Parity Barriers for Decoupling Inequalities:
why no comparison functional of bounded marginal order can certify uniform decoupling

Lluis Eriksson

July 2026 — v4

Abstract

For every $r \geq 1$, the uniform measure on the even-parity subset of $\{\pm 1\}^{r+1}$ is r-wise independent, yet the last coordinate has unit variance while being an a.s. function of the others. This example is classical — parity-check codes are the standard construction of k-wise independent distributions in the pseudorandomness literature [1, 2, 3, 4] — and we claim no novelty for it. What we record here is a consequence we have found useful and have not seen isolated as a statement: any "comparison functional" whose value depends only on marginal data of order $\leq r$, with constants uniform over finite measures, takes identical values on the parity measure and on the uniform product measure, and is therefore consistent with perfect decoupling on a measure where decoupling fails maximally. Hence no inequality built from bounded-order functionals whose right-hand side recognises perfect decoupling (a certification condition we make explicit) can certify uniform decoupling principles — Dobrushin-type mixing, approximate tensorisation with measure-free constants, covariance decay — on any class of measures containing the parity family. The case $r = 1$ recovers, and explains structurally, the failure of raw-oscillation/Doob and Efron–Stein-type steps found repeatedly in an adversarial audit of a constructive Yang–Mills programme: those inequalities lie in the order-1 class, and no repair within bounded-order data can succeed, because the barrier recurs at every order. Statements (a) and (b) are machine-checked in Lean 4/Mathlib *parametrically in* r (no `sorry`; standard axioms only), and the abstract certifying-barrier schema — order-r equality of the right-hand side on the pair, certification, and separation jointly exclude any uniform constant — is formalized as well (`certifying_barrier_schema`); finite `decide` instances for $r \leq 4$ and exact rational arithmetic for $r \leq 6$ serve as independent audits, and the general proof is a one-line character computation, included for completeness.

1 The classical example, and what is taken from it

Fix $r \geq 1$ and $n = r + 1$. Let u_n be the uniform measure on $\{\pm 1\}^n$ and let π_n be the uniform measure on the even-parity set $\{x \in \{\pm 1\}^n : \prod_i x_i = 1\}$. Two facts:

(a) every subset of at most r coordinates of π_n is distributed exactly as under u_n (π_n is r-*wise independent*);

(b) $f(x) = x_n$ satisfies $\mathrm{Var}_{\pi_n}(f) = 1$ and $f = \prod_{i \leq r} x_i$ π_n-almost surely.

Both are classical. The support of π_n is the parity-check code; distributions supported on codes of dual distance $> k$ are k-wise independent, and the parity example is the standard witness that $(n-1)$-wise independence does not imply independence [1, 2, 3, 4]. For completeness, the Fourier proof: $d\pi_n/du_n = 1 + \chi_{[n]}$ where $\chi_S(x) = \prod_{i \in S} x_i$; for $1 \leq |S| \leq r$, character orthogonality gives $\mathbb{E}_{\pi_n}[\chi_S] = \mathbb{E}_{u_n}[\chi_S] + \mathbb{E}_{u_n}[\chi_{[n]\setminus S}] = 0$, which is (a); (b) holds because $x_n = \prod_{i\le r} x_i$ on the support and the one-dimensional marginal of x_n is uniform by (a).

The purpose of this note is the *barrier* that (a)+(b) impose on a specific family of inequalities that arise in constructive quantum field theory, which we now formalise.

2 The barrier

Definition 1 (comparison functional of order r)**.** *Let $\mathcal{M}_n$ be the finite probability measures on $\prod_{i=1}^n E_i$ (finite alphabets E_i). A functional $\Phi_b(\mu, f)$ at site b* has order r *if it depends on (μ, f) only through the joint marginals of μ of order $\le r$ together with the single-site data of f at b (oscillation, conditional variance given order-$\le r$ marginal information). An* order-r comparison inequality *is a bound of the form $\mathcal{D}(\mu, f) \le c\sum_b \Phi_b(\mu, f)$ with c uniform over $\mathcal{M}_n$, where $\mathcal{D}$ is a decoupling functional (variance defect, covariance, martingale increment). We call the inequality* certifying *if the right-hand side recognises perfect decoupling:*

$$\sum_b \Phi_b(u, f) = 0 \qquad \textit{whenever } u \textit{ is a product measure and } f \textit{ depends on a single coordinate.} \tag{1}$$

All influence- and cross-oscillation-type functionals in the applications we know of (Dobrushin row entries, Doob cross increments, covariance-decorated variants) satisfy (1)*; a functional violating it (e.g. the Efron–Stein single-site variance proxy, whose sum stays positive under product measures) yields true but non-certifying bounds — its right-hand side can never force a decoupling conclusion.*

Theorem 1 (parity barrier)**.** *Fix $r \ge 1$.* No *certifying order-r comparison inequality holds on any class of measures containing $\{\pi_{r+1}\} \cup \{u_{r+1}\}$ for a decoupling functional $\mathcal{D}$ that separates them (i.e. $\mathcal{D}(u_{r+1}, f) = 0$ and $\mathcal{D}(\pi_{r+1}, f) > 0$ for $f(x) = x_{r+1}$). In particular no bounded-order data can certify Dobrushin-type mixing, approximate tensorisation with measure-independent constants, or covariance decay on such a class.*

Proof. Take $f(x) = x_{r+1}$. By (a), every order-r functional takes identical values on (π_{r+1}, f) and (u_{r+1}, f): $\sum_b \Phi_b(\pi_{r+1}, f) = \sum_b \Phi_b(u_{r+1}, f)$. Under u_{r+1} the coordinate f is independent of the others and f depends on a single coordinate, so the certification condition (1) gives $\sum_b \Phi_b(u_{r+1}, f) = 0$. If the comparison inequality held on the class, then $\mathcal{D}(\pi_{r+1}, f) \le c\sum_b \Phi_b(\pi_{r+1}, f) = c \cdot 0 = 0$, contradicting separation: e.g. for $\mathcal{D} = \mathrm{Var}(f) - \sum_b \mathbb{E}\,\mathrm{Var}_b(f \mid \text{rest})$ one has $\mathcal{D}(u_{r+1}, f) = 0$ but $\mathcal{D}(\pi_{r+1}, f) = 1$ by (b). □

The construction generalises to arbitrary finite abelian alphabets (take density $1 + \mathrm{Re}\,\chi$ for any full-support character χ: the same barrier at order $n - 1$), and vector-valued observables change nothing (apply the theorem coordinatewise).

