An Elementary Recurrence–Amos Proof of the Unit-Step Order-Monotonicity of $(\log I_\nu)'$, with a Feynman–Hellmann Application to Two-Dimensional Lattice Gauge Theory

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Abstract

Let I_ν denote the modified Bessel function of the first kind and, for $x > 0$, let $\rho_\nu(x) = I_{\nu+1}(x)/I_\nu(x)$. We give a four-step, fully elementary proof of the sharp difference inequality

$$0 \;<\; \rho_\nu(x) - \rho_{\nu+1}(x) \;<\; \frac{1}{x}, \qquad x > 0,\; \nu \geq 0,$$

whose right-hand inequality is exactly the strict increase of the logarithmic derivative $(\log I_\nu)'(x)$ under the unit shift $\nu \mapsto \nu + 1$; consequently $\nu \mapsto (\log I_\nu)'(x)$ is strictly increasing along every unit-spaced grid $\nu_0 + \mathbb{N}$, $\nu_0 \geq 0$ — in particular on the integer and half-integer orders that arise in the application below. The stronger statement that $(\log I_\nu)'(x)$ increases in the *continuous* order ν is known — it is Lemma 10 of Freitas and Laugesen [7], proved there through the Mittag–Leffler expansion of xI_ν'/I_ν over Bessel zeros — and we make no elementary claim about fractional steps. The interest of the proof given here is that it uses no information about Bessel zeros: it combines the three-term recurrence with the classical Amos-type upper bound $\rho_\nu < x/\big(a + \sqrt{a^2 + x^2}\big)$, $a = \nu + \frac{1}{2}$ [1, 3], and rests on the observation that this bound is *exactly calibrated* for the problem: writing U for the bound, $1/U - U = 2a/x$ is an algebraic identity, and $(2\nu + 1)/x$ is precisely the threshold the unit step requires. In fact the unit-step monotonicity and the Amos bound are equivalent. As an application we record the following consequence in two-dimensional lattice gauge theory: for the Wilson action, every mass gap between character sectors of the 2D transfer operator — for $U(1)$ and $SU(2)$ alike, where only integer-spaced Bessel orders occur — is a strictly decreasing function of the bare coupling β, by the Feynman–Hellmann identity. The algebraic core of the proof (the calibration lemma and the unit-step chain) is machine-checked in Lean 4/Mathlib — no `sorry`, axiom oracle exactly Lean's three standard axioms — and we report an independent high-precision numerical audit of every inequality used.

1 Introduction

The ratios $\rho_\nu(x) = I_{\nu+1}(x)/I_\nu(x)$ of modified Bessel functions of consecutive orders appear throughout applied analysis, probability and statistical physics, and a substantial literature is devoted to bounding them by algebraic functions of ν and x [1, 2, 3, 4, 5, 6]. This note concerns the behaviour under order shifts of the logarithmic derivative

$$\frac{\partial}{\partial x} \log I_\nu(x) \;=\; \frac{I_\nu'(x)}{I_\nu(x)}.$$

Theorem 1 (unit-step order-monotonicity)**.** *For every $x > 0$ and every real $\nu \geq 0$,*

$$\frac{I'_{\nu+1}(x)}{I_{\nu+1}(x)} - \frac{I'_{\nu}(x)}{I_{\nu}(x)} \;=\; \frac{1}{x} - \big(\rho_\nu(x) - \rho_{\nu+1}(x)\big) \;>\; 0. \tag{1}$$

Equivalently, the sharp two-sided bound

$$0 \;<\; \rho_\nu(x) - \rho_{\nu+1}(x) \;<\; \frac{1}{x} \tag{2}$$

holds. The two sides have different provenance: the left inequality is the classical Turán-type inequality [8], which we import; the elementary content of this note is the right *inequality, i.e. the unit-step monotonicity* (1)*. Consequently, for every $\nu_0 \geq 0$ the map $k \mapsto I'_{\nu_0+k}(x)/I_{\nu_0+k}(x)$ is strictly increasing on $k \in \mathbb{N}$; in particular $\nu \mapsto I'_\nu(x)/I_\nu(x)$ is strictly increasing on the nonnegative integers and on the positive half-odd-integers.*

