Ratio Monotonicity for a Killed von Mises Bridge:

exact bridge structure, certified negative results, a single-Bessel reduction, and a two-scale closure map for a surface expansion in two-dimensional lattice gauge theory

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Abstract

For $\beta > 0$ let $I_m = I_m(\beta)$ denote modified Bessel functions of the first kind, and set $a_m = I_m^2\,[(m-1)I_{m-1}^2 + (m+1)I_{m+1}^2]$, $b_m = m\,I_m^4$, $F_A(t) = \sum_{m\geq 1} a_m \sin(mt)$, $F_B(t) = \sum_{m\geq 1} b_m \sin(mt)$, and $E = F_A/(2F_B)$. The global ratio-monotonicity problem for the surface expansion of a two-dimensional $SU(2)$ lattice gauge observable is: (i) $F_B > 0$ on $(0, \pi)$, and (ii) $E' < 0$ on $(0, \pi)$, for every $\beta > 0$. We prove (i) twice, and prove (ii) for $0 < \beta \leq 3$ via an exact pair-identity skeleton plus an archived interval certificate executed by two independent interval-arithmetic implementations. The structural core is exact: E is, as an algebraic identity, the mean of $\cos\psi$ under the midpoint law of a four-step killed von Mises bridge; the generating kernels reduce, via the Neumann addition theorem, to two-dimensional integrals of a *single* Bessel function whose saddle deficit is an exact sum of two squares; and exact saddle cancellations yield the coefficients of the (verified) closed second-order law $E = \cos(t/2)(1 - c(t)/\beta) + O(\beta^{-2})$, $c(t) = (4\cos^2(t/4) - 1)/(2\cos(t/4)\cos(t/2))$. Three certified negative results (interval arithmetic, two implementations, nested enclosures) kill every monotone full-path coupling, with an exact mechanism at threshold $\beta|\cos t| = 3/2$. For the remaining range we give a two-scale closure map with a verified candidate threshold $\beta_0 = 15$ and the complete π-endpoint package for the cubic coefficient c_3. Every claim is labelled exact / certified / verified; the machine-checked lemmas are Lean 4/Mathlib, machine-checked modulo classical Bessel inputs carried as named hypotheses.

1 Introduction

This paper consolidates the resolution status of a two-part positivity problem that arises as the last analytic gap in a surface (character) expansion for two-dimensional $SU(2)$ lattice gauge theory [5, 6, 7]. The problem is stated entirely in terms of modified Bessel functions and trigonometric series, and we treat it as such; no claim of physical consequence beyond the stated expansion is made, and importance is not inherited by thematic proximity.

Throughout, $\beta > 0$, $I_m = I_m(\beta)$, and

$$a_m = I_m^2\big[(m-1)I_{m-1}^2 + (m+1)I_{m+1}^2\big], \qquad b_m = m\,I_m^4, \qquad m \geq 1. \tag{1}$$

Both sequences decay super-exponentially, so

$$F_A(t) = \sum_{m\geq 1} a_m \sin(mt), \qquad F_B(t) = \sum_{m\geq 1} b_m \sin(mt), \qquad E(t) = \frac{F_A(t)}{2F_B(t)}, \tag{2}$$

are real-analytic on $\mathbb{R}$, and the Wronskian $W = 2(F_A' F_B - F_A F_B')$ satisfies $W = 4F_B^2\,(E)'$ wherever $F_B \neq 0$ [7].

Conjecture 1 (global ratio monotonicity)**.** *For every $\beta > 0$:* (i) $F_B > 0$ *on* $(0, \pi)$*;* (ii) $E' < 0$ *on* $(0, \pi)$.

Part (i) is proved below (twice). Part (ii) is proved for $0 < \beta \leq 3$ (Theorem 7; exact skeleton plus the archived certificate `certify_thmB.py` and its Arb twin), is supported by an exact structural reduction and a verified truth map for $\beta \geq 15$, and is otherwise open; Section 8 states the precise two-scale closure map. The conjecture was born inside this programme (it is the global form of an ε-window positivity required by the surface expansion); no external pedigree is claimed.

Convention 2 (tricotomy)*.* Every assertion below carries one of three labels. *Exact*: a proved identity or inequality. *Certified*: a numerical statement enclosed in interval arithmetic with outward rounding, nested re-computation at higher precision, and (for the load-bearing negatives) a second independent implementation in Arb. *Verified*: high-precision numerics with stated grids and precision policies, not a proof. No verified statement is used as a hypothesis of an exact one.

2 The spectral framework

Convention 3 (normalizations)*.* On $(0, \pi)$ define the kernel, entry law, and smoothing operator

$$q_1(u, v) = 2\, e^{\beta \cos u \cos v} \sinh(\beta \sin u \sin v) \;=\; 4 \sum_{m \geq 1} I_m \sin(mu) \sin(mv), \tag{3}$$

$$s_1(u) = \tfrac{\beta}{2} \sin u\, e^{\beta \cos u} \;=\; \sum_{m \geq 1} m\, I_m \sin(mu), \tag{4}$$

$$(Qf)(v) = \frac{1}{2\pi} \int_0^\pi q_1(v, u) f(u)\, du, \tag{5}$$

so that $Q \sin(m\cdot) = I_m \sin(m\cdot)$. Both closed forms are classical consequences of the generating function $e^{\beta \cos \theta} = I_0 + 2 \sum_{m \geq 1} I_m \cos(m\theta)$; (4) follows by differentiation. All ten verification artifacts in the repository use exactly these normalizations.

Since $q_1 > 0$ on $(0, \pi)^2$ and $s_1 > 0$ on $(0, \pi)$, the walk with step kernel q_1 killed on $\{0, \pi\}$ is well defined; $H(\psi) = \beta \sin(\psi/2) I_1(2\beta \cos(\psi/2))$ and $K_2(\varphi) = I_0(2\beta \cos(\varphi/2))$ appear below. Note $F_B = Q^3 s_1$ and, by the sine orthogonality computation of Lemma 10, F_A is the companion cosine-weighted integral.

3 Theorem A: positivity of F_B

Theorem 4 (exact)**.** $F_B(t) > 0$ *for all* $t \in (0, \pi)$ *and all* $\beta > 0$.

