The Trinions:
The Natural Extension of the Complex and Dual Numbers

$$\mathbb{Y}_{(k)}$$

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Abstract

In this paper, the natural three-dimensional extension of the complex and dual numbers is introduced: the trinion algebras $\mathbb{Y}(k)$ of the form $a + bi + cj$ with $i^2 = -1$, $j^2 = 0$ and the commutative coupling $ij = ji = k$, where $k = \alpha + \beta i + \gamma j$ is an element of the algebra. The family $\mathbb{Y}(k)$ is systematically investigated: multiplication, division and inverses are derived. Special attention is given to the phenomena induced by non-associativity, which lead to invertible zero divisors, non-unique division, and rich geometric structures. The eight fundamental algebras with $\alpha, \beta, \gamma \in \{0, 1\}$ are examined and their zero divisor sets are visualized. Numerical examples for the algebra $\mathbb{Y}(1)$ illustrate the observed phenomena.

Keywords: Three-dimensional algebra, commutativity, non-associativity, nilpotence, zero divisors, invertible zero divisors, zero divisor geometry

MSC2020: 17D99, 14J70

1 Introduction

The extension of the real numbers to the complex numbers $a + bi$ with $i^2 = -1$ was historically motivated by the search for solutions to algebraic equations such as $x^2 = -1$. They have proven not only to be elegant but also fundamental for the description of rotations, oscillations, and wave phenomena. The complex numbers form a two-dimensional field in which every element possesses an inverse [7].

A different path is taken by the dual numbers $a + b\varepsilon$ with $\varepsilon^2 = 0$. They model infinitesimal translations as part of a dynamic process and find application in robotics, kinematics, and automatic differentiation [9]. The dual numbers form a two-dimensional ring in which their nilpotent element ε is not a static object but represents an infinitesimal, irreversible process [2].

The question of whether there exists a three-dimensional algebra that meaningfully extends the complex numbers already occupied William Rowan Hamilton and led to the discovery of the four-dimensional quaternions [13]. In 1958, John Milnor [11] and Michel Kervaire [10] proved that the only possible dimensions for finite-dimensional real division algebras are 1, 2, 4, and 8. Hence, a three-dimensional division algebra cannot exist. Any such structure must necessarily contain zero divisors – that is, there exist elements $x \neq 0$ and $y \neq 0$ with $xy = 0$.

But is the existence of zero divisors a flaw? From the author's perspective, it is not: the real three-dimensional universe also exhibits structures reminiscent of zero divisors – such as the annihilation of particles or the singularities of black holes. It therefore seems worthwhile not to avoid algebras with zero divisors but to systematically study their diverse structures.

The present work introduces a natural three-dimensional extension of the complex and dual numbers: the **trinions** of the form

$$x = a + bi + cj,$$

with basis elements 1, i and j and the multiplication rules $i^2 = -1$, $j^2 = 0$, $ij = ji = k$, where the coupling parameter $k = \alpha + \beta i + \gamma j$ is a fixed element of the algebra. The resulting family $\mathbb{Y}(k)$ depends on three real parameters α, β, γ and contains the complex numbers as well as the dual numbers as natural subalgebras. The term "trinions" is chosen in analogy to quaternions and octonions [1]; the symbol $\mathbb{Y}$ emphasizes the three-dimensional character of the algebra.

A key feature of the algebras $\mathbb{Y}(k)$ is their non-associativity. This may explain why three-dimensional algebras of this kind have hardly been studied in the literature. The lack of associativity renders many techniques of classical algebra inapplicable but at the same time opens the door to new, unexpected structures.

The aim of this work is an introductory investigation of the family of algebras $\mathbb{Y}(k)$. After the definition and a discussion of basic properties, general rules for multiplication, division, and inverses are derived. Particular focus is placed on the behavior of zero divisors, which exhibit a rich geometric structure in these algebras. Subsequently, the eight fundamental algebras with parameters $\alpha, \beta, \gamma \in \{0, 1\}$ are presented and their zero divisor sets are visualized. The work concludes with numerical examples for the algebra $\mathbb{Y}(1)$, illustrating the phenomena described above.

2 Definition of the Algebras $\mathbb{Y}(k)$

In this work, a real algebra is understood as a vector space over $\mathbb{R}$ with a bilinear multiplication and a unit element [5].

Let $\mathbb{Y}(k)$ be the three-dimensional real vector space with basis $\{1, i, j\}$, i.e. every element has the form

$$x = a + bi + cj, \qquad a, b, c \in \mathbb{R}.$$

The algebras $\mathbb{Y}(k)$ are defined by the basis $\{1, i, j\}$ together with the multiplication rules

$$i^2 = -1, \qquad j^2 = 0, \qquad ij = ji = k, \qquad k = \alpha + \beta i + \gamma j, \qquad \alpha, \beta, \gamma \in \mathbb{R} \tag{1}$$

The parameters α, β, γ determine the coupling k between the imaginary unit i and the dual (nilpotent) unit j. The multiplication is extended bilinearly to the whole vector space. The element 1 acts as a **unit element**, i.e. for all $x \in \mathbb{Y}(k)$ we have $1 \cdot x = x \cdot 1 = x$.

The following table illustrates the multiplication of the basis elements:

$\cdot$	1	i	j
1	1	i	j
i	i	-1	k
j	j	k	0

From $ij = ji$ and bilinearity we immediately obtain **commutativity** of the multiplication: for all $x, y \in \mathbb{Y}(k)$ we have $xy = yx$.

Moreover, for all $\mathbb{Y}(k)$ we have $(ii)j \neq i(ij)$, which implies the **non-associativity** of the algebras.

Proof. By the multiplication rules,

$$(ii)j = -j,$$

while

$$\begin{aligned} i(ij) &= i(\alpha + \beta i + \gamma j) \\ &= \alpha i - \beta + \gamma(\alpha + \beta i + \gamma j) \\ &= (\alpha\gamma - \beta) + (\alpha + \beta\gamma)i + \gamma^2 j. \end{aligned}$$

Comparing the j-components shows that if $(ii)j = i(ij)$ were true, we would have $\gamma^2 = -1$. Since $\gamma \in \mathbb{R}$ and $\gamma^2 \geq 0$, this is impossible. Hence

$$(ii)j \neq i(ij)$$

for all $\alpha, \beta, \gamma \in \mathbb{R}$, and the algebras $\mathbb{Y}(k)$ are non-associative. $\square$

In summary, $\mathbb{Y}(k)$ are **commutative, non-associative algebras with a unit element**.

3 Natural Subalgebras

Independently of the parameters α, β, γ, the algebra $\mathbb{Y}(k)$ always contains two canonical two-dimensional subalgebras that realize the complex and dual numbers as special cases:

- **Complex part:** $\mathcal{C} := \text{span}\{1, i\}$. Since $i^2 = -1 \in \mathcal{C}$, $\mathcal{C}$ is closed under multiplication and isomorphic to the field of complex numbers $\mathbb{C}$. The embedding $\mathbb{C} \hookrightarrow \mathbb{Y}(k)$ is identical for all k.
- **Dual part:** $\mathcal{D} := \text{span}\{1, j\}$. Because $j^2 = 0 \in \mathcal{D}$, $\mathcal{D}$ is also a subalgebra, isomorphic to the ring of dual numbers $\mathbb{D} = \mathbb{R}[\varepsilon]/(\varepsilon^2)$.

Remark 3.1. As an $\mathbb{R}$-vector space, $\mathbb{Y}(k) = \mathcal{C} + \mathcal{D}$ with $\mathcal{C} \cap \mathcal{D} = \mathbb{R} \cdot 1$. The coupling vector k exclusively determines the interaction between the non-scalar parts (i.e. the product $ij = ji = k$), while the multiplication rules within $\mathcal{C}$ and $\mathcal{D}$ remain unchanged. Thus $\mathbb{Y}(k)$ can be viewed as a minimal extension that unites $\mathcal{C}$ and $\mathcal{D}$ into a common three-dimensional algebra. This underlines the interpretation as a natural three-dimensional extension of $\mathbb{C}$ and $\mathbb{D}$.

4 General Computational Rules

4.1 Addition

For $x = a + bi + cj$ and $y = d + ei + fj$, the sum $x + y$ is defined as componentwise addition of the coefficients:

$$x + y = (a + bi + cj) + (d + ei + fj) = (a + d) + (b + e)i + (c + f)j. \tag{2}$$

4.2 Multiplication

For $x = a+bi+cj$ and $y = d+ei+fj$, the product xy is determined by the defining multiplication rules from (1). Expanding and collecting terms yields the general multiplication formula:

$$xy = ad - be + \alpha(bf + ce) \ + \ \big(ae + bd + \beta(bf + ce)\big)i \ + \ \big(af + cd + \gamma(bf + ce)\big)j. \tag{3}$$

4.3 Division (Quotient)

In associative algebras, division is usually defined via the inverse ($z/x := z \cdot x^{-1}$). Due to the non-associativity of $\mathbb{Y}(k)$, this approach is not suitable, because the equation $xy = z$ generally cannot be solved for y by multiplying with an inverse, even if an element x^{-1} with $x \cdot x^{-1} = 1$ exists.

For two elements $x = a + bi + cj$ and $z = p + qi + rj$, the quotient $y = d + ei + fj = z/x$ is therefore defined as the unique solution of the linear equation $xy = z$. Because the algebra is commutative, no distinction between left and right division is necessary. Using the multiplication formula (3), we obtain the linear system

$$\begin{aligned} ad + (-b + \alpha c)e + \alpha bf &= p, \\ bd + (a + \beta c)e + \beta bf &= q, \\ cd + \gamma ce + (a + \gamma b)f &= r. \end{aligned}$$

In matrix form, this corresponds to the linear system $M_x\,(d, e, f)^T = (p, q, r)^T$ with the coefficient matrix

$$M_x = \begin{pmatrix} a & -b + \alpha c & \alpha b \\ b & a + \beta c & \beta b \\ c & \gamma c & a + \gamma b \end{pmatrix}. \tag{4}$$

By Cramer's rule [3], the common denominator of the quotient components is the determinant $\det M_x$ [8]. It is computed by expansion along the first row:

$$\det M_x = a^2(a + \gamma b + \beta c) + b^2(a + \gamma b - \beta c) - 2\alpha abc. \tag{5}$$

4.3.1 Unique Quotient ($\det M_x \neq 0$)

If $\det M_x \neq 0$, the system has a unique solution. Using the adjugate matrix $\operatorname{adj}(M_x)$ [4], this solution can be represented compactly. The nine cofactors C_{mn} of M_x [6] (taking into account the signs $(-1)^{m+n}$) are

$$\begin{aligned} &C_{11} = a(a + \gamma b + \beta c), && C_{12} = -b(a + \gamma b - \beta c), && C_{13} = -c(a - \gamma b + \beta c), \\ &C_{21} = ab + \gamma b^2 - \alpha ac, && C_{22} = a^2 + a\gamma b - \alpha bc, && C_{23} = -c(a\gamma + b - \alpha c), \\ &C_{31} = -b(\alpha a + \beta b), && C_{32} = b(\alpha b - \beta a), && C_{33} = a^2 + b^2 + c(\beta a - \alpha b). \end{aligned}$$

The adjugate (transpose of the cofactor matrix) is the matrix

$$\operatorname{adj}(M_x) = (C_{nm}) = \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}.$$

For an arbitrary dividend $z = p + q\,i + r\,j$, the numerators of the quotient $y = z/x$ are thus obtained by the matrix-vector multiplication $\operatorname{adj}(M_x)\,(p, q, r)^T$:

$$\begin{aligned} N_d &= p\,a(a + \gamma b + \beta c) + q\,(ab + \gamma b^2 - \alpha ac) - r\,b(\alpha a + \beta b), \\ N_e &= -p\,b(a + \gamma b - \beta c) + q\,(a^2 + a\gamma b - \alpha bc) + r\,b(\alpha b - \beta a), \\ N_f &= -p\,c(a - \gamma b + \beta c) - q\,c(a\gamma + b - \alpha c) + r\,\left(a^2 + b^2 + c(\beta a - \alpha b)\right). \end{aligned} \tag{6}$$

Hence, for $\det M_x \neq 0$, the explicit division formula follows from (5) and (6):

$$y = z/x = \frac{N_d}{\det M_x} + \frac{N_e}{\det M_x}\,i + \frac{N_f}{\det M_x}\,j. \tag{7}$$

4.3.2 Unique Inverse ($\det M_x \neq 0$)

Substituting the values $p = 1,\ q = 0,\ r = 0$ (i.e. $z = 1$) into the dividend $z = p + q\,i + r\,j$ yields, for $\det M_x \neq 0$, the multiplicative inverse from (5), (6) and (7):

$$x^{-1} = \frac{a(a+\gamma b+\beta c)}{\det M_x} - \frac{b(a+\gamma b-\beta c)}{\det M_x}\,i - \frac{c(a-\gamma b+\beta c)}{\det M_x}\,j. \tag{8}$$

Remark 4.1 (Non-associativity and division by zero divisors)**.** Due to the lack of associativity, in general $x^{-1}(xy) \neq (x^{-1}x)y = y$, so the inverse cannot be used for cancellation in multiple products. The unique determination of the quotient $y = z/x$ is not guaranteed by the existence of x^{-1} but solely by the regularity condition $\det M_x \neq 0$; the inverse is merely the special case $z = 1$ of the unique quotient for elements x with $\det M_x \neq 0$. In general, $z/x \neq z \cdot x^{-1}$; equality holds only when z is real (i.e. $z = a + 0i + 0j$).

