The diagonal Amos-type family at real order: a machine-checked quantitative crossing classification

Lluis Eriksson*

July 2026

Abstract

For the one-parameter family of Amos-type expressions

$$\mathrm{B}_{\nu,c}(x) \;=\; \frac{x}{\nu + c + \sqrt{(\nu+c)^2 + x^2}}, \qquad c \in \mathbb{R},$$

whose member $c = \frac{1}{2}$ is the classical Amos-type upper bound for the modified Bessel ratio $\rho_\nu = I_{\nu+1}/I_\nu$, we formalize in Lean 4, at every real order $\nu \geq 0$ over the Γ-power series, the classification of the parameter: $\mathrm{B}_{\nu,c}$ is a uniform upper bound for ρ_ν (all $\nu \geq 0$, $x > 0$) exactly when $c \leq \frac{1}{2}$, and a uniform lower bound for every $c \geq 1$, with explicit counterexample witnesses. The classification statement itself is equivalent to known results of Ruiz-Antolín and Segura; for it we claim only the machine-checking. The contribution is the regime between the ends: for every $\nu \geq 0$ and every $c \in (\frac{1}{2}, 1)$ we prove that the fixed diagonal member $\mathrm{B}_{\nu,c}$ crosses ρ_ν *exactly once* on $(0,\infty)$ — a transversal crossing located in the explicit window $[\sqrt{1-c},\, 2\hat{D}/(2c-1)]$ with $\hat{D} = (\nu + \frac{3}{2})^2 - (\nu+c)^2$, obeying the strict threshold $x_* > x_\dagger = 2(\nu+c)\sqrt{c(1-c)}/(2c-1)$, with globally determined sign on both sides and the two-sided scale law $2(\nu+c)\sqrt{c(1-c)} \;\leq\; (2c-1)\,x_* \;\leq\; 2\hat{D}$. Degenerate (critical) contact is *excluded by theorem*, through the exact identity $(\nu+c)^2\,D''(x_\dagger) = (2c-1)^3\,\mathrm{B}_{\nu,c}(x_\dagger)$ for the gap $D = \rho_\nu - \mathrm{B}_{\nu,c}$ at any hypothetical critical contact. The entire chain carries the axiom oracle [`propext`, `Classical.choice`, `Quot.sound`] and no analytic hypothesis beyond $\nu \geq 0$, $x > 0$; a pre-registered certified interval-arithmetic companion verifies the crossing phenomenon independently of the C5 crossing theorems at 30 parameter pairs spanning the hard regimes, all passing at 128 bits.

1 Introduction

The ratio $\rho_\nu(x) = I_{\nu+1}(x)/I_\nu(x)$ of modified Bessel functions of the first kind admits algebraically calibrated bounds of the shape $x/(\lambda + \sqrt{\mu^2 + x^2})$; the literature on such bounds — Amos [1], Segura [2, 5, 6], Hornik–Grün [3, 7], Ruiz-Antolín–Segura [4] — is rich, and the one-parameter *diagonal* family studied here,

$$\mathrm{B}_{\nu,c}(x) \;=\; \frac{x}{\nu + c + \sqrt{(\nu+c)^2 + x^2}},$$

*This manuscript and parts of the formal development were produced with AI assistance under the repository's audit protocol. Every theorem below is machine-checked in Lean 4 against a pinned Mathlib; the independent numerical companion of Section 6 is a committed interval-arithmetic transcript; remaining numerics are labelled paper-level where they occur. Repository: https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME.

is, up to reparametrization, the family whose sharp ends [4] identifies: under $\mu = \nu+1$, $\alpha = 2c-1$, the family b_α of [4] is exactly $\mathrm{B}_{\nu,c}$. *No novelty is claimed for the family, nor for the qualitative observation that its non-bounding members must cross the ratio* — [4] already notes that for parameters between the sharp ends the graphs cross. Three companion papers of this series formalized the consequence calculus of the $c = \frac{1}{2}$ bound at integer order [11], the bound itself at integer order [12], and the bound at every real order $\nu \ge 0$ over the Γ-power series [13]; the present paper builds directly on the last.

The contribution here is twofold, and the two halves carry different claims:

1. **The classification, machine-checked.** For every real $\nu \ge 0$ and $x > 0$: $\rho_\nu < \mathrm{B}_{\nu,c}$ holds uniformly iff $c \le \frac{1}{2}$ (Theorem 3.1), and for $c \ge 1$ the reverse inequality holds uniformly (Theorem 3.2). As mathematics this is equivalent to results of [4]; the claim is the formalization — uniform in real order, from the series, with explicit rational witnesses, inside one pinned axiom-clean development.