First proof (kernel positivity). $F_B = Q^3 s_1$ by (5) and $b_m = m I_m^4$. Each application of Q integrates a positive function against the strictly positive kernel (3); $s_1 > 0$ on $(0, \pi)$. Hence $F_B > 0$ pointwise. □

Second proof (symmetric-decreasing convolution). With $K_2(\varphi) = I_0(2\beta \cos(\varphi/2))$ and $P_4 = K_2 * K_2$ (circle convolution; Section 6), $F_B = -\frac{1}{2} P_4'$. The function K_2 is even and *strictly decreasing in* $|\varphi|$ *on* $[0, \pi]$ (I_0 strictly increasing, $\cos(\varphi/2)$ strictly decreasing). By the layer-cake representation, $K_2 = \int_0^\infty \mathbf{1}_{A_\lambda}\, d\lambda$ with A_λ centered arcs whose half-lengths $a(\lambda)$ fill $(0, \pi)$ continuously (strict decrease). For centered arcs, $(\mathbf{1}_A * \mathbf{1}_B)(t) = |A \cap (t+B)|$ is even and nonincreasing in $|t|$ on $[0, \pi]$, and *strictly* decreasing on $\big||a-b|, \min(a+b, 2\pi - a - b)\big|$. Hence $P_4(t) = \iint |A_\lambda \cap (t + A_\mu)|\, d\lambda\, d\mu$ is even, nonincreasing on $[0, \pi]$, and strictly decreasing on $(0, \pi)$ because for every $t \in (0, \pi)$ a positive measure of pairs (λ, μ) has $|a(\lambda) - a(\mu)| < t < \min(a+b, 2\pi - a - b)$ (the half-lengths fill $(0, \pi)$). At the derivative level: each overlap $t \mapsto |A_\lambda \cap (t + A_\mu)|$ is Lipschitz with a.e. derivative -1 in the strict regime and 0 otherwise (on $[0, \pi]$); since P_4 is analytic, dominated convergence gives $P_4'(t) = \iint \partial_t |A_\lambda \cap (t + A_\mu)|\, d\lambda\, d\mu \le -\mu\{(\lambda, \mu) : \text{strict at } t\} < 0$ pointwise on $(0, \pi)$. Thus $F_B = -\frac{1}{2} P_4' > 0$ on $(0, \pi)$. $\square$

4 Theorem B: ratio monotonicity for $0 < \beta \le 3$

Lemma 5 (exact; Wronskian identity)**.** *For real arrays with* $\sum m(|a_m| + |b_m|) < \infty$,

$$W := 2(F_A' F_B - F_A F_B') = \sum_{1 \le m < n} c_{mn}\, T_{mn}(t), \qquad c_{mn} = a_m b_n - a_n b_m, \tag{6}$$

where $T_{mn}(t) = (m-n)\sin((m+n)t) + (m+n)\sin((n-m)t)$.

Proof. Product-to-sum: $T_{mn} = 2[m\cos(mt)\sin(nt) - n\sin(mt)\cos(nt)] =: 2G(m,n)$, antisymmetric with $G(n,n) = 0$; since c_{mn} is antisymmetric, $\sum_{m<n} c_{mn} 2G = 2\sum_{m,n} a_m b_n G(m,n) = 2[F_A' F_B - F_A F_B']$ (first recorded in [7]). $\square$

The coefficient input is the weighted Turán-type lemma (Appendix A, self-contained): $\varphi_m = a_m / b_m$ is strictly increasing in $m \ge 1$ for every $\beta > 0$, hence *every minor is negative*: $c_{mn} < 0$ for $m < n$ (exact).

Lemma 6 (exact; the pair identities and bounds)**.** *For* $t \in (0, \pi)$, *with* $p = n - m$, $q = n + m$:

- *(a)* $T_{12} = 4\sin^3 t$; $T_{13} = 16 \sin^3 t \cos t$; $T_{24} = 8\sin^3(2t)$, *so* $|T_{13}| \le 16\sin^3 t$ *and* $|T_{24}| \le 64 \sin^3 t$.
- *(b)* $T_{m,m+1} \ge 0$ *for every* $m \ge 1$.
- *(c)* $|T_{mn}| \le \dfrac{\pi^3}{48}\, pq(q^2 + p^2)\, \sin^3 t$ *for all* $m < n$.

Proof. (a) $3\sin u - \sin 3u = 4\sin^3 u$ gives T_{12} (at $u = t$) and T_{24} (at $u = 2t$); $T_{13} = 4\sin 2t - 2\sin 4t = 4\sin 2t(1 - \cos 2t) = 16\sin^3 t \cos t$. (b) $T_{m,m+1} = (2m+1)\sin t - \sin((2m+1)t) \ge 0$ since $|\sin(qt)| \le q \sin t$ on $(0, \pi)$ (induction on q). (c) T_{mn} and its first two derivatives vanish at $t = 0$ and $t = \pi$ (the parities of $m \pm n$ agree), and $|T_{mn}'''| \le pq^3 + qp^3 = pq(q^2 + p^2)$; hence $|T_{mn}(t)| \le \frac{pq(q^2+p^2)}{6} \min(t, \pi - t)^3$, and $\sin t \ge \frac{2}{\pi}\min(t, \pi - t)$ gives the claim. $\square$

Theorem 7 (exact modulo the certified computation `certify_thmB.py`)**.** *For* $0 < \beta \le 3$: $W < 0$ *on* $(0, \pi)$, *hence* $E' < 0$ *on* $(0, \pi)$.

Proof. Drop the adjacent pairs $m \ge 2$ (each contributes $c_{m,m+1} T_{m,m+1} \le 0$ by Lemma 6(b) and $c < 0$); keep T_{12}, T_{13}, T_{24} exactly and majorize every other pair by Lemma 6(c). Pointwise in t,

$$\frac{W(t)}{\sin^3 t} \le \operatorname{Crit}(\beta) := -4|c_{12}| + 16|c_{13}| + 64|c_{24}| + \frac{\pi^3}{48} \sum_{\substack{p \ge 2 \\ (m,n) \ne (1,3),(2,4)}} pq(q^2 + p^2)\, |c_{mn}|.$$

The certificate `certify_thmB.py` (Section 9) encloses Crit in interval arithmetic over adaptive β-boxes covering $[1/20, 3]$ (I_m increasing in β gives rigorous boxes; cutoff $M = 60$ with the tail itself computed in intervals and closed by a derived, per-box-asserted geometric ratio) and certifies Crit < 0 on all of $[1/20, 3]$; the script then re-certifies *every* box at precision +70 before reporting success (the box count is invocation- and implementation-dependent; the independent Arb implementation `certify_thmB_arb.py` certifies the same coverage with 86 boxes and a full stability pass, transcript archived). For $0 < \beta \le 1/20$, Lemma 8 applies. $\square$

Lemma 8 (exact; small β)**.** *For* $0 < \beta \le 1/20$*:* $\mathrm{Crit}(\beta) < 0$.