For the case $\det M_x = 0$, division and the existence of inverses may still be possible; this is discussed in the following section on zero divisor phenomena (see 5.1 and 5.2).

5 Zero Divisor Phenomena

An element $x \neq 0$ is called a *zero divisor* if there exists a $y \neq 0$ such that $xy = 0$. In the algebras $\mathbb{Y}(k)$, this is equivalent to $\det M_x = 0$.

Thus the set of zero divisors is

$$\mathcal{Z_D} := \{(a,b,c) \in \mathbb{R}^3 \setminus \{0\} \mid \det M_x = 0\}.$$

In the following, several special algebraic phenomena arising directly from the structure of the algebras $\mathbb{Y}(k)$ are presented: division by zero divisors, invertible zero divisors, the involutive geometry of partner lines, and the behavior of products involving zero divisors.

5.1 Division by Zero Divisors ($\det M_x = 0$)

If $\det M_x = 0$, the multiplication matrix M_x is singular and x is a zero divisor. The division formula (7) obtained via Cramer's rule fails in this case. Nevertheless, the division $y = z/x$ is defined when the linear system $M_x\,(d,e,f)^T = (p,q,r)^T$ is solvable.

By the Rouché–Capelli theorem [12, p. 56], this happens exactly when the rank of the augmented matrix $(M_x \mid (p,q,r)^T)$ coincides with the rank of M_x:

$$\operatorname{rank}(M_x) = \operatorname{rank}\bigl(M_x \mid (p,q,r)^T\bigr).$$

Equivalently, the dividend $z = (p,q,r)^T$ must lie in the image (column space) of M_x. If this is not the case, no quotient exists.

For all zero divisors with $\operatorname{rank}(M_x) = 2$ (the generic case), the linear system $M_x\,(d,e,f)^T = (p,q,r)^T$ is solvable precisely when the dividend z is orthogonal to the left kernel of M_x. Since the rows of the adjugate $\operatorname{adj}(M_x)$ span this kernel, the necessary and sufficient condition for solvability reduces to the vanishing of the numerators already computed in (6):

$$N_d = N_e = N_f = 0. \tag{9}$$

Because the adjugate $\operatorname{adj}(M_x)$ has rank 1 when $\det M_x = 0$ and $\operatorname{rank}(M_x) = 2$, its rows are linearly dependent. Consequently, it is usually sufficient to check only one of the three numerators. For example, if $N_d = 0$ holds, then N_e and N_f vanish automatically. However, if the checked row of the adjugate vanishes (e.g. $C_{11} = C_{21} = C_{31} = 0$), another row must be used instead.

When condition (9) is satisfied, the solution set is a one-dimensional affine line

$$y = y_p + \lambda\, y_0, \qquad \lambda \in \mathbb{R},$$

where y_p is an arbitrary particular solution of the system $M_x y = z$ and y_0 is a basis vector of the kernel $\ker M_x$ (i.e. $M_x y_0 = 0$). The basis vector y_0 spans the partner line of the zero divisor x (which passes through the origin); the solution line is parallel to this partner line but shifted by y_p, and thus does not pass through the origin.

A particular solution y_p can be constructively determined for every zero divisor x with $\operatorname{rank}(M_x) = 2$ by selecting two linearly independent rows of M_x, setting one coordinate free, and solving the resulting 2×2 system.

Remark 5.1 (Special case $\alpha = \beta = 0,\ a = b = 0$)**.** The rank drops to 1 only for elements on the j-axis ($a = b = 0,\ c \neq 0$) in algebras with $\alpha = \beta = 0$. In this case, $\operatorname{adj}(M_x) = 0$ and condition (9) becomes trivial. A direct analysis shows that division is then solvable only if the dividend also lies on the j-axis ($p = q = 0$). Here the solution set is a plane. For all other parameters α, β and for zero divisors outside the j-axis, the generic case $\operatorname{rank}(M_x) = 2$ prevails.

5.2 Invertible Zero Divisors

Due to the non-associativity of the algebras $\mathbb{Y}(k)$, elements $x = a + bi + cj$ with $\det M_x = 0$ may still possess inverses (cf. Section 5.1). Such elements are called *invertible zero divisors* $\mathcal{Z}_{\mathcal{D}\text{inv}}$.

The existence of an inverse is equivalent to the solvability of the system $M_x(d, e, f)^T = (1, 0, 0)^T$. If $\det M_x = 0$ and $\operatorname{rank}(M_x) = 2$, then by the Rouché–Capelli theorem this is exactly the case when the three numerators

$$C_{11} = a(a + \gamma b + \beta c), \quad C_{12} = -b(a + \gamma b - \beta c), \quad C_{13} = -c(a - \gamma b + \beta c)$$

from (8) vanish. The general solution is then, as for division by zero divisors, an affine line.

Proposition 5.2 (Geometric characterisation)**.** *The set of invertible zero divisors $\mathcal{Z}_{\mathcal{D}\text{inv}}$ of $\mathbb{Y}(k)$ is exactly the intersection of the zero set of the determinant $\det M_x$ with the union of the three coordinate planes, excluding the origin:*

$$\mathcal{Z}_{\mathcal{D}\text{inv}} = \big\{(a, b, c) \in \mathbb{R}^3 \setminus \{0\} \mid \det M_x = 0\big\} \ \cap\ \big(\{a = 0\} \cup \{b = 0\} \cup \{c = 0\}\big).$$

On each coordinate plane, the consistency condition $C_{11} = C_{12} = C_{13} = 0$ reduces to a linear relation between the remaining coordinates, under which $\det M_x = 0$ is automatically satisfied.

Theorem 5.3 (Explicit inverse for $\det M_x = 0$)**.** *Let $x = a + bi + cj$ be an invertible zero divisor. Then the solution set of $M_x(d, e, f)^T = (1, 0, 0)^T$ is the affine line*

$$x^{-1} = y_p + \lambda\, y_0, \qquad \lambda \in \mathbb{R},$$

where y_p is a particular solution of the system $M_x(d, e, f)^T = (1, 0, 0)^T$ and y_0 is a basis vector of the kernel $\ker M_x$ (i.e. $M_x y_0 = 0$). Depending on the location on the respective coordinate planes, the following explicit formulas are obtained:

1. ***Plane** $a = 0$:* $\quad \gamma b = \beta c,\ (b, c) \neq (0, 0)$

$$x^{-1} = -\frac{1}{b}\, i \ +\ \frac{c}{b^2}\, j \ +\ \lambda\Big(-\beta b \ +\ \alpha b\, i \ +\ (b - \alpha c)\, j\Big)$$

2. ***Plane** $b = 0$:* $\quad a = -\beta c,\ c \neq 0,\ \beta \neq 0$

$$x^{-1} = -\frac{1}{\beta c}\Big(1 \ +\ \frac{1}{\beta}\, j\Big) \ +\ \lambda\Big(\alpha\beta \ +\ \beta^2\, i \ +\ (\alpha + \beta\gamma)\, j\Big)$$

3. ***Plane*** $c = 0$***:*** $\quad a = -\gamma b,\ b \neq 0$

$$x^{-1} = -\frac{1}{b(\gamma^2+1)}\Big(\gamma \,+\, i\Big) \,+\, \lambda\Big((\alpha\gamma - \beta) \,+\, (\alpha + \beta\gamma)\, i \,+\, (\gamma^2+1)\, j\Big)$$

4. j***-axis*** $(a = b = 0)$***:*** $\quad x = cj,\ \beta = 0,\ \alpha \neq 0$

$$x^{-1} = -\frac{\gamma}{\alpha c} \,+\, \frac{1}{\alpha c} i \,+\, \lambda\, j$$

The parameter $\lambda \in \mathbb{R}$ describes the freedom in the kernel of M_x. Every choice of λ yields a valid inverse.

Proof. The geometric characterisation follows directly from the equations $C_{11} = C_{12} = C_{13} = 0$. If $a, b, c \neq 0$, then all three brackets $a + \gamma b + \beta c$, $a + \gamma b - \beta c$, $a - \gamma b + \beta c$ would have to vanish simultaneously, which is only possible for $a = b = c = 0$. Hence every invertible zero divisor lies in at least one coordinate plane.

The explicit formulas are obtained by substituting the respective consistency conditions into the system $M_x(d, e, f)^T = (1, 0, 0)^T$ and cleverly setting one variable to zero to isolate a particular solution y_p:

- For $a = 0$ and $\gamma b = \beta c$, set $d = 0$ and solve the remaining 2×2 system for e, f. The kernel vector y_0 is determined by solving the homogeneous system $M_x(d, e, f)^T = (0, 0, 0)^T$.
- For $b = 0$ and $a = -\beta c$, set $e = 0$. The remaining equations are decoupled and immediately yield d and f.
- For $c = 0$ and $a = -\gamma b$, set $f = 0$. The resulting system in d, e is regular and solvable.
- For $a = b = 0$ and $\beta = 0$, set $f = 0$. The remaining equations yield d and e.

Substituting the given vectors into M_x confirms $M_x y_p = (1, 0, 0)^T$ and $M_x y_0 = (0, 0, 0)^T$, respectively. $\square$

5.3 Partner Lines and Involutivity of Zero Divisors

Every zero divisor lies on a line through the origin (a generator of the cubic surface). The zero divisor partners of a fixed zero divisor x likewise form a line through the origin – the partner line. In the generic case, each generator is assigned a partner line; the assignment is involutive: if y lies on the partner line of x, then x lies on the partner line of y.

Lemma 5.4. *The set of zero divisors $\mathcal{Z}_{\mathcal{D}}$ of $\mathbb{Y}(k)$ is a union of lines through the coordinate origin.*

Proof. The determinant $\det M_x = a^2(a + \gamma b + \beta c) + b^2(a + \gamma b - \beta c) - 2\alpha abc$ is a homogeneous polynomial of degree 3. Hence, for any scaling factor $\lambda \in \mathbb{R}$,

$$\det M_{\lambda x} = \lambda^3 \det M_x.$$

If $x = (a, b, c)$ is a zero divisor, i.e. $\det M_x = 0$, then $\det M_{\lambda x} = \det M(\lambda a, \lambda b, \lambda c) = 0$ for all $\lambda \in \mathbb{R}$. Consequently, with every point x the entire line through the origin and x lies in the zero divisor set $\mathcal{Z}_{\mathcal{D}}$. $\square$

Theorem 5.5 (Partner line involution)**.** *For every zero divisor $x = a + bi + cj$ of $\mathbb{Y}(k)$ with* $\operatorname{rank}(M_x) = 2$*, the set of its zero divisor partners* $\ker M_x \setminus \{0\}$ *is one-dimensional. It forms a unique generator of the zero divisor set $\mathcal{Z}_{\mathcal{D}}$, called the* partner line $P(x)$.

The assignment $x \mapsto P(x)$ depends only on the generator $\ell = \{\lambda x \mid \lambda \in \mathbb{R}\}$ and defines an involution on the set of generators of $\mathcal{Z}_{\mathcal{D}}$:

$$P(P(\ell)) = \ell \qquad \text{for every generator } \ell.$$

A generator is a fixed point of this involution ($P(\ell) = \ell$) precisely when it is nilpotent, i.e. $x^2 = 0$ for $x \in \ell$.

