A Machine-Checked Exact Evaluation of the Two-Dimensional $SU(2)$ Heat-Kernel Lattice Model on Certified Finite Combinatorial Disk Cellulations

From Haar Measure to Conditioned Original-Edge Amplitudes

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1 August 2026

Abstract

We present an end-to-end Lean 4/Mathlib formalization of the exact evaluation of the two-dimensional $SU(2)$ heat-kernel lattice model on certified finite combinatorial disk cellulations. The development starts from normalized Haar probability on the concrete matrix group $SU(2)$. It identifies its transport to S^3 with the canonical spherical measure, proves an all-order orbital integration formula, derives translated character convolution, and passes from finite character sums to the infinite heat-kernel semigroup by dominated convergence. A genuine shared-edge integral then yields the two-face Migdal move.

The geometric layer is independent of any reduction tree. A cellulation stores vertices, paired half-edges, cyclic face words, incidence, Euler characteristic, and positive face areas. Connected dual graphs admit certified elimination schedules, every valid schedule reduces to the heat kernel at total area, and all schedules give the same amplitude. For the original edge model, a rooted spanning tree produces a measurable, product-Haar-preserving gauge equivalence $SU(2)^E \simeq SU(2)^{V\setminus\{r\}} \times SU(2)^{E\setminus T}$. A compatible tree–cotree construction then retains the exterior holonomy rather than integrating it out. For every certified physical disk cellulation, the boundary-conditioned original-edge amplitude is exactly the $SU(2)$ heat kernel at the total face area. Coefficient extraction gives, for every irreducible label n, the normalized exterior-boundary identity

$$\mathbb{E}_P[W_n(H_\partial)] = \exp\left[-\frac{n(n+2)}{4}\sum_f t_f\right],$$

where H_∂ is the retained holonomy of the complete exterior boundary word. The universal record is demonstrably inhabited: a concrete three-spoke disk has $(V, E, F) = (4, 6, 3)$ and derived dual graph K_3. A reproduced audit covers 177 audited declarations, explicitly including both headline theorems, and finds only `propext`, `Classical.choice`, and `Quot.sound` in their dependency cones. The analytic solution is classical. The contribution is a concrete kernel-checked composition from Haar measure and characters to physical edge variables, gauge fixing, tree–cotree elimination, and the exact boundary-observable endpoint.

1 Introduction

Two-dimensional Yang–Mills theory is exactly soluble, but its familiar closed formulas compress several logically distinct constructions: Haar integration on the gauge group, harmonic analysis of characters, heat-kernel convolution, gauge reduction of edge variables, independence of lattice subdivision, and extraction of Wilson-loop coefficients. Each step is classical; their end-to-end compatibility is usually left implicit.

This paper makes that chain explicit in a proof assistant. The formalization does not begin from an abstract package that assumes the desired harmonic analysis or a preassembled area law. It constructs the relevant $SU(2)$ objects, proves the analytic identities consumed downstream, defines a finite oriented disk cellulation independently of an elimination syntax, and closes the original-edge model through explicit measure-preserving gauge coordinates. The result is a theorem about the physical finite lattice integral with a retained exterior holonomy, not merely about a reduced evaluator.

1.1 Main contribution

The mathematical content may be summarized as follows. Let P be a certified finite disk cellulation with primal and dual connectedness, a once-around boundary enumeration for every bounded face, positive face areas (t_f), and a distinguished exterior boundary. Let $A_P(g)$ be the original-edge heat-kernel amplitude conditioned on exterior holonomy $g \in SU(2)$, and let H_∂ denote the ordered holonomy of that complete exterior boundary word. Throughout this paper, "loop" means this certified exterior boundary; no arbitrary embedded plane curve is part of the formal input.

Here "disk cellulation" is the name of the certified finite combinatorial record—incidence, cyclic face words, Euler relation, and connected primal and dual graphs—not a theorem that every record has been realized as an embedded planar cell complex. Moreover, "conditioned" refers to the explicit product-Haar chart integral defined in Section 2; it is not an appeal to a conditional measure on a measure-zero holonomy fibre.

Theorem 1.1 (Machine-checked disk amplitude and exterior-boundary law)**.** *For every such P and every $g \in SU(2)$,*

$$A_P(g) = K_{\sum_f t_f}(g).$$

The value is independent of the compatible gauge tree, dual elimination tree, and elimination schedule used in its construction. Consequently, for every $n \in \mathbb{N}$, the normalized irreducible character $W_n = \chi_n/(n+1)$ satisfies

$$\mathbb{E}_P[W_n(H_\partial)] = \exp\left[-\frac{n(n+2)}{4}\sum_f t_f\right].$$

The formal endpoints are:

```
SU2BoundaryDiskCellulation.conditionedEdgeModelAmplitude_eq_heatKernel
su2ConnectedDisk_simpleLoop_areaLaw_exact
```

The first endpoint is the stronger statement: it retains the complete boundary holonomy and therefore determines every character coefficient at once.

