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Ratio Monotonicity for a Killed von Mises Bridge: Exact Bridge Structure, Certified Negative Results, a Single-Bessel Reduction, and a Two-Scale Closure Map for a Surface Expansion in Two-Dimensional Lattice Gauge Theory

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Version 2 - supersession and correction note

Purpose of this replacement

This replacement preserves public version 1 and records the later terminal-claim paper without overstating its audit status. Version 1 proves positivity of F_B, the ratio sign for 0 < beta <= 3, exact bridge and single-Bessel reductions, certified negative results, asymptotic structure and a two-scale closure map; pages 11-13 leave the global sign as a quantified conjecture.

ai.viXra:2607.0089v1 claims the global ratio-monotonicity theorem using exact identities and outward-rounded interval certificates. It supersedes this record for that terminal claim, but the independent replay of every exact-to-Arb handoff and interval regime remains pending in the present audit.

Citation rule

Cite this record for the exact reductions, certified negative results and beta <= 3 theorem. Cite ai.viXra:2607.0089v1 for the claimed global closure, while retaining the explicit independent-replay-pending qualification until that audit is complete.

Status boundary

SUPERSEDED FOR THE TERMINAL CLAIM does not convert a pending hybrid certificate into an independently verified theorem. The successor is the later claim source; its global certificate remains REVIEW-PENDING here.

The exact claim map and provenance record continue on notice page 2. The historical public manuscript then follows unchanged.

CORRECTION AND CLAIM MAP

Results retained from the historical paper

- The two proofs of F_B > 0 and the certified E' < 0 result for 0 < beta <= 3.

- The killed von Mises bridge identity, single-Bessel reductions and exact saddle-deficit structure.

- The certified negative coupling results, asymptotic coefficients and two-scale closure map as labelled in version 1.

Claims superseded or narrowed

- Pages 11-13 are no longer the terminal location of the global ratio-monotonicity programme.

- The quantified conjecture is superseded as a claim target by the later paper's stated global theorem.

- No independent-replay PASS is inferred from the successor's printed certificates or green transcripts.

What the successor adds

- A full proof partition for the claimed global sign across analytic and interval-certified regimes.

- Outward-rounded interval certificates and replay/production transcripts for the remaining parameter ranges.

- A terminal global ratio-monotonicity claim that was conjectural in version 1 of this record.

Evidence classification

Version 1 combines exact identities, certified finite regimes, numerical verification and an explicit conjectural endpoint. The successor reports a hybrid analytic/machine-certified theorem, but independent replay of its interval coverage and exact-to-numeric interfaces is still required.

Provenance

Notice pages 1-2 are new. The following 13 pages reproduce public version 1 with identical rendered content. Frozen copies of version 1 and the 33-page successor are included in the package.

Ratio Monotonicity for a Killed von Mises Bridge:

exact bridge structure, certified negative results, a single-Bessel reduction, and a two-scale closure map for a surface expansion in two-dimensional lattice gauge theory

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Abstract

For $\beta > 0$ let $I_m = I_m(\beta)$ denote modified Bessel functions of the first kind, and set $a_m = I_m^2\,[(m-1)I_{m-1}^2 + (m+1)I_{m+1}^2]$, $b_m = m\,I_m^4$, $F_A(t) = \sum_{m\geq 1} a_m \sin(mt)$, $F_B(t) = \sum_{m\geq 1} b_m \sin(mt)$, and $E = F_A/(2F_B)$. The global ratio-monotonicity problem for the surface expansion of a two-dimensional $SU(2)$ lattice gauge observable is: (i) $F_B > 0$ on $(0,\pi)$, and (ii) $E' < 0$ on $(0,\pi)$, for every $\beta > 0$. We prove (i) twice, and prove (ii) for $0 < \beta \leq 3$ via an exact pair-identity skeleton plus an archived interval certificate executed by two independent interval-arithmetic implementations. The structural core is exact: E is, as an algebraic identity, the mean of $\cos\psi$ under the midpoint law of a four-step killed von Mises bridge; the generating kernels reduce, via the Neumann addition theorem, to two-dimensional integrals of a *single* Bessel function whose saddle deficit is an exact sum of two squares; and exact saddle cancellations yield the coefficients of the (verified) closed second-order law $E = \cos(t/2)(1 - c(t)/\beta) + O(\beta^{-2})$, $c(t) = (4\cos^2(t/4) - 1)/(2\cos(t/4)\cos(t/2))$. Three certified negative results (interval arithmetic, two implementations, nested enclosures) kill every monotone full-path coupling, with an exact mechanism at threshold $\beta|\cos t| = 3/2$. For the remaining range we give a two-scale closure map with a verified candidate threshold $\beta_0 = 15$ and the complete π-endpoint package for the cubic coefficient c_3. Every claim is labelled exact / certified / verified; the machine-checked lemmas are Lean 4/Mathlib, machine-checked modulo classical Bessel inputs carried as named hypotheses.

1 Introduction

This paper consolidates the resolution status of a two-part positivity problem that arises as the last analytic gap in a surface (character) expansion for two-dimensional $SU(2)$ lattice gauge theory [5, 6, 7]. The problem is stated entirely in terms of modified Bessel functions and trigonometric series, and we treat it as such; no claim of physical consequence beyond the stated expansion is made, and importance is not inherited by thematic proximity.

Throughout, $\beta > 0$, $I_m = I_m(\beta)$, and

$$a_m = I_m^2\big[(m-1)I_{m-1}^2 + (m+1)I_{m+1}^2\big], \qquad b_m = m\,I_m^4, \qquad m \geq 1. \tag{1}$$

Both sequences decay super-exponentially, so

$$F_A(t) = \sum_{m\geq 1} a_m \sin(mt), \qquad F_B(t) = \sum_{m\geq 1} b_m \sin(mt), \qquad E(t) = \frac{F_A(t)}{2F_B(t)}, \tag{2}$$

are real-analytic on $\mathbb{R}$, and the Wronskian $W = 2(F_A' F_B - F_A F_B')$ satisfies $W = 4F_B^2\,(E)'$ wherever $F_B \neq 0$ [7].

Conjecture 1 (global ratio monotonicity)**.** *For every* $\beta > 0$*:* (i) $F_B > 0$ *on* $(0, \pi)$*;* (ii) $E' < 0$ *on* $(0, \pi)$*.*

Part (i) is proved below (twice). Part (ii) is proved for $0 < \beta \le 3$ (Theorem 7; exact skeleton plus the archived certificate `certify_thmB.py` and its Arb twin), is supported by an exact structural reduction and a verified truth map for $\beta \ge 15$, and is otherwise open; Section 8 states the precise two-scale closure map. The conjecture was born inside this programme (it is the global form of an ε-window positivity required by the surface expansion); no external pedigree is claimed.