Proof. Write $\kappa = \beta/2 \le 1/40$, $L_m = \kappa^m/m!$, $U_m = e^* L_m$ with $e^* = e^{\beta^2/4} \le e^{1/1600} < 1.001$; then $L_m \le I_m \le U_m$ (positive series). *Leading minor:* $a_2 b_1 \ge (I_1^2 I_2^2)(I_1^4) \ge L_1^6 L_2^2 = \kappa^{10}/4$ and $a_1 b_2 = (2I_1^2 I_2^2)(2I_2^4) \le 4U_1^2 U_2^6 = (e^*)^8 \kappa^{14}/16$, so

$$|c_{12}| \;\ge\; \frac{\kappa^{10}}{4}\Big(1 - \frac{(e^*)^8\kappa^4}{4}\Big) \;\ge\; \frac{\kappa^{10}}{4}\,(1 - 10^{-6}).$$

The three visible positive terms: $a_3 \le I_3^2 \cdot (2I_2^2 + 4I_4^2) \le (e^*)^4\kappa^{10}/24 \cdot (1 + 10^{-4})$ and $b_1 \le (e^*)^4\kappa^4$ give $16|c_{13}| \le 16(a_3 b_1 + a_1 b_3) \le 0.70\,\kappa^{14}$; $64|c_{24}| \le 64(a_4 b_2 + a_2 b_4) \le 10^{-3}\kappa^{14}$ (both products have κ-degree ≥ 22); the largest remaining weighted pair is $(1,4)$: $\frac{\pi^3}{48} \cdot 510 \cdot (a_4 b_1 + a_1 b_4) \le 0.2\,\kappa^{18} \le 10^{-6}\kappa^{14}$. *All other pairs* (those with $s := m + n \ge 6$, $p \ge 2$): using $a_n b_m + a_m b_n \le 2mn(e^*)^8\kappa^{4s-2}[((m-1)!m!)^{-2} n!^{-4} + ((n-1)!n!)^{-2} m!^{-4}] \le s^2 (e^*)^8 \kappa^{4s-2}$ (the factorial bracket is ≤ 2 and $2mn \le s^2/2$), $pq(q^2 + p^2) \le 2s^4$, and at most s pairs per level,

$$\frac{\pi^3}{48}\sum_{s\ge 6}(\text{level } s) \le \frac{\pi^3}{48}(e^*)^8 \sum_{s \ge 6} 2s^7 \kappa^{4s-2} \le 0.66\,\kappa^{22} \cdot 6^7 \sum_{j \ge 0}\big[(7/6)^7\kappa^4\big]^j \le 3.7 \times 10^5\,\kappa^{22} \le 10^{-7}\kappa^{14},$$

where the level ratio $s \mapsto s+1$ is at most $(7/6)^7\kappa^4 < 10^{-5} < \frac{1}{2}$ (so the geometric closure is valid), and $\kappa^8 \le 40^{-8}$ absorbs the constant. Collecting:

$$\mathrm{Crit} \le -\kappa^{10}(1 - 10^{-6}) + 0.71\,\kappa^{14} = -\kappa^{10}\big(1 - 10^{-6} - 0.71\,\kappa^4\big) < 0,$$

since $\kappa^4 \le 40^{-4} < 4 \cdot 10^{-7}$. (All numbered constants above are crude upper counts; each step loses factors, none gains.) $\square$

Remark 9 (verified; the sharp family and saturation)*.* The sharper per-pair bound $|T_{mn}| \le \frac{pq(q^2-p^2)}{6}\sin^3 t$ and the adjacent floor $T_{m,m+1} \ge 2m \sin^3 t$ (note the sign: the floor is a *lower* bound on a positive quantity) are verified far beyond the range used ($q \le 151$; 13 pairs to $(10, 30)$, max ratio 1.0) and would extend Theorem 7 to $\beta \approx 3.5$; their per-pair certification is a scheduled upgrade. Beyond that the decomposition is *saturated*: with the exact far block (cancellation included) the criterion fails at $\beta = 6$ by $+2.2 \times 10^{14}$ while the true $W/\sin^3 t$ remains negative (-6.9×10^{12}) — a verified adversarial finding about the method, and the reason the large-β closure uses different machinery (Sections 6–8).

5 The bridge structure of E (exact)

Let $k = q_1 \circ q_1$ denote the two-step kernel, $k(\psi, t) = 4\sum_{m\ge 1} I_m^2 \sin(m\psi)\sin(mt)$, and define

$$\omega_t(\,d\psi) \;\propto\; (Qs_1)(\psi)\,k(\psi, t)\,d\psi \qquad \text{on } (0, \pi), \tag{7}$$

the law of the time-2 position (midpoint) of the four-step bridge entering from 0 with law s_1 and conditioned to end at t.

Lemma 10 (exact)**.** $\int_0^\pi (Qs_1)\,k(\cdot,t)\,d\psi \;=\; 2\pi F_B(t)$ *and* $\int_0^\pi \cos\psi\,(Qs_1)\,k(\cdot,t)\,d\psi \;=\; \pi F_A(t)$. *Hence, as an identity,*

$$E(t) \;=\; \mathbb{E}_{\omega_t}[\cos\psi] \;=\; \frac{F_A(t)}{2F_B(t)}.$$

Proof. $(Qs_1)(\psi) = \sum mI_m^2 \sin(m\psi)$. By sine orthogonality the first integral picks the diagonal, giving $2\pi \sum mI_m^4 \sin(mt)$. In the second, $\cos\psi\,\sin(m\psi)\sin(n\psi)$ integrates to $\frac{\pi}{4}(\delta_{n,m+1}+\delta_{n,m-1})$, which produces exactly the weights $(n-1)I_{n-1}^2 + (n+1)I_{n+1}^2$ of a_n. ☐

Lemma 11 (exact; entry law)**.** $\partial_t k(0^+,\psi) = 2H(\psi)$, *where* $H(\psi) = \beta\sin(\psi/2)\,I_1(2\beta\cos(\psi/2))$.

