Machine-Checked CMP116 Fluctuation Reduction: Physical Constraint Coordinates and the Interacting-Hessian Frontier

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Abstract

We give a machine-checked reduction of the finite-dimensional fluctuation integral in Bałaban's CMP116 large-field analysis. Starting from the physical block constraint Q, the formal development constructs a sparse right inverse E and the constraint-elimination operator $C = I - EQ$. It proves $QE = I$, $QC = 0$, $C^2 = C$, the exact sparse norm $\|EB\| = M^{d-1}\|B\|$, and the volume-independent bound $\|C\| \leq 1 + M^{d-1}$ for $d \geq 3$. An exact physical/CMP116 isometry transports C to finite Gaussian coordinates without norm loss. The same development constructs the physical localization projector P_{Z_0}, evaluates the complex quadratic Gaussian, localizes its determinant to rank $|I(Z_0)|$, performs the outer Gaussian integration, and absorbs both costs into an explicit $\exp(c|Z_0|)$ factor.

Two corrections exposed by formalization are central. First, the useful domination occurs after Gaussian integration rather than through an unavailable pointwise supremum in the fluctuation field. Second, the localized quadratic matrix is $A = -\alpha_5 P_{Z_0}$. In the exactly identified trivial-background sector the terminal Lean theorem inserts the concrete C, the flat Hessian, the complement localization, and the covariance root directly into the printed source $\Gamma_k = C^T \Delta_k (C P_{Z_0^c})(C^{(k)})^{1/2}$ and returns an explicit Cauchy bound with no ambient-volume factor. CMP116, however, requires the base Hessian at a generally nontrivial small background $\bar{U}$. We do not construct $D^2 S_{\text{Wilson}}(\bar{U})$ or the random-walk estimate (2.16), and therefore do not prove the physical domination, (2.26), `hraw`, `hRpoly`, a continuum limit, or a mass gap. The contribution is an auditable reduction that closes the constraint and Gaussian layers and identifies the first genuinely missing interacting construction.

1 Introduction

Rigorous renormalization-group analyses of lattice gauge theory separate small- and large-field regions, integrate finite-dimensional fluctuation fields, and reorganize the result into polymer activities. Bałaban's series [1, 2, 3, 4] and Dimock's exposition [5, 6, 7] provide the mathematical source for the present formalization. The specific target is the fluctuation-integral step around equations (2.14), (2.20)–(2.26) of the CMP116 large-field analysis.

The purpose of the development is not to replace an analytic estimate by an interface bearing the same estimate as a hidden hypothesis. Instead, it asks which parts of the source argument are finite, algebraic, geometric, or measure-theoretic and can therefore be closed independently, and which part is the genuine remaining Yang–Mills estimate.

*This manuscript and parts of the formal development were produced with AI assistance under the repository's audit protocol. Every result labelled *machine-checked* is a Lean 4 theorem against a pinned Mathlib. Repository: https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME.

The machine-checked chain established here is

$$\begin{aligned} (D,P) &\longmapsto Z_0 \longmapsto P_{Z_0} \\ Q &\longmapsto E \longmapsto C = I - EQ \longmapsto C_{\mathrm{matrix}} \\ (C_{\mathrm{matrix}}, \Delta_{\mathrm{flat}}, R_{\mathrm{matrix}}) &\longmapsto \Gamma_{\mathrm{flat}} \longmapsto \text{localized Gaussian cost} \longmapsto \text{Cauchy term bound.} \end{aligned} \tag{1}$$

Here D denotes the family of large-field components, P the selected large-field bonds, Z_0 their least admissible localization region, R_{matrix} is the finite matrix of the physical covariance root, while C_{matrix} is the transported constraint elimination. The terminal theorem in the flat sector leaves the physical domination as its substantive input; all source-norm, determinant, cardinality, and outer Gaussian costs are produced internally from explicit operator bounds.

This separation is useful even before those estimates are proved. It prevents three common sources of ambiguity: choosing the localization projector by hand, placing an impossible supremum before a Gaussian integration, and using the wrong sign in the shifted quadratic form.

