A machine-checked proof of the Amos bound for modified Bessel function ratios

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Abstract

Amos's upper bound for the modified Bessel function ratio,

$$\rho_n(x) \;=\; \frac{I_{n+1}(x)}{I_n(x)} \;<\; \frac{x}{n+\frac{1}{2}+\sqrt{(n+\frac{1}{2})^2+x^2}} \;=\; \mathrm{B}_n(x),$$

is a classical theorem, and its derivation through the qualitative theory of the associated Riccati equation is an established technique. This paper contributes, to our knowledge, the first formalization: a complete, machine-checked Lean 4 proof of the bound for every integer order $n \geq 0$ and every $x > 0$, over the power-series definition of I_n carried in the same pinned development, with axiom oracle [`propext`, `Classical.choice`, `Quot.sound`] and no analytic hypothesis of any kind. The formalized route runs through the Riccati equation $\rho_n' = 1 - \frac{2n+1}{x}\rho_n - \rho_n^2$ (itself derived from the formalized series calculus), the observation that B_n is exactly the positive root of the Riccati quadratic, a small-argument zone bound uniform in n obtained from pure geometric tail estimates, and a first-crossing barrier argument in a transformed variable $\psi_n = x(1/\rho_n - \rho_n)$ whose structural feature — every touch of the critical level forces $\rho_n' = 0$, so the barrier never needs to be differentiated — is the simplification this formalization contributes. As corollaries, the unit-step inequality, the strict monotonicity of the logarithmic derivative across orders (in `deriv` form), and a φ-monotonicity step used by a lattice-gauge surface expansion all become unconditional theorems. The theorem is proved for the in-core power-series definition of the integer-order modified Bessel function; no formal identification with an external special-functions library object, and no extension to noninteger order, is claimed.

1 Introduction

The ratio $\rho_\nu = I_{\nu+1}/I_\nu$ of modified Bessel functions of the first kind is the workhorse of character expansions in two-dimensional lattice gauge theory and of many statistical applications. Amos [1] established the upper bound

$$\rho_\nu(x) \;<\; \mathrm{B}_\nu(x) \;=\; \frac{x}{\nu+\frac{1}{2}+\sqrt{(\nu+\frac{1}{2})^2+x^2}} \qquad (x > 0); \tag{1}$$

*This manuscript and parts of the formal development were produced with AI assistance under the repository's audit protocol. Every theorem below is machine-checked in Lean 4 against a pinned Mathlib; the independent numerical cross-check of Section 6 is a committed interval-arithmetic transcript; remaining numerics are labelled paper-level where they occur. Repository: https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME.

sharp forms and systematic families were developed by Segura [2], Hornik and Grün [3, 5], and Ruiz-Antolín and Segura [4]; the derivation of such bounds through the qualitative theory of the Riccati equation satisfied by ρ_ν is an established technique described explicitly in [4]. *No novelty is claimed for the bound or for the Riccati approach.* What this paper contributes is, to our knowledge, the first *formalization*: a complete machine-checked proof in Lean 4, from the power series upward, with every analytic ingredient a theorem of the same pinned development — and, within the formalization, one structural simplification (the ψ-formulation of the barrier, Section 4.3) that eliminates the need to differentiate the barrier function at all.

The companion paper of this series [7] formalized the *consequence calculus* of (1) — calibration engine, unit step, log-derivative monotonicity, φ-step — with the bound itself carried as the single named hypothesis (`AmosBound`) over the series-defined `besselI`, together with its derivative interface. The present paper removes that hypothesis: `amosBound_holds` proves it, and every consumer — every statement that carried the bound as a hypothesis — becomes unconditional. The exact scope is stated in the abstract's final sentence and in Section 5.

Statuses: **exact** means a Lean theorem in the pinned central repository with clean axiom oracle; **certified** means a committed interval-arithmetic transcript; **paper-level** means ordinary prose, always flagged. The repository's largest namespace (`YangMills/`) reflects its original motivation in lattice gauge theory; no reduction to any continuum or gauge-theoretic problem is claimed or used.

2 Setting

All objects live in module `AmosClosure` of the pinned central repository. From the companion development [7]: `besselI` $n\ x = \sum_{k\geq 0}(x/2)^{n+2k}/(k!\,(n+k)!)$ as a `tsum`; positivity for $x > 0$; the three-term recurrence and the derivative identity $I_n' = I_{n+1} + (n/x)I_n$ as theorems; $\mathrm{B}_n(x) =$ `amosRHS` $n\ x$ and the predicate `AmosBound` $n\ x\ r \iff r < \mathrm{B}_n(x)$. Write $\rho_n(x) =$ `besselI` $(n{+}1)\ x/$`besselI` $n\ x$ (`besselRatio`). All eighteen statements of that layer carry the clean axiom oracle and are inherited here unchanged.