Proof. Differentiate in ψ the squared Graf identity $I_0^2 + 2\sum_{m\ge 1} I_m^2 \cos(m\psi) = I_0(2\beta\cos(\psi/2))$ [4, §10.23(ii)] and compare with $\partial_t k(0^+,\psi) = 4\sum mI_m^2 \sin(m\psi)$. ☐

Proposition 12 (certified negative; no monotone full-path coupling)**.** *At $\beta = 20$ the time-3 marginal $\rho_t(u) \propto (Q^2 s_1)(u)\,q_1(u,t)$ is not stochastically monotone in t: with $a = \frac{4802}{10^4}$, $t_1 = \frac{29621}{10^4}$, $t_2 = \frac{30070}{10^4}$, the tail masses satisfy $T(t_2,a) < T(t_1,a)$, with certified enclosure*

$$\begin{aligned} T(t_2,a) - T(t_1,a) \in [&-2.5299150980081786690028107,\\ &-2.5299150980081786690028074] \times 10^{-10}. \end{aligned}$$

Consequently no monotone coupling of full bridge paths exists at $\beta = 20$. The certificate uses exact termwise tails ($J(m,n,a) = \int_a^\pi \sin mu \sin nu\,du$ in closed form: no quadrature error), positive-series Bessel enclosures with geometric tails, explicit truncation bounds, nested passes $(M,\mathrm{prec}) = (120,350) \subset (140,500)$*, and two independent interval implementations (`mpmath.iv`; Arb via `python-flint`).*

Proposition 13 (certified negatives; the weight–kernel matrix)**.** *At $\beta = 20$, for every entry weight $Q^{k-1}s_1$ ($k = 1,2,3,4$) the marginal one q_1-step from the moving endpoint fails stochastic monotonicity, with certified witnesses sharing $(t_1,t_2) = (3, 61/20)$: $k = 1$ (two-step bridge midpoint, the minimal counterexample), $a = 9/10$, margin $\approx -1.34\times 10^{-2}$; $k = 2$, $a = 3/4$, $\approx -1.80\times 10^{-6}$; $k = 4$, $a = 3/10$, $\approx -4.75\times 10^{-13}$ ($k = 3$ is Proposition 12). Two smoothing steps repair all tested weights (verified); one never does. The statement is relative to the natural weight family: k_2 possesses its own total-positivity corner ($\beta \gtrsim 8$) reachable by two-atom adversarial weights.*

Lemma 14 (exact; the 3/2 mechanism)**.** $\lim_{u\downarrow 0} \dfrac{\partial_u\partial_t \log q_1(u,t)}{\beta\sin t\; u} = 1 + \dfrac{2\beta}{3}\cos t$. *For $t > \pi/2$ the corner sign change of $\partial_u\partial_t \log q_1$ occurs exactly when $\beta|\cos t| > 3/2$ — the same threshold as the TP_2 failure of q_1.*

Proof. $\partial_t \log q_1 = -\beta\sin t\cos u + \beta\cos t\sin u\coth(\beta\sin t\sin u)$ (direct differentiation of (3)); expand $\coth z = z^{-1} + z/3 + O(z^3)$ and differentiate in u; the stated limit is a one-line computation (symbolic residual zero). ☐

Remark 15 (verified; the mechanism reading)*.* The identification of this local defect as *the* engine of Propositions 12–13 — sine-type entry weights, vanishing linearly at 0 with mass on both sides of the non-monotone region, flip the covariance $\mathrm{Cov}_{\rho_t}(\mathbf{1}_{u\ge a}, \partial_t \log q_1)$, while the flat weight does not — is supported by the full weight–kernel matrix and the β-scan of the violations (onset at $\beta \gtrsim 12$, consistent with the threshold), but is a verified reading, not a theorem.

6 The single-Bessel reduction and the master formula

Lemma 16 (exact; Neumann product formulas)**.** *For $x, y \geq 0$, with $w = \sqrt{x^2 + y^2 + 2xy\cos\alpha}$,*

$$I_0(x)I_0(y) = \frac{1}{2\pi}\int_0^{2\pi} I_0(w)\,d\alpha, \qquad I_0(x)I_1(y) = \frac{1}{2\pi}\int_0^{2\pi} I_1(w)\,\frac{y + x\cos\alpha}{w}\,d\alpha.$$

Proof. The first is the integrated Graf/Neumann addition theorem [4, §10.23(ii)]; the second is its ∂_y-derivative. □

Remark 17 (signs). Both formulas extend to all real x, y, as used below (on $[-\pi, \pi]^2$ one has $c_1 = \cos(\varphi/2) \geq 0$ but c_2 may be negative): $w(x, -y, \alpha) = w(x, y, \pi - \alpha)$, and the substitution $\alpha \mapsto \pi - \alpha$ over a full period maps the first identity to itself (both sides even in y) and the second to its negative on both sides simultaneously (I_1 odd; $(-y + x\cos(\pi - \alpha)) = -(y + x\cos\alpha)$).

With $K_2(\varphi) = I_0(2\beta\cos(\varphi/2)) = \sum_{m\in\mathbb{Z}} I_m^2 e^{im\varphi}$ and $P_4 = K_2 * K_2$ (circle convolution), $P_4 = \sum_m I_m^4 e^{imt}$ and $F_B = -\frac{1}{2}P_4'$. Applying Lemma 16 to both convolution factors:

Lemma 18 (exact; master representation)**.** *Let $s = \varphi - t/2$, $c_1 = \cos(\varphi/2)$, $c_2 = \cos((t-\varphi)/2)$, $P = \sin^2(s/2)$, $Q = \sin^2(\alpha/2)$, and*

$$R^2 = c_1^2 + c_2^2 + 2c_1c_2\cos\alpha = 4\big[\cos^2(t/4)\,(1 - P - Q) + PQ\big]. \tag{8}$$

Then

$$F_B(t) = \frac{\beta\sin(t/2)}{8\pi^2}\iint_{[-\pi,\pi]^2} \frac{I_1(2\beta R)}{R}\,\big[\cos^2(s/2) - \sin^2(\alpha/2)\big]\,ds\,d\alpha, \tag{9}$$

and, under the common positive kernel $K = I_1(2\beta R)/R$,

$$E(t) = \frac{\big\langle \cos(t/2)\cos 2s + \cos\alpha\,[\cos(t/2)\cos s - \sin^2 s] \big\rangle}{\big\langle \cos s + \cos\alpha \big\rangle}, \tag{10}$$

where $\langle\cdot\rangle$ denotes the $K\,ds\,d\alpha$-integral over $[-\pi, \pi]^2$. The prefactor collapse behind (9) is the sum-to-product symmetrization $\frac{1}{2}[\sin(\frac{t-\varphi}{2})(c_2 + c_1\cos\alpha) + (\varphi \leftrightarrow t - \varphi)] = \frac{1}{2}\sin(t/2)(\cos s + \cos\alpha)$, and $\cos s + \cos\alpha = 2[\cos^2(s/2) - \sin^2(\alpha/2)]$.