Corollary 1 (the audit failures, explained)**.** *The raw-oscillation/Doob increment bound and the Efron–Stein covariance step — found, and refuted by the $r = 1$ case, in four papers of the 2602-series adversarial audit of a constructive Yang–Mills programme [7] — are order-*1 *comparison inequalities of certifying type (their right-hand sides are built from cross-site influence data vanishing under product measures); the "XOR" counterexample used there is the $r = 2$ barrier. No repair that stays within bounded-order data — including correlation-decorated variants using pair moments — can succeed: the barrier recurs at every order.*

Remark 1 (sharpness: what the barrier does *not* forbid). Inequalities whose constants consume the *conditional* structure of the measure (Dobrushin interdependence coefficients, log-Sobolev constants of the true single-site conditionals) are of unbounded order in the sense of Definition 1, and are untouched: indeed the classical positive results (Dobrushin uniqueness $\Rightarrow$ tensorisation with constant $(1-\delta)^{-1}$ [5]) are of exactly this type. The moral is a division of labour: bounded-order data can *falsify* decoupling, never *certify* it; certification must consume conditional information. We believe this is the precise structural reason why the decoupling step is the irreducible analytic core of constructive Yang–Mills programmes, and why it cannot be bypassed by moment or oscillation bookkeeping of any fixed order.

3 Machine verification

Statements (a) and (b) are finite for each r and hence decidable by exhaustive computation. They are kernel-checked in Lean 4/Mathlib via `decide` for $r = 1, 2, 3, 4$ (file `ParityBarrier.lean` accompanying this note: theorems `parity_kwise_r1`–`r4` for the vanishing of all character sums of nonempty support and order $\leq r$ over the parity set, `parity_support_r1`–`r4` for the support identity, and mean-zero statements giving the unit variance), and independently verified in exact rational arithmetic for $r \leq 6$ (script `verify_parity.py`). The Lean file was checked with `lake env lean` against Lean `4.30.0-rc2`, Mathlib commit `cd3b69b`; the axiom oracle returns Lean's three standard axioms for every theorem in the file, e.g.

```
'parity_kwise_r4' depends on axioms:
    [propext, Classical.choice, Quot.sound]
'parity_support_r4' depends on axioms:
    [propext, Classical.choice, Quot.sound]
'parity_mean_zero_r4' depends on axioms:
    [propext, Classical.choice, Quot.sound]
```

(exit code 0). Moreover the *parametric-in-r* statement is formalized in the same file: `sum_chi_eq_zero` (the full-space character sum vanishes for every nonempty support, every n, by the flip involution), `parity_kwise` (statement (a) in Fourier form for *all* n and all nonempty proper S, via $2\sum_\pi \chi_S = \sum_x \chi_S + \sum_x \chi_{[n]\setminus S} = 0$) and `parity_support` (statement (b) for all n) — each with the same standard-axiom oracle and no `sorry`. The finite `decide` instances are retained as independent kernel checks of the parametric theorems. Finally, the barrier schema itself is formalized abstractly (`certifying_barrier_schema`): from the three hypotheses [Φ order-r on the pair, i.e. $\Phi(\pi) = \Phi(u)$] + [certification, $\Phi(u) = 0$] + [separation, $\mathcal{D}(\pi) > 0$], no uniform constant exists; for the parity pair the first hypothesis is exactly what `parity_kwise` discharges. The bridge from concrete influence functionals to the order-r hypothesis remains, deliberately, a natural-language step (Definition 1).

Acknowledgements

This note is part of an audit-first research programme whose records, Lean sources and numerical suites are public at https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME. The parity example itself is classical and no priority is claimed for it; the debt to the k-wise independence literature is recorded in the references.

References

[1] A. Joffe, *On a set of almost deterministic k-independent random variables*, Ann. Probab. **2** (1974), 161–162.

[2] N. Alon, L. Babai and A. Itai, *A fast and simple randomized parallel algorithm for the maximal independent set problem*, J. Algorithms **7** (1986), 567–583.

[3] N. Alon, O. Goldreich and Y. Mansour, *Almost k-wise independence versus k-wise independence*, Inform. Process. Lett. **88** (2003), 107–110.

[4] N. Alon and J. H. Spencer, *The Probabilistic Method*, 4th ed., Wiley, 2016.

[5] R. L. Dobrushin, *The description of a random field by means of conditional probabilities and conditions of its regularity*, Theory Probab. Appl. **13** (1968), 197–224.

[6] B. Efron and C. Stein, *The jackknife estimate of variance*, Ann. Statist. **9** (1981), 586–596.

[7] L. Eriksson, *THE MASTER MAP: an audit-first navigation guide to the conditional construction of 4D $SU(N)$ Yang–Mills with mass gap*, ai.viXra:2602.0096, and the audited 2602-series; repository `https://github.com/lluiseriksson/THE-ERIKSSON-PROGRAMME`.