2. **The quantitative crossing theorem.** For $\nu \ge 0$ and $\frac{1}{2} < c < 1$, the fixed diagonal member $\mathrm{B}_{\nu,c}$ crosses ρ_ν exactly once; the crossing is transversal, satisfies an explicit strict threshold, determines the sign of $\rho_\nu - \mathrm{B}_{\nu,c}$ globally on both sides, lies in a constructive finite window, and obeys an explicit two-sided scale law (Theorem 3.3). Degenerate contact is excluded by an exact second-derivative identity (Theorem 3.4). We are not aware of a previous theorem giving this complete quantitative crossing classification for the diagonal family; Section 7 compares against the closest antecedents parameter by parameter.

Statuses: **exact** means a Lean theorem in the pinned central repository with clean axiom oracle; **certified** means a committed interval-arithmetic transcript; **paper-level** means ordinary prose, always flagged; **verified** marks an ordinary high-precision computation used only as a cross-check of a certified result, never as evidence. The repository's largest namespace (`YangMills/`) reflects its original motivation in lattice gauge theory; no reduction to any continuum or gauge-theoretic problem is claimed or used.

2 Setting

All objects live in module `AmosClosure` of the pinned central repository, over the real-order interface of [13]: $I_\nu =$ `besselIReal` ν is the Γ-power series $\sum_{k\ge 0}(x/2)^{\nu+2k}/(k!\,\Gamma(\nu+k+1))$ (real exponents via `rpow`), with positivity, the three-term recurrence, termwise differentiability, the Riccati equation

$$\rho_\nu'(x) \;=\; Q_x(\rho_\nu(x)), \qquad Q_x(y) := 1 - \frac{2\nu+1}{x}\,y - y^2, \tag{1}$$

the barrier $\psi_\nu > 2\nu + 1$ for $\psi_\nu(x) = x(1/\rho_\nu - \rho_\nu)$, and the $c = \frac{1}{2}$ bound $\rho_\nu < \mathrm{B}_{\nu,1/2}$ (`amosBoundReal_holds`). New objects of this paper: the family $\mathrm{B}_{\nu,c} =$ `amosFamily` ν c, the lower bound $L_\nu =$ `amosLower` ν of Section 4.2, and the crossing gap

$$D(x) \;=\; \rho_\nu(x) - \mathrm{B}_{\nu,c}(x).$$

Throughout, $a = \nu + c$ and $s = \sqrt{a^2 + x^2}$. Everything is stated at $x > 0$, the `rpow` domain of the development.

3 Main results

Theorem 3.1 (classification of the upper-bound range; `amosFamily_uniform_upper_iff`). *For every $c \in \mathbb{R}$:*

$$\Big(\forall\, \nu \geq 0,\ \forall\, x > 0 :\ \rho_\nu(x) < \mathrm{B}_{\nu,c}(x)\Big) \iff c \leq \tfrac{1}{2}.$$

The failure witness for $c > \frac{1}{2}$ is explicit and rational: $x_0(c) = 2/(2c-1)$ at $\nu = 0$ (`amosFamily_lt_amosLower_witness`).

Theorem 3.2 (the uniform lower half; `amosFamily_lower_of_one_le`). *For every $c \geq 1$, every $\nu \geq 0$ and every $x > 0$, $\mathrm{B}_{\nu,c}(x) < \rho_\nu(x)$.*

Theorems 3.1 and 3.2 are machine-checked forms of known mathematics [4]; between them lies the regime $\frac{1}{2} < c < 1$, where $\mathrm{B}_{\nu,c}$ bounds on neither side. There — with the strictness of the threshold in item (ii), and through it the unconditional uniqueness and orientation, delivered by Theorem 3.4 below:

Theorem 3.3 (the global crossing theorem; `amosFamily_global_crossing`, `amosFamily_crossing_orientation`). *Fix $\nu \geq 0$ and $\frac{1}{2} < c < 1$. There is exactly one $x_* > 0$ with $\rho_\nu(x_*) = \mathrm{B}_{\nu,c}(x_*)$, and:*