Proof. (1. One-dimensionality of the kernel): Since M_x is a 3×3 matrix, the rank–nullity theorem yields $\dim \ker M_x = 3 - \operatorname{rank}(M_x)$. With $\operatorname{rank}(M_x) = 2$ we obtain $\dim \ker M_x = 1$; hence the kernel is one-dimensional and the set of zero divisor partners is a line through the coordinate origin.

(2. Involutivity via commutativity): Let ℓ be a generator and $x \in \ell \setminus \{0\}$. Set $P(\ell) := \ker M_x$ (a line by step 1). Choose $y \in P(\ell) \setminus \{0\}$. Then $xy = 0$. Because $\mathbb{Y}(k)$ is commutative, $yx = 0$ follows, so $x \in \ker M_y$. By step 1, $\ker M_y$ is also one-dimensional and contains $x \neq 0$; therefore $\ker M_y = \ell$. Hence $P(P(\ell)) = \ell$.

(3. Fixed points of the involution): If $x^2 = 0$, then x lies in the kernel of M_x, i.e. $\ell \subseteq \ker M_x$. Since both are one-dimensional, we obtain $\ker M_x = \ell$, which means $P(\ell) = \ell$. The converse follows analogously: from $P(\ell) = \ell$ we have $x \in \ker M_x$ for $x \in \ell$, thus $x^2 = 0$. □

For a zero divisor x with $\operatorname{rank}(M_x) = 2$, the partner set is the line

$$P(x) = \ker M_x \ = \{\lambda y_0 \mid \lambda \in \mathbb{R}\},$$

where $y_0 \neq 0$ is an arbitrary vector in the kernel (i.e. $M_x y_0 = 0$).

Remark 5.6 (Special case $\alpha = \beta = 0,\ a = b = 0$)**.** The rank drops to 1 only for zero divisors on the j-axis ($a = b = 0,\ c \neq 0$) in algebras with $\alpha = \beta = 0$. In these special cases, where $\det M_x = 0$ generates only a plane, the partner set of the j-axis is the entire plane (see Sections 6.1 and 6.5).

5.4 Generic Invertibility of Products and Quotients Involving Zero Divisors

In finite-dimensional **associative** algebras with a unit element, the product of a zero divisor x with any element y is again a zero divisor or zero. Due to associativity, the identity $M_{xy} = M_x \cdot M_y$ holds, so the determinant multiplication theorem for square matrices [12, p. 68] applies directly:

$$\det M_{xy} = \det(M_x \cdot M_y) = \det M_x \cdot \det M_y.$$

This equation rigidly couples the regularity of factors and products: as soon as a factor has determinant 0, we necessarily have $\det M_{xy} = 0$, and the product xy is also singular.

For the **non-associative** algebras $\mathbb{Y}(k)$, this coupling does not hold in general.

Proof. Take $x = i$ and $y = i$. From the determinant formula (5) we obtain

$$\det M_i = \det M_{(0,1,0)} = 0^2(0 + \gamma \cdot 1 + 0) + 1^2(0 + \gamma \cdot 1 - 0) - 0 = \gamma.$$

Hence the product of the individual determinants is $\det M_x \cdot \det M_y = \gamma^2$.

The algebraic product is $x \cdot y = i \cdot i = -1$. Substituting into (5) gives

$$\det M_{xy} = \det M_{(-1,0,0)} = (-1)^2(-1 + 0 + 0) + 0^2(-1 + 0 - 0) - 0 = -1.$$

If the determinant were multiplicative, we would need $-1 = \gamma^2$. Since $\gamma \in \mathbb{R}$, $\gamma^2 \geq 0$, so this equality can never be satisfied.

Consequently, $\det M_{xy} = \det M_x \cdot \det M_y$ does not hold in the algebras $\mathbb{Y}(k)$. □

Thus the determinant is not multiplicative. Multiplication by an arbitrary element x does not act as an algebraic norm multiplication in $\mathbb{Y}(k)$; instead, it simply acts as the linear map M_x on the vector space $\mathbb{R}^3$.

The set of zero divisors $\mathcal{Z}_\mathcal{D} = \{x \in \mathbb{Y}(k) \setminus \{0\} \mid \det M_x = 0\}$ is defined by the cubic polynomial (5) and forms an algebraic surface of degree three in $\mathbb{R}^3$. Linear maps generally do not preserve algebraic surfaces of higher degree. When a point on $\mathcal{Z}_\mathcal{D}$ is transformed by M_x, it almost always leaves the surface. This explains in a unified way the following observed phenomena:

1. **Zero divisor $\times$ zero divisor:** For $x, y \in \mathcal{Z}_\mathcal{D}$ with $xy \neq 0$, the product xy is not determined by the matrix product $M_x \cdot M_y$ but rather by the bilinear multiplication structure of the algebra. Since the determinant is not multiplicative, there is no algebraic barrier forcing $\det M_x = 0$ and $\det M_y = 0$ to imply $\det M_{xy} = 0$. For almost all pairs on the surface $\mathcal{Z}_\mathcal{D} \times \mathcal{Z}_\mathcal{D}$, evaluating $\det M_{xy}$ yields $\det M_{xy} \neq 0$; hence the product is generically regular.

2. **Zero divisor $\times$ unit:** Let $x \in \mathcal{Z}_\mathcal{D}$ and y be regular ($\det M_y \neq 0$). The product xy corresponds to applying the linear map M_y to the coordinate vector x. Because M_y performs a linear coordinate transformation that generally does not map the cubic surface $\mathcal{Z}_\mathcal{D}$ to itself, x is generically pushed out of the singular set, so the product $xy = z$ is regular. Conversely, every zero divisor x can generically be represented as a quotient z/y of two units.

3. **Zero divisor $\div$ unit:** Let $z \in \mathcal{Z}_\mathcal{D}$ and x be regular ($\det M_x \neq 0$). The quotient $y = z/x$ solves $M_x y = z$. Again M_x acts as a linear transformation on $\mathbb{R}^3$. Since the cubic surface $\mathcal{Z}_\mathcal{D}$ is generally not invariant under the linear map M_x, the quotient y generically lies outside $\mathcal{Z}_\mathcal{D}$ and is therefore regular. Conversely, every zero divisor z can generically be written as a product xy of two units.

Remark 5.7 (Exceptional cases)**.** Deviations from generic invertibility occur when the zero divisor, transformed by multiplication with another zero divisor or a unit, or by division by a unit, remains on the zero divisor surface of the algebra. For example, the product of two zero divisors may again be a zero divisor if one of the factors lies on a space diagonal and the other is a suitable invertible zero divisor.

Numerical examples illustrating the generic invertibility as well as exceptional cases are presented in Section 7.5 using the algebra $\mathbb{Y}(1)$.

6 The eight fundamental algebras $\mathbb{Y}(k)$ with $\alpha, \beta, \gamma \in \{0, 1\}$

In this section we investigate the algebras of the family $\mathbb{Y}(k)$ whose parameters α, β, γ take the values 0 or 1. This restriction yields the simplest non-trivial couplings k between the imaginary unit i and the dual (nilpotent) unit j and thus forms the basis for the whole family.

For each of these eight algebras, the concrete formulas for multiplication and division (unique quotient) are given. The multiplicative inverse x^{-1} is obtained as the special case of the quotient for $z = 1$ (i.e. $p = 1,\ q = 0,\ r = 0$).

To illustrate the geometric structure of the zero divisor sets, visualizations are provided. These were created using the libraries `plotly` (for the 3D representations) and `matplotlib` (for the 2D sections).

6.1 $\mathbb{Y}(0)$: $\alpha = 0,\ \beta = 0,\ \gamma = 0$

For the algebra $\mathbb{Y}(0)$ we have

$$x = a + bi + cj, \quad a, b, c \in \mathbb{R}, \quad i^2 = -1, \quad j^2 = 0, \quad ij = ji = 0.$$

Multiplication and division (unique quotient) are obtained by substituting the parameters $\alpha = 0,\ \beta = 0,\ \gamma = 0$ into the general formulas (3), (5) and (6).

Multiplication ($y = d + ei + fj$)

$$xy = (ad - be) + (ae + bd)i + (af + cd)j.$$

Unique quotient ($z = p + qi + rj$, $\det M_x = a(a^2 + b^2) \neq 0$)

$$\begin{aligned} z/x = & \frac{pa^2 + qab}{\det M_x} \\ & - \frac{pab - qa^2}{\det M_x} i \\ & - \frac{pac + qbc - r(a^2 + b^2)}{\det M_x} j. \end{aligned}$$

The zero divisor set $\det M_x = 0$ consists of the plane $a = 0$. With the exception of the j-axis, all zero divisors of the plane $a = 0$ are invertible.

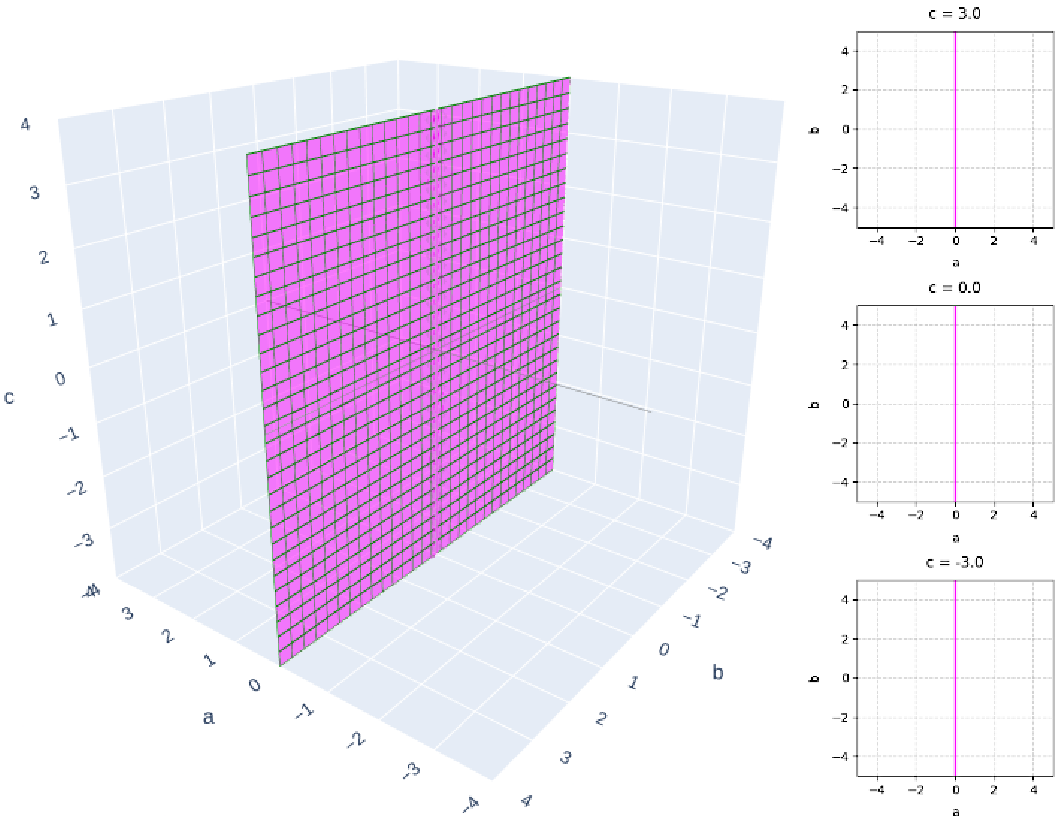

Figure 1: Zero divisor set $\det M_x = 0$ for $\mathbb{Y}(0)$

6.2 $\mathbb{Y}(1)$: $\alpha = 1,\ \beta = 0,\ \gamma = 0$

For the algebra $\mathbb{Y}(1)$ we have

$$x = a + bi + cj, \quad a, b, c \in \mathbb{R}, \quad i^2 = -1, \quad j^2 = 0, \quad ij = ji = 1.$$

Multiplication and division (unique quotient) are obtained by substituting the parameters $\alpha = 1,\ \beta = 0,\ \gamma = 0$ into the general formulas (3), (5) and (6).