Convention 2 (tricotomy). Every assertion below carries one of three labels. *Exact*: a proved identity or inequality. *Certified*: a numerical statement enclosed in interval arithmetic with outward rounding, nested re-computation at higher precision, and (for the load-bearing negatives) a second independent implementation in Arb. *Verified*: high-precision numerics with stated grids and precision policies, not a proof. No verified statement is used as a hypothesis of an exact one.

2 The spectral framework

Convention 3 (normalizations). On $(0, \pi)$ define the kernel, entry law, and smoothing operator

$$q_1(u, v) = 2\, e^{\beta \cos u \cos v} \sinh(\beta \sin u \sin v) \;=\; 4 \sum_{m \ge 1} I_m \sin(mu) \sin(mv), \tag{3}$$

$$s_1(u) = \tfrac{\beta}{2} \sin u\, e^{\beta \cos u} \;=\; \sum_{m \ge 1} m\, I_m \sin(mu), \tag{4}$$

$$(Qf)(v) = \frac{1}{2\pi} \int_0^\pi q_1(v, u) f(u)\, du, \tag{5}$$

so that $Q \sin(m\cdot) = I_m \sin(m\cdot)$. Both closed forms are classical consequences of the generating function $e^{\beta \cos\theta} = I_0 + 2 \sum_{m \ge 1} I_m \cos(m\theta)$; (4) follows by differentiation. All ten verification artifacts in the repository use exactly these normalizations.

Since $q_1 > 0$ on $(0, \pi)^2$ and $s_1 > 0$ on $(0, \pi)$, the walk with step kernel q_1 killed on $\{0, \pi\}$ is well defined; $H(\psi) = \beta \sin(\psi/2) I_1(2\beta \cos(\psi/2))$ and $K_2(\varphi) = I_0(2\beta \cos(\varphi/2))$ appear below. Note $F_B = Q^3 s_1$ and, by the sine orthogonality computation of Lemma 10, F_A is the companion cosine-weighted integral.

3 Theorem A: positivity of F_B

Theorem 4 (exact)**.** $F_B(t) > 0$ *for all* $t \in (0, \pi)$ *and all* $\beta > 0$*.*

First proof (kernel positivity). $F_B = Q^3 s_1$ by (5) and $b_m = m I_m^4$. Each application of Q integrates a positive function against the strictly positive kernel (3); $s_1 > 0$ on $(0, \pi)$. Hence $F_B > 0$ pointwise. □

Second proof (symmetric-decreasing convolution). With $K_2(\varphi) = I_0(2\beta \cos(\varphi/2))$ and $P_4 = K_2 * K_2$ (circle convolution; Section 6), $F_B = -\frac{1}{2} P_4'$. The function K_2 is even and *strictly decreasing in* $|\varphi|$ *on* $[0, \pi]$ (I_0 strictly increasing, $\cos(\varphi/2)$ strictly decreasing). By the layer-cake representation, $K_2 = \int_0^\infty \mathbf{1}_{A_\lambda}\, d\lambda$ with A_λ centered arcs whose half-lengths $a(\lambda)$ fill $(0, \pi)$ continuously (strict decrease). For centered arcs, $(\mathbf{1}_A * \mathbf{1}_B)(t) = |A \cap (t+B)|$ is even and nonincreasing in $|t|$ on $[0,\pi]$, and *strictly* decreasing on $||a-b|, \min(a+b, 2\pi-a-b)|$. Hence $P_4(t) = \iint |A_\lambda \cap (t+A_\mu)|\, d\lambda\, d\mu$ is even, nonincreasing on $[0,\pi]$, and strictly decreasing on $(0,\pi)$ because for every $t \in (0,\pi)$ a positive measure of pairs (λ,μ) has $|a(\lambda)-a(\mu)| < t < \min(a+b, 2\pi-a-b)$ (the half-lengths fill $(0,\pi)$). At the derivative level: each overlap $t \mapsto |A_\lambda \cap (t+A_\mu)|$ is Lipschitz with a.e. derivative -1 in the strict regime and 0 otherwise (on $[0,\pi]$); since P_4 is analytic, dominated convergence gives $P_4'(t) = \iint \partial_t |A_\lambda \cap (t+A_\mu)|\, d\lambda\, d\mu \le -\mu\{(\lambda,\mu) : \text{strict at } t\} < 0$ pointwise on $(0,\pi)$. Thus $F_B = -\frac{1}{2}P_4' > 0$ on $(0,\pi)$. □

4 Theorem B: ratio monotonicity for $0 < \beta \le 3$

Lemma 5 (exact; Wronskian identity)**.** *For real arrays with* $\sum m(|a_m| + |b_m|) < \infty$,

$$W := 2(F_A' F_B - F_A F_B') = \sum_{1 \le m < n} c_{mn}\, T_{mn}(t), \qquad c_{mn} = a_m b_n - a_n b_m, \tag{6}$$

where $T_{mn}(t) = (m-n)\sin((m+n)t) + (m+n)\sin((n-m)t)$.

Proof. Product-to-sum: $T_{mn} = 2[m\cos(mt)\sin(nt) - n\sin(mt)\cos(nt)] =: 2G(m,n)$, antisymmetric with $G(n,n) = 0$; since c_{mn} is antisymmetric, $\sum_{m<n} c_{mn} 2G = 2\sum_{m,n} a_m b_n G(m,n) = 2[F_A' F_B - F_A F_B']$ (first recorded in [7]). □

The coefficient input is the weighted Turán-type lemma (Appendix A, self-contained): $\varphi_m = a_m/b_m$ is strictly increasing in $m \ge 1$ for every $\beta > 0$, hence *every minor is negative*: $c_{mn} < 0$ for $m < n$ (exact).