2 Finite physical localization

Let Z_0 be a finite family of coarse blocks. A physical site belongs to the interior when the site and all its forward and backward nearest neighbours belong to the union of blocks. A positively oriented physical bond is interior when both endpoints are interior. This two-endpoint convention is important: a one-endpoint convention does not represent the source boundary condition faithfully.

The development constructs a least region

$$Z_0 = \mathrm{LocalizationCore}(D,P)$$

that contains the geometric seed induced by D and makes every $e \in P$ interior. It proves the universal property

$$\mathrm{LocalizationCore}(D,P) \subseteq W \quad\Longleftrightarrow\quad W \text{ contains the seed and every } e \in P \text{ is interior to } W. \tag{2}$$

The principal Lean endpoints are `cmp116LocalizationCore_subset_iff_admissible` and `cmp116LocalizationCore_admissible`.

The physical-to-scalar dictionary contains an equivalence

$$\mathrm{coordEquiv} : (q,a) \simeq (e,b)$$

between CMP116 scalar coordinates and physical bond/Lie coordinates. It defines the canonical finite carrier

$$I(Z_0) = \{qa : \mathrm{bond}(\mathrm{coordEquiv}(qa)) \text{ is interior to } Z_0\}. \tag{3}$$

The projector P_{Z_0} is the diagonal matrix with entries 1 on $I(Z_0)$ and 0 elsewhere. The following theorem is machine-checked.

Theorem 2.1 (Physical localization projector)**.** *For every scalar field x and coordinate qa,*

$$(P_{Z_0}x)_{qa} = \begin{cases} x_{qa}, & qa \in I(Z_0), \\ 0, & qa \notin I(Z_0), \end{cases} \qquad x^T P_{Z_0} x = \sum_{qa \in I(Z_0)} x_{qa}^2.$$

If Z_0 is admissible for (D,P), then for every $e \in P$ and every Lie coordinate a, $\mathrm{coordEquiv}^{-1}(e,a) \in I(Z_0)$*. The same holds for the least region* $\mathrm{LocalizationCore}(D,P)$*.*

This is formalized in `BalabanCMP116Eq223PhysicalLocalizationProjector.lean`. In particular, the shifted Hessian below no longer accepts an arbitrary user-supplied set of coordinates.

3 Physical constraint elimination

Let Q be the flat block constraint from fine one-cochains to coarse one-cochains. For every coarse bond the formalization chooses one distinguished fine pivot bond and defines a sparse insertion E supported on those pivots. The normalization is fixed so that

$$QE = I. \tag{4}$$

The physical elimination operator is then

$$C = I - EQ. \tag{5}$$

This is the padded square realization of the rectangular free-coordinate map: on the full fine space it is the projection onto $\ker Q$, while on inputs vanishing at the pivot bonds it parametrizes the constrained field by the remaining bonds. The distinction matters; no Gaussian Jacobian statement is inferred merely from the padded realization.

Theorem 3.1 (Constraint elimination and uniform norm)**.** *The operators E, Q, C satisfy*

$$QE = I, \qquad QC = 0, \qquad Cx = x \ (Qx = 0), \qquad C^2 = C.$$

Every nonpivot coordinate is preserved, and each pivot coordinate is changed by exactly the coarse constraint value. Moreover

$$\|EB\| = M^{d-1}\|B\|, \qquad \|C\| \leq 1 + M^{d-1} \quad (d \geq 3), \tag{6}$$

independently of the periodic volume N'.

The first equality in (6) follows from the disjoint support of the pivot insertion. The second combines it with the certified contraction of Q and the triangle inequality in $C = I - EQ$. The exact isometric physical/CMP116 dictionary defines C_{matrix} and proves

$$\|C_{\mathrm{matrix}}\| = \|C\|, \qquad \text{coordinates}(C_{\mathrm{matrix}}x) = C(\text{coordinates}(x)). \tag{7}$$

Thus the strongest flat-sector endpoint receives neither an arbitrary matrix C, nor a constant C_{norm}, nor a proof bounding it.