Proof. (i) The squared Graf identity gives $K_2(\varphi) = \sum_m I_m^2 e^{im\varphi}$; convolution multiplies Fourier coefficients, so $P_4 = \sum_m I_m^4 e^{im\varphi}$ and, termwise (super-exponential decay), $F_B = -\frac{1}{2}P_4'$. (ii) Insert Lemma 16 (first formula) with $x = 2\beta c_1$, $y = 2\beta c_2$ into $P_4(t) = \frac{1}{2\pi}\int K_2(\varphi)K_2(t - \varphi)\,d\varphi$: a two-dimensional integral of $I_0(2\beta R)$ with R as in (8); the (P, Q) form of R^2 is product-to-sum. (iii) Differentiate under the integral (smooth integrand, compact domain): $\partial_t I_0(2\beta R) = I_1(2\beta R)\cdot 2\beta\,\partial_t R$ and $2R\,\partial_t R = 2(c_2 + c_1\cos\alpha)\,\partial_t c_2$ with $\partial_t c_2 = -\frac{1}{2}\sin((t-\varphi)/2)$. (iv) Symmetrize over $\varphi \leftrightarrow t - \varphi$ (R invariant): with $\sin(x/2)\cos(x/2) = \frac{1}{2}\sin x$ and the angle-addition formula,

$$\begin{aligned}\tfrac{1}{2}\big[\sin(\tfrac{t-\varphi}{2})(c_2 + c_1\cos\alpha) + \sin(\tfrac{\varphi}{2})(c_1 + c_2\cos\alpha)\big] &= \tfrac{1}{4}[\sin(t-\varphi) + \sin\varphi] + \tfrac{1}{2}\sin(\tfrac{t}{2})\cos\alpha \\ &= \tfrac{1}{2}\sin(\tfrac{t}{2})(\cos s + \cos\alpha),\end{aligned}$$

using $\sin(t - \varphi) + \sin\varphi = 2\sin(t/2)\cos s$ and $\sin(\frac{t-\varphi}{2})c_1 + \sin(\frac{\varphi}{2})c_2 = \sin(t/2)$. Collecting constants yields (9). The same substitution in the companion representation of F_A (Lemma 10 with Lemma 16, second formula) and cancellation of the common factor $\beta\sin(t/2)$ yields (10). As an end-to-end check, (9) and (10) match the quartic series with ratio 1.000000000000 at $t \in \{0.4, 1.1, 2.3, 2.9\}$, $\beta = 4$. □

Lemma 19 (exact; geometry of the (P,Q) square, $0<t<\pi$)**.** *On $[0,1]^2$: the main saddle is $(0,0)$ with $R=2\cos(t/4)$; the secondary saddle is $(1,1)$ with $R=2\sin(t/4)$ and gap $2(\cos(t/4)-\sin(t/4))>0$ degenerating only at $t=\pi$; the zero set $\{R=0\}$ consists of exactly the two corners $(1,0)$ and $(0,1)$, and at both of them* both *brackets of (9) and (10) vanish, so the singular points of K are switched off (the extension $I_1(z)/z\to\frac{1}{2}$ makes K continuous with value β). At $t=0$ the zero set degenerates to the edges $P=1$, $Q=1$. The saddle deficit is the exact sum of two squares*

$$4\cos^2(t/4)-R^2=4\cos^2(t/4)\sin^2(s/2)+4c_1c_2\sin^2(\alpha/2),\qquad c_1c_2=\cos^2(t/4)-\sin^2(s/2),\tag{11}$$

with sign region $c_1c_2\ge 0\iff |s|\le\pi-t/2$.

Proof. (8) and (11) are product-to-sum computations (symbolic zeros). $R^2\ge(|c_1|-|c_2|)^2$ with equality forcing $|c_1|=|c_2|$ (i.e. $s=0$ or $s=\pm\pi$) and $\cos\alpha=\mp\mathrm{sign}(c_1c_2)$, which lands exactly on $(P,Q)\in\{(0,1),(1,0)\}$. Bracket vanishing at both points is direct substitution. The Jacobian of $(s,\alpha)\mapsto(P,Q)$ carries the arcsine weight $ds\,d\alpha=dP\,dQ/\sqrt{P(1-P)Q(1-Q)}$ (per quadrant), singular at all four corners; near the main saddle one therefore works in (s,α), and in (P,Q) only for the global geometry. □

Lemma 20 (exact; saddle cancellations)**.** *Write N,D for numerator and denominator brackets of (10), $C=\cos(t/2)$, $c_0=\cos(t/4)$. At the saddle $(s,\alpha)=(0,0)$:*

$$\frac{N}{D}\Big|_{(0,0)}=C,\qquad N_{\alpha\alpha}-C\,D_{\alpha\alpha}=0,\qquad N_{ss}-C\,D_{ss}=-4C-2,$$

and for $N_t=-\frac{1}{2}\sin(t/2)[\cos 2s+\cos\alpha\cos s]$:

$$N_{t,\alpha\alpha}-m\,D_{\alpha\alpha}=0\quad(m=-\tfrac{1}{2}\sin(t/2)),\qquad N_{t,ss}-m\,D_{ss}=2\sin(t/2).$$

Also exact: $2C+1=4c_0^2-1$ and $C\,c(t)=2c_0-1/(2c_0)$ for $c(t)=(4c_0^2-1)/(2c_0C)$.