Remark 2 (scope, and relation to the known continuous statement). The stronger statement — $\nu \mapsto I'_\nu(x)/I_\nu(x)$ strictly increasing in the *continuous* order on $[0, \infty)$ — is Lemma 10 of Freitas and Laugesen [7], established as a technical ingredient in the spectral geometry of the Robin Laplacian via the infinite-product (Mittag–Leffler) expansion $xI'_\nu/I_\nu = \nu + 2\sum_{m\geq 1} x^2/(x^2 + j^2_{\nu,m})$ and the increase of the Bessel zeros $j_{\nu,m}$ in ν. We emphasize that Theorem 1 does *not* recover this: unit-step increase along every grid does not imply increase between grids, and the fractional-step inequality $\rho_\mu - \rho_\nu < (\nu - \mu)/x$, $0 < \nu - \mu < 1$, does not follow from our method. Indeed it is asymptotically saturated — $\rho_\nu(x) = 1 - (2\nu+1)/(2x) + O(x^{-2})$, so $\rho_\mu - \rho_\nu = (\nu-\mu)/x + O(x^{-2})$ — which rules out any proof by splitting the difference across the first-order Amos-type upper/lower pair: already at order $1/x$ the split bound overshoots by $1/(2x)$. Closing the fractional step elementarily would need two-sided bounds sharp to $O(x^{-2})$ uniformly in ν, e.g. those of [5, Thm. 5], at the price of the four-line character of the present proof; we do not pursue it. For the application in Section 3 the grid statement is exactly what is needed, since only integer-spaced orders occur.

The purpose of this note is threefold.

First, the proof of Theorem 1 is completely elementary — four steps, using only the three-term recurrence, the derivative identity, and one classical Amos-type bound. No properties of Bessel zeros, no integral representations, and no Sturm theory are needed. The engine is a curious exact calibration (Lemma 3): the classical upper bound

$$\rho_\nu(x) \;<\; U_\nu(x) := \frac{x}{a + \sqrt{a^2 + x^2}}, \qquad a = \nu + \tfrac{1}{2},$$

which goes back to Amos [1] and was proved best of its form by Segura [3, 5], satisfies the *algebraic identity*

$$\frac{1}{U_\nu(x)} - U_\nu(x) \;=\; \frac{2a}{x} \;=\; \frac{2\nu+1}{x},$$

and $(2\nu + 1)/x$ is exactly the threshold at which the unit step closes. The bound is, in this precise sense, machined for the lock; in fact the two statements are equivalent (Remark 4).

Second, the proof yields the upper bound of (2) directly in quantitative form, with an explicit strict margin at every (ν, x) inherited from the strict Amos margin $U_\nu - \rho_\nu > 0$; the lower bound (positivity of the difference) is not reproved here but imported from the Turán literature [8].

Third, we record a consequence in two-dimensional lattice gauge theory (we are not aware of an explicit statement in the literature, though it may well be folklore): by the Feynman–Hellmann identity — exact here, because the 2D transfer operator is diagonal in a β-independent character basis — the grid case of Theorem 1 is precisely the strict decrease in the bare coupling β of every mass gap between character sectors, for the $U(1)$ and $SU(2)$ Wilson actions simultaneously (Section 3). We emphasize the modest scope: this is a statement about the exactly soluble two-dimensional theory; nothing here is claimed about four-dimensional gauge theory.

Section 4 presents the machine-checked Lean 4/Mathlib verification of the algebraic core (no `sorry`; standard axioms only). Section 5 reports an independent numerical audit.

2 The elementary proof

Throughout, $x > 0$, $\nu \geq 0$, I_ν is the modified Bessel function of the first kind, all values $I_\nu(x)$ are strictly positive, and $\rho_\nu(x) = I_{\nu+1}(x)/I_\nu(x) \in (0, 1)$. We use two classical facts, valid for all real orders (see [11, 10.29.1–10.29.2]):

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_\nu(x), \tag{3}$$

$$I'_\nu(x) = \tfrac{1}{2}\big(I_{\nu-1}(x) + I_{\nu+1}(x)\big). \tag{4}$$

Step 1: a closed form for the logarithmic derivative

Dividing (4) by I_ν and using (3) in the form $I_{\nu-1}/I_\nu = \rho_\nu + 2\nu/x$:

$$\frac{I'_\nu(x)}{I_\nu(x)} = \frac{1}{2}\left(\frac{I_{\nu-1}}{I_\nu} + \frac{I_{\nu+1}}{I_\nu}\right) = \frac{1}{2}\left(\rho_\nu + \frac{2\nu}{x} + \rho_\nu\right) = \rho_\nu(x) + \frac{\nu}{x}. \tag{5}$$

