A Machine-Checked Reduction of Balaban's CMP116 Fluctuation Integral to Its Physical Analytic Frontier

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Abstract

We give a machine-checked reduction of the finite-dimensional fluctuation integral appearing in the large-field renormalization analysis of lattice gauge theory, following the CMP116 formulation of Bałaban and its later exposition by Dimock. The formal development constructs the localization region from the physical large-field data, identifies the scalar projector P_{Z_0} induced by that region, transports a certified physical covariance root to the finite Gaussian coordinate space without norm loss, evaluates the resulting complex quadratic Gaussian exactly, and proves the shifted Hessian positive from the source smallness condition $\alpha_5\|\mathcal{C}\| < 1/2$.

Two corrections exposed by formalization are central. First, a pointwise uniform bound on the bilinear fluctuation factor is generally unavailable; the correct interface bounds its Gaussian integral after completing the square. Second, the localized exponential enters the standard Gaussian identity with matrix $A = -\alpha_5 P_{Z_0}$, not with a positive-semidefinite Hessian. The terminal Lean theorem combines the physical localization, covariance certificate, Gaussian identity, and Cauchy extraction. Its only substantive analytic premises are the physical domination inequalities corresponding to equations (2.20)–(2.22) and a uniform bound on the explicit determinant/source majorant of equation (2.24). We do not prove those premises, equation (2.26), the polymer estimate `hRpoly`, a continuum limit, or a Yang–Mills mass gap. The contribution is an auditable reduction theorem that fixes the types, signs, measures, and remaining proof frontier.

1 Introduction

Rigorous renormalization-group analyses of lattice gauge theory separate small- and large-field regions, integrate finite-dimensional fluctuation fields, and reorganize the result into polymer activities. Bałaban's series [1, 2, 3, 4] and Dimock's exposition [5, 6, 7] provide the mathematical source for the present formalization. The specific target is the fluctuation-integral step around equations (2.14), (2.20)–(2.26) of the CMP116 large-field analysis.

The purpose of the development is not to replace an analytic estimate by an interface bearing the same estimate as a hidden hypothesis. Instead, it asks which parts of the source argument are finite, algebraic, geometric, or measure-theoretic and can therefore be closed independently, and which part is the genuine remaining Yang–Mills estimate.

The machine-checked chain established here is

$$\begin{aligned}(D, P) &\longmapsto Z_0 \longmapsto P_{Z_0} \longmapsto R_{\text{matrix}} \\ &\longmapsto \text{positive shifted Hessian} \longmapsto M_{2.24} \longmapsto \text{Cauchy term bound.}\end{aligned} \tag{1}$$

*This manuscript and parts of the formal development were produced with AI assistance under the repository's audit protocol. Every result labelled *machine-checked* is a Lean 4 theorem against a pinned Mathlib. Repository: `https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME`.

Here D denotes the family of large-field components, P the selected large-field bonds, Z_0 their least admissible localization region, R_{matrix} the finite matrix of the physical covariance root, and $M_{2.24}$ the explicit determinant/source majorant. The terminal theorem leaves two named analytic inputs: a domination of the physical integrand by a real Gaussian, and a uniform bound on $M_{2.24}$.

This separation is useful even before those estimates are proved. It prevents three common sources of ambiguity: choosing the localization projector by hand, placing an impossible supremum before a Gaussian integration, and using the wrong sign in the shifted quadratic form.

2 Finite physical localization

Let Z_0 be a finite family of coarse blocks. A physical site belongs to the interior when the site and all its forward and backward nearest neighbours belong to the union of blocks. A positively oriented physical bond is interior when both endpoints are interior. This two-endpoint convention is important: a one-endpoint convention does not represent the source boundary condition faithfully.

The development constructs a least region

$$Z_0 = \text{LocalizationCore}(D, P)$$

that contains the geometric seed induced by D and makes every $e \in P$ interior. It proves the universal property

$$\text{LocalizationCore}(D, P) \subseteq W \iff W \text{ contains the seed and every } e \in P \text{ is interior to } W. \tag{2}$$

The principal Lean endpoints are `cmp116LocalizationCore_subset_iff_admissible` and `cmp116LocalizationCore_admissible`.

The physical-to-scalar dictionary contains an equivalence

$$\text{coordEquiv} : (q, a) \simeq (e, b)$$

between CMP116 scalar coordinates and physical bond/Lie coordinates. It defines the canonical finite carrier

$$I(Z_0) = \{qa : \text{bond}(\text{coordEquiv}(qa)) \text{ is interior to } Z_0\}. \tag{3}$$

The projector P_{Z_0} is the diagonal matrix with entries 1 on $I(Z_0)$ and 0 elsewhere. The following theorem is machine-checked.