1.2 Contribution ledger

Table 1 distinguishes the proved statements from the geometric and continuum assertions that are deliberately outside scope.

1.3 Relation to the classical theory

Migdal's recursion [1], Rusakov's loop and partition-function formulas [2], Witten's exact solution [3], and Driver's heat-kernel treatment [4] supply the classical analytic setting. Stochastic and rigorous constructions on compact surfaces were developed by Gross, King, and Sengupta [5], Fine [6], Sengupta [7], and Lévy [8]; the latter treats discrete and continuum holonomy fields far beyond the finite disk objects considered here. Lévy's planar master

Layer	Formalized result	Status in this paper
Haar geometry	Concrete $SU(2) \simeq S^3$ measure identification and all-order orbital law	proved
Harmonic analysis	Translated character convolution and infinite heat-kernel semigroup	proved
Local lattice move	Shared-edge Migdal integration for two faces	proved
Combinatorics	Elimination schedules for every connected finite dual graph; schedule independence	proved
Gauge reduction	Global tree gauge coordinates preserving product Haar; simultaneous facial covariance	proved
Physical endpoint	Conditioned original-edge amplitude equals one heat kernel at total area	proved
Exterior-boundary observable	Exact normalized Casimir law for every irreducible label	proved
Nonvacuity	Explicit triangle and three-spoke boundary disks; the latter has $(V, E, F) = (4, 6, 3)$ and dual K_3	proved
Planar topology	Identification with embedded planar isotopy classes	not claimed
Beyond simple loops	Self-intersections and Makeenko–Migdal equations	not claimed
Continuum theory	Continuum holonomy field or four-dimensional theory	not claimed

Table 1: Claim ledger. "Proved" refers to declarations in the audited Lean snapshot; the last three rows delimit the frontier.

field [10] and the spherical master-field analysis of Dahlqvist and Norris [11] locate the exact finite-lattice calculation within the modern large-N literature. The Makeenko–Migdal equation for compact surfaces is established in substantially greater geometric generality by Driver, Gabriel, Hall, and Kemp [9]. We do not reprove that equation.

The novelty claimed here is therefore not a new analytic solution. It is the kernel-checked composition of the concrete $SU(2)$ analytic layer with a finite original-edge lattice model, including global gauge fixing, tree–cotree compatibility, schedule elimination, and the boundary-conditioned endpoint. A separate strong-coupling formalization [16] proves a volume-uniform inequality for finite $SU(N_c)$ Wilson-action lattices; it neither implies nor is used to instantiate the exact heat-kernel identity.

We also performed a targeted related-formalization search on 1 August 2026 across arXiv, GitHub/Lean, the Lean Zulip archive, Isabelle's Archive of Formal Proofs, and Coq/Rocq indexes. The closest items found were an axiomatized Lean encoding of Seiberg–Witten physical reasoning [14] and a Lean repository formalizing parameterized statements of the Millennium problems [15]; neither formalizes this finite $SU(2)$ heat-kernel lattice evaluation. Subject to the limits of that search, we found no prior end-to-end formalization matching the theorem proved here. This is a qualified novelty statement, not a claim that the classical mathematics is new.

1.4 Organization

Section 2 fixes conventions and the claim boundary. Sections 3, 4, 5, and 6 construct the analytic chain. Sections 7–8 pass from physical edge variables to a schedule-independent amplitude. Section 9 extracts the exact exterior-boundary coefficient, Section 10 records the trust boundary, and Section 11 states what remains open.

2 Model, main chain, and claim boundary

Let dg be normalized Haar probability on $SU(2)$, let χ_n be the character of the irreducible representation of dimension $d_n = n + 1$, and put

$$c_n = \frac{n(n+2)}{4}, \qquad K_t(g) = \sum_{n\geq 0} d_n e^{-tc_n} \chi_n(g), \qquad t > 0.$$

With convolution

$$(f * h)(g) = \int_{SU(2)} f(x)h(x^{-1}g)\, dx,$$

the formalized chain is

$$\begin{aligned}\text{Haar on } S^3 &\Longrightarrow \text{all-order orbital law} \Longrightarrow \chi_n * \chi_m \\ &\Longrightarrow K_s * K_t = K_{s+t} \Longrightarrow \text{Migdal} \Longrightarrow \text{exterior-boundary coefficient.}\end{aligned}$$

The conditioned amplitude. Let B_P be the physical boundary-elimination chart selected by the Lean definition. Write $G_P = SU(2)^{n_P}$ for its $V - 1$ pure-gauge coordinates and $R_P = SU(2)^{O_P}$ for the remaining chord coordinates after the exterior holonomy has been split off. If $U_{B_P}(q, r, g)$ is the original-edge configuration reconstructed by the chart from $(q, r, g) \in G_P \times R_P \times SU(2)$, define

$$A_P(g) := \int_{G_P\times R_P} \prod_f K_{t_f}(H_f(U_{B_P}(q, r, g)))\, d\mu_{G_P}(q)\, d\mu_{R_P}(r), \tag{1}$$

where both measures are finite product Haar probabilities. Equation (1) is the literal mathematical content of `SU2BoundaryDiskCellulation.conditionedEdgeModelAmplitude`: the Lean definition chooses B_P from the proved nonempty chart type and calls its ordinary `conditionedEdgeIntegral`. The theorem later proves that every compatible chart has the same value, so the classical choice disappears from the answer.

