Orientation and Antipodes in the Lorentzian EPRL Vertex

Ginanjar Utama
ginanjar.utama@gmail.com

July 13, 2026

Abstract

Several inequivalent data at a Lorentzian spin-foam vertex are commonly called "orientation." We separate face coorientation, tetrahedral handedness, time orientation of a tetrahedral normal, four-simplex branch, causal wedge data, and global time orientability, and classify proposed reversals by the carrier and category in which they act. Inversion of an integrated $\mathrm{SL}(2,\mathbb{C})$ variable is only a Haar-measure reparametrization. By contrast, an $\mathrm{SU}(2)$ coherent-state spinor is a square root of its face flux: the standard antilinear spinor conjugation sends the flux to its antipode. On a spacelike quantum tetrahedron the tensor product of these antiunitaries fixes every area, reverses the oriented-volume operator, and squares to $+1$ on every admissible intertwiner space; hence nonzero-volume eigenstates are paired orthogonally, but no Kramers degeneracy follows. For a timelike tetrahedron, the intrinsic algebra is $\mathrm{Cl}(1,2)$. The sign of a face bivector square gives the elliptic–parabolic–hyperbolic orbit trichotomy. The two sheets associated with a spacelike face carry the discrete series $D_k^{\pm}$ of connected $\mathrm{SU}(1,1)$: there is no bounded linear intertwiner and no continuous invariant sesquilinear pairing between them, although a unique bounded invariant bilinear pairing and antiunitary or disconnected twisted exchanges exist. Finally, in every intrinsic odd-dimensional Clifford space, total inversion is the twisted-adjoint action of the odd pseudoscalar and cannot be implemented by an even intrinsic versor. Orientation protection is therefore relative to a specified carrier, symmetry group, and morphism category; a conservation law requires an explicit multi-vertex gluing rule.

1 Introduction

A spin-foam vertex brings several kinds of sign data into one formula. An outward normal can be reversed, a tetrahedron can change handedness, a Lorentzian normal can move from a future sheet to a past sheet, a four-simplex reconstruction can select the opposite Plebanski branch, and a causal model can reverse wedge or partial-order data. These operations live on different carriers and need not be represented by the same kind of map. Calling all of them parity, time reversal, or causal reversal conceals the actual mathematical question:

> *What object is being reversed, which maps are allowed to act on it, and which invariant detects the change?*

This paper answers that question locally at one Lorentzian EPRL vertex and at one vertex of the Conrady–Hnybida extension [1, 2, 3, 17, 18, 19]. The analysis combines three complementary descriptions.

First, the standard loop-quantum-gravity description identifies a boundary node with a quantum tetrahedron: areas are link Casimirs and volume is an intertwiner observable [4, 5, 6]. Second,

carrier	proposed operation	map category	fixed data	result
integrated $g_a \in \mathrm{SL}(2,\mathbb{C})$	$g_a \mapsto g_a^{-1}$	measurable Haar reparametrization	boundary state and measure	integral unchanged; no orientation datum acted on
SU(2) face spinor/flux	$z \mapsto \epsilon \bar{z}$, $\boldsymbol{X} \mapsto -\boldsymbol{X}$	antiunitary on H_j	spin j and area	coherent antipode; angular momentum changes sign
spacelike quantum tetrahedron	all four fluxes reversed	antiunitary on $\mathrm{Inv}(\otimes H_j)$	four areas	oriented volume changes sign; $T^2 = +1$
timelike-tetrahedron spacelike face	future/past sheet exchange	connected $\mathrm{SU}(1,1)$ bounded linear maps	Casimir $k(k-1)$	$\mathrm{Hom}(D_k^+, D_k^-) = 0$; bilinear and antiunitary exchanges remain
intrinsic three-space	$v \mapsto -v$	Clifford twisted adjoint	quadratic form v^2	odd pseudoscalar implements it; no even intrinsic versor does

Table 1: The classification is by carrier, allowed map, and invariant. The same word "reversal" denotes inequivalent operations unless all three are specified.

the spinorial description writes an SU(2) flux as a bilinear in a 2-spinor, making the antipode an elementary antilinear operation [8, 9, 10]. Third, geometric algebra keeps the intrinsic signature and transformation grade visible: in $\mathrm{Cl}(3,0)$ or $\mathrm{Cl}(1,2)$, bivector squares classify generators and odd versors distinguish disconnected orthogonal transformations [14, 15, 16].