- *(i)* (window) $\sqrt{1-c} \leq x_* \leq \dfrac{2\hat{D}}{2c-1}$, *where* $\hat{D} = (\nu + \frac{3}{2})^2 - (\nu + c)^2$ *(`amosFamily_crossing_exists`);*
- *(ii)* (strict threshold) $x_* > x_\dagger := \dfrac{2(\nu+c)\sqrt{c(1-c)}}{2c-1}$ *(`crossing_threshold_strict`);*
- *(iii)* (transversality) *at the crossing,* $D'(x_*) = \dfrac{\mathrm{B}_{\nu,c}(x_*)}{x_*}\Big(2c - 1 - \dfrac{\nu+c}{\sqrt{(\nu+c)^2 + x_*^2}}\Big) > 0$ *(`crossing_hasDerivAt`, whose factor is positive by `crossing_threshold_strict`);*
- *(iv)* (orientation) $\rho_\nu < \mathrm{B}_{\nu,c}$ *on* $(0, x_*)$ *and* $\rho_\nu > \mathrm{B}_{\nu,c}$ *on* (x_*, ∞) *(`amosFamily_crossing_orientation`); conversely — an immediate trichotomy consequence — the sign of D locates x relative to x_*;*
- *(v)* (scale law) $2(\nu+c)\sqrt{c(1-c)} \leq (2c-1)\, x_* \leq 2\hat{D}$ *(`crossing_scale`).*

Theorem 3.4 (no degenerate contact; `crossing_no_critical_contact`). *Fix $\nu \geq 0$ and $\frac{1}{2} < c < 1$. No crossing of ρ_ν and $\mathrm{B}_{\nu,c}$ is a critical point of D: at any hypothetical contact with $D = D' = 0$ — which the slope identity forces to lie exactly at $x_\dagger$ — the second derivative obeys the exact identity*

$$(\nu + c)^2\, D''(x_\dagger) \;=\; (2c-1)^3\, \mathrm{B}_{\nu,c}(x_\dagger) \;>\; 0,$$

and a critical contact with positive curvature contradicts the below-threshold orientation. Consequently the threshold in Theorem 3.3 is strict and uniqueness and orientation hold unconditionally.

Theorem 3.5 (the Turán inequality at real order; `besselIReal_turan`). *For every $\nu \geq 0$ and $x > 0$, $I_{\nu+1}(x)^2 > I_\nu(x)\, I_{\nu+2}(x)$; equivalently $\psi_\nu(x) < 2\nu + 2$, completing the sandwich $2\nu + 1 < \psi_\nu(x) < 2\nu + 2$ whose lower half is the barrier of [13]. This inequality is classical (Turán-type inequalities for I_ν go back to Thiruvenkatachar–Nanjundiah; see Baricz–Ponnusamy [9]); it is proved here at real order over the Γ-series and used as supporting structure.*

4 The proof

The formal proofs are the authority; this section presents the mathematics in full, at the level of the checklist a referee needs.

4.1 Family algebra: calibration and the nullcline trichotomy

Since $s = \sqrt{a^2 + x^2} > |a| \geq -a$, the denominator $a + s$ is positive for *every* real c, and the conjugate form $\mathrm{B}_{\nu,c} = (s-a)/x$ follows from $(s-a)(s+a) = x^2$ (`amosFamily_pos`). The general calibration identity

$$\frac{1}{\mathrm{B}_{\nu,c}(x)} - \mathrm{B}_{\nu,c}(x) \;=\; \frac{2(\nu + c)}{x} \qquad (\texttt{amosFamily\_calibration}) \tag{2}$$

extends the $c = \frac{1}{2}$ calibration of [13]. Substituting into the Riccati quadratic of (1): $1 - \mathrm{B}^2 = (2a/x)\mathrm{B}$, so

$$Q_x\big(\mathrm{B}_{\nu,c}(x)\big) \;=\; \frac{2a - (2\nu + 1)}{x}\,\mathrm{B}_{\nu,c}(x) \;=\; \frac{2c - 1}{x}\,\mathrm{B}_{\nu,c}(x) \qquad (\texttt{riccatiQReal\_amosFamily}), \tag{3}$$

with the sign trichotomy: zero iff $c = \frac{1}{2}$ (the member *on* the nullcline — exactly the calibration that drives the proof of the $c = \frac{1}{2}$ bound), positive iff $c > \frac{1}{2}$, negative iff $c < \frac{1}{2}$. Since $a \mapsto x/(a + \sqrt{a^2 + x^2})$ is decreasing (`amosFamily_anti`; the non-strict form is all the chain uses), $c \leq \frac{1}{2}$ gives $\mathrm{B}_{\nu,c} \geq \mathrm{B}_{\nu,1/2} > \rho_\nu$ by [13]: the sufficiency half of Theorem 3.1 (`amosFamily_upper_of_le_half`).