4 Exact Gaussian calculus and physical covariance

Let γ denote the standard Gaussian probability measure on a finite real coordinate space. For a real symmetric matrix A with $I + A > 0$ and a complex source c, the formal development proves

$$\int \exp\left(-\frac{1}{2}x^T Ax + c^T x\right) d\gamma(x) = \frac{1}{\sqrt{\det(I + A)}} \exp\left(\frac{1}{2}c^T (I + A)^{-1} c\right). \tag{8}$$

The source pairing is bilinear, not Hermitian: there is no complex conjugation on c. The proof proceeds through the one-dimensional integral, finite products, orthogonal invariance, spectral diagonalization, determinant reconstruction, and the inverse quadratic form. Coupled two-stage integrals and their Schur complement are also formalized.

For a real covariance-root matrix R, the pushforward measure $R_*\gamma$ is a positive probability measure. Substituting $x = Ru$ in (8) gives the majorant

$$M_{2.24}(R, A, c) = \det(I + R^T AR)^{-1/2} \exp\left(\frac{1}{2}\operatorname{Re}\left[c^T (I + R^T AR)^{-1} c\right]\right). \tag{9}$$

The equality between the norm of the complex Gaussian integral and $M_{2.24}$ is machine-checked. So is the domination principle

$$|f(u)| \le e^{-u^T Au/2 + r^T u} \text{ a.e.} \quad \Longrightarrow \quad \left| \int f\, d(R_* \gamma) \right| \le M_{2.24}(R, A, r). \tag{10}$$

The matrix part of this majorant is also quantitatively closed. Put $q = \alpha_5 \|R\|^2_{2\to 2} < 1$, let $S = I(Z_0)$, and take $A = -\alpha_5 P_S$. A rectangular Sylvester identity moves the determinant to the localized coordinate subtype and yields

$$(1-q)^{|S|} \le \det(I - \alpha_5 R^T P_S R). \tag{11}$$

The exponent is the physical localized rank, not the dimension of the ambient Gaussian space. The inverse source form is bounded spectrally, while the remaining outer Gaussian moment is evaluated exactly:

$$\int \exp\left(\beta \sum_{i\in S} x_i^2\right) d\gamma(x) = \big(1 - 2\beta\big)^{-|S|/2}, \qquad 2\beta < 1. \tag{12}$$

Finally, the explicit geometric estimate

$$|S| \le |Z_0| M^d d(N_c^2 - 1) \tag{13}$$

absorbs both (11) and (12) into

$$\text{localized determinant cost} \times \text{outer moment} \le \exp[(c_{\mathrm{det}} + c_G)|Z_0|]\,, \tag{14}$$

where both rates are explicit logarithmic expressions in the corresponding small parameters. There is no hidden factor depending on the total volume.

The physical covariance certificate supplies an operator C, a root $C^{1/2}$, self-adjointness, positivity, the exact identity $(C^{1/2})^2 = C$, and localization/norm bounds. The scalar dictionary is proved to be an exact L^2 isometry. Therefore its matrix realization R_{matrix} satisfies

$$\|R_{\mathrm{matrix}}\|_{2\to 2} = \|C^{1/2}\|_{2\to 2}, \qquad \|R_{\mathrm{matrix}}\|^{,2}_{2\to 2} = \|C\|_{2\to 2}. \tag{15}$$

There is no dimension-dependent condition number in this transport.

5 Two corrections forced by formalization

5.1 Integrate before taking the norm

A tempting interface asks for a pointwise bound

$$|\,\mathrm{innerIntegrand}(X, B)| \le M \quad \text{uniformly in } B.$$

For the source bilinear factor $\exp(-\langle B, \Gamma X\rangle)$ this is generally false: along a direction of B with the appropriate sign, the real part of the exponent grows. Such a hypothesis would make a downstream theorem formally true but physically unusable.

The repaired interface integrates against the conditioned Gaussian first:

$$\left| \int \mathrm{innerIntegrand}(X, B), d\mu_{\sigma,\tau}(B) \right| \le M_{\mathrm{inner}}. \tag{16}$$

Equation (10), followed by the exact evaluation (9), produces this bound while retaining the Gaussian damping and completing the square. The outer integration is then controlled by its probability measure and the outer weight.