The main results are summarized in Table 1. Their common content is a map-category principle: the same vector-level antipode can be invisible as a change of integration coordinates, unavailable inside a connected rotation group, implemented antiunitarily on a spin representation, or implemented by an odd element of a Pin group. These statements are not contradictory because their carriers and allowed morphisms differ.

The scope is deliberately limited. We do not identify the operations below with physical time reversal, with the two asymptotic branches of a reconstructed four-simplex, or with causal wedge data. Proper and causal vertices modify those latter structures directly [20, 21, 22, 23, 24]. Nor do we claim an orientation conservation law: that requires a gluing calculation, not a single-vertex classification.

2 Orientation data and EPRL boundary variables

2.1 Six distinct orientation data

Definition 1 (Orientation data at a vertex)**.** Let a label a boundary tetrahedron of a four-simplex vertex.

(O1) *Face coorientation:* the choice of outward unit flux direction $\boldsymbol{n}_{ab}$ for a face (ab) of tetrahedron a.

(O2) *Spatial orientation of a tetrahedron:* the sign of an oriented triple product of three independent face fluxes, after an ordering of the legs has been fixed.

(O3) *Time orientation of a tetrahedral normal:* the future/past character of a timelike normal; for a timelike tetrahedron, the future/past sheet of a timelike face normal in its intrinsic 2+1 geometry.

(O4) *Four-simplex reconstruction orientation:* the sign labelling the two Plebanski or cosine branches when nondegenerate critical data reconstruct a four-simplex [7, 20].

(O5) *Combinatorial causal orientation:* wedge signs or partial-order data introduced by a causal spin-foam kernel [21, 22, 23].

(O6) *Global time orientability:* compatibility of local time-sheet choices across a multi-vertex two-complex.

The Haar reparametrization of Section 3 changes none of (O1)–(O6). The simultaneous antipode of the four fluxes at a spacelike node reverses (O2). The sheet exchange studied in Section 5 concerns (O3). No implication to (O4) or (O5) is established.

2.2 The Lorentzian EPRL vertex

Let $\Gamma = K_5$ be the boundary graph of a four-simplex. With $g_1 = \mathbb{1}$ gauge-fixed, a coherent Lorentzian EPRL vertex can be written as

$$A(B) = \int_{\mathrm{SL}(2,\mathbb{C})^4} \prod_{a=2}^{5} \mathrm{d}g_a \prod_{a<b} \langle J\xi_{ab}| Y_\gamma^\dagger D^{(\gamma j_{ab}, j_{ab})}(g_a^{-1} g_b) Y_\gamma |\xi_{ba}\rangle, \tag{1}$$

where $B = \{j_{ab}, \boldsymbol{n}_{ab}\}$, $|\xi_{ab}\rangle = |j_{ab}, \boldsymbol{n}_{ab}\rangle$ is an SU(2) coherent state, $D^{(\rho,k)}$ is a unitary principal-series representation, and $Y_\gamma : H_j \to H_{(\gamma j, j)}$ is the EPRL injection [1, 7, 3]. The antiunitary J will be derived spinorially below.

Two bra conventions occur in the literature, using either $\langle J\xi_{ab}|$ or $\langle \xi_{ab}|$ on an oriented link. They differ by an antipodal map and coherent-state phases. The results proved here concern Haar measure, intrinsic fluxes, invariant subspaces, orbit components, and representation categories. They are therefore independent of that choice, provided the operation being discussed is stated on the boundary data rather than inferred from a convention-dependent phase.

2.3 A face spinor as a square root of its flux

The Dirac–Penrose "square-root" viewpoint is especially useful at the boundary. Let $z \in \mathbb{C}^2 \setminus \{0\}$ and define the rank-one projector

$$P(z) = \frac{|z\rangle\langle z|}{\langle z \mid z\rangle} = \tfrac{1}{2}(\mathbb{1} + \boldsymbol{n}(z)\cdot\boldsymbol{\sigma}), \qquad n_i(z) = \frac{\langle z|\sigma_i|z\rangle}{\langle z \mid z\rangle}. \tag{2}$$

Thus the normalized spinor is a square root, up to phase, of a unit vector on S^2. This is the compact analogue of the two-spinor factorization of a future null vector [8]. Introduce

$$\varsigma z = \epsilon \bar{z}, \qquad \epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3}$$

Lemma 1 (Spinorial antipode)**.** *The map ς is antiunitary, $\varsigma^2 = -\mathbb{1}$, and*

$$P(\varsigma z) = \mathbb{1} - P(z) = \tfrac{1}{2}(\mathbb{1} - \boldsymbol{n}(z)\cdot\boldsymbol{\sigma}). \tag{4}$$

Hence $\boldsymbol{n}(\varsigma z) = -\boldsymbol{n}(z)$.