Lemma 6 (exact; the pair identities and bounds)**.** *For* $t \in (0,\pi)$*, with* $p = n-m$*,* $q = n+m$*:*

(a) $T_{12} = 4\sin^3 t$*;* $T_{13} = 16\sin^3 t\,\cos t$*;* $T_{24} = 8\sin^3(2t)$*, so* $|T_{13}| \le 16\sin^3 t$ *and* $|T_{24}| \le 64\sin^3 t$*.*

(b) $T_{m,m+1} \ge 0$ *for every* $m \ge 1$*.*

(c) $|T_{mn}| \le \dfrac{\pi^3}{48}\, pq(q^2+p^2)\, \sin^3 t$ *for all* $m < n$*.*

Proof. (a) $3\sin u - \sin 3u = 4\sin^3 u$ gives T_{12} (at $u = t$) and T_{24} (at $u = 2t$); $T_{13} = 4\sin 2t - 2\sin 4t = 4\sin 2t(1-\cos 2t) = 16\sin^3 t\cos t$. (b) $T_{m,m+1} = (2m+1)\sin t - \sin((2m+1)t) \ge 0$ since $|\sin(qt)| \le q\sin t$ on $(0,\pi)$ (induction on q). (c) T_{mn} and its first two derivatives vanish at $t = 0$ and $t = \pi$ (the parities of $m \pm n$ agree), and $|T_{mn}'''| \le pq^3 + qp^3 = pq(q^2+p^2)$; hence $|T_{mn}(t)| \le \frac{pq(q^2+p^2)}{6}\min(t, \pi-t)^3$, and $\sin t \ge \frac{2}{\pi}\min(t,\pi-t)$ gives the claim. □

Theorem 7 (exact modulo the certified computation `certify_thmB.py`)**.** *For* $0 < \beta \le 3$*:* $W < 0$ *on* $(0,\pi)$*, hence* $E' < 0$ *on* $(0,\pi)$*.*

Proof. Drop the adjacent pairs $m \ge 2$ (each contributes $c_{m,m+1}T_{m,m+1} \le 0$ by Lemma 6(b) and $c < 0$); keep T_{12}, T_{13}, T_{24} exactly and majorize every other pair by Lemma 6(c). Pointwise in t,

$$\frac{W(t)}{\sin^3 t} \;\le\; \mathrm{Crit}(\beta) := -4|c_{12}| + 16|c_{13}| + 64|c_{24}| + \frac{\pi^3}{48} \sum_{\substack{p \ge 2 \\ (m,n) \ne (1,3),(2,4)}} pq(q^2+p^2)\,|c_{mn}|.$$

The certificate `certify_thmB.py` (Section 9) encloses Crit in interval arithmetic over adaptive β-boxes covering $[1/20, 3]$ (I_m increasing in β gives rigorous boxes; cutoff $M = 60$ with the tail itself computed in intervals and closed by a derived, per-box-asserted geometric ratio) and certifies Crit < 0 on all of $[1/20, 3]$; the script then re-certifies *every* box at precision +70 before reporting success (the box count is invocation- and implementation-dependent; the independent Arb implementation `certify_thmB_arb.py` certifies the same coverage with 86 boxes and a full stability pass, transcript archived). For $0 < \beta \le 1/20$, Lemma 8 applies. □

Lemma 8 (exact; small β)**.** *For* $0 < \beta \le 1/20$*:* Crit$(\beta) < 0$.

Proof. Write $\kappa = \beta/2 \le 1/40$, $L_m = \kappa^m/m!$, $U_m = e^* L_m$ with $e^* = e^{\beta^2/4} \le e^{1/1600} < 1.001$; then $L_m \le I_m \le U_m$ (positive series). *Leading minor:* $a_2 b_1 \ge (I_1^2 I_2^2)(I_1^4) \ge L_1^6 L_2^2 = \kappa^{10}/4$ and $a_1 b_2 = (2I_1^2 I_2^2)(2I_2^4) \le 4U_1^2 U_2^6 = (e^*)^8 \kappa^{14}/16$, so

$$|c_{12}| \;\ge\; \frac{\kappa^{10}}{4}\Big(1 - \frac{(e^*)^8\kappa^4}{4}\Big) \;\ge\; \frac{\kappa^{10}}{4}\,(1 - 10^{-6}).$$

The three visible positive terms: $a_3 \le I_3^2 \cdot (2I_2^2 + 4I_4^2) \le (e^*)^4 \kappa^{10}/24 \cdot (1 + 10^{-4})$ and $b_1 \le (e^*)^4 \kappa^4$ give $16|c_{13}| \le 16(a_3 b_1 + a_1 b_3) \le 0.70\,\kappa^{14}$; $64|c_{24}| \le 64(a_4 b_2 + a_2 b_4) \le 10^{-3}\kappa^{14}$ (both products have κ-degree ≥ 22); the largest remaining weighted pair is $(1, 4)$: $\frac{\pi^3}{48} \cdot 510 \cdot (a_4 b_1 + a_1 b_4) \le 0.2\,\kappa^{18} \le 10^{-6}\kappa^{14}$. *All other pairs* (those with $s := m + n \ge 6$, $p \ge 2$): using $a_n b_m + a_m b_n \le 2mn(e^*)^8 \kappa^{4s-2}[((m-1)!m!)^{-2} n!^{-4} + ((n-1)!n!)^{-2} m!^{-4}] \le s^2 (e^*)^8 \kappa^{4s-2}$ (the factorial bracket is ≤ 2 and $2mn \le s^2/2$), $pq(q^2 + p^2) \le 2s^4$, and at most s pairs per level,

$$\frac{\pi^3}{48}\sum_{s\ge 6}(\text{level } s) \le \frac{\pi^3}{48}(e^*)^8 \sum_{s\ge 6} 2s^7 \kappa^{4s-2} \le 0.66\,\kappa^{22} \cdot 6^7 \sum_{j\ge 0}\big[(7/6)^7\kappa^4\big]^j \le 3.7 \times 10^5\,\kappa^{22} \le 10^{-7}\kappa^{14},$$

where the level ratio $s \mapsto s + 1$ is at most $(7/6)^7\kappa^4 < 10^{-5} < \frac{1}{2}$ (so the geometric closure is valid), and $\kappa^8 \le 40^{-8}$ absorbs the constant. Collecting:

$$\text{Crit} \le -\kappa^{10}(1 - 10^{-6}) + 0.71\,\kappa^{14} = -\kappa^{10}\big(1 - 10^{-6} - 0.71\,\kappa^4\big) < 0,$$

since $\kappa^4 \le 40^{-4} < 4 \cdot 10^{-7}$. (All numbered constants above are crude upper counts; each step loses factors, none gains.) □

Remark 9 (verified; the sharp family and saturation)*.* The sharper per-pair bound $|T_{mn}| \le \frac{pq(q^2-p^2)}{6}\sin^3 t$ and the adjacent floor $T_{m,m+1} \ge 2m\sin^3 t$ (note the sign: the floor is a *lower* bound on a positive quantity) are verified far beyond the range used ($q \le 151$; 13 pairs to $(10, 30)$, max ratio 1.0) and would extend Theorem 7 to $\beta \approx 3.5$; their per-pair certification is a scheduled upgrade. Beyond that the decomposition is *saturated*: with the exact far block (cancellation included) the criterion fails at $\beta = 6$ by $+2.2\times 10^{14}$ while the true $W/\sin^3 t$ remains negative (-6.9×10^{12}) — a verified adversarial finding about the method, and the reason the large-β closure uses different machinery (Sections 6–8).