3 Main results

Theorem 3.1 (the Amos bound; `amosBound_holds`). *For every $n \in \mathbb{N}$ and every real $x > 0$,*

$$\frac{\texttt{besselI}\ (n{+}1)\ x}{\texttt{besselI}\ n\ x} < \frac{x}{n + \frac{1}{2} + \sqrt{(n + \frac{1}{2})^2 + x^2}}.$$

No hypothesis remains: the statement is closed over the series definition.

Theorem 3.2 (unconditional consumers). *For every n and $x > 0$: $\rho_n(x) - \rho_{n+1}(x) < 1/x$; the derivative of $\log\circ$`besselI` n at x is strictly below that of $\log\circ$`besselI` $(n{+}1)$ (in `deriv` form); and the φ-monotonicity step of [7] holds at every order ≥ 1 (every order at which that step is defined) — each now a theorem with no analytic hypothesis. Lean names:*

`besselI_unit_step_unconditional`,
`besselI_logDeriv_lt_unconditional`, `besselI_phi_step_unconditional`.

4 The proof

The formal proofs are the authority; this section presents the mathematics in full, at the level of the checklist a referee needs.

4.1 The Riccati equation and the nullcline

From the formalized derivative identity and the recurrence, the quotient rule gives

$$\rho_n'(x) \;=\; Q_x(\rho_n(x)), \qquad Q_x(y) := 1 - \frac{2n+1}{x}\, y - y^2 \tag{2}$$

(Lean: `besselRatio_hasDerivAt`). The bound B_n satisfies the exact calibration identity

$$\frac{1}{\mathrm{B}_n(x)} - \mathrm{B}_n(x) \;=\; \frac{2n+1}{x} \qquad \text{(Lean: } \texttt{amosRHS\_calibration}\text{)}, \tag{3}$$

which after multiplication by $\mathrm{B}_n > 0$ is precisely $Q_x(\mathrm{B}_n(x)) = 0$ (`riccatiQ_amosRHS`): *the Amos bound is the positive root of the Riccati quadratic* — the nullcline of the flow (2). Moreover $Q_x(y) > 0$ for $0 \le y < \mathrm{B}_n(x)$, since $Q_x(y) - Q_x(\mathrm{B}_n) = (\mathrm{B}_n - y)\big(\frac{2n+1}{x} + y + \mathrm{B}_n\big)$ (`riccatiQ_pos_of_lt`).

4.2 The zone: $x \le 1/4$, uniformly in n

Set $\psi_n(x) = x\,(1/\rho_n(x) - \rho_n(x))$. By (3) and the strict antitonicity of $t \mapsto 1/t - t$ on $(0, \infty)$, the target $\rho_n < \mathrm{B}_n$ is equivalent to $\psi_n > 2n+1$. On the zone we prove the latter directly from the series. Write $t_{n,k}(x) = (x/2)^{n+2k}/(k!\,(n+k)!)$ for the series terms. Three elementary facts, all formalized in product form (no inverse comparisons):

- *geometric domination* (`besselTerm_le_geometric`): $t_{n,k} \le t_{n,0}\, q_n^k$ with $q_n = (x/2)^2/(n+1)$, by induction on k; hence the product-form upper bound $(1-q_n)\, I_n(x) \le t_{n,0}$ for $q_n < 1$ (`besselI_mul_le`);
- *strict two-term lower bound* (`besselTerm_zero_lt_besselI`): $t_{n,0} < I_n(x)$, since the $k=1$ term is positive;
- *exact first-term identity* (`besselTerm_zero_succ`): $2(n+1)\, t_{n+1,0} = x\, t_{n,0}$.

For $0 < x \le 1/4$ the geometric ratio at order $n+1$ satisfies the exact bound $2(n+1)\, q_{n+1} = \frac{(n+1)x^2}{2(n+2)} \le \frac{x^2}{2}$ — the n-dependence cancels in the ratio $(n+1)/(n+2) \le 1$, which is what makes the zone uniform in n — and with $x^2 \le 1/16$ the coefficient inequality $x^2 + 2(n+1)q_{n+1} \le \frac{3}{2}x^2 \le \frac{3}{32} < 1$ yields

$$2n+1+x^2 \;\le\; 2(n+1)\,(1-q_{n+1}).$$

Chaining through the three series facts (upper bound at order $n+1$, identity, strict lower bound at order n) gives the decisive strict inequality