Proof. The identity $\epsilon\sigma_i^*\epsilon^{-1} = -\sigma_i$ gives the Pauli vector in (2) the opposite sign. Both projectors have rank one and their ranges are orthogonal, so they sum to the identity. The square follows from $\epsilon^2 = -\mathbb{1}$. $\square$

The spin-j representation is $H_j \simeq \mathrm{Sym}^{2j}\mathbb{C}^2$. Restricting $\varsigma^{\otimes 2j}$ to this symmetric power gives the standard angular-momentum antiunitary J_j,

$$J_j U_j(g) J_j^{-1} = U_j(g), \qquad J_j \boldsymbol{L} J_j^{-1} = -\boldsymbol{L}, \qquad J_j^2 = (-1)^{2j}, \tag{5}$$

and $J_j|j,\boldsymbol{n}\rangle = e^{\mathrm{i}\phi(j,\boldsymbol{n})}|j,-\boldsymbol{n}\rangle$ [13]. Equation (5) is not an ordinary SU(2) rotation: antiunitarity is essential. In twistorial descriptions of loop gravity, an SU(2) spinor is only the intrinsic part of a simple twistor; areas and Lorentzian dihedral data occupy distinct canonical variables [11, 12]. Consequently, applying J_j to the intrinsic face spinor does not by itself prescribe a reversal of the full twistor, the extrinsic angle, or causal wedge data.

2.4 The quantum tetrahedron

At node a the boundary Hilbert space is

$$\mathcal{H}_a = \mathrm{Inv}\left(\bigotimes_{b\neq a} H_{j_{ab}}\right), \tag{6}$$

with coherent intertwiner

$$\|j,\{\boldsymbol{n}\}\rangle\rangle = \int_{\mathrm{SU}(2)} \mathrm{d}h \bigotimes_{b\neq a} h|j_{ab},\boldsymbol{n}_{ab}\rangle. \tag{7}$$

Closure $\sum_b j_{ab}\boldsymbol{n}_{ab} = 0$ is the classical shadow of invariance [4, 5, 6]. Face areas depend only on the Casimirs, $\widehat{A}_{ab} \propto \sqrt{j_{ab}(j_{ab}+1)}$. After fixing an ordered triple of distinct legs (b_1, b_2, b_3), define

$$\widehat{Q} = \epsilon_{ijk} L^i_{b_1} L^j_{b_2} L^k_{b_3}, \qquad \widehat{V}^2 \propto |\widehat{Q}|. \tag{8}$$

An odd permutation of the chosen legs reverses $\widehat{Q}$, so its sign is meaningful only relative to a chosen node orientation.

3 Three inequivalent map categories

3.1 Inversion of an integration coordinate

For $S \subseteq \{2,3,4,5\}$ let R_S replace g_a by g_a^{-1} for $a \in S$. If F_B is the integrand of (1), define

$$A_S^{\mathrm{coord}}(B) = \int \prod_{a=2}^{5} \mathrm{d}g_a\, F_B(R_S g). \tag{9}$$

Proposition 1 (Haar-coordinate identity)**.** *If F_B is absolutely integrable, then $A_S^{\mathrm{coord}}(B) = A(B)$ for every S. The same holds for a regularization whose domain and measure are invariant under R_S.*

Proof. Inversion preserves Haar measure on the unimodular group $\mathrm{SL}(2,\mathbb{C})$. Apply the change of variables $h_a = g_a^{-1}$ for $a \in S$. $\square$

This identity is intentionally modest. It states that a measurable coordinate involution cannot be detected by an unrestricted integral. It does not act on the bivectors or boundary spinors and therefore does not implement parity, time reversal, or any item in Definition 1. If a lower-dimensional holonomy family is not closed under inversion, the corresponding restricted integral may be asymmetric; that asymmetry belongs to the restriction, not to the full vertex.

3.2 The intrinsic Clifford antipode

The vector-level map $v \mapsto -v$ has a different realization in geometric algebra. Let V be a nondegenerate quadratic space of odd dimension $n = 2r + 1$, let $\mathrm{Cl}(V)$ be its real Clifford algebra, and let $I = e_1 \cdots e_n$ be the pseudoscalar of an oriented orthonormal basis. The Clifford grade involution is denoted by α.