This result is distinct from the programme's earlier strong-coupling area-law theorem [16]. The earlier theorem is a volume-uniform *inequality* for finite lattices and $SU(N_c)$ in an explicit convergence window, obtained from a Kotecký–Preiss cluster expansion. The present theorem is an *exact identity* in the two-dimensional $SU(2)$ heat-kernel model, obtained by Haar integration and subdivision. The former is cited context; it is not used to discharge any exact two-dimensional statement here.

No claim is made here about four-dimensional Yang–Mills, a continuum construction in four dimensions, or a new proof of the classical exact solution. Table 1 is the authoritative summary of this claim boundary.

3 From spherical Haar measure to Weyl integration and the orbital law

Write an element of $SU(2)$ in its standard form

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1.$$

The formal half-trace coordinate is therefore $x = \frac{1}{2}\operatorname{tr} g = \operatorname{Re}(a)$. The development proves the $SU(2)$ Weyl integration formula in the equivalent semicircle form: for every continuous $F\colon \mathbb{R} \to \mathbb{R}$,

$$\int_{SU(2)} F\Big(\tfrac{1}{2}\operatorname{tr} g\Big)\, dg = \frac{2}{\pi}\int_{-1}^{1} F(x)\sqrt{1-x^2}\, dx. \tag{2}$$

Thus the half-trace pushforward has density $\frac{2}{\pi}\sqrt{1-x^2}\,\mathbf{1}_{[-1,1]}(x)$. Earlier modules construct a homeomorphism $SU(2) \simeq S^3$, realize left multiplication as an ambient real-linear isometry, prove preservation of the canonical `Measure.toSphere` measure, and use uniqueness of normalized Haar measure to obtain literal equality of the two probability measures.

The all-order step studies $u = |a|^2$. In dimension four, the squared mass in one complex rail is uniform on $[0, 1]$. Rather than assume this distribution, the formal proof establishes its moment recurrence, proves

$$\int_{SU(2)} u^k \, dg = \frac{1}{k+1} \qquad (k \geq 0),$$

and then identifies measures on the compact interval using polynomial density. The resulting literal endpoint is:

```
theorem su2FirstRailMassMeasure_eq_uniform :
  su2FirstRailMassMeasure = su2UnitIntervalMeasure
```

Together with `su2HaarHalfTraceMeasure_eq_su2SemicircleMeasure` and `integral_su2SemicircleMeasure`, this gives (2) without postulating an abstract Weyl formula.

For $x, y \in [-1, 1]$, choose representatives with half-traces x, y and average their product over conjugation. The preceding uniform law reduces the orbital integral to a one-variable Chebyshev integral. For the Chebyshev polynomial U_m of the second kind, the formalized formula is

$$\int_{SU(2)} U_m\left(\tfrac{1}{2}\operatorname{tr}(kak^{-1}b)\right) dk = \frac{U_m(\frac{1}{2}\operatorname{tr} a)U_m(\frac{1}{2}\operatorname{tr} b)}{m+1}.$$

The proof includes the endpoint cases $x = \pm 1$; no division by a vanishing sine is hidden in the statement.

4 All-order translated character convolution

Characters are implemented concretely as

$$\chi_n(g) = U_n\left(\tfrac{1}{2}\operatorname{tr} g\right).$$

Conjugacy invariance, left and right Haar invariance, Fubini, and the orbital law give the translated orthogonality relation

$$\int_{SU(2)} \chi_n(x)\chi_m(x^{-1}g) \, dx = \delta_{nm}\frac{\chi_n(g)}{n+1}. \tag{3}$$

This is stronger than orthogonality at the identity and is exactly the consumer needed by the heat-kernel product.

Theorem 4.1 (Machine-checked character convolution). *For every $n, m \in \mathbb{N}$ and $g \in SU(2)$,*

$$(\chi_n * \chi_m)(g) = \begin{cases} \chi_n(g)/(n+1), & n = m, \\ 0, & n \neq m. \end{cases}$$

The public Lean endpoints are listed below. They are theorems about the actual normalized Haar integral, not fields projected from an abstract character package.

```
integral_su2CharacterReal_mul_translate
su2CharacterChebyshev_convolution
```

5 Infinite heat-kernel semigroup

For a cutoff N, expand both finite kernels, interchange the finite sums and the Haar integral, and apply (3). The Kronecker selector leaves a single sum and the Casimir weights multiply:

$$e^{-sc_n}e^{-tc_n} = e^{-(s+t)c_n}.$$

This proves the finite convolution identity. The infinite theorem then uses the already established uniform Casimir majorant for positive times. Partial kernels converge pointwise and uniformly; the product is bounded by the product of two summable majorants, so dominated convergence passes the Haar integral to the limit.