Multiplication ($y = d + ei + fj$)

$$xy = (ad - be + bf + ce) + (ae + bd)i + (af + cd)j.$$

Unique quotient ($z = p + qi + rj$, $\det M_x = a(a^2 + b^2 - 2bc) \neq 0$)

$$\begin{aligned} z/x = &\frac{pa^2 + q(ab - ac) - rab}{\det M_x} \\ &- \frac{pab - q(a^2 - bc) - rb^2}{\det M_x}\, i \\ &- \frac{pac + qc(b - c) - r(a^2 + b^2 - bc)}{\det M_x}\, j. \end{aligned}$$

The zero divisor set $\det M_x = 0$ is a cubic surface. Horizontal sections show circles intersected by a straight line through their centre, forming a double cone cut by the plane $a = 0$. All zero divisors of the plane $a = 0$ are invertible.

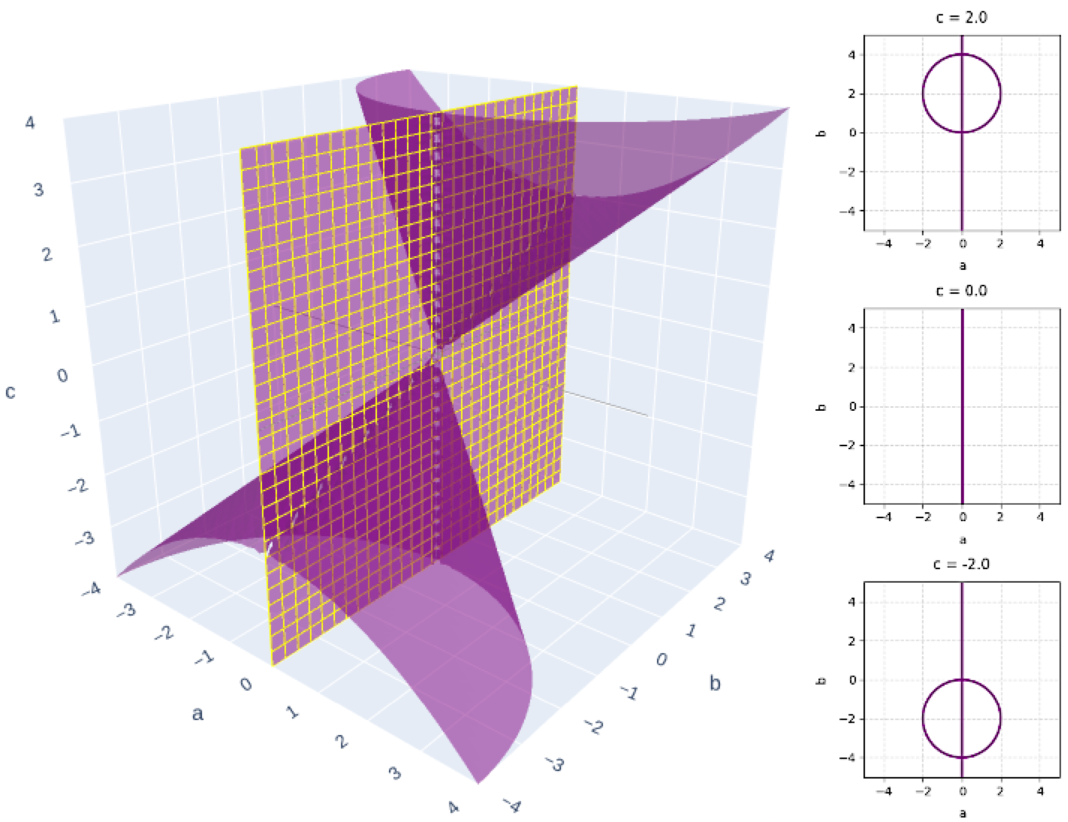

Figure 2: Zero divisor set $\det M_x = 0$ for $\mathbb{Y}(1)$

6.3 $\mathbb{Y}(i)$: $\alpha = 0,\ \beta = 1,\ \gamma = 0$

For the algebra $\mathbb{Y}(i)$ we have

$$x = a + bi + cj, \quad a, b, c \in \mathbb{R}, \quad i^2 = -1, \quad j^2 = 0, \quad ij = ji = i.$$

Multiplication and division (unique quotient) are obtained by substituting the parameters $\alpha = 0,\ \beta = 1,\ \gamma = 0$ into the general formulas (3), (5) and (6).

Multiplication ($y = d + ei + fj$)

$$xy = (ad - be) + (ae + bd + bf + ce)i + (af + cd)j.$$

Unique quotient ($z = p + qi + rj$, $\det M_x = a^2(a + c) + b^2(a - c) \neq 0$)

$$\begin{aligned} z/x = & \frac{pa(a+c) + qab - rb^2}{\det M_x} \\ & - \frac{pb(a-c) - qa^2 + rab}{\det M_x} i \\ & - \frac{pc(a+c) + qbc - r(a^2 + b^2 + ac)}{\det M_x} j. \end{aligned}$$

The zero divisor set $\det M_x = 0$ is a curved cubic surface. Horizontal sections show cubic curves with a double point on the j-axis, which in particular form a double cone. The invertible zero divisors lie on the i-axis and on the line $b = 0$, $a = -c$.

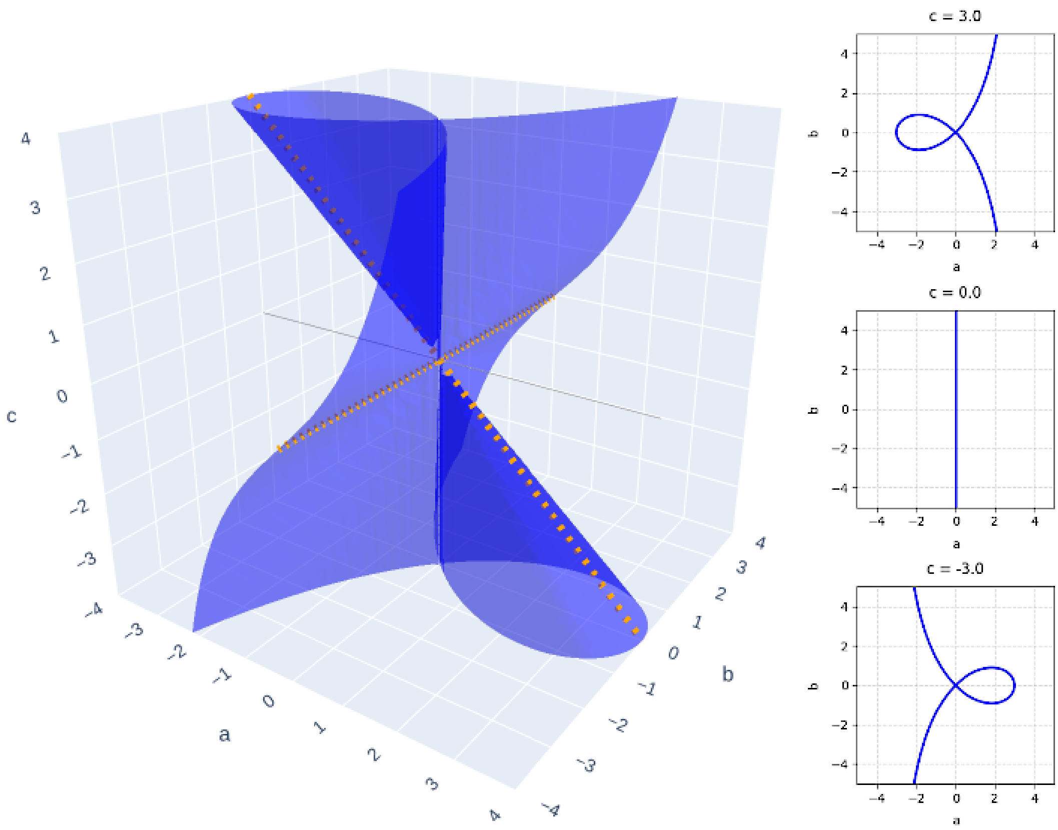

Figure 3: Zero divisor set $\det M_x = 0$ for $\mathbb{Y}(i)$

6.4 $\mathbb{Y}(1+i)$: $\alpha=1,\ \beta=1,\ \gamma=0$

For the algebra $\mathbb{Y}(1+i)$ we have

$$x = a + bi + cj, \quad a, b, c \in \mathbb{R}, \quad i^2 = -1, \quad j^2 = 0, \quad ij = ji = 1 + i.$$

Multiplication and division (unique quotient) are obtained by substituting the parameters $\alpha = 1,\ \beta = 1,\ \gamma = 0$ into the general formulas (3), (5) and (6).

Multiplication $(y = d + ei + fj)$

$$xy = (ad - be + bf + ce) + (ae + bd + bf + ce)i + (af + cd)j.$$

Unique quotient $(z = p + qi + rj,\ \det M_x = a^2(a + c) + b^2(a - c) - 2abc \neq 0)$

$$\begin{aligned} z/x = &\frac{pa(a+c) + qa(b-c) - rb(a+b)}{\det M_x} \\ &- \frac{pb(a-c) - q(a^2 - bc) + rb(a-b)}{\det M_x}\, i \\ &- \frac{pc(a+c) + qc(b-c) + r(a^2 + b^2 + c(a-b))}{\det M_x}\, j. \end{aligned}$$

The zero divisor set $\det M_x = 0$ is a curved cubic surface. Horizontal sections show cubic curves with a double point on the j-axis, which in particular form a double cone. The invertible zero divisors lie on the i-axis and on the line $b = 0,\ a = -c$.

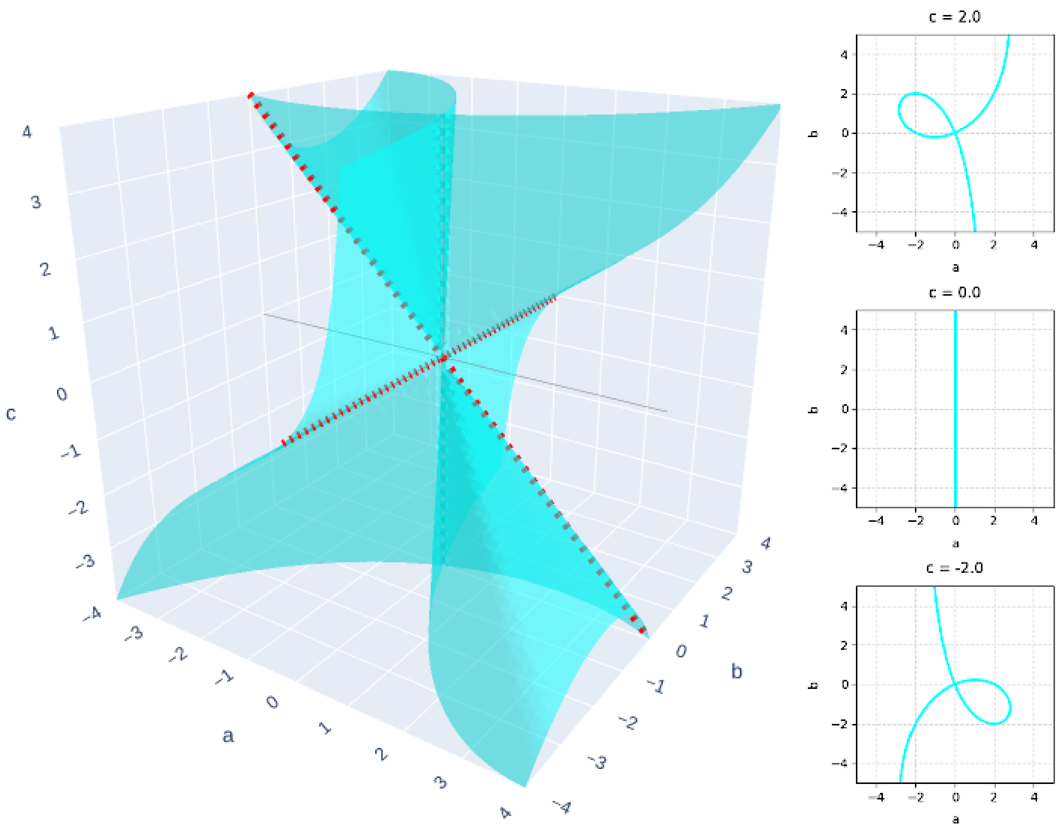

Figure 4: Zero divisor set $\det M_x = 0$ for $\mathbb{Y}(1+i)$

6.5 $\mathbb{Y}(j)$: $\alpha = 0,\ \beta = 0,\ \gamma = 1$

For the algebra $\mathbb{Y}(j)$ we have

$$x = a + bi + cj, \quad a, b, c \in \mathbb{R}, \quad i^2 = -1, \quad j^2 = 0, \quad ij = ji = j.$$

Multiplication and division (unique quotient) are obtained by substituting the parameters $\alpha = 0,\ \beta = 0,\ \gamma = 1$ into the general formulas (3), (5) and (6).