4.2 The lower bound by recurrence inversion

The ratio recurrence $1/\rho_\nu = 2(\nu + 1)/x + \rho_{\nu+1}$ (`besselIReal_ratio_recurrence`) turns the $c = \frac{1}{2}$ *upper* bound at order $\nu{+}1$ into a *lower* bound at order ν: with $b = \nu{+}\frac{3}{2}$ and $s' = \sqrt{b^2 + x^2}$,

$$\frac{1}{\rho_\nu(x)} \;<\; \frac{2(\nu+1)}{x} + \frac{s' - b}{x} \;=\; \frac{\nu + \frac{1}{2} + s'}{x}, \quad \text{i.e.} \quad \rho_\nu(x) \;>\; \frac{x}{\nu + \frac{1}{2} + \sqrt{(\nu + \frac{3}{2})^2 + x^2}} \;=:\; L_\nu(x),$$

where the middle step is the algebraic collapse $x^2/(b + s') = s' - b$ (`besselLowerReal_holds`). This is the sharp-at-infinity lower bound of [4]; here it is a three-line corollary of the real-order upper bound, which is the point of recording it. L_ν matches ρ_ν to first order at infinity ($L_\nu \sim 1 - (\nu + \frac{1}{2})/x$), while $\mathrm{B}_{\nu,c} \sim 1 - (\nu + c)/x$ (paper-level asymptotics, not used in the proofs); for $c > \frac{1}{2}$ the family member falls below L_ν at explicit finite x: at $\nu = 0$ and $x_0(c) = 2/(2c - 1)$ the inequality $\mathrm{B}_{0,c}(x_0) < L_0(x_0)$ reduces, after clearing the calibrated denominators with $(2c - 1)x_0 = 2$, to the rational inequality $\frac{9}{4} - c^2 < 2$ for $c \in (\frac{1}{2}, 1]$ and its routine extension beyond (`amosFamily_lt_amosLower_witness`); chaining $\mathrm{B}_{0,c}(x_0) < L_0(x_0) < \rho_0(x_0)$ kills uniformity. That is the necessity half of Theorem 3.1. For Theorem 3.2: $c \geq 1$ gives $\mathrm{B}_{\nu,c} \leq \mathrm{B}_{\nu,1} < L_\nu < \rho_\nu$, where $\mathrm{B}_{\nu,1} < L_\nu$ is the root comparison $\sqrt{(\nu + \frac{3}{2})^2 + x^2} < \frac{1}{2} + \sqrt{(\nu + 1)^2 + x^2}$, i.e. $\nu + 1 < \sqrt{(\nu + 1)^2 + x^2}$ after squaring (`amosFamily_one_lt_amosLower`).

4.3 The Turán sandwich

$\rho_\nu > \mathrm{B}_{\nu,1}$ (from Section 4.2) plus the calibration (2) at $c = 1$ gives $\psi_\nu < 2(\nu+1)$ (`besselPsiReal_lt`); together with the barrier $\psi_\nu > 2\nu + 1$ of [13] this is the sandwich $2\nu + 1 < \psi_\nu < 2\nu + 2$. The upper half converts, through the recurrence, into $\rho_\nu > \rho_{\nu+1}$ and thence into the Turán inequality of Theorem 3.5 (`besselIReal_turan`). We emphasize provenance: the mathematical content of the sandwich upper half is classical [9]; the formal statement at real order over the Γ-series is what is new here, and the crossing results below do not depend on it.

4.4 Existence: the explicit window

Fix $\frac{1}{2} < c < 1$. At the left end, the product-form upper bound $\rho_\nu(x) < x/\big(2(\nu+1)(1-q_\nu)\big)$ with $q_\nu = (x/2)^2/(\nu+2)$, valid for $q_\nu < 1$ (`besselRatioReal_lt_product`, from the geometric-domination toolkit of [13]) beats the family member below $\sqrt{1-c}$: using $\sqrt{a^2+x^2} \le a + x^2/(2a)$ one checks $D(x) < 0$ for all $0 < x \le \sqrt{1-c}$, uniformly in ν (`crossing_witness_small_le`). At the right end, at $X = 2\hat{D}/(2c-1)$ with $\hat{D} = (\nu+\frac{3}{2})^2 - (\nu+c)^2$, the two calibrated roots separate by less than $(2c-1)/4$ and $L_\nu(X) > \mathrm{B}_{\nu,c}(X)$, so $D(X) > 0$ (`crossing_witness_large`). Continuity of D (from differentiability) and the intermediate value theorem give a crossing in the window (`amosFamily_crossing_exists`).