Theorem 5.1 (Concrete heat semigroup)**.** *For $s, t > 0$ and every $g \in SU(2)$,*

$$\int_{SU(2)} K_s(x)K_t(x^{-1}g)\,dx = K_{s+t}(g).$$

Equivalently, as functions, $K_s * K_t = K_{s+t}$.

The corresponding Lean declarations, including the function-level consumer, are:

```
su2HeatKernelPartial_convolution
su2HeatKernel_convolution
su2HeatKernel_convolution_consumer
```

6 A concrete Migdal move

Consider two adjacent faces of areas s and t. Let a, b encode the external boundary holonomies and integrate their shared oriented edge x. The actual two-face density in the development is

$$D^{a,b}_{s,t}(x) = K_s(ax)K_t(x^{-1}b).$$

Left invariance changes variables $y = ax$ and the heat semigroup gives

$$\int_{SU(2)} D^{a,b}_{s,t}(x)\,dx = K_{s+t}(ab). \tag{4}$$

Theorem 6.1 (Two-face Migdal subdivision)**.** *Equation* (4) *holds for all $a, b \in SU(2)$ and all $s, t > 0$. It is proved by a genuine Haar integration over the shared edge and invokes the concrete infinite heat-kernel semigroup.*

This closes the first nontrivial subdivision move: two holonomy variables, two faces, and one integrated shared edge. The endpoints below contain no package hypothesis.

```
su2Migdal_twoFace_merge
su2Migdal_subdivision_invariant
```

7 Gauge fixing on cyclic and general cellulations

The two-face move alone does not identify a reduced evaluator with the original lattice integral when the face-dual graph has cycles. We therefore formalize the first cyclic test explicitly. Consider a disk split into three bounded faces by three spokes meeting at one interior vertex. Its face dual is K_3, equivalently the three-cycle. Write $x, y, z \in SU(2)$ for the original spoke-edge variables and $a, b, c \in SU(2)$ for the fixed boundary arcs. The three face holonomies are

$$xay^{-1}, \qquad ybz^{-1}, \qquad zcx^{-1}.$$

Thus the unreduced density and measure are

$$K_{t_1}(xay^{-1})K_{t_2}(ybz^{-1})K_{t_3}(zcx^{-1})\,dx\,dy\,dz. \tag{5}$$

Gauge transformations at the interior vertex act by simultaneous left multiplication. Lean constructs the measurable equivalence

$$\Phi(x,y,z) = (x, x^{-1}y, x^{-1}z), \qquad \Phi^{-1}(u,v,w) = (u, uv, uw).$$

The skew-product theorem and left invariance of normalized Haar prove the literal measure identity

$$\Phi_*(dx\,dy\,dz) = dx\,dy\,dz.$$

After substitution, conjugation invariance of the heat kernel removes the first coordinate and gives

$$K_{t_1}(av^{-1})K_{t_2}(vbw^{-1})K_{t_3}(wc).$$

The pure-gauge integral has mass one. Applying the reversed two-face Migdal identity first to w and then to v yields the fully unreduced result

$$\int_{SU(2)^3} K_{t_1}(xay^{-1})K_{t_2}(ybz^{-1})K_{t_3}(zcx^{-1})\,dx\,dy\,dz = K_{t_1+t_2+t_3}(abc). \tag{6}$$

The proof does not store a Jacobian certificate or assume an abstract gauge package. The map, its inverse, measurability, product-Haar preservation, density factorization, Fubini integrability, and both edge integrations are all kernel checked. The principal endpoints are:

```
su2ThreeSpokeFaceDual_connected
su2ThreeSpokeGaugeFix_measurePreserving
su2ThreeSpoke_density_gaugeFactorization
su2ThreeSpoke_unreducedIntegral_eq_gaugeFixedIntegral
su2ThreeSpoke_unreducedIntegral_eq_heatKernel
```

Equation (6) is the first concrete product-Haar instance. It is now subsumed by the uniform original-edge theorem below, whose spanning-tree gauge fixing is constructed directly from the incidence data of the independent disk object.

7.1 Uniform finite-valence theorem

The preceding three-spoke map is not intrinsically three-dimensional. Let I be any finite type and let

$$d\mu_I = \bigotimes_{i\in I} dg_i$$

be the normalized product Haar probability on $SU(2)^I$. The formalization constructs, without choosing an enumeration of I, the measurable equivalence

$$\begin{aligned} \Phi_I \colon SU(2)\times SU(2)^I &\to SU(2)\times SU(2)^I, \\ \Phi_I(x,y) &= (x, (x^{-1}y_i)_{i\in I}). \end{aligned}$$

with inverse $(u,v)\mapsto(u,(uv_i)_i)$. The product measure is Mathlib's `Measure.pi`. Using finite-product Haar invariance and the skew-product theorem, Lean proves

$$(\Phi_I)_*(dg\otimes d\mu_I) = dg\otimes d\mu_I \tag{7}$$

for every finite I. It also proves that the diagonal action $(x,y_i)\mapsto(ux,uy_i)$ preserves the same measure.

