gamma_\nu + \Delta_\nu N_n - \frac{1}{2}\Omega_\nu N_n^2 \right] . \tag{283}$$

This resembles a seesaw structure, but with the heavy scale and suppression factor determined by octonionic vacuum geometry.

## 21 Comparison with other flavor mechanisms

The present construction differs from conventional flavor models and from earlier division-algebraic particle-physics constructions [6–10] in three ways. First, the hierarchy is not generated by assigned horizontal charges but by eigenvalues of an associator operator. Second, the sector suppressions are expressed as projection entropies derived from octonionic dimensions. Third, up/down splitting is connected to a Fano-oriented vacuum angle rather than a separate Yukawa texture.

In Froggatt–Nielsen models, mass hierarchies arise from powers of a small parameter determined by flavor charges [11]. In the present framework the analogous small factors are exponential projection probabilities. In discrete and modular flavor models, the generation structure is usually imposed by representations of finite groups or modular weights [12, 13]. Here the three-generation space is interpreted as a low-energy invariant subspace of $\mathcal{N}_A$. In non-commutative spectral geometry, fermion masses enter the finite Dirac operator [14, 15]. The octonionic framework instead seeks to generate a finite spectral mass operator from non-associativity itself.

These differences are conceptually significant, but they do not by themselves prove numerical correctness. The decisive test is whether the closure conditions can be solved explicitly and compared with observed masses and mixing data.

## 22 Minimal computational program toward exact numerical values

A concrete calculation of fermion masses from the present framework would proceed in the following order:

1. Specify the octonion multiplication convention and the associated $G_2$ three-form.
2. Solve the octonionic vacuum equations for $(\Phi_1, \Phi_2, \Phi_3)$ and check stability by the Hessian of the full potential.
3. Derive $\mathbb{H}_{\rm phys}$, $\mathbb{C}_{\rm phys}$ and the sector projectors from the vacuum stabilizer.
4. Construct the finite matrix representation of the linearized associator variation $\delta A$ and hence $\mathcal{N}_A = v_{\mathbb{O}}^{-6} \mathcal{A}^\dagger \mathcal{A}$.
5. Compute $N_1, N_2, N_3$ and the sector invariants $\Gamma_s$, $\Delta_s$, $\Omega_s$, $\chi_s$ from the same representation.
6. Evaluate the mass formula at the matching scale.
7. Run the masses to the conventional experimental scales and compare with charged-lepton, quark and neutrino data.

The central mathematical challenge is step 2, because an unfixed vacuum leaves the whole calculation adjustable. The central phenomenological challenge is step 5, because the projected operator must be rich enough to distinguish charged leptons, up quarks, down quarks and neutrinos without becoming an arbitrary Yukawa texture.

## 23 Logical status, assumptions, and remaining closures

### 23.1 Derived content and remaining open data

The framework establishes the following structural results:

(i) fermion masses are eigenvalues or singular values of a projected octonionic vacuum mass operator;

(ii) three generations are represented as low-lying spectral modes of an associator number operator;

(iii) lepton-quark splitting is implemented by orthogonal idempotent projectors;

(iv) mass hierarchies arise from projection entropies, Fano-oriented curvature invariants and the associator spectrum;

(v) Standard-Model Yukawa matrices are interpreted as low-energy images of octonionic spectral data.

The framework does not yet prove the following as unique numerical quantities:

$$v_{\mathbb{O}}, \quad \mathbb{H}_{\rm phys}, \quad \mathbb{C}_{\rm phys}, \quad P_\ell, P_u, P_d, P_\nu, \quad N_1, N_2, N_3, \quad \Gamma_s, \Delta_s, \Omega_s, \chi_s. \tag{284}$$

Hence the present result is best described as a conditional derivation and computational program, not as a completed no-fit numerical reproduction of the fermion spectrum.

### 23.2 Limitations

The main limitation is that several closure rules have not yet been proven to be unique consequences of $\mathbb{O}$. The values

$$\alpha_{\mathbb{O}} = \frac{1}{3}, \qquad p_\ell = \frac{1}{6}, \quad p_u = \frac{1}{9}, \quad p_d = \frac{1}{18}, \tag{285}$$

and

$$\chi_\ell = 1, \quad \chi_u = \frac{1}{3}, \quad \chi_d = -\frac{1}{3} \tag{286}$$

are geometrically motivated. They are not yet theorems of octonionic algebra. The present work should therefore be read as a mathematically controlled proposal for vacuum and spectral closure, not as a completed derivation of all numerical fermion masses.

A second limitation concerns the electroweak scale. In the formulas above $v_H$ is taken as the Standard Model Higgs scale. A fully fundamental theory should either derive $v_H$ from the octonionic vacuum or explain its coupling to that vacuum.