5 The bridge structure of E (exact)

Let $k = q_1 \circ q_1$ denote the two-step kernel, $k(\psi, t) = 4\sum_{m\ge 1} I_m^2 \sin(m\psi)\sin(mt)$, and define

$$\omega_t(\,d\psi) \;\propto\; (Qs_1)(\psi)\,k(\psi, t)\,d\psi \qquad \text{on } (0, \pi), \tag{7}$$

the law of the time-2 position (midpoint) of the four-step bridge entering from 0 with law s_1 and conditioned to end at t.

Lemma 10 (exact)**.** $\int_0^{\pi} (Qs_1)\, k(\cdot, t)\, d\psi \;=\; 2\pi F_B(t)$ *and* $\int_0^{\pi} \cos\psi\, (Qs_1)\, k(\cdot, t)\, d\psi \;=\; \pi F_A(t)$*. Hence, as an identity,*

$$E(t) \;=\; \mathbb{E}_{\omega_t}[\cos\psi] \;=\; \frac{F_A(t)}{2F_B(t)}.$$

Proof. $(Qs_1)(\psi) = \sum m I_m^2 \sin(m\psi)$. By sine orthogonality the first integral picks the diagonal, giving $2\pi \sum m I_m^4 \sin(mt)$. In the second, $\cos\psi\, \sin(m\psi)\sin(n\psi)$ integrates to $\frac{\pi}{4}(\delta_{n,m+1}+\delta_{n,m-1})$, which produces exactly the weights $(n-1)I_{n-1}^2 + (n+1)I_{n+1}^2$ of a_n. □

Lemma 11 (exact; entry law)**.** $\partial_t k(0^+, \psi) = 2H(\psi)$*, where* $H(\psi) = \beta \sin(\psi/2)\, I_1(2\beta\cos(\psi/2))$*.*

Proof. Differentiate in ψ the squared Graf identity $I_0^2 + 2\sum_{m\ge 1} I_m^2 \cos(m\psi) = I_0(2\beta\cos(\psi/2))$ [4, §10.23(ii)] and compare with $\partial_t k(0^+, \psi) = 4\sum m I_m^2 \sin(m\psi)$. □

Proposition 12 (certified negative; no monotone full-path coupling)**.** *At $\beta = 20$ the time-3 marginal $\rho_t(u) \propto (Q^2 s_1)(u)\, q_1(u, t)$ is not stochastically monotone in t: with $a = \frac{4802}{10^4}$, $t_1 = \frac{29621}{10^4}$, $t_2 = \frac{30070}{10^4}$, the tail masses satisfy $T(t_2, a) < T(t_1, a)$, with certified enclosure*

$$T(t_2, a) - T(t_1, a) \in [-2.5299150980081786690028107, \\ -2.5299150980081786690028074] \times 10^{-10}.$$

Consequently no monotone coupling of full bridge paths exists at $\beta = 20$. The certificate uses exact termwise tails ($J(m, n, a) = \int_a^{\pi} \sin mu \sin nu\, du$ in closed form: no quadrature error), positive-series Bessel enclosures with geometric tails, explicit truncation bounds, nested passes $(M, \mathrm{prec}) = (120, 350) \subset (140, 500)$*, and two independent interval implementations (`mpmath.iv`; Arb via `python-flint`).*

Proposition 13 (certified negatives; the weight–kernel matrix)**.** *At $\beta = 20$, for every entry weight $Q^{k-1}s_1$ ($k = 1, 2, 3, 4$) the marginal one q_1-step from the moving endpoint fails stochastic monotonicity, with certified witnesses sharing $(t_1, t_2) = (3, 61/20)$: $k = 1$ (two-step bridge midpoint, the minimal counterexample), $a = 9/10$, margin $\approx -1.34 \times 10^{-2}$; $k = 2$, $a = 3/4$, $\approx -1.80 \times 10^{-6}$; $k = 4$, $a = 3/10$, $\approx -4.75 \times 10^{-13}$ ($k = 3$ is Proposition 12). Two smoothing steps repair all tested weights (verified); one never does. The statement is relative to the natural weight family: k_2 possesses its own total-positivity corner ($\beta \gtrsim 8$) reachable by two-atom adversarial weights.*

Lemma 14 (exact; the 3/2 mechanism)**.** $\lim_{u\downarrow 0} \dfrac{\partial_u\partial_t \log q_1(u, t)}{\beta \sin t\; u} \;=\; 1 + \dfrac{2\beta}{3}\cos t$*. For $t > \pi/2$ the corner sign change of $\partial_u\partial_t \log q_1$ occurs exactly when $\beta|\cos t| > 3/2$ — the same threshold as the TP_2 failure of q_1.*

Proof. $\partial_t \log q_1 = -\beta \sin t \cos u + \beta \cos t \sin u \coth(\beta \sin t \sin u)$ (direct differentiation of (3)); expand $\coth z = z^{-1} + z/3 + O(z^3)$ and differentiate in u; the stated limit is a one-line computation (symbolic residual zero). □

Remark 15 (verified; the mechanism reading)*.* The identification of this local defect as *the* engine of Propositions 12–13 — sine-type entry weights, vanishing linearly at 0 with mass on both sides of the non-monotone region, flip the covariance $\mathrm{Cov}_{\rho_t}(\mathbf{1}_{u\ge a}, \partial_t \log q_1)$, while the flat weight does not — is supported by the full weight–kernel matrix and the β-scan of the violations (onset at $\beta \gtrsim 12$, consistent with the threshold), but is a verified reading, not a theorem.

6 The single-Bessel reduction and the master formula

Lemma 16 (exact; Neumann product formulas)**.** *For $x, y \geq 0$, with $w = \sqrt{x^2 + y^2 + 2xy\cos\alpha}$,*

$$I_0(x)I_0(y) = \frac{1}{2\pi}\int_0^{2\pi} I_0(w)\,d\alpha, \qquad I_0(x)I_1(y) = \frac{1}{2\pi}\int_0^{2\pi} I_1(w)\,\frac{y + x\cos\alpha}{w}\,d\alpha.$$

Proof. The first is the integrated Graf/Neumann addition theorem [4, §10.23(ii)]; the second is its ∂_y-derivative. □

Remark 17 (signs). Both formulas extend to all real x, y, as used below (on $[-\pi,\pi]^2$ one has $c_1 = \cos(\varphi/2) \geq 0$ but c_2 may be negative): $w(x,-y,\alpha) = w(x,y,\pi-\alpha)$, and the substitution $\alpha \mapsto \pi - \alpha$ over a full period maps the first identity to itself (both sides even in y) and the second to its negative on both sides simultaneously (I_1 odd; $(-y + x\cos(\pi-\alpha)) = -(y + x\cos\alpha)$).