Theorem 2.1 (Physical localization projector). *For every scalar field x and coordinate qa,*

$$(P_{Z_0}x)_{qa} = \begin{cases} x_{qa}, & qa \in I(Z_0), \\ 0, & qa \notin I(Z_0), \end{cases} \qquad x^T P_{Z_0} x = \sum_{qa \in I(Z_0)} x_{qa}^2.$$

If Z_0 is admissible for (D, P), then for every $e \in P$ and every Lie coordinate a, $\text{coordEquiv}^{-1}(e, a) \in I(Z_0)$. *The same holds for the least region* $\text{LocalizationCore}(D, P)$.

This is formalized in `BalabanCMP116Eq223PhysicalLocalizationProjector.lean`. In particular, the shifted Hessian below no longer accepts an arbitrary user-supplied set of coordinates.

3 Exact Gaussian calculus and physical covariance

Let γ denote the standard Gaussian probability measure on a finite real coordinate space. For a real symmetric matrix A with $I + A > 0$ and a complex source c, the formal development proves

$$\int \exp\left(-\frac{1}{2}x^T Ax + c^T x\right) d\gamma(x) = \frac{1}{\sqrt{\det(I + A)}} \exp\left(\frac{1}{2}c^T (I + A)^{-1} c\right). \tag{4}$$

The source pairing is bilinear, not Hermitian: there is no complex conjugation on c. The proof proceeds through the one-dimensional integral, finite products, orthogonal invariance, spectral diagonalization, determinant reconstruction, and the inverse quadratic form. Coupled two-stage integrals and their Schur complement are also formalized.

For a real covariance-root matrix R, the pushforward measure $R_*\gamma$ is a positive probability measure. Substituting $x = Ru$ in (4) gives the majorant

$$M_{2.24}(R, A, c) = \det(I + R^T AR)^{-1/2} \exp\left(\frac{1}{2}\,\mathrm{Re}\big[c^T (I + R^T AR)^{-1} c\big]\right). \tag{5}$$

The equality between the norm of the complex Gaussian integral and $M_{2.24}$ is machine-checked. So is the domination principle

$$|f(u)| \le e^{-u^T Au/2 + r^T u} \text{ a.e.} \quad \Longrightarrow \quad \left|\int f\, d(R_*\gamma)\right| \le M_{2.24}(R, A, r). \tag{6}$$

The physical covariance certificate supplies an operator C, a root $C^{1/2}$, self-adjointness, positivity, the exact identity $(C^{1/2})^2 = C$, and localization/norm bounds. The scalar dictionary is proved to be an exact L^2 isometry. Therefore its matrix realization R_{matrix} satisfies

$$\|R_{\mathrm{matrix}}\|_{2\to 2} = \|C^{1/2}\|_{2\to 2}, \qquad \|R_{\mathrm{matrix}}\|_{2\to 2}^{2} = \|C\|_{2\to 2}. \tag{7}$$

There is no dimension-dependent condition number in this transport.

4 Two corrections forced by formalization

4.1 Integrate before taking the norm

A tempting interface asks for a pointwise bound

$$|\,\mathrm{innerIntegrand}(X, B)| \le M \quad \text{uniformly in } B.$$

For the source bilinear factor $\exp(-\langle B, \Gamma X\rangle)$ this is generally false: along a direction of B with the appropriate sign, the real part of the exponent grows. Such a hypothesis would make a downstream theorem formally true but physically unusable.

The repaired interface integrates against the conditioned Gaussian first:

$$\left|\int \mathrm{innerIntegrand}(X, B), d\mu_{\sigma,\tau}(B)\right| \le M_{\mathrm{inner}}. \tag{8}$$

Equation (6), followed by the exact evaluation (5), produces this bound while retaining the Gaussian damping and completing the square. The outer integration is then controlled by its probability measure and the outer weight.

4.2 The localized quadratic matrix has negative sign

The localized source factor contains

$$\exp\left(\frac{\alpha_5}{2}\|P_{Z_0}B\|^2\right).$$

In the convention of (4), this corresponds to

$$A = -\alpha_5 P_{Z_0}, \tag{9}$$

not to a positive-semidefinite A. Hence the actual positivity obligation is

$$I - \alpha_5 R_{\text{matrix}}^T P_{Z_0} R_{\text{matrix}} > 0. \tag{10}$$

By (7), the source condition

$$0 \le \alpha_5, \qquad \alpha_5 \,\text{covNormBound} < \frac{1}{2}, \qquad \|C\|_{2\to2} \le \text{covNormBound}, \tag{11}$$

implies (10). The corresponding Lean endpoint is

`posDef_physicalRootMatrix_of`
`_alpha5_covariance_half_small`
`_physicalZ0`.

Its canonical-core specialization takes only (D, P) as localization data.

5 The main reduction theorem

Let G denote the finite CMP116 data for the contour variables, weights, cutoffs, field evaluation, and interaction exponent. The construction `withPhysicalRootMatrix` replaces its covariance-root field by the canonical matrix R_{matrix} and changes no other field.

