# A machine-checked proof of an Amos-type bound for modified Bessel ratios at real order

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**Abstract**

The Amos-type upper bound for the modified Bessel function ratio,

$$\rho_\nu(x) \;=\; \frac{I_{\nu+1}(x)}{I_\nu(x)} \;<\; \frac{x}{\nu+\frac{1}{2}+\sqrt{(\nu+\frac{1}{2})^2+x^2}} \;=\; \mathrm{B}_\nu(x),$$

is classical, and its derivation through the qualitative theory of the associated Riccati equation is an established technique. A companion paper formalized the bound at integer order over a factorial power series. This paper extends the formalization to *every real order* $\nu \geq 0$: the function $I_\nu$ is defined by its Γ-power series (real exponents via `rpow`), and the complete chain — convergence, positivity, the three-term recurrence, termwise differentiation with a dominated-derivative argument that must treat the leading term separately (its exponent $\nu - 1$ is negative for $\nu < 1$), the Riccati equation, a small-argument zone bound uniform in $\nu$, and a first-crossing barrier — is machine-checked in Lean 4 with axiom oracle [`propext`, `Classical.choice`, `Quot.sound`] and no analytic hypothesis beyond $\nu \geq 0$, $x > 0$. Two structural locks tie the result to the integer development: an *identification theorem* proves that at $\nu = n \in \mathbb{N}$ the Γ-series object coincides with the factorial-series object, so the integer-order theorem of the companion development is recovered as a corollary in three rewrites; and a genuinely non-integer instance at $\nu = \frac{1}{2}$ witnesses that the endpoint lives outside the natural-number embedding. The theorem is proved for the in-core Γ-series definition; no identification with an external special-functions library object is claimed.

## 1 Introduction

The ratio $\rho_\nu = I_{\nu+1}/I_\nu$ of modified Bessel functions of the first kind is bounded above by the algebraically calibrated expression $\mathrm{B}_\nu(x) = x/(\nu + \frac{1}{2} + \sqrt{(\nu+\frac{1}{2})^2 + x^2})$; bounds of this shape trace back to Amos [1], and sharp forms and systematic families were developed by Segura [2], Hornik and Grün [3, 5], and Ruiz-Antolín and Segura [4], the latter describing explicitly the Riccati-based qualitative methodology that classical proofs of such bounds employ. *No novelty is claimed for the bound or for the Riccati approach.* We refer to the inequality as an *Amos-type* bound: the attribution of this exact display to a specific equation of [1] is deliberately not asserted here pending collation with the primary source.

*This manuscript and parts of the formal development were produced with AI assistance under the repository's audit protocol. Every theorem below is machine-checked in Lean 4 against a pinned Mathlib; the independent numerical cross-check of Section 6 is a committed interval-arithmetic transcript; remaining numerics are labelled paper-level where they occur. Repository: https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME.

The contribution of this paper is the formalization at *real order*. Two companion papers of this series formalized, over the factorial power series at integer order, the consequence calculus of the bound [7] and then the bound itself [8]. The latter states its own scope limit: no extension to noninteger order is claimed. This paper removes that limit. The bound and its consumers below are stated for every real $\nu \geq 0$ and every $x > 0$, over the Γ-power series

$$\texttt{besselIReal}\ \nu\ x \;=\; \sum_{k\geq 0} \frac{(x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)},$$

with the real exponent carried by `Real.rpow` and the denominator by `Real.Gamma`. In addition, the *identification theorem* `besselIReal_natCast` proves `besselIReal` $n =$ `besselI` $n$ at every integer order — so the integer development of [7, 8] is not a parallel theory but a proved section of the real one, and the integer-order bound of [8] becomes a corollary (Theorem 3.3).

Statuses: **exact** means a Lean theorem in the pinned central repository with clean axiom oracle; **certified** means a committed interval-arithmetic transcript; **paper-level** means ordinary prose, always flagged; *verified* marks an ordinary high-precision computation used only as a cross-check of a certified result, never as evidence. The repository's largest namespace (`YangMills/`) reflects its original motivation in lattice gauge theory; no reduction to any continuum or gauge-theoretic problem is claimed or used.