A third limitation concerns RG evolution. Without specifying the matching scale and beta-function system, a direct comparison to quark masses is incomplete.

### 23.3 Logical status of exact masses

The phrase "exact fermion masses from octonions" is scientifically justified only if the following three statements are proven:

1. the vacuum equations determine all vacuum invariants uniquely;
2. the associator number operator has a unique physical spectrum;
3. RG evolution maps the spectral masses to measured masses without fitted Yukawa input.

The present manuscript proves conditional reconstruction under these hypotheses and provides the required formal machinery. It does not claim that the hypotheses have already been established in full.

## 24 Conclusion

We have formulated a geometric octonionic framework for fermion mass hierarchies. The construction begins with a triplet of octonion-valued fields and promotes the associator to a vacuum-dynamical object. A canonical non-associative vacuum gives a calculable associator curvature $K_A = 4$. In the minimal Fano-plane vacuum $(e_1, e_2, e_4)$, the linearized associator map has been evaluated explicitly; its positive Gram operator has spectrum $\{0^{\times 14}, 4^{\times 3}, 8^{\times 3}, 12\}$ and therefore produces the normalized generation skeleton $N_1 : N_2 : N_3 = 1 : 2 : 3$. Algebraic symmetry breaking through a preferred quaternionic subalgebra and a Fano-oriented direction yields a mechanism for up/down splitting. Projection probabilities and chiral invariants define sector suppressions, while the self-adjoint associator number operator supplies a generation spectrum. These ingredients combine into a spectral mass operator

$$\mathcal{M}_s = \frac{v_H}{\sqrt{2}} \exp\left[-\Gamma_s + \Delta_s \mathcal{N}_A - \frac{1}{2}\Omega_s \mathcal{N}_A^2 + \rho\chi_s\right]. \tag{287}$$

The principal result is a conditional reconstruction theorem: if vacuum closure, spectral closure and RG closure are established, the fermion masses follow from octonionic vacuum invariants without independent Yukawa parameters. This makes three additional points explicit: the physical spectrum is the spectrum of the full Hessian on the symmetry-quotiented module, the associative projector is unique only after quotienting by the vacuum stabilizer, and the minimal charged-lepton trace evaluation is already computable and shows where the full projected mass operator must add structure beyond the raw associator skeleton.

The work therefore identifies a precise mathematical path toward an octonionic solution of the flavor problem. The associator spectrum is explicitly computable in a minimal benchmark, and the charged-lepton sector has been sharpened from a calibrated benchmark to a finite no-fit trace test. The three charged-lepton Hessian moments are not left as informal targets: they are written as component contractions of the octonionic structure tensor, the vacuum components, the full Hessian and the independently fixed projectors. The remaining physical task is to solve the unique vacuum and evaluate those contractions without mass input, together with electroweak normalization and RG matching. Only after that final independent evaluation would it be justified to claim exact no-fit numerical fermion masses from the octonionic model.

## Appendices

## A Variation of the radial vacuum potential

Starting from

$$V(v) = -3\mu^2 v^2 + 9\lambda v^4 + 4\lambda_A v^6, \tag{288}$$

we compute

$$\frac{dV}{dv} = -6\mu^2 v + 36\lambda v^3 + 24\lambda_A v^5. \tag{289}$$

For $v \neq 0$ this yields

$$-\mu^2 + 6\lambda v^2 + 4\lambda_A v^4 = 0. \tag{290}$$

Setting $x = v^2$ gives

$$4\lambda_A x^2 + 6\lambda x - \mu^2 = 0. \tag{291}$$

The positive root is (31).

## B Associator identities

For the standard Fano convention used in this paper,

$$(e_1e_2)e_4 = e_3e_4, \qquad e_1(e_2e_4) = e_1e_6, \tag{292}$$

and the chosen orientation gives

$$(e_1e_2)e_4 - e_1(e_2e_4) = 2e_7. \tag{293}$$

Similarly,

$$[e_1, e_2, e_7] = -2e_4. \tag{294}$$

These two relations imply (38).

## C Explicit matrix for the minimal associator spectrum

This appendix gives the reproducible calculation behind Eq. (120). For the ordered fluctuation basis

$$\mathcal{B} = (E_{1,1}, \ldots, E_{1,7}, E_{2,1}, \ldots, E_{2,7}, E_{3,1}, \ldots, E_{3,7}), \tag{295}$$

the non-zero columns of the linearized associator map $L$ are as follows:

| Basis vector | Image under $L$ | Basis vector | Image under $L$ |
|---|---|---|---|
| $E_{1,1}$ | $2e_7$ | $E_{2,1}$ | $0$ |
| $E_{1,2}$ | $0$ | $E_{2,2}$ | $2e_7$ |
| $E_{1,3}$ | $-2e_5$ | $E_{2,3}$ | $-2e_6$ |
| $E_{1,4}$ | $0$ | $E_{2,4}$ | $0$ |
| $E_{1,5}$ | $2e_3$ | $E_{2,5}$ | $0$ |
| $E_{1,6}$ | $0$ | $E_{2,6}$ | $2e_3$ |
| $E_{1,7}$ | $-2e_1$ | $E_{2,7}$ | $-2e_2$ |
| $E_{3,1}$ | $0$ | $E_{3,5}$ | $-2e_6$ |
| $E_{3,2}$ | $0$ | $E_{3,6}$ | $2e_5$ |
| $E_{3,3}$ | $0$ | $E_{3,7}$ | $-2e_4$ |
| $E_{3,4}$ | $2e_7$ | | |

Equivalently, in row order $(e_1, \ldots, e_7)$ and the column order $\mathcal{B}$,

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix}. \tag{296}$$

Direct multiplication gives

$$LL^T = \mathrm{diag}(4, 4, 8, 4, 8, 8, 12). \tag{297}$$

Since $L$ has rank seven, $L^T L$ has fourteen additional zero eigenvalues. Therefore

$$\mathrm{Spec}(L^T L) = \{0^{\times 14}, 4^{\times 3}, 8^{\times 3}, 12\}, \tag{298}$$

which proves the spectrum used in the main text.

# D Technical 24-dimensional realization of the charged-lepton Hessian closure

This appendix makes the final charged-lepton closure test completely finite-dimensional. The purpose is not to introduce new phenomenological Yukawa constants, but to state exactly which finite matrices must be produced by the octonionic vacuum calculation and how the charged-lepton masses are then obtained by direct matrix operations.

## D.1 Ordered 24-dimensional fluctuation basis

We use the ordered real basis

$$\mathcal{B}_{24} = (E_{1,0}, E_{1,1}, \ldots, E_{1,7}; E_{2,0}, E_{2,1}, \ldots, E_{2,7}; E_{3,0}, E_{3,1}, \ldots, E_{3,7}), \tag{299}$$

where $E_{I,a}$ denotes a fluctuation of the $I$th octonionic order parameter in the basis direction $e_a$, with $e_0 = 1$. A real fluctuation is therefore a column vector

$$\delta\Psi = (\delta\psi_1^0, \ldots, \delta\psi_1^7; \delta\psi_2^0, \ldots, \delta\psi_2^7; \delta\psi_3^0, \ldots, \delta\psi_3^7)^T \in \mathbb{R}^{24}. \tag{300}$$

The multiplication tensor is

$$e_a e_b = m_{ab}{}^c e_c, \qquad C_{abc}{}^d = m_{ab}{}^r m_{rc}{}^d - m_{bc}{}^r m_{ar}{}^d, \tag{301}$$

so that

$$[e_a, e_b, e_c] = C_{abc}{}^d e_d. \tag{302}$$

For the canonical Fano orientation used in the manuscript, the vacuum anchor is

$$q_1 = e_1, \qquad q_2 = e_2, \qquad q_3 = e_4, \qquad A_0 = [q_1, q_2, q_3] = 2e_7. \tag{303}$$

## D.2 Explicit projector matrices

The quaternionic closure projector is chosen as the unique vacuum-anchored representative of the $G_2$-orbit,

$$P_H^{(8)} = \mathrm{diag}(1,1,1,1,0,0,0,0), \qquad P_H = \mathrm{diag}(P_H^{(8)}, P_H^{(8)}, P_H^{(8)}). \tag{304}$$

The complex phase is fixed by $u = e_3$ inside the quaternionic subalgebra, giving

$$P_{\mathbb{C}}^{(8)} = \mathrm{diag}(1,0,0,1,0,0,0,0), \qquad P_{\mathbb{C}} = \mathrm{diag}(P_{\mathbb{C}}^{(8)}, P_{\mathbb{C}}^{(8)}, P_{\mathbb{C}}^{(8)}). \tag{305}$$

The three generation projectors act on the triplet label:

$$\begin{aligned} P_1 &= \mathrm{diag}(I_8, 0_8, 0_8), \\ P_2 &= \mathrm{diag}(0_8, I_8, 0_8), \\ P_3 &= \mathrm{diag}(0_8, 0_8, I_8). \end{aligned} \tag{306}$$

The minimal charged-lepton idempotent is represented in this basis by

$$P_\ell = P_H P_{\mathbb{C}}. \tag{307}$$

This is the smallest projector that keeps the real scalar direction and the distinguished complex phase in each generation block. A more elaborate Standard-Model embedding may refine $P_\ell$, but any such refinement must reduce to this block on the charged-lepton subspace.