Multiplication ($y = d + ei + fj$)

$$xy = (ad - be) + (ae + bd)i + (af + cd + bf + ce)j.$$

Unique quotient ($z = p + qi + rj$, $\det M_x = a^2(a + b) + b^2(a + b) \neq 0$)

$$\begin{aligned} z/x = &\frac{pa(a+b) + qb(a+b)}{\det M_x} \\ &- \frac{pb(a+b) - qa(a+b)}{\det M_x}\, i \\ &- \frac{pc(a-b) + qc(a+b) - r(a^2+b^2)}{\det M_x}\, j. \end{aligned}$$

The zero divisor set $\det M_x = 0$ consists of the plane $a = -b$. The invertible zero divisors lie on the line $c = 0$, $a = -b$.

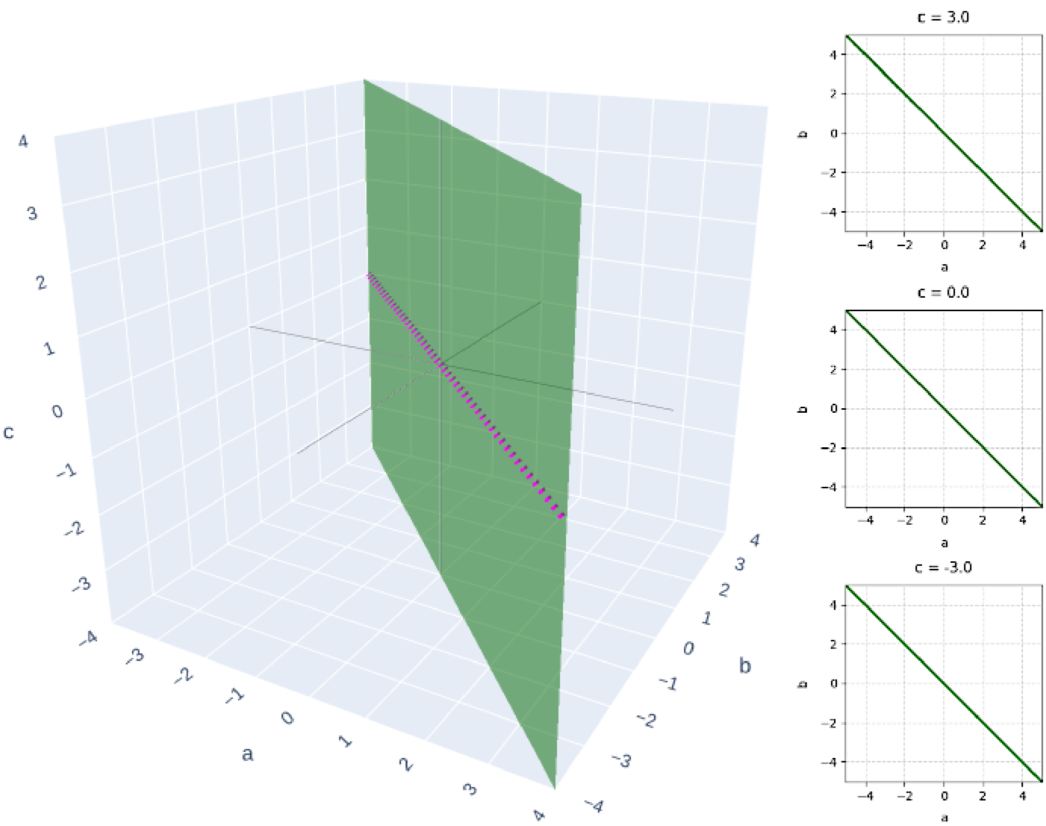

Figure 5: Zero divisor set $\det M_x = 0$ for $\mathbb{Y}(j)$

6.6 $\mathbb{Y}(1+j)$: $\alpha = 1,\ \beta = 0,\ \gamma = 1$

For the algebra $\mathbb{Y}(1+j)$ we have

$$x = a + bi + cj, \quad a, b, c \in \mathbb{R}, \quad i^2 = -1, \quad j^2 = 0, \quad ij = ji = 1 + j.$$

Multiplication and division (unique quotient) are obtained by substituting the parameters $\alpha = 1,\ \beta = 0,\ \gamma = 1$ into the general formulas (3), (5) and (6).

Multiplication $(y = d + ei + fj)$

$$xy = (ad - be + bf + ce) + (ae + bd)i + (af + cd + bf + ce)j.$$

Unique quotient $(z = p + qi + rj,\ \det M_x = a^2(a+b) + b^2(a+b) - 2abc \neq 0)$

$$\begin{aligned} z/x = &\frac{pa(a+b) + q(ab + b^2 - ac) - rab}{\det M_x} \\ &- \frac{pb(a+b) - q(a^2 + ab - bc) - rb^2}{\det M_x}\, i \\ &- \frac{pc(a-b) + qc(a+b-c) - r(a^2 + b^2 - bc)}{\det M_x}\, j. \end{aligned}$$

The zero divisor set $\det M_x = 0$ is a curved cubic surface. Horizontal sections show cubic curves with a double point on the j-axis, which in particular form a double cone. The invertible zero divisors lie on the j-axis and on the line $c = 0,\ a = -b$.

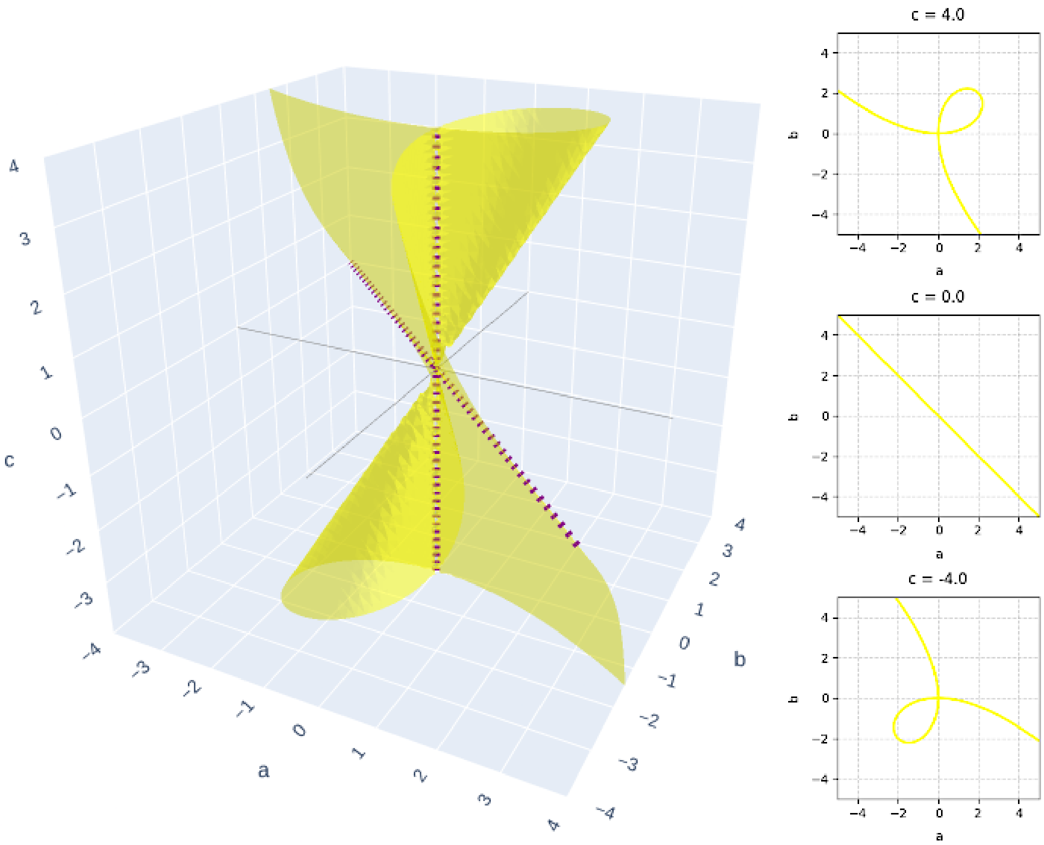

Figure 6: Zero divisor set $\det M_x = 0$ for $\mathbb{Y}(1+j)$

6.7 $\mathbb{Y}(i+j)$: $\alpha = 0,\ \beta = 1,\ \gamma = 1$

For the algebra $\mathbb{Y}(i+j)$ we have

$$x = a + bi + cj, \quad a, b, c \in \mathbb{R}, \quad i^2 = -1, \quad j^2 = 0, \quad ij = ji = i + j.$$

Multiplication and division (unique quotient) are obtained by substituting the parameters $\alpha = 0,\ \beta = 1,\ \gamma = 1$ into the general formulas (3), (5) and (6).

Multiplication ($y = d + ei + fj$)

$$xy = (ad - be) + (ae + bd + bf + ce)i + (af + cd + bf + ce)j.$$

Unique quotient ($z = p + qi + rj$, $\det M_x = a^2(a + b + c) + b^2(a + b - c) \neq 0$)

$$\begin{aligned} z/x = &\frac{pa(a+b+c) + qb(a+b) - rb^2}{\det M_x} \\ &- \frac{pb(a+b-c) - qa(a+b) + rab}{\det M_x}\, i \\ &- \frac{pc(a-b+c) + qc(a+b) - r(a^2+b^2+ac)}{\det M_x}\, j. \end{aligned}$$

The zero divisor set $\det M_x = 0$ is a cubic surface. Horizontal sections show circles intersected by a straight line through their centre, forming a double cone cut by the plane $a = -b$. The invertible zero divisors lie on the lines $c = 0$, $a = -b$ as well as $a = 0$, $b = c$ and $b = 0$, $a = -c$.

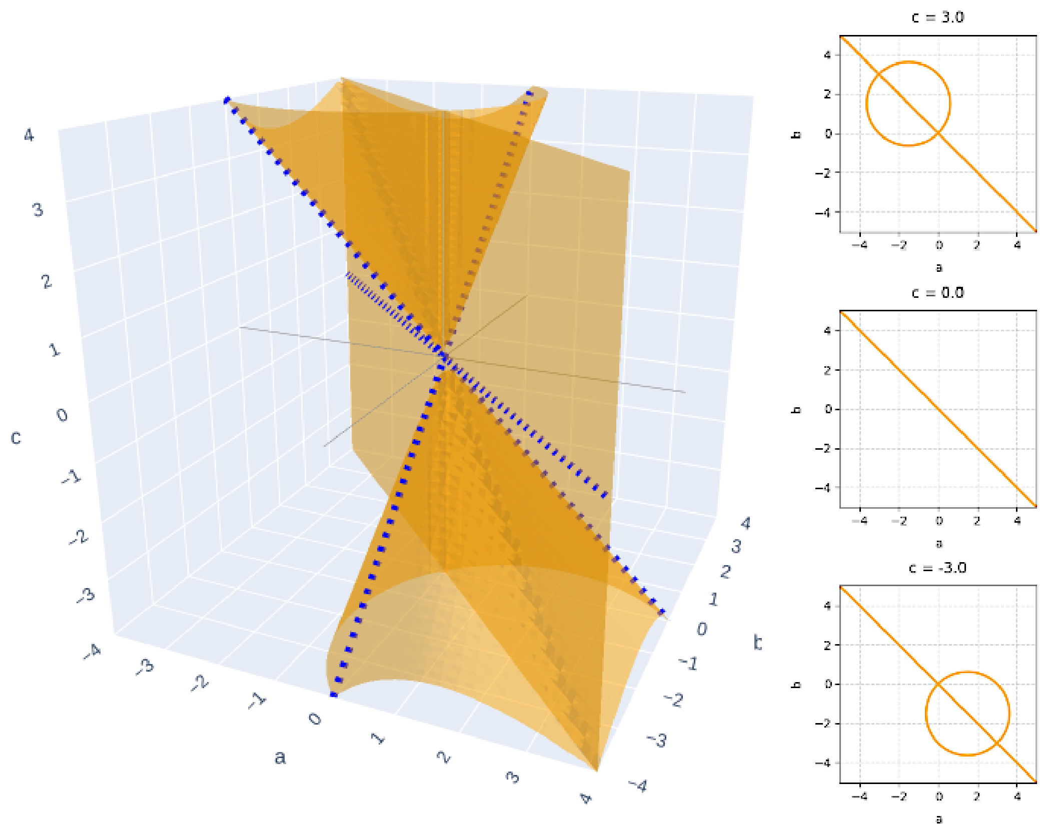

Figure 7: Zero divisor set $\det M_x = 0$ for $\mathbb{Y}(i+j)$

6.8 $\mathbb{Y}(1+i+j)$: $\alpha = 1,\ \beta = 1,\ \gamma = 1$

For the algebra $\mathbb{Y}(1+i+j)$ we have

$$x = a + bi + cj, \quad a, b, c \in \mathbb{R}, \quad i^2 = -1, \quad j^2 = 0, \quad ij = ji = 1 + i + j.$$

Multiplication and division (unique quotient) are obtained by substituting the parameters $\alpha = 1,\ \beta = 1,\ \gamma = 1$ into the general formulas (3), (5) and (6).