5.2 The localized quadratic matrix has negative sign

The localized source factor contains

$$\exp\Big(\frac{\alpha_5}{2}\|P_{Z_0}B\|^2\Big).$$

In the convention of (8), this corresponds to

$$A = -\alpha_5 P_{Z_0}, \tag{17}$$

not to a positive-semidefinite A. Hence the actual positivity obligation is

$$I - \alpha_5 R_{\text{matrix}}^T P_{Z_0} R_{\text{matrix}} > 0. \tag{18}$$

By (15), the source condition

$$0 \le \alpha_5, \qquad \alpha_5\,\text{covNormBound} < \frac{1}{2}, \qquad \|C\|_{2\to2} \le \text{covNormBound}, \tag{19}$$

implies (18). The corresponding Lean endpoint is

`posDef_physicalRootMatrix_of`
`_alpha5_covariance_half_small`
`_physicalZ0.`

Its canonical-core specialization takes only (D, P) as localization data.

6 The main reduction theorem

Let G denote the finite CMP116 data for the contour variables, weights, cutoffs, field evaluation, and interaction exponent. In the exactly identified trivial-background sector define

$$\Delta_{\text{flat}} = K_0 + aQ^*Q, \tag{20}$$

$$\Gamma_{\text{flat}} = C_{\text{matrix}}^T \Delta_{\text{flat}}(C_{\text{matrix}} P_{Z_0^c}) R_{\text{matrix}}. \tag{21}$$

The finite-range kernel and bond-ball count give the volume-independent bound

$$\|\Delta_{\text{flat}}\|_{2\to2} \le \big((4d)^2 + 4 + |a|M^2\big)\big(2(3M+1)\big)^d d. \tag{22}$$

Since coordinate transport is isometric and $\|CP_{Z_0^c}\|_{2\to2} \le \|C\|_{2\to2}$, equations (6) and (22) produce an explicit squared source rate for (21). The only substantive functional premise left in the strongest endpoint is the physical Gaussian domination

$$|\,\text{innerIntegrand}_G(b)| \le \exp\Big(\frac{\alpha_5}{2} b^T P_{Z_0} b + \langle \Gamma_{\text{flat}} P_{Z_0} x, b\rangle\Big) \qquad \text{(Dom)} \tag{23}$$

for $(R_{\text{matrix}})_*\gamma$-almost every b on the Cauchy contours.

Theorem 6.1 (Physical-constraint Gaussian reduction, flat sector)**.** *Assume $d \ge 3$, a physical localized covariance-root certificate, the smallness conditions needed for the two Gaussian moments, positive Cauchy radii, a nonnegative uniform outer-weight bound M_{out}, and (23). Then the literal finite equation-(2.14) term satisfies*

$$\|\text{term}_G(Y_0, P; \psi, \phi)\| \le \text{CauchyRate}_\Delta(\text{CauchyRate}_Y(M_{\text{out}} \exp(c_{\text{tot}}|Z_0|))), \tag{24}$$

where c_{tot} is the explicit total Gaussian cardinality rate formed from M, d, N_c, α_5, the covariance-root norm, the bound $1 + M^{d-1}$ for C, and the right-hand side of (22). No arbitrary C, C-norm certificate, Hessian matrix, Hessian-norm certificate, ambient determinant dimension, uniform-in-x source bound, or free Gaussian-majorant premise occurs in the endpoint.

The Lean endpoint is

`norm_term_le_cauchyRate_of_physicalConstraintGamma _flatDelta_expCard`

in `BalabanCMP116Eq214PhysicalConstraintC.lean`.

Remark 6.2 (What the theorem does not say)**.** Theorem 6.1 is a reduction theorem. It does not prove the concrete interacting-background kernel hypotheses used to obtain (23), the random-walk estimate (2.16), the subsequent termwise factorization of equation (2.26), or `hRpoly`. Consequently it implies neither a lattice mass gap nor a continuum Yang–Mills construction.

7 Formal architecture and audit

Table 1 maps the mathematical chain to the principal modules. All entries abbreviate filenames with prefix `BalabanCMP116` and suffix `.lean`.