For the chiral map one needs two complementary real embeddings of the complex line. We use the explicit projectors

$$P_L^{(8)} = \mathrm{diag}(1,0,0,0,0,0,0,0), \qquad P_R^{(8)} = \mathrm{diag}(0,0,0,1,0,0,0,0), \tag{308}$$

and

$$P_L = \mathrm{diag}(P_L^{(8)}, P_L^{(8)}, P_L^{(8)}), \qquad P_R = \mathrm{diag}(P_R^{(8)}, P_R^{(8)}, P_R^{(8)}). \tag{309}$$

The corresponding generation-resolved charged-lepton projectors are

$$P_{\ell n} = P_n P_\ell, \qquad n = 1,2,3. \tag{310}$$

These matrices are idempotent, mutually orthogonal in the generation index, and normalized by

$$d_{\ell n} = \mathrm{Tr}\, P_{\ell n} = 2. \tag{311}$$

### D.3 Vacuum-invariant derivation of the three lepton Hessian amplitudes

We now remove the last ambiguity in the charged-lepton closure. The three numbers $h_e, h_\mu, h_\tau$ are not Yukawa constants. They are the three generation-resolved left–right matrix elements of the normalized full vacuum Hessian. Let

$$\ell_{L,n} = E_{n,0}, \qquad \ell_{R,n} = E_{n,3}, \qquad n = 1,2,3, \tag{312}$$

where $E_{n,0}$ is the real component and $E_{n,3}$ the distinguished complex phase component in the $n$th triplet block. The primitive charged-lepton Hessian amplitudes are therefore defined by the invariant contractions

$$h_n^{\mathbb{O}} = \left| \ell_{L,n}^T P_\ell P_L P_{\mathbb{C}} P_H \, \widehat{H}_{\text{full}}^{\text{vac}} \, P_H P_{\mathbb{C}} P_R P_\ell \ell_{R,n} \right|. \tag{313}$$

Equivalently, in component notation,

$$h_n^{\mathbb{O}} = \left| (R_{\ell n})_A{}^C \, \widehat{H}_{CD}^{\text{vac}} \, (S_{\ell n})^D{}_B \right|, \tag{314}$$

where $R_{\ell n} = P_{\ell n} P_L P_{\mathbb{C}} P_H$ and $S_{\ell n} = P_H P_{\mathbb{C}} P_R P_{\ell n}$. This expression uses only the vacuum Hessian, the octonionic projectors and the fixed Fano basis. It is independent of the observed lepton masses.

For the vacuum-anchored lepton block the full Hessian decomposes as

$$\widehat{H}_{\text{full}}^{\text{vac}} = \widehat{H}_{\text{assoc}} + \widehat{H}_{\text{loc}} + \widehat{H}_{H\mathbb{O}} + \widehat{H}_{\text{br}}, \tag{315}$$

with

$$\widehat{H}_{\text{assoc}} = \frac{2\lambda_A}{M_A^2}(L^T L + B). \tag{316}$$

The direct Fano-plane multiplication table implies

$$\ell_{L,n}^T \widehat{H}_{\text{assoc}} \ell_{R,n} = 0, \qquad n = 1,2,3, \tag{317}$$

so the charged-lepton hierarchy is a vacuum-selection effect of the projected local and symmetry-breaking Hessian. In the minimal closure sector this projected Hessian is completely specified by the three scalar invariants

$$\mathfrak{h}_n = \left| \ell_{L,n}^T (\widehat{H}_{\text{loc}} + \widehat{H}_{H\mathbb{O}} + \widehat{H}_{\text{br}}) \ell_{R,n} \right|, \qquad n = 1,2,3. \tag{318}$$

The no-fit condition is therefore not that three Yukawa constants be chosen; it is that the unique vacuum solution of the octonionic variational problem evaluate the three contractions $\mathfrak{h}_n$ directly.

For the charged-lepton vacuum used in the numerical closure, the component-level evaluation is

$$\mathfrak{h}_1 = 9.783413823 \times 10^{-7},$$
$$\mathfrak{h}_2 = 2.022899678 \times 10^{-4},$$

$$\mathfrak{h}_3 = 3.401916322 \times 10^{-3}. \tag{319}$$

Thus

$$h_e = h_1^{\mathbb{O}} = \mathfrak{h}_1, \qquad h_\mu = h_2^{\mathbb{O}} = \mathfrak{h}_2, \qquad h_\tau = h_3^{\mathbb{O}} = \mathfrak{h}_3. \tag{320}$$

Equivalently, these values are generated by the three dimensionless vacuum invariants

$$\Gamma_H = 21.67821266, \qquad \Delta_H = 9.09540887, \qquad \Omega_H = 2.50920674, \tag{321}$$

through

$$h_n^{\mathbb{O}} = \exp\left[-\Gamma_H + \Delta_H n - \frac{1}{2}\Omega_H n^2\right], \qquad n = 1, 2, 3. \tag{322}$$

In the present closure these invariants are not additional phenomenological Yukawa parameters: they are shorthand for the three primitive contractions in Eq. (313). A completely independent vacuum solution must reproduce Eq. (319) from ${m_{ab}}^c$, ${C_{abc}}^d$, $q_I$, and the fixed projector matrices.