The result is consumed at the integral level. If $F : SU(2) \times SU(2)^I \to \mathbb{C}$ is invariant under that diagonal action, then

$$\int F(x, y)\, dg(x)\, d\mu_I(y) = \int F(1, y)\, d\mu_I(y). \tag{8}$$

More generally, the source contains an explicit factorization theorem for any density whose pullback through Φ_I^{-1} loses its first coordinate. The principal uniform endpoints are:

```
su2FiniteStarDiagonalLeft_measurePreserving
su2FiniteStarGaugeFix_measurePreserving
su2FiniteStar_integral_eq_gaugeFixed
su2FiniteStar_integral_eq_identitySlice
```

Equations (7)–(8) remove the fixed-valence limitation of the local gauge step. The next two subsections compose them with an ordered spanning tree, first for vertex coordinates and then for all original edge variables and all facial holonomies simultaneously.

7.2 Global rooted-tree vertex coordinates

The development now performs the global composition at the level of vertex coordinates. A value of `SU2RootedTreeOrder n` records a rooted tree on $n+1$ vertices by adjoining one leaf at a time, together with the already present parent of each new leaf. Its parent–child list is embedded in the final index type. For every finite connected simple graph, Lean constructs an equivalence

$$e : \mathrm{Fin}(n+1) \simeq V$$

and such an order for which every recorded parent–child pair is an actual graph edge. The proof removes a vertex whose complement remains connected, recurses on the induced graph, and reattaches the vertex across a witnessed edge.

For vertex variables $x_0, \ldots, x_n \in SU(2)$, the associated single global map retains the root and replaces each later variable by its tree increment:

$$\Psi_T(x)_0 = x_0, \qquad \Psi_T(x)_k = x_{p(k)}^{-1} x_k \quad (k > 0). \tag{9}$$

It is implemented recursively by splitting the last coordinate, applying the triangular Haar shear, transforming the older coordinates, and reassembling the finite product. The inverse is part of the measurable equivalence, not an existence assertion. Induction and finite-product Haar invariance prove

$$(\Psi_T)_* \left(\bigotimes_{v \in V} dg_v \right) = \bigotimes_{v \in V} dg_v. \tag{10}$$

For a `SU2FiniteDiskCellulation`, primal adjacency is derived from an actual oriented half-edge and made into a simple graph. Under the explicit hypothesis that this primal graph is connected, Lean chooses the certified tree internally, transports (9) through the vertex enumeration, and returns a global measure-preserving equivalence on `C.Vertex -> SU2`. The principal endpoints are:

```
SU2RootedTreeOrder.exists_validForGraph_of_connected
SU2RootedTreeOrder.coordinateEquiv_measurePreserving
SU2FiniteDiskCellulation.exists_rootedSpanningTree_of_primal_connected
SU2FiniteDiskCellulation.RootedSpanningTree.vertexCoordinateEquiv_measurePreserving
SU2FiniteDiskCellulation.exists_rootedTreeVertexCoordinateEquiv_measurePreserving
```

The edge module consumes this tree and proves the stronger decomposition

$$SU(2)^E \simeq SU(2)^{V \setminus \{r\}} \times SU(2)^{E \setminus T},$$

as a literal measurable equivalence. Each construction step chooses an oriented parent–child half-edge; strict ordering of parents before children proves that the resulting physical tree edges are injective. The complement $E \setminus T$ is therefore an actual finite subtype, not a cardinality-only placeholder. After correcting the stored orientation of each tree edge, the map reconstructs potentials $g_r = 1$ and $g_k = g_{p(k)} a_k$, and sends each chord to

$$X_e = g_{s(e)} U_e g_{t(e)}^{-1}.$$

Finite-product relabelling, coordinatewise inversion and a Haar-preserving skew product prove preservation of normalized product Haar.

The physical edge record `SU2EdgeConnectedDiskCellulation` additionally requires an explicit once-around, duplicate-free boundary enumeration for each bounded face. This condition is stated because the older combinatorial record did not exclude a formally declared face with no incident half-edge. For every well-formed face f, Lean proves simultaneously

$$H_f(g \cdot U) = g_{v_f} H_f(U) g_{v_f}^{-1}.$$

Consequently the complete heat-kernel density is gauge invariant. Under the inverse edge equivalence it factors only through the chord block, and the normalized gauge coordinates integrate to one:

$$\int_{SU(2)^E} \prod_f K_{t_f}(H_f(U))\, dU = \int_{SU(2)^{E \setminus T}} \prod_f K_{t_f}(H_f(U_T = 1, X))\, dX. \tag{11}$$

Both sides of (11) integrate every physical edge, including the exterior boundary edges, and are therefore scalars. The boundary-dependent theory requires one further construction: the exterior cycle is certified once around, one physical boundary edge is retained, and the remaining variables are integrated in a gauge chart compatible with a dual elimination tree. This is formalized by `SU2BoundaryDiskCellulation` and its physical boundary-elimination charts.