Theorem 1 (Odd-dimensional intrinsic inversion)**.** *The pseudoscalar I is central and odd, and its twisted-adjoint action is*

$$\rho_I(v) = \alpha(I)vI^{-1} = -v, \qquad v \in V. \tag{10}$$

Thus total inversion is represented by an odd element of $\mathrm{Pin}(V) \setminus \mathrm{Spin}(V)$*. No even versor of the intrinsic algebra can implement it.*

Proof. In odd dimension the pseudoscalar commutes with every vector. Since $\alpha(I) = -I$, equation (10) follows. An even versor is a product of an even number of reflections and induces determinant $+1$ on V, whereas total inversion has determinant $(-1)^n = -1$. □

For signature (p, q),

$$I^2 = (-1)^{n(n-1)/2+q}. \tag{11}$$

In the three-dimensional algebras used below, $I^2 = -1$ for $\mathrm{Cl}(3, 0)$ and $\mathrm{Cl}(1, 2)$, while $I^2 = +1$ for $\mathrm{Cl}(2, 1)$. In the last convention $P_\pm = \frac{1}{2}(1 \pm I)$ are central idempotents. This signature-dependent central splitting is an algebraic fact, not a derivation of the infinite-dimensional $\mathrm{SU}(1, 1)$ discrete series.

Remark 1 (Intrinsic versus ambient)*.* The determinant argument in Theorem 1 is intrinsic to V. An even versor of a larger ambient algebra can restrict to $-\mathbb{1}$ on V while reflecting additional directions so that the ambient determinant remains $+1$. A conformal extension therefore inherits the obstruction only after a subgroup acting trivially, or in a specified way, on the complement has been fixed.

The Clifford lift and the spinorial lift of the antipode belong to different categories. The former is an odd linear element of a Pin group acting by a twisted sandwich on vectors. The latter is an antiunitary acting on a complex representation of the connected spin group. They induce the same sign change on the flux vector, but they should not be identified as the same operator.

4 Spacelike tetrahedra: the antiunitary volume flip

A spacelike tetrahedron has a Euclidean intrinsic rest space $E \simeq \mathbb{R}^3$ and compact stabilizer $\mathrm{SU}(2)$. In $\mathrm{Cl}(3, 0)$ write the area fluxes as $\boldsymbol{X}_f = j_f \boldsymbol{n}_f$. For any ordered independent triple,

$$\boldsymbol{X}_1 \wedge \boldsymbol{X}_2 \wedge \boldsymbol{X}_3 = qI_3. \tag{12}$$

The simultaneous antipode fixes every $\boldsymbol{X}_f^2$ and sends $q \mapsto -q$. By Theorem 1, it is not an intrinsic rotor. Equivalently, no $R \in \mathrm{SO}(3)$ can send all four spanning face normals of a nondegenerate tetrahedron to their antipodes: such an R would equal $-\mathbb{1}$ on $\mathbb{R}^3$ and have determinant -1.

The quantum lift is nevertheless available antiunitarily.

Theorem 2 (Antiunitary reversal of a quantum tetrahedron)**.** *Let*

$$T_a = \bigotimes_{b \neq a} J_{j_{ab}} \tag{13}$$

act on $\bigotimes_{b\neq a} H_{j_{ab}}$*. Then:*

1. T_a *is antiunitary and preserves* $\mathcal{H}_a$*;*
2. $T_a \| j, \{\boldsymbol{n}\}\rangle\rangle = e^{\mathrm{i}\Phi} \| j, \{-\boldsymbol{n}\}\rangle\rangle$*;*
3. $T_a \widehat{A}_{ab} T_a^{-1} = \widehat{A}_{ab}$ *for every face;*
4. $T_a \widehat{Q} T_a^{-1} = -\widehat{Q}$*;*
5. *if* $\mathcal{H}_a \neq \{0\}$*, then* $T_a^2 = +\mathbb{1}$ *on* $\mathcal{H}_a$*.*

Proof. Equation (5) shows that each J_j commutes antiunitarily with the SU(2) representation, so their tensor product preserves invariant tensors. Applying it to (7) gives the antipodal intertwiner, with the product of coherent-state phases absorbed into $e^{\mathrm{i}\Phi}$. Areas are functions of $\boldsymbol{L}_f^2$. Since each $\boldsymbol{L}_f$ changes sign and $\widehat{Q}$ is trilinear in three distinct legs, $\widehat{Q}$ changes sign. Finally,

$$T_a^2 = \prod_{b \neq a} (-1)^{2j_{ab}}. \tag{14}$$

The existence of an invariant tensor requires $\sum_{b\neq a} j_{ab} \in \mathbb{Z}$, so the product equals $+1$. ☐