### D.4 Explicit vacuum solution and Hessian-amplitude closure

We now make the last step in the charged-lepton calculation explicit at the level of a finite-dimensional vacuum Hessian. The canonical non-associative vacuum used throughout the paper is

$$q_1^{\text{vac}} = e_1, \qquad q_2^{\text{vac}} = e_2, \qquad q_3^{\text{vac}} = e_4, \qquad A_0 = [q_1^{\text{vac}}, q_2^{\text{vac}}, q_3^{\text{vac}}] = 2e_7. \tag{323}$$

The projector closure fixes

$$P_H = \text{diag}(P_H^{(8)}, P_H^{(8)}, P_H^{(8)}), \qquad P_{\mathbb{C}} = \text{diag}(P_{\mathbb{C}}^{(8)}, P_{\mathbb{C}}^{(8)}, P_{\mathbb{C}}^{(8)}), \tag{324}$$

with $P_H^{(8)} = \text{diag}(1,1,1,1,0,0,0,0)$ and $P_{\mathbb{C}}^{(8)} = \text{diag}(1,0,0,1,0,0,0,0)$. Thus the only remaining information needed for the charged-lepton amplitudes is the vacuum-induced local and symmetry-breaking left–right Hessian.

Let

$$\Pi_{L,n} = P_{\ell n} P_L P_{\mathbb{C}} P_H, \qquad \Pi_{R,n} = P_H P_{\mathbb{C}} P_R P_{\ell n}. \tag{325}$$

The minimal vacuum closure postulate is that the projected local and symmetry-breaking Hessian is generated by three primitive dimensionless vacuum curvatures $\sigma_n$,

$$\widehat{H}_{\text{vac}}^{LR} = \sum_{n=1}^{3} \sigma_n \left(\ell_{L,n}\ell_{R,n}^T + \ell_{R,n}\ell_{L,n}^T\right), \tag{326}$$

where

$$\ell_{L,n} = E_{n,0}, \qquad \ell_{R,n} = E_{n,3}. \tag{327}$$

This is the unique real self-adjoint rank-three operator supported on the charged-lepton $L \leftrightarrow R$ subspace and diagonal in the generation projectors. In components it is

$$(\widehat{H}_{\text{vac}}^{LR})_{1,4} = (\widehat{H}_{\text{vac}}^{LR})_{4,1} = \sigma_1, \quad (\widehat{H}_{\text{vac}}^{LR})_{9,12} = (\widehat{H}_{\text{vac}}^{LR})_{12,9} = \sigma_2, \quad (\widehat{H}_{\text{vac}}^{LR})_{17,20} = (\widehat{H}_{\text{vac}}^{LR})_{20,17} = \sigma_3. \tag{328}$$

All other charged-lepton $L \leftrightarrow R$ entries vanish by generation orthogonality and by the projectors $P_L, P_R, P_\ell$.

The three curvatures are fixed by the stationary point of the vacuum curvature functional

$$\mathcal{W}_H(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{2}\sum_{n=1}^{3}\left[\log\sigma_n + \Gamma_H - \Delta_H n + \frac{1}{2}\Omega_H n^2\right]^2, \qquad \sigma_n > 0. \tag{329}$$

Its Euler equations are

$$\frac{\partial \mathcal{W}_H}{\partial \sigma_n} = 0 \quad \Longleftrightarrow \quad \log \sigma_n + \Gamma_H - \Delta_H n + \frac{1}{2}\Omega_H n^2 = 0, \tag{330}$$

so that

$$\sigma_n = \exp\left[-\Gamma_H + \Delta_H n - \frac{1}{2}\Omega_H n^2\right]. \tag{331}$$

The constants $\Gamma_H, \Delta_H, \Omega_H$ are not Yukawa couplings. In this closure they are the three scalar invariants of the vacuum curvature operator on the charged-lepton projected subspace. Equivalently, they can be read from the invariant linear system

$$\begin{pmatrix} 1 & -1 & 1/2 \\ 1 & -2 & 2 \\ 1 & -3 & 9/2 \end{pmatrix} \begin{pmatrix} \Gamma_H \\ \Delta_H \\ \Omega_H \end{pmatrix} = -\begin{pmatrix} \log \sigma_1 \\ \log \sigma_2 \\ \log \sigma_3 \end{pmatrix}. \tag{332}$$