For such a cellulation P, write $A_P(g)$ for the integral in original edge variables with exterior holonomy fixed to g. A tree–cotree connectivity theorem first constructs a primal spanning tree avoiding the selected dual edges. The resulting chart splits off the retained boundary coordinate, proves the conditioned edge and chord integrals equal, and then performs the physical Migdal eliminations. Thus

$$A_P(g) = K_{P.\mathrm{totalArea}}(g). \tag{12}$$

Every compatible chart computes the same function because both sides reduce to (12); the auxiliary tree–cotree choices have therefore disappeared from the answer. The principal endpoints are:

```
SU2FiniteDiskCellulation.RootedSpanningTree.treeEdge_injective
SU2FiniteDiskCellulation.RootedSpanningTree.globalEdgeGaugeEquiv_measurePreserving
SU2FiniteDiskCellulation.RootedSpanningTree.gaugeTransform_eq_gaugeFixedEdgeConfiguration
SU2EdgeConnectedDiskCellulation.allFaceHolonomies_gaugeTransform
SU2EdgeConnectedDiskCellulation.edgeHeatKernelDensity_globalEdgeFactorization
SU2EdgeConnectedDiskCellulation.unreducedEdgeIntegral_eq_chordGaugeFixedIntegral
SU2FiniteDiskCellulation.exists_rootedSpanningTree_avoiding_of_connected
SU2DualRootedEliminationTree.primalGraphAvoiding_selectedEdgeRange_connected
SU2BoundaryDiskCellulation.nonempty_physicalBoundaryEliminationChart
SU2PhysicalBoundaryEliminationChart.conditionedChordIntegral_eq_heatKernel
SU2PhysicalBoundaryEliminationChart.conditionedEdgeIntegral_eq_heatKernel
SU2BoundaryDiskCellulation.conditionedEdgeModelAmplitude_eq_heatKernel
```

7.3 Concrete nonvacuity and a cyclic boundary disk

The universal quantifier above is not vacuous. The module `SU2BoundaryExamples.lean` constructs two complete inhabitants of `SU2BoundaryDiskCellulation`. The first is a single triangular face, with $(V, E, F) = (3, 3, 1)$. The second is the promised three-spoke disk: three triangular faces meet at one interior vertex, so

$$(V, E, F) = (4, 6, 3), \qquad V - E + F = 1,$$

and its dual graph is proved equal to the complete graph on three vertices, hence to the three-cycle K_3. All face areas are one. Instantiating the physical theorem gives the fully concrete equality

$$A_{P_{3\mathrm{sp}}}(g) = K_{t=3}(g) \qquad (g \in SU(2)). \tag{13}$$

Equation (13) uses $P_{3\mathrm{sp}}$ for the cellulation; K_3 continues to denote its dual graph, whereas $K_{t=3}$ denotes the heat kernel at total area three. The declarations `nonempty_su2BoundaryDiskCellulation`, `SU2ThreeSpokeBoundaryExample.boundaryDisk_card_data`, `SU2ThreeSpokeBoundaryExample.boundaryDisk_dualGraph_eq_top`, and `SU2ThreeSpokeBoundaryExample.boundaryDisk_conditionedEdgeModelAmplitude_eq_heatKernel` are themselves included in the kernel audit. This example exercises the original six-edge record and its derived incidence data; it is not merely the earlier three-variable gauge-fixed calculation.

8 Independent finite cellulations and elimination

The analytic evaluator used by the development is a labelled binary schedule. If a schedule node has left and right subproblems L, R, its amplitude is defined recursively by

$$A_C(a, b) = \int_{SU(2)} A_L(a, x) A_R(x^{-1}, b)\, dx.$$

The earlier tree module proves by structural induction that

$$A_C(a, b) = K_{T(C)}(ab).$$

At each node the induction hypotheses expose two heat kernels and the proof invokes the concrete two-face theorem.

The new point is that the geometric input is no longer this evaluator syntax. The new independent cellulation structure stores finite types of vertices, edges, half-edges, and bounded faces; a fixed-point-free involution pairs the two half-edges of every edge; source and target incidence is linked to a cyclic face-successor permutation; `none` denotes the exterior face; and the disk Euler relation $V - E + F = 1$ is part of the certificate. The dual face graph is derived from the two sides of internal edges.

A labelled elimination schedule is valid when its leaves list every bounded face exactly once and each binary merge is witnessed by a genuine adjacency in the derived dual graph. Lean proves the following existence theorem for an arbitrary finite connected dual graph. If the graph has one vertex, the schedule is a leaf. Otherwise, a vertex whose deletion leaves a connected induced subgraph is removed, the proof recurses on the smaller finite type, and the vertex is reattached across an actual adjacent edge. Thus no acyclicity assumption is used.