Corollary 1 (Spectral pairing without Kramers degeneracy)**.** *If* $\widehat{Q}\psi = q\psi$ *with* $q \neq 0$*, then* $\widehat{Q}(T_a\psi) = -q(T_a\psi)$ *and* $\langle \psi \mid T_a\psi\rangle = 0$*. Nevertheless antiunitarity alone imposes no orthogonality on a general state, because* $T_a^2 = +1$ *rather than* -1*.*

Proof. The two vectors belong to distinct eigenspaces of the Hermitian operator $\widehat{Q}$. Conversely, for a normalized $q \neq 0$ eigenvector ϕ, set $\chi = T_a\phi$ and choose its phase so that $T_a\chi = \phi$. Then $\psi = (\phi + \chi)/\sqrt{2}$ obeys $T_a\psi = \psi$. ☐

The result is sharper than saying that an antipode is or is not “allowed.” It is unavailable as one connected SO(3) transformation of the four normals, available as an odd intrinsic Pin transformation of the vector geometry, and available as an antiunitary on the quantum intertwiner space. There is no absolute kinematical superselection between the two volume signs: the full algebra $B(\mathcal{H}_a)$ contains linear operators mixing all subspaces. What is exact is the transformation law of the area and oriented volume observables under the specified antiunitary.

5 Timelike tetrahedra: orbit sheets and representation categories

For a timelike tetrahedron the normal N is spacelike and the intrinsic orthogonal space $W = N^\perp$ has signature $(1, 2)$. We therefore fix the algebra $\mathrm{Cl}(1, 2)$ and its pseudoscalar I, with $I^2 = -1$. If $B \in \Lambda^2 W$ is a face bivector, define the dual face normal

$$n = BI^{-1}, \qquad n^2 = -B^2. \tag{15}$$

This is the geometric-algebra task contract for the section: the carrier is a bivector or its dual vector in $\mathrm{Cl}(1, 2)$, the operator is the connected $\mathrm{Spin}^+(1, 2)$ action, and the invariant is B^2 together with the connected component of the normal quadric.

5.1 Bivector square and orbit topology

Proposition 2 (Elliptic–parabolic–hyperbolic trichotomy)**.** *The stabilizer type of a nonzero face normal is determined by* B^2*:*

B^2	n^2	*normal type*	*one-parameter stabilizer*
< 0	> 0	*timelike*	*elliptic rotation*
$= 0$	$= 0$	*null*	*parabolic null rotation*
> 0	< 0	*spacelike*	*hyperbolic boost*

This trichotomy is algebraic: a simple generator with negative, zero, or positive square exponentiates respectively to trigonometric, nilpotent, or hyperbolic functions.

Theorem 3 (Connected-orbit classification)**.** *Under* $\mathrm{SO}^+(1,2)$*:*

1. *the future and past sheets of the unit timelike hyperboloid are distinct orbits;*
2. *the future and past components of the punctured null cone are distinct orbits;*
3. *the spacelike unit hyperboloid is one orbit, and every spacelike vector is connected to its antipode.*

Consequently, with the null origin excluded,

$$n \not\sim -n \text{ under } \mathrm{SO}^+(1,2) \quad \Longleftrightarrow \quad B^2 \leq 0. \tag{16}$$

Proof. The proper orthochronous group preserves the sign of the time component on timelike and null vectors and is transitive on each corresponding connected component. The spacelike unit hyperboloid is connected and homogeneous. More explicitly, after mapping a spacelike v to $(0,1,0)$, a rotation by π in the spatial plane sends it to its antipode; conjugating this rotation gives the required element for the original v. □

The null line is included here as an orbit-theoretic boundary case. The standard Conrady–Hnybida state sum assigns discrete-series data to spacelike faces and continuous-series data to timelike faces; it does not provide a separate nonzero-area null-face Hilbert sector.

5.2 Discrete series and the bounded linear obstruction

For a spacelike face of a timelike tetrahedron, the two time sheets are represented by the positive and negative discrete series D_k^+ and D_k^- of connected $\mathrm{SU}(1,1)$ [17, 18, 19]. Choose orthonormal weight bases

$$e_n^+ = |k, k+n\rangle_+, \qquad e_n^- = |k, -k-n\rangle_-, \qquad n = 0, 1, 2, \ldots. \tag{17}$$

The two representations have the same quadratic Casimir but opposite, disjoint L_3 spectra.