Step 2: reduction to a ratio inequality

By (5), for the unit step $\nu \to \nu + 1$,

$$\frac{I'_{\nu+1}}{I_{\nu+1}} - \frac{I'_\nu}{I_\nu} = \rho_{\nu+1} - \rho_\nu + \frac{1}{x},$$

which is (1). Writing the recurrence (3) at order $\nu+1$ as $\rho_{\nu+1} = 1/\rho_\nu - 2(\nu+1)/x$, the inequality $\rho_\nu - \rho_{\nu+1} < 1/x$ is equivalent to

$$\frac{1}{\rho_\nu(x)} - \rho_\nu(x) \; > \; \frac{2\nu + 1}{x}. \tag{6}$$

Step 3: the Amos-type bound

We use the classical upper bound (Amos [1]; proved best of its form and valid for all $\nu \geq 0$ by Segura [5, Thm. 2], after the index shift $\nu \mapsto \nu + 1$):

$$\rho_\nu(x) \; < \; U_\nu(x) := \frac{x}{a+s}, \qquad a = \nu + \tfrac{1}{2}, \quad s = \sqrt{a^2 + x^2}. \tag{7}$$

Step 4: the exact calibration

Lemma 3 (calibration)**.** *Let* $a, x > 0$ *and* $s = \sqrt{a^2 + x^2}$, $U = x/(a + s)$. *Then*

$$\frac{1}{U} - U = \frac{2a}{x},$$

and for every t *with* $0 < t < U$,

$$\frac{1}{t} - t \; > \; \frac{2a}{x}.$$

Proof. Since $(s - a)(s + a) = s^2 - a^2 = x^2$, we have $U = x/(a + s) = (s - a)/x$, hence

$$\frac{1}{U} - U = \frac{a + s}{x} - \frac{s - a}{x} = \frac{2a}{x}.$$

The map $t \mapsto 1/t - t$ is strictly decreasing on $(0, \infty)$, so $t < U$ gives $1/t - t > 1/U - U = 2a/x$. ☐

Applying Lemma 3 with $t = \rho_\nu(x) < U_\nu(x)$ and $a = \nu + \frac{1}{2}$ yields (6) strictly, hence (1) and (2); the grid statements follow by summing the strict unit-step increments. ☐

Remark 4 (equivalence with the Amos bound)*.* The chain of Step 2 consists of equivalences: since $t \mapsto 1/t - t$ is a strictly decreasing bijection of $(0, \infty)$ onto $\mathbb{R}$, and U_ν is precisely the positive root of $1/t - t = (2\nu + 1)/x$ (Lemma 3), the inequality (6) is *equivalent* to $\rho_\nu < U_\nu$. Therefore: *the unit-step order-monotonicity* (1) *is equivalent to the classical Amos upper bound* (7). Any of the many proofs of the latter [1, 3, 4, 5] yields the former, and conversely the continuous monotonicity of Freitas–Laugesen [7] yields, in particular, a Bessel-zero proof of the Amos bound. The bound U_ν is also recognizable as the b_0-type bound of [3]: the positive root of the Riccati-derived quadratic $t^2 + \frac{2\nu+1}{x} t - 1 = 0$.

3 Application: bare-coupling monotonicity of all 2D sector gaps

Consider lattice gauge theory on a two-dimensional rectangular lattice with Wilson action and compact structure group G, at bare coupling $\beta > 0$. We fix notation carefully, separating the three objects involved.

(a) Character coefficients. The one-plaquette Boltzmann weight expands as (see, e.g., [12, 13])

$$e^{\beta \operatorname{Re} \chi_f(U)} \; = \; \sum_r d_r \, c_r(\beta) \, \chi_r(U), \qquad c_r(\beta) = \frac{1}{d_r} \int_G \overline{\chi_r(U)} \, e^{\beta \operatorname{Re} \chi_f(U)} \, dU,$$

the sum over irreducible representations r of G, $d_r = \chi_r(e)$. For the two classical cases ($n \in \mathbb{Z}_{\geq 0}$, $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$; verified independently to 30 digits in the audit of Section 5):

$$G = U(1): \quad c_n(\beta) = I_n(\beta), \qquad\qquad G = SU(2): \quad c_j(\beta) = \frac{I_{2j+1}(2\beta)}{\beta},$$

the $SU(2)$ formula being the statement that the Haar class integral equals $\int_G \chi_j(U) \, e^{\beta \chi_f(U)} \, dU = (2j + 1) \, I_{2j+1}(2\beta)/\beta$, i.e. $d_j \, c_j(\beta)$ with $d_j = 2j + 1$ (all $SU(2)$ characters are real, so $\operatorname{Re} \chi_f = \chi_f = 2 \cos \theta$).