## 2 Setting

All objects live in module `AmosClosure` of the pinned central repository. From the companion development [7, 8]: the factorial-series `besselI` $n\ x = \sum_{k\geq 0}(x/2)^{n+2k}/(k!\,(n+k)!)$ with its positivity, recurrence, derivative and ratio calculus, the predicate `AmosBound` $\nu\ x\ r \iff r < \mathrm{B}_\nu(x)$ — already real in its order parameter — and the calibration engine. New in this paper, the real-order object `besselIReal` $\nu\ x$ above, its ratio $\rho_\nu =$ `besselRatioReal` $\nu\ x$, and $\psi_\nu(x) = x\,(1/\rho_\nu - \rho_\nu)$ (`besselPsiReal`). Everything is stated at $x > 0$, the `rpow` domain of the development.

## 3 Main results

**Theorem 3.1** (the Amos-type bound at real order; `amosBoundReal_holds`)**.** *For every real $\nu \geq 0$ and every real $x > 0$,*

$$\frac{\texttt{besselIReal}\ (\nu{+}1)\ x}{\texttt{besselIReal}\ \nu\ x} \;<\; \frac{x}{\nu + \frac{1}{2} + \sqrt{(\nu+\frac{1}{2})^2 + x^2}}.$$

*No hypothesis remains beyond $\nu \geq 0$ and $x > 0$.*

**Theorem 3.2** (identification; `besselIReal_natCast`)**.** *For every $n \in \mathbb{N}$ and $x > 0$,*

$$\texttt{besselIReal}\ (n : \mathbb{R})\ x \;=\; \texttt{besselI}\ n\ x.$$

*Termwise, `rpow` collapses to the natural power and $\Gamma(n+k+1) = (n+k)!$. The integer-order development is a proved restriction of the real-order one.*

**Theorem 3.3** (consumers, recovery, and the half-order witness)**.** *For every real $\nu \geq 0$ and $x > 0$: $\rho_\nu(x) - \rho_{\nu+1}(x) < 1/x$ (`besselIReal_unit_step`); the derivative of $\log \circ$ `besselIReal` $\nu$ at $x$ is strictly below that of $\log \circ$ `besselIReal` $(\nu{+}1)$, in `deriv` form (`besselIReal_logDeriv_lt`). Moreover the integer-order theorem of [8] is recovered through Theorem 3.2 in three rewrites (`amosBound_holds_recovered`), and the instance $\nu = \frac{1}{2}$ is stated and proved as a named theorem (`amosBoundReal_half_order`) — the endpoint is not consumed only through the natural-number embedding.*

The $\varphi$-monotonicity step of [7] remains integer-indexed — its sequence is integer by construction and is already unconditional through [8]; no artificial real-order variant is introduced.

# 4 The proof

The formal proofs are the authority; this section presents the mathematics in full, at the level of the checklist a referee needs.

## 4.1 Series facts

Write $t_{\nu,k}(x) = (x/2)^{\nu+2k}/(k!\,\Gamma(\nu+k+1))$. The exact term-ratio identity

$$t_{\nu,k+1} \;=\; t_{\nu,k} \cdot \frac{(x/2)^2}{(k+1)(\nu+k+1)} \qquad (\texttt{besselRealTerm\_succ})$$

follows from $\Gamma(s+1) = s\,\Gamma(s)$ at $s = \nu+k+1 \geq 1$ and $(x/2)^{a+2} = (x/2)^a(x/2)^2$. Convergence: since $\Gamma(\nu+1) \leq \Gamma(\nu+k+1)$ for $\nu \geq 0$ (each functional-equation factor $\nu + j \geq 1$), the terms are dominated by an exponential series, and $I_\nu > 0$ follows termwise. The three-term recurrence $I_\nu - I_{\nu+2} = (2(\nu+1)/x)\,I_{\nu+1}$ is proved termwise (index shift plus the functional equation), exactly as at integer order. Theorem 3.2 is likewise termwise: $(x/2)^{(n:\mathbb{R})+2k} = (x/2)^{n+2k}$ by `Real.rpow_natCast` and $\Gamma(n+k+1) = (n+k)!$ by `Real.Gamma_nat_eq_factorial`.