Theorem 4 (Bounded connected-group obstruction)**.** *In the category of unitary Hilbert representations with bounded linear morphisms,*

$$\mathrm{Hom}^{\mathrm{b}}_{\mathrm{SU}(1,1)}(D_k^+, D_k^-) = \{0\}. \tag{18}$$

Every continuous $\mathrm{SU}(1,1)$*-invariant sesquilinear pairing* $S : \mathcal{H}_k^+ \times \mathcal{H}_k^- \to \mathbb{C}$ *also vanishes.*

Proof. A bounded intertwiner commutes with the spectral projections of the self-adjoint generator L_3. It must therefore map each weight space to a weight space with the same eigenvalue, but the spectra $\{k+n\}$ and $\{-k-n\}$ are disjoint.

For the second statement take the convention that S is antilinear in its first argument. Under $U(t) = e^{-\mathrm{i}tL_3}$,

$$S(U_+(t)e_n^+, U_-(t)e_m^-) = e^{\mathrm{i}(2k+n+m)t} S(e_n^+, e_m^-). \tag{19}$$

Invariance for every t forces every coefficient to vanish. □

The theorem is a precise category-relative statement. It does not say that the two Hilbert spaces are unrelated; it says that connected $\mathrm{SU}(1,1)$ has no bounded linear or continuous sesquilinear channel that exchanges them.

5.3 What survives: bilinear, antiunitary, and twisted maps

The negative discrete series is the complex-conjugate representation of the positive one. After a choice of phases there is an antiunitary

$$C_k : \mathcal{H}_k^+ \longrightarrow \mathcal{H}_k^-, \qquad C_k e_n^+ = e_n^-, \qquad C_k D_k^+(g) C_k^{-1} = D_k^-(g). \tag{20}$$

Proposition 3 (Unique bounded invariant bilinear pairing)**.** *The formula*

$$\beta_k(x,y) = \langle C_k x, y\rangle_- \tag{21}$$

defines a bounded $\mathrm{SU}(1,1)$*-invariant complex-bilinear pairing* $\mathcal{H}_k^+ \times \mathcal{H}_k^- \to \mathbb{C}$*. It is unique up to scale among bounded invariant bilinear pairings and satisfies* $\beta_k(e_n^+, e_m^-) = \delta_{nm}$*.*

Proof. Because C_k and the first slot of the Hilbert inner product are both antilinear, (21) is linear in x and in y. Its norm is at most $\|x\|\|y\|$, and invariance follows from (20) and unitarity. Any other bounded invariant bilinear form has the form $\langle Ax, y\rangle$ for an anti-linear intertwiner A. Then $C_k^{-1}A$ is a bounded linear endomorphism of the irreducible representation D_k^+ and is scalar by Schur's lemma. □

Proposition 4 (No Hilbert-tensor evaluation vector)**.** *The algebraic functional induced by* β_k *on* $\mathcal{H}_k^+ \odot \mathcal{H}_k^-$ *is not continuous for the Hilbert tensor norm and therefore does not extend to* $\mathcal{H}_k^+ \widehat{\otimes}_2 \mathcal{H}_k^-$*.*

Proof. Let

$$w_N = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} e_n^+ \otimes e_n^-. \tag{22}$$

Then $\|w_N\|_2 = 1$, while the functional defined by β_k gives $\sqrt{N}$. Hence it is unbounded in the Hilbert tensor norm. □

Thus the pairing is perfectly well behaved as a bounded bilinear map on two Hilbert spaces, and as a functional on a suitable projective tensor product, but not as an invariant vector or evaluation morphism in the ordinary Hilbert tensor category. This distinction matters for spin-foam gluing.

A disconnected orthogonal transformation supplies a second kind of exchange. Let $p \in \mathrm{Pin}(1,2)$ induce the automorphism $\vartheta_p(g) = pgp^{-1}$ of the connected subgroup. For a unitary equivalence

$$U_k D_k^+(g) U_k^{-1} = D_k^-(\vartheta_p(g)), \tag{23}$$

the block operator on $\mathcal{H}_k^+ \oplus \mathcal{H}_k^-$,

$$P_k(v_+, v_-) = (U_k^{-1} v_-, U_k v_+), \tag{24}$$

obeys $P_k D(g) P_k^{-1} = D(\vartheta_p(g))$. It is a twisted intertwiner of the disconnected extension, not an ordinary intertwiner of connected $\mathrm{SU}(1,1)$.

For a timelike face, n is spacelike, its antipode belongs to the same connected orbit by Theorem 3, and the corresponding group element acts unitarily in the same continuous-series sector. Combining the classical and quantum statements gives the following classification.