For any valid schedule S, permutation of the face list gives $T(S) = \sum_f t_f$. The tree reduction then proves

$$A_{C,S}(a, b) = K_{\sum_f t_f}(ab). \tag{14}$$

Consequently, for all valid S_1, S_2,

$$A_{C,S_1}(a, b) = A_{C,S_2}(a, b).$$

The public structure SU2ConnectedDiskCellulation stores only the cellulation and connectedness of its dual. A schedule is chosen internally from the existence theorem; the preceding equality proves that every other valid choice is observationally identical. Its principal declarations are:

```
SU2FaceEliminationSchedule.exists_schedule_of_connected_graph
SU2FaceEliminationSchedule.exists_valid_schedule_of_dual_connected
SU2FaceEliminationSchedule.amplitude_eq_of_valid_schedules
SU2ConnectedDiskCellulation.amplitude_eq_heatKernel
SU2ConnectedDiskCellulation.amplitude_eq_any_valid_schedule
```

For each representation label n, the formalization defines an exact-area-law package whose "loops" are connected, independently specified disk cellulations equipped with their complete exterior boundary word. Their area parameter is $\sum_f t_f$, their string tension is $n(n+2)/4$, and their Wilson functional integrates the schedule-independent amplitude. This terminology does not identify the input with an arbitrary plane curve. Equation (14) followed by coefficient extraction gives a nontrivial concrete instance `su2ConnectedDiskExactAreaLawPackage`. The generic public interface is then consumed by `su2ConnectedDisk_simpleLoop_areaLaw_exact`; no field of that package remains assumed. The trivial-label case also proves the normalization

$$\int_{SU(2)} A_C(1, g)\, dg = 1.$$

For a general dual graph with cycles, the schedule theorem concerns the explicitly defined post-gauge-fixed amplitude. Equation (11) supplies the uniform original-edge gauge reduction for every facially well-formed, primal- and dual-connected disk cellulation; the earlier three-cycle computation remains a concrete diagnostic instance. The boundary-conditioned construction of Section 7 closes the remaining type distinction. A compatible tree–cotree chart realizes the schedule in physical chord variables, and (12) identifies the retained-holonomy edge integral with the same heat kernel as (14). Hence the final theorem is not merely schedule independence of an evaluator: it is an exact evaluation of the conditioned original-edge model.

9 Exact exterior-boundary coefficient

The same repository proves, for every label n and positive area t, that the normalized Wilson character $W_n = \chi_n/(n+1)$ extracts the corresponding heat coefficient:

$$\int_{SU(2)} W_n(g) K_t(g)\, dg = e^{-tc_n} = \exp\left[-t\frac{n(n+2)}{4}\right].$$

The proof first establishes the identity for partial character sums and then passes to the full kernel by dominated convergence. Its final consumer is:

```
su2_exact_simpleLoop_areaLaw
```

Together with the iterated Migdal reduction and (12), this is the exact two-dimensional route from original edge variables on a certified finite combinatorial disk to the exterior-boundary coefficient. The choice of compatible chart and elimination schedule does not change the boundary amplitude, and the effective face has the Casimir coefficient prescribed by its total area.

10 Audit and reproducibility

We reproduced the proof audit locally on 1 August 2026 with Lean `4.29.0-rc6` and Mathlib revision `07642720480157414db592fa85b626dafb71355b`. The commands

```
lake build Lean2dYangMills.AuditMigdalClosure
lake build Lean2dYangMills
rg -n "\bsorry\b|\badmit\b|^\s*axiom\b" Lean2dYangMills -g '*.lean'
```

completed with both builds successful and no source-placeholder match. The source search and the kernel audit are distinct controls: the former excludes unfinished proof syntax, while the latter asks Lean for the actual axiom cone of each selected theorem.

The revised `AuditMigdalClosure.lean` applies `#print axioms` to 177 audited declarations. This count explicitly includes both headline results, `su2ConnectedDisk_simpleLoop_areaLaw_exact` and `SU2BoundaryDiskCellulation.conditionedEdgeModelAmplitude_eq_heatKernel`, as well as the concrete K_3 endpoint. Every printed cone contains only

`propext`, `Classical.choice`, `Quot.sound`.

The transcripts contain no dependency on `native_decide`; the finite incidence certificates of the examples are discharged by kernel reduction. The four newly included analytic and cyclic declarations are `su2HaarHalfTraceMeasure_eq_su2SemicircleMeasure`, `integral_su2SemicircleMeasure`, `su2HeatKernel_convolution_consumer`, and `su2ThreeSpokeFaceDual_connected`. Table 2 maps the principal mathematical statements to their audited declarations and source modules.

Authoritative proof artifact. The public repository `lluiseriksson/lean-2d-yang-mills` [17] contains the complete proof chain at base commit `a1fbea97cbe673d383dbb4bc5e2a2fb70dbf190a`. The reviewed companion archive for this manuscript is based on that commit and additionally contains `SU2BoundaryExamples.lean` plus ten additional audit entries. Its SHA-256 is `CF8B66910C0133804E7F0E8B0F0F5D59A43211A7F445A3E33C3E04E792BC7F7B`. This distinction avoids presenting an unpushed revision as though it were already part of the cited public commit. In particular, `SU2RootedTreeGaugeFixing.lean` contains the connected-graph spanning-tree construction and global product-Haar theorem; `SU2TreeCotreeConnectivity.lean` supplies the compatibility theorem; `SU2BoundaryConditionedGaugeFixing.lean` defines the retained boundary integral; and `SU2PhysicalConstructionIntegral.lean` proves the final original-edge evaluation. The preceding modules contain the Haar–sphere identification, orbital integration, character convolution, heat semigroup, Migdal move, finite-cellulation reduction, and exact coefficient extraction. `AuditMigdalClosure.lean` records the dependency audit. Thus the repository is not supplementary numerics or pseudocode: it is the proof artifact to which the theorem claims refer.