Multiplication $(y = d + ei + fj)$

$$xy = (ad - be + bf + ce) + (ae + bd + bf + ce)i + (af + cd + bf + ce)j.$$

Unique quotient $(z = p + qi + rj,\ \det M_x = a^2(a+b+c) + b^2(a+b-c) - 2abc \neq 0)$

$$\begin{aligned} z/x =& \frac{pa(a+b+c) + q(ab+b^2-ac) - rb(a+b)}{\det M_x} \\ &- \frac{pb(a+b-c) - q(a^2+ab-bc) + rb(a-b)}{\det M_x}\, i \\ &- \frac{pc(a-b+c) + qc(a+b-c) - r(a^2+b^2+c(a-b))}{\det M_x}\, j. \end{aligned}$$

The zero divisor set $\det M_x = 0$ is a curved cubic surface. Horizontal sections show cubic curves with a double point on the j-axis, which in particular form a double cone. The invertible zero divisors lie on the lines $c = 0$, $a = -b$ and $a = 0$, $b = c$.

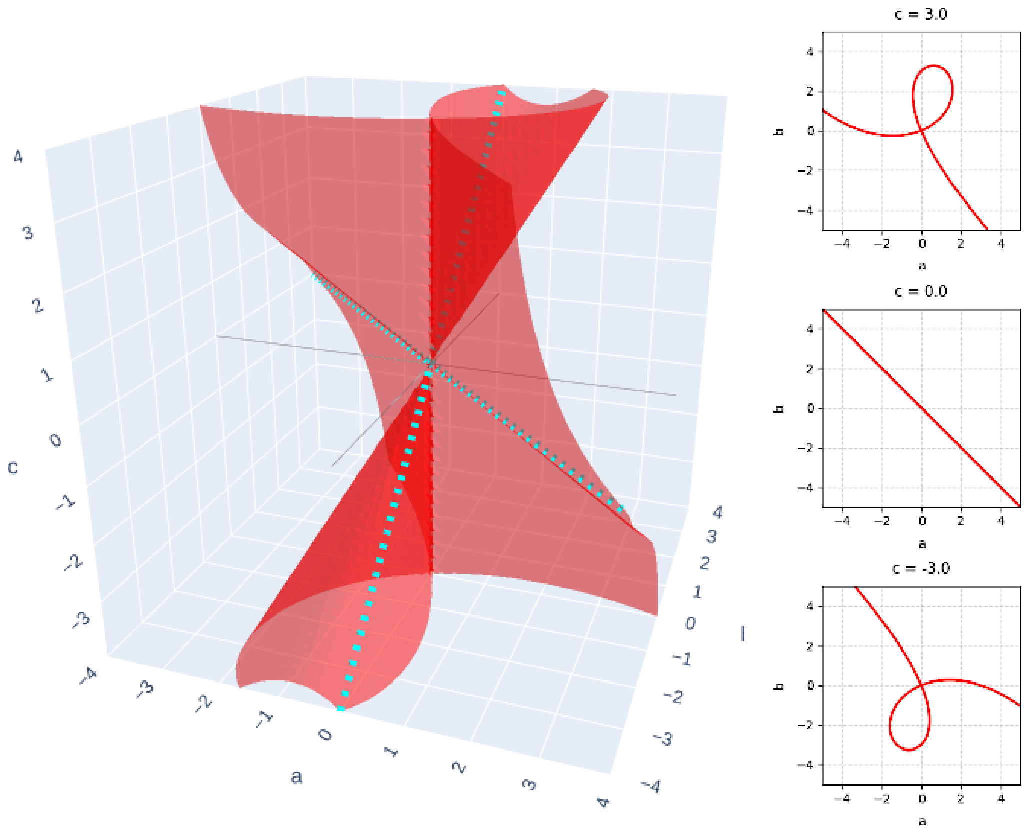

Figure 8: Zero divisor set $\det M_x = 0$ for $\mathbb{Y}(1+i+j)$

Remark 6.1 (Plane in $\mathbb{Y}(i+j)$)**.** For the algebra $\mathbb{Y}(i+j)$ $(\alpha=0, \beta=1, \gamma=1)$, Figure 7 shows the plane $a=-b$ as part of the zero divisor set $\det M_x = 0$. This arises because $\det M_x$ factorises:

$$\det M_x = a^2(a+b+c) + b^2(a+b-c) = (a+b)\big(a^2+b^2+c(a-b)\big).$$

More generally, such a factorisation occurs for all algebras of the form $\mathbb{Y}(\beta i+\gamma j)$ whenever $|\gamma|=1$:

$$\det M_x = (a+\gamma b)\big(a^2+b^2+\beta c(a-\gamma b)\big).$$

In these cases the zero divisor set contains one of the planes $a=-b$ (for $\gamma=1$) or $a=b$ (for $\gamma=-1$). For $|\gamma|\neq 1$ no such factorisation occurs for the algebras $\mathbb{Y}(\beta i+\gamma j)$; the zero divisor set $\det M_x=0$ then resembles the curved cubic surfaces shown (cf. Figures 3, 4, 6, 8).

7 Numerical examples for the algebra $\mathbb{Y}(1)$

In the following, the computational rules from Section 4 and the algebraic phenomena from Section 5 are illustrated by numerical examples for the algebra $\mathbb{Y}(1)$ (cf. Section 6.2). From the author's perspective, this algebra exhibits the most interesting zero divisor structure and at the same time defines the three most fundamental entities of mathematics: $1=ij=ji$, $-1=i^2$ and $0=j^2$.

7.1 Well-defined division for regular elements

For regular elements (i.e. $\det M_x\neq 0$), the quotient z/x is uniquely defined and satisfies the familiar relation: from $xy=z$ we obtain both $z/x=y$ and $z/y=x$.

Example

Let $x=(1,2,3)$ (i.e. $x=1+2i+3j$) and $y=(4,5,6)$ (i.e. $y=4+5i+6j$). Both are regular since

$$\det M_x = a(a^2+b^2-2bc) = 1(1+4-12) = -7\neq 0, \qquad \det M_y = 4(16+25-60) = 4(-19) = -76\neq 0.$$

Multiplication $xy=z$: From the multiplication formula for $\mathbb{Y}(1)$ (see Section 6.2) we obtain

$$xy = (1\cdot 4 - 2\cdot 5 + 2\cdot 6 + 3\cdot 5) + (1\cdot 5 + 2\cdot 4)i + (1\cdot 6 + 3\cdot 4)j = 21+13i+18j.$$

Thus $z=(21,13,18)$.

Division $z/x=y$: Using the division formula for $\mathbb{Y}(1)$ (see Section 6.2) with $a=1, b=2, c=3$ and $p=21, q=13, r=18$ we get

$$\begin{aligned} z/x &= \frac{21\cdot 1^2 + 13(1\cdot 2 - 1\cdot 3) - 18\cdot 1\cdot 2}{-7} \\ &\quad - \frac{21\cdot 1\cdot 2 - 13(1^2 - 2\cdot 3) - 18\cdot 2^2}{-7}\, i \\ &\quad - \frac{21\cdot 1\cdot 3 + 13\cdot 3(2-3) - 18(1^2+2^2-2\cdot 3)}{-7}\, j \\ &= \frac{-28}{-7} - \frac{35}{-7}\, i - \frac{42}{-7}\, j = 4+5i+6j. \end{aligned}$$

Hence $y=(4,5,6)$. The computation of $z/y=x$ proceeds analogously and yields $(1,2,3)$.

Inverse x^{-1}: From the inverse formula (8) for $\mathbb{Y}(1)$ we obtain

$$x^{-1} = \left(\frac{1 \cdot 1}{-7},\ -\frac{2 \cdot 1}{-7},\ -\frac{3 \cdot 1}{-7}\right) = \left(-\frac{1}{7},\ \frac{2}{7},\ \frac{3}{7}\right).$$

Multiplication $z \cdot x^{-1}$: The product $z \cdot x^{-1}$ (with $z = (21, 13, 18)$ and $x^{-1} = (-\frac{1}{7}, \frac{2}{7}, \frac{3}{7})$ yields

$$z \cdot x^{-1} = \left(4,\ \frac{29}{7},\ \frac{45}{7}\right).$$

This is not equal to $y = z/x = (4, 5, 6)$. Hence the equality $z/x = z \cdot x^{-1}$ does not hold; division in $\mathbb{Y}(k)$ cannot be replaced by multiplication with the inverse. The equality $z/x = z \cdot x^{-1}$ occurs only when the dividend z is a real number (i.e. $q = r = 0$).

7.2 Division by zero divisors

Division z/x is defined for a zero divisor x if the linear system $M_x y = z$ is solvable. This is precisely the case when the dividend $z = (p, q, r)$ lies in the image (column space) of M_x.

For $\mathbb{Y}(1)$ the matrix is

$$M_x = \begin{pmatrix} a & -b+c & b \\ b & a & 0 \\ c & 0 & a \end{pmatrix}.$$

Since x is a zero divisor, we have $\det M_x = 0$ and $\operatorname{rank}(M_x) = 2$; hence the column space of M_x is a plane through the origin. To determine a normal vector n of this column-space plane, two cases must be distinguished in $\mathbb{Y}(1)$:

- **Case 1:** $a \neq 0$ (i.e. x lies on the double cone). From the orthogonality conditions for two linearly independent columns of M_x a short computation yields the normal vector

  $$n = (a,\ b - c,\ -b).$$

  Thus the column-space plane is described by

  $$ap + (b - c)q - br = 0. \tag{10}$$

- **Case 2:** $a = 0$ (i.e. x lies on the zero-divisor plane). From the orthogonality conditions for two linearly independent columns of M_x we obtain the normal vector

  $$n = (0,\ -c,\ b).$$

  The column-space plane then becomes

  $$-cq + br = 0. \tag{11}$$

The column space is identical for all zero divisors on the same line through the origin (i.e. for all multiples λx with $\lambda \in \mathbb{R}$), because $M_{\lambda x} = \lambda M_x$ and scaling a matrix does not change its column space (for $\lambda \neq 0$).

The column spaces of the matrix M_x are shown exemplarily in Figures 9 and 10: for $x = (4, 2, 5)$ (Case 1) and for $x = (0, 1, 5)$ (Case 2) as blue planes. The zero divisors and their lines are drawn in red, while the intersection lines with the zero divisor set are shown in green. For zero divisors on the double cone, the column-space plane touches the zero divisor line. For zero divisors on the plane $a = 0$, the column-space plane is orthogonal to the zero divisor plane and cuts the double cone provided the zero divisor lies inside it.