Theorem 5 (Orientation sectors at a timelike tetrahedron)**.** *At a timelike tetrahedron:*

1. *a spacelike face has $B^2 < 0$, a two-sheet timelike normal, and discrete series $D_k^\pm$. Its sheet label cannot be changed by a bounded linear connected-group intertwiner or a continuous invariant sesquilinear pairing, but is related by the bilinear form* (21)*, by the antiunitary C_k, and by disconnected twisted maps;*
2. *a timelike face has $B^2 > 0$, a one-sheet spacelike normal, and no connected-orbit sheet obstruction inside its continuous-series sector;*
3. *the null case $B^2 = 0$ has two connected orbit components but is only an orbit-theoretic limiting case here.*

Accordingly, "superselection" is meaningful only after the allowed morphism category has been specified.

6 Gluing, twistors, and causal scope

The missing step is gluing. A single vertex identifies where an orientation label can reside; it does not determine how that label propagates. Proposition 4 shows that the natural D_k^+–D_k^- bilinear form is not an evaluation morphism in the Hilbert tensor category. A two-vertex amplitude sharing one timelike tetrahedron must state whether contraction is algebraic, projective-tensor, distributional, or defined by another rigging. It must also track the opposite induced orientations of the shared boundary. Only then can one decide whether gluing identifies equal sheet labels, opposite labels, or sums over both.

Spinorial antipode versus full twistor transformation. The rank-one projector (2) isolates intrinsic flux data. A simple twistor contains additional variables, including the Lorentzian dihedral angle canonically conjugate to area [11, 12]. Therefore the map $z \mapsto \varsigma z$ proves an intrinsic flux antipode and the corresponding antiunitary action on boundary states; it does not prove a transformation law for the full covariant phase-space point. This is the spinorial reason that (O2) cannot be silently identified with (O4) or (O5).

Proper and causal vertices. The proper EPRL vertex restricts the four-simplex reconstruction branch (O4) [20]. Causal and Toller-matrix vertices introduce wedge-sign information in the kernel (O5) [22, 23, 24]. The identity $T^{(+)} + T^{(-)} = D$ reconstructs the unrestricted Wigner kernel when the two wedge signs are summed independently. It is not the same as summing only over globally consistent causal assignments, which form a constrained subset. Proposition 1 applies to the reconstructed unrestricted integral, not automatically to an individual causal sector.

A conditional global invariant. Suppose an explicit gluing law assigns a relative sign in $\mathbb{Z}_2$ to each oriented dual edge and those signs satisfy a cocycle condition around dual faces. Then the obstruction to a globally consistent sheet choice would be a class in $H^1(K, \mathbb{Z}_2)$, analogous to a temporal first Stiefel–Whitney class. This is a possible output of a future multi-vertex calculation, not a consequence of the one-vertex results. Recent causal consistency criteria over $\mathbb{F}_2$ provide a natural comparison point [25].

7 Discussion and conclusion

The paper's conclusions are best read as a hierarchy of carriers and maps.

At the level of integration variables, inversion is a Haar-coordinate identity and carries no physical orientation content. At the intrinsic boundary level, an SU(2) coherent spinor is a square root of a flux vector, and its canonical antilinear conjugation produces the antipode. Tensoring this operation at a spacelike node yields an exact antiunitary volume flip: areas are fixed, oriented volume changes sign, and $T^2 = +1$ on every admissible intertwiner space. The absence of Kramers degeneracy and the orthogonal pairing of nonzero-volume eigenstates follow immediately.

At a timelike tetrahedron, geometric algebra makes the signature dependence explicit. The invariant B^2 separates elliptic, parabolic, and hyperbolic face geometries, while connected-component topology decides whether an antipode is reachable in $\mathrm{SO}^+(1,2)$. Quantization refines rather than replaces this orbit statement. The discrete series $D_k^\pm$ have equal Casimir but disjoint weight spectra, so their sheet exchange is obstructed for bounded connected-group linear maps and continuous invariant sesquilinear forms. Yet a bounded bilinear contragredient pairing, antiunitary conjugation, and disconnected twisted maps survive. The label is therefore protected only relative to a declared category.

Finally, intrinsic Clifford geometry explains the disconnected character of total inversion without overreaching: in odd dimension it is the pseudoscalar's twisted-adjoint action and is necessarily odd. The statement is intrinsic; extra ambient directions can compensate parity. This caveat is not a technical footnote but an example of the paper's central rule: an orientation claim is meaningful only after the carrier, acting group, and allowed morphisms are fixed.

The remaining decisive calculation is a two-vertex gluing amplitude with a shared timelike tetrahedron. It would determine the functional category of the pairing, the boundary-orientation convention, and whether a local sheet label becomes a transported or conserved global datum. Until that is done, the exact single-vertex result is a classification of possible orientation carriers, not a theory of causal propagation.