4.5 Slope identity, threshold, uniqueness, orientation

The family derivative is $\mathrm{B}'_{\nu,c}(x) = a/\big(s\,(a+s)\big)$ (`amosFamily_hasDerivAt`). At any zero of D, ρ_ν takes the family value, so by (1) and (3) $\rho'_\nu = \frac{2c-1}{x}\mathrm{B}_{\nu,c}$ there, while $a/(s(a+s)) = a\mathrm{B}_{\nu,c}/(sx)$ by the conjugate form; hence the *slope identity*

$$D'(\text{zero}) \;=\; \frac{\mathrm{B}_{\nu,c}}{x}\left(2c-1-\frac{a}{s}\right) \qquad (\texttt{crossing\_hasDerivAt}). \tag{4}$$

The factor $2c-1-a/s$ is negative below, zero at, and positive above $x_\dagger = 2a\sqrt{c(1-c)}/(2c-1)$ (where $s = a/(2c-1)$). Three consequences, in order: every crossing satisfies $x \ge x_\dagger$, since a crossing strictly below the threshold would be a strictly *downward* crossing yet $D < 0$ must persist from the left window endpoint (`crossing_threshold`); above the threshold every crossing is strictly upward (`crossing_pos_right`), and two strictly upward zeros of a differentiable function cannot occur without an intermediate downward zero — excluded — so the crossing is unique (`crossing_unique`); and the sign of D is then globally determined on both sides (`crossing_orientation_after`, `crossing_orientation_below`).

4.6 Exclusion of the degenerate contact

One case survives Section 4.5: a contact exactly at $x_\dagger$, where the slope factor vanishes identically and the crossing would be a critical point of D. At such a contact every quantity is an explicit algebraic function of (ν, c): writing $t = 2c-1 \in (0,1)$,

$$s = \frac{a}{t}, \qquad \rho_\nu = \mathrm{B}_{\nu,c} = \sqrt{\frac{1-c}{c}}, \qquad \mathrm{B}_{\nu,c}\,x = s - a = \frac{a(1-t)}{t}, \qquad x^2 = \frac{a^2(1-t)(1+t)}{t^2},$$

and $2\nu + 1 = 2a - t$. Differentiating the Riccati equation along the flow and the family derivative once more (`crossing_hasDerivAt_everywhere` provides D's derivative as an explicit function everywhere; that function is differentiated once more inside the tangency proof), the two curvatures collapse to

$$\rho_\nu'' \;=\; \frac{\mathrm{B}}{x^2}\big[(2\nu{+}1)(1{-}t){-}2t\,\mathrm{B}x\big] \;=\; -\frac{t^3\,\mathrm{B}}{a^2(1+t)}, \qquad \mathrm{B}'' \;=\; -\frac{a\,(x/s)(a+2s)}{s^2(a+s)^2} \;=\; -\frac{t^3\,\mathrm{B}\,(t+2)}{a^2(t+1)},$$

whence the exact identity of Theorem 3.4: $D'' = t^3\mathrm{B}/a^2$, i.e. $(\nu+c)^2 D'' = (2c-1)^3\,\mathrm{B}_{\nu,c} > 0$. In the formal development the identity is proved division-free in exactly this form; positivity needs only $2c-1>0$, $\nu+c>0$, $\mathrm{B}_{\nu,c}>0$. The contradiction: at a critical contact, $D'=0$ and $D''>0$ force $D'<0$ on a left interval, so D is strictly decreasing into the contact and $D(y) > D(x_\dagger) = 0$ just left of it; but $y < x_\dagger$ is strictly below the threshold, where the orientation of Section 4.5 gives $D(y)<0$ (`crossing_no_critical_contact`). No critical contact exists; the threshold is strict (`crossing_threshold_strict`); uniqueness and orientation are unconditional (`crossing_unique_unconditional`); and the packaged existence-and-uniqueness statement is `amosFamily_global_crossing`. The scale law of Theorem 3.3(v) is then immediate from the threshold and the window (`crossing_scale`).

5 Scope

All results are for the in-core Γ-power-series definition of I_ν at real order $\nu \geq 0$, $x>0$; no identification with an external special-functions library object is claimed (none exists at the pinned Mathlib to identify with), though the identification theorem of [13] ties the object to the factorial series at every integer order. Theorems 3.1 and 3.2 are machine-checked forms of known mathematics [4], and the crossing *phenomenon* for intermediate parameters is likewise observed there; the claims of this paper are (i) the formalization, uniform in real order, and (ii) the quantitative crossing package of Theorems 3.3 and 3.4: unique transversal crossing, strict explicit threshold, bilateral orientation, constructive window, two-sided scale law, degenerate contact excluded. We are not aware of a previous theorem giving this complete quantitative crossing classification for the diagonal family.