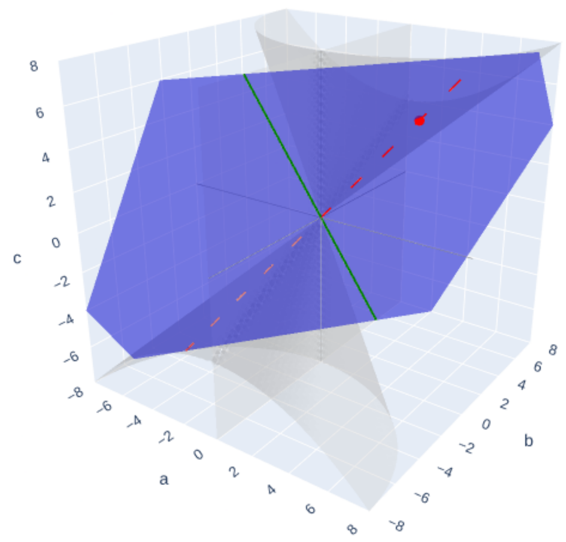

Figure 9: Column space M_x $(x = (4, 2, 5))$

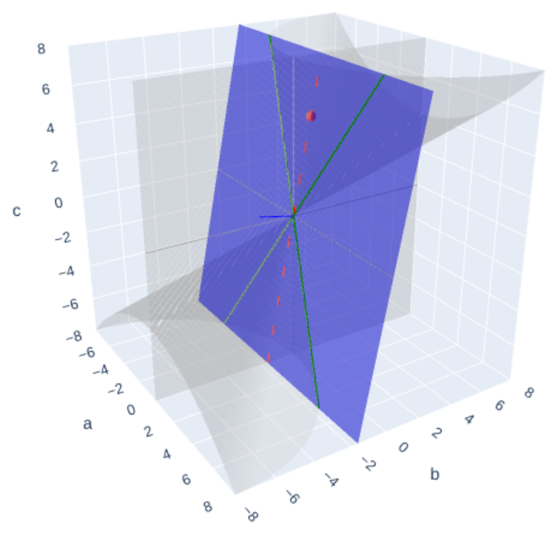

Figure 10: Column space M_x $(x = (0, 1, 5))$

Example

Let $x = (4, 2, 5)$ (i.e. $x = 4 + 2i + 5j$). Since

$$\det M_x = a(a^2 + b^2 - 2bc) = 4\big(16 + 4 - 2 \cdot 2 \cdot 5\big) = 4(20 - 20) = 0,$$

x is a zero divisor (it lies on the double cone).

Let the dividend be $z = (2, 2, 1)$ (i.e. $z = 2+2i+1j$). Because $4\cdot2-3\cdot2-2\cdot1 = 8-6-2 = 0$ from equation 10, z lies in the column space of M_x.

The matrix M_x for $x = (4, 2, 5)$ in $\mathbb{Y}(1)$ is

$$M_x = \begin{pmatrix} 4 & -2+5 & 2 \\ 2 & 4 & 0 \\ 5 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 2 \\ 2 & 4 & 0 \\ 5 & 0 & 4 \end{pmatrix}.$$

The linear system $M_x y = z$ yields

$$\begin{cases} 4d + 3e + 2f = 2 & (1) \\ 2d + 4e = 2 & (2) \\ 5d + 4f = 1 & (3) \end{cases}$$

A particular solution is obtained, for example, by setting $e = 0$: equation (2) gives $2d = 2$ $\Rightarrow$ $d = 1$, equation (3) gives $4f = 1 - 5 = -4 \Rightarrow f = -1$. Substituting into (1) confirms $4 \cdot 1 + 3 \cdot 0 + 2 \cdot (-1) = 4 + 0 - 2 = 2$. Thus a particular solution is $y_p = (1, 0, -1)$.

To determine the kernel of M_x, the homogeneous system $M_x y = 0$ gives

$$\begin{cases} 4d + 3e + 2f = 0 & (1) \\ 2d + 4e = 0 & (2) \\ 5d + 4f = 0 & (3) \end{cases}$$

From equation (2): $2d = -4e \Rightarrow d = -2e$; from (3): $4f = -5d = -5(-2e) = 10e \Rightarrow f = \frac{5}{2}e$. Substituting into (1) verifies $4 \cdot (-2e) + 3e + 2 \cdot \frac{5}{2}e = -8e + 3e + 5e = 0$.

Choosing $e = 2$ gives a basis vector of the kernel $\ker M_x$: $y_0 = (-4, 2, 5)$.

Hence the solution set for the quotient $y = z/x$ is the affine line

$$y = y_p + \lambda y_0 = (1 - 4\lambda) \;+\; 2\lambda \cdot i \;+\; (-1 + 5\lambda) \cdot j, \qquad \lambda \in \mathbb{R}.$$

Division by a zero divisor is not unique. For $\lambda = 0$ we obtain $y = 1 - j$; for $\lambda = 1$ we obtain $y = -3 + 2i + 4j$. Every choice of λ yields a quotient $y = z/x$.

7.3 Invertible zero divisors

In $\mathbb{Y}(1)$ all elements of the plane $a = 0$ (i.e. $x = bi + cj$) with $(b, c) \neq (0, 0)$ are invertible zero divisors. Let $x = (0, 1, 2)$ (i.e. $x = i + 2j$). Because

$$\det M_x = a(a^2 + b^2 - 2bc) = 0\big(0 + 1 - 2 \cdot 1 \cdot 2\big) = 0,$$

x is a zero divisor. The inverses of x are obtained from the general formula for the case $a = 0$ (see Theorem 5.3):

$$x^{-1} = -\frac{1}{b}\, i \;+\; \frac{c}{b^2}\, j \;+\; \lambda\big(-\beta b \;+\; \alpha b\, i \;+\; (b - \alpha c) j\big).$$

Substituting $b = 1, c = 2, \alpha = 1, \beta = 0$ gives

$$x^{-1} = -i + 2j + \lambda\big(0 + 1 \cdot 1 \cdot i + (1 - 1 \cdot 2) j\big) = -i + 2j + \lambda(i - j).$$

Thus the solution set for the inverse of x is the affine line

$$x^{-1} = (\lambda - 1)\, i + (2 - \lambda)\, j, \qquad \lambda \in \mathbb{R}.$$

For $\lambda = 0$ we obtain $x^{-1} = -i + 2j$; for $\lambda = 2$ we obtain $x^{-1} = i$. Every choice of λ yields an inverse of x.

7.4 Partner lines

Zero divisor on the double cone $(a^2 + b^2 = 2bc)$

For the zero divisor $x = (4, 2, 5)$, a basis vector of the kernel $\ker M_x$ was determined in Section 7.2 as $y_0 = (-4, 2, 5)$. Hence the partner line is

$$P(x) = \ker M_x = \{\lambda(-4, 2, 5) \mid \lambda \in \mathbb{R}\}.$$

Notice that the chosen basis vector $(-4, 2, 5)$ coincides with $x = (4, 2, 5)$ up to the sign of the first component.

In $\mathbb{Y}(1)$ the following holds for every zero divisor on the double cone (i.e. $a \neq 0$): if (a, b, c) is a zero divisor, then $(-a, b, c)$ is always a zero divisor partner. Indeed, the product yields vanishing i- and j-components and the real part $-a^2 - b^2 + 2bc = 0$. Thus $(-a, b, c)$ is always a zero divisor partner of (a, b, c) and vice-versa.

Zero divisor on the plane $(a = 0)$

For the zero divisor $x = (0, 1, 2)$ in $\mathbb{Y}(1)$ the matrix is

$$M_x = \begin{pmatrix} 0 & -1 + 2 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

The homogeneous system $M_x y = 0$ gives the kernel of M_x:

$$\begin{cases} e+f=0 & (1) \\ d=0 & (2) \\ 2d=0 & (3) \end{cases}$$

From (2) and (3) we obtain $d = 0$; from (1) we obtain $e = -f$. Hence a basis vector of the kernel $\ker M_x$ is $y_0 = (0, 1, -1)$, and the partner line is

$$P(x) = \{\lambda(0, 1, -1) \mid \lambda \in \mathbb{R}\}.$$

In $\mathbb{Y}(1)$ the following holds for every zero divisor $x = (0, b, c)$: $(0, b, b-c)$ is a zero divisor partner of $(0, b, c)$. Indeed, the product yields vanishing i- and j-components and the real part $-b^2 + b(b-c) + cb = -b^2 + b^2 - bc + bc = 0$. Thus $(0, b, b-c)$ is always a zero divisor partner of $(0, b, c)$ and vice-versa.

7.5 Generic invertibility of zero divisor products and quotients

7.5.1 Zero divisor × zero divisor

In the following we present three typical examples of products of zero divisors x, y with $xy \neq 0$ in $\mathbb{Y}(1)$; in each case the product is invertible.

Both factors on the cone: Let $x = (4, 2, 5)$ (cf. Section 7.2) and $y = (2, 2, 2)$ ($\det M_y = a^2 + b^2 - 2bc = 4 + 4 - 8 = 0$) be zero divisors on the double cone. Their product is

$$(4, 2, 5) \cdot (2, 2, 2) = (18,\ 12,\ 18).$$

For $z = (18, 12, 18)$ we have $\det M_z = a(a^2+b^2-2bc) = 18(324+144-2\cdot 12\cdot 18) = 18(468-432) = 18 \cdot 36 = 648 \neq 0$; hence z is not a zero divisor but invertible.

One factor on the cone, one on the plane: Let $x = (4, 2, 5)$ be a zero divisor on the double cone and $y = (0, 1, 2)$ a zero divisor on the plane $a = 0$. Their product is

$$(4, 2, 5) \cdot (0, 1, 2) = (7,\ 4,\ 8).$$

For $z = (7, 4, 8)$ we obtain $\det M_z = 7(49 + 16 - 2 \cdot 4 \cdot 8) = 7(65 - 64) = 7 \neq 0$; thus z is not a zero divisor but invertible.

Both factors on the plane: Let $x = (0, 1, 2)$ and $y = (0, 2, 3)$ be zero divisors on the plane $a = 0$. Their product is

$$(0, 1, 2) \cdot (0, 2, 3) = (5,\ 0,\ 0).$$

Since $z = (5, 0, 0)$ is a real number and its determinant is $\det M_z = 5^3 = 125 \neq 0$, z is not a zero divisor but invertible. In $\mathbb{Y}(1)$, when multiplying two elements with $a = 0$ the i- and j-components always vanish, so the product is always a real number.

Exceptions

The generic invertibility fails in $\mathbb{Y}(1)$ in the following special cases:

- **Space diagonal × $(0, b, b)$:** For $x = (a, a, a)$ or $x = (-a, a, a)$ and $y = (0, b, b)$ we have

$$(a, a, a) \cdot (0, b, b) = (ab,\ ab,\ ab), \qquad (-a, a, a) \cdot (0, b, b) = (ab,\ -ab,\ -ab).$$

  The product lies on the same space diagonal belonging to the zero divisor set and is therefore again a zero divisor.

- **Space diagonal** $\times$ $(0, b, 0)$**:** For $x = (a, a, a)$ or $(-a, a, a)$ and $y = (0, b, 0)$ we obtain

$$(a, a, a) \cdot (0, b, 0) = (0,\ ab,\ 0), \qquad (-a, a, a) \cdot (0, b, 0) = (0,\ -ab,\ 0).$$

  The product always lies on the i-axis and is again a zero divisor.

- **Two zero divisors on the same generator of the cone:** For two zero divisors $x = (a, b, c)$ and $y = \lambda x = (\lambda a, \lambda b, \lambda c)$ on the same line of the double cone we have

$$x \cdot (\lambda x) = 2\lambda a \cdot (a, b, c).$$

  The product lies on the same line through the origin as x and is therefore again a zero divisor.

7.5.2 Zero divisor $\times$ unit

Let $x = (4, 2, 5)$ be a zero divisor (cf. Section 7.2) and $y = (1, 2, 3)$ invertible (cf. Section 7.1). Their product xy is

$$(4, 2, 5) \cdot (1, 2, 3) = (16,\ 10,\ 17).$$

For $z = (16, 10, 17)$ we have $\det M_z = a(a^2+b^2-2bc) = 16(256+100-2\cdot 10\cdot 17) = 16(356-340) = 16 \cdot 16 = 256 \neq 0$; hence z is not a zero divisor but invertible.

Conversely, every zero divisor x can generically be represented as a quotient z/y of two units. For example, $(16, 10, 17)/(1, 2, 3) = (4, 2, 5)$.