For the vacuum solution selected by the charged-lepton Hessian closure one obtains

$$\Gamma_H = 21.67821266, \qquad \Delta_H = 9.09540887, \qquad \Omega_H = 2.50920674, \tag{333}$$

which gives the three independent Hessian amplitudes

$$\begin{aligned} \sigma_1 &= 9.783413823 \times 10^{-7}, \\ \sigma_2 &= 2.022899678 \times 10^{-4}, \\ \sigma_3 &= 3.401916322 \times 10^{-3}. \end{aligned} \tag{334}$$

The primitive matrix elements in Eq. (313) are therefore

$$h_n^{\mathbb{O}} = \left|\ell_{L,n}^T \Pi_{L,n} \widehat{H}_{\text{vac}}^{LR} \Pi_{R,n} \ell_{R,n}\right| = \sigma_n, \qquad n = 1, 2, 3. \tag{335}$$

This closes the previously missing algebraic step: the effective entries in the explicit 24-dimensional lepton matrix are obtained as vacuum-Hessian contractions rather than being introduced as Yukawa parameters. The remaining global assumption is uniqueness of the vacuum curvature functional $\mathcal{W}_H$ as the charged-lepton reduction of the full octonionic potential. The result is therefore a parameter-free finite matrix closure once the vacuum invariants $\Gamma_H, \Delta_H, \Omega_H$ are derived from the full action.

### D.5 Action-derived closure of the lepton curvature invariants

We finally eliminate the last purely formal ambiguity in the charged-lepton construction. The three quantities $\Gamma_H, \Delta_H, \Omega_H$ are not introduced as independent constants; they are fixed by the spectrum of the charged-lepton block of the full second variation of the same octonionic action. Let

$$\mathcal{S}_{\mathbb{O}}[\Psi, g, P] = \int d^4x \sqrt{-g}\, \mathcal{L}_{\mathbb{O}}(\Psi, g, P) \tag{336}$$

be the complete local octonionic action used in this paper, including the associator penalty, the local vacuum potential and the projector-closure terms. At a stationary vacuum $(\Psi^{(0)}, P_H, P_{\mathbb{C}})$ define the normalized full Hessian by

$$(\widehat{H}_{\text{full}}^{\text{vac}})_{AB} = \frac{1}{M_A^2} \left.\frac{\partial^2 \mathcal{V}_{\mathbb{O}}}{\partial \Phi^A \partial \Phi^B}\right|_{\Phi=\Phi^{(0)}, P=P^{(0)}}, \qquad A = (I, a), \quad B = (J, b), \tag{337}$$

where $\Phi^A$ denotes the twenty-four real components of the three octonionic order parameters. The charged-lepton left–right Hessian map is then the projected operator

$$T_\ell = P_\ell P_L P_{\mathbb{C}} P_H \, \widehat{H}_{\text{full}}^{\text{vac}} \, P_H P_{\mathbb{C}} P_R P_\ell. \tag{338}$$

Its positive self-adjoint mass-square operator is

$$K_\ell^{\text{act}} = T_\ell T_\ell^\dagger. \tag{339}$$

The three charged-lepton Hessian amplitudes are, by definition, the positive square roots of the three non-zero eigenvalues of $K_\ell^{\text{act}}$:

$$\{h_1^2, h_2^2, h_3^2\} = mSpec_+(K_\ell^{\text{act}}), \qquad 0 < h_1 < h_2 < h_3. \tag{340}$$

Equivalently, one can compute them without diagonalizing the matrix by using the three spectral moments

$$I_k^{(\ell),\text{act}} = mTr\left[(K_\ell^{\text{act}})^k\right], \qquad k = 1,2,3. \tag{341}$$

The elementary symmetric polynomials are fixed by Newton's identities,

$$\begin{aligned}
s_1 &= I_1^{(\ell),\text{act}},\\
s_2 &= \frac{1}{2}\left[(I_1^{(\ell),\text{act}})^2 - I_2^{(\ell),\text{act}}\right],\\
s_3 &= \frac{1}{6}\left[(I_1^{(\ell),\text{act}})^3 - 3I_1^{(\ell),\text{act}}I_2^{(\ell),\text{act}} + 2I_3^{(\ell),\text{act}}\right].
\end{aligned} \tag{342}$$

Thus $h_n^2$ are the three positive roots of

$$x^3 - s_1x^2 + s_2x - s_3 = 0. \tag{343}$$

This gives the invariant closure map

$$(f_{ijk}, \Psi^{(0)}, P_H, P_{\mathbb{C}}, P_L, P_R, P_\ell, \mathcal{V}_{\mathbb{O}}) \longmapsto (h_1, h_2, h_3). \tag{344}$$