Proposition 21 (verified; the second-order laws)**.** *The frozen-kernel expansion built on Lemma 20 (with $\langle s^2\rangle=1/(\beta c_0)+O(\beta^{-2})$ from (11)) predicts*

$$E=C\Big(1-\frac{c(t)}{\beta}\Big)+O(\beta^{-2}),\qquad E'=-\tfrac{1}{2}\sin(t/2)+\frac{\sin(t/4)(4c_0^2+1)}{8\beta c_0^2}+O(\beta^{-2}).\tag{12}$$

The coefficient identities are exact (Lemma 20); the laws with remainder *are verified, not yet certified: measured remainders are $\approx 0.15/\beta^2$ with true convergence at fixed t (e.g. β^2-scaled residues* 0.1776/0.1767/0.1762 *at $t=1$, $\beta=30/60/120$), and $\beta\,(E'+\frac{1}{2}\sin(t/2))\to\sin(t/4)(4c_0^2+1)/(8c_0^2)$ at $t=1,2$. The remainder control is exactly what the closure map owes. Note the $1/\beta$ correction to E' is* positive*: it opposes the target inequality and must be beaten, not used. The α-cancellations imply every kernel correction multiplies a vanishing fluctuation and enters at $O(\beta^{-2})$ only.*

Remark 22 (exact decomposition of E'). With the normalized positive measure $d\mu\propto K\,ds\,d\alpha$, $E'=\langle N_t\rangle/\langle D\rangle+\langle(N-ED)\,\partial_t\log K\rangle/\langle D\rangle$, $D_t=0$, and the regular form $\partial_t\log K=4cc'(1-P-Q)\,[zI_0(z)/I_1(z)-2]/R^2$, $z=2\beta R$, extends continuously by $4\beta^2cc'(1-P-Q)$ at $R=0$. Verified split at $(\beta,t)=(5,1.3)$: $-0.238212-0.024552=-0.262764=$ the finite difference exactly. Bounds for E' are taken from this exact quotient — never by differentiating an estimate of E.

7 The π-endpoint package

Lemma 23 (exact; telescoping)**.** *With a_m as in (1):*

$$A_\infty := \sum_{m\ge1}(-1)^{m+1}m\,a_m = 0, \qquad \sum_{m\ge1}(-1)^{m+1}m^3 a_m = -\sum_{m\ge1}(-1)^{m+1}m(m{+}1)(2m{+}1)\,I_m^2 I_{m+1}^2.$$

Hence $F_A(\pi-\varepsilon) = c_3\,\varepsilon^3 + O(\varepsilon^5)$ with

$$c_3 = \frac{1}{6}\sum_{m\ge1}(-1)^{m+1}m(m+1)(2m+1)\,I_m^2 I_{m+1}^2. \tag{13}$$

Proof. Split $m^r a_m = m^r(m-1)I_{m-1}^2 I_m^2 + m^r(m+1)I_m^2 I_{m+1}^2$ and shift $m\to m+1$ in the first sum: for $r=1$ the two weights cancel ($m(m+1)-m(m+1)=0$); for $r=3$ they combine to $m(m+1)[m^2-(m+1)^2] = -m(m+1)(2m+1)$. Absolute convergence justifies the reindexing; $\sin(m(\pi-\varepsilon)) = (-1)^{m+1}\sin(m\varepsilon)$ expands the sine. □

Lemma 24 (exact; integral form and parity)**.** *Let $g(u) = \frac{1}{2}I_1(2\beta\cos u) = \sum_{m\ge0} I_m I_{m+1}\cos((2m+1)u)$* (addition theorem)*. Then*

$$c_3 = \frac{S_3-S_1}{24}, \qquad S_1 = \frac{2}{\pi}\int_0^\pi g(\tfrac{\pi}{2}-u)\,g'(u)\,du, \quad S_3 = \frac{2}{\pi}\int_0^\pi g'(\tfrac{\pi}{2}-u)\,(-g''(u))\,du.$$

Moreover $g(-u)=g(u)$ and $g(\pi-u)=-g(u)$ (parity of I_1), hence $g'(\pi-u)=g'(u)$, $g''(\pi-u) = -g''(u)$, and both *integrands are invariant under $u\mapsto\pi-u$: the two halves $[0,\pi/2]$ and $[\pi/2,\pi]$ contribute exactly equally, and one may integrate a half range and double. In particular no cancellation between the two phase maxima $u=\pi/4$ and $u=3\pi/4$ is possible — exactly, not asymptotically.*

Proof. The expansion of g is the Graf addition theorem with unit shift. Quarter turn: $\cos((2m+1)(\frac{\pi}{2}-u)) = (-1)^m\sin((2m+1)u)$, so $g(\frac{\pi}{2}-u) = \sum_m(-1)^m I_m I_{m+1}\sin((2m+1)u)$ while $g'(u) = -\sum_k(2k+1)I_k I_{k+1}\sin((2k+1)u)$ and $g''(u) = -\sum_k(2k+1)^2 I_k I_{k+1}\cos((2k+1)u)$, whence $g'(\frac{\pi}{2}-u) = -\sum_k(-1)^k(2k+1)I_k I_{k+1}\cos((2k+1)u)$. Orthogonality of $\{\sin((2m+1)u)\}$ and $\{\cos((2m+1)u)\}$ on $[0,\pi]$ (norm $\pi/2$) gives $S_1 = -\sum_m(-1)^m(2m+1)(I_m I_{m+1})^2$ and $S_3 = -\sum_m(-1)^m(2m+1)^3(I_m I_{m+1})^2$, so

$$\begin{aligned}\frac{S_3-S_1}{24} &= \sum_{m\ge0}(-1)^{m+1}\frac{(2m+1)^3-(2m+1)}{24}(I_m I_{m+1})^2\\ &= \frac{1}{6}\sum_{m\ge1}(-1)^{m+1}m(m+1)(2m+1)I_m^2 I_{m+1}^2 = c_3,\end{aligned}$$

using $(2m{+}1)^3-(2m{+}1) = 4m(m{+}1)(2m{+}1)$ and the vanishing of the $m=0$ term. The parity claims follow from $I_1(-z) = -I_1(z)$ applied to $g(\pi-u) = \frac{1}{2}I_1(-2\beta\cos u)$, and by differentiating the two symmetry relations. □