$$(2n+1+x^2)\; I_{n+1}(x) \;<\; x\, I_n(x) \qquad (0 < x \le 1/4), \tag{4}$$

Two further lines convert (4) into the zone claim: since $2n+1+x^2 \ge 1$, (4) also gives the crude bound $I_{n+1}(x) \le x\, I_n(x)$, hence $x\, I_{n+1}^2 \le x^2\, I_{n+1} I_n$; multiplying (4) by $I_n > 0$ and inserting this,

$$(2n+1)\, I_{n+1} I_n + x\, I_{n+1}^2 \;\le\; (2n+1)\, I_{n+1} I_n + x^2\, I_{n+1} I_n \;<\; x\, I_n^2,$$

which, by the identity $\psi_n \cdot (I_{n+1} I_n) = x\,(I_n^2 - I_{n+1}^2)$, is precisely $\psi_n(x) > 2n+1$ on the zone — these are the two auxiliary steps inside `besselPsi_zone`.

4.3 The barrier: every touch is an upward crossing

The structural simplification of this formalization is that the barrier argument runs entirely in ψ_n and never differentiates B_n. Suppose $\psi_n(c) = 2n+1$ at some $c > 0$. Then $1/\rho_n(c) - \rho_n(c) = (2n+1)/c$, and substituting this into the Riccati quadratic annihilates it:

$$Q_c(\rho_n(c)) = 1 - \Big(\frac{1}{\rho_n(c)} - \rho_n(c)\Big)\rho_n(c) - \rho_n(c)^2 = 1 - 1 + \rho_n(c)^2 - \rho_n(c)^2 = 0,$$

so $\rho_n'(c) = 0$ by (2) (`riccati_zero_of_touch`). The product rule then collapses the derivative of ψ_n at the touch to

$$\psi_n'(c) \;=\; \Big(\frac{1}{\rho_n(c)} - \rho_n(c)\Big) + c \cdot 0 \;=\; \frac{2n+1}{c} \;>\; 0:$$

every touch of the critical level is an upward crossing.

Lemma 4.1 (first crossing; the human form of `besselPsi_gt`). *Suppose, for contradiction, $\psi_n(x_1) \le 2n+1$ for some $x_1 > 0$. By the zone theorem $x_1 > 1/4$. The set $S = \{y \in [1/4, x_1] : \psi_n(y) \le 2n+1\}$ is nonempty, bounded below, and closed (continuity of ψ_n on the interval, from differentiability); let $c = \inf S \in S$. Since $\psi_n(1/4) > 2n+1$, in fact $c > 1/4$, and by minimality $\psi_n > 2n+1$ on $(1/4, c)$ (at $1/4$ itself this holds by the zone theorem); left-continuity forces the touch $\psi_n(c) = 2n+1$. But then $\psi_n'(c) = (2n+1)/c > 0$, so the difference quotient $\big(\psi_n(y) - \psi_n(c)\big)/(y-c)$ is eventually positive as $y \uparrow c$; since $y - c < 0$ there, $\psi_n(y) < \psi_n(c) = 2n+1$ just left of c — contradicting $\psi_n > 2n+1$ on $(1/4, c)$. Hence $\psi_n(x) > 2n+1$ for all $x > 0$.*

Finally, $\psi_n > 2n+1$ converts into $\rho_n < \mathrm{B}_n$ by the strict antitonicity of $t \mapsto 1/t - t$, verified algebraically rather than through an abstract monotonicity lemma: with $s = 1/\rho_n - \rho_n$ and $t = 1/\mathrm{B}_n - \mathrm{B}_n$,

$$(s - t) \cdot \rho_n \mathrm{B}_n = (\mathrm{B}_n - \rho_n)\,(1 + \rho_n \mathrm{B}_n),$$

and the left-hand side is positive — both its factors are: $s > t$ is the barrier conclusion $\psi_n > 2n+1$ divided by x, and $\rho_n \mathrm{B}_n > 0$ — so, since $1 + \rho_n \mathrm{B}_n > 0$, positivity passes to $\mathrm{B}_n - \rho_n$; this is the closing step of `amosBound_holds`. That completes Theorem 3.1; Theorem 3.2 is direct application.

5 Scope

The theorem is proved for the in-core power-series definition of the integer-order modified Bessel function. No formal identification with an external special-functions library object, and no extension to noninteger order, is claimed. In particular, results elsewhere in this programme that consume Amos-type bounds for the classical I_ν of the analysis literature — such as the surface-track φ-lemma conditional discussed in [7] — do not change verification status through this paper; connecting the two would require the (routine but unformalized) identification of the series object with the library object consumed there.

6 Independent numerical cross-check

The companion development committed a certified interval-arithmetic transcript (256-bit precision, self-contained series-plus-tail enclosures with a corrected, audited remainder majorant) verifying (1) with certified strict interval enclosures at all 1206 points of a pre-registered grid [7]. Relative to the present paper that artifact is exactly what its name says — an *independent numerical cross-check* of a now-proved theorem, with measured slack floor 2.782×10^{-6} at $(\nu, x) = (0, 300)$. It is *not* part of the proof, and the proof does not cite it.