(b) Transfer eigenvalues. In two dimensions, gauge fixing reduces the theory to a product of independent plaquette weights, and the transfer operator (one lattice row, temporal gauge) is diagonal in the character basis $\{\chi_r\}$ — the standard exact solubility of 2D lattice gauge theory

[12, 9]; see [10] for a textbook treatment of the character expansion — a basis independent of β — with eigenvalue in sector r proportional to $c_r(\beta)$, the proportionality constant being a sector-independent normalization. Ratios of transfer eigenvalues between sectors are therefore exactly $c_r(\beta)/c_{r'}(\beta)$, with every normalization convention cancelling.

(c) Sector gaps. For two sectors with $c_r(\beta) > c_{r'}(\beta) > 0$ (in both models the coefficient is strictly decreasing in the sector label, so "heavier" sectors have smaller coefficients), the mass gap of sector r' relative to r is

$$m_{r,r'}(\beta) \;=\; \log \frac{c_r(\beta)}{c_{r'}(\beta)} \;>\; 0,$$

e.g. for $U(1)$ the fundamental string tension is $m_{0,1} = \log(I_0/I_1)$ and the $v87$ bracket instance is $m_{1,2} = \log(I_1/I_2)$. Because the eigenbasis is β-independent, $\frac{d}{d\beta} m_{r,r'}$ acts on the eigenvalues alone — the Feynman–Hellmann identity in its exact, diagonal form; no perturbation theory is involved.

Corollary 5. *For 2D Wilson lattice gauge theory:*

(i) $G = U(1)$*: for all integers* $0 \le n < n'$*,*

$$\frac{d}{d\beta}\, m_{n,n'}(\beta) = \frac{d}{d\beta}\,\log \frac{I_n(\beta)}{I_{n'}(\beta)} = (\rho_n - \rho_{n'})(\beta) - \frac{n'-n}{\beta} \;<\; 0.$$

Every sector gap — in particular the string tension $\log(I_0/I_1)$ *and the bracket instance* $\log(I_1/I_2)$ *— is strictly decreasing in* β*.*

(ii) $G = SU(2)$*: for all* $0 \le j < j'$ *in* $\frac{1}{2}\mathbb{Z}$*,*

$$\frac{d}{d\beta}\, m_{j,j'}(\beta) = 2\left[(\log I_{2j+1})'(2\beta) - (\log I_{2j'+1})'(2\beta)\right] \;<\; 0,$$

the $1/\beta$ *prefactors cancelling in the ratio* $c_j/c_{j'} = I_{2j+1}(2\beta)/I_{2j'+1}(2\beta)$*, which is free of representation-dimension factors.*

In both cases the orders involved are integer-spaced — $n'-n \in \mathbb{N}$ *in (i),* $(2j'+1)-(2j+1) = 2(j'-j) \in \mathbb{N}$ *in (ii) — so the grid case of Theorem 1 applies directly, with no appeal to continuous-order monotonicity.*

Proof. (i) By (5), $\frac{d}{d\beta}\log(I_n/I_{n'}) = (\rho_n + n/\beta) - (\rho_{n'} + n'/\beta) = (\rho_n - \rho_{n'}) - (n'-n)/\beta$, and iterating (2) over the $n'-n$ unit steps gives $\rho_n - \rho_{n'} < (n'-n)/\beta$ strictly. (ii) Identical, at argument $x = 2\beta$ and orders $2j+1 < 2j'+1$; the chain rule contributes the factor 2. □

Remark 6 (scope). Corollary 5 is a statement about the exactly soluble two-dimensional theory. The analogous monotonicity $m'(\beta) \le 0$ for the four-dimensional mass gap on a weak-to-strong coupling corridor is an open problem (universally expected in lattice physics, covered by no theorem); the present corollary is the exact 2D instance of that bracket and nothing more. In the terminology of the audit-first programme to which this note belongs, the 2D instance is *signed*; the 4D bracket remains a named hypothesis.