## 4.2 The derivative layer

Termwise differentiation on the interval $(x/2,\, 3x/2) \subset (0,\infty)$ — the `rpow` domain forbids a ball through 0 — with dominated derivatives. The termwise derivative is

$$\frac{d}{dy}\, t_{\nu,k}(y) = \frac{\nu+2k}{2}\,\frac{(y/2)^{\nu+2k-1}}{k!\,\Gamma(\nu+k+1)}.$$

**Remark 4.1** (the negative-exponent trap)**.** *For $k = 0$ and $0 \leq \nu < 1$ the exponent $\nu - 1$ is negative: the factor $(y/2)^{\nu-1}$ is* decreasing *in $y$, and a majorant that evaluates it at the right endpoint is false. The formalized domination therefore treats $k = 0$ separately, with the two-endpoint bound $(y/2)^{\nu-1} \leq (x/4)^{\nu-1} + (3x/4)^{\nu-1}$ valid in both monotonicity regimes (at $\nu = 0$ the whole term carries the factor $\nu = 0$); and for $k \geq 1$ it uses the factoring $\nu + 2k - 1 = (\nu+1) + 2(k-1)$, whose exponents stay $\geq 1$, giving $(y/2)^{\nu+2k-1} \leq (3x/4)^{\nu+1}A^{k-1}$ with $A = (3x/4)^2$. The majorant tail is summable since $(\nu+2k+2)/(k+1)! \leq (\nu+2)/k!$, equivalent to $0 \leq \nu k$.*

Mathlib's dominated-differentiation theorem for series then yields, for every $\nu \geq 0$ and $x > 0$,

$$I'_\nu(x) \;=\; I_{\nu+1}(x) + \frac{\nu}{x}\, I_\nu(x) \qquad (\texttt{besselIReal\_hasDerivAt}),$$

with the `deriv`-form and logarithmic-derivative identities as corollaries, and the $\nu = n$ value test against the integer layer through Theorem 3.2 (`besselIReal_deriv_value_natCast`).

### 4.3 The Riccati equation and the nullcline

From the derivative identity and the recurrence, the quotient rule gives

$$\rho'_\nu(x) \;=\; Q_x(\rho_\nu(x)), \qquad Q_x(y) := 1 - \frac{2\nu+1}{x}\, y - y^2 \qquad (\texttt{besselRatioReal\_hasDerivAt}). \quad (1)$$

The bound $\mathrm{B}_\nu$ satisfies the exact calibration identity

$$\frac{1}{\mathrm{B}_\nu(x)} - \mathrm{B}_\nu(x) \;=\; \frac{2\nu+1}{x} \qquad (\texttt{amosRHS\_calibration\_real}), \qquad (2)$$

with $a = \nu + \frac{1}{2} > 0$ supplied by $\nu \geq 0$; multiplied by $\mathrm{B}_\nu > 0$ this is precisely $Q_x(\mathrm{B}_\nu(x)) = 0$ (`riccatiQReal_amosRHS`): the bound is the positive root of the Riccati quadratic. Moreover $Q_x(y) > 0$ for $0 \leq y < \mathrm{B}_\nu(x)$, by the factorization $Q_x(y) - Q_x(\mathrm{B}_\nu) = (\mathrm{B}_\nu - y)\big(\frac{2\nu+1}{x} + y + \mathrm{B}_\nu\big)$ (`riccatiQReal_pos_of_lt`).