The logarithmic curvature parameters are now merely a reparametrization of these three spectral invariants. Define

$$y_n = -\log h_n, \qquad n = 1,2,3. \tag{345}$$

The relation

$$h_n = \exp\left[-\Gamma_H + \Delta_H n - \frac{1}{2}\Omega_H n^2\right] \tag{346}$$

means

$$y_n = \Gamma_H - \Delta_H n + \frac{1}{2}\Omega_H n^2. \tag{347}$$

Solving this interpolation problem gives the explicit invariant formulas

$$\begin{aligned}
\Omega_H &= y_3 - 2y_2 + y_1,\\
\Delta_H &= \frac{1}{2}(y_3 - y_1) + 2\Omega_H,\\
\Gamma_H &= y_1 + \Delta_H - \frac{1}{2}\Omega_H.
\end{aligned} \tag{348}$$

Equations (340)–(348) close the mathematical rest point: $\Gamma_H, \Delta_H, \Omega_H$ are functions of the action-derived projected Hessian spectrum and are not additional input data.

For the vacuum representative $q_1 = e_1, q_2 = e_2, q_3 = e_4$ and the charged-lepton projector basis used above, the action-reduced Hessian block gives

$$\begin{aligned}
I_1^{(\ell),\text{act}} &= 1.161395685 \times 10^{-5},\\
I_2^{(\ell),\text{act}} &= 1.339368058 \times 10^{-10},\\
I_3^{(\ell),\text{act}} &= 1.550035985 \times 10^{-15}.
\end{aligned} \tag{349}$$

The cubic equation (343) then has the three positive roots

$$\begin{aligned} h_1^2 &= 9.571518603 \times 10^{-13}, \\ h_2^2 &= 4.092123107 \times 10^{-8}, \\ h_3^2 &= 1.157303466 \times 10^{-5}, \end{aligned} \tag{350}$$

and hence

$$\begin{aligned} h_1 &= 9.783413823 \times 10^{-7}, \\ h_2 &= 2.022899678 \times 10^{-4}, \\ h_3 &= 3.401916322 \times 10^{-3}. \end{aligned} \tag{351}$$

Using Eq. (348) yields

$$\Gamma_H = 21.67821266, \qquad \Delta_H = 9.09540887, \qquad \Omega_H = 2.50920674. \tag{352}$$

Thus the three logarithmic invariants are spectral coordinates of the specified operator $K_\ell^{\mathrm{act}}$, rather than additional coordinates once that operator is fixed. They become independent theory outputs only after $K_\ell^{\mathrm{act}}$ has itself been obtained from the stationary vacuum without charged-lepton mass input. The charged-lepton masses follow from

$$m_{\ell n} = 3\frac{v_H}{\sqrt{2}} h_n, \qquad n = 1, 2, 3, \tag{353}$$

which gives

$$(m_e, m_\mu, m_\tau) = (0.00051099895,\ 0.1056583755,\ 1.77686)\,\mathrm{GeV}. \tag{354}$$

**Closure status.** Mathematically, the dependence of $\Gamma_H, \Delta_H, \Omega_H$ on the underlying theory is now closed by the finite spectral map (344). Physically, an independent calculation of the stationary vacuum and of the complete potential $\mathcal{V}_{\mathbb{O}}$ must still reproduce the moment values in Eq. (349). This is no longer a missing mathematical definition but a checkable dynamical prediction of the full model.

### D.6 Full-Hessian block required for the charged-lepton masses

The normalized full Hessian is

$$\widehat{H} = \frac{1}{M_A^2}\left(H_{\mathrm{loc}} + H_{\mathrm{break}} + H_{H\mathbb{O}} + 2\lambda_A L^T L + 2\lambda_A B\right). \tag{355}$$

The universal associator part $L^T L$ is fixed by the Fano-plane calculation in the previous appendix. The remaining terms are not optional: they encode the vacuum-selected lepton idempotent, chirality, and symmetry-breaking information. The charged-lepton closure calculation requires the following $L \to R$ block of the full Hessian:

$$D_\ell = P_\ell P_L P_{\mathbb{C}} P_H \widehat{H} P_H P_{\mathbb{C}} P_R P_\ell. \tag{356}$$

In the ordered basis $\mathcal{B}_{24}$ the minimal left–right target representative of this block is the $24 \times 24$ matrix whose only non-zero entries are

$$(D_\ell)_{1,4} = h_e, \qquad (D_\ell)_{9,12} = h_\mu, \qquad (D_\ell)_{17,20} = h_\tau, \tag{357}$$

when one starts counting indices at one in the ordered basis $\mathcal{B}_{24}$. The row indices $1, 9, 17$ are the three left real components $E_{n,0}$ and the column indices $4, 12, 20$ are the three right complex-phase components $E_{n,3}$. All other entries vanish, and

$$h_e = h_1^{\mathbb{O}} = 9.783413823 \times 10^{-7},$$

$$
\begin{aligned}
h_\mu &= h_2^{\mathbb{O}} = 2.022899678 \times 10^{-4}, \\
h_\tau &= h_3^{\mathbb{O}} = 3.401916322 \times 10^{-3}.
\end{aligned} \tag{358}
$$

These entries are the target values of the primitive contractions defined in Eq. (313). Their independent derivation requires evaluation of those contractions on the stationary vacuum.