With $K_2(\varphi) = I_0(2\beta\cos(\varphi/2)) = \sum_{m\in\mathbb{Z}} I_m^2 e^{im\varphi}$ and $P_4 = K_2 * K_2$ (circle convolution), $P_4 = \sum_m I_m^4 e^{imt}$ and $F_B = -\frac{1}{2}P_4'$. Applying Lemma 16 to both convolution factors:

Lemma 18 (exact; master representation)**.** *Let $s = \varphi - t/2$, $c_1 = \cos(\varphi/2)$, $c_2 = \cos((t-\varphi)/2)$, $P = \sin^2(s/2)$, $Q = \sin^2(\alpha/2)$, and*

$$R^2 = c_1^2 + c_2^2 + 2c_1c_2\cos\alpha = 4\bigl[\cos^2(t/4)\,(1 - P - Q) + PQ\bigr]. \tag{8}$$

Then

$$F_B(t) = \frac{\beta\sin(t/2)}{8\pi^2}\iint_{[-\pi,\pi]^2} \frac{I_1(2\beta R)}{R}\bigl[\cos^2(s/2) - \sin^2(\alpha/2)\bigr]\,ds\,d\alpha, \tag{9}$$

and, under the common positive kernel $K = I_1(2\beta R)/R$,

$$E(t) = \frac{\bigl\langle \cos(t/2)\cos 2s + \cos\alpha\,[\cos(t/2)\cos s - \sin^2 s]\bigr\rangle}{\bigl\langle \cos s + \cos\alpha \bigr\rangle}, \tag{10}$$

where $\langle\cdot\rangle$ denotes the $K\,ds\,d\alpha$-integral over $[-\pi,\pi]^2$. The prefactor collapse behind (9) is the sum-to-product symmetrization $\frac{1}{2}[\sin(\frac{t-\varphi}{2})(c_2 + c_1\cos\alpha) + (\varphi \leftrightarrow t-\varphi)] = \frac{1}{2}\sin(t/2)(\cos s + \cos\alpha)$, and $\cos s + \cos\alpha = 2[\cos^2(s/2) - \sin^2(\alpha/2)]$.

Proof. (i) The squared Graf identity gives $K_2(\varphi) = \sum_m I_m^2 e^{im\varphi}$; convolution multiplies Fourier coefficients, so $P_4 = \sum_m I_m^4 e^{im\varphi}$ and, termwise (super-exponential decay), $F_B = -\frac{1}{2}P_4'$. (ii) Insert Lemma 16 (first formula) with $x = 2\beta c_1$, $y = 2\beta c_2$ into $P_4(t) = \frac{1}{2\pi}\int K_2(\varphi)K_2(t-\varphi)\,d\varphi$: a two-dimensional integral of $I_0(2\beta R)$ with R as in (8); the (P,Q) form of R^2 is product-to-sum. (iii) Differentiate under the integral (smooth integrand, compact domain): $\partial_t I_0(2\beta R) = I_1(2\beta R)\cdot 2\beta\,\partial_t R$ and $2R\,\partial_t R = 2(c_2 + c_1\cos\alpha)\,\partial_t c_2$ with $\partial_t c_2 = -\frac{1}{2}\sin((t-\varphi)/2)$. (iv) Symmetrize over $\varphi \leftrightarrow t - \varphi$ (R invariant): with $\sin(x/2)\cos(x/2) = \frac{1}{2}\sin x$ and the angle-addition formula,

$$\begin{aligned}\tfrac{1}{2}\bigl[\sin(\tfrac{t-\varphi}{2})(c_2 + c_1\cos\alpha) + \sin(\tfrac{\varphi}{2})(c_1 + c_2\cos\alpha)\bigr] &= \tfrac{1}{4}[\sin(t-\varphi) + \sin\varphi] + \tfrac{1}{2}\sin(\tfrac{t}{2})\cos\alpha \\ &= \tfrac{1}{2}\sin(\tfrac{t}{2})(\cos s + \cos\alpha),\end{aligned}$$

using $\sin(t-\varphi) + \sin\varphi = 2\sin(t/2)\cos s$ and $\sin(\frac{t-\varphi}{2})c_1 + \sin(\frac{\varphi}{2})c_2 = \sin(t/2)$. Collecting constants yields (9). The same substitution in the companion representation of F_A (Lemma 10 with Lemma 16, second formula) and cancellation of the common factor $\beta\sin(t/2)$ yields (10). As an end-to-end check, (9) and (10) match the quartic series with ratio 1.000000000000 at $t \in \{0.4, 1.1, 2.3, 2.9\}$, $\beta = 4$. □

Lemma 19 (exact; geometry of the (P,Q) square, $0<t<\pi$)**.** *On $[0,1]^2$: the main saddle is $(0,0)$ with $R=2\cos(t/4)$; the secondary saddle is $(1,1)$ with $R=2\sin(t/4)$ and gap $2(\cos(t/4)-\sin(t/4))>0$ degenerating only at $t=\pi$; the zero set $\{R=0\}$ consists of exactly the two corners $(1,0)$ and $(0,1)$, and at both of them* both *brackets of (9) and (10) vanish, so the singular points of K are switched off (the extension $I_1(z)/z\to\frac{1}{2}$ makes K continuous with value β). At $t=0$ the zero set degenerates to the edges $P=1$, $Q=1$. The saddle deficit is the exact sum of two squares*

$$4\cos^2(t/4)-R^2=4\cos^2(t/4)\sin^2(s/2)+4c_1c_2\sin^2(\alpha/2),\qquad c_1c_2=\cos^2(t/4)-\sin^2(s/2),\tag{11}$$

with sign region $c_1c_2\geq 0 \iff |s|\leq\pi-t/2$.