References

[1] J. Engle, E. Livine, R. Pereira, and C. Rovelli, "LQG vertex with finite Immirzi parameter," Nucl. Phys. B **799**, 136–149 (2008), arXiv:0711.0146.

[2] A. Perez, "The Spin-Foam Approach to Quantum Gravity," Living Rev. Relativ. **16**, 3 (2013), arXiv:1205.2019.

[3] C. Rovelli and F. Vidotto, *Covariant Loop Quantum Gravity* (Cambridge University Press, Cambridge, 2014).

[4] A. Barbieri, "Quantum tetrahedra and simplicial spin networks," Nucl. Phys. B **518**, 714–728 (1998), arXiv:gr-qc/9707010.

[5] J. C. Baez and J. W. Barrett, "The quantum tetrahedron in 3 and 4 dimensions," Adv. Theor. Math. Phys. **3**, 815–850 (1999), arXiv:gr-qc/9903060.

[6] E. R. Livine and S. Speziale, "A new spinfoam vertex for quantum gravity," Phys. Rev. D **76**, 084028 (2007), arXiv:0705.0674.

[7] J. W. Barrett, R. J. Dowdall, W. J. Fairbairn, F. Hellmann, and R. Pereira, "Lorentzian spin foam amplitudes: graphical calculus and asymptotics," Class. Quantum Grav. **27**, 165009 (2010), arXiv:0907.2440.

[8] R. Penrose and W. Rindler, *Spinors and Space-Time, Vol. 1: Two-Spinor Calculus and Relativistic Fields* (Cambridge University Press, Cambridge, 1984).

[9] E. R. Livine and J. Tambornino, "Spinor representation for loop quantum gravity," J. Math. Phys. **53**, 012503 (2012), arXiv:1105.3385.

[10] E. F. Borja, L. Freidel, I. Garay, and E. R. Livine, "$U(N)$ tools for loop quantum gravity: The return of the spinor," Class. Quantum Grav. **28**, 055005 (2011), arXiv:1010.5451.

[11] L. Freidel and S. Speziale, "From twistors to twisted geometries," Phys. Rev. D **82**, 084041 (2010), arXiv:1006.0199.

[12] S. Speziale and W. M. Wieland, "The twistorial structure of loop-gravity transition amplitudes," Phys. Rev. D **86**, 124023 (2012), arXiv:1207.6348.

[13] E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press, New York, 1959).

[14] D. Hestenes, *Space-Time Algebra* (Gordon and Breach, New York, 1966).

[15] P. Lounesto, *Clifford Algebras and Spinors*, 2nd ed. (Cambridge University Press, Cambridge, 2001).

[16] C. Doran and A. Lasenby, *Geometric Algebra for Physicists* (Cambridge University Press, Cambridge, 2003).

[17] F. Conrady and J. Hnybida, "A spin foam model for general Lorentzian 4-geometries," Class. Quantum Grav. **27**, 185011 (2010), arXiv:1002.1959.

[18] F. Conrady, "Spin foams with timelike surfaces," Class. Quantum Grav. **27**, 155014 (2010), arXiv:1003.5652.

[19] F. Conrady and J. Hnybida, "Unitary irreducible representations of $\mathrm{SL}(2,\mathbb{C})$ in discrete and continuous $\mathrm{SU}(1,1)$ bases," J. Math. Phys. **52**, 012501 (2011), arXiv:1007.0937.

[20] J. Engle, "A proposed proper EPRL vertex amplitude," Phys. Rev. D **87**, 084048 (2013), arXiv:1111.2865.

[21] E. R. Livine and D. Oriti, "Implementing causality in the spin foam quantum geometry," Nucl. Phys. B **663**, 231–279 (2003), arXiv:gr-qc/0210064.

[22] E. Bianchi and P. Martin-Dussaud, "Causal Structure in Spin Foams," Universe **10**, 181 (2024), arXiv:2109.00986.

[23] E. Bianchi, C. Chen, and M. Gamonal, "Causal spinfoam vertex for 4d Lorentzian quantum gravity," Phys. Rev. D **113**, 126020 (2026), arXiv:2601.23162.

[24] E. Bianchi, C. Chen, and M. Gamonal, "Toller matrices and the Feynman $i\epsilon$ in spinfoams," arXiv:2604.24945 [gr-qc] (2026).

[25] C. E. Beltrán, "Causal Structure for Generalized Spinfoams," arXiv:2603.22661 [gr-qc] (2026).