Proposition 25 (verified; prefactor law)**.** *$c_3>0$ for $\beta\in\{1,5,20,40,80,120\}$ at cancellation-adjusted precision* (the alternating sum cancels at scale $e^{-(4-2\sqrt{2})\beta}$; evaluations use dps $\ge 0.6\beta+50$)*, and*

$$c_3 \;\sim\; A\,\beta^{3/2}e^{2\sqrt{2}\beta}, \qquad A = \frac{1}{24\cdot 2^{1/4}\pi^{3/2}} = 0.0062922\ldots,$$

with measured ratios 0.0058646, 0.0060049, 0.0060996, 0.0061475 *at* $\beta = 40, 60, 90, 120$ *and* $O(1/\beta)$ *gaps. The certified large-*β *branch (pending) must produce the signed prefactor with an explicit* B*:* $I_{\mathrm{half}} \geq \frac{A}{2}\beta^{3/2}e^{2\sqrt{2}\beta}(1 - B/\beta) - T(\beta)$*,* $c_3 = 2I_{\mathrm{half}}$*, read* β^**, and close* $[15, B_{\mathrm{fin}}]$*,* $B_{\mathrm{fin}} = \max\{\beta^*, 10C\}$*, by intervals.*

8 The closure map and the candidate threshold

Proposition 26 (verified truth map; all statements on the stated finite grids)**.** *Let* $g(t,\beta) = E' + \frac{1}{4}\sin(t/2)$*. At* $\beta = 15$*,* $g < 0$ *at every point of a* 13*-point grid spanning* $[0.05, \pi - 0.09]$ *(independently reproduced on a much finer mesh at* 110 *digits), with thinnest margin* -0.00573 *at* $t = 0.05$ *(on the seal window* $[0.05, \pi - 0.10]$*) and* -0.00216 *at* $\pi - 0.09$*; the crossing sits at* $\pi - t^* = 0.0890502225$*. At* $\beta = 12$ *the truth itself fails at* $\pi - 0.10$ *(*$g = +0.025$*); crossings scale as the mirror window:* $\pi - t^* = 0.1155, 0.0891, 0.0644$ *at* $\beta = 12, 15, 20$*, matching* $\sqrt{2}/\beta$*. A sweep* $\beta = 15 \ldots 60$ *keeps the maximum at* $t = 0.05$*, drifting to* -0.0061*: strong uniformity evidence, not proof.*

Conjecture 27 (quantified closure; the seal)**.** *Part* (ii) *of Conjecture 1 holds with the two-scale decomposition:* (a) $0 < \beta \leq 3$ *by Theorem 7;* (b) $\beta \in [3, 15]$ *by certified evaluation* (the holonomic structure of Section 9 makes the compact cheap)*;* (c) $\beta \geq 15$ *by three uniform blocks: left edge* $(0, 0.05]$ *(*$e_2 > 0$*, proved by the Chebyshev-sum corollary of the Turán lemma, plus a remainder radius* 0.05*); bulk* $[0.05, \pi - 0.10]$ *(enclosure of* $\langle s^2 \rangle$ *and of the covariance of Remark 22, with the five prototype-to-certificate conversions); right edge with* moving *boundary: assembly to* $\pi - C/\beta$*, then the quadratic regime of* c_3 *on* $[\pi - C/\beta, \pi]$ *with* $C_{\mathrm{mirror}} \leq C \leq C_{\mathrm{quad}}$ *(a fixed quadratic window is provably impossible:* $\kappa \approx 0.11\beta$ *grows while* $E(\pi - 0.10)$ *stays bounded; retention* $E/\kappa d^2$ *at* $d = 2/\beta$ *is* 0.896, 0.892, 0.890 *at* $\beta = 20, 40, 80$ *— the validity radius scales as* $1/\beta$*). The named constants still owed:* B*,* β^**,* C_{quad}*,* C_{mirror}*, and the remainder function* $T(\beta)$*.*

The first-order competition is not the obstacle: the two-term law (12) requires only $\beta > (1 + 1/(4c_0^2))/c_0 \leq 3\sqrt{2}/2 \approx 2.121$; everything above that is rigor slack. The candidate $\beta_0 = 15$ is near truth-optimal for its edge window (Proposition 26), not padded.

9 Verification, certificates, and reproducibility

All numerical artifacts live in the repository

https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME

(directories `scripts/`, `docs/`); the immutable commit hash for the version audited with this manuscript is recorded in the repository's hash registry (`docs/SURFACE-CLOSURE-NOTES.md`) and in the arXiv comment field on submission. Each certificate embeds its pinned statement and self-verifies on every execution.

Certificates (interval arithmetic, outward rounding, nested or stability passes). `certify_time3_negative.py` and its Arb twin (Proposition 12); `certify_bridge_matrix.py` and twin (Proposition 13); `certify_thmB.py` (Theorem 7: adaptive β-boxes covering $[1/20, 3]$; tails computed in intervals and closed at the level of *complete shells*: each pair maps into the next shell with term ratio r derived from the termwise-exact $I_{n+1}(x) \leq xI_n(x)/(2(n+1))$, and the one new pair per shell is bounded by a double index shift, giving shell ratio $2r$, asserted $< \frac{1}{2}$ per box before the geometric closure; a built-in full stability pass at precision +70; the Arb twin `certify_thmB_arb.py` independently certifies the full coverage — 86 boxes, both passes, transcript archived; the `mpmath.iv` canonical transcript is committed on completion of its run, as recorded in the repository hash registry). Bessel enclosures use the positive series with an exact-integer stopping rule and geometric tails; floats appear only in loop-termination heuristics. Nested enclosures prevent silent numerical or truncation degradation; statement auditing and the independent Arb implementation guard against common-mode errors.

Verified computations. The master-formula and reduction identities (ratios 1.0 to 12 digits); the truth map of Proposition 26 (dps 80, confirmed independently at 110 digits); the c_3 ladder (Proposition 25); the term-split coefficients of (12) at $\beta = 60, 120, 240$.

Precision policies (each traceable to a caught artifact): dps $\geq 2.2\beta + 40$ for any evaluation with $t \geq 2$, $\beta > 30$ (mirror cancellation); dps $\geq 0.6\beta + 50$ for c_3-type alternating sums; series cutoffs scale with the maximal argument, and scaled-asymptotic evaluation of $e^{-z}I_n(z)$ is the default for $z > 35$; every measured coefficient carries a β-scaling sanity check; whatever can be computed from a positive integral representation is.