Proof. (8) and (11) are product-to-sum computations (symbolic zeros). $R^2\geq(|c_1|-|c_2|)^2$ with equality forcing $|c_1|=|c_2|$ (i.e. $s=0$ or $s=\pm\pi$) and $\cos\alpha=\mp\mathrm{sign}(c_1c_2)$, which lands exactly on $(P,Q)\in\{(0,1),(1,0)\}$. Bracket vanishing at both points is direct substitution. The Jacobian of $(s,\alpha)\mapsto(P,Q)$ carries the arcsine weight $ds\,d\alpha = dP\,dQ/\sqrt{P(1-P)Q(1-Q)}$ (per quadrant), singular at all four corners; near the main saddle one therefore works in (s,α), and in (P,Q) only for the global geometry. □

Lemma 20 (exact; saddle cancellations)**.** *Write N,D for numerator and denominator brackets of (10), $C=\cos(t/2)$, $c_0=\cos(t/4)$. At the saddle $(s,\alpha)=(0,0)$:*

$$\frac{N}{D}\Big|_{(0,0)}=C,\qquad N_{\alpha\alpha}-C\,D_{\alpha\alpha}=0,\qquad N_{ss}-C\,D_{ss}=-4C-2,$$

and for $N_t=-\frac{1}{2}\sin(t/2)[\cos 2s+\cos\alpha\cos s]$:

$$N_{t,\alpha\alpha}-m\,D_{\alpha\alpha}=0\ \ (m=-\tfrac{1}{2}\sin(t/2)),\qquad N_{t,ss}-m\,D_{ss}=2\sin(t/2).$$

Also exact: $2C+1=4c_0^2-1$ and $C\,c(t)=2c_0-1/(2c_0)$ for $c(t)=(4c_0^2-1)/(2c_0C)$.

Proposition 21 (verified; the second-order laws)**.** *The frozen-kernel expansion built on Lemma 20 (with $\langle s^2\rangle=1/(\beta c_0)+O(\beta^{-2})$ from (11)) predicts*

$$E=C\Big(1-\frac{c(t)}{\beta}\Big)+O(\beta^{-2}),\qquad E'=-\tfrac{1}{2}\sin(t/2)+\frac{\sin(t/4)(4c_0^2+1)}{8\beta c_0^2}+O(\beta^{-2}).\tag{12}$$

The coefficient identities are exact (Lemma 20); the laws with remainder *are verified, not yet certified: measured remainders are $\approx 0.15/\beta^2$ with true convergence at fixed t (e.g. β^2-scaled residues* 0.1776/0.1767/0.1762 *at $t=1$, $\beta=30/60/120$), and $\beta\,(E'+\frac{1}{2}\sin(t/2))\to\sin(t/4)(4c_0^2+1)/(8c_0^2)$ at $t=1,2$. The remainder control is exactly what the closure map owes. Note the $1/\beta$ correction to E' is* positive*: it opposes the target inequality and must be beaten, not used. The α-cancellations imply every kernel correction multiplies a vanishing fluctuation and enters at $O(\beta^{-2})$ only.*

Remark 22 (exact decomposition of E'). With the normalized positive measure $d\mu\propto K\,ds\,d\alpha$, $E'=\langle N_t\rangle/\langle D\rangle+\langle(N-ED)\,\partial_t\log K\rangle/\langle D\rangle$, $D_t=0$, and the regular form $\partial_t\log K=4cc'(1-P-Q)\,[zI_0(z)/I_1(z)-2]/R^2$, $z=2\beta R$, extends continuously by $4\beta^2cc'(1-P-Q)$ at $R=0$. Verified split at $(\beta,t)=(5,1.3)$: $-0.238212-0.024552=-0.262764=$ the finite difference exactly. Bounds for E' are taken from this exact quotient — never by differentiating an estimate of E.

7 The π-endpoint package

Lemma 23 (exact; telescoping)**.** *With a_m as in (1):*

$$A_\infty := \sum_{m\geq 1}(-1)^{m+1}m\,a_m = 0, \qquad \sum_{m\geq 1}(-1)^{m+1}m^3 a_m = -\sum_{m\geq 1}(-1)^{m+1}m(m+1)(2m+1)\,I_m^2 I_{m+1}^2.$$

Hence $F_A(\pi-\varepsilon) = c_3\,\varepsilon^3 + O(\varepsilon^5)$ with

$$c_3 = \frac{1}{6}\sum_{m\geq 1}(-1)^{m+1}m(m+1)(2m+1)\,I_m^2 I_{m+1}^2. \tag{13}$$

Proof. Split $m^r a_m = m^r(m-1)I_{m-1}^2 I_m^2 + m^r(m+1)I_m^2 I_{m+1}^2$ and shift $m \to m+1$ in the first sum: for $r=1$ the two weights cancel ($m(m+1)-m(m+1)=0$); for $r=3$ they combine to $m(m+1)[m^2-(m+1)^2] = -m(m+1)(2m+1)$. Absolute convergence justifies the reindexing; $\sin(m(\pi-\varepsilon)) = (-1)^{m+1}\sin(m\varepsilon)$ expands the sine. □

Lemma 24 (exact; integral form and parity)**.** *Let $g(u) = \frac{1}{2}I_1(2\beta\cos u) = \sum_{m\geq 0} I_m I_{m+1}\cos((2m+1)u)$* (addition theorem)*. Then*

$$c_3 = \frac{S_3 - S_1}{24}, \qquad S_1 = \frac{2}{\pi}\int_0^\pi g(\tfrac{\pi}{2}-u)\,g'(u)\,du, \quad S_3 = \frac{2}{\pi}\int_0^\pi g'(\tfrac{\pi}{2}-u)\,(-g''(u))\,du.$$

Moreover $g(-u) = g(u)$ and $g(\pi-u) = -g(u)$ (parity of I_1), hence $g'(\pi-u) = g'(u)$, $g''(\pi-u) = -g''(u)$, and both *integrands are invariant under $u \mapsto \pi-u$: the two halves $[0,\pi/2]$ and $[\pi/2,\pi]$ contribute exactly equally, and one may integrate a half range and double. In particular no cancellation between the two phase maxima $u=\pi/4$ and $u=3\pi/4$ is possible — exactly, not asymptotically.*

Proof. The expansion of g is the Graf addition theorem with unit shift. Quarter turn: $\cos((2m+1)(\frac{\pi}{2}-u)) = (-1)^m\sin((2m+1)u)$, so $g(\frac{\pi}{2}-u) = \sum_m(-1)^m I_m I_{m+1}\sin((2m+1)u)$ while $g'(u) = -\sum_k(2k+1)I_k I_{k+1}\sin((2k+1)u)$ and $g''(u) = -\sum_k(2k+1)^2 I_k I_{k+1}\cos((2k+1)u)$, whence $g'(\frac{\pi}{2}-u) = -\sum_k(-1)^k(2k+1)I_k I_{k+1}\cos((2k+1)u)$. Orthogonality of $\{\sin((2m+1)u)\}$ and $\{\cos((2m+1)u)\}$ on $[0,\pi]$ (norm $\pi/2$) gives $S_1 = -\sum_m(-1)^m(2m+1)(I_m I_{m+1})^2$ and $S_3 = -\sum_m(-1)^m(2m+1)^3(I_m I_{m+1})^2$, so

$$\begin{aligned}\frac{S_3-S_1}{24} &= \sum_{m\geq 0}(-1)^{m+1}\frac{(2m+1)^3-(2m+1)}{24}(I_m I_{m+1})^2\\ &= \frac{1}{6}\sum_{m\geq 1}(-1)^{m+1}m(m+1)(2m+1)I_m^2 I_{m+1}^2 = c_3,\end{aligned}$$

using $(2m+1)^3-(2m+1) = 4m(m+1)(2m+1)$ and the vanishing of the $m=0$ term. The parity claims follow from $I_1(-z) = -I_1(z)$ applied to $g(\pi-u) = \frac{1}{2}I_1(-2\beta\cos u)$, and by differentiating the two symmetry relations. □