4 The machine-checked algebraic core

The proof was deliberately arranged so that its arithmetic core is formalizable without any Bessel analysis, and this has been carried out: the calibration lemma and the unit-step chain are *proved* in Lean 4/Mathlib, with no `sorry` and no project axioms. The verification file `FHBesselAmos.lean` accompanies this note in the repository and was checked with `lake env lean` against Lean `4.30.0-rc2`, Mathlib commit `cd3b69b`. The axiom oracle returns, for each of the three theorems, exactly Lean's three standard axioms:

```
'amos_calibration' depends on axioms:
    [propext, Classical.choice, Quot.sound]
'unit_step_of_recurrence_and_amos' depends on axioms:
    [propext, Classical.choice, Quot.sound]
'logderiv_unit_step_increase' depends on axioms:
    [propext, Classical.choice, Quot.sound]
```

The proved signatures, verbatim:

```
theorem amos_calibration (a b t : Real)
    (ha : 0 < a) (hb : 0 < b) (ht : 0 < t)
    (hU : t < b / (a + Real.sqrt (a ^ 2 + b ^ 2))) :
    2 * a / b < 1 / t - t

theorem unit_step_of_recurrence_and_amos
    (x nu I0 I1 I2 : Real) (hx : 0 < x) (hnu : 0 <= nu)
    (hI0 : 0 < I0) (hI1 : 0 < I1)
    (hrec  : I0 - I2 = 2 * (nu + 1) / x * I1)
    (hamos : I1 / I0 <
      x / ((nu + 1/2) + Real.sqrt ((nu + 1/2) ^ 2 + x ^ 2))) :
    I1 / I0 - I2 / I1 < 1 / x
```

together with a third theorem, `logderiv_unit_step_increase`, expressing the conclusion in the closed form of Step 1: $(\rho_{\nu+1} + (\nu+1)/x) - (\rho_\nu + \nu/x) > 0$. Positivity of `I2` turned out not to be needed and is not assumed.

Honest scope of the formalization. What is machine-checked is the ordered-field implication [three-term recurrence + Amos bound ⇒ unit step], with the two analytic inputs entering as the named hypotheses `hrec` and `hamos`; these are the classical facts (3) and (7) [11, 1, 3, 5], to be discharged inside Lean if and when a modified-Bessel library exists in Mathlib. The file is archived both alongside this note and in the `lean-transfer-matrix` satellite of the programme (`https://github.com/lluiseriksson/lean-transfer-matrix`).

5 Numerical audit

All inequalities were checked independently at 50 decimal digits (mpmath), on the grid $\beta \in \{10^{-2}, \ldots, 10^4\}$ (16 values, geometrically spread), $\nu \in \{0, 1, \ldots, 29\} \cup \{50, 100, 500\}$: the closed form (5) holds to $1.4 \cdot 10^{-18}$ (central finite difference at step 10^{-20}); the Amos margin $U_\nu - \rho_\nu$ is strictly positive throughout, with minimum $2.5 \cdot 10^{-9}$ attained at $(\beta, \nu) = (10^4, 0)$ (the bound is asymptotically sharp, as it must be, being extremal of its form); the quantitative slack $1/\beta - (\rho_\nu - \rho_{\nu+1})$ is strictly positive throughout, with minimum $1.5 \cdot 10^{-8}$ at $(10^4, 1)$; and $m(\beta) = \log(I_1/I_2)$ has $m'(\beta) < 0$ at every grid point, ranging from -1.96 at $\beta = \frac{1}{2}$ to $-1.9 \cdot 10^{-5}$ at $\beta = 278$. The $SU(2)$ Haar class integral $\int_G \chi_j \, e^{\beta \chi_f} \, dU = \frac{2}{\pi} \int_0^\pi \sin\theta \, \sin((2j+1)\theta) \, e^{2\beta \cos\theta} \, d\theta$ was verified to equal $(2j+1) \, I_{2j+1}(2\beta)/\beta$ to 30 digits at $j \in \{0, 1, 2\}$, $\beta \in \{0.7, 2\}$ — hence $c_j = I_{2j+1}(2\beta)/\beta$ after dividing by $d_j = 2j+1$. The verification script and the sources of this note are archived in the programme repository:

https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME/tree/main/papers/bessel-amos-fh

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This note is part of an audit-first research programme whose records, Lean sources and numerical suites are public at https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME (machine-checked core) and the satellite repositories linked there. The author thanks the adversarial-audit process itself, which located [7] and thereby corrected the novelty claim of an earlier draft — what is new here is the proof route, the quantitative unit-step form and the application, not continuous order-monotonicity — and a referee report on that draft, which correctly insisted that the elementary argument closes the unit-step statement only; the present version claims exactly that.

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