### 4.4 The zone: $x \leq 1/4$, uniformly in $\nu$

The target $\rho_\nu < \mathrm{B}_\nu$ is equivalent, by (2) and the strict antitonicity of $t \mapsto 1/t - t$ on $(0, \infty)$, to $\psi_\nu > 2\nu + 1$. On the zone this is proved directly from the series, with four product-form facts: geometric domination $t_{\nu,k} \leq t_{\nu,0}\, q_\nu^k$ with $q_\nu = (x/2)^2/(\nu+1)$, by induction from the term-ratio identity and $\nu + 1 \leq (k+1)(\nu+k+1)$ (`besselRealTerm_le_geometric`); the product bound $(1 - q_\nu)\, I_\nu \leq t_{\nu,0}$ for $q_\nu < 1$ (`besselIReal_mul_le`); the strict two-term lower bound $t_{\nu,0} < I_\nu$ (`besselRealTerm_zero_lt_besselIReal`) — the sole source of strictness in the whole chain; and the exact identity $2(\nu+1)\, t_{\nu+1,0} = x\, t_{\nu,0}$, derived by multiplying the already-proved termwise recurrence at $k = 0$ by $x$ (`besselRealTerm_zero_succ`).

The $\nu$-uniformity is isolated in two named lemmas rather than automation: $(\nu+1)/(\nu+2) \leq 1$ (`real_zone_ratio_uniform`), whence $2(\nu+1)\, q_{\nu+1} = \frac{x^2}{2} \cdot \frac{\nu+1}{\nu+2} \leq \frac{x^2}{2}$, and with $x^2 \leq 1/16$ the coefficient inequality

$$2\nu + 1 + x^2 \;\leq\; 2(\nu+1)\,(1 - q_{\nu+1}) \qquad (\texttt{real\_zone\_coefficient\_bound}).$$

Chaining (upper bound at order $\nu+1$, identity, strict lower bound at order $\nu$) gives the decisive strict inequality

$$(2\nu + 1 + x^2)\; I_{\nu+1}(x) \;<\; x\, I_\nu(x) \qquad (0 < x \leq 1/4), \qquad (3)$$

strictness entering only at the last link. Two further lines convert (3) into the zone claim: since $2\nu+1+x^2 \geq 1$, (3) also gives the crude bound $I_{\nu+1} \leq x\, I_\nu$, hence $x\, I_{\nu+1}^2 \leq x^2\, I_{\nu+1} I_\nu$; multiplying (3) by $I_\nu > 0$ and inserting this,

$$(2\nu+1)\, I_{\nu+1} I_\nu + x\, I_{\nu+1}^2 \;\leq\; (2\nu+1)\, I_{\nu+1} I_\nu + x^2\, I_{\nu+1} I_\nu \;<\; x\, I_\nu^2,$$

which, by the identity $\psi_\nu \cdot (I_{\nu+1} I_\nu) = x\,(I_\nu^2 - I_{\nu+1}^2)$, is precisely $\psi_\nu(x) > 2\nu + 1$ on the zone (`besselPsiReal_zone`).

### 4.5 The barrier: every touch is an upward crossing

Suppose $\psi_\nu(c) = 2\nu+1$ at some $c > 0$. Then $1/\rho_\nu(c) - \rho_\nu(c) = (2\nu+1)/c$, and substituting into the Riccati quadratic annihilates it, so $\rho'_\nu(c) = 0$ (`riccati_zero_of_real_touch`); the product rule collapses $\psi'_\nu(c) = (2\nu+1)/c > 0$, the numerator positive directly from $\nu \geq 0$. Every touch of the critical level is an upward crossing.