Proposition 25 (verified; prefactor law)**.** *$c_3 > 0$ for $\beta \in \{1,5,20,40,80,120\}$ at cancellation-adjusted precision* (the alternating sum cancels at scale $e^{-(4-2\sqrt{2})\beta}$; evaluations use dps $\geq 0.6\beta+50$)*, and*

$$c_3 \;\sim\; A\,\beta^{3/2}e^{2\sqrt{2}\beta}, \qquad A = \frac{1}{24\cdot 2^{1/4}\pi^{3/2}} = 0.0062922\ldots,$$

with measured ratios 0.0058646, 0.0060049, 0.0060996, 0.0061475 *at* $\beta = 40, 60, 90, 120$ *and* $O(1/\beta)$ *gaps. The certified large-*β *branch (pending) must produce the signed prefactor with an explicit* B*:* $I_{\text{half}} \geq \frac{A}{2}\beta^{3/2}e^{2\sqrt{2}\beta}(1 - B/\beta) - T(\beta)$*,* $c_3 = 2I_{\text{half}}$*, read* β^**, and close* $[15, B_{\text{fin}}]$*,* $B_{\text{fin}} = \max\{\beta^*, 10C\}$*, by intervals.*

8 The closure map and the candidate threshold

Proposition 26 (verified truth map; all statements on the stated finite grids)**.** *Let* $g(t, \beta) = E' + \frac{1}{4}\sin(t/2)$*. At* $\beta = 15$*,* $g < 0$ *at every point of a* 13*-point grid spanning* $[0.05, \pi - 0.09]$ *(independently reproduced on a much finer mesh at* 110 *digits), with thinnest margin* -0.00573 *at* $t = 0.05$ *(on the seal window* $[0.05, \pi - 0.10]$*) and* -0.00216 *at* $\pi - 0.09$*; the crossing sits at* $\pi - t^* = 0.0890502225$*. At* $\beta = 12$ *the truth itself fails at* $\pi - 0.10$ *(*$g = +0.025$*); crossings scale as the mirror window:* $\pi - t^* = 0.1155, 0.0891, 0.0644$ *at* $\beta = 12, 15, 20$*, matching* $\sqrt{2}/\beta$*. A sweep* $\beta = 15 \ldots 60$ *keeps the maximum at* $t = 0.05$*, drifting to* -0.0061*: strong uniformity evidence, not proof.*

Conjecture 27 (quantified closure; the seal)**.** *Part* (ii) *of Conjecture 1 holds with the two-scale decomposition:* (a) $0 < \beta \leq 3$ *by Theorem 7;* (b) $\beta \in [3, 15]$ *by certified evaluation* (the holonomic structure of Section 9 makes the compact cheap)*;* (c) $\beta \geq 15$ *by three uniform blocks: left edge* $(0, 0.05]$ *(*$e_2 > 0$*, proved by the Chebyshev-sum corollary of the Turán lemma, plus a remainder radius* 0.05*); bulk* $[0.05, \pi - 0.10]$ *(enclosure of* $\langle s^2 \rangle$ *and of the covariance of Remark 22, with the five prototype-to-certificate conversions); right edge with* moving *boundary: assembly to* $\pi - C/\beta$*, then the quadratic regime of* c_3 *on* $[\pi - C/\beta, \pi]$ *with* $C_{\text{mirror}} \leq C \leq C_{\text{quad}}$ *(a fixed quadratic window is provably impossible:* $\kappa \approx 0.11\beta$ *grows while* $E(\pi - 0.10)$ *stays bounded; retention* $E/\kappa d^2$ *at* $d = 2/\beta$ *is* $0.896, 0.892, 0.890$ *at* $\beta = 20, 40, 80$ *— the validity radius scales as* $1/\beta$*). The named constants still owed:* B*,* β^**,* C_{quad}*,* C_{mirror}*, and the remainder function* $T(\beta)$*.*

The first-order competition is not the obstacle: the two-term law (12) requires only $\beta > (1 + 1/(4c_0^2))/c_0 \leq 3\sqrt{2}/2 \approx 2.121$; everything above that is rigor slack. The candidate $\beta_0 = 15$ is near truth-optimal for its edge window (Proposition 26), not padded.

9 Verification, certificates, and reproducibility

All numerical artifacts live in the repository

`https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME`

(directories `scripts/`, `docs/`); the immutable commit hash for the version audited with this manuscript is recorded in the repository's hash registry (`docs/SURFACE-CLOSURE-NOTES.md`) and in the arXiv comment field on submission. Each certificate embeds its pinned statement and self-verifies on every execution.

Certificates (interval arithmetic, outward rounding, nested or stability passes). `certify_time3_negative.py` and its Arb twin (Proposition 12); `certify_bridge_matrix.py` and twin (Proposition 13); `certify_thmB.py` (Theorem 7: adaptive β-boxes covering $[1/20, 3]$; tails computed in intervals and closed at the level of *complete shells*: each pair maps into the next shell with term ratio r derived from the termwise-exact $I_{n+1}(x) \leq xI_n(x)/(2(n+1))$, and the one new pair per shell is bounded by a double index shift, giving shell ratio $2r$, asserted $< \frac{1}{2}$ per box before the geometric closure; a built-in full stability pass at precision +70; the Arb twin `certify_thmB_arb.py` independently certifies the full coverage — 86 boxes, both passes, transcript archived; the `mpmath.iv` canonical transcript is committed on completion of its run, as recorded in the repository hash registry). Bessel enclosures use the positive series with an exact-integer stopping rule and geometric tails; floats appear only in loop-termination heuristics. Nested enclosures prevent silent numerical or truncation degradation; statement auditing and the independent Arb implementation guard against common-mode errors.

Verified computations. The master-formula and reduction identities (ratios 1.0 to 12 digits); the truth map of Proposition 26 (dps 80, confirmed independently at 110 digits); the c_3 ladder (Proposition 25); the term-split coefficients of (12) at $\beta = 60, 120, 240$.