6 The certified companion

A certified interval-arithmetic companion verifies the crossing phenomenon *independently of the C5 crossing theorems* — without consulting any C5 crossing, uniqueness, orientation, or scale-law theorem: its protocol — registered in the development's charter in its own commits, strictly before the certified run and before any certified result was observed — mandates that verdicts be decided *only* by ball signs of D and D', never by consulting the crossing theorems; that the bisection search interval come from the fixed rule $[\,(1{-}c)/4,\;8(\nu{+}2)^2/(2c{-}1)\,]$, deliberately *wider* than the Lean window, with the theoretical window checked afterwards rather than searched; and that failures escalate a precision ladder (128 to 4096 bits) rather than widen intervals. The grid of 30 pairs, $\nu \in \{0, \frac{1}{2}, 1, \pi, 10, 100\} \times c \in \{0.501, 0.6, 0.75, 0.9, 0.999\}$ (π a certified ball), forces the hard regimes: $c \downarrow \frac{1}{2}$ (the crossing escapes to infinity — at $(\nu, c) = (100, 0.501)$ the search interval reaches 4.2×10^7), $c \uparrow 1$ (the crossing value $\mathrm{B}(x_\dagger) \to 0$), $\nu = 0$, large order, and the doubly-extreme corners. Per pair, four certificates, all by ball sign: a bracketing box $X = [\mathrm{lo}, \mathrm{hi}]$ with certified opposite signs of D at its ends (certified bisection; midpoints are outward-rounded balls — for non-dyadic parameters the registered exact-dyadic midpoints are not representable — a strictly conservative refinement, since wider endpoint balls make every certificate harder, never easier); transversality $\inf_{x\in X} D'(x) > 0$ by interval evaluation over the hull; window inclusion $X \subset [x_\dagger, x_+]$; and both sides of the scale law. The ratio is evaluated by the backward recurrence $\rho_\alpha = x/(2(\alpha+1)+x\rho_{\alpha+1})$ seeded at order $\nu+N$ with the *interval* $[0,1]$ — valid by positivity and the $c = \frac{1}{2}$ bound of [13], both outside the forbidden theorem set — so no exponentially large value of I_ν is ever formed; the seed contraction is certified a posteriori by the final ball width, and a too-wide ball can only produce an undecided verdict, never a wrong one. All 30 pairs pass all four certificates at the first ladder rung (128 bits); an independent ordinary high-precision pass (labelled *verified*, not certified) agrees at all 30. Two illustrative rows: at $(100, 0.501)$ the certified box $[50250.8995\ldots]$ (width $\sim 3.5\times10^{-14}$) clears the threshold $x_\dagger = 50250.3995\ldots$ with certified D' ball $= 3.95\times10^{-13} \pm 4.1\times10^{-16} > 0$; at $(\frac{1}{2}, 0.501)$ the box sits at $x_* = 501.000000000000000\ldots$, reflecting the closed (hyperbolic-cotangent) form of $\rho_{1/2}$: by the calibration (2), the crossing equation is $\psi_{1/2}(x_*) = 2(\nu+c)$, and $\psi_{1/2}(x) = 2 + 1/(x-1)$ up to exponentially small error, so $x_* \approx 1 + \frac{1}{2c-1} = \frac{2c}{2c-1} = 501$ — a paper-level remark; the certified box needs no such structure. The companion is *not* part of the proof, and the proof does not cite it.

7 Related work

The comparison below follows the development's pre-registered collation; the antecedents are compared claim by claim.

Ruiz-Antolín–Segura [4] is the exact antecedent of the *family* and of *qualitative existence*: under $\mu = \nu+1$, $\alpha = 2c-1$, their b_α family is $\mathrm{B}_{\nu,c}$; they identify the sharp ends ($\alpha \le 0$, i.e. $c \le \frac{1}{2}$, at $x \to \infty$; $\alpha \ge 1$, i.e. $c \ge 1$, at $x \to 0$) and observe that for intermediate parameters the graphs must cross (asymptotic end signs). We declare expressly: the family and the mere existence of some crossing are prior art. [4] contains no uniqueness, orientation, threshold, tangency exclusion, or quantitative localization for the intermediate members.

Hornik–Grün [3] is the closest antecedent in spirit: it characterizes which two-parameter Amos-type expressions $G_{\alpha,\beta}$ are upper or lower bounds, and describes minimal upper bounds as those tangent to the ratio at a single point. The statements are about the *boundary of the bounding region* in the (α,β)-plane; classifying the fixed diagonal $G_{\alpha,\alpha}$-type member $\mathrm{B}_{\nu,c}$ — which for $c \in (\frac{1}{2}, 1)$ is *not* a bound at all — is a different question. Parameter by parameter: [3] yields, for the diagonal, membership/non-membership in the bounding classes; it does not state, for non-bounding diagonal members, the unique simple crossing, its transversality, the explicit threshold $x_\dagger$, bilateral orientation, or a scale law.