Layer	Lean module
Physical incidence and least Z_0	`Eq214Interior`, `Eq214LocalizationCore`
Cutoff and Cauchy extraction	`Eq23Cutoff`, `Eq214Cauchy`
Exact Gaussian identities	`QuadraticGaussianMatrix`, `CorrelatedGaussian`
Domination and (2.24)	`Eq223GaussianDomination`, `Eq224GaussianBound`, `Eq224DeterminantBound`, `Eq224SourceBound`
Physical root and norm transport	`Eq223PhysicalRootNormBridge`
Physical P_{Z_0}	`Eq223PhysicalLocalizationProjector`
Constraint right inverse and $C = I - EQ$	`CMP96ConstraintRightInverse`, `CMP96ConstraintElimination`
Uniform norm and CMP116 transport of C	`CMP96ConstraintNorm`, `Eq214PhysicalConstraintC`
Localized determinant, outer moment	`Eq224LocalizedDeterminant`, `Eq225OuterGaussianMoment`, `Eq226GaussianCardinality`
Physical source and flat Hessian	`Eq214GammaComplement`, `Eq214FlatDeltaBound`
Terminal reduction	`Eq214PhysicalConstraintC`

Table 1: Principal theorem-to-artifact map.

The repository pins Lean and Mathlib. The root build after the physical constraint checkpoint completes 8,467 jobs. The 32 new mathematical endpoints across the construction, norm, and terminal-consumption checkpoints were audited separately; each prints only `propext`, `Classical.choice`, and `Quot.sound`. The added modules contain no `sorry`, `admit`, local `axiom`, `unsafe`, or `opaque` declaration.

The reproducible checkpoint is maintained on the repository branch `codex/hrpoly-b3-card-tilt`, pull request #28. The immutable release tag is `hrpoly-cmp116-main-reduction-v0.3-2026-07-16`. The exact commands are

```
lake exe cache get
lake build YangMillsCore
lake env lean oracle_check.lean
```

8 Remaining analytic frontier

The formalization localizes the remaining difficulty rather than reducing its mathematical importance. CMP116 does not identify the contour Hessian with the flat operator (20). Its base point is $(\sigma(Z), U, J) = (0, \bar{U}, 0)$, where $\bar{U}$ is a generally nontrivial small background. The differences in the quadratic forms, covariance, and mixed source produce R_1, R_2, R_3. The required estimate has the schematic form

$$|R_i(b, b')| \le \left[c_0 e^{-\delta_0 M/3} + c_1(\alpha_0 + \alpha_1)\right] e^{-\delta_0 d(b,b')/2}. \tag{25}$$

The repository contains consumers of such a bound, but no load-bearing construction of the second variation of the nonabelian Wilson action at $\bar{U}$ and no random-walk expansion proving (25).

Thus the remaining route is

$$\begin{aligned} S_{\text{Wilson}} \longrightarrow D^2 S_{\text{Wilson}}(\bar{U}) \longrightarrow \text{random-walk expansion} \longrightarrow (2.16) \\ \longrightarrow h_{\text{dom}} \longrightarrow (2.26) \longrightarrow h_{\text{raw}} \longrightarrow \text{KP}. \end{aligned}$$

The determinant, inverse-source, cardinality, constraint-coordinate, and flat source-norm analyses are already internal. Introducing (2.16) as a renamed hypothesis would not close this frontier; the next valuable endpoint must derive it from the physical Wilson Hessian.

9 Conclusion

The formal development closes the finite geometry, physical constraint elimination, coordinate transport, Gaussian evaluation, determinant localization, outer integration, sign, and positivity steps between CMP116 localization data and an explicit polymer-volume Cauchy bound. It also records why three superficially convenient formulations are wrong: bounding the bilinear inner factor pointwise before Gaussian integration, assigning a positive sign to the localized quadratic matrix, and replacing the small interacting background by the trivial background. The terminal theorem is both useful and deliberately incomplete: it exposes, without wrappers, the physical Wilson-Hessian and random-walk estimates that still carry the analytic content of (2.16), equation (2.26), and `hRpoly`.

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