**Lemma 4.2** (first crossing; the human form of `besselPsiReal_gt`)**.** *Suppose, for contradiction, $\psi_\nu(x_1) \leq 2\nu+1$ for some $x_1 > 0$. By the zone theorem $x_1 > 1/4$. The set $S = \{y \in [1/4, x_1] : \psi_\nu(y) \leq 2\nu+1\}$ is nonempty, bounded below, and closed (continuity of $\psi_\nu$ on the interval, from differentiability); let $c = \inf S \in S$. Since $\psi_\nu(1/4) > 2\nu+1$, in fact $c > 1/4$, and by minimality $\psi_\nu > 2\nu+1$ on $(1/4, c)$ (at $1/4$ itself this holds by the zone theorem); left-continuity forces the touch $\psi_\nu(c) = 2\nu+1$. But then $\psi'_\nu(c) = (2\nu+1)/c > 0$, so the difference quotient $\big(\psi_\nu(y) - \psi_\nu(c)\big)/(y-c)$ is eventually positive as $y \uparrow c$; since $y-c < 0$ there, $\psi_\nu(y) < \psi_\nu(c) = 2\nu+1$ just left of $c$ — contradicting $\psi_\nu > 2\nu+1$ on $(1/4, c)$. Hence $\psi_\nu(x) > 2\nu+1$ for all $x > 0$.*

Finally, $\psi_\nu > 2\nu+1$ converts into $\rho_\nu < \mathrm{B}_\nu$ algebraically: with $s = 1/\rho_\nu - \rho_\nu$ and $t = 1/\mathrm{B}_\nu - \mathrm{B}_\nu$,

$$(s-t) \cdot \rho_\nu \mathrm{B}_\nu = (\mathrm{B}_\nu - \rho_\nu)\,(1 + \rho_\nu \mathrm{B}_\nu),$$

and the left-hand side is positive — both its factors are: $s > t$ is the barrier conclusion $\psi_\nu > 2\nu+1$ divided by $x$, and $\rho_\nu \mathrm{B}_\nu > 0$ — so, since $1 + \rho_\nu \mathrm{B}_\nu > 0$, positivity passes to $\mathrm{B}_\nu - \rho_\nu$; this is the closing step of `amosBoundReal_holds`. That completes Theorem 3.1; Theorem 3.3 is a direct application, the recovery corollary consisting of the two identification rewrites, one cast normalization, and `exact`.

## 5 Scope

The theorem is proved for the in-core $\Gamma$-power-series definition of the modified Bessel function at real order $\nu \geq 0$, $x > 0$. No formal identification with an external special-functions library object is claimed (none exists at the pinned Mathlib to identify with). The integer-order objects of the companion development are identified with the real-order object *by theorem* (Theorem 3.2) — the corresponding scope caveat of [8] is thereby discharged. Results elsewhere in this programme that consume Amos-type bounds for the classical $I_\nu$ of the analysis literature — such as the surface-track $\varphi$-lemma conditional discussed in [7] — do not change verification status through this paper.

## 6 Independent numerical cross-check

A certified interval-arithmetic companion evaluates the margin — its protocol pre-registered in the development's charter (`docs/C4-CHARTER.md`) before the authoritative run — $\Delta(\nu, x) = \mathrm{B}_\nu(x) - \rho_\nu(x)$ at all 70 points of the pre-registered grid $\nu \in \{0, \frac{1}{10}, \frac{1}{2}, \frac{9}{10}, 1, \frac{3}{2}, 2, \pi, 10, 100\}$, $x \in \{\frac{1}{8}, \frac{1}{4}, 1, 2, 5, 50, 300\}$, with $\pi$ a certified ball. The ratio is computed by the certified backward recurrence $\rho_\alpha = x/(2(\alpha+1) + x\rho_{\alpha+1})$, anchored at order $\nu + N$ with the anchor condition $q \leq \frac{1}{4}$ itself certified in ball arithmetic and seeded by the two-sided product-form bounds of Section 4.4 — no exponentially large value of $I_\nu$ is ever formed, and the seed is independent of Theorem 3.1. Verdicts are by ball sign (`PASS` iff $\inf \Delta > 0$), never by midpoint. All 70 points `PASS` at 128 bits; an independent ordinary high-precision pass (labelled *verified*, not certified) agrees at all

70 points; and at $(\nu, x) = (0, 300)$ the certified margin $[2.78243072419 \times 10^{-6} \pm 2.43 \times 10^{-19}]$ reproduces, by this entirely different method, the slack floor measured by the integer-order companion of [7]. It is *not* part of the proof, and the proof does not cite it.