Segura [6] studies bounds $(\alpha + \sqrt{\beta^2+\gamma^2x^2})/x$-type with coefficients *optimized as functions of a prescribed tangency point*: there the bounding function is a global bound whose graph is tangent to the ratio (osculatory contact by construction). Here the family member is *fixed in advance*, is not a bound, the meeting is transversal, and tangential contact is *excluded by theorem*. Close in machinery, not equivalent in statement.

Hornik–Grün [7] proves single-sign-change results *between two bounds* within their generalized family (bound-versus-bound comparisons); the present single-crossing statement is *ratio-versus-family-member*, a different pair of curves — we flag the terminological near-collision explicitly.

Segura [5] (Riccati/nullcline monotonicity methods) and **Salazar** [8] (Bernstein-positivity certificates for Riccati reductions) supply methodology adjacent to ours; neither presents the quantitative diagonal crossing package.

On the formalization side, the companion papers [11, 12, 13] formalized the $c = \frac{1}{2}$ bound and its consumers; we know of no other machine-checked development of Amos-type bounds or of any crossing classification. The recurrence and derivative identities are [10, 10.29]. The development builds on Mathlib [14].

8 Reproducibility

Toolchain: Lean `leanprover/lean4:v4.29.0-rc6`; Mathlib pinned to commit

`07642720480157414db592fa85b626dafb71355b`

(lakefile and manifest agree); `tectonic` 0.15.0; `python-flint` 0.9.0 for the certified companion.

Recorded build facts: at the final mathematics commit

`f1c87fef` *(short id; the release manifest records the full hashes)*

`lake build AmosClosure` completed successfully (**8175 jobs**); the axiom oracle printed, for all **110** registered statements (the 74 of the `c4-v1.0` release registry plus the 36 of this paper), exactly

`[propext, Classical.choice, Quot.sound]`.

One proof-engineering disclosure: the division-free identity inside the tangency-exclusion proof (`crossing_no_critical_contact`) is closed by `linear_combination` with cofactor $K = -2c^2(2\nu + 1)$; the cofactor was found by polynomial division (SymPy `div`, exact, remainder 0) against the `ring_nf`-normalized goal, and is *replayed entirely inside Lean's ring normalizer* — the computer-algebra system plays no trusted role. The full run history of this development — seven green build transcripts and the failed-attempt record, each committed the day it was produced with its diagnosis in the commit message; the certified companion script-and-transcript pair; and eight charter amendments recording every external audit round, every score checkpoint, and every method deviation (including the companion's anchor deviation and depth correction, both registered *before* the certified run) — is committed under `scripts/` and `docs/C5-CHARTER.md`; per the repository's audit protocol it is archival material and plays no role in the mathematics above.

Headline-theorem-to-artifact map (all in module `AmosClosure`; every row carries a direct oracle witness; the 74 inherited statements are mapped in [11, 12, 13]):

Result	Lean name	File	Status
family, positivity, Eq. (2)	`amosFamily`, `amosFamily_calibration`	`AmosFamily.lean`	exact
Eq. (3) + trichotomy	`riccatiQReal_amosFamily` etc.	`AmosFamily.lean`	exact
sufficiency half	`amosFamily_upper_of_le_half`	`AmosFamily.lean`	exact
ratio recurrence	`besselIReal_ratio_recurrence`	`AmosLowerReal.lean`	exact
lower bound L_ν	`besselLowerReal_holds`	`AmosLowerReal.lean`	exact
witness $x_0(c)$	`amosFamily_lt_amosLower_witness`	`AmosLowerReal.lean`	exact
Thm. 3.1	`amosFamily_uniform_upper_iff`	`AmosLowerReal.lean`	exact
Thm. 3.2	`amosFamily_lower_of_one_le`	`AmosCrossing.lean`	exact
Thm. 3.5	`besselIReal_turan`	`AmosCrossing.lean`	exact
window (§4.4)	`amosFamily_crossing_exists`	`AmosCrossing.lean`	exact
Eq. (4), threshold	`crossing_hasDerivAt`, `crossing_threshold`	`AmosCrossing.lean`	exact
uniqueness, orientation	`crossing_unique`, `crossing_orientation_*`	`AmosCrossing.lean`	exact
scale law	`crossing_scale`	`AmosCrossing.lean`	exact
Thm. 3.4	`crossing_no_critical_contact`	`AmosTangency.lean`	exact
Thm. 3.3 package	`amosFamily_global_crossing`	`AmosTangency.lean`	exact
§6 grid	`c5_crossing_arb.py` + transcript	`scripts/`	certified

From a clean clone at the release revision (on Windows, clone with `core.autocrlf=false` — LF bytes reproduce the companion script's runtime self-hash — and `core.longpaths=true`):

```
lake exe cache get
lake build AmosClosure
lake env lean AmosClosure/Oracle.lean
python scripts/c5_crossing_arb.py
tectonic -X compile papers/c5-crossing/c5_crossing.tex
```

At the release revision, canonical artifact hashes will be recorded — tag-scoped, in `RELEASE-MANIFEST.md` in this paper's directory — committed together with the release revision of this manuscript; compare against `git show <rev>:path`, never a worktree checkout. The annotated tag `c5-v1.0` will mark that release commit after the pre-release audits close, with the audit registry in `docs/C5-CHARTER.md`; the release-tree audit is recorded in the manifest, post-tag by sequencing, with any fixes in post-tag commits; tags never move.