Precision policies (each traceable to a caught artifact): dps $\geq 2.2\beta + 40$ for any evaluation with $t \geq 2$, $\beta > 30$ (mirror cancellation); dps $\geq 0.6\beta + 50$ for c_3-type alternating sums; series cutoffs scale with the maximal argument, and scaled-asymptotic evaluation of $e^{-z} I_n(z)$ is the default for $z > 35$; every measured coefficient carries a β-scaling sanity check; whatever can be computed from a positive integral representation is.

Audit trail. Fifteen numerical or formulation artifacts ("ghosts") were caught during fabrication and writing by the three-way review protocol; Appendix B tabulates them. We consider the table part of the result: it is the reproducibility record that makes the certificates credible.

Machine-checked lemmas. The unit-step and Turán-type coefficient inequalities behind Theorem 7 are formalized in Lean 4/Mathlib (`PhiMonotone.lean`, `FHBesselAmos.lean`, `WronskianIdentity.lean`; axiom oracle `[propext, Classical.choice, Quot.sound]`, no `sorry`). Mathlib has no Bessel functions: what is machine-checked is the ordered-field implication, with the analytic inputs (three-term recurrence [4, 10.29.1]; the Amos-type bound of [1, 2, 3], cited verbatim, shift $\nu \to \nu+1$ as in Appendix A) carried as named hypotheses. Correct claim everywhere: *machine-checked modulo classical Bessel inputs carried as named hypotheses.*

Acknowledgements and provenance

The conjecture, the bridge reading, the reduction, and all certificates were produced inside an audit-first programme in which three independent reviewing voices verify every claim before acceptance; scores, objections, and the complete ghost ledger are archived with the repository history. Nothing in this paper rests on trust in any single computation: every load-bearing negative has two witnesses, and every identity marked exact has a symbolic-zero transcript.

A The weighted Turán-type lemma, self-contained

Theorem 28 (exact)**.** *For every $x > 0$ and every real $m \geq 1$, with $\varphi_m = a_m/b_m$ built from (1) at argument x: $\varphi_m < \varphi_{m+1}$. Equivalently, with $\rho_\mu = I_{\mu+1}/I_\mu$,*

$$\frac{1}{m}\Big(\frac{m-1}{\rho_{m-1}^2} + (m+1)\rho_m^2\Big) < \frac{1}{m+1}\Big(\frac{m}{\rho_m^2} + (m+2)\rho_{m+1}^2\Big).$$

Proof. Two classical inputs, valid for all real orders: the three-term recurrence $\rho_{m-1}^{-1} = \rho_m + 2m/x$, $\rho_{m+1} = \rho_m^{-1} - 2(m+1)/x$ [4, 10.29.1], and the Amos-type bound

$$\rho_m(x) < U_m(x) := \frac{x}{a + \sqrt{a^2 + x^2}}, \qquad a = m + \tfrac{1}{2},$$

proved best of its form for all real orders ≥ 0 by Segura [2]; we use [3, Thm. 2], which bounds $I_\nu(x)/I_{\nu-1}(x)$ for $\nu \geq \frac{1}{2}$ by $x/(\nu - \frac{1}{2} + \sqrt{(\nu+\frac{1}{2})^2 + x^2}\,)$; setting $\nu = m+1$ bounds $\rho_m = I_{m+1}/I_m$ by $x/(m+\frac{1}{2}+\sqrt{(m+\frac{3}{2})^2+x^2}\,) \leq x/(a+\sqrt{a^2+x^2}\,)$ with $a = m+\frac{1}{2}$, which is the displayed bound (the last step uses $\sqrt{(m+\frac{3}{2})^2+x^2} \geq \sqrt{a^2+x^2}$). Substitute the recurrences, write $u = \rho_m \in (0,1)$, $c = 1/x$, $S = u + 1/u$, $P = 1/u - u$; clearing the positive denominators, the claim becomes $\Delta_m > 0$ where the *exact factorization*

$$\frac{\Delta_m}{2m(m+1)} = (S-3c)\big(P-(2m+1)c\big) + (2m+1)c^2$$

holds as an algebraic identity (expand and collect; the coefficients of $u^{\pm 2}$ are $\pm 2m(m+1)$, the linear-in-c terms are $-4m(m+1)c[(m-1)u + (m+2)/u]$, and $(m-1)u + (m+2)/u = \frac{1}{2}[(2m+1)S + 3P]$, $1/u^2 - u^2 = PS$). The second factor is positive because U_m is the positive root of $1/w - w = (2m+1)/x$ (the calibration $1/U_m - U_m = 2a/x$) and $w \mapsto 1/w - w$ is strictly decreasing, so $\rho_m < U_m$ gives $P > (2m+1)c$. The first factor is positive because $\sqrt{a^2+x^2} > a$ gives $U_m < x/(2a)$, so $u < x/(2m+1) \leq x/3$ and $S > 1/u > (2m+1)/x \geq 3c$. The third term is positive. Strictness is strict throughout. (This is the proof of [6], reproduced so that Theorem 7 is self-contained; the ordered-field implication is machine-checked in `PhiMonotone.lean`, modulo the two classical inputs carried as named hypotheses.) □

B Audit table (the ghost ledger)

Each row: what was wrong, how it was caught, and the rule it produced. All entries are documented with transcripts in `docs/SURFACE-CLOSURE-NOTES.md`.

#	artifact	catch / rule produced
1	random TP_2 scan missed adversarial corners	targeted corner search; adversarial grids mandatory
2	verification cell with unbound parameter	explicit binding rule for inline prints
3	near-π scan at low precision, false violation	precision rule dps $\geq 2.2\beta + 40$ for $t \geq 2$, $\beta > 30$
4	stale-PDF claim misattributed between voices	provenance ledger: attribute to exact source
5	float64 CDF increments ± 11 (impossible)	positive-representation preference
6	t=2, β=240 residue 0.363 vs 0.317	boundary $t \geq 2$ included in the dps rule
7	I_1 series truncated at 60 terms, $z \leq 480$	cutoffs scale with argument; scaled asymptotics for $z > 35$; β-scaling checks
8	c_3 sign flip at $\beta = 120$ (cancellation)	alternating-sum rule dps $\geq 0.6\beta + 50$
9	$R = 0$ corner misplaced at $(1, 1)$	complete critical-point census required in prose too
10	one-saddle prefactor off by exactly 2	census rule; the factor 2 later proved an exact parity
11	I_1 lower bound dropped the negative tail	sign audits are a separate mandatory pass
12	fixed quadratic window at π was provably false	interrogate obligations before attempting them
13	Thm. A second proof: rearrangement misquoted	layer-cake arc proof substituted (this revision)
14	Lemma sign error: floor stated as upper bound	corrected; per-pair statements now carry signs explicitly
15	"exact" labels on verified inequalities	tricotomy re-audit of every label (this revision)

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