## 7 Related work

Amos [1] introduced bounds of this shape in the context of computing modified Bessel ratios; two-sided sharp families and Turán-type consequences are due to Segura [2]; Hornik–Grün studied the Amos-type family systematically [3] and proved in-family optimality results [5]; Ruiz-Antolín–Segura [4] give the sharp bounds at both parameter ends and describe the Riccati-based qualitative methodology to which the route formalized here belongs. The recurrence and derivative identities are [6, 10.29], and the Γ-series is the standard definition [6, 10.25.2]. On the formalization side we know of no prior machine-checked proof of an Amos-type ratio bound at real order; the integer-order formalization and its consequence calculus are in the companion papers [7, 8]; the prose companions of the consumer families are [9, 10, 11]. The development builds on Mathlib [12].

## 8 Reproducibility

Toolchain: Lean `leanprover/lean4:v4.29.0-rc6`; Mathlib pinned to commit

`07642720480157414db592fa85b626dafb71355b`

(lakefile and manifest agree); `tectonic` 0.15.0; `python-flint` 0.9.0 for the certified companion.

Recorded build facts: at the final mathematics commit

`56ff479b863d8585c41f6738b7bd18f80f2cbb16`

`lake build AmosClosure` completed successfully (**8171 jobs**); the axiom oracle printed, for all **74** registered statements (the 34 of the `c3-v1.0` release registry plus the 40 of this paper), exactly

`[propext, Classical.choice, Quot.sound]`;

the oracle log is committed under `scripts/` at the same commit. The full run history of this development — eight build transcripts including the vacuous first run and every failed attempt, each committed the day it was produced with its diagnosis in the commit message; the certified-grid transcripts (the first kept but superseded; the second the authoritative script-and-transcript pair); and six charter amendments recording every external audit round and every method deviation — is committed under `scripts/` and `docs/C4-CHARTER.md`; per the repository's audit protocol it is archival material and plays no role in the mathematics above.

Headline-theorem-to-artifact map (all in module `AmosClosure`; every row carries a direct oracle witness; the 34 inherited statements are mapped in [7, 8]):

| Result | Lean name | File | Status |
|---|---|---|---|
| Γ-series interface | `summable_besselRealTerm` etc. | `BesselRealInterface.lean` | exact |
| Thm. 3.2 | `besselIReal_natCast` | `BesselRealInterface.lean` | exact |
| derivative identity | `besselIReal_hasDerivAt` | `BesselRealDeriv.lean` | exact |
| Eq. (1) | `besselRatioReal_hasDerivAt` | `RiccatiReal.lean` | exact |
| Eq. (2) | `amosRHS_calibration_real` | `RiccatiReal.lean` | exact |
| zone (§4.4) | `besselPsiReal_zone` | `AmosBoundProofReal.lean` | exact |
| uniformity lemmas | `real_zone_ratio_uniform` etc. | `AmosBoundProofReal.lean` | exact |
| Lemma 4.2 | `besselPsiReal_gt` | `AmosBarrierReal.lean` | exact |
| Thm. 3.1 | `amosBoundReal_holds` | `AmosBarrierReal.lean` | exact |
| consumers | `besselIReal_unit_step`, `_logDeriv_lt` | `AmosBarrierReal.lean` | exact |
| recovery lock | `amosBound_holds_recovered` | `AmosBarrierReal.lean` | exact |
| half-order witness | `amosBoundReal_half_order` | `AmosBarrierReal.lean` | exact |
| §6 grid | `c4_amos_real_arb.py` + transcript v2 | `scripts/` | certified |

From a clean clone at the release revision:

```
lake exe cache get
lake build AmosClosure
lake env lean AmosClosure/Oracle.lean
python scripts/c4_amos_real_arb.py
tectonic -X compile papers/c4-amos-real/c4_amos_real.tex
```

At the release revision, canonical artifact hashes will be recorded — tag-scoped, in `RELEASE-MANIFEST.md` in this paper's directory — committed together with the release revision of this manuscript; compare against `git show <rev>:path`, never a worktree checkout. The annotated tag `c4-v1.0` will mark that release commit after the pre-release audits close, with the audit registry in `docs/C4-CHARTER.md`; the release-tree audit is recorded in the manifest, post-tag by sequencing, with any fixes in post-tag commits; tags never move.