9 Conclusion

For the diagonal Amos-type family at real order, the pinned formal development now contains the parameter classification — uniform upper bound iff $c \le \frac{1}{2}$, uniform lower bound for $c \ge 1$, with explicit rational witnesses — and, in the intermediate regime $\frac{1}{2} < c < 1$, a global quantitative crossing theorem: exactly one crossing, transversal, strictly above an explicit threshold, with bilateral orientation, a constructive window, a two-sided scale law, and degenerate contact excluded by an exact curvature identity. Limitations: the in-core series definition only, and $\nu \ge 0$ (Section 5); the family, the classification as mathematics, and the crossing phenomenon are prior art [4] — the quantitative crossing package is the candidate-new content, stated with the priority prudence of Section 5. Natural next steps: the analogous classification for the second-kind ratio $K_{\nu+1}/K_\nu$ (where [6] suggests the tangency structure differs), asymptotics of $x_*(\nu, c)$ beyond the scale law, and upstreaming the modified-Bessel API.

References

[1] D. E. Amos, *Computation of modified Bessel functions and their ratios*, Math. Comp. **28** (1974), 239–251. doi:10.1090/S0025-5718-1974-0333287-7.

[2] J. Segura, *Bounds for ratios of modified Bessel functions and associated Turán-type inequalities*, J. Math. Anal. Appl. **374** (2011), 516–528. doi:10.1016/j.jmaa.2010.09.030.

[3] K. Hornik and B. Grün, *Amos-type bounds for modified Bessel function ratios*, J. Math. Anal. Appl. **408** (2013), 91–101. doi:10.1016/j.jmaa.2013.05.070.

[4] D. Ruiz-Antolín and J. Segura, *A new type of sharp bounds for ratios of modified Bessel functions*, J. Math. Anal. Appl. **443** (2016), 1232–1246. doi:10.1016/j.jmaa.2016.06.011. arXiv:1606.02008.

[5] J. Segura, *Monotonicity properties for ratios and products of modified Bessel functions and sharp trigonometric bounds*, Results Math. **76** (2021), no. 4, Paper No. 221. doi:10.1007/s00025-021-01531-1. arXiv:2105.02524.

[6] J. Segura, *Simple bounds with best possible accuracy for ratios of modified Bessel functions*, J. Math. Anal. Appl. **526** (2023), no. 1, 127211. doi:10.1016/j.jmaa.2023.127211. arXiv:2207.02713.

[7] K. Hornik and B. Grün, *Generalized Amos-type bounds for modified Bessel function ratios*, Math. Inequal. Appl. **27** (2024), 775–787. doi:10.7153/mia-2024-27-55.

[8] D. S. P. Salazar, *Riccati reductions for modified Bessel ratios: Bernstein positivity, exact certificates, and transfer obstructions*, arXiv:2607.05538, 2026.

[9] Á. Baricz and S. Ponnusamy, *On Turán type inequalities for modified Bessel functions*, Proc. Amer. Math. Soc. **141** (2013), 523–532. doi:10.1090/S0002-9939-2012-11325-5. arXiv:1010.3346.

[10] NIST Digital Library of Mathematical Functions, F. W. J. Olver et al., eds., https://dlmf.nist.gov/.

[11] L. Eriksson, *One Amos bound, three consumer sites: machine-checked Bessel-ratio calculus for lattice gauge expansions*, companion paper, same repository, release `c2-v1.2.1`; ai.viXra.org:2607.0033, July 2026.

[12] L. Eriksson, *A machine-checked proof of the Amos bound for modified Bessel function ratios*, companion paper, same repository, release `c3-v1.0.1`; ai.viXra.org:2607.0032, July 2026.

[13] L. Eriksson, *A machine-checked proof of an Amos-type bound for modified Bessel ratios at real order*, companion paper, same repository, release `c4-v1.0`; ai.viXra.org:2607.0030, July 2026.

[14] The mathlib Community, *The Lean mathematical library*, in: Proc. CPP 2020, ACM, 2020, pp. 367–381. doi:10.1145/3372885.3373824.