Write

$$A_{Z_0} = -\alpha_5 P_{Z_0}, \qquad g_{A,r}(b) = \exp\left(-\frac{1}{2}b^T A b + r^T b\right).$$

The final analytic premises are:

$$|\,\text{innerIntegrand}_G(b)| \le g_{A_{Z_0},r}(b) \quad \text{for } (R_{\text{matrix}})_*\gamma\text{-a.e. } b, \tag{Dom}$$

$$M_{2.24}(R_{\text{matrix}}, A_{Z_0}, r) \le M_{\text{inner}} \quad \text{uniformly on the Cauchy contours.} \tag{Maj}$$

Theorem 5.1 (Machine-checked CMP116 Gaussian reduction)**.** *Assume a physical localized covariance-root certificate, the source smallness (11), positive Cauchy radii, and a uniform outer-weight bound $|W_{\text{out}}| \le M_{\text{out}}$. If (Dom) and (Maj) hold, then the literal finite equation-(2.14) term satisfies*

$$\|\text{term}_G(Y_0, P; \psi, \phi)\| \le \text{CauchyRate}_\Delta(\text{CauchyRate}_Y(M_{\text{out}} M_{\text{inner}})).$$

All Gaussian integrability, covariance-coordinate transport, projector construction, and shifted-Hessian positivity are discharged internally.

The Lean endpoint is `norm_term_le_cauchyRate_of_physicalGaussianReduction` in `BalabanCMP116Eq214MainReduction.lean`. A pointwise companion theorem controls the inner Gaussian integral directly by M_{inner}.

Remark 5.2 (What the theorem does not say)**.** Theorem 5.1 is a reduction theorem. It does not prove (Dom), (Maj), their source-uniform constants, the subsequent termwise factorization of equation (2.26), or `hRpoly`. Consequently it implies neither a lattice mass gap nor a continuum Yang–Mills construction.

6 Formal architecture and audit

Table 1 maps the mathematical chain to the principal modules. All entries abbreviate filenames with prefix `BalabanCMP116` and suffix `.lean`.

Layer	Lean module
Physical incidence and least Z_0	`Eq214Interior`, `Eq214LocalizationCore`
Cutoff and Cauchy extraction	`Eq23Cutoff`, `Eq214Cauchy`
Exact Gaussian identities	`QuadraticGaussianMatrix`, `CorrelatedGaussian`
Domination and (2.24)	`Eq223GaussianDomination`, `Eq224GaussianBound`
Physical root and norm transport	`Eq223PhysicalRootNormBridge`
Physical P_{Z_0}	`Eq223PhysicalLocalizationProjector`
Terminal reduction	`Eq214MainReduction`

Table 1: Principal theorem-to-artifact map.

The repository pins Lean and Mathlib. The root build after the terminal reduction completes 8,447 jobs. The three public endpoints in the terminal module, and the supporting endpoints audited during the campaign, print only `propext`, `Classical.choice`, and `Quot.sound`. The added modules contain no `sorry`, `admit`, or project `axiom`.

The reproducible checkpoint is maintained on the repository branch `codex/hrpoly-b3-card-tilt`, pull request #28. The immutable release tag is `hrpoly-cmp116-main-reduction-v0.1-2026-07-15`. The exact commands are

```
lake exe cache get
lake build YangMillsCore
lake env lean oracle_check.lean
```

7 Remaining analytic frontier

The formalization localizes the remaining difficulty rather than reducing its mathematical importance. The next source theorem must instantiate the physical Hessian, source, cutoffs, and interactions strongly enough to prove (Dom). It must then control the determinant and inverse-source term in (5) uniformly over the contours, volumes, and scales. The resulting term bound must be factored and resummed into the metric decay of equation (2.26), producing the raw activity estimate consumed by the already formalized $H^\#$ and Kotecký–Preiss chain.

Thus the remaining route is

$$h_{\mathrm{dom}} \longrightarrow h_{\mathrm{majorant}} \longrightarrow (2.26) \longrightarrow h_{\mathrm{raw}} \longrightarrow \mathrm{KP}.$$

A first nontrivial physical instance of h_{dom}, even for a restricted contour sector or a simplified interaction, would strengthen the present reduction from a formal-methods result toward a mathematical-physics contribution. It is the natural next target.

8 Conclusion

The formal development closes the finite geometry, coordinate dictionary, Gaussian evaluation, integrability, sign, and positivity steps between the physical CMP116 localization data and an explicit Cauchy term bound. It also records why two superficially convenient formulations are wrong: bounding the bilinear inner factor pointwise before Gaussian integration, and assigning a positive sign to the localized quadratic matrix. The terminal theorem is both useful and deliberately incomplete: it exposes, without wrappers, the physical domination and majorant estimates that still carry the analytic content of equation (2.26) and `hRpoly`.

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