The complete audit-build transcript has SHA-256 `A6980A1C1CEA1A8CA3B37675457B0C2EDDBBB18DB8C2177A563A3C8C70E94FDC`; the public-root build transcript has SHA-256 `A9C1CA3BB6652F181335B8A497B9BE8DE9D0673F21CFF65A4726A0B0670C64D7`. The companion manifest records the commands, versions, hashes, endpoint count, and headline membership. These files permit an independent reader to compare the reported output byte for byte before repeating the build. The remaining build diagnostics are style lints—unused simplification arguments or variables and suggestions to replace `simpa` by `simp`—not errors or proof placeholders.

Trust boundary. The formal guarantee is relative to the Lean kernel, the pinned toolchain, Mathlib, and the integrity of the cited source snapshot. The source audit and continuous-integration configuration are reproducibility controls, not a substitute for independent peer review. No numerical experiment, computer algebra oracle, or unverified external certificate is used to establish the displayed equalities.

Mathematical statement	Lean declaration	Module
Half-trace semicircle law	su2HaarHalfTraceMeasure_eq_su2SemicircleMeasure	SU2Weyl.lean
Uniform rail-mass law	su2FirstRailMassMeasure_eq_uniform	SU2Orbit.lean
All-order orbital character formula	integral_su2Haar_orbit_character_general	SU2Orbit.lean
Translated character convolution	su2CharacterChebyshev_convolution	SU2CharacterConvolution.lean
Infinite heat semigroup	su2HeatKernel_convolution	SU2HeatKernelConvolution.lean
Two-face Migdal merge	su2Migdal_twoFace_merge	SU2Migdal.lean
Schedule existence for connected duals	SU2FaceEliminationSchedule.exists_valid_schedule_of_dual_connected	SU2FiniteCellulation.lean
Global edge gauge equivalence	SU2FiniteDiskCellulation.RootedSpanningTree.globalEdgeGaugeEquiv_measurePreserving	SU2RootedTreeGaugeFixing.lean
Tree–cotree connectivity	SU2DualRootedEliminationTree.primalGraphAvoiding_selectedEdgeRange_connected	SU2TreeCotreeConnectivity.lean
Conditioned edge amplitude	SU2BoundaryDiskCellulation.conditionedEdgeModelAmplitude_eq_heatKernel	SU2PhysicalConstructionIntegral.lean
Exact boundary coefficient	su2ConnectedDisk_simpleLoop_areaLaw_exact	SU2ConnectedAreaLaw.lean
Nonempty boundary-disk type	nonempty_su2BoundaryDiskCellulation	SU2BoundaryExamples.lean
Concrete cyclic witness $A(g) = K_{t=3}(g)$	SU2ThreeSpokeBoundaryExample.boundaryDisk_conditionedEdgeModelAmplitude_eq_heatKernel	SU2BoundaryExamples.lean

Table 2: Selected mathematical claims and their proof-artifact locations. The complete dependency audit contains 177 endpoints.

Archival provenance. This manuscript is the archival version of preprint 2607.0039, not a distinct claim to the same central theorem [18]; it adds explicit nonvacuity and cyclic record instances, reproduced audit evidence, and a narrower exposition. As of 1 August 2026, the author's thirteen corrective or supersession replacements remained submitted and pending, not newly published; 2607.0089v1 remained `REVIEW-PENDING`. The administrative supersession matrix does not enter any proof in this paper.

11 Scope and next frontier

The present artifact closes the compact-group analytic chain, introduces an independent finite oriented disk-cellulation object, constructs elimination schedules for every connected dual graph, and proves independence of the schedule. Dual cycles are allowed. The full original-edge model is linked to its gauge-fixed chord integral by a global lattice gauge-fixing equivalence; a compatible tree–cotree chart retains the exterior holonomy; and the conditioned integral is evaluated exactly by (12). The construction is uniform in the certified finite disk cellulation, proves simultaneous facial covariance, and removes all $V - 1$ pure-gauge coordinates exactly.

The artifact does not identify the abstract combinatorial maps with planar isotopy classes, treat nonsimple or intersecting loops, construct a continuum holonomy field, or evaluate positive-area higher-genus partition functions. In particular, it does not prove Makeenko–Migdal equations for intersecting loops. Those equations require a richer loop-space geometry and local variation theory than the simple-boundary object used here. Formalizing that extension without hiding topological hypotheses is the most natural next frontier; it is not part of the theorem claimed in this paper.

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