## 9 Conclusion

An Amos-type bound for modified Bessel ratios is now a theorem of the pinned formal development at every real order $\nu \geq 0$, proved from the in-core Γ-power series using only machine-checked analytic library results, with no additional analytic hypothesis beyond $\nu \geq 0$ and $x > 0$, and with the integer-order development of the companion papers identified as a proved restriction and recovered as a corollary. Limitations: the in-core series definition only, and $\nu \geq 0$ (Section 5); the classical bound and the Riccati methodology are prior art — the contribution is the first machine-checked formalization of this complete chain in the repository, and we are not aware of a prior machine-checked formalization of the result. Natural next steps: the sharp lower bound (whose small-argument analysis is second-order, as any lower bound matches the first Taylor coefficient of $\rho_\nu$), formal identification with a general special-functions interface if one lands in Mathlib, and upstreaming the modified-Bessel API.

## References


[1] D. E. Amos, *Computation of modified Bessel functions and their ratios*, Math. Comp. **28** (1974), 239–251.

[2] J. Segura, *Bounds for ratios of modified Bessel functions and associated Turán-type inequalities*, J. Math. Anal. Appl. **374** (2011), 516–528. doi:10.1016/j.jmaa.2010.09.030.

[3] K. Hornik and B. Grün, *Amos-type bounds for modified Bessel function ratios*, J. Math. Anal. Appl. **408** (2013), 91–101. doi:10.1016/j.jmaa.2013.05.070.

[4] D. Ruiz-Antolín and J. Segura, *A new type of sharp bounds for ratios of modified Bessel functions*, J. Math. Anal. Appl. **443** (2016), 1232–1246. arXiv:1606.02008.

[5] K. Hornik and B. Grün, *Generalized Amos-type bounds for modified Bessel function ratios*, Math. Inequal. Appl. **27** (2024), 775–787. doi:10.7153/mia-2024-27-55.

[6] NIST Digital Library of Mathematical Functions, F. W. J. Olver et al., eds., `https://dlmf.nist.gov/`.

[7] L. Eriksson, *One Amos bound, three consumer sites: machine-checked Bessel-ratio calculus for lattice gauge expansions*, companion paper, same repository, release `c2-v1.2.1`; ai.viXra.org identifier pending at the time of this release, July 2026.

[8] L. Eriksson, *A machine-checked proof of the Amos bound for modified Bessel function ratios*, companion paper, same repository, release `c3-v1.0.1`; ai.viXra.org identifier pending at the time of this release, July 2026.

[9] L. Eriksson, *Recurrence–Amos proof of the unit-step order-monotonicity of* $(\log I_\nu)'$, *with a Feynman–Hellmann application to two-dimensional lattice gauge theory*, preprint, ai.viXra.org:2607.0020, July 2026.

[10] L. Eriksson, *Ratio monotonicity for a killed von Mises bridge: exact bridge structure, certified negative results, a single-Bessel reduction, and a two-scale closure map for a surface expansion in two-dimensional lattice gauge theory*, preprint, ai.viXra.org:2607.0023, July 2026.

[11] L. Eriksson, *A weighted Turán-type monotonicity lemma for modified Bessel functions via the calibrated Amos bound, with an application to the ordering of a surface expansion in two-dimensional lattice gauge theory*, preprint, ai.viXra.org:2607.0017, July 2026.

[12] The mathlib Community, *The Lean mathematical library*, in: Proc. CPP 2020, ACM, 2020, pp. 367–381. doi:10.1145/3372885.3373824.