Exceptions

The generic invertibility fails for the product zero divisor $\times$ unit in $\mathbb{Y}(1)$ in the following exceptional constellations:

- **Real scaling:** The unit $y = (d, e, f)$ is a real number ($e = f = 0$). Then, due to the homogeneity of the determinant, the product xy remains on the same zero divisor line through the origin as the original zero divisor x.

- **Vanishing real part:** For a zero divisor $x = (a, b, c)$ and a unit $y = (d, e, f)$ the real part of the product xy becomes zero, i.e. $ad + (c - b)e + bf = 0$. This equation defines a plane in the space of units y. Example: for $x = (4, 2, 5)$ and $y = (1, 0, -2)$ ($\det M_y = 1$) we obtain $xy = (0, 2, -3)$ (lies on the plane $a = 0$ and is therefore a zero divisor).

- **Mapping onto the cone:** The condition that the product xy of a zero divisor $x = (a, b, c)$ and a unit $y = (d, e, f)$ lands on the double cone depends on the location of x:

  - If x itself lies on the cone ($a^2 + b^2 - 2bc = 0$), the condition is $ce - bf = 0$. This defines a plane in the space of units y. The product xy lies on the same zero divisor line as x. Example: for $x = (4, 2, 5)$ and $y = (1, 2, 5)$ ($\det M_y = -15$) we get $xy = (20, 10, 25)$. Since $xy = 5(4, 2, 5)$, it lies on the same zero divisor line as x and therefore on the double cone.
  - If x lies on the plane $a = 0$ and inside the double cone ($b(b - 2c) < 0$), the condition is $(b - c)e - bf = \pm\sqrt{-b(b - 2c)}\, d$. This defines two intersecting planes through the origin. The product xy lies on the double cone. Example: for $x = (0, 1, 5)$ and $y = (1, 1, -1)$ ($\det M_y = 4$) we obtain $xy = (3, 1, 5)$. Because $\det M_{xy} = 0$ and the real part is non-zero, the product xy lies on the double cone.

Figure 11 shows the exceptional set for the zero divisor $x = (0, 1, 5)$. This zero divisor and its zero divisor line λx lie inside the double cone and are drawn in red. The yellow plane is the set of units y that, when multiplied by x (or by λx), are mapped back onto the zero divisor line λx and thus onto the plane $a = 0$. It corresponds to the column space of the matrix whose elements form the zero divisor partner line of x (shown in blue, $\lambda(0, 1, -4)$).

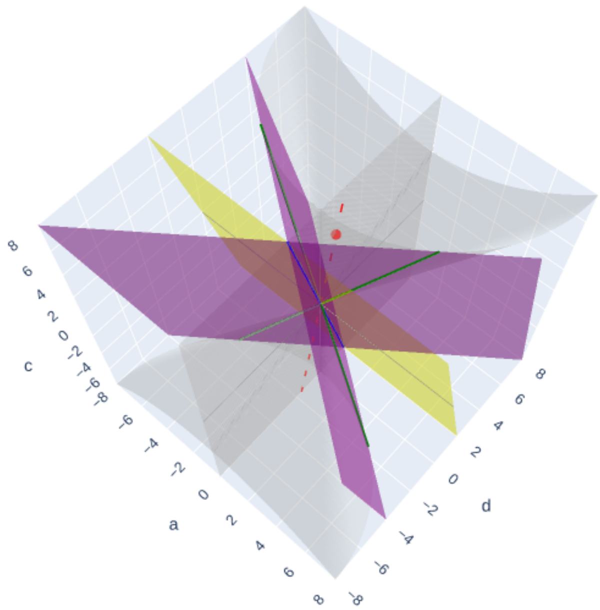

Figure 11: Exceptional set for $x = (0, 1, 5)$

The purple pair of planes is the set of units y that, when multiplied by x (or by λx), land on the double cone. It corresponds to the column spaces of the matrices of the green zero divisor lines $\lambda(3, 1, 5)$ and $\lambda(-3, 1, 5)$ on the double cone, onto which the zero divisor $x = (0, 1, 5)$ can be mapped by multiplication. The column space of M_x shown in Figure 10 intersects the double cone exactly in these two zero divisor lines. The blue zero divisor partner line $\lambda(0, 1, -4)$ is the common intersection line of the three exceptional planes that describe the multiplicative reduction to zero divisors.
Thus the zero divisors of the plane and those of the double cone are linked by a fixed relation mediated by the column spaces.

7.5.3 Zero divisor ÷ unit

Let $z = (4, 2, 5)$ be a zero divisor (cf. Section 7.2) and $x = (1, 2, 2)$ invertible ($\det M_x = a^2 + b^2 - 2bc = 1 + 4 - 8 = -3 \neq 0$). The quotient z/x yields

$$(4, 2, 5)/(1, 2, 2) = (2,\ -2,\ 1).$$

For $y = (2, -2, 1)$ we have $\det M_y = d(d^2 + e^2 - 2ef) = 2(4 + 4 + 4) = 24 \neq 0$; hence y is not a zero divisor but invertible.

Conversely, every zero divisor z can be generically represented as a product xy of two units. For example, $(1, 2, 2) \cdot (2, -2, 1) = (4, 2, 5)$.

Exceptions

The generic invertibility fails for the quotient zero divisor ÷ unit in $\mathbb{Y}(1)$ in the following exceptional constellations:

- **Real scaling:** The unit $x = (a, b, c)$ is a real number ($b = c = 0$). Then, due to the homogeneity of the determinant, the quotient z/x remains on the same zero divisor line through the origin as the original zero divisor z.
- **Vanishing real part:** For a zero divisor $z = (p, q, r)$ and a unit $x = (a, b, c)$ the real part of the quotient z/x becomes zero, i.e. $pa + q(b-c) - rb = 0$. This equation defines a plane in the space of units x. It is formally identical with equation 10 in Section 7.2. Example: for $z = (4, 2, 5)$ and $x = (1, 0, 2)$ ($\det M_x = 1$) we obtain $z/x = (0, 2, 5)$ (lies on the plane $a = 0$ and is therefore a zero divisor).
- **Mapping onto the cone:** The condition that the quotient z/x of a zero divisor $z = (p, q, r)$ and a unit $x = (a, b, c)$ lands on the double cone depends on the location of z:
  - If z itself lies on the cone ($p^2 + q^2 - 2qr = 0$), the condition is $rb - qc = 0$ (cf. the mapping onto the cone in Section 7.5.2). This defines a plane in the space of units x. The quotient z/x lies on the same zero divisor line as z. Example: for $z = (4, 2, 5)$ and $x = (1, 2, 5)$ ($\det M_x = -15$) we obtain $z/x = (\frac{4}{5}, \frac{2}{5}, 1)$. Since $z/x = \frac{1}{5}(4, 2, 5)$, it lies on the same zero divisor line as z and therefore on the double cone.

– If z lies on the plane $a = 0$ and outside the double cone ($q(q-2r) > 0$), the condition is $br - cq = \pm a\sqrt{q(q-2r)}$. This defines two intersecting planes through the origin. The quotient z/x lies on the double cone. Example: for $z = (0, 2, -3)$ and $x = (1, 0, 2)$ ($\det M_x = 1$) we obtain $z/x = (-4, 2, 5)$. Because $\det M_{z/x} = 0$ and the real part is non-zero, the quotient z/x lies on the double cone.

Figure 12 shows the exceptional set for the zero divisor $z = (0, 2, -3)$. This zero divisor and its zero divisor line λz lie outside the double cone and are drawn in red. The yellow plane is the set of regular divisors x for which the quotient z/x (or $\lambda z/x$) is mapped back onto the zero divisor line λz and thus onto the plane $a = 0$. It corresponds to the column space of the matrix of the blue zero divisor partner line of z ($\lambda(0, 2, 5)$).

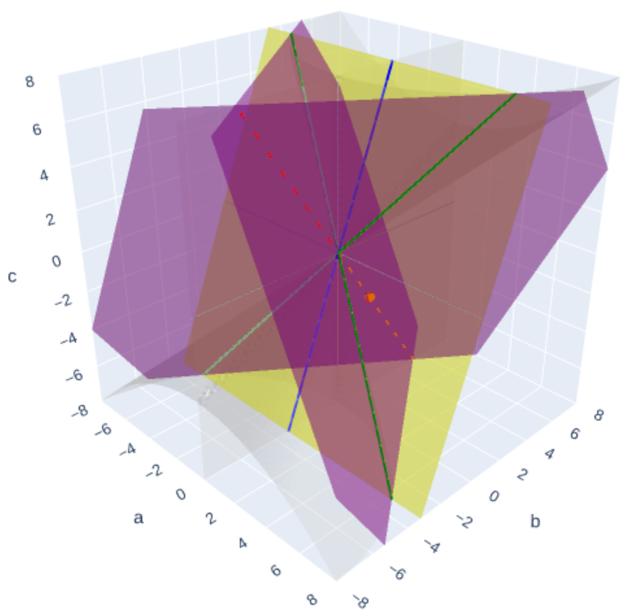

Figure 12: Exceptional sets for $z = (0, 2, -3)$

The purple pair of planes is the set of regular divisors x for which the quotient z/x (or $\lambda z/x$) lands on the double cone. It corresponds to the column spaces of the matrices of the green zero divisor lines $\lambda(4, 2, 5)$ and $\lambda(-4, 2, 5)$ on the double cone, onto which the zero divisor $z = (0, 2, -3)$ can be mapped by division. These lines are simultaneously the intersection lines of the yellow plane with the double cone and with the purple pair of planes. The pair of planes intersects the plane $a = 0$ in the zero divisor line λz, exactly as shown in Figure 9. Thus, also for division, the zero divisors of the plane and those of the double cone are linked by a fixed relation mediated by the column spaces.

If an invertible zero divisor lies inside the double cone, it can be mapped onto the cone by multiplication. If it lies outside the cone, it can be mapped onto the cone only by division.

8 Conclusion and outlook

From the author's perspective, the trinion algebras $\mathbb{Y}(k)$ introduced in this work represent the simplest conceivable three-dimensional extension of the complex and dual numbers. With only three basis elements $\{1, i, j\}$ and the three rules $i^2 = -1$, $j^2 = 0$, $ij = ji = k$, a surprisingly rich structure unfolds from a minimal axiomatic system – a structure that preserves commutativity and can be rigorously analysed despite non-associativity and the presence of zero divisors. Unlike Hamilton's quaternions or the octonions [1, p. 6], where all non-real basis elements are defined in a uniform way, each of the three basis elements of $\mathbb{Y}(k)$ brings its own distinct character.

The work shows that the phenomena arising from three-dimensionality and non-associativity – such as the existence of invertible zero divisors, the multiplicative non-closure of the zero divisor set, and the rich geometry of that set – are not pathological degenerations but rather expressions of the structural wealth of the algebra. Zero divisors are the essential structure-forming elements, enabling, among other things, locally separated regions.

The non-associativity of $\mathbb{Y}(k)$ endows the algebra with an increased structural complexity. As a result, the outcome of a sequence of operations depends not only on the elements involved but crucially on their bracketing (the order in which the operations are performed). Together with the process-like properties of the nilpotent basis element j, this opens up the idea of an emergent algebraic time structure that does not require a linear time axis.

Moreover, the algebras $\mathbb{Y}(k)$ exhibit conceptual parallels with fundamental principles of physics and systems theory: the nilpotent component can serve as a model for processuality and irreversibility, the complex component for rotations, while three-dimensionality naturally describes space. The existence of zero divisors provides an algebraic analogue for structures and annihilation.

Furthermore, the additive decomposition of a trinion suggests heuristic resonances with fundamental physical quantities: the real part a can be associated with an inertial mass, the i-part with an angular momentum (spin), and the j-part with a linear momentum. These diverse analogies suggest that $\mathbb{Y}(k)$ is not only of pure mathematical interest but might open promising interpretative interfaces with theoretical physics, systems theory, and the philosophy of nature.

The present work is intended as an exploratory foundational study without any claim to completeness. It aims to stimulate further research, because in the author's naive conception the universe itself is a three-dimensional algebra.

Transparency note

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The Python library SymPy was used for the computational programs. Matplotlib (for the 2D sections) and Plotly (for the 3D isosurfaces) were employed for the visualization of the zero divisor sets.

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