The associated positive charged-lepton mass block is

$$
K_\ell = D_\ell D_\ell^\dagger. \tag{359}
$$

Thus the only nonzero eigenvalues of $K_\ell$ in the charged-lepton blocks are

$$
\begin{aligned}
\Lambda_e &= h_e^2 = 9.571518603 \times 10^{-13}, \\
\Lambda_\mu &= h_\mu^2 = 4.092123107 \times 10^{-8}, \\
\Lambda_\tau &= h_\tau^2 = 1.157303466 \times 10^{-5}.
\end{aligned} \tag{360}
$$

The three spectral moments are therefore obtained directly from the explicit $24 \times 24$ matrix:

$$
\begin{aligned}
I_1^{(\ell)} &= \operatorname{Tr} K_\ell = 1.161395685 \times 10^{-5}, \\
I_2^{(\ell)} &= \operatorname{Tr}(K_\ell^2) = 1.339368058 \times 10^{-10}, \\
I_3^{(\ell)} &= \operatorname{Tr}(K_\ell^3) = 1.550035985 \times 10^{-15}.
\end{aligned} \tag{361}
$$

These values generate the characteristic polynomial

$$
\chi_\ell(x) = x^3 - s_1 x^2 + s_2 x - s_3, \tag{362}
$$

where

$$
\begin{aligned}
s_1 &= I_1^{(\ell)}, \\
s_2 &= \frac{1}{2}\left[(I_1^{(\ell)})^2 - I_2^{(\ell)}\right], \\
s_3 &= \frac{1}{6}\left[(I_1^{(\ell)})^3 - 3 I_1^{(\ell)} I_2^{(\ell)} + 2 I_3^{(\ell)}\right].
\end{aligned} \tag{363}
$$

Its three positive roots are exactly the three values

$$
\Lambda_e, \qquad \Lambda_\mu, \qquad \Lambda_\tau. \tag{364}
$$

### D.7 Numerical charged-lepton masses

With the electroweak normalization used in the main text,

$$
M_0 = 3\frac{v_H}{\sqrt{2}}, \qquad v_H = 246.21965 \text{ GeV}, \tag{365}
$$

we obtain

$$
\begin{aligned}
m_e &= M_0 \sqrt{\Lambda_e} = 0.00051099895 \text{ GeV}, \\
m_\mu &= M_0 \sqrt{\Lambda_\mu} = 0.1056583755 \text{ GeV}, \\
m_\tau &= M_0 \sqrt{\Lambda_\tau} = 1.77686 \text{ GeV}.
\end{aligned} \tag{366}
$$

Thus the explicit finite matrix calculation reconstructs the charged-lepton target spectrum at the stated normalization. It is an independent prediction only when the entries of $D_\ell$ are obtained from the action-derived vacuum rather than fixed to the target amplitudes.

### D.8 Interpretation and remaining independence requirement

The calculation above is the requested explicit 24-dimensional matrix closure. It removes the ambiguity about which trace, which projectors, and which finite-dimensional block are being diagonalized. The three amplitudes $h_e, h_\mu, h_\tau$ are now identified with the primitive vacuum contractions in Eq. (313), so the logical status is sharper than a fitted Yukawa ansatz. The remaining physical audit is the independent solution of the vacuum equations themselves: once $\Psi_I^{(0)}$, $P_H$, $P_\ell$, and the local symmetry-breaking Hessian are computed from the action, Eq. (313) gives a direct yes-or-no test of the charged-lepton sector. Thus the manuscript now reduces exact charged-lepton masses to three explicitly defined Hessian contractions rather than three adjustable Yukawa couplings.

## E Dimensional analysis

In four spacetime dimensions the Lagrangian density has mass dimension four. If $[\Psi] = 1$, then

$$[A] = [\Psi^3] = 3, \qquad [\|A\|^2] = 6. \tag{367}$$

Therefore

$$[\lambda_A] = -2. \tag{368}$$

A natural parametrization is

$$\lambda_A = \frac{\eta_A}{M_*^2}, \tag{369}$$

where $M_*$ is a high non-associative scale and $\eta_A$ is dimensionless.


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## Data availability

No data were created or analyzed in this theoretical study.

## Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work, the author used ChatGPT (OpenAI) for language editing, assistance with the organization of references, and consistency checking. The author subsequently reviewed and edited all AI-assisted content as needed and takes full responsibility for the scientific content, accuracy, and integrity of the published article.

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