Audit trail. Fifteen numerical or formulation artifacts ("ghosts") were caught during fabrication and writing by the three-way review protocol; Appendix B tabulates them. We consider the table part of the result: it is the reproducibility record that makes the certificates credible.

Machine-checked lemmas. The unit-step and Turán-type coefficient inequalities behind Theorem 7 are formalized in Lean 4/Mathlib (`PhiMonotone.lean`, `FHBesselAmos.lean`, `WronskianIdentity.lean`; axiom oracle `[propext, Classical.choice, Quot.sound]`, no `sorry`). Mathlib has no Bessel functions: what is machine-checked is the ordered-field implication, with the analytic inputs (three-term recurrence [4, 10.29.1]; the Amos-type bound of [1, 2, 3], cited verbatim, shift $\nu \to \nu + 1$ as in Appendix A) carried as named hypotheses. Correct claim everywhere: *machine-checked modulo classical Bessel inputs carried as named hypotheses.*

Acknowledgements and provenance

The conjecture, the bridge reading, the reduction, and all certificates were produced inside an audit-first programme in which three independent reviewing voices verify every claim before acceptance; scores, objections, and the complete ghost ledger are archived with the repository history. Nothing in this paper rests on trust in any single computation: every load-bearing negative has two witnesses, and every identity marked exact has a symbolic-zero transcript.

A The weighted Turán-type lemma, self-contained

Theorem 28 (exact)**.** *For every $x > 0$ and every real $m \geq 1$, with $\varphi_m = a_m/b_m$ built from (1) at argument x: $\varphi_m < \varphi_{m+1}$. Equivalently, with $\rho_\mu = I_{\mu+1}/I_\mu$,*

$$\frac{1}{m}\Big(\frac{m-1}{\rho_{m-1}^2} + (m+1)\rho_m^2\Big) < \frac{1}{m+1}\Big(\frac{m}{\rho_m^2} + (m+2)\rho_{m+1}^2\Big).$$

Proof. Two classical inputs, valid for all real orders: the three-term recurrence $\rho_{m-1}^{-1} = \rho_m + 2m/x$, $\rho_{m+1} = \rho_m^{-1} - 2(m+1)/x$ [4, 10.29.1], and the Amos-type bound

$$\rho_m(x) < U_m(x) := \frac{x}{a + \sqrt{a^2 + x^2}}, \qquad a = m + \tfrac{1}{2},$$

proved best of its form for all real orders ≥ 0 by Segura [2]; we use [3, Thm. 2], which bounds $I_\nu(x)/I_{\nu-1}(x)$ for $\nu \geq \frac{1}{2}$ by $x/(\nu - \frac{1}{2} + \sqrt{(\nu+\frac{1}{2})^2 + x^2}\,)$; setting $\nu = m+1$ bounds $\rho_m = I_{m+1}/I_m$ by $x/(m + \frac{1}{2} + \sqrt{(m+\frac{3}{2})^2 + x^2}\,) \leq x/(a + \sqrt{a^2+x^2}\,)$ with $a = m + \frac{1}{2}$, which is the displayed bound (the last step uses $\sqrt{(m+\frac{3}{2})^2 + x^2} \geq \sqrt{a^2+x^2}$). Substitute the recurrences, write $u = \rho_m \in (0,1)$, $c = 1/x$, $S = u + 1/u$, $P = 1/u - u$; clearing the positive denominators, the claim becomes $\Delta_m > 0$ where the *exact factorization*

$$\frac{\Delta_m}{2m(m+1)} = (S - 3c)\big(P - (2m+1)c\big) + (2m+1)c^2$$

holds as an algebraic identity (expand and collect; the coefficients of $u^{\pm 2}$ are $\pm 2m(m+1)$, the linear-in-c terms are $-4m(m+1)c[(m-1)u + (m+2)/u]$, and $(m-1)u + (m+2)/u = \frac{1}{2}[(2m+1)S + 3P]$, $1/u^2 - u^2 = PS$). The second factor is positive because U_m is the positive root of $1/w - w = (2m+1)/x$ (the calibration $1/U_m - U_m = 2a/x$) and $w \mapsto 1/w - w$ is strictly decreasing, so $\rho_m < U_m$ gives $P > (2m+1)c$. The first factor is positive because $\sqrt{a^2+x^2} > a$ gives $U_m < x/(2a)$, so $u < x/(2m+1) \leq x/3$ and $S > 1/u > (2m+1)/x \geq 3c$. The third term is positive. Strictness is strict throughout. (This is the proof of [6], reproduced so that Theorem 7 is self-contained; the ordered-field implication is machine-checked in `PhiMonotone.lean`, modulo the two classical inputs carried as named hypotheses.) □

B Audit table (the ghost ledger)

Each row: what was wrong, how it was caught, and the rule it produced. All entries are documented with transcripts in `docs/SURFACE-CLOSURE-NOTES.md`.

#	artifact	catch / rule produced
1	random TP_2 scan missed adversarial corners	targeted corner search; adversarial grids mandatory
2	verification cell with unbound parameter	explicit binding rule for inline prints
3	near-π scan at low precision, false violation	precision rule dps $\geq 2.2\beta + 40$ for $t \geq 2$, $\beta > 30$
4	stale-PDF claim misattributed between voices	provenance ledger: attribute to exact source
5	float64 CDF increments ± 11 (impossible)	positive-representation preference
6	t=2, β=240 residue 0.363 vs 0.317	boundary $t \geq 2$ included in the dps rule
7	I_1 series truncated at 60 terms, $z \leq 480$	cutoffs scale with argument; scaled asymptotics for $z > 35$; β-scaling checks
8	c_3 sign flip at $\beta = 120$ (cancellation)	alternating-sum rule dps $\geq 0.6\beta + 50$
9	$R = 0$ corner misplaced at $(1, 1)$	complete critical-point census required in prose too
10	one-saddle prefactor off by exactly 2	census rule; the factor 2 later proved an exact parity
11	I_1 lower bound dropped the negative tail	sign audits are a separate mandatory pass
12	fixed quadratic window at π was provably false	interrogate obligations before attempting them
13	Thm. A second proof: rearrangement misquoted	layer-cake arc proof substituted (this revision)
14	Lemma sign error: floor stated as upper bound	corrected; per-pair statements now carry signs explicitly
15	"exact" labels on verified inequalities	tricotomy re-audit of every label (this revision)

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