7 Related work

The bound is Amos's [1]; two-sided sharp families and Turán-type consequences are due to Segura [2]; Hornik–Grün studied the Amos-type family systematically [3] and proved in-family optimality results for the generalized family [5]; Ruiz-Antolín–Segura [4] give the sharp bounds at both parameter ends and describe the Riccati-based qualitative methodology that classical proofs of such bounds employ — the mathematical route formalized here belongs to that tradition. The recurrence and derivative identities are [6, 10.29]. On the formalization side we know of no prior machine-checked proof of (1) or of any comparable Bessel-ratio bound; the formalized consequence calculus and the series interface are in the companion paper [7]; the prose companions of the consumer families are [8, 9, 10]. The development builds on Mathlib [11].

8 Reproducibility

Toolchain: Lean `leanprover/lean4:v4.29.0-rc6`; Mathlib pinned to commit

`07642720480157414db592fa85b626dafb71355b`

(lakefile and manifest agree); `tectonic` 0.15.0.

Recorded build facts: at the Lean-integration commit

`538562dcdc06a1d6c6ea2522ba57f991c5bdd847`

`lake build AmosClosure` completed successfully (**8166 jobs**); the axiom oracle printed, for all **28** registered statements (the eighteen of [7] plus the ten headline statements of this paper), exactly

`[propext, Classical.choice, Quot.sound]`;

the oracle log is committed under `scripts/` at the same commit. At the release revision the oracle registry was extended by the six supporting lemmas named in the map below, so that every "exact" row carries a direct rather than transitive oracle witness: **34** entries, every one printing the same axiom list, with the green rebuild (run 9, again 8166 jobs) and the extended log committed under `scripts/`. The full run history of this development — nine build transcripts including every failed attempt, each committed the day it was produced with its diagnosis in the commit message, and one re-registration of an attempt budget recorded in the audit charter (`docs/C3-CHARTER.md`, Amendment 1) — is committed under `scripts/` and in that charter file; per the repository's audit protocol it is archival material and plays no role in the mathematics above.

New-theorem-to-artifact map (all in module `AmosClosure`; the eighteen inherited statements are mapped in [7]):

Result	Lean name	File	Status
Eq. (2)	`besselRatio_hasDerivAt`	`Riccati.lean`	exact
Eq. (3)	`amosRHS_calibration`	`Riccati.lean`	exact
$Q(\mathrm{B}) = 0$	`riccatiQ_amosRHS`	`Riccati.lean`	exact
$Q > 0$ below root	`riccatiQ_pos_of_lt`	`Riccati.lean`	exact
series bounds	`besselTerm_le_geometric` etc.	`AmosBoundProof.lean`	exact
ψ derivative	`besselPsi_hasDerivAt`	`AmosBoundProof.lean`	exact
zone (§4.2)	`besselPsi_zone`	`AmosBoundProof.lean`	exact
touch lemma	`riccati_zero_of_touch`	`AmosBarrier.lean`	exact
Lemma 4.1	`besselPsi_gt`	`AmosBarrier.lean`	exact
Thm. 3.1	`amosBound_holds`	`AmosBarrier.lean`	exact
Thm. 3.2	`besselI_*_unconditional` ($\times 3$)	`AmosBarrier.lean`	exact
§6 grid	—	[7] artifacts	certified

From a clean clone at the release revision:

```
lake exe cache get
lake build AmosClosure
lake env lean AmosClosure/Oracle.lean
tectonic -X compile papers/c3-amos-proof/c3_amos_proof.tex
```

Canonical artifact hashes are recorded, tag-scoped, in `RELEASE-MANIFEST.md` in this paper's directory, committed together with this manuscript; compare against `git show <rev>:path`, never a worktree checkout. The release tag `c3-v1.0` marks the release commit of this manuscript, placed after the pre-release audit desks of the repository's protocol closed (audit registry: `docs/C3-CHARTER.md`); the release-tree audit is recorded in the manifest, post-tag by sequencing, with any fixes in post-tag commits; tags never move.

9 Conclusion

The Amos bound is now a theorem of the pinned formal development, proved from the power series with no analytic input, and every consumer statement of the companion development is unconditional. Limitations: integer orders and the in-core series definition only (Section 5); the classical bound and the Riccati methodology are prior art — the contribution is the formalization and its ψ-shaped barrier. Natural next steps: formal identification with a general special-functions interface, extension to real orders, and upstreaming the modified-Bessel